Some methods and results of analytic studies of the problem of three nuclear particles

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The classical and modern methods and the results of analytic studies of the structure of the stationary Schrödinger and Faddeev equations and their solutions describing states of three-particle systems with central and S-wave interactions are reviewed. Special attention is given to the angular analysis of these equations and the exact transformation laws of the three-particle coordinates, operators, basis functions, and wave functions under cyclic permutations of the particles. The problem of spurious solutions is analyzed. © *1999 American Institute of Physics.* [S1063-7796(99)00506-9]

1. INTRODUCTION

Unfortunately, in the physics literature it is not only common to encounter widely differing notation for the same quantity, an operator or a function, but also the definitions and concepts introduced are often inadequate. Therefore, the discussion of any particular problem should begin with some discussion of precisely what meaning the various terms and symbols have.

As a rule, in this review we shall use the terminology and notation of the well known textbooks[1–5] and handbooks[6–9] on various topics in mathematics, the classic quantum-mechanics text,[10] books on the theory of potential scattering[11–21] and angular momentum,[22–24] and monographs devoted to the method of hyperspherical harmonics.[25–27] All terms denoting relative concepts and all compact author notation will be explained in detail before they are used. Nearly all such notation is the same as that used in the series of previous studies[28–74] performed entirely or in part by the present author. Everywhere below, the symbol $\equiv$ will connect a symbol and its definition or the left- and right-hand sides of an identity, and vectors, matrices, and sets will be denoted by boldface and upper-case letters.

The author knows well from long experience that for many physicists the term "analytic results of studying a problem which is correctly stated mathematically" is associated only with explicit, closed expressions for the exact solutions of the problem possessing a particular physical meaning. This interpretation of the term "analytic results..." is too narrow. In mathematics, the analytic results of the analysis of a posed problem often include all the statements and symbolic expressions proved without any calculations and describing widely varying features such as the problem itself and all its solutions, and also the global and local properties of auxiliary operators and functions.

In the present review we analyze the Schrödinger and Faddeev formulations of the problem of three quantum-mechanical particles with pair interactions in the form of arbitrary superpositions of Coulomb and nuclear potentials. It is assumed that each pair interaction depends only on the relative coordinates of the particles in the pair. For brevity, this problem will be referred to as the problem of three nuclear particles.

Let us list the objects in the quantum theory of several particles whose analytic study is fundamental for the further development of the theory. A special role is played by the exact transformation laws of the operators, the basis functions, the auxiliary functions, and the wave functions in geometrical transformations of various coordinate systems. No less important are the analytical methods for projecting multidimensional dynamical equations onto various complete bases in order to decrease the number of independent variables. Such a reduction necessarily leads to matrix elements of the interactions and various types of overlap integrals between the basis functions and the desired functions. Knowledge of the analytic properties of these objects allows a significant simplification of the analysis of the reduced equations and their numerical solution. From the theoretical and practical viewpoints it is certainly interesting to study the equivalence of different formulations of the same multiparticle problem. One of the problems of this type is the existence of regular solutions of the reformulated problem which are devoid of physical meaning. Proofs of the criteria for the existence of such special solutions, their classification, and methods of exactly eliminating them are certainly important for the construction of a mathematically correct theory. Obviously, such a theory is incomplete unless it contains asymptotic methods allowing the explicit construction of complete asymptotic expansions of the wave functions in physically interesting limits of the arguments and parameters.

The main goal of the present study is the detailed description of some fundamental methods for analytically studying the above-mentioned objects in the problem of three nuclear particles and the analysis of the reliability of some of the exact results obtained by using the theory of Faddeev equations and the method of hyperspherical harmonics in coordinate space.

This choice has been made because the author has performed a series of studies[57–72,74] of the problem of three nuclear particles, using just these methods. Discussion of

other approaches to the exact solution of this problem is impossible, owing to restrictions on the size of this review, and, moreover, appears to be unnecessary, owing to the existence of other reviews and monographs. Let us mention the most important of these in order to recall the foundations of the most commonly used analytical methods and, at the same time, explain which analytical results have not been summarized and therefore need to be in the present review.

One of the most widely used methods of studying multidimensional dynamical equations for several particles is based on the representation of the desired functions and/or operators as series in suitable bases and the subsequent projection of the equations onto such bases. Complete sets of eigenfunctions of the squared angular-momentum operators contained in the free Hamiltonian are called angular bases. The problem of projecting the equations onto angular bases is frequently referred to as the angular analysis of these equations. The angular basis functions are known explicitly; they have been well studied and, by definition, do not depend on the interactions. For these reasons angular bases are used most often. By projection onto a suitable angular basis, the original equations are exactly reduced to equations with a small number of independent variables. The analytic and numerical analysis of the equations in a space of small dimension is a less complicated problem which in many cases can be solved fairly completely. An example of such a problem is the two-particle Schrödinger equation with a central (spherically symmetric) interaction.[14] This equation was originally formulated in the three-dimensional space $\mathcal{R}_x^3$ of the vectors $\mathbf{x}$ characterizing the relative positions of the two particles in some fixed coordinate system S_3. The spherical harmonics[23] or linear combinations of them are often used as the angular basis. The spherical harmonics $Y_{b\beta}(\hat{x})$ depend on the two spherical angles $\hat{x}=(\theta_x,\varphi_x)$ of the vector $\mathbf{x}$, they are the eigenfunctions of the squared angular-momentum operator $\mathbf{l}_x$, and they form a complete orthonormal basis on the unit sphere $\mathcal{S}_x^2$ in $\mathcal{R}_x^3$. This basis is therefore often termed spherical. By projection onto the spherical basis, the original three-dimensional Schrödinger equation is reduced to a system of uncoupled ordinary second-order differential equations for the unknown functions of a single variable x. Such equations are generally referred to as radial Schrödinger equations.

The general theory of ordinary differential equations[1,6,8] is quite well developed. The radial Schrödinger equation can be solved exactly for a series of model potentials by using the methods of this theory. Examples of such potentials are the Coulomb and oscillator potentials, the Morse potential, and the square well,[10] and also the family of Bargmann potentials.[18]

For most *realistic* potentials reproducing the experimental data on the two-particle spectra and phase shifts, the radial Schrödinger equations are not exactly solvable. Various approaches have been developed for studying them analytically and obtaining approximate solutions: the method of approximating local interactions by separable potentials of finite rank,[17] the variational method,[20] the method of zero-range potentials,[16] perturbation theory,[14,19] the semiclassical approximation,[4,11] and various versions[73,74] of the variable-phase method.[12,13]

The generalization of all the methods listed above for the analytic study of the three-particle problem and the development of other approaches to this problem are difficult for just two reasons: the relatively large dimension of the three-particle space and the commonly used method of describing interactions. Let us discuss these two features to see which difficulties can be overcome and which in principle cannot be.

In a coordinate system S_3 attached in $\mathcal{R}^3$ to the center of mass of the three-particle system, the relative positions of the three particles are characterized by not three coordinates, as in the case of two particles, but six coordinates. These are often taken to be the three-dimensional Jacobi vectors[17] $\mathbf{x}$ and $\mathbf{y}$ or their associated hyperspherical coordinates[1] (r,Ω): the hyperradius $r\equiv\sqrt{x^2+y^2}$ and the set of five hyperangles $\Omega\equiv(\hat{x},\hat{y},\varphi)$, where $\varphi\equiv\arctan(y/x)$. The lengths x and y are associated with the distances between the particles in the selected pair and the distance from the third particle to the center of mass of this pair. Clearly, for the coordinate description of the three-particle configuration it is necessary to introduce the six-dimensional coordinate space $\mathcal{R}^6\equiv\mathcal{R}_x^3\oplus\mathcal{R}_y^3$. The bispherical (hyperspherical) harmonics form an angular basis on the four-dimensional torus $\mathcal{T}_{xy}^4\equiv\mathcal{S}_x^2\oplus\mathcal{S}_y^2$ (the unit five-dimensional sphere $\mathcal{S}^5$) in $\mathcal{R}^6$ (Refs. 1 and 23). The bispherical harmonics $\mathcal{Y}_{ab}^{\ell m}(\hat{x},\hat{y})$ depend on four variables and are the eigenfunctions of the squared angular-momentum operators $\mathbf{l}_x$ and $\mathbf{l}_y$ conjugate to the vectors $\mathbf{x}$ and $\mathbf{y}$, and of the total angular momentum $\mathbf{l}\equiv\mathbf{l}_x+\mathbf{l}_y$. The hyperspherical harmonics (hyperharmonics) $Y_{Lab}^{\ell m}(\Omega)$ depend on five angles and are the eigenfunctions of all the squares of the above-listed operators, and also of the square of the six-dimensional angular momentum (or the hyper-angular momentum) L. The expansion of a given function in a series in bispherical harmonics (hyperharmonics) will be referred to as its bispherical (hyperspherical) series. The coefficients of such series will be called the bispherical or hyperspherical components of the expanded function. By definition, the bispherical components are functions of two arguments x, y or r, φ, and the hyperspherical components depend only on the single coordinate r. The region of allowed values of the variables x, y or r, φ is the first quadrant of the two-dimensional plane R_2^+, and the hyperradius r always belongs to the non-negative semiaxis R_1^+.

If the functions obeying the studied three-particle equations in $\mathcal{R}^6$ are replaced by their bispherical (hyperspherical) series and then the equations are projected onto the bispherical (hyperspherical) basis, exact two-dimensional (one-dimensional) equations are obtained in $\mathcal{R}_+^2$ ($\mathcal{R}_+^1$) for the unknown bispherical (hyperspherical) components. In the vast majority of cases the sets of equations for the components are infinite systems of coupled equations. The coupling is effected by integral operators, functions, and numerical coefficients.

For example, three-particle Schrödinger equations in R_+^1 are coupled by the elements of the total interaction matrix in the hyperharmonics basis,[25] and the Faddeev equations in R_+^2 are coupled also by the matrices of the pair interactions

in the bispherical basis and by integral operators.

Let us recall the main reasons for the coupling. Let (ijk) be the triad of indices labeling the three particles and taking different values 1, 2, and 3. There are exactly three groupings $i(jk)$ into a pair of particles labeled j and k and a particle labeled i. Each $(i=1,2,3)$ grouping $i(jk)$ corresponds to its own pair of Jacobi vectors $\mathbf{x}_i$, $\mathbf{y}_j$ or their associated hyperspherical coordinates (r,Ω_i). In a cyclic permutation of the indices $i(jk)\to k(ij)$ the pair $\mathbf{x}_i$, $\mathbf{y}_i$ becomes the pair $\mathbf{x}_k$, $\mathbf{y}_k$. The coordinates labeled by index i are called the proper coordinates for the grouping $i(jk)$ and improper ones for the two other groupings. The interactions in the three-particle system are *a priori* assumed to be pair interactions. In addition, it is postulated that the total interaction V is the sum of all three pair interactions V_i, and that the interactions V_i in each pair (jk) are described by a function $V_i(\mathbf{x}_i)$ specified by definition in the proper coordinates for this pair $\mathbf{x}_i$. This postulate of additive representation of the total interaction is confirmed by various experimental data. Determination of the functional dependence of the pair interactions in improper coordinates is possible in principle by experimental analysis of three-particle processes. Such experiments are rather complicated to perform, and their analysis introduces many theoretical uncertainties. Therefore, pair interactions are reconstructed experimentally in their proper coordinates and enter into the theory in this form. Obviously, if the total interaction is written down in a single system of Jacobi coordinates, it is the proper interaction for one pair interaction and an improper one for the other two. In going from the proper coordinates to improper ones, the interactions and functions usually lose many of their properties, and their functional dependence, as a rule, becomes more complicated. For example, the central interaction $V_i(x_i)$ in improper coordinates is described by a function of complicated argument $V_i(x_i(\mathbf{x}_k,\mathbf{y}_k))$ which does not possess spherical symmetry.

The interactions must therefore be assumed to be pair ones given in their proper coordinates. The development of exact methods of solving the reduced three-particle equations is thus impossible without knowledge of all the properties of the objects effecting the coupling of these equations and the methods of minimizing the couplings. A powerful trick to minimize the coupling is the simultaneous use of different coordinate systems and splitting of the desired function into auxiliary components. This trick is often used in molecular, atomic, and nuclear physics and allows a considerable simplification of the mathematical formulation of the few-particle problem, and also of its subsequent analytical and numerical study.

A clear example of this is the Faddeev splitting[21] of the three-particle T matrix into three equations for its components $T_i(\mathbf{p}_i,\mathbf{q}_i)$ written in their proper coordinates: the Jacobi momenta $\mathbf{p}_i$, $\mathbf{q}_i$ conjugate to the Jacobi coordinates $\mathbf{x}_i$, $\mathbf{y}_i$. It is this splitting and notation which allowed Faddeev to give the first mathematically correct formulation[75] of the three-particle problem with short-range pair potentials.

In the method of hyperharmonics,[25–27] the simultaneous use of three systems of hyperspherical coordinates (r,Ω_i) allows a considerable simplification and unification of the calculation of the matrix elements of pair interactions in the hyperspherical basis.[76]

The simultaneous use of several coordinate systems is not possible without knowledge of the transformation laws of the operators, the basis functions, the auxiliary functions, and the wave functions in going from one coordinate system to another. The transformations of the pair-interaction and angular-momentum operators and of the spherical functions in reflections, translations, and rotations of the coordinate systems in $\mathcal{R}^3_x$ have been carefully studied.[22,23] Explicit expressions are known for the matrix elements of many two-particle operators acting in $\mathcal{R}^3_x$, and effective algebraic methods of calculating them have been developed.[22–24] The transformations of the three-particle operators, the angular bases, and the various auxiliary functions in cyclic permutations of the particles in $\mathcal{R}^6$ (in going from one set of Jacobi coordinates to another) have not been studied in such detail. Therefore, in Sec. 2 below we give an original analysis of a kinematical transformation of the coordinates, operators, and angular bases more general than a cyclic permutation. Special attention is paid to the study of the general and local properties of the kernels of the integral operators of the Faddeev equations in $\mathcal{R}^2_+$ and the coefficients of the unitary transformation of the hyperharmonics in the transformation $(r,\Omega_i)\to(r,\Omega_k)$. It is worth noting that explicit expressions for these coefficients with $\ell=L=0,1,2,3$ were derived by Bogoslovskiĭ and Klepikov,[77] and later in the general case by Raynal and Revai.[78] The Raynal–Revai coefficients are six-fold systems containing $3j$ and $9j$ symbols and trigonometric functions of the kinematical angle γ. This complicated representation proved to be extremely inconvenient from the computational point of view. Therefore, various recursion relations,[25,79–81] systems of linear equations,[72,79,82] and integral and differential representations[65,72] were later derived for the Raynal–Revai coefficients.

However, the recursion relations are too complicated for studying the analytic properties of the Raynal–Revai coefficients as functions of the kinematical angle γ. Knowledge of these properties is necessary not only for accelerating the calculations of various matrix elements in the hyperharmonics basis[25] and for constructing functions with *a priori* specified symmetry,[81,83–86] but also for explicitly summing the various series containing Raynal–Revai coefficients. The problem of the existence of spurious[67–71] and some exact[66] solutions of the Faddeev equations reduces to the problem of determining the values of the parameter γ for which the algebraic systems of equations containing the Raynal–Revai coefficients are solvable. The problem of summing the series containing these coefficients arises when studying the asymptotic form of the three-particle wave functions and their Faddeev components in the vicinity of a triple-collision point.[64,68]

Thus, there is a wide range of problems which cannot be solved without preliminary study of the general and local properties of the Raynal–Revai coefficients as functions of the kinematical angle. These properties have been studied in Refs. 58, 65, and 72. In Sec. 2 below we discuss their main results and several original numerical algorithms.

The following section begins with a detailed and unified

description of the construction of the three-particle Schrödinger and Faddeev equations in the bispherical and hyperspherical bases and the general analytic properties of the solutions of these equations. Such a description cannot be found in the literature.

The mathematically correct formulation of the quantum problem of three particles with short-range pair potentials was first given by Faddeev in momentum space.[75] Later on, along with the Faddeev integral equations for the components of the T matrix, the Faddeev differential equations for the three components Ψ_i of the wave function $\Psi=\Psi_1+\Psi_2+\Psi_3$ in six-dimensional coordinate space $\mathcal{R}^6$ came to be widely used. The main problem in the quantum theory of three particles in coordinate space is the derivation of the physical boundary conditions in the limit of large relative separations between the particles, which guarantee the existence and uniqueness of the solutions of the Faddeev equations. An important contribution to solving this complicated problem was made by Merkur'ev and coworkers. Numerous references to these studies and alternative investigations are given in Ref. 21 and in detailed reviews.[87–91]

The Faddeev equations in $\mathcal{R}^2_+$ are integro-differential equations for the unknown bispherical components depending on the two arguments x, y or r, φ. Integro-differential equations have been studied intensively and used in practice for two reasons. The first is that the existence and uniqueness of the solutions of such equations have been established for a large class of local pair interactions[21] for both neutral and charged particles. The second is that the integral operators act only on a single variable φ. Therefore, discretization reduces the integro-differential equations to a system of linear equations with a band matrix.[59,88,92] It is the sparseness of the matrix of the linear problem in combination with simple boundary conditions that ensures the broad practical application of integro-differential equations for studying the various properties of three-particle systems. A variety of numerical algorithms have been developed for this purpose on the basis of the finite-difference approximation[88] and the approximation by cubic Hermitian splines[92–94] and cubic splines possessing continuous second derivatives.[59] The method of strong channel coupling has recently been proposed for the Faddeev equations.[95] The advantages of this approach over the method of finite-dimensional approximations of the subsystem Hamiltonians, developed in Refs. 29–38 by using the multiparticle Lippmann–Schwinger equation, is worth explaining in more detail. The method of finite-dimensional approximations for this equation has a very narrow region of applicability,[35] bounded by negative values of the total energy of the system. Moreover, calculations by this method are possible only when the pair interactions are described by finite-rank separable potentials, and the continuous spectrum of the subsystems cannot be taken into account. These three constraints are unimportant in the method of strong channel coupling for the Faddeev equations,[95] which is based on expansion of the components of the wave functions in bases of eigenfunctions of the Hamiltonians of the two-particle subsystems.[96] This method is based on the Faddeev equations and therefore possesses a number of advantages over the method of strong channel coupling for the Schrödinger equation.[14] As was noted in Ref. 95, the main ones are the absence in the final expressions of terms related to the nonorthogonality of the scattering channels,[97,98] and the possibility of rigorously including the continuum contributions of the interacting clusters.

The effectiveness of all methods of discretizing the Faddeev integro-differential equations in $\mathcal{R}^2_+$ essentially depends on the choice of the two-dimensional grid. The optimal choice is determined by the qualitative dependence of the required solutions on their arguments. Therefore, knowledge of the analytical properties, primarily the complete asymptotic expansions near all the boundaries of the region $\mathcal{R}^2_+$, is especially important also for economizing in computational algorithms. It is difficult to study the analytic properties of the bispherical Faddeev components because they depend on two variables and obey equations containing nonlocal operators. The following reduction step suggested by Mandelzweig[99] is therefore important and completely natural. It amounts to separating the variables r and φ by expanding the Faddeev bispherical components in the most convenient basis functions of the angular variable φ. The main problem in this expansion is to choose the most convenient angular basis and to study the analytic properties of the kernels of the integral operators. These problems were solved fairly completely in Refs. 58, 67, and 72. There the passage from the Faddeev equations in the bispherical basis to the Faddeev equations in the hyperspherical basis was studied. The analysis of this passage as the initial step in combining the Faddeev theory and the hyperharmonics method is performed in Sec. 3 below.

This unification opens up real possibilities for the exact solution of the Faddeev equations with arbitrary pair interactions. Among the exact solutions of the three-particle Faddeev equations with given pair interactions, a special role is played by solutions which do not carry any information about the interactions. Such nontrivial ($\Psi_i\neq 0$) solutions of the Faddeev equations correspond to the trivial wave function $\Psi\equiv 0$ and are called spurious solutions or ghost solutions. The existence of spurious solutions has been pointed out by many authors of reviews[100,101] and original studies.[102–107] The general properties of the three-particle Hamiltonian generating spurious solutions and such general questions as the completeness of the space of spurious and physical solutions have been studied in Refs. 102–105. The statement most important for the spectral theory of the three-particle Faddeev equations has been proved by Yakovlev[105] in the case of the full three-particle Hamiltonian H with a purely discrete spectrum. In this case the matrix operators corresponding to the three-particle Faddeev equations and the conjugate equations have two types of invariant space. In spaces of the first type the operator spectrum coincides with the spectrum of the full Hamiltonian, and in spaces of the second type these operators are equivalent to the free Hamiltonian.

The spurious solutions for a system of three identical particles interacting via S-wave pair potentials and located in a state with total angular momentum $\ell=0$ were found explicitly in Refs. 67, 106, and 107. The spurious solutions for the same system but in the $\ell=1$ state were referred to in

Refs. 66 and 68. For an arbitrary three-particle system with pair central potentials, the criterion for the existence of spurious solutions possessing *a priori* specified ℓ was first proved in Ref. 71. The spurious solutions for systems of three identical bosons or fermions in states with $\ell=0$ were obtained explicitly in Ref. 108.

The equations determining the spurious solutions do not contain potentials. Therefore, spurious solutions found analytically can be used as universal reference functions for testing algorithms for numerically solving the three-particle equations with arbitrary interactions. The existence of spurious solutions is sometimes erroneously treated as an inequivalence of the three-particle Schrödinger equation and the Faddeev equations corresponding to it. This idea is mainly due to the fact that the Faddeev equations are not uniquely solvable when they are not supplemented by physical boundary conditions or when they contain potentials for which the existence and uniqueness of the Faddeev boundary-value problem are not established. It is clear from these remarks that the discussion of the known properties of spurious solutions and the problems generated by these properties is extremely important and interesting. Section 4 below, devoted to exact solutions of the Faddeev equations, begins with an analysis of the general properties of spurious solutions. Exact solutions are known only for certain potentials. Examples are the harmonic oscillator[57,106] and the potential falling off as the square of the inverse distance.[66,109] The theorem of the existence and uniqueness of solutions of the Faddeev problem with pair potentials which grow at infinity or which have singularities of the centrifugal type at small distances has not been proved. Moreover, in these cases the asymptotic form of the Faddeev components at large distances is not known. Therefore, the analysis of the exact solutions of the problem of three identical bosons with oscillator and centrifugal potentials is interesting for generalizing the Faddeev theory. Another reason for including this analysis in Sec. 4 is that there are other facts just as important for nuclear physics. Let us list them.

The study of the problem of N quantum particles with pair interactions having singularities of the type αx^{-2}, where α is a coefficient and $x\to 0$ is the relative separation between the particles, is quite interesting, both for generalizing scattering theory and for the mathematically correct solution of this problem. The point is that many nucleon–nucleon (NN) potentials obtained from modern field-theoretic models of the NN interaction,[74] for example, the Bonn potential,[110] contain short-range terms with asymptotic form $\sim x^{-2}$ for $x\to 0$. However, the criterion for the existence of regular solutions of the N-particle Schrödinger equation with such singular potentials is known only in special cases. For $N=2$ such solutions exist[10] if $\alpha > -1/4$. In Ref. 109 Avishai made a numerical study of the problem of three identical bosons with total angular momentum $\ell=0$ and S-wave pair potentials αx_i^{-2}. The result was the first numerical proof of the fact that at certain values of the parameter α the Faddeev equations have regular solutions in the form of a product of a Bessel function $J_\nu(r)$ and a function of the hyperangle φ. The same three-particle problem was studied in Ref. 66, but for the case of arbitrary ℓ. The criterion for the existence of exact solutions representable as a product of a Bessel function $J_\nu(r)$ and a finite linear combination of hyperharmonics was proved in this general case. Section 4 concludes with the proof of this criterion and a comparison of the results of Refs. 66 and 109.

Just as important for the theory is knowledge of the asymptotic form of the wave functions in physically interesting limits of their parameters. One of these is the total energy. Several analytical methods of constructing low-energy expansions in the problems of two and three nuclear particles have been described in Refs. 39–56. The current status of the theory of low-energy expansions for such problems has been analyzed in earlier reviews.[73,74]

The present review is written in the same methodological spirit as the earlier ones. Special attention is given to the derivation of the most important relations, to comparison with the known results, and to discussion of their physical and mathematical meaning. In order to make this description accessible and not too tedious for a wide range of readers, we make special efforts to give simple explanatory examples, and we present the material in order of increasing difficulty, with detailed explanations. The most important statements are first formulated and then proved and discussed. This way of presenting the material seems optimal to the author, because it offers the reader the choice of acquiring only an understanding of the essence of the problem as a whole, or becoming acquainted with all the hidden obstacles in the complete proof and discussion of special cases. Each section begins with a brief overview of the contents of the subsections, except for Sec. 5, which contains conclusions and final remarks.

2. THE KINEMATICAL TRANSFORMATION IN THE THREE-BODY PROBLEM

This section is devoted to the kinematical transformation in the three-body problem more general than the well known cyclic permutation of particles. In Sec. 2.1 we introduce the Jacobi Cartesian and hyperspherical coordinates, and then we explain the geometrical meaning of their kinematical transformation in the general case and in two special cases: when all the particles are located in the coordinate plane, and when they are located on a single line. In Sec. 2.2 we use the operator methods of the theory of angular momentum to analyze systematically the kinematical transformation of functions of arbitrary form, the angular bases, and the expansions in these bases. The analysis begins in Sec. 2.2.1 with the definition of the permutation, reflection, and rotation operators, and the construction of the kinematical transformation operators. Then in Sec. 2.2.2 we list the main properties of the angular bases, and in Sec. 2.2.3 we explain the connection between addition theorems and the kinematical transformation. Finally, in Secs. 2.2.4 and 2.2.5 we describe the kinematical transformation of the bispherical and hyperspherical series.

2.1. The kinematical coordinate transformation

Let us consider a system of three particles of masses m_i, $i=1,2,3$. In three-dimensional coordinate space $\mathcal{R}^3$ we intro-

duce the Cartesian coordinate system S_3 with axes $\hat{\mathbf{e}}_i$. Let $\mathbf{a}_i$ be the radius vector of a particle labeled i in the system S_3. We introduce three systems of relative Jacobi coordinates:

$$\mathbf{x}_k \equiv \hbar^{-1}\left(\frac{2m_im_j}{m_i+m_j}\right)^{1/2}(\mathbf{a}_j-\mathbf{a}_i),$$

$$\mathbf{y}_k \equiv \hbar^{-1}\left(\frac{2m_k(m_i+m_j)}{m_1+m_2+m_3}\right)^{1/2}\left(\frac{m_i\mathbf{a}_i+m_j\mathbf{a}_j}{m_i+m_j}-a_k\right), \qquad (1)$$

where the indices i, j, and k form a cyclic permutation of the triad of indices (1,2,3): the index i becomes k; j becomes i; and k becomes j. The Jacobi vectors $\mathbf{x}_i$ and $\mathbf{y}_i$ are combined to form two-component column vectors $(\mathbf{x}_i,\mathbf{y}_i)^T$ and a six-dimensional vector $\mathbf{r}_i \equiv (\mathbf{x}_i,\mathbf{y}_i)$. We arbitrarily choose two of the three such vectors and introduce the abbreviated notation

$$\mathbf{r}=(\mathbf{x},\mathbf{y})\equiv(\mathbf{x}_i,\mathbf{y}_i), \quad \mathbf{r}'=(\mathbf{x}',\mathbf{y}')\equiv(\mathbf{x}_k,\mathbf{y}_k). \qquad (2)$$

By definition, the vector $\mathbf{r}$ with three-dimensional components $\mathbf{x}$ and $\mathbf{y}$ belongs to the six-dimensional coordinate space $\mathcal{R}^6 \equiv \mathcal{R}^3_x \oplus \mathcal{R}^3_y$, which is the direct sum of the three-dimensional spaces $\mathcal{R}^3_x$ and $\mathcal{R}^3_y$ of the vectors $\mathbf{x}$ and $\mathbf{y}$. We choose the Cartesian coordinate system S_6 in $\mathcal{R}^6$ such that projections r_ν of the vector $\mathbf{r}$ onto the axes $\hat{\mathbf{n}}_\nu$ of this system are related to the coordinates $x_\mu \equiv \mathbf{x}\cdot\hat{\mathbf{e}}_\mu$ and $y_\mu \equiv \mathbf{y}\cdot\hat{\mathbf{e}}_\mu$, $\mu = 1,2,3$, of the vectors $\mathbf{x}$ and $\mathbf{y}$ in the system S_3 as follows: $r_\nu \equiv x_\nu$ if $\nu = 1,2,3$ and $r_\nu \equiv y_{\nu-3}$ if $\nu = 4,5,6$. For this definition of the system S_6 the hyperspherical angles $\Omega = (\hat{\mathbf{x}},\hat{\mathbf{y}},\varphi)$ of the vector $\mathbf{r}$ in $\mathcal{R}^6$ admit a clear geometrical interpretation in three-dimensional terms: the pair $\hat{q} \equiv (\theta_a,\varphi_a)$ of spherical angles of the vector $\mathbf{q}$ determines the direction of the vector $\mathbf{q}=\mathbf{x},\mathbf{y}$ in S_3, and the value of the hyperangle $\varphi \equiv \arctan(y/x) \in [0,\pi/2]$ fixes the ratio of the three-dimensional lengths x and y of the vectors $\mathbf{x}$ and $\mathbf{y}$. Therefore, in contrast to the angles $\hat{x}$ and $\hat{y}$, the angle φ does not change in an arbitrary three-dimensional rotation of the vectors $\mathbf{x}$ and $\mathbf{y}$ or of the coordinate system S_3. The relative positions of the three particles in their center-of-mass frame are characterized by six numbers. We shall take this set, hereafter denoted by the Dirac ket symbol $|\mathbf{r}\rangle$, to be the Cartesian coordinates x_μ and y_μ of the three-dimensional components $\mathbf{x}$ and $\mathbf{y}$ of the vector $\mathbf{r}$ in the system S_3, its Cartesian coordinates r_ν in the system S_6, and the corresponding hyperspherical coordinates (r,Ω): the hyperradius $r \equiv (x^2+y^2)^{1/2}$ and the set of angles Ω. The same three-particle configuration $|\mathbf{r}\rangle$ can be described in the coordinates x'_μ, y'_μ; r'_ν or r', $\Omega' \equiv (\hat{x}',\hat{y}',\varphi')$ of the vector $\mathbf{r}'$, because there is a one-to-one correspondence between the vectors $\mathbf{r}$ and $\mathbf{r}'$. Actually, it follows from the definition (1) that the columns $(\mathbf{x},\mathbf{y})^T$ and $(\mathbf{x}',\mathbf{y}')^T$ of selected Jacobi vectors corresponding to the vectors $\mathbf{r}$ and $\mathbf{r}'$ are related by a linear orthogonal, one-parameter transformation:

$$\begin{pmatrix}\mathbf{x}'\\ \mathbf{y}'\end{pmatrix} = -\begin{pmatrix}+\cos\gamma & \sin\gamma\\ -\sin\gamma & \cos\gamma\end{pmatrix}\begin{pmatrix}\mathbf{x}\\ \mathbf{y}\end{pmatrix}. \qquad (3)$$

The absolute value of the parameter γ for the three-particle system under study is fixed only by the particle-mass ratios, and the sign is determined by the parity of the permutation of the triad of indices (i,j,k) labeling the particles into the triad (1,2,3). In our case, $\mathbf{x}=\mathbf{x}_i$, $\mathbf{y}=\mathbf{y}_i$, and $\mathbf{x}'=\mathbf{x}_k$, $\mathbf{y}'=\mathbf{y}_k$, and

$$\gamma \equiv \gamma_{ki} \equiv \varepsilon_{ijk}\arctan(m_j(m_1+m_2+m_3)/m_im_k)^{1/2}, \qquad (4)$$

where $\varepsilon_{ijk} = -\varepsilon_{kji} = 1$ for $(k,i) = (3,1),(1,2),(2,3)$. Since the parameter γ has the meaning of an angle determined only by the kinematical characteristics of the three-particle system, it is called the kinematical angle, and Eq. (3) is called the kinematical transformation of the pair of Jacobi vectors and the six-dimensional vectors associated with them.

Let us describe the region of allowed kinematical angles and a set of special values of them. According to the definition (4), three kinematical angles with indices from any triad (i,j,k) are linearly dependent:

$$\gamma_{ij}+\gamma_{ik}+\gamma_{jk} = \pi, \quad (i,j,k) = (1,2,3),(1,3,2),(2,3,1), \qquad (5)$$

and for any nonzero and finite values of the particle masses the values of any of the six kinematical angles lie in the range $(-\pi/2,\pi/2)$. The points 0, $\pm\pi/4$, and $\pm\pi/2$ are limiting points in the following sense. For three-particle systems consisting of one light and two heavy particles of comparable mass, the values of the kinematical angles are close to 0 and $\pm\pi/2$. If the system consists of a single heavy particle and two light ones with equal masses, the values of the kinematical angles are close to $\pm\pi/4$ and $\pm\pi/2$.

For example,

$$\gamma_{12}\to 0; \quad \gamma_{31}, \ \gamma_{23}\to\pi/2, \quad \text{if } m_3/m_1\to 0, \ m_1 = O(m_2),$$

$$\gamma_{12}\to\pi/2; \quad \gamma_{31}, \ \gamma_{23}\to\pi/4, \quad \text{if } m_3/m_1\to\infty, \ m_1 = m_2.$$

In these limits ($\gamma=\gamma_{12}\to 0,\pi/2$), Eq. (3) takes the corresponding asymptotic form:

$$\mathbf{x}' = -\mathbf{x}+\mathbf{y}O(\gamma), \quad \mathbf{y}' = -\mathbf{y}+\mathbf{x}O(\gamma), \quad \gamma\to 0; \qquad (6)$$

$$\mathbf{x}' = -\mathbf{y}+\mathbf{x}O(\gamma'), \quad \mathbf{y}' = \mathbf{x}+\mathbf{y}O(\gamma'),$$

$$\gamma' \equiv (\pi/2-\gamma)\to 0. \qquad (7)$$

For all γ the linear combinations $\mathbf{x}'$ and $\mathbf{y}'$ (3) of noncollinear vectors $\mathbf{x}$ and $\mathbf{y}$ lie in the same plane $\mathcal{P}$ passing through the three particles. For any γ the kinematical transformation (3) also preserves the normal $\mathbf{N}$,

$$\mathbf{N} \equiv \mathbf{x}\times\mathbf{y} = \mathbf{x}'\times\mathbf{y}', \qquad (8)$$

to this plane and the length ($r=r'$) of the six-dimensional vector $\mathbf{r}$. For this and many other reasons it is convenient to use the hyperspherical coordinates (r,Ω) and (r,Ω') of the vectors $\mathbf{r}$ and $\mathbf{r}'$.

We shall describe the kinematical transformation of the hyperspherical angles in the general case and in two special cases. Using (3) and the definition of these angles, we can express the three-dimensional projections x'_μ and y'_μ of the vectors $\mathbf{x}'$ and $\mathbf{y}'$ onto the axes $\hat{\mathbf{e}}_\mu$ of the system S_3 and the vectors $\mathbf{x}$ and $\mathbf{y}$ in terms of the length r of the vector $\mathbf{r}$ and its hyperspherical angles Ω, and then find the explicit dependence $\Omega' = \Omega'(\Omega,\gamma)$ of the angles Ω' on the angles Ω and the parameter γ. The equations describing this dependence will have a compact form if the combination of trigonometric

functions of the angles $\hat{x}$ and $\hat{y}$ equal to the cosine of the angle θ between the vectors $\mathbf{x}$ and $\mathbf{y}$ is denoted by the symbol

$$u \equiv \cos\theta \equiv u_{xy} = \cos\theta_x \cos\theta_y + \sin\theta_x \sin\theta_y \cos(\varphi_x - \varphi_y). \tag{9}$$

For example, the angle φ' and the cosines u_{ab}, invariant under three-dimensional rotations of the coordinate system S_3, of the angles θ_{ab} between different vectors $\mathbf{a},\mathbf{b}=\mathbf{x},\mathbf{y},\mathbf{x}',\mathbf{y}'$ and the angle φ' are represented as functions $\varphi'=\varphi'(\varphi,u;\gamma)$ and $u_{ab}=u_{ab}(\varphi,u;\gamma)$ of the two arguments φ and u and the parameter γ:

$$\cos 2\varphi' = \cos 2(\gamma-\varphi) + (u-1)\sin 2\gamma \sin 2\varphi, \tag{10}$$

$$u_{xx'} \equiv \cos\theta_{xx'} = -\sec\varphi'(\cos\gamma\cos\varphi + u\sin\gamma\sin\varphi),$$

$$u_{xy'} \equiv \cos\theta_{xy'} = \operatorname{cosec}\varphi'(\sin\gamma\cos\varphi - u\cos\gamma\sin\varphi),$$

$$u_{yx'} \equiv \cos\theta_{yx'} = -\sec\varphi'(\sin\gamma\sin\varphi + u\cos\gamma\cos\varphi),$$

$$u_{yy'} \equiv \cos\theta_{yy'} = -\operatorname{cosec}\varphi'(\cos\gamma\sin\varphi - u\sin\gamma\cos\varphi),$$

$$u_{x'y'} \equiv \cos\theta_{x'y'} = -\operatorname{cosec} 2\varphi'(\sin 2\gamma\cos 2\varphi - u\cos 2\gamma\sin 2\varphi). \tag{11}$$

The functional dependence $\Omega'=\Omega'(\Omega,\gamma)$ is considerably simplified for several special values of the hyperangles Ω corresponding to special configurations of the three-particle system. We shall describe two such configurations.

Let the plane $\mathcal{P}$ of the vectors $\mathbf{x}$ and $\mathbf{y}$ coincide with the plane $\mathcal{P}_{13}$ of the axes $\hat{\mathbf{e}}_1$ and $\hat{\mathbf{e}}_3$, and the vector $\mathbf{x}$ be collinear with the axis $\hat{\mathbf{e}}_3$. In other words, let the normal (8) and the vector $\mathbf{x}$ be oriented so that $\mathbf{N}\cdot\hat{\mathbf{e}}_2=N$ and $\mathbf{x}\cdot\hat{\mathbf{e}}_3=x$. Then, owing to Eqs. (3), (9), and (11), we have

$$\theta_x,\ \varphi_x=0;\quad \theta_y=\theta,\quad \varphi_y=0.$$

$$\theta_{x'}=\arccos u_{xx'},\quad \varphi_{x'}=\pi(\operatorname{sign}\gamma+1)/2;$$

$$\theta_{y'}=\arccos u_{xy'},\quad \varphi_{y'}=\pi. \tag{12}$$

The sets Ω and Ω' of such hyperspherical angles describing the kinematical transformation in the plane $\mathcal{P}_{13}$ is labeled by the index p, i.e., we introduce the notation $\Omega_p \equiv (0,0,\theta,0,\varphi)$ and $\Omega'_p(\theta_{xx'},\varphi_{x'},\theta_{xy'},0,\varphi')$.

Now let all three particles lie on the same line $\mathcal{L}_3$ as the axis $\hat{\mathbf{e}}_3$ with the vector $\mathbf{y}$ collinear to this axis, i.e., $\mathbf{y}\cdot\hat{\mathbf{e}}_3=y$. Then according to the definition (9) we have $u=1$. For $u=1$ and any φ and γ, the functions (11) reach their extremal values, and

$$u_{xx'}=-\operatorname{sign}(\pi/2+\gamma-\varphi),\quad u_{xy'}=\operatorname{sign}(\gamma-\varphi), \tag{13}$$

and from (10) we find the algebraic relations

$$\varphi'=|\varphi-\gamma| \quad \gamma\geqslant 0;$$

$$\varphi'=\pi/2+(\varphi-\gamma-\pi/2)\operatorname{sign}(\pi/2-\varphi+\gamma),\quad \gamma<0. \tag{14}$$

Using (12) and (13), we find that the configuration in question corresponds to the extremal values of the spherical angles:

$$\theta_x,\varphi_x,\quad \theta_y,\varphi_y=0;$$

$$\theta_{x'}=\pi(1+\operatorname{sign}(\pi/2+\gamma-\varphi))/2,\quad \varphi_{x'}=\pi(\operatorname{sign}\gamma+1)/2;$$

$$\theta_{y'}=\pi(1-\operatorname{sign}(\gamma-\varphi))/2,\quad \varphi_{y'}=0. \tag{15}$$

The sets Ω and Ω' of angles (14) and (15) describing the kinematical transformation on the line $\mathcal{L}_3$ coinciding with the axis $\hat{\mathbf{e}}_3$ are labeled by the index l, i.e., we define $\Omega_l \equiv (0,0,0,0,\varphi)$ and $\Omega'_l \equiv (0,\theta_{xy'},\theta_{xx'},\varphi')$, where according to (13) and (15) the angles $\theta_{xy'}$ and $\theta_{xx'}$ coincide with the angles $\theta_{y'}$ and $\theta_{x'}$ and are equal to 0 or π.

Let us give the geometrical interpretation of the kinematical transformation on the line $\mathcal{L}_3$. For this we introduce the auxiliary two-dimensional space $\mathcal{R}^2_q$ of vectors $\mathbf{q}$ with coordinates $q_1\equiv r\cos\varphi$ and $q_2\equiv r\sin\varphi$ in some fixed Cartesian coordinate system S_2. It follows from (3) and (14) that the kinematical transformation (3) on the line $\mathcal{L}_3$ consists of reflection of the vector $\mathbf{q}$ and rotation of the resulting vector $-\mathbf{q}$ by the angle $u\gamma$ in $\mathcal{R}^2_q$.

We shall show that for an arbitrary kinematical transformation, except in the above-mentioned special case, in R^3 there always exists a coordinate system S_{p3} in which this transformation occurs in the coordinate plane $\mathcal{P}_{p13}$ of its axes $\hat{\mathbf{e}}_{p1}$ and $\hat{\mathbf{e}}_{p3}$.

Let $\mathbf{x}$ and $\mathbf{y}$ be arbitrary but not collinear vectors. In R^3 we introduce the coordinate system S_{p3} with axes $\hat{\mathbf{e}}_{p2}$ and $\hat{\mathbf{e}}_{p3}$ collinear to the normal (8) and the vector $\mathbf{x}$. The system S_{p3} is obtained by rotating the system S_3. In S_3 this rotation is defined by the set ω of three Euler angles,[24]

$$\omega \equiv (\varphi_x,\theta_x,\tilde{\varphi}_y),\quad \tilde{\varphi}_y \equiv \varphi_y - \varphi_x, \tag{16}$$

and is described by the three-dimensional matrix $\mathbf{R}(\omega)$: the components of any vector $\mathbf{q}$ in S_{3p} and S_3 are related by the matrix equations

$$(q_{1p},q_{2p},q_{3p})^T = \mathbf{R}(\omega)(q_1,q_2,q_3)^T,$$

$$\mathbf{R}(\omega)=\mathbf{R}(\varphi_x,\hat{\mathbf{e}}_3)\mathbf{R}(\theta_x,\hat{\mathbf{e}}_2)\mathbf{R}(\tilde{\varphi}_y,\hat{\mathbf{e}}_3), \tag{17}$$

in which $\mathbf{R}(\alpha,\hat{\mathbf{e}}_i)$ is the matrix for rotation by an angle α about the axis $\hat{\mathbf{e}}_i$:

$$\mathbf{R}(\alpha,\hat{\mathbf{e}}_2)=\begin{pmatrix} \cos\alpha & 0 & \sin\alpha \\ 0 & 1 & 0 \\ -\sin\alpha & 0 & \cos\alpha \end{pmatrix},$$

$$\mathbf{R}(\alpha,\hat{\mathbf{e}}_3)=\begin{pmatrix} \cos\alpha & -\sin\alpha & 0 \\ \sin\alpha & \cos\alpha & 0 \\ 0 & 0 & 1 \end{pmatrix}. \tag{18}$$

In the system S_{p3} the pairs $(\mathbf{x},\mathbf{y})$ and $(\mathbf{x}',\mathbf{y}')$ of Jacobi vectors are related by a kinematical transformation in the plane $\mathcal{P}_{p13}$ of the vectors $\hat{\mathbf{e}}_{p1}$ and $\hat{\mathbf{e}}_{p3}$, and the hyperangles of the vectors $\mathbf{r}$ and $\mathbf{r}'$ corresponding to these pairs are Ω_p and Ω'_p.

Now let us describe the kinematical transformation (3) in operator language. We introduce the operators P_x, P_y, and T_{xy} for permutation and reflection of the components $\mathbf{x}$ and $\mathbf{y}$ of the vector $\mathbf{r}$ and the operators P_r and $\mathbf{R}_{r\nu}(\gamma)$ for reflection and rotation of this vector in the hyperplanes of the axes $\hat{\mathbf{n}}_\nu$ and $\hat{\mathbf{n}}_{\nu+3}$ by an angle γ about the normal $\mathbf{N}_\nu \equiv \hat{\mathbf{n}}_\nu \times \hat{\mathbf{n}}_{\nu+3}$:

$$P_x|\mathbf{x},\mathbf{y}\rangle \equiv |-\mathbf{x},\mathbf{y}\rangle,\quad P_y|\mathbf{x},\mathbf{y}\rangle \equiv |\mathbf{x},-\mathbf{y}\rangle,$$

$$P_r \equiv P_x P_y;\quad T_{xy}|\mathbf{r}\rangle = T_{xy}|\mathbf{x},\mathbf{y}\rangle \equiv |\mathbf{y},\mathbf{x}\rangle;$$

$$\mathbf{R}_{r\nu}(\gamma)\begin{pmatrix} r_\mu \\ r_{3+\mu}\end{pmatrix}\equiv\left((1-\delta_{\mu\nu})+\delta_{\mu\nu}\begin{pmatrix}\cos\gamma & -\sin\gamma\\ \sin\gamma & +\cos\gamma\end{pmatrix}\right)\times\begin{pmatrix} r_\mu \\ r_{3+\mu}\end{pmatrix},\quad \mu,\nu=1,2,3. \tag{19}$$

The transformation (3) of the vector $\mathbf{r}$ into the vector $\mathbf{r}'$ is described by the operator

$$K_r(\gamma):|\mathbf{r}\rangle\to|\mathbf{r}'\rangle\equiv K_r(\gamma)|\mathbf{r}\rangle. \tag{20}$$

Let us express it in terms of the operators (19) in the limits (6) and (7) and in the general case:

$$K_r(0)=P_r,\quad K_r(\pi/2)=P_xT_{xy}=T_{xy}P_y, \tag{21}$$

$$K_r(\gamma)=P_rR_r(-\gamma)=R_r(-\gamma)P_r,$$

$$R_r(\gamma)\equiv R_{r1}(\gamma)R_{r2}(\gamma)R_{r3}(\gamma). \tag{22}$$

Using (3), (19), (20), and (22), we derive the addition formula

$$K_r(\gamma_1+\gamma_2)=P_rK_r(\gamma_1)K_r(\gamma_2),\quad \forall\gamma_1,\gamma_2, \tag{23}$$

and a series of identities in the angle γ:

$$K_r(-\gamma)=P_xK_r(\gamma)P_x=P_yK_r(\gamma)P_y=T_{xy}K_r(\gamma)T_{xy},$$

$$K_r(\pi/2-\gamma)=T_{xy}K_r(\gamma)P_x=P_yK_r(\gamma)T_{xy}. \tag{24}$$

Let us explain the geometrical meaning of the kinematical transformation in easy-to-understand three-dimensional terms. As can be seen from (21), for the special values of the angle $\gamma=0$ and $\pi/2$ the kinematical transformation (3) reduces to reflection and permutation of the three-dimensional components of the vector $\mathbf{r}$. Equations (22) imply that in general the kinematical transformation in $\mathcal{R}^6$ consists of reflection of the vector $\mathbf{r}$ followed by rotation of the resulting vector in the hyperplane $\mathcal{P}$. Reflection and rotation are commuting operations. Rotation is described by the operator $\mathbf{R}(-\gamma)$ and is represented by an arbitrary sequence of three commuting rotations about the normal N_ν, $\nu=1,2,3$, by identical angles $-\gamma$.

It should be noted that the definition of the Jacobi vectors (1) is not very convenient, although this is the definition used most often in the literature and in books.[21,25] This is the only reason not to use the alternative definition of the Jacobi vectors[26] for which the kinematical transformation corresponds to rotation in the hyperplane $\mathcal{P}$ by the angle γ.

2.2. The kinematical transformation of functions

The transformation of functions depending on several or all of the six-dimensional components of the vector $\mathbf{r}$ under a kinematical transformation of their arguments is called a kinematical transformation of the function. In what follows we shall consider only the kinematical transformation of *scalar* functions given, in general, in $\mathcal{R}^6$. The set of all such functions is denoted by $\mathcal{A}$. If necessary, we shall use additional constraints on the smoothness of the functions to separate a subset from this set and indicate on which subset a given relation is valid.

2.2.1. The permutation, reflection, rotation, and kinematical-transformation operators

Let I be the operator for the identity transformation: $I\Psi\equiv\Psi$. We introduce the operators P_1, P_2, P, T, and $K(\gamma)$ describing the transformation of an arbitrary function $\Psi\in\mathcal{A}$ for the corresponding operation (19) and (20) on its arguments:

$$P_1\Psi(\mathbf{r})\equiv\Psi(P_x\mathbf{r})=\Psi(-\mathbf{x},\mathbf{y}),$$

$$P_2\Psi(\mathbf{r})\equiv\Psi(P_y\mathbf{r})=\Psi(\mathbf{x},-\mathbf{y}),$$

$$P\Psi(\mathbf{r})\equiv P_1P_2\Psi(\mathbf{r})\equiv\Psi(P_r\mathbf{r})=\Psi(-\mathbf{r}),$$

$$T\Psi(\mathbf{r})\equiv\Psi(T_{xy}\mathbf{r})=\Psi(\mathbf{y},\mathbf{x}), \tag{25}$$

$$\Psi(r')\equiv K(\gamma)\Psi(\mathbf{r})\equiv\Psi(K_r(\gamma)\mathbf{r}). \tag{26}$$

We use the standard method from the theory of angular momentum[24] to construct the operator for the kinematical transformation of the functions $K(\gamma)$ in differential form. First we use (21) and (26) to express this operator at the special values $\gamma=0,\pi/2$ in terms of the reflection and permutation operators (25):

$$K(0)=P,\quad K(\pi/2)=P_1T=TP_2. \tag{27}$$

Furthermore, we note that (23) and (26) lead to the operator identity

$$K(\gamma_1+\gamma_2)\equiv PK(\gamma_1)K(\gamma_2),\quad \forall\gamma_1,\gamma_2. \tag{28}$$

Now let us consider the case $\gamma\to0$. Using the representation $\mathbf{r}=(\mathbf{x},\mathbf{y})$, we expand the function $\Psi(\mathbf{r}')$ in a Taylor series at the point $-\mathbf{r}$. Taking into account (6), we obtain the transformation formula for the function $\Psi(\mathbf{r})$ for an infinitesimal kinematical variation of its arguments:

$$K(\gamma)\Psi(\mathbf{r})=P(I-i\gamma J)\Psi(\mathbf{r})+O(\gamma^2),\quad \gamma\to0. \tag{29}$$

Here the symbol J denotes an infinitesimal rotation operator, independent of the form of the function, in the hyperplane $\mathcal{P}$ of the space $\mathcal{R}^6$:

$$J\equiv-i(\mathbf{x}\cdot\nabla_y-\mathbf{y}\cdot\nabla_x)=-i\sum_{\mu=1}^{3}(x_\mu\partial_{y_\mu}-y_\mu\partial_{x_\mu}). \tag{30}$$

Let us express the operator $K(\gamma)$ in terms of the operators P and J. For this we set $\gamma=\gamma_1$ and $\delta\gamma=\gamma_2\to0$ in the identity (28). Taking into account (29), we obtain the asymptotic relation

$$K(\gamma+\delta\gamma)=PK(\gamma)K(\delta\gamma)=K(\gamma)(I-i\gamma J)+O(\delta\gamma^2),$$

$$\delta\gamma\to0. \tag{31}$$

It generates the operator differential equation

$$dK(\gamma)/d\gamma=-iJK(\gamma). \tag{32}$$

We supplement this by the boundary condition (27) at the point $\gamma=0$. The only solution of the resulting boundary-value problem is represented as the series

$$K(\gamma)=P\exp(-i\gamma J)\equiv P\sum_{n=0}^{\infty}(-i\gamma)^nJ^n/\Gamma(n+1). \tag{33}$$

We begin the study of this series by analyzing the properties of the operator J independent of the form of the function $\Psi \in \mathcal{A}$. The commutation relations are an important example of such properties. Using the definitions (25) and (30), we verify that the operator J anticommutes with the operator T and commutes with the operator P. Now we rewrite (30) in the coordinates $r_1, \ldots, r_6$ as the sum

$$J=\sum_{\mu=1}^{3} L_{\mu,\mu+3}, \quad L_{\mu\nu}\equiv -i(r_\mu \partial_{r_\nu}-r_\nu \partial_{r_\mu}) \tag{34}$$

of three of the fifteen ($\mu>\nu$; $\mu,\nu=1,2,\ldots,6$) components $L_{\mu\nu}$ of the hyper-angular-momentum operator L obeying the well known commutation relations[26]

$$[L_{k\ell},L_{mn}]_-=i(\delta_{nk}L_{m\ell}+\delta_{m\ell}L_{nk}-\delta_{mk}L_{n\ell}-\delta_{n\ell}L_{mk}). \tag{35}$$

We use the operators $L_{\mu\nu}$ to express the angular-momentum operators $\mathbf{l}_x$, $\mathbf{l}_y$ conjugate to the vectors $\mathbf{x}$ and $\mathbf{y}$ and the total angular-momentum operator $\mathbf{l}$:

$$\mathbf{l}_x\equiv -i\mathbf{x}\times\nabla_x=\sum_{i=1}^{3} l_{xi}\hat{\mathbf{e}}_i=L_{23}\hat{\mathbf{e}}_1+L_{31}\hat{\mathbf{e}}_2+L_{12}\hat{\mathbf{e}}_3,$$

$$\mathbf{l}_y\equiv -i\mathbf{y}\times\nabla_y=\sum_{i=1}^{3} l_{yi}\hat{\mathbf{e}}_i=L_{56}\hat{\mathbf{e}}_1+L_{64}\hat{\mathbf{e}}_2+L_{45}\hat{\mathbf{e}}_3,$$

$$\mathbf{l}\equiv\mathbf{l}_x+\mathbf{l}_y=\sum_{i=1}^{3} l_i\hat{\mathbf{e}}_i, \quad l_i=l_{xi}+l_{yi}. \tag{36}$$

We recall that

$$L^2=-\sum_{\mu>\nu} L_{\mu\nu}^2, \tag{37}$$

and in hyperspherical coordinates

$$\mathbf{l}_q^2=-\operatorname{cosec}^2\theta_q(\sin\theta_q\partial_{\theta_q}(\sin\theta_q\partial_{\theta_q}+\partial_{\varphi_z}^2),$$

$$l_{q3}=i\partial_{\varphi_q}, q\text{-}x,y, \tag{38}$$

$$L^2=-\partial_\varphi^2-4\cot 2\varphi\partial_\varphi+\mathbf{l}_x^2/\cos^2\varphi+\mathbf{l}_y^2/\sin^2\varphi. \tag{39}$$

Using Eqs. (34)–(39), we can show that the operator J does not commute with the operators $\mathbf{l}_q^2$, l_{q3}, $q=x,y$, but it does commute with the operators $\mathbf{l}^2$, l_3, and L^2. We sum all the above-mentioned commutation relations:

$$[J,T]_+=[J,Q]_-=0, \quad Q=P,L^2,\mathbf{l}^2,l_3;$$

$$[J,q]_-\neq 0, q=\mathbf{l}_x^2,\mathbf{l}_y^2,l_{x3},l_{y3}. \tag{40}$$

Let us construct the unitary-like operator J with respect to the three-dimensional rotation (16), (17) from the coordinate system S_3 to the system S_{p3}. We change to hyperspherical coordinates (r,Ω) in the definition (30) of the operator J. The resulting combinations of trigonometric functions are expressed in terms of the derivatives of the function $u=u(\hat{x},\hat{y})$ given by (9). Then we find the equation

$$J=-i\left(\cot\varphi\left(\frac{\partial u}{\partial\theta_y}\partial_{\theta_y}+\operatorname{cosec}^2\theta_y\frac{\partial u}{\partial\varphi_y}\partial_{\varphi_y}\right)\right)$$
$$+i\left(\tan\varphi\left(\frac{\partial u}{\partial\theta_x}\partial_{\theta_x}+\operatorname{cosec}^2\theta_x\frac{\partial u}{\partial\varphi_x}\partial_{\varphi_x}\right)\right)-iu\partial_\varphi. \tag{41}$$

Using this and the definition (9) of the function u, we prove the equivalence

$$JF(r,u,\varphi)=J_pF(r,u,\varphi), \quad J_p\equiv 2\tan 2\varphi(1-u^2)\partial_u+u\partial_\varphi, \tag{42}$$

of the operators J and J_p on the set of functions $\Psi(r,\Omega)=F(r,u,\varphi)$ depending on the spherical angles of the vectors $\mathbf{x}$ and $\mathbf{y}$ only through their combination u.

We note that any function $\Psi(r,\Omega)$ specified in the coordinate system S_3 in the special case $\Omega=\Omega_p$ becomes a function $\Psi(r,\Omega_p)$ of the type $F(r,u,\varphi)$.

As is well known,[23] in going from the coordinate system S_3 to the system S_{p3} in (17), functions of the class $\mathcal{A}$ are transformed as

$$\Psi(r,\Omega)=D^{-1}(\omega)\Psi(r,\Omega_p)=D^{-1}(\omega)F(r,u,\varphi), \tag{43}$$

where the inverse Wigner operator is expressed in terms of the projections $l_2=L_{13}+L_{46}$ and $l_3=L_{12}+L_{45}$ of the total angular-momentum operator (36) onto the axes $\hat{\mathbf{e}}_2$ and $\hat{\mathbf{e}}_3$ by the expression

$$D^{-1}(\omega)\equiv\exp(-i\varphi_x l_3)\exp(-i\theta_x l_2)\exp(-i\bar{\varphi}_y l_3). \tag{44}$$

Using (42) and (43), we write the chain of equalities

$$J\Psi(r,\Omega_p)=J(D^{-1}\Psi(r,\Omega))$$
$$=J_p\Psi(r,\Omega_p)=J_pD^{-1}\Psi(r,\Omega) \tag{45}$$

and thereby prove the unitary similarity of the operators J and J_p:

$$J=D(\omega)J_pD^{-1}(\omega). \tag{46}$$

Let us list the properties of the operator $K(\gamma)$ independent of the form of the function Ψ. From the representations (30) and (33) we find the unitarity relations

$$K(\gamma)K(-\gamma)=I, \quad K^+(\gamma)=K^{-1}(\gamma)=K(-\gamma) \tag{47}$$

and the differential identities

$$\partial_\gamma^n K(\gamma)=(-iJ)^nK(\gamma), n=0,1,\ldots. \tag{48}$$

Owing to (33) and (40), we have the commutation relations

$$[K(\gamma),Q]_-=0, Q=P,L^2,\mathbf{l}^2,l_3;$$

$$[K(\gamma),q]_-\neq 0, q=\mathbf{l}_x^2,\mathbf{l}_y^2,L_{12},L_{45} \tag{49}$$

and the unitary similarity of the operators K and K_p with respect to a three-dimensional rotation (43), (44):

$$K(\gamma)=D(\omega)K_p(\gamma)D^{-1}(\omega), \quad K_p(\gamma)\equiv\exp(-i\gamma J_p). \tag{50}$$

Equations (24)–(26) generate the following identities in the angle γ:

$$K(-\gamma)=P_1K(\gamma)P_1=P_2K(\gamma)P_2=TK(\gamma)T,$$

$$K(\pi/2-\gamma)=TK(\gamma)P_1=P_2K(\gamma)T. \tag{51}$$

Let us show how for a given three-particle system the operators of any particle permutation can be expressed in terms of the operators P_1 and $K(\gamma)$. In our notation (2) the reflection operator P_1 is equivalent to the operator P_{ij} permuting the particles labeled i and j. According to the definition (1), in a cyclic permutation of particles labeled i,j,k, the vectors $\mathbf{x}_i,\mathbf{y}_i$ become the vectors $\mathbf{x}_k,\mathbf{y}_k$. It therefore follows from (3) and (26) that the operator $K(\gamma)$ is equivalent to the particle cyclic permutation operator P_{ijk} only for a definite value of the parameter γ:

$$K(\gamma_{ki})=P_{ijk}\equiv P_{ki}P_{jk}. \tag{52}$$

Therefore, the operator $K(\gamma)$ describes a transformation more general than a cyclic permutation of particles. Using (52), we write the operators S^- and S^+ for antisymmetrization and symmetrization with respect to any particle permutation in a form invariant with respect to the choice of the index i:

$$S_\pm=(I\pm(P_{ij}+P_{ik}+P_{jk})+P_{ij}P_{jk}+P_{ik}P_{ki})/6,$$

$$S^\pm=S_iS^\pm_{jk},\quad S_i\equiv I+\sum_{k\neq i}K(\gamma_{ki}),\quad S^\pm_{jk}\equiv(1\pm P_{jk})/6. \tag{53}$$

We note that the image $S^\pm_{jk}\Psi(\mathbf{r}_i)$ of any function $\Psi(\mathbf{r}_i)$ is an (anti)symmetric function under permutation of the particles labeled j and k, and the operator S_i maps a function already possessing this symmetry into a function which is (anti)symmetric with respect to any permutation.

2.2.2. Fundamental properties of the angular bases

Let us recall some of the well known properties of spherical functions, bispherical harmonics,[23,24] and three-particle hyperharmonics.[1,25,26]

We write the three-particle hyperharmonics as products

$$|L\ell mab\rangle\equiv|\ell mab\rangle|Lab\rangle,$$

$$\langle\Omega|L\ell mab\rangle\equiv Y^{\ell m}_{Lab}(\Omega)\equiv W_{Lab}(\varphi)\mathcal{Y}^{\ell m}_{ab}(\hat{x},\hat{y}) \tag{54}$$

of the functions $|Lab\rangle$,

$$\langle\varphi|Lab\rangle\equiv W_{Lab}(\varphi)$$
$$\equiv N_{Lab}(\sin\varphi)^a(\cos\varphi)^bP_n^{(a+1/2,b+1/2)}(\cos 2\varphi), \tag{55}$$

with normalization factors

$$N_{Lab}\equiv\left(\frac{(2L+4)\Gamma(n+1)\Gamma(L-n+2)}{\Gamma(n+a+3/2)\Gamma(n+b+3/2)}\right)^{1/2}, \tag{56}$$

and the bispherical harmonics

$$\langle\hat{x},\hat{y}|\ell mab\rangle\equiv\mathcal{Y}^{\ell m}_{ab}(\hat{x},\hat{y})\equiv\sum_\alpha C^{\ell m}_{a\alpha b\beta}Y_{a\alpha}(\hat{y})Y_{b\beta}(\hat{x}), \tag{57}$$

containing the Clebsch–Gordan coefficients $C^{\ell m}_{a\alpha b\beta}$ and the spherical functions

$$\langle\hat{q}|c\delta\rangle\equiv Y_{c\delta}(\hat{q})$$
$$=(2\pi)^{-1/2}\exp(i\delta\varphi_q)\Theta_{c\delta}(\cos\theta_q),\quad \hat{q}=\hat{x},\hat{y}. \tag{58}$$

In the $\mathcal{C}^2$ class of functions, the countable sets of functions

$$Y_{b\beta}(\hat{x}),\quad b=0,1,...,\quad \beta=-b,-b+1,...,b;$$

$$\mathcal{Y}^{\ell m}_{ab}(\hat{x},\hat{y}),\quad Y^{\ell m}_{Lab}(\Omega),$$

$$\ell=0,1....;\quad -\ell\leq m\leq\ell;\quad |a-b|\leq\ell\leq a+b;$$

$$L=a+b,a+b+2n,\quad n=0,1,... \tag{59}$$

form complete and orthonormal bases, respectively, on the unit two-dimensional sphere $\mathcal{S}^2_x$ in $\mathcal{R}^3_x$, the four-dimensional torus $\mathcal{T}^4\equiv\mathcal{S}^2_x\otimes\mathcal{S}^2_y$, and the five-dimensional unit sphere $\mathcal{S}^5$ in $\mathcal{R}^6$. The orthonormality of the bases (59) implies that for any allowed values of the indices, the corresponding integral relations are satisfied:

$$\langle b\beta|b'\beta'\rangle\equiv\int_{S^2_x}d\hat{x}(Y_{b\beta}(\hat{x}))^*Y_{b'\beta'}(\hat{x})=\delta_{bb'}\delta_{\beta\beta'},$$

$$\int_{S^2_x}d\hat{x}\equiv\int_0^\pi d\theta_x\sin\theta_x\int_0^{2\pi}d\varphi_x; \tag{60}$$

$$\langle\ell mab|\ell'm'a'b'\rangle\equiv\int_{\mathcal{T}^4_{xy}}d\hat{x}d\hat{y}(\mathcal{Y}^{\ell m}_{ab}(\hat{x},\hat{y}))^*\mathcal{Y}^{\ell'm'}_{a'b'}(\hat{x},\hat{y})$$
$$=\delta_{\ell\ell'}\delta_{m,m'}\delta_{aa'}\delta_{bb'},$$

$$\int_{\mathcal{T}^4_{xy}}d\hat{x}d\hat{y}\equiv\int_{S^2_x}d\hat{x}\int_{S^2_y}d\hat{y}; \tag{61}$$

$$\langle L\ell mab|L'\ell'm'a'b'\rangle\equiv\int_{S^5}d\omega(Y^{\ell m}_{Lab}(\Omega))^*Y^{\ell'm'}_{L'a'b'}(\Omega)$$
$$=\delta_{L'L}\delta_{\ell\ell'}\delta_{mm'}\delta_{aa'}\delta_{bb'},$$

$$\int_{S^5_{xy}}d\Omega\equiv\int_0^{\pi/2}d\varphi\rho^2(\varphi)\int_{\mathcal{T}^4_{xy}}d\hat{x}d\hat{y},\quad \rho(\varphi)\equiv\sin\varphi\cos\varphi. \tag{62}$$

Completeness of the bases (59) implies that an arbitrary function $\Psi(\mathbf{r})\in\mathcal{C}^2$ can be represented on the corresponding set $\mathcal{S}^2_x$, $\mathcal{T}^4_{xy}$, and $\mathcal{S}^5$ by the series

$$\Psi(\mathbf{r})=\sum_{b\beta}\Psi_{b\beta}(r,\hat{y},\varphi)Y_{b\beta}(\hat{x}),$$

$$\Psi_{b\beta}(r,\hat{y},\varphi)\equiv\langle b\beta|\Psi\rangle\equiv\int_{S^2_x}d\hat{x}(Y_{b\beta}(\hat{x}))^*\Psi(\mathbf{r}); \tag{63}$$

$$\Psi(\mathbf{r})=\sum_{\ell m}\sum_{ab}\Psi^{\ell m}_{ab}(r,\varphi)\mathcal{Y}^{\ell m}_{ab}(\hat{x},\hat{y}),$$

$$\Psi^{\ell m}_{ab}(r,\varphi)\equiv\langle\ell mab|\Psi\rangle\equiv\int_{\mathcal{T}^4_{xy}}d\hat{x}d\hat{y}(\mathcal{Y}^{\ell m}_{ab}(\hat{x},\hat{y}))^*\Psi(\mathbf{r}); \tag{64}$$

$$\Psi(\mathbf{r})=\sum_{L}\sum_{\ell m}\sum_{ab}\Psi^{\ell m}_{Lab}(r)Y^{\ell m}_{Lab}(\Omega),$$

$$\Psi^{\ell m}_{Lab}(r)\equiv\langle L\ell mab|\Psi\rangle\equiv\int_{S^5_{xy}}d\Omega(Y^{\ell m}_{Lab}(\Omega))^*\Psi(\mathbf{r}), \quad (65)$$

which can be differentiated twice term by term with respect to any argument. In the series (63)–(65) the indices take only the values indicated in (59). The projections $\Psi^{\ell m}_{ab}$ and $\Psi^{\ell m}_{Lab}$ of the function Ψ onto the bispherical harmonic $|\ell mab\rangle$ and hyperharmonic $|L\ell mab\rangle$ are often called, respectively, the partial component[17] of this function and the hyperradial function,[25] or the coefficients of the expansions of the function Ψ in the bispherical or hyperspherical bases. The more suitable and less awkward terms "bispherical and hyperspherical components" are used below. The bispherical components $\Psi^{\ell m}_{ab}$ are functions of the two variables x,y or r,φ belonging to the region

$$\mathcal{R}^2_+\equiv\{x,y:0\leqslant x,y\leqslant\infty\}=\{r,\varphi:0\leqslant r\leqslant\infty,\quad 0\leqslant\varphi\leqslant\pi/2\}.$$

Owing to the relations $r=(x^2+y^2)^{1/2}$ and $\tan\varphi=y/x$, in this region r and φ are polar coordinates. The only argument r of the hyperspherical components $\Psi^{\ell m}_{Lab}(r)$ always belongs to the non-negative semiaxis:

$$\mathcal{R}^+_1\equiv\{r:0\leqslant r\leqslant\infty\}.$$

The following remarks are especially important for understanding the sense in which the functions (55) are basis functions. The functions $|Lab\rangle$ and $|L'ab\rangle$ with identical indices a and b are orthonormal on the segment $\varphi\in[0,\pi/2]$:

$$\langle Lab|L'ab\rangle\equiv\int_0^{\pi/2}d\varphi\rho^2(\varphi)W_{Lab}(\varphi)W_{L'ab}(\varphi)=\delta_{LL'}, \quad (66)$$

where the weight ρ^2 is given by (62). Since each hyperharmonic $|L\ell mab\rangle$ is a product (55) of the functions $|Lab\rangle$ and $|\ell mab\rangle$, and the bases of hyperharmonics and bispherical harmonics are complete and orthonormal, on the segment $\varphi\in[0,\pi/2]$ the countable $(L=a+b,a+b+2,\ldots)$ set of functions $|Lab\rangle$ is a complete and orthonormal [with weight $\rho^2(\varphi)$] basis for the bispherical component $\Psi^{\ell m}_{ab}(r,\varphi)$ of the given function $\Psi(\mathbf{r})$. In other words, for each bispherical component we have the exact expansion

$$\Psi^{\ell m}_{ab}(r,\varphi)=\sum_{L}\Psi^{\ell m}_{Lab}(r)W_{Lab}(\varphi),\quad\forall(r,\varphi)\in\mathcal{R}^2_+,$$

$$\Psi^{\ell m}_{Lab}(r,\varphi)\equiv\langle Lab|\Psi^{\ell m}_{ab}\rangle\equiv\int_0^{\pi/2}d\varphi\rho^2(\varphi)W_{Lab}(\varphi)\Psi^{\ell m}_{ab}(r,\varphi), \quad (67)$$

where the index L is an integer $(L=2n+a+b,\ n=0,1,\ldots)$ of the same parity $\sigma=(-1)^{a+b}=(-1)^L$ as for the sum $a+b$.

The reflection and permutation operators (25), like the angular-momentum operators (36) and (37) and the three-dimensional rotation operator (44), transform the bases (55) and (59) according to very simple formulas. Let us comment on them.

The matrix of the squared hyper-angular momentum (37), (39) is diagonal in the bispherical basis. Each element on its diagonal is a differential operator

$$L^2_{ab}(\varphi)\equiv\langle\ell mab|L^2|\ell mab\rangle=-\delta^2_\varphi-4\cot\varphi\,\delta_\varphi+\frac{a(a+1)}{\sin^2\varphi}+\frac{b(b+1)}{\cos^2\varphi}, \quad (68)$$

for which the function $|Lab\rangle$ is the eigenfunction:

$$(L^2_{ab}-L(L+4))|Lab\rangle=0. \quad (69)$$

All the functions $|Lab\rangle$ are invariant under reflections (25) and rotations (43). In the permutation $x\to y$, $y\to x$ the angle φ transforms as $\varphi\to\pi/2-\varphi$, and the function $\langle\varphi|Lab\rangle$ transforms according to the rule

$$T|Lab\rangle=(-1)^{(L-a-b)/2}|Lba\rangle. \quad (70)$$

The spherical functions (58) of the arguments $\hat{x}$ are eigenfunctions of the operators $\mathbf{I}^2_x$, ℓ_{x3}, and P_1:

$$(\mathbf{I}^2_x-b(b+1))|b\beta\rangle,\quad(l_{x_3}-\beta)|b\beta\rangle,$$

$$(P_1-(-)^b)|b\beta\rangle=0. \quad (71)$$

The spherical functions of the arguments $\hat{y}$ possess analogous properties with respect to the operators $\mathbf{I}^2_y$, l_{y3}, and P_2:

$$(\mathbf{I}^2_y-a(a+1))|a\alpha\rangle,\quad(l_{y^3}-\alpha)|a\alpha\rangle,$$

$$(P_2-(-1)^a)|a\alpha\rangle=0. \quad (72)$$

Using the properties (70)–(72) of the factors of the products (55) and (57) and the well known equation for the Clebsch–Gordan coefficients

$$C^{c\delta}_{a\alpha b\beta}=(-1)^{c-a-b}C^{c\delta}_{b\beta a\alpha}, \quad (73)$$

it is easily shown that any bispherical harmonic $|\ell mab\rangle$ and its corresponding hyperharmonic $|L\ell mab\rangle$ are simultaneously eigenfunctions of the operators P_1, P_2, P, $\mathbf{I}^2_x$, $\mathbf{I}^2_y$, $\mathbf{I}^2$, and l_3, and the action of the operator T on such harmonics reduces to permutation of the indices a and b and multiplication by a definite phase factor. Owing to (68) and (69), the hyperharmonics are eigenfunctions of the operator L^2. We can summarize all these properties in the following compact manner:

$$(P_1-(-1)^b)|Q^{\ell m}_{ab}\rangle,\quad(P_2-(-1)^a)|Q^{\ell m}_{ab}\rangle,$$

$$(P-(-1)^{a+b})|Q^{\ell m}_{ab}\rangle=0,$$

$$\mathbf{I}^2_x-b(b+1))|Q^{\ell m}_{ab}\rangle,\quad(\mathbf{I}^2_y-a(a+1))|Q^{\ell m}_{ab}\rangle=0,$$

$$(\mathbf{I}^2-\ell(\ell+1))|Q^{\ell m}_{ab}\rangle,\quad(l_3-m)|Q^{\ell m}_{ab}\rangle,$$

$$|Q^{\ell m}_{ab}\rangle=|\ell mab\rangle,\ |L\ell mab\rangle;$$

$$T|\ell mab\rangle=(-1)^{\ell-a-b}|\ell mba\rangle,$$

$$T|L\ell mab\rangle=(-1)^{(L-2\ell+a+b)/2}|L\ell mba\rangle,$$

$$(L^2-L(L+4))|L\ell mab\rangle=0. \quad (74)$$

The operator $D(\omega)$ (43) which rotates the coordinate system S_{3p} into the system S_3 relates the functions (55), (57), and (58), written in the system S_3 in the coordinates Ω or Ω', to linear combinations of the corresponding functions written in the system S_{3p} in the coordinates Ω_p or Ω'_p and the Wigner functions $D^{\ell}_{mm_p}(\omega)$. The basis functions of the hyperangles Ω transform according to the rules

$$Y_{c\delta}(\hat{q})=D^{-1}(\omega)Y_{c\delta}(\hat{q}_p)=\sum_{\delta_p} D^{c}_{\delta_p\delta}(\omega)Y_{c\delta_p}(\hat{q}_p),$$

$$q=x,y, \tag{75}$$

$$\mathcal{Y}^{\ell m}_{ab}(\hat{x},\hat{y})=D^{-1}(\omega)\mathcal{Y}^{\ell m}_{ab}(\hat{x}_p,\hat{y}_p)$$

$$=\sum_{m_p} D^{\ell}_{m_pm}(\omega)\mathcal{Y}^{\ell m_p}_{ab}(\hat{x}_p,\hat{y}_p), \tag{76}$$

$$Y^{\ell m}_{ab}(\Omega)=D^{-1}(\omega)Y^{\ell m}_{ab}(\Omega_p)=\sum_{m_p} D^{\ell}_{m_pm}(\omega)Y^{\ell m_p}_{Lab}(\Omega_p), \tag{77}$$

and the transformation rules of the same basis functions but with argument Ω' are obtained by the replacement $\Omega\to\Omega'$ and $\Omega_p\to\Omega'_p$.

Let us redefine the hyperharmonics (55) as follows:[67]

$$Y^{\ell m}_{Lab}(\Omega)=\rho_{-1}(\varphi)\mathcal{Y}^{\ell m}_{ab}(\hat{x},\hat{y})\tilde{W}_{Lab}(\varphi), \tag{78}$$

$$\langle\varphi|L\tilde{a}b\rangle=\tilde{W}_{Lab}(\varphi)\equiv\rho(\varphi)W_{Lab}(\varphi). \tag{79}$$

From these definitions and (66)–(69), two facts follow. First, the countable ($L=a+b,\ a+b+2,\dots$) set of functions $|L\tilde{a}b\rangle$ is a complete and orthonormal (with *unit* weight) basis for the expansion of the bispherical component $\Psi^{\ell m}_{ab}$ of an arbitrary function Ψ on the segment $\varphi\in[0,\pi/2]$:

$$\langle L\tilde{a}b|L'\tilde{a}b\rangle=\int_0^{\pi/2}d\varphi\tilde{W}_{Lab}(\varphi)\tilde{W}_{L'ab}=\delta_{LL'}, \tag{80}$$

$$\Psi^{\ell m}_{ab}(r,\varphi)=\sum_L \Psi^{\ell m}_{Lab}(r)\tilde{W}_{Lab}(\varphi),$$

$$\Psi^{\ell m}_{Lab}(r,\varphi)\equiv\langle L\tilde{a}b|\Psi^{\ell m}_{ab}\rangle\equiv\int_0^{\pi/2}d\varphi\tilde{W}_{Lab}(\varphi)\Psi^{\ell m}_{ab}(r,\varphi). \tag{81}$$

Second, the functions $|L\tilde{a}b\rangle$ with different $L=a+b,\ a+b+2,\dots$ but fixed a and b are eigenfunctions of the operator $\tilde{L}^2_{ab}$:

$$(\tilde{L}^2_{ab}-(L+2)^2)|L\tilde{a}b\rangle=0, \tag{82}$$

$$\tilde{L}^2_{ab}\equiv-\delta^2_{\varphi}+a(a+1)/\sin^2\varphi+b(b+1)/\cos^2\varphi. \tag{83}$$

Let us describe the expansion of functions of the class $\mathcal{C}^2$ written as

$$\Psi^{\varepsilon}(\mathbf{r})=(xy)^{-1}U^{\varepsilon}(\mathbf{r})=2r^{-2}\operatorname{cosec}2\varphi U^{\varepsilon}(\mathbf{r}), \tag{84}$$

which are eigenfunctions of the three operators $\mathbf{l}^2$, l_3, and P:

$$(\mathbf{l}^2-\ell(\ell+1))|\Psi^{\varepsilon}\rangle,\quad (l_3-m)|\Psi^{\varepsilon}\rangle,$$

$$(P-\sigma)|\Psi^{\varepsilon}\rangle=0\quad \varepsilon\equiv(\ell,m,\sigma). \tag{85}$$

Here and below, the eigenfunctions of a definite set of operators will be labeled by the multiple index ε, the components of which are the eigenvalues of these operators. The set of such functions in the class $\mathcal{C}^2$ will be denoted by $\mathcal{A}^{\varepsilon}$.

Owing to Eqs. (74) for the functions (84), the expansions (64) and (65) in bases of the spherical functions (57) and hyperharmonics (55) or (78) degenerate into double and triple sums:

$$\Psi^{\varepsilon}(\mathbf{r})=(xy)^{-1}\sum_{ab} U^{\ell}_{ab}(r,\varphi)\mathcal{Y}^{\ell m}_{ab}(\hat{x},\hat{y}),$$

$$U^{\ell}_{ab}(r,\varphi)\equiv\langle\ell mab|U^{\varepsilon}\rangle, \tag{86}$$

$$\Psi^{\varepsilon}(\mathbf{r})=r^{-2}\sum_{Lab} U^{\ell}_{Lab}(r,\varphi)Y^{\ell m}_{Lab}(\Omega),$$

$$U^{\ell}_{Lab}(r)\equiv\langle L\ell mab|U^{\varepsilon}\rangle, \tag{87}$$

in which the indices a and b satisfy the condition $(-1)^{a+b}=\sigma$. For completeness, let us describe an alternative derivation of the representation (87). Taking into account the equation $xy=r^2\rho(\varphi)$, we expand the bispherical components $\rho^{-1}U^{\ell}_{ab}$ of the function $r^2\Psi^{\varepsilon}$ in series in the bases (55) or (79). We then obtain the series

$$U^{\ell}_{ab}(r,\varphi)=\rho(\varphi)\sum_L U^{\ell}_{Lab}(r)W_{Lab}(\varphi)$$

$$=\sum_L U^{\ell}_{Lab}(r)\tilde{W}_{Lab}(\varphi),$$

$$U^{\ell}_{Lab}(r)\equiv\int_0^{\pi/2}d\varphi\rho(\varphi)W_{Lab}(\varphi)U^{\ell}_{ab}(r,\varphi)$$

$$=\int_0^{\pi/2}d\varphi\tilde{W}_{Lab}(\varphi)U^{\ell}_{ab}(r,\varphi). \tag{88}$$

Using these, we transform the series (86) into the series (87). Using the definitions (39), (68), and (83), we derive the identities

$$(L^2+4)(xy)^{-1}\mathcal{Y}^{\ell m}_{ab}(\hat{x},\hat{y})Q(r,\varphi)$$

$$\equiv\mathcal{Y}^{\ell m}_{ab}(\hat{x},\hat{y})(L^2_{ab}+4)(xy)^{-1}Q(r,\varphi)$$

$$\equiv(xy)^{-1}\mathcal{Y}^{\ell m}_{ab}(\Omega)\tilde{L}^2_{ab}Q(r,\varphi),\forall Q(r,\varphi). \tag{89}$$

In conclusion, we note that economical algorithms for calculating bispherical functions with $\ell\leqslant3$ have been proposed in Ref. 111.

2.2.3. The kinematical transformation and addition theorems

We begin our analysis of the kinematical transformation of the angular bases with some very simple examples. Let us consider the well known addition theorems for the Legendre polynomials P_n and the spherical harmonics and bispherical functions related to the hyperharmonics (55) as[24]

$$\mathcal{Y}_{b\beta}(\mathbf{x})\equiv x^bY^{b\beta}_{b0b}(\Omega);\quad \mathcal{Y}^{\ell m}_{ab}(\mathbf{X},\mathbf{Y}\equiv r^{\ell}Y^{\ell m}_{a+b,ab}(\Omega). \tag{90}$$

The term "addition theorem" (or formula) is often used to describe the expansion of a function of a sum of two vectors in a series of functions depending only on the coordinates of one or the other vector. An example of such a factorization is the addition theorem[23] for the spherical harmonics (90):

$$\mathcal{Y}_{b\beta}(\mathbf{z}+\mathbf{v})=\sum_{c+d=b} A^{b}_{cd}\mathcal{Y}^{b\beta}_{cd}(\mathbf{z},\mathbf{v}),$$

$$A^{b}_{cd}\equiv\left(\frac{4\pi\Gamma(2b+2)}{\Gamma(2c+2)\Gamma(2d+2)}\right)^{1/2}. \tag{91}$$

This term (addition theorem or formula) is often used also for an expansion of a function of an argument, which is itself a function of two vectors, into a sum like the one above with separated variables. A well known example of such a representation is the addition theorem for the Legendre polynomials $P_n(u)$ with an argument that is a function (9) of the spherical coordinates of two vectors $\mathbf{x}$ and $\mathbf{y}$. We rewrite this theorem as an addition theorem for the functions $\Theta_{n0}(u)$ forming a complete, orthonormal basis[23] on the segment $-1\leqslant u\leqslant 1$:

$$\Theta_{n0}(u)=\sqrt{n+1/2}P_n(u)=(-1)^n 2\pi\mathcal{Y}^{00}_{nn}(\hat{x},\hat{y}),\quad n=0,1,\ldots. \tag{92}$$

The kinematical transformation (3) of the Jacobi coordinates (1) simultaneously transforms the vector $\mathbf{x}$ into the sum $\mathbf{x}'$ of the vectors $-\mathbf{x}\cos\gamma$ and $\mathbf{y}\sin\gamma$, the vector $\mathbf{y}$ into the sum $\mathbf{y}'$ of the vectors $\mathbf{x}\sin\gamma$ and $-\mathbf{y}\cos\gamma$, and the angle φ into the function $\varphi'(\varphi,u;\gamma)$ (10) of the variables u, φ, and γ. Therefore, the construction of expansions of the kinematical image $\Psi(\Omega'(\Omega;\gamma))=K(\gamma)\Psi(\Omega)$ in the angular bases (59) of arguments Ω implies a proof of the addition theorem. The first method of proving such a theorem is to work straightforwardly through the known relations in the proper way. As an example, let us expand the kinematical image

$$\langle\Omega|K(\gamma)|Y_{b'\beta'}(\hat{x})\rangle=Y_{b'\beta'}(\hat{x}') \tag{93}$$

of the spherical function $Y_{b'\beta'}(\hat{x})$ in a series in functions of the variables Ω. We apply (91) to the right-hand side of (93), and rewrite the result in the coordinates Ω. We express powers of trigonometric functions of the angles φ, φ', and γ in terms of the functions (55). In the end we obtain the desired expansion:

$$\langle\Omega|K(\gamma)|Y_{b'\beta'}(\hat{x})\rangle=Y_{b'\beta'}(\hat{x}')=\frac{(-1)^{b'}N_{b'0b'}}{W_{b'0b'}(\varphi'(\varphi,u;\gamma))}\times\sum_{c+d=b'}A^{b'}_{cd}N^{-2}_{b'cd}W_{b'cd}(\varphi)\times W_{b'cd}(\gamma)\mathcal{Y}^{b'\beta'}_{cd}(\hat{x},\hat{y}). \tag{94}$$

The following expression can be proved similarly:

$$\langle\Omega|K(\gamma)|Y_{a'\alpha'}(\hat{y})\rangle=Y_{a'\alpha'}(\hat{y}')=\frac{N_{a'a'0}}{W_{a'a'0}(\varphi'(\varphi,u;\gamma))}\times\sum_{c+d=a'}(-1)^c A^{a'}_{cd}N^{-2}_{a'cd}W_{a'cd}\times(\varphi)W_{a'dc}(\gamma)\mathcal{Y}^{a'\alpha'}_{cd}(\hat{x},\hat{y}). \tag{95}$$

Applying the kinematical-transformation rules (94) and (95) to each spherical function contained in the bispherical harmonic $\mathcal{Y}^{\ell m}_{a'b'}(\hat{x}',\hat{y}')$, we obtain its expansion in functions of the variables Ω:

$$\begin{aligned}\langle\Omega|K(\gamma)|\mathcal{Y}^{\ell m}_{a'b'}(\hat{x},\hat{y})\rangle&=\mathcal{Y}^{\ell m}_{a'b'}(\hat{x}',\hat{y}')\\&=(2a'+1)^{1/2}(2b'+1)^{1/2}\frac{(-1)^{b'}N_{a'+b',a'b'}}{4\pi W_{a'+b',a'b'}(\varphi'(\varphi,u;\gamma))}\\&\times\sum_{c+d=a'}(-1)^c A^{a'}_{cd}\sum_{e+f=b'}A^{b'}_{ef}\frac{W_{a'+b',d+e,c+f}(\gamma)}{N_{a'+b',d+e,c+f}}\\&\times\frac{W_{a'+b',c+e,d+f}(\varphi)}{N_{a'+b',c+e,d+f}}\sum_{gh}B^{g}_{ce}B^{h}_{df}\begin{Bmatrix}c&e&g\\d&f&h\\a'&b'&\ell\end{Bmatrix}\\&\times\mathcal{Y}^{\ell m}_{gh}(\hat{x},\hat{y}),\end{aligned} \tag{96}$$

$$B^{c}_{ab}\equiv((2a+1)(2b+1))^{1/2}C^{c0}_{a0b0}. \tag{97}$$

We note that, owing to (73), the coefficients B^{c}_{ab} are equal to zero when $(-1)^{a+b+\ell}=1$, and the proved relations (94)–(96) are true for all values of the five angles Ω and the parameter γ, i.e., they are five-dimensional identities in these angles.

The following proof of the addition theorem for the bases (59) is more elegant than the awkward chain of constructions described above. It consists of reducing the known multidimensional identities relating angular functions of the angles Ω and $\Omega'(\Omega,\gamma)$ to identities of lower dimension. This construction can be realized by choosing special values of the angles Ω for which the functions contained in the original identity take a simpler form or vanish. As an example, let us consider the identities (94)–(96) for special values of their arguments (12) and (13).

First we give the auxiliary formulas. At extremal values of their arguments, the spherical functions $Y_{c\delta}(\hat{q})$ vanish if $\delta\neq 0$ or become constant:

$$\begin{aligned}&Y_{c\delta}(0,0)=\delta_{\delta 0}\sqrt{(2c+1)/4\pi},\\&Y_{c\delta}(\theta_q,\varphi_q)=Y_{c\delta}(0,0)(-1)^{c(2-\theta_q/\pi)}\times(-1)^{\delta(2-\varphi_q/\pi)},\ \theta_q,\varphi_q=0,\pi.\end{aligned} \tag{98}$$

Therefore, in the case (12), when $\varphi_x,\theta_y,\theta_y=0$ and $\varphi_y=0,\pi$,

$$Y_{b\beta}(\hat{x})=Y_{b\beta}(0,0),$$

$$Y_{a\alpha}(\hat{y})=(-1)^{\alpha(2-\varphi_y/\pi)}(2\pi)^{-1/2}\Theta_{a\alpha}(u);$$

$$Y_{b'\beta'}(\hat{x}')=(-\operatorname{sign}\gamma)^{\beta'}$$
$$(-1)^{\beta'\varphi_y/\pi}(2\pi)^{1/2}\Theta_{b'\beta'}(u_{xx'}),$$
$$Y_{a'\alpha'}(\hat{y}')=(-1)^{\alpha'(1-\varphi_y/\pi)}(2\pi)^{-1/2}\Theta_{a'\alpha'}(u_{xy'}), \quad (99)$$

and in the case (13), when $\varphi_x,\theta_x,\theta_y=0$ and $\varphi_y=0,\pi$, we have

$$Y_{c\delta}(\hat{q})=Y_{c\delta}(0,0), \quad \hat{q}=\hat{x},\hat{y}; W_{Lab}(\varphi')=W_{Lab}(\varphi-\gamma);$$
$$Y_{b'\beta'}(\hat{x}')=(-1)^m Y_{b'\beta'}(0,0),$$
$$Y_{a'\alpha'}(\hat{y}')=(-1)^m Y_{a'\alpha'}(0,0). \quad (100)$$

Using (99), we can prove that in the case (12), Eqs. (94)–(96) degenerate into two-dimensional identities in the variables u and φ:

$$\Theta_{b'\beta'}(u_{xx'})=\frac{(-1)^{b'}N_{b'0b'}}{2\pi W_{b'0b'}(\varphi' u;\gamma))}$$
$$\times\sum_{c+d=b'}\sqrt{d+1/2}\frac{A^{b'}_{cd}}{N_{b'cd}}C^{b'\beta'}_{c\beta' do}W_{b'cd}(\varphi)$$
$$\times W_{b'cd}(\gamma)\Theta_{c\beta'}(u), \quad (101)$$

$$\Theta_{a'\alpha'}(u_{xy'})=\frac{N_{a'a'0}}{2\pi W_{a'a'0}(\varphi'(\varphi,u;\gamma))}$$
$$\times\sum_{c+d=a'}\sqrt{d+1/2}\frac{(-1)^c A^{a'}_{cd}}{N_{a'cd}}C^{a'\alpha}_{ca' do}$$
$$\times W_{a'cd}(\varphi)W_{a'dc}(\gamma)\Theta_{c\alpha'}(u), \quad (102)$$

$$\sum_{a'}(-\operatorname{sign}\gamma)^m C^{\ell m}_{a\alpha' b\beta'}\Theta_{a\alpha'}(u_{xy'})\Theta_{b'\beta'}(u_{xx'})$$
$$=((2a'+1)(2b'+1))^{1/2}\frac{(-1)^{b'}N_{a'+b',a'b'}}{W_{a'+b',a'b'}(\varphi'(\varphi,u;\gamma))}$$
$$\times\sum_{c+d=a'}(-1)^c A^{a'}_{cd}\sum_{e+f=b'}A^{b'}_{ef}\frac{W_{a'+b',d+e,c+f}(\gamma)}{N_{a'+b',d+e,c+f}}$$
$$\times\frac{W_{a'+b',c+e,d+f}(\varphi)}{N_{a'+b',c+e,d+f}}\sum_{gh}B^g_{ce}B^h_{df}\begin{Bmatrix} c & e & g\\ d & f & h\\ a' & b' & \ell\end{Bmatrix}$$
$$\times\sqrt{h+1/2}C^{\ell m}_{gmh0}\Theta_{gm}(u). \quad (103)$$

Using (73) and (100), we can show that in the case (13) the identity (94) with $\beta'\neq 0$, the identity (95) with $\alpha'\neq 0$, and the identity (96) with $m\neq 0$, $(-1)^{a'+b'}\neq(-1)^{\ell}$ degenerate into the trivial equation $0\equiv 0$. If β', α', m, $(-1)^{a'+b'-\ell}-1=0$, then in this case (13) the identities (94)–(96) generate the following addition theorems for the functions $W_{La'b'}$ with index $L=a'+b'$:

$$W_{b'0b'}(\varphi-\gamma)=(2\pi)^{-1}N_{b'0b'}\sum_{c+d=b'}A^{b'}_{cd}B^{b'}_{cd}N^{-2}_{b'cd}$$
$$\times W_{b'cd}(\varphi)W_{b'cd}(\gamma),$$

$$W_{a'a0}(\varphi-\gamma)=(2\pi)^{-1}N_{a'a'0}\sum_{c+d=a'}(-1)^c A^{a'}_{cd}B^{a'}_{cd}$$
$$\times N^{-2}_{a'cd}W_{a'cd}(\varphi)W_{a'dc}(\gamma),$$
$$W_{a'+b',a'b'}(\varphi-\gamma)=(-1)^{b'}(2a'+1)^{1/2}(2b'$$
$$+1)^{1/2}N_{a'+b',a'b'}\times\sum_{c+d=a'}$$
$$(-1)^c A^{a'}_{cd}\sum_{e+f=b'}A^{b'}_{ef}$$
$$\times\frac{W_{a'+b',d+e,c+f}(\gamma)}{N_{a'+b',d+e,c+f}}$$
$$\times\frac{W_{a'+b',c+e,d+f}(\varphi)}{N_{a'+b',c+e,d+f}}$$
$$\times\sum_{gh}B^g_{ce}B^h_{df}\begin{Bmatrix} c & e & g\\ d & f & h\\ a' & b' & \ell\end{Bmatrix}B^{\ell}_{gh}. \quad (104)$$

The proved relations (104) can also be viewed as the transformation rules for the function $W_{a+b,ab}(\varphi)$ when its argument is shifted by the angle γ. It is useful to show that the operator $K(\gamma)$ in the special case (13) effects just this transformation. For this we study the kinematical transformation (26) of the function $\Psi=W_{Lab}(\varphi)$. Owing to (42), (46), and (50), we have the following identities bidimensional in the arguments u and φ:

$$\langle\Omega|J|W_{Lab}(\varphi)\rangle=\langle\Omega|J_p|W_{Lab}(\varphi)\rangle u\partial_\varphi W_{Lab}(\varphi),$$
$$\langle\Omega|K(\gamma)|W_{Lab}(\varphi)\rangle=\langle\Omega|K_p(\gamma)W_{Lab}(\varphi)\rangle$$
$$=W_{Lab}(\varphi'(\varphi,u;\gamma)), \quad (105)$$

which describe the kinematical transformation of the function $W_{Lab}(\varphi)$ into a function of the variables u and φ. Setting $u=1$ in the identities (105), we pass to the case (13) and obtain operator formulas for differentiating and shifting the argument of the function $W_{Lab}(\varphi)$ by the angle γ:

$$J_p W_{Lab}(\varphi)=\partial_\varphi W_{Lab}(\varphi), \quad K_p(\gamma)W_{Lab}(\varphi)=W_{Lab}(\varphi-\gamma). \quad (106)$$

The third and most general approach to proving the addition theorems for the angular bases (59) is to study the overlap integrals between the basis functions and their kinematical images. This method will be demonstrated in the following two subsections by describing the known and new methods of reducing the dimension of such integrals.

2.2.4. The kinematical transformation of bispherical series

In order to reduce the calculations significantly, we proceed as follows. First we study the properties of the kinematical transformation (26), independent of the form of the factor $Q(r,\varphi)$, of the products

$$|\Psi\rangle\equiv|Q\ell ma'b'\rangle,\quad \Psi(\mathbf{r})=Q(r,\varphi)\mathcal{Y}^{\ell m}_{a'b'}(\hat{x},\hat{y}). \quad (107)$$

Setting $Q\equiv 1$, $\Psi^{\ell m}_{a'b'}$, $U^{\ell}_{a'b'}$, $W_{La'b'}$ in the expressions that we have obtained, we shall prove the rules for the kinematical transformation of the bispherical basis, the series (64) and (86), and the representation of the matrix elements of the operator $K(\gamma)$ in the hyperspherical basis in the form of double integrals.

According to Eqs. (25), (38), (40), and (49), the operators P, $\mathbf{l}^2$, and l_3 do not act on the variable φ and commute with the operators J and $K(\gamma)$. This means that functions of the type (107) and their kinematical images $J\Psi$ and $K(\gamma)\Psi$ are eigenfunctions of the operators P, ℓ^2, and ℓ_3 and correspond to the eigenvalues $\ell(\ell+1)$, m, and $\sigma'=(-1)^{a'+b'}$. However, the operators J and $K(\gamma)$ do not commute with the operators $\mathbf{l}_x^2$ and $\mathbf{l}_y^2$. Therefore, the coefficients $\langle\ell mab|K(\gamma)|\Psi\rangle$ of the expansion of the kinematical image of the function (107) in the bispherical basis calculated by the rules (64) are diagonal only in the quantum numbers l, m, and σ:

$$\langle\ell'm'ab|K(\gamma)|Q\ell ma'b'\rangle \equiv\delta_{\ell'\ell}\delta_{m'm}\delta_{\sigma'\sigma}\langle ab|K(\gamma)|Qa'b'\rangle_\ell(\varphi),$$

$$\langle ab|K(\gamma)|Qa'b'\rangle_\ell(\varphi)\equiv\int_{\mathcal{T}^4_{xy}}d\hat{x}d\hat{y}(\mathcal{Y}^{\ell m}_{ab}(\hat{x},\hat{y}))^*Q \times(r,\varphi')\mathcal{Y}^{\ell m}_{a'b'}(\hat{x}',\hat{y}'). \quad (108)$$

Here the variables $\hat{x}'$, $\hat{y}'$, and φ' are functions of the arguments $\hat{x}$, $\hat{y}$, φ and the parameter γ. The function $Q(\varphi')$ therefore cannot be removed from the integral, and the integral itself depends on the index ℓ and is a function of φ and γ. Let us study these functions.

Inserting the identities (47) and (51) between $\langle\ell mab|$ and $|Q\ell ma'b'\rangle$, and using the properties (74) of the operators (25), we prove the symmetry relations

$$\begin{aligned}\langle ab|K(\gamma)|Qa'b'\rangle_\ell(\varphi)&=\langle Qa'b'|K(-\gamma)|ab\rangle_\ell(\varphi)\\&=\langle ba|K(\gamma)|Qb'a'\rangle_\ell(\pi/2-\varphi)\\&=(-1)^a\langle ab|K(-\gamma)|Qa'b'\rangle_\ell(\varphi)\\&=(-1)^{\ell+a'}\langle ba|K(\pi/2-\gamma)\\&\quad\times|Qa'b'\rangle_\ell(\pi/2-\varphi)\\&=(-1)^{\ell+a}\langle ab|K(\pi/2-\gamma)\\&\quad\times|Qb'a'\rangle_\ell(\pi/2-\varphi). \quad (109)\end{aligned}$$

They imply that the functions $\langle ab|K(\gamma)|Qa'b'\rangle_\ell(\varphi)$ with any indices are even or odd with respect to the point $\gamma=0$ on the interval $\gamma\in(-\pi/2,\pi/2)$, and for $a=b$ and/or $a'=b'$ such functions possess definite parity relative to the point $\gamma=\pi/4$ on the interval $\gamma\in(0,\pi/2)$.

Using Eqs. (27) and (74), we obtain

$$\langle ab|K(0)|Qa'b'\rangle_\ell(\varphi)=(-1)^{a'+b'}\delta_{aa'}\delta_{bb'}Q(\varphi),$$

$$\langle ab|K(\pi/2)|Qa'b'\rangle_\ell(\varphi)=(-1)^{\ell-b'}\delta_{ab'}\delta_{a'b}Q(\pi/2-\varphi). \quad (110)$$

Owing to (108), the bispherical series (64) for the function $\Psi=K(\gamma)|Q\ell ma'b'\rangle$ degenerates into the double sum

$$\begin{aligned}K(\gamma)Q(r,\varphi)\mathcal{Y}^{\ell m}_{a'b'}(\hat{x},\hat{y})&=Q(r,\varphi')\mathcal{Y}^{\ell m}_{a'b'}(\hat{x}',\hat{y}')\\&=\sum_{ab}\langle ab|K(\gamma)|Qa'b'\rangle_\ell(\varphi)\\&\quad\times\mathcal{Y}^{\ell m}_{ab}(\hat{x},\hat{y}), \quad (111)\end{aligned}$$

where a and b take all possible values for given ℓ and σ. For $\gamma=0,\pi/2$ this sum contains a single term, owing to (110).

Equation (111) is valid for all values of the angles Ω, i.e., it is a five-dimensional identity in the variables $\hat{x}$, $\hat{y}$, and φ on which the variables $\hat{x}'$, $\hat{y}'$, and φ' depend. Equations (99) hold in the special case (12), and so the identity (111) becomes a two-dimensional identity in the variables φ and u:

$$\begin{aligned}Q(r,\varphi')\sum_{\alpha'}(-\operatorname{sign}\gamma)^{a'}C^{\ell m}_{a'\alpha'b'\beta'}\Theta_{a'\alpha'}(u_{xy'})\Theta_{b'\beta'}(u_{xx'})\\=\sum_{ab}(-\operatorname{sign}\gamma)^m\sqrt{b+1/2}C^{\ell m}_{amb0}\Theta_{am}(u)\\\times\langle ab|K(gg)|Qa'b'\rangle_\ell(\varphi), \quad (112)\end{aligned}$$

where u_{ab} and φ' are functions (10) and (11) of the variables φ, u and the parameter γ. In the special case (13) the identity (111), owing to (100) and the definitions (97) of the coefficients B^c_{ab}, degenerates into the trivial equation $0=0$ if m, $(-1)^{a+b+\ell}-1\neq 0$. If the latter are satisfied, this identity becomes a one-dimensional identity in the variable φ and the parameter γ:

$$B^{\ell}_{a'b'}Q(\varphi-\gamma)=\sum_{ab}B^{\ell}_{ab}\langle ab|K(\gamma)|Qa'b'\rangle_\ell(\varphi). \quad (113)$$

The four-dimensional integral (108) can be reduced to a one-dimensional one. First let us describe the simplest way of performing this reduction. We multiply each ($m=-\ell,\ldots,\ell$) identity (112) by the corresponding function $C^{\ell m}_{amb0}\Theta_{am}(u)$, integrate the resulting equations over the variable u on the segment $-1\leq u\leq 1$, and, finally, sum the resulting integral relations over the index m. The sum can be simplified by using the orthogonality conditions for the functions $\Theta_{\ell m}$ and the Clebsch–Gordan coefficients:[23]

$$\int_{-1}^{1}du\Theta_{am}(u)\Theta_{bm}(u)=\delta_{ab},$$

$$\sum_{m=-\ell}^{\ell}C^{\ell m}_{amb0}C^{\ell m}_{amc0}=\delta_{bc}\frac{2\ell+1}{2b+1}. \quad (114)$$

We thus obtain the desired representation for the integral (108):

$$\langle ab|K(\gamma)|Qa'b'\rangle_\ell(\varphi)=\int_{-1}^{1}duK^{\ell}_{aba'b'}(\varphi,u;\gamma)\times Q(\varphi'(\varphi,u;\gamma)), \quad (115)$$

$$K^{\ell}_{aba'b'}(\varphi,u;\gamma)=\frac{\sqrt{4b+2}}{2\ell+1}\times\sum_m(-\operatorname{sign}\gamma)^m C^{\ell m}_{amb0}\Theta_{am}(u)\times\sum_{\alpha'}(\operatorname{sign}\gamma)^{\alpha'}C^{\ell m}_{a'\alpha'b'\beta'}\Theta_{a'\alpha'}\times(u_{xy'})\Theta_{b'\beta'}(u_{xx'}). \quad (116)$$

In the five special cases where $\ell=0$ or $\ell\neq 0$ and two of the four indices (a,b,a',b') are zero, the double sum (116) becomes noticeably simpler:

$$K^{0}_{aaa'a'}(\varphi,u;\gamma)=(-1)^{a+b}((2a+1)(2a'+1))^{1/2}\times P_a(u)P_{a'}(u_{x'y'}),$$

$$K^{\ell}_{0\ell0\ell}(\varphi,u;\gamma)=P_{\ell}(u_{xx'}),$$

$$K^{\ell}_{\ell0\ell0}(\varphi,u;\gamma)=P_{\ell}(u_{yy'}),$$

$$K^{\ell}_{0\ell\ell0}(\varphi,u;\gamma)=P_{\ell}(u_{xy'}),\quad K^{\ell}_{\ell00\ell}(\varphi,u;\gamma)=P_{\ell}(u_{yx'}). \quad (117)$$

We recall that the arguments of the functions Θ_{am} and P_n in (116) and (117) are functions (11) of the independent variables u and φ.

We shall give another derivation[67] of (115)–(117), based on the transformation rules for bispherical functions under three-dimensional rotations. In the integral (108) we replace these functions by the sums (76) and change from the variables $\hat{x}$, $\hat{y}$ to the variables ω, u. Then in the integrand only the products

$$(D^{\ell}_{mm_p}(\omega))^{*}D^{\ell}_{m'm_p}(\omega)$$

will depend on the set ω of three Euler angles. If we use the orthogonality relations for the D functions[23] and calculate the integrals over angles ω of such a product, we obtain the one-dimensional integral (115) with kernel (116).

Another, very awkward, method of transforming the integral (108) into a one-dimensional integral (115) and deriving a representation of the function $K^{\ell}_{aba'b'}$ in the form of a sevenfold sum is described in detail in Ref. 21 for the case of equal particle masses. We shall describe this method for arbitrary masses. In the integral (108) we replace the function $\mathcal{Y}^{\ell m}_{a'b'}$ by the series (96), and expand the function $W^{-1}_{a'+b',a'b'}Q$ in a series in the basis (92):

$$W^{-1}_{a'+b',a'b'}(\varphi')Q(\varphi')=2\pi\sum_{n=0}^{\infty}(-1)^n\mathcal{Y}^{00}_{nn}(\hat{x},\hat{y})\times\int_{-1}^{1}du\Theta_{n0}(u)W^{-1}_{a'+b',a'b'}\times(\varphi'(u,\varphi;\gamma))Q(\varphi'(\varphi,u;\gamma)). \quad (118)$$

Then we integrate the products of three bispherical functions over the torus $\mathcal{T}^4_{xy}$, using the equation[23]

$$\mathcal{Y}^{00}_{nn}(\hat{x},\hat{y})\mathcal{Y}^{\ell m}_{ab}(\hat{x},\hat{y})=\frac{(-1)^{b+\ell+n}}{4\pi(2n+1)}\sum_{cd}(-1)^c B^c_{an}B^d_{bn}\times\begin{Bmatrix} d & c & \ell\\ a & b & n\end{Bmatrix}\mathcal{Y}^{\ell m}_{cd}(\hat{x},\hat{y}), \quad (119)$$

where the coefficients B^a_{cd} are given by (97). Using the resulting equations

$$\Gamma^{\ell n}_{aba'b'}\equiv\int_{\mathcal{T}^4_{xy}}d\hat{x}d\hat{y}(\mathcal{Y}^{\ell m}_{ab}(\hat{x},\hat{y}))^{*}\mathcal{Y}^{00}_{nn}(\hat{x},\hat{y})\mathcal{Y}^{\ell m}_{a'b'}(\hat{x},\hat{y})=\frac{(-1)^{a+b'+\ell+n}}{4\pi(2n+1)}B^a_{a'n}B^b_{b'n}\begin{Bmatrix} b & a & \ell\\ a' & b' & n\end{Bmatrix}, \quad (120)$$

we reduce the integral of interest (108) to a one-dimensional integral (115) with kernel

$$\begin{aligned}K^{\ell}_{aba'b'}(\varphi,u;\gamma)=&(-1)^{a'+b'+\ell}(2a+1)(2b+1)\\&\times(\Gamma(2a+1)\Gamma(2b+1))^{1/2}N_{a'+b',a'b'}\\&\times W^{-1}_{a'+b',a'b'}(\varphi'(\varphi,u;\gamma))\\&\times\sum_{n=0}^{(a+b+a'+b')/2}(-1)^n\sqrt{4n+2}\Theta_{n0}(u)\\&\times\sum_{d+e=a}\sum_{f+g=b}(-1)^{d+f}(\Gamma(2d+1)\Gamma\\&\times(2e+1)\Gamma(2f+1)\Gamma(2g+1))^{1/2}\\&\times N^{-1}_{a+b,e+f,d+g}N^{-1}_{a+b,d+f,e+g}\\&\times W_{a+b,e+f,d+g}(\gamma)W_{a+b,d+f,e+g}(\varphi)\\&\times\sum_{s,t}B^s_{df}B^t_{eg}B^{a'}_{ns}B^{b'}_{nt}\begin{Bmatrix} b' & a' & \ell\\ s & t & n\end{Bmatrix}\\&\times\begin{Bmatrix} d & e & a\\ f & g & b\\ s & t & \ell\end{Bmatrix}\end{aligned} \quad (121)$$

Up to now, the variables φ and u have been the arguments of the kernels $K^{\ell}_{aba'b'}$ of the integrals (115) and the arbitrary function $Q(r,\varphi'(\varphi,u;\gamma))$, and the variable u has been the integration variable. We shall describe in more detail than in Ref. 21 the variable replacement $u\to\varphi'$. We use (10) to express the variable u as a function of the arguments φ and φ':

$$u(\varphi,\varphi';\gamma)=(\cos 2\varphi'-\cos 2\gamma\cos 2\varphi)\operatorname{cosec}2\gamma\operatorname{cosec}2\varphi. \quad (122)$$

Substituting u in this form into (11), we obtain the functions $u_{ab}=u_{ab}(\varphi,\varphi';\gamma)$ of the new variables φ and φ':

$$u_{xx'}=-\sec\varphi'(\cos 2\varphi+\cos 2\varphi'+\cos 2\gamma+1)/4,$$

$$u_{xy'}=-\operatorname{cosec}\varphi'(\cos 2\varphi-\cos 2\varphi'-\cos 2\gamma+1)/4,$$

$$u_{yx'}=\sec\varphi'(\cos 2\varphi-\cos 2\varphi'+\cos 2\gamma-1)/4,$$

$$u_{yy'}=\operatorname{cosec}\varphi'(\cos 2\varphi+\cos 2\varphi'-\cos 2\gamma-1)/4,$$

$$u_{x'y'}=\operatorname{cosec}2\varphi'\operatorname{cosec}2\gamma(\cos 2\varphi+\cos 2\varphi'\cos 2\gamma)/4. \tag{123}$$

We calculate the Jacobian of the transformation $u\to\varphi'$,

$$\partial_{\varphi'}u(\varphi,\varphi';\gamma)=t(\varphi,\varphi';\gamma)\operatorname{sign}\gamma,$$

$$t(\varphi,\varphi';\gamma)\equiv\frac{\sin 2\varphi'}{\sin 2\varphi|\sin 2\gamma|}, \tag{124}$$

and the limits C_- and C_+ of the integration over the new variable φ' corresponding to the boundary values $u=-1$ and $u=1$ of the variable u. If $\gamma>0$, then we always have $\partial_{\varphi'}u\geqslant 0$, and the upper limit C_+ is not smaller than the lower limit C_-. If $\gamma<0$, then $\partial_{\varphi'}u\leqslant 0$, and the upper limit C_+ does not exceed the lower limit C_-. For $\gamma<0$ we can interchange the integration limits and change the sign of the integral. Such an identity transformation is not necessary if the derivative (124) is replaced by its modulus t, and the lower and upper limits C_- and C_+ are redefined as

$$C_-(\varphi;\gamma)\equiv|\varphi-|\gamma||,\quad C_+(\varphi;\gamma)\equiv\min\{\varphi+\gamma,\pi-\varphi-|\gamma|\}. \tag{125}$$

Using Eqs. (123)–(125), we write the integral (115) as

$$\langle ab|K(\gamma)|Qa'b'\rangle_\ell(\varphi)=\int_{C_-(\varphi;\gamma)}^{C_+(\varphi;\gamma)}d\varphi'\times K^\ell_{aba'b'}(\varphi,\varphi';\gamma)Q(r,\varphi'), \tag{126}$$

$$K^\ell_{aba'b'}(\varphi,\varphi';\gamma)\equiv t(\varphi,\varphi';\gamma)K^\ell_{aba'b'}(\varphi,u(\varphi,\varphi';\gamma)). \tag{127}$$

Let $Q\equiv 1$. Then the function (107) is a bispherical harmonic, and Eqs. (109) and (110) describe the symmetry and limiting values of the matrix elements (108) of the operator $K(\gamma)$ in the bispherical basis. These matrix elements satisfy the sum rules (112) and (113) and are represented by integrals (115) of the functions (116), (117), or (121).

For $Q\equiv\Psi^{\ell m}_{a'b'}$, Eqs. (115), (116) and (126), (127) become the rules for kinematical transformation of the term $\Psi^{\ell m}_{a'b'}\mathcal{Y}^{\ell m}_{a'b'}$ of the bispherical series (64) of the arbitrary function Ψ. Applying them to each term of this series, we obtain its expansion in bispherical functions $\mathcal{Y}^{\ell m}_{ab}(\hat{x},\hat{y})$:

$$\begin{aligned}K(\gamma)\sum_{\ell m}\sum_{a'b'}&\Psi^{\ell m}_{a'b'}(r,\varphi)\mathcal{Y}^{\ell m}_{a'b'}(\hat{x},\hat{y})\\&=\sum_{\ell m}\sum_{a'b'}\Psi^{\ell m}_{a'b'}(r,\varphi')\mathcal{Y}^{\ell m}_{a'b'}(\hat{x}',\hat{y}')\\&=\sum_{\ell m}\sum_{ab}\mathcal{Y}^{\ell m}_{ab}(\hat{x},\hat{y})\int_{-1}^{1}duK^\ell_{aba'b'}(\varphi,u;\gamma)\Psi^{\ell m}_{a'b'}\\&\quad\times(r,\varphi'(\varphi,u;\gamma))\\&=\sum_{\ell m}\sum_{ab}\mathcal{Y}^{\ell m}_{ab}(\hat{x},\hat{y})\int_{C_-(\varphi;\gamma)}^{C_+(\varphi;\gamma)}d\varphi'K^\ell_{aba'b'}(\varphi,\varphi';\gamma)\\&\quad\times\Psi^{\ell m}_{a'b'}(r,\varphi').\end{aligned} \tag{128}$$

We note that according to these expressions, the kinematical image of the function $\Psi(x)$ of the modulus of the vector $\mathbf{x}$, i.e., the bispherical series from one term

$$\Psi(x)=\mathcal{Y}^{00}_{00}(\hat{x},\hat{y})\Psi^{00}_{00}(x),\quad\Psi^{00}_{00}(x)=(4\pi)\Psi(x),$$

is an infinite sum over all bispherical functions with ℓ, $m=0$:

$$K(\gamma)\Psi(x)=\sum_{ab}\mathcal{Y}^{00}_{ab}(\hat{x},\hat{y})\int_{-1}^{1}duK^0_{ab00}(\varphi,u;\gamma)\Psi\times(r\cos\varphi'(\varphi,u;\gamma)). \tag{129}$$

If $Q\equiv U^\ell_{a'b'}$, Eqs. (115), (116) and (126), (127) describe the kinematical transformation of the term $U^\ell_{a'b'}\mathcal{Y}^{\ell m}_{a'b'}$ of the series (86) of the function (84) satisfying the conditions (85). For this function

$$\begin{aligned}K(\gamma)\psi^\varepsilon(\mathbf{r})&=K(\gamma)(xy)^{-1}\sum_{a'b'}U^\ell_{a'b'}(r,\varphi)\mathcal{Y}^{\ell m}_{a'b'}(\hat{x},\hat{y})\\&=(x'y')^{-1}\sum_{a'b'}U^\ell_{a'b'}(r,\varphi')\mathcal{Y}^{\ell m}_{a'b'}(\hat{x}',\hat{y}')\\&=(xy)^{-1}\sum_{ab}\sum_{a'b'}\mathcal{Y}^{\ell m}_{ab}(\hat{x},\hat{y})\\&\quad\times\langle r,\varphi|h^\ell_{aba'b'}|U^\ell_{a'b'}\rangle.\end{aligned} \tag{130}$$

Here we have introduced the integral operator $h^\ell_{aba'b'}$. By definition, it maps an arbitrary function $Q(r,\varphi)$ specified in $\mathcal{R}^2_+$ into the one-dimensional integrals

$$\begin{aligned}\langle r,\varphi|h^\ell_{aba'b'}|Q\rangle&\equiv\int_{-1}^{1}duh^\ell_{aba'b'}(\varphi,u;\gamma)Q(r,\varphi'(\varphi,u;\gamma))\\&=\int_{C_-(\varphi;\gamma)}^{C_+(\varphi;\gamma)}d\varphi'h^\ell_{aba'b'}(\varphi,\varphi';\gamma)Q(r,\varphi'),\end{aligned} \tag{131}$$

the kernels of which are related to the functions (116) and (127) as

$$h^\ell_{aba'b'}(\varphi,u;\gamma)\equiv(xy/x'y')K^\ell_{aba'b'}(\varphi,u;\gamma),$$

$$h^\ell_{aba'b'}(\varphi,\varphi';\gamma)\equiv(\sin 2\varphi/\sin 2\varphi')K^\ell_{aba'b'}(\varphi,\varphi';\gamma). \tag{132}$$

Now, using the trick described in Ref. 67, we reduce the matrix elements of the operator $K(\gamma)$ in the basis of hyperharmonics to two-dimensional integrals with kernels $K^\ell_{aba'b'}$. For this we set $Q=W_{La'b'}(\varphi)$. We multiply the integral (108) by the function $\rho(\varphi)W_{Lab}(\varphi)$. The resulting expression is integrated over the variable φ on the segment $[0,\pi/2]$. The resulting integral will be the integral of a product of hyperharmonics $\langle L\ell mab|\Omega\rangle$ and $\langle\Omega|K(\gamma)|L\ell ma'b'\rangle$ over the hypersphere $\mathcal{S}^5$. Integrating over the torus $\mathcal{T}^4_{xy}$ by any of the methods described above, we obtain the desired representation for this integral:

$$\langle L\ell mab|K(\gamma)|L\ell ma'b'\rangle$$
$$\equiv \int_{S^5} d\Omega (Y^{\ell m}_{Lab}(\Omega))^* Y^{\ell m}_{La'b'}(\Omega')$$
$$= \int_0^{\pi/2} \rho(\varphi) W_{Lab}(\varphi)$$
$$\times \int_{-1}^{1} du K^{\ell}_{aba'b'}(\varphi,u;\gamma) W_{La'b'}(\varphi'). \quad (133)$$

Using the definitions (79), (131), and (132), we rewrite this representation as a relation between the matrix elements of the operators $K(\gamma)$ and $h^{\ell}_{aba'b'}$ in the basis of hyperharmonics and functions $\tilde{W}_{Lab}$:

$$\langle L\ell mab|K(\gamma)|L\ell ma'b'\rangle = \langle L\tilde{a}b|h^{\ell}_{aba'b'}(\gamma)|L\tilde{a}'b'\rangle$$
$$\equiv \int_{\pi/2}^{0} d\varphi \tilde{W}_{Lab}(\varphi) \int_{C_-(\varphi;\gamma)}^{C_+(\varphi;\gamma)}$$
$$\times d\varphi' h^{\ell}_{aba'b'}(\varphi,\varphi';\gamma)$$
$$\times \tilde{W}_{La'b'}(\varphi'). \quad (134)$$

In principle, the representations (133) and (134) allow the properties of all the matrix elements of the operator $K(\gamma)$ in the hyperspherical basis to be reproduced in terms of the known properties of the kernels $K^{\ell}_{aba'b'}$ or $h^{\ell}_{aba'b'}$. This method of studying the kinematical transformation of the hyperharmonics has so far not been realized, owing to the lack of a complete analysis of these kernels as functions of the two variables φ, u or φ, φ'; moreover, it appears too complicated. A simpler approach based on the use of the operator methods of the theory of angular momentum applied directly to the hyperharmonics is described below.

2.2.5. The kinematical transformation of hyperspherical series

Let us first describe the kinematical transformation (26) of the hyperharmonics (55) in operator language. In the hyperharmonics basis the unit operator has the very simple spectral representation

$$I = \sum_{L\ell mab} |L\ell mab\rangle\langle L\ell mab|, \quad (135)$$

and the matrices of the operators J and K are fairly sparse, owing to the commutation relations (40) and (49). Because of these relations, the images $J|L\ell mab\rangle$ and $K(\gamma)|L\ell mab\rangle$ of any hyperharmonic $|L\ell mab\rangle$ are eigenfunctions of the operators P, $\mathbf{l}^2$, l_3, and L^2, but not of the operators $\mathbf{l}_x^2$ and $\mathbf{l}_y^2$. In other words, in a kinematical transformation of the hyperharmonics $|L\ell mab\rangle$, both by an infinitesimal angle and by any finite angle γ, the quantum numbers L, ℓ, m, and $\sigma=(-1)^L$ are conserved, but the numbers a and b are not conserved. This means that the representations

$$\langle L'\ell'm'a'b'|Q|L\ell mab\rangle = \delta_{L'L}\delta_{\ell'\ell}\delta_{m'm}\langle a'b'|Q|ab\rangle_{L\ell},$$
$$Q=J,K(\gamma) \quad (136)$$

are valid. According to the definition (30), the operator J is the scalar product of the three-dimensional vectors $\mathbf{x}$ and ∇_y, antisymmetrized with respect to the interchange $\mathbf{x}\leftrightarrow\mathbf{y}$. As is well known,[23] in general the operators for multiplication by the vector $\mathbf{q}$ and the gradient operator ∇_q map a spherical harmonic $Y_{\ell m}(\hat{\mathbf{q}})$ onto a linear combination of two spherical harmonics $Y_{\ell\pm1,m}(\hat{\mathbf{q}})$. Therefore, in general, the function $J|L\ell mab\rangle$ is a linear combination of the type

$$J|L\ell mab\rangle = \sum_{\mu,\nu=\pm1} \langle a+\mu,b+\nu|J|mab\rangle_{L\ell}$$
$$\times |L\ell m,a+\mu,b+\nu\rangle. \quad (137)$$

Let us act with the operator J on this combination. We express the images $J|L\ell m,a+\mu,b+\nu\rangle$ as sums of the same type (137). We repeat this construction $n-2$ times. Then the image $J^n|L\ell mab\rangle$ is a linear combination of hyperharmonics $|L\ell m.a',b'\rangle$ with indices a' and b' given by the expressions $a'=a-n+2\mu$ and $b'=b-n+2\nu$, where μ and ν take only those integer values from 0 to n for which $|a'-b'|\leqslant\ell\leqslant a'+b'$. This means that the following selection rules hold for the operator J^n:

$$\langle a'b'|J^n|ab\rangle_{L\ell}=0, \quad n<\max\{|a-a'|,|b-b'|\}, \quad (138)$$

while all the other matrix elements are, in general, nonzero. Using these selection rules and the fact that for $|\gamma|\neq0,\pi/2$ the operator $K(\gamma)$ is an infinite series (33) in integer powers of the operator J, we prove that in general

$$K(\gamma)|L\ell ma'b'\rangle = \sum_{ab} \langle ab|K(\gamma)|a'b'\rangle_{L\ell}|L\ell mab\rangle, \quad (139)$$

where the indices a and b take all possible values for given L and ℓ. Using (27) and (74), we can show that for $\gamma=0,\pi/2$ each row of the matrix of the operator $K(\gamma)$ has only one nonzero element:

$$\{ab|K(0)|a'b'\}_{L\ell}=(-1)^L\delta_{aa'}\delta_{bb'}, \quad (140)$$
$$\langle ab|K(\pi/2)|a'b'\rangle_{L\ell}=(-1)^{(L-2\ell+a'-b')/2}\delta_{ab'}\delta_{a'b}. \quad (141)$$

Therefore, for $\gamma=0,\pi/2$ the sum (139) consists of a single term.

Now let us describe the kinematical transformation of hyperharmonics in coordinate space $\langle r,\Omega|$. In general, the relations (139) between the hyperharmonics before and after the kinematical transformation are identities with respect to the five independent variables $\Omega=(\theta_x,\varphi_x,\theta_y,\varphi_y,\varphi)$ and the indices a' and b':

$$\langle\Omega|K(\gamma)|L\ell ma'b'\rangle = Y^{\ell m}_{La'b'}(\Omega'(\Omega;\gamma))$$
$$= \sum_{ab} \langle ab|K(\gamma)|a'b'\rangle_{L\ell} Y^{\ell m}_{Lab}(\Omega). \quad (142)$$

Equations (99) are satisfied in the plane $\mathcal{P}$ [the case (12)], and so the hyperharmonics and their kinematical images become functions of two instead of five variables. Therefore, in the plane $\mathcal{P}$ the relations (142) degenerate into two-dimen-

sional identities with respect to the independent variables u and φ, the parameter γ, and the indices a' and b':

$$\sum_{ab}(-\mathrm{sign}\gamma)^m\sqrt{b+1/2}C^{\ell m}_{amb0}\Theta_{am}(u)W_{lab}(\varphi)\times\langle ab|K(\gamma)|a'b'\rangle_{L\ell}=\sum_{\alpha}(\mathrm{sign}\gamma)^{\alpha}C^{\ell m}_{a'ab'\beta}\Theta_{a'\alpha}(u_{xy'})\Theta_{b'\beta}\times(u_{xx'})W_{La'b'}(\varphi'). \quad (143)$$

Equations (100) are valid on the line $\mathcal{L}_3$ [the case (13)], and according to them the hyperharmonics and their kinematical images are identically zero if $m\neq 0$ or $(-1)^{\ell+L}\neq 1$, while otherwise they degenerate into functions of a single variable. Therefore, on the line $\mathcal{L}_3$ the identities (142) and (143) degenerate into trival relations $0=0$ if $m\neq 0$ or $(-1)^{\ell+L}=-1$, while in the case $m=(-1)^{\ell+L}-1=0$ they take the form of one-dimensional identities in the hyperangle φ, the parameter γ, and the indices a' and b':

$$\sum_{ab}B^{\ell}_{ab}W_{Lab}(\varphi)\langle ab|K(\gamma)|a'b'\rangle_{L\ell}=(-1)^L B^{\ell}_{a'b'}W_{La'b'}(\varphi-\gamma), \quad (144)$$

where the coefficients B^c_{ab} are given by (97).

The hyperradius does not change in a kinematical transformation of coordinates. Therefore, the expressions for the kinematical transformation (26) of the hyperspherical series (65) for an arbitrary function Ψ can be derived by applying the rule (142) to each hyperharmonic of this series. Accordingly,

$$\left\langle r,\Omega|K(\gamma)|\Psi(\mathbf{r})=\sum_{\ell m}\sum_{La'b'}\Psi^{\ell m}_{La'b'}(r)Y^{\ell m}_{La'b'}(\Omega')\right\rangle=\sum_{\ell m}\sum_{Lab}Y^{\ell m}_{Lab}(\Omega)\times\sum_{a'b'}\langle ab|K(\gamma)|a'b'\rangle_{L\ell}\Psi^{\ell m}_{La'b'}(r). \quad (145)$$

Since this chain of equalities is valid for any function of the class C^2, and the hyperharmonics form an orthonormal basis in this class, the spectral representation

$$K(\gamma)=\sum_{\ell m}\sum_{aba'b'}\sum_{L}|L\ell mab\rangle\langle ab|K(\gamma)\times|a'b'\rangle_{L\ell}\langle L\ell ma'b'| \quad (146)$$

is valid, where the action of the bra and ket operators is given by (65). Using the definition (55) of these operators as products of the operators $|\ell mab\rangle$ and $|L\ell mab\rangle$, we separate the subsum $K^{\ell}_{aba'b'}(\gamma)$ over the index L from the sum (146). Then we obtain

$$K(\gamma)=\sum_{\ell m}\sum_{ab}|\ell mab\rangle\sum_{a'b'}k^{\ell}_{aba'b'}(\gamma)\langle\ell ma'b'|, \quad (147)$$

$$K^{\ell}_{aba'b'}(\gamma)\equiv\sum_{L}|Lab\rangle\langle ab|K(\gamma)|a'b'\rangle_{L\ell}\langle La'b'|. \quad (148)$$

Owing to (128) and (145), any of the operators $K^{\ell}_{aba'b'}(\gamma)$ acts on the bispherical component $\Psi^{\ell m}_{a'b'}(r,\varphi)$ exactly like the integral operator (126) with the kernel (121):

$$\langle\mathbf{r}|k^{\ell}_{aba'b'}|\Psi^{\ell m}_{a'b'}(r,\varphi)\rangle=\int_{C_-(\varphi;\gamma)}^{C_+(\varphi;\gamma)}d\varphi' K^{\ell}_{aba'b'}\times(\varphi,\varphi';\gamma)\Psi^{\ell m}_{a'b'}(r,\varphi'). \quad (149)$$

To complete our description of the kinematical transformation of hyperspherical series, we still need to list the properties of the matrix elements of the operators J and K and the methods of calculating them.

Matrix elements of the operator J. Owing to the unitary similarity (46), the operators J and J_p have identical matrix elements (136) in the basis of hyperharmonics. Using (77), (99), and the variable replacement $\hat{x},\hat{y}\to\omega,u$, we reduce the matrix elements of the operator J_p to double integrals:

$$\langle ab|J|a'b'\rangle_{L\ell}=\langle ab|J_p|a'b'\rangle_{L\ell},$$

$$\langle ab|J_p|a'b'\rangle_{L\ell}=C^{\ell m}_{amb0}C^{\ell m}_{a'mb'0}\int_0^{\pi/2}d\varphi\rho(\varphi)W_{Lab}(\varphi)\times\int_{-1}^{1}du\Theta_{am}(u)\times J_p(\sigma,u)W_{La'b'}(\varphi)\Theta_{a'm}(u). \quad (150)$$

To calculate these integrals, we transform the integrands as

$$J_p\Theta_{c\delta}(u)W_{lab}(\varphi)=(1-u^2)\delta_u\Theta_{c\delta}(u)\mathrm{ctg}2\varphi W_{Lab}(\varphi)+u\Theta_{c\delta}(u)\partial_{\varphi}W_{Lab}. \quad (151)$$

First, let us rewrite the known relations[23] in the compact form

$$(1-u^2)\partial_u\Theta_{c\delta}(u)=\sum_{r=\pm 1}(c+1)(2c)^{\tau}C^{c\delta}_{\tau}\Theta_{c+\tau,\delta}(u),$$

$$u\Theta_{c\delta}(u)=\sum_{r=\pm 1}C^{c\delta}_{\tau}\Theta_{c+\tau,\delta}(u),$$

$$C^{c\delta}_{\tau}\equiv\left(\frac{(c+1/2+\tau/2)^2-\delta^2}{4(c+1/2+\tau/2)^2-1}\right)^{1/2}. \quad (152)$$

Now we shall prove a series of identities for the functions W_{Lab}. We shall express the operators $\cot 2\varphi$ and ∂_{φ} contained in Eq. (151) in terms of more convenient ones. Using the well known[6] differentiation formula and the relations for the Jacobi polynomials $P_n^{(a+1/2,b+1/2)}$ with superscripts differing by ± 1, we verify that the operators

$$W^{ab}_{++}=\partial_{\varphi}-a\cot\varphi+b\tan\varphi,$$

$$W^{ab}_{-+}\equiv\partial_\varphi+(a+1)\cot\varphi+b\tan\varphi,$$

$$W^{ab}_{+-}\equiv\partial_\varphi-a\cot\varphi-(b+1)\tan\varphi,$$

$$W^{ab}_{--}\equiv\partial_\varphi+(a+1)\cot\varphi-(b+1)\tan\varphi, \quad (153)$$

change the indices a and b of the function W_{Lab} by unity, but conserve the index L:

$$W^{ab}_{\mu\nu}W_{Lab}(\varphi)=\langle a+\mu,b+\nu|W^{ab}_{\mu\nu}|ab\rangle_L W_{L,a+\mu,b+\nu}(\varphi),$$

$$\mu,\nu=\pm1;$$

$$\langle a+1,b+1|W^{ab}_{++}|ab\rangle_L=-2(n(n+a+b+2))^{1/2},$$

$$\langle a-1,b+1|W^{ab}_{-+}|ab\rangle_L=((2n+2a+1)(2n+2b+3))^{1/2},$$

$$\langle a+1,b-1|W^{ab}_{+-}|ab\rangle_L=-((2n+2b+1)(2n+2a+3))^{1/2},$$

$$\langle a-1,b-1|W^{ab}_{--}|ab\rangle_L=2((n+1)(n+a+b+1))^{1/2}. \quad (154)$$

Since the four operators $W^{ab}_{\pm1,\pm1}$ are linearly dependent,

$$\sum_{\nu=\pm1}(W^{ab}_{\nu\nu}-W^{ab}_{\nu,-\nu})=0, \quad (155)$$

the operators $2\cot2\varphi$ and ∂_φ can be written as linear combinations of four or three such operators. For example,

$$\partial_\varphi=W^{ab}_{++}-b(2b+1)^{-1}W^{ab}_{-+}-a(2a+1)^{-1}W^{ab}_{+-}+\left(1-\frac{a+b+1}{(2a+1)(2b+1)}\right)W^{ab}_{--}$$

$$=\frac{a+b+1}{(2a+1)(2b+1)}W^{ab}_{++}+b(2b+1)^{-1}W^{ab}_{-+}+a(2a+1)^{-1}W^{ab}_{+-}, \quad (156)$$

$$2\cot2\varphi=-(2a+1)^{-1}W^{ab}_{++}+(2b+1)^{-1}W^{ab}_{--}+\frac{2(b-a)}{(2a+1)(2b+1)}(W^{ab}_{-+}-W^{ab}_{+-})$$

$$=(2a+1)^{-1}(W^{ab}_{-+}-W^{ab}_{++})+(2b+1)^{-1}(W^{ab}_{+-}-W^{ab}_{++}). \quad (157)$$

Using Eqs. (153)–(155), from (156) we derive two equivalent expressions for differentiating the functions W_{Lab}:

$$\partial_\varphi W_{Lab}(\varphi)=2\left[1-\frac{a+b+1}{(2a+1)(2b+1)}\right]\times[(n+1)(n+a+b+1)]^{1/2}W_{L,a-1,b-1}(\varphi)-\frac{b}{b+1}[(2n+2a+1)(2n+2b+3)]^{1/2}\times W_{L,a-1,b+1}(\varphi)+\frac{a}{a+1}[(2n+2b+1)(2n+2a+3)]^{1/2}W_{L,a+1,b-1}(\varphi)-2[n(n+a+b+2)]^{1/2}W_{L,a+1,b+1}(\varphi), \quad (158)$$

$$\partial_\varphi W_{Lab}(\varphi)=\frac{a}{2a+1}[(2n+2a+1)(2n+2b+3)]^{1/2}\times W_{L,a-1,b+1}(\varphi)-\frac{a}{2b+1}[(2n+2b+1)(2n+2a+3)]^{1/2}W_{L,a+1,b-1}(\varphi)-2\frac{a+b+1}{(2a+1)(2b+1)}[n(n+a+b+2)]^{1/2}\times W_{L,a+1,b+1}(\varphi). \quad (159)$$

Owing to the equivalence, the functions $W_{L,a\pm1,b\pm1}$ are linearly dependent.

Using the expansions (152), (156), and (157), we reduce (151) to the sum

$$J_p\Theta_{c\delta}W_{Lab}(\varphi)=\sum_{\tau,u,\nu=\pm1}J^{Labc}_{\tau\mu\nu}\Theta_{c,\delta+\tau}(u)\times W_{L,a+\mu,b+\nu}(\varphi), \quad (160)$$

$$J^{Labc\delta}_{\tau\mu\nu}=\left(\frac{(c+1/2+\tau/2)^2-\delta^2}{4(c+1/2+\tau/2)^2-1}\right)^{1/2}\cdots, \quad (161)$$

implying that the operator J maps functions of the type $\Theta_{c\delta}(u)W_{Lab}(\varphi)$ into finite linear combinations of four functions of the same type. We calculate the integrals (150) by using this fact and the known relations[23] between Clebsch–Gordan coefficients with angular-momentum indices differing by ±1. In the end we prove the selection rules (138) for the operator J and find all its nonzero matrix elements explicitly. To describe them completely we use the expressions

$$\langle a+1,b+1|J|ab\rangle_{L\ell}=i(n(n+a+b+2))^{1/2}\times\left\{\frac{(\ell+a+b+2)(\ell+a+b+3)(\ell-a-b-1)(\ell-a-b-2)}{(2a+1)(2a+3)(2b+1)(2b+3)}\right\}^{1/2}$$

$$\times\langle a-1,b+1|J|ab\rangle_{L\ell}=(i/2)((2n+2a+1)(2n+2b+1))^{1/2}\times\left\{\frac{(\ell+a-b+1)(\ell+b-a+2)(\ell+a-b-1)(\ell+a-b)}{(2a-1)(2a+1)(2b+1)(2b+3)}\right\}^{1/2}, \quad (162)$$

in which $n=(L-a-b)/2$, and we list the symmetry properties

$$\langle a'b'|J|ab\rangle_{L\ell}=-\langle ab|J|a'b'\rangle_{L\ell},$$

$$\langle a+1,b-1|J|ab\rangle_{L\ell}=-\langle a-1,b+1|J|ab\rangle_{L\ell},$$

$$\langle a+\mu,b+\nu|J|ab\rangle_{L\ell}=(-1)^{\mu+\nu+1}\langle b+\nu,\ a+\mu|J|ba\rangle_{L\ell} \quad (163)$$

and the additional selection rules

$$\langle a+\nu,a-\nu|J|aa\rangle_{L\ell}=0,\quad \nu=\pm1,\quad \ell=0,1. \quad (164)$$

The coefficients $\langle ab|J^n|a'b'\rangle_{L\ell}$ with $n>1$ can be expressed in terms of a sum of matrix elements of the operator J by using the identity $J^n\equiv JIJ\ldots JIJ$ and Eqs. (135), (162), and (163). For example, for $n=2$ we have

$$\langle ab|J^2|a'b'\rangle_{L\ell}=\sum_{\mu,\nu=\pm 1}\langle ab|J|a'+\mu,b'+\nu\rangle_{L\ell}\times\langle a'+\mu,b'+\nu|J|a'b'\rangle_{L\ell}.$$

Matrix elements of the operator K. Let us begin by analyzing the global properties of the matrix elements of the operator $K(\gamma)$. Writing the identities (47) and (51) in the hyperharmonics basis and taking into account the properties (74) of the operators (25), we prove the symmetry relations

$$\begin{aligned}\langle ab|K(\gamma)|a'b'\rangle_{L\ell}&=\langle a'b'|K(-\gamma)|ab\rangle_{L\ell}\\&=(-1)^{(a-b+a'-b')/2}\langle ba|K(\gamma)|b'a'\rangle_{L\ell}\\&=(-1)^{a+a'}\langle ab|K(-\gamma)|a'b'\rangle_{L\ell}\\&=(-1)^{b+b'}\langle ab|K(-\gamma)|a'b'\rangle_{L\ell}\\&=(-1)^{\ell+b'+(L+a+b)/2}\\&\quad\times\langle ba|K(\pi/2-\gamma)|a'b'\rangle_{L\ell}\\&=(-1)^{\ell+a+(L+a'+b')/2}\\&\quad\times\langle ab|K(\pi/2-\gamma)|b'a'\rangle_{L\ell}.\end{aligned}\tag{165}$$

These equations imply that all the functions $\langle ab|K(\gamma)|a'b'\rangle_{L\ell}$ are even or odd with respect to the point $\gamma=0$ on the interval $(-\pi/2,\pi/2)$, and in the special cases $a=b$ and/or $a'=b'$ such functions possess definite parity relative to the point $\gamma=\pi/4$ on the interval $(0,\pi/2)$.

Let us write down the operator equations (28), (47), and (48) in the hyperharmonics basis. Inserting the unit operator (135) between all the operator factors and taking into account the rule (137), we derive the unitarity relation and the addition and differentiation formulas for the functions $\langle ab|K(\gamma)|a'b'\rangle_{L\ell}$:

$$\sum_{ab}\langle a'b'|K(\gamma)|ab\rangle_{L\ell}\langle ab|K(-\gamma)|a''b''\rangle_{L\ell}=\delta_{a'a''}\delta_{b'b''},\tag{166}$$

$$\langle a'b'|K(\gamma_1+\gamma_2)|a''b''\rangle_{L\ell}=(-1)^L\sum_{ab}\langle a'b'|K(\gamma_1)|ab\rangle_{L\ell}\langle ab|K(\gamma_2)|a''b''\rangle_{L\ell},\tag{167}$$

$$i\partial_\gamma\langle ab|K(\gamma)|a'b'\rangle_{L\ell}=\sum_{\mu,\nu=\pm 1}\langle ab|J|a+\mu,b+\nu\rangle_{L\ell}\times\langle a+\mu,b+\nu|K(\gamma)|a'b'\rangle_{L\ell}.\tag{168}$$

Using the differentiation formula (168) and the boundary values (140) and (141), we can prove the useful integral relation

$$\begin{aligned}&i\sum_{\mu,\nu=\pm 1}\langle ab|J|a+\mu,b+\nu\rangle_{L\ell}\\&\times\int_0^{\pi/2}d\gamma\langle a+\mu,b+\nu|K(\gamma)|a'b'\rangle_{L\ell}\\&=(-1)^L\delta_{aa'}\delta_{bb'}-(-1)^{(L-2\ell+a'-b')/2}\delta_{ab'}\delta_{a'b}.\end{aligned}\tag{169}$$

We note that (165)–(169) are valid in the two cases $(-1)^{\ell+L}=\pm 1$.

In the case $(-1)^{\ell+L}=1$ the coefficients (97) do not vanish, and the identities (144) are nontrivial. We can derive from these identities a series of auxiliary relations for the matrix elements of the operator $K(\gamma)$. According to the definition (55), at the points $\varphi=0,\pi/2$ all the functions $W_{Lab}(\varphi)$ with nonzero index a or, respectively, b, vanish. Therefore, in the limits $\varphi\to 0,\pi/2$ the identities (144) become the corresponding ones valid for any a' and b':

$$\langle 0\ell|K(\gamma)|a'b'\rangle_{L\ell}=(-1)^L B^{\ell}_{a'b'}W_{La'b'}(\gamma)/((2\ell+1)\times W_{L0\ell}(0)),\tag{170}$$

$$\langle \ell 0|K(\gamma)|a'b'\rangle_{L\ell}(\gamma)=(-1)^L B^{\ell}_{a'b'}W_{La'b'}(\gamma)/((2\ell+1)W_{L\ell 0}(\pi/2)).\tag{171}$$

Owing to (156), the derivative $\partial_\varphi W_{Lab}(\varphi)$ is nonzero for $\varphi=0$ only when $a=1$. Therefore, differentiating the identity (144) with respect to the variable φ and then setting $\varphi=0$, we obtain the equation

$$\sum_b B^{\ell}_{1b}\partial_\varphi W_{L1b}(0)\langle 1b|K(\gamma)|a'b'\rangle_{L\ell}=(-1)^{L+1}B^{\ell}_{a'b'}\partial_\gamma W_{La'b'}(\gamma),\tag{172}$$

in which the sum is calculated over all the values of the index b allowed for $a=1$. Similarly, we derive the identity

$$\sum_{a=0,2}\sum_b B^{\ell}_{ab}\partial^2_\varphi W_{Lab}(0)\langle ab|K(\gamma)|a'b'\rangle_{L\ell}(\gamma)=(-1)^L B^{\ell}_{a'b'}\partial^2_\gamma W_{La'b'}(\gamma),\tag{173}$$

in which the index b takes all the values allowed for $a=0$ and $a=2$. Then by induction we prove that in the case $(-1)^{L+\ell}=1$ we have the series of identities

$$\sum_{k=0}^{[n/2]}\sum_b B^{\ell}_{n-2k,b}\partial^n_\varphi W_{L,n-2k,b}(0)\langle n-2k,b|K(\gamma)|a'b'\rangle_{L\ell}=(-1)^{L+n}B^{\ell}_{a'b'}\partial^n_\gamma W_{La'b'}(-\gamma),\ n=0,1,\ldots,\tag{174}$$

where the index b takes all the values allowed for a given index $a=n-2k$, and the symbol $[n]$ denotes the integer part of the number n.

Let us continue our description of the global properties of the matrix elements of the operator $K(\gamma)$ by analyzing their known representations valid in the two cases $(-1)^{L+\ell}=\pm 1$. According to (139) and (142), such matrix elements describe the relation between hyperharmonics whose arguments are related by a kinematical transforma-

tion. Coefficients of this type are called Raynal–Revai coefficients. Our notation for these coefficients is related to that used in Refs. 25 and 78 simply as

$$\langle ab|K(\gamma)|a'b'\rangle_{L\ell}=i^{k}\langle ba|b'a'\rangle^{k}_{L\ell}$$
$$=(-1)^{a+a'}\langle ba|b'a'|\rangle^{\gamma}_{L\ell},\quad \gamma=\gamma_{ki}. \tag{175}$$

Using these expressions, we can rewrite the explicit expressions for the Raynal–Revai coefficients obtained in Ref. 78 as the awkward sums

$$\langle ab|K(\gamma)|a'b'\rangle_{L\ell}(\gamma)=\frac{\pi}{4}(C^{n}_{ab}C^{n'}_{a'b'})^{-1/2}$$
$$\times\sum_{cdef}(-1)^{c+d}i^{e+f+b-b'}$$
$$\times B^{a'}_{ce}B^{b'}_{df}B^{a}_{cf}B^{b}_{ed}\begin{Bmatrix} c & e & a\\ f & d & b\\ a' & b' & \ell\end{Bmatrix}$$
$$\times\sum_{gh}(-1)^{g}C^{g}_{df}C^{h}_{cd}(\cos\gamma)^{2h+c+d}$$
$$\times(\sin\gamma)^{2g+e+f}, \tag{176}$$

containing the coefficients B^{c}_{ab} and C^{c}_{ab} given by (97) and

$$C^{c}_{ab}\equiv\frac{\Gamma(2c+a+b+2)}{\Gamma(c+1)\Gamma(a+b+c+2)\Gamma(c+a+3/2)\Gamma(c+b+3/2)}.$$

Now, developing the idea of Ref. 72, let us treat (168) as a system of equations for the unknown functions $\langle ab|K(\gamma)|a'b'\rangle_{L\ell}$ with fixed indices L, ℓ, a', and b' and all possible values of a and b. The symbol N will denote the number of such pairs of indices a and b. We supplement this system by the boundary conditions (140). Let us study the resulting boundary-value problem, well known in the theory of differential equations,[8] by solving systems of ordinary differential equations with constant coefficients. We use the index $i=1,\ldots,N$ to label all possible pairs of indices a and b. We assume that the spectral problem

$$\sum_{\mu,\nu=\pm1}\langle ab|J|a+\mu,b+\nu\rangle_{L\ell}X_{a+\mu,b+\nu}=-\lambda X_{ab},\forall a,b \tag{177}$$

has N different eigenvalues $\lambda=\lambda_j$ and the corresponding eigensolutions $X_{ab}=X^{j}_{ab}$, $j=1,\ldots,N$. Then the original problem (140), (168) is satisfied by the sums

$$\langle ab|K(\gamma)|a'b'\rangle_{L\ell}=\sum_{j=1}^{N}C_jX^{j}_{ab}\exp(i\lambda_j\gamma) \tag{178}$$

with coefficients C_j satisfying the system of linear equations

$$\sum_{j=1}^{N}C_jX^{j}_{ab}=(-1)^{L}\delta_{aa'}\delta_{bb'}, \tag{179}$$

ensuring that the boundary conditions (140) are satisfied.

We shall show that all the eigenvalues of the problem (177) are nondegenerate. We assume the contrary: suppose that at least one of them, namely, λ_1, occurs $s+1>1$ times. As is well known,[8] this value corresponds to the fundamental solution of the system of equations (168),

$$Z_{ab}(\gamma)=X^{1}_{ab}P_s(\gamma)\exp(I\lambda_1\gamma),$$

where P_s is a polynomial of degree s in the variable γ. Therefore, the general solution of this system can be written as a sum containing this polynomial:

$$\langle ab|K(\gamma)|a'b'\rangle_{L\ell}=C_1Z_{ab}(\gamma)+\sum_{j=2}^{N}C_jX^{j}_{ab}\exp(i\lambda_j\gamma). \tag{180}$$

However, in general the Raynal–Revai coefficient (175) is a series in integer powers of the functions $\cos\gamma$ and $\sin\gamma$. This series reduces to a finite linear combination of the functions $\exp(\pm in\gamma)$ with integer indices $n=0,1,\ldots$. This combination has no terms with polynomial dependence on γ, and so it cannot be represented in the form (180). This contradiction implies that all the eigenvalues λ_j of the problem (177) are nondegenerate and integers.

The solutions of the problem (168), (140) can also be represented as single or double integrals. In order to derive the first integral representation, we multiply each ($m=-\ell,\ldots,\ell$) identity (143) by the corresponding function $C^{\ell m}_{amb0}\Theta_{am}(u)$. We integrate the resulting $(2\ell+1)$ equations over the variable u on the segment $-1\leq u\leq 1$ and sum over the index m. Using (114) and (116), we reduce the resulting equation to a one-dimensional integral identity in the variable φ:

$$\langle ab|K(\gamma)|a'b'\rangle_{L\ell}(\gamma)=W^{-1}_{Lab}(\varphi)\int_{-1}^{1}duK^{\ell}_{aba'b'}(\varphi,u;\gamma)$$
$$\times W_{La'b'}(\varphi'(\varphi,u;\gamma)). \tag{181}$$

Replacing the integration variable u by φ' according to the rules (122)–(125), we rewrite this identity as

$$\langle ab|K(\gamma)|a'b'\rangle_{L\ell}(\gamma)=W^{-1}_{Lab}(\varphi)\int_{C_+(\varphi;\gamma)}^{C_-(\varphi;\gamma)}d\varphi'K^{\ell}_{aba'b'}$$
$$\times(\varphi,\varphi';\gamma)W_{La'b'}(\varphi'). \tag{182}$$

Using the definitions (79) and (131), we represent the resulting expression as the equation

$$\langle\varphi|h^{\ell}_{aba'b'}(\gamma)|\tilde{W}_{La'b'}(\varphi)\rangle=\langle ab|K(\gamma)|a'b'\rangle_{L\ell}\tilde{W}_{Lab}(\varphi). \tag{183}$$

Now let us multiply the identity (181) by the function $\rho(\varphi)W_{Lab}(\varphi)$. Using the orthonormality conditions (66) for such functions, we integrate both sides of the resulting equation over the variable φ on the segment $[0,\pi/2]$. As a result, the matrix element $\langle ab|K(\gamma)|a'b'\rangle_{L\ell}$ is written as a double integral (133), equal to the integral (134).

We shall now derive the discrete representations of the functions $\langle ab|K(\gamma)|a'b'\rangle_{L\ell}$ in the form of single or double sums over arbitrary nodal values of the sets of arguments Ω or, accordingly, u and φ.

Let N be the number of hyperharmonics (55) with given indices L, ℓ, and m. We write down the five-dimensional identity (142) at N arbitrary but different points $\Omega=\Omega_i$,

$i=1,\ldots,N$, of the unit hypersphere $\mathcal{S}^5$, different from the points with coordinates Ω_p [the case (12)] or Ω_l [the case (14), (15)]. Then we obtain a system of N linear equations for the unknown functions $\langle ab|K(\gamma)|a'b'\rangle_{L\ell}$ with fixed indices L, ℓ, a', and b' and all indices a and b allowed by this condition:

$$Y^{\ell m}_{La'b'}\Big(\Omega'(\Omega_i;\gamma)=\sum_{ab}\langle ab|K(\gamma)|a'b'\rangle_{L\ell}Y^{\ell m}_{Lab}(\Omega_i),$$

$$i=1,\ldots,N. \tag{184}$$

We label the indices a and b by the index $j=1,\ldots,N$ in order to rewrite (184) in a form convenient for numerical solution:

$$\sum_{j=1}^{N} Y_{ij}X_j=Y^{\ell m}_{La'b'}(\Omega'(\Omega_i;\gamma)),$$

$$X_j\equiv\langle a_jb_j|K(\gamma)|a'b'\rangle_{L\ell},\quad Y_{ij}\equiv Y^{\ell m}_{La_jb_j}(\Omega_i),i,j=1,\ldots,N. \tag{185}$$

Since the hyperharmonics $Y^{\ell m}_{a_jb_j}(\Omega)$ with $j=1,\ldots,N$ are linearly independent on $\mathcal{S}^5$, the matrix $\mathbf{Y}$ of the system (185) is not degenerate for any choice of the points Ω_i. Therefore, this system always has a unique solution. It is invariant with respect to the choice of nodes Ω_i and can be represented as

$$\langle a_jb_j|K(\gamma)|a'b'\rangle_{L\ell}=\sum_{i=1}^{N} Y^{-1}_{ji}Y^{\ell m}_{La'b'}(\Omega'(\Omega_i;\gamma)),$$

$$j=1,\ldots,N. \tag{186}$$

The idea of the discretization of the identity (142) and the use of the representation (186) to calculate the Raynal–Revai coefficients is due to Efros.[82] In developing this idea, we shall derive a discrete representation for these coefficients which is more convenient computationally.

Now let N be the number of possible values of the index a of hyperharmonics with given L, ℓ, and m, and let M be the number of possible values of the index b of such hyperharmonics for fixed index a. We shall arrange all possible values of the index a in increasing order and label them by an integer: $a=a_i$, $i=1,\ldots,N$, $a_1<a_2<\ldots<a_N$. We introduce a similar correspondence for all the values of the index b possible for a given a: $b=b_j$, $j=1,\ldots,M$, $b_1<b_2<\ldots<b_M$. Let us fix the values of the indices a' and b'. To simplify the calculations we set $m=[1-(-1)^{\ell+L}]/2$. We write the identity (143) with selected a', b', and m as

$$\sum_{j=1}^{N}\Theta_{a_jm}(u)X_j(\varphi;\gamma)=Q_{a'b'}(u,\varphi;\gamma), \tag{187}$$

$$X_j(\varphi;\gamma)\equiv\sum_{t=1}^{M}(b_t+1/2)^{1/2}C^{\ell m}_{a_jmb_t0}W_{La_jb_t}(\varphi)\times\langle a_jb_t|K(\gamma)|a'b'\rangle. \tag{188}$$

We specify a grid of nodes u_i, $i=1,\ldots,N$, on the interval $(-1,1)$, and write the identity (187) on this grid. As a result of this discretization we obtain a system of linear equations for the unknown functions (188):

$$\sum_{j=1}^{N}\Theta_{ij}X_j(\varphi;\gamma)=Q_{a'b'}(u_i,\varphi;\gamma),$$

$$\Theta_{ij}\equiv\Theta_{a_jm}(u_i),\quad i,j=1,\ldots,N. \tag{189}$$

The functions $\Theta_{am}(u)$ with fixed index m and different indices a are linearly independent on the interval $(-1,1)$.[6] Therefore, for any choice of grid the matrix Θ_{ij} of the system (189) is nondegenerate. Inverting this matrix, for each j we obtain the corresponding identity in the variable φ:

$$\sum_{t=1}^{M}(b_t+1/2)^{1/2}C^{\ell m}_{a_jmb_t0}W_{La_jb_t}(\varphi)\langle a_jb_t|K(\gamma)|a'b'\rangle=\sum_{i=1}^{N}\Theta^{-1}_{ji}Q_{a'b'}(u_i,\varphi;\gamma). \tag{190}$$

Writing down each such identity at the nodes φ_s, $s=1,\ldots,M$, of the arbitrary grid specified on the interval $(0,\pi/2)$, we obtain the corresponding system of linear equations for the desired functions $\langle ab_t|K(\gamma)|a'b'\rangle_{L\ell}$ with identical indices $a=a_j$ and a', b', but different indices b_t:

$$\sum_{t=1}^{M}W_{st}\langle a_jb_t|K(\gamma)|a'b'\rangle_{L\ell}=\sum_{i=1}^{N}\Theta^{-1}_{jt}Q_{a'b'}(u_i,\varphi_s;\gamma),$$

$$s=1,\ldots,M. \tag{191}$$

We shall prove that the matrix $\mathbf{W}$,

$$W_{st}\equiv(b_t+1/2)^{1/2}C^{\ell m}_{a_jmb_t0}W_{La_jb_t}(\varphi_s),\quad s,t=1,\ldots,M, \tag{192}$$

of the system of equations (191) is nondegenerate. Introducing the notation $\alpha\equiv a_j$, $\beta\equiv b_1$, $z\equiv\cos 2\varphi$, and using the definition (55) of the functions W_{Lab}, we write all the matrix elements (192) as

$$W_{st}=2^{t-1}(b_t+1/2)^{1/2}C^{\ell m}_{a_jmb_t0}N_{La_jb_t}(\sin\varphi_s)^{a_j}(\cos\varphi_s)^{b_1}F_{st}, \tag{193}$$

where the matrix $\mathbf{F}$ is constructed from the nodal values of the functions

$$f_t(z)\equiv(1+z)^{t-1}P^{(\alpha,\beta+2(t-1))}_{M-t}(z),\quad t=1,\ldots,M, \tag{194}$$

according to the rules

$$F_{st}\equiv f_t(z_s),\quad z_s\equiv\cos 2\varphi_s,\quad s,t=1,\ldots,M. \tag{195}$$

Using the well known recursion relations for the Jacobi polynomials,[6] we can represent all the functions (194) as expansions

$$f_t(z)=\sum_{k=0}^{M-1}C_{tk}P^{(\alpha,\beta+M-1)}_{k}(z),\quad t=1,\ldots,M, \tag{196}$$

in the system of M functions $P^{(\alpha,\beta+M-1)}_{k}(z)$, $k=1,..,M$. It is known[6] that such polynomials are linearly independent on the interval $z\in(-1,1)$. Owing to the representation (196), the system of M functions (194) also possesses this property. Therefore, for any choice of nodes φ_s, the matrix (195) and also the matrix (193) have inverses F^{-1} and W^{-1}. The sys-

tem (191) thus always has a unique solution. It is independent of the choice of nodes u_i and φ_s, and can be represented as

$$\langle a_j b_t|K(\gamma)|a'b'\rangle_{L\ell}=\sum_{s=1}^{M} W_{ts}^{-1}\sum_{i=1}^{N}\Theta_{ji}^{-1}Q_{a'b'}(u_i,\varphi_s;\gamma),$$

$$t=1,...,M. \tag{197}$$

We shall derive expressions which can be used to study such local properties of the functions $\langle ab|K(\gamma)|a'b'\rangle_{L\ell}$ as the asymptotes at a chosen point γ_0 and the locations of the extrema.

Using Eqs. (48) and (138), we expand the function $\langle ab|K(\gamma_0+\gamma)|a'b'\rangle_{L\ell}$ in a Taylor series with center at the point γ_0:

$$\langle ab|K(\gamma_0+\gamma)|a'b'\rangle_{L\ell}$$
$$=\sum_{n=0}^{\infty}(-i\gamma)^n\sum_{\mu,\nu=0}^{n}\langle ab|J^n|a-n+2\mu,b-n+2\nu\rangle_{L\ell}$$
$$\times\langle a-n+2\mu,b-n+2\nu|K(\gamma_0)|a'b'\rangle_{L\ell}. \tag{198}$$

For $\gamma\to 0$ this series can be approximated by its finite subsum, and the latter can be used to describe qualitatively the behavior of the function $\langle ab|K(\gamma_0+\gamma)|a'b'\rangle_{L\ell}$ in a small neighborhood of the point γ_0. As simple examples let us consider the cases $\gamma_0=0,\pi/2$. We expand first for $\gamma_0=0$ and then for $\gamma_0=\pi/2$ in a series (198). Taking into account (138), (140), and (141), we obtain the expansions

$$\langle ab|K(\gamma)|a'b'\rangle_{L\ell}=(-1)^L\sum_{n=0}^{\infty}(-i\gamma)^{N+n}$$
$$\times\langle ab|J^{N+n}|a'b'\rangle_{L\ell},$$
$$\langle ab|K(\pi/2+\gamma)|a'b'\rangle_{L\ell}=(-1)^{(L+2\ell+a'-b')/2}$$
$$\times\sum_{n=0}^{\infty}(-i\gamma)^{N'+n}$$
$$\times\langle ab|J^{N'+n}|b'a'\rangle_{L\ell},$$
$$N\equiv\max\{|a'-a|,|b'-b|\},\quad N'\equiv\max\{|a'-b|,|b'-a|\}, \tag{199}$$

according to which for $\gamma\to 0,\pi/2$ the functions $\langle ab|K(\gamma)|ab\rangle_{L\ell}$ tend to ± 1 if $a'=a$, $b'=b$ or $a'=b$, $b'=a$, while otherwise they converge to zero no more slowly than the corresponding power-law functions γ^N and $(\pi/2-\gamma)^{N'}$.

The generating function $\langle ab|K(\gamma)|a'b'\rangle_{L\ell}$ vanishes if and only if the right-hand side of (168) is zero. Therefore, the location of the local extrema of this function is given by the equation

$$\sum_{\mu,\nu=\pm 1}\langle ab|J|a+\mu,b+\nu\rangle_{L\ell}\langle a+\mu,b$$
$$+\nu|K(\gamma)|a'b'\rangle_{L\ell}=0. \tag{200}$$

We now prove the addition theorem for the Jacobi polynomials with half-integer superscript. The addition theorem for the Jacobi polynomials with integer superscript is given in Ref. 23 in the form of an addition theorem for the Wigner D functions. The addition theorem for the Jacobi polynomials in the case of half-integer superscript is not mentioned in that book or in specialized mathematical handbooks.[6,7] It is therefore important to study this case. Using the definitions (55) of the functions W_{Lab}, we rewrite the identity (144) in terms of Jacobi polynomials. We thus prove the following theorem.

Theorem 1. *For any integer index*

$$\ell,a,b,a',b';L=2n+a+b=2n'+a'+b';(-1)^{a+b+\ell}=1$$

we have the addition formula

$$P_{n'}^{(a'+1/2,b'+1/2)}(\cos 2\varphi')=(-1)^{a'+b'}\frac{B_{ab}^{\ell}}{B_{a'b'}^{\ell}N_{La'b'}}$$
$$\times\sum_{ab}\langle ab|K(\gamma)|a'b'\rangle_{L\ell}$$
$$\times\frac{(\sin\varphi)^a(\cos\varphi)^b}{(\sin\varphi')^{a'}(\cos\varphi')^{b'}}$$
$$\times P_n^{(a+1/2,b+1/2)}(\cos 2\varphi),$$

where $\varphi'=\varphi-\gamma$, and the coefficients N_{Lab} and B_{ab}^{ℓ} are given by (56) and (97).

Let us discuss the new methods of calculating the Raynal–Revai coefficients. These coefficients (175) can be found by integrating the system of differential equations (168) with the boundary condition (140) or by solving the equivalent matrix problem (177)–(179), and also by calculating the one-dimensional integrals (181) and (182) or by solving the set of finite systems of linear equations (187)–(192). The dimension of the matrix $\mathbf{Y}$ of the system (185) increases for $L\to\infty$ as L^4. Therefore, at large L instead of the representation (186) it is better to use (197), in which the dimensions of the matrices $\mathbf{\Theta}$ and $\mathbf{W}$ grow much more slowly. In fact, the calculation of all the Raynal–Revai coefficients $\langle a_j b_t|K(\gamma)|a'b'\rangle_{L\ell}$ with given indices L, ℓ and a', b' according to Eqs. (187)–(192) and (197) requires a single inversion of the matrix $\mathbf{\Theta}$ of dimension $N\sim L$, the construction of N inverse matrices $\mathbf{W}$ of dimension $M\sim L$, and the calculation of the sum (197) for all $j=1,\ldots,N$ and $t=1,\ldots,M$.

The number of mathematical operations needed to calculate the Raynal–Revai coefficients by any of the above methods can be decreased considerably by using the symmetry relations (165) and the addition formula (167). From the latter we can calculate the Raynal–Revai coefficients for the kinematical angle $\gamma=\gamma_1+\gamma_2$ by using the known values of these coefficients for $\gamma=\gamma_1$ and $\gamma=\gamma_2$. In the limiting cases (6) and (7) the Raynal–Revai coefficients can be approximated by finite sums of the corresponding series (199). The accuracy of calculating these coefficients by any of the methods mentioned above can be monitored carefully by substituting the results of the calculations into the left-hand sides of (143), (144), (166), (169), (172)–(174), and (200) and then comparing the resulting values with the values of the corresponding right-hand sides of these expressions.

We conclude by listing the equations proved earlier in other studies, but using methods more complicated than those discussed above. Equations (162)–(164) were proved in Ref. 79 by taking the limit $\gamma\to 0$ in the explicit expressions for the Raynal–Revai coefficients (176). In that study, systems of linear equations of the type (168) were derived for these coefficients, and it was proved that the eigenvalues of the problem (177) are integers. Equations (170) and (171) were obtained in Ref. 80 as a special case of the complicated recursion relations. The integral representation (181) was first proved in Ref. 65.

3. THE SCHRÖDINGER AND FADDEEV EQUATIONS

The goal of this section is to use the concept and properties of the kinematical transformation to describe the reduction of the Schrödinger and Faddeev equations, originally written in $\mathcal{R}^6$, to systems of equations in $\mathcal{R}^2_+$ or $\mathcal{R}^1_+$. In Sec. 3.1 we list some of the known operator and spectral properties of the free three-particle Hamiltonian. In Sec. 3.2 we explain the matrix representations of the central and S-wave interactions in the angular bases. Sections 3.3 and 3.4 are devoted to the structure and angular analysis of the Schrödinger and Faddeev equations.

3.1. Basic properties of the free Hamiltonian

In quantum mechanics,[10] the states $|\varepsilon\rangle$ of a system of particles are classified by sets ε of conserved quantum numbers. A state $|\varepsilon\rangle$ is associated with a regular wave function $\Psi^\varepsilon(\mathbf{r})\equiv\langle\mathbf{r}|\varepsilon\rangle$. The wave function Ψ^ε of three free particles obeys the Schrödinger equation

$$(H_0(\mathbf{r})-E)\Psi^\varepsilon(\mathbf{r})=0, \tag{201}$$

where the numerical parameter E has the meaning of the total energy, and the free three-particle Hamiltonian H_0 in the representation $|\mathbf{r}\rangle=|\mathbf{x},\mathbf{y}\rangle$ is equivalent to the sum of two Laplace operators:

$$H_0(\mathbf{x},\mathbf{y})\equiv-\Delta_x-\Delta_y,\quad \Delta_q\equiv\sum_{\mu=1}^{3}\partial^2_{q_\mu},\quad \mathbf{q}=\mathbf{x},\mathbf{y}. \tag{202}$$

Here and below, for clarity in the brackets following the symbol for an operator we shall indicate the variables on which the operator acts.

Let us list the other commonly used representations of the Hamiltonian H_0. Using the definitions of the angular-momentum operators (36) and (37), we can rewrite the Hamiltonian (202) as a sum of operators acting on the moduli of the vectors $\mathbf{x}$, $\mathbf{y}$ and their spherical angles, or on the hyperradius r and the hyperspherical angles Ω:

$$H_0(\mathbf{x},\mathbf{y})=-x^{-2}\partial_x(x^2\partial_x)-y^{-2}\partial_y(y^2\partial_y)+x^{-2}\mathbf{l}^2_x(\hat{x})+y^{-2}\mathbf{l}^2_y(\hat{y}), \tag{203}$$

$$H_0(r,\Omega)=-r^{-5}\partial_r(r^5\partial_r)-r^{-2}L^2(\Omega). \tag{204}$$

Using these representations and the properties (89) of the operator L^2, it is easily shown that for any function Q of the variables x, y, r, φ or r we have the representations

$$H_0(\mathbf{x},\mathbf{y})(xy)^{-1}\mathcal{Y}^{\ell m}_{ab}(\hat{x},\hat{y})Q(x,y)=(xy)^{-1}\mathcal{Y}^{\ell m}_{ab}(\hat{x},\hat{y})\tilde{H}_{0ab}(x,y)Q(x,y),$$

$$\tilde{H}_{0ab}(x,y)\equiv\partial^2_x+\partial^2_y-a(a+1)y^{-2}-b(b+1)x^{-2}; \tag{205}$$

$$H_0(r,\Omega)(xy)^{-1}\mathcal{Y}^{\ell m}_{ab}(\hat{x},\hat{y})Q(x,y)=(xy)^{-1}\mathcal{Y}^{\ell m}_{ab}(\hat{x},\hat{y})\tilde{H}_{0ab}(r,\varphi)Q(r,\varphi),$$

$$\tilde{H}_{0ab}(r,\varphi)\equiv\partial^2_r+r^{-1}\partial_r+r^{-2}\tilde{L}^2_{ab}(\varphi); \tag{206}$$

$$H_0(r,\Omega)(xy)^{-1}Y^{\ell m}_{Lab}(\Omega)Q(r)=r^{-2}Y^{\ell m}_{Lab}(\Omega)\tilde{H}_{0L}(r)Q(r),$$

$$\tilde{H}_{0L}(r)\equiv\partial^2_r+r^{-1}\partial_r-r^{-2}(L+2)^2. \tag{207}$$

We shall describe two methods of separating the variables in (201) and its particular and general solutions. Owing to Eqs. (25), (36), (37), (202), and (204), we have

$$[H_0,Q]_-=0,\quad Q=\mathbf{l}^2,l_3,P,\mathbf{l}^2_x,\mathbf{l}^2_y,L^2,P_1,P_2,P,T. \tag{208}$$

Because of these commutation relations, the eigenfunctions of the operators $\mathbf{l}^2$, l_3, and P are also eigenfunctions of the operator H_0. We therefore seek the solution $\Psi^\varepsilon\equiv\Psi^{\ell m\sigma}$ of Eq. (201) with the set of quantum numbers $\varepsilon=(\ell,m,\sigma)$ in the form of the function (84) satisfying the conditions (85) and represented by the bispherical series (86). We replace it by the function Ψ^ε in (201). Using (206), we project the resulting equation onto the bispherical basis. Then for the desired bispherical components U^ℓ_{ab} we obtain an infinite system of equations uncoupled in the indices a and b:

$$(\tilde{H}_{0ab}(r,\varphi)-E)U^\ell_{ab}(r,\varphi)=0. \tag{209}$$

The desired solution U^ℓ_{ab}, $(-1)^{a+b}=\sigma$, of each such equation is replaced by its series (88) in the basis functions $\tilde{W}_{Lab}$. The resulting equation is projected onto this basis by using the property (82) of the operator $\tilde{L}^2_{ab}$. Then for each unknown function U^ℓ_{Lab}, $L=a+b$, $a+b+2,\ldots$ we obtain the equation

$$(\tilde{H}_{0L}(r)-E)U^\ell_{Lab}(r)=0. \tag{210}$$

Using the substitution

$$z=\sqrt{E}r,\quad \nu=L+2,\quad U^\ell_{Lab}(r)=Z_\nu(z),$$

we reduce it to the canonical equation for the cylindrical Bessel functions:[6]

$$(z^2\partial^2_z+z\partial_z+(z^2-\nu^2))Z_\nu(z)=0. \tag{211}$$

This equation, and therefore the original equation (210), has a nontrivial regular solution J_{L+2} only for $E>0$. Consequently, for any allowed indices a, b, and L, we have

$$U^\ell_{Lab}(r)=J_{L+2}(\sqrt{E}r),\quad E>0, \tag{212}$$

and the general regular solution U^ℓ_{ab} of Eq. (210) is described by the equations

$$U^\ell_{ab}(r,\varphi)=\begin{cases}0, & E<0\\ \Sigma_L C^L J_{L+2}(\sqrt{E}r)\tilde{W}_{Lab}(\varphi), & E>0\end{cases}, \tag{213}$$

where C^L are arbitrary numbers. By construction, the solution (212) of (210) corresponds to the eigenenergy of the operator H_0 and all the operators (208) commuting with it:

$$\Psi_0^{\varepsilon}(\mathbf{r}) = r^{-2} J_{L+2}(\sqrt{E}r) Y_{Lab}^{\ell m}(\Omega), \quad \varepsilon = (L,a,b,\ell,m,\sigma). \tag{214}$$

Thus, by construction, a linear combination of such functions with all possible indices L, a, and b is the desired solution Ψ^{ε} of (201).

Thus, in the free Schrödinger equation (201) it is possible to separate successively the variables $\hat{x}$, $\hat{y}$ from the variables r, φ, and then the variables r, φ from each other. The variable separation is achieved by projecting the original equation onto the bispherical basis and then projecting each resulting equation (209) onto the corresponding basis of the functions W_{Lab}.

An alternative but equivalent method is based on the use of the hyperspherical ansatz (84), (87) for the desired function $\Psi^{\varepsilon} = \Psi^{\ell m \sigma}$, and consists of the one-stage separation of the variables Ω from r by projecting Eq. (201) directly onto the hyperspherical basis. Using (207), it is easily checked that this projection leads to the same equations (210) for the desired hyperspherical components U_{Lab} as the method described earlier.

It is known[14,19] that the set of functions (214) with all possible components of the multiple index ε is a fundamental system of regular solutions (FSRS) of the free Schrödinger equation: any other regular solution can be represented as a finite or infinite linear combination of the functions (214) and arbitrary numerical coefficients. For example, the solution describing a plane wave and corresponding to the set of conserved quantum numbers $\varepsilon = (\mathbf{p},\mathbf{q})$ from the six coordinates of the two three-dimensional scattering momenta $\mathbf{p}$ and $\mathbf{q}$ can be represented as[25]

$$\Psi_0^{\mathbf{p},\mathbf{q}}(\mathbf{r}) = \exp(i(\mathbf{p}\cdot\mathbf{x})(\mathbf{q}\cdot\mathbf{y})) = \frac{8\pi^3}{E} \sum_{\ell m} \sum_{ab} \sum_{L} i^L (Y_{Lab}^{\ell m}(\Omega_s))^* \times r^{-2} J_{L+2}(\sqrt{E}r) Y_{Lab}^{\ell m}(\Omega),$$

$$\mathbf{s} \equiv (\mathbf{p},\mathbf{q}), \quad E = p^2 + q^2. \tag{215}$$

3.2. Matrix representation of interaction operators

A well known method for elegantly solving the problem of projections of operator relations onto angular bases is based on the use of matrix representations of operators. We shall present and comment on such representations for two types of pair interaction operator $V_i(\mathbf{r}_i)$, $i = 1,2,3$, acting in the space $\mathcal{R}^6$ only on the relative coordinates $\mathbf{x}_i$ corresponding to a pair of particles labeled j and k.

First we recall the relative concepts of a proper (improper) coordinate representation and angular basis. A coordinate representation $\langle \mathbf{r}_i|$ or a system of coordinates $\mathbf{r}_i$ is said to be proper for some operator V_i or function Ψ_i labeled by the same index i. The other two representations $\langle \mathbf{r}_k|$, $k \neq i$, will not be proper for such an operator or such a function. Proper and improper angular bases are defined similarly: an angular basis composed of functions of the arguments $\mathbf{r}_i$ ($\mathbf{r}_k$) is said to be proper (improper) for V_i or Ψ_i.

Below, for brevity we shall omit the indices i and k, and assume that the representation $\langle \mathbf{r}| \equiv \langle \mathbf{r}_i|$ or $\langle \mathbf{r}'| \equiv \langle \mathbf{r}_k|$ is proper or improper.

An operator $V(\mathbf{r})$ acting in a proper coordinate representation $\langle \mathbf{r}|$ on an arbitrary function $\Psi(\mathbf{r})$ like an ordinary multiplication operator on a potential function $V(x)$ is called a central or spherically symmetric pair interaction.[10] In $\mathcal{R}^6$ the operator $V(\mathbf{r})$ does not act on the variables $\mathbf{y}$, and the function $V(x)$ is independent of the angles $\hat{x}$. Therefore, the nonzero matrix elements of the operator $V(\mathbf{r})$ in its proper angular bases are given by

$$\langle b\beta|V(\mathbf{r})|b\beta\rangle = \langle \ell mab|V(\mathbf{r})|\ell mab\rangle = V(x),$$

$$\langle L\ell mab|V(\mathbf{r})|L'\ell mab\rangle = \langle L\tilde{a}b|V(r\cos\varphi)|L'\tilde{a}b\rangle = \int_0^{\pi/2} d\varphi \tilde{W}_{Lab}(\varphi) V(r\cos\varphi) \times \tilde{W}_{L'ab}(\varphi) \equiv V_{ab}^{LL'}(r). \tag{216}$$

Using these selection rules and the angular bra and ket operators defined by (63)–(65), we write the central interaction as an infinite sum of b-wave operators V^b:

$$V(\mathbf{r}) = \sum_{b=0}^{\infty} V^b(\mathbf{r}), \tag{217}$$

$$V^b(\mathbf{r}) = \sum_{\beta=-b}^{b} |Y_{b\beta}(\hat{x})\rangle V(x) \langle Y_{b\beta}(\hat{x})|, \tag{218}$$

$$V^b(\mathbf{r}) = \sum_{\ell m} \sum_{a} |\mathcal{Y}_{ab}^{\ell m}(\hat{x},\hat{y})\rangle V(x) \langle \mathcal{Y}_{ab}^{\ell m}(\hat{x},\hat{y})|, \tag{219}$$

$$V^b(\mathbf{r}) = \sum_{\ell m} \sum_{a} \sum_{LL'} |Y_{Lab}^{\ell m}(Q)\rangle V_{ab}^{LL'}(r) \langle Y_{L'ab}^{\ell m}(\Omega)|. \tag{220}$$

It follows from the matrix representations (218)–(220) that each operator V^b is not a projector ($V^b V^b \neq I$), but possesses the projection properties

$$(V^b(\mathbf{r}) - \delta_{bb'} V(x))|Q_{b'\beta}\rangle = 0,$$

$$\forall |Q_{b'\beta}\rangle = |b'\beta\rangle, \ |\ell ma'b'\rangle, \ |L\ell mab'\rangle \tag{221}$$

and therefore maps an arbitrary function $\Psi(\mathbf{r})$ onto an eigenfunction of the operator $\mathbf{l}_x^2$ corresponding to eigenvalue $b(b+1)$.

In nuclear physics, operators V^b with index $b = 0,1$ are called S- and P-wave interactions. In the series (218)–(220) for the S- and P-wave interactions $b = 0$, $a = \ell$, and, accordingly, $b = 1$, $a = \ell, \ell \pm 1$.

3.3. The structure and angular analysis of the Schrödinger equation

The Schrödinger equation for the wave function Ψ^{ε} of the state $|\varepsilon\rangle$ of three particles with pair interactions V_i is usually written in the compact (operator) form[10]

$$(H-E)\Psi^{\varepsilon}=0, \quad H\equiv H_0+V, \quad V\equiv\sum_{k=1}^{3} V_k, \tag{222}$$

and it is understood that all pair interactions are given in their proper coordinates, and all operators and the desired solution Ψ^{ε} are given in some coordinate representation $\langle \mathbf{r}_i|$. Clearly, for any choice of one of the three such representations it is improper for two interactions, namely, for V_j and V_k. Let us explain why this fact, unavoidable in the Schrödinger formulation of the three-particle problem, significantly complicates the analytic study of this problem. We shall consider the case of central interactions.

We write down Eq. (222), taking into account the commonly used rules listed above and the definition (26) of the operator $K(\gamma)$. Then in the space $\mathcal{R}^6$ we obtain the differential Schrödinger equation in its canonical form:

$$\left(H_0(\mathbf{x}_i,\mathbf{y}_i)+V_i(x_i)-E+\sum_{k\neq i} K(\gamma_{ik})V_k(x_i)\right)\Psi^{\varepsilon}(\mathbf{r}_i)=0. \tag{223}$$

Here $V_i(x_i)$ is a function of three arguments $\mathbf{x}_i$, and the two kinematical images are functions of the six variables $\mathbf{r}_i$:

$$K(\gamma_{ik})V_k(x_i)=V_k(x_k(\mathbf{r}_i)), \quad k\neq i. \tag{224}$$

It is this functional dependence which makes the construction of the exact solution of the original Schrödinger equation (223) more difficult, even in the case of pair interactions described in proper coordinates by elementary functions.

Let us recall what hinders the construction of the exact solution by the method of variable separation. In the case of central interactions, the total Hamiltonian H satisfies the equations

$$[H,Q]_-=0, \quad Q=\mathbf{l}^2,l_3,P; \quad [H,q]_-\neq 0, \quad q=L^2,\mathbf{l}_x^2,\mathbf{l}_y^2, \tag{225}$$

and so the set of quantum numbers $\varepsilon=(\ell,m,\sigma)$ is conserved also for the desired function Ψ^{ε}, and (84)–(86) are true. In Eq. (223) we replace the function Ψ^{ε}, the kinematical images (224), and the interaction $V_i(\mathbf{r}_i)$ by their bispherical series, written according to the corresponding rules (86), (129), and (219) in the representation $\langle \mathbf{r}_i|$. Using Eqs. (119), (120), and (206), we project the resulting equation onto its proper bispherical basis of the functions $\mathcal{Y}_{ab}^{\ell m}(\hat{x}_i,\hat{y}_i)$. Then for the unknown bispherical components U_{ab}^{ℓ} we obtain the system of differential equations

$$(\tilde{H}_{0ab}(r,\varphi_i)+V_i(r\cos\varphi_i)-E)U_{ab}^{\ell}(r,\varphi_i)$$
$$=\sum_{k\neq i}\sum_{a'b'}\Gamma_{aba'b'}^{\ell m}\left(\int_{C_-(\varphi_i,\gamma_{ik})}^{C_+(\varphi_i,\gamma_{ik})} d\varphi_k K_{aba'b'}^{\ell}(\varphi_i,\varphi_k;\gamma_{ik})\right.$$
$$\left.\times V_k(r\cos\varphi_k)\right)U_{ab}^{\ell}(r,\varphi_i). \tag{226}$$

The kinematical images (224) in their improper bispherical basis have nondiagonal matrices. Therefore, the number of equations of the system (226) is infinite in the indices a and b, and the equations themselves are coupled in the indices a' and b' and contain the coefficients $\Gamma_{aba'b'}^{\ell m}$ [the integrals (120) of three bispherical functions] and rather complicated integrals of two potentials.

We shall describe a nontraditional method of reducing the equations (223). We write the operator H_0, the function Ψ^{ε} in the expression $H_0\Psi^{\varepsilon}$, and both factors of the product $V_i\Psi^{\varepsilon}$ in the representation $\langle \mathbf{r}_i|$, and the factors of the two $(k\neq i)$ products $V_k\Psi^{\varepsilon}$ in the corresponding proper representations $\langle \mathbf{r}_k|$. Then we obtain the equation

$$(H_0(\mathbf{x}_i,\mathbf{y}_i)+V_i(x_i)-E)\Psi^{\varepsilon}(\mathbf{r}_i)$$
$$+\sum_{k\neq i} K(\gamma_{ik})(V_k)(x_k)\Psi^{\varepsilon}(\mathbf{r}_k))=0. \tag{227}$$

Using the ansatz (84), (86) and Eqs. (130), (217), and (219), we reduce the resulting equation (227) to the system of equations

$$(\tilde{H}_{0ab}(r,\varphi_i)+V_i(r\cos\varphi_i)-E)U_{ab}^{\ell}(r,\varphi_i)$$
$$=\sum_{k\neq i}\sum_{a'b'}\int_{C_-(\varphi_i,\gamma_{ik})}^{C_+(\varphi_i,\gamma_{ik})} d\varphi_k(h_{aba'b'}^{\ell}(\varphi_i,\varphi_k;\gamma_{ik})V_{ik}$$
$$\times(r\cos\varphi_k)U_{a'b'}^{\ell}(r,\varphi_k)). \tag{228}$$

These equations contain both the potentials and the desired functions U_{ab}^{ℓ} inside the integrals and are therefore even more inconvenient for exact solution than Eqs. (226) considered above.

By trying all the methods of representing the terms of the operator Schrödinger equation (222) different from those described above, we can verify that the projection of the resulting Schrödinger differential equations onto the bispherical basis leads to systems of integro-differential equations.

Let us consider the case of three identical particles, which is important in nuclear physics. Here all the particles have the same mass, by the definition (4) the moduli of all the kinematical angles are $\pi/3$, and all the pair interactions and their proper coordinate systems are described by the same operator V:

$$m_i=m, \quad |\gamma_{ki}|=\pi/3, \quad V_i(\mathbf{r}_i)=V(\mathbf{r}_i), \quad \forall k\neq i=1,2,3. \tag{229}$$

According to the Pauli principle, the wave function Ψ^{ε} must be (anti)symmetric under any permutations of identical (fermions) bosons:

$$S^{\pm}\Psi^{\varepsilon}(\mathbf{r}_i)=\Psi^{\varepsilon}(\mathbf{r}_i), \quad i=1,2,3. \tag{230}$$

Here the minus or plus sign is taken for fermions or bosons, respectively, and the operators $S^{\pm}$ are defined by (52) and (53).

The auxiliary condition of being identical (230) leads only to an insignificant simplification of the system of Schrödinger equations (226) or (227): owing to (229), instead of different interaction matrices, these systems contain only a single matrix $\mathbf{V}$, but they remain systems of infinite rank.

The Schrödinger equation therefore contains the sum of all pair interactions, each specified by definition in its proper coordinate representation. Projection of an equation with this

structure onto the bispherical basis leads to infinite systems of differential equations (226) with awkward coefficients (120) and integrals over potentials, or to infinite systems of integro-differential equations like (228). This conclusion remains valid in the case of identical particles and, as is easily checked, in the case of S-wave interactions. In all these cases and for the same reason it is just as complicated to deal with the systems of differential equations obtained by projecting the Schrödinger equation (223) or (227) onto the hyperspherical basis.

These facts suggest how to reformulate the Schrödinger equation so as to obtain a differential problem with a simpler structure. The equation should be reduced to a system of three equations. Each equation should contain only one interaction V_i and should be written in the representation $\langle \mathbf{r}_i|$ proper for this interaction. The Faddeev theory was constructed in this manner.[21]

3.4. The structure and angular analysis of the Faddeev equations

Let us analyze the structure of the Faddeev differential equations in the space $\mathcal{R}^6$, for now without imposing any restrictions on the particle masses and the form of the pair interactions. The key mathematical objects of the Faddeev theory, which however still have no clear physical interpretation, are the Faddeev components

$$\Psi_i^\varepsilon = G_0(Z)V_i\Psi^\varepsilon + \Psi_{i0}^\varepsilon, \quad G_0(Z) \equiv (H_0 - Z)^{-1},$$
$$Z \equiv E + i0, \quad i = 1,2,3, \tag{231}$$

of the wave function Ψ^ε of the three-particle state $|\varepsilon\rangle$ in question,

$$\Psi^\varepsilon = \sum_{k=1}^{3} \Psi_k^\varepsilon. \tag{232}$$

These components obey a system of three ($i = 1,2,3$) operator equations

$$(H_0 - E)\Psi_i^\varepsilon = -V_i\Psi^\varepsilon = -V_i \sum_{k=1}^{f3} \Psi_k^\varepsilon \tag{233}$$

and possess an important property. It follows from (231) that if Ψ^ε is an eigenfunction of some operator commuting with H_0 and V_i, then Ψ_i^ε will be an eigenfunction of this operator. The converse is not always true: if Ψ_i^ε is an eigenfunction of some operator commuting with H_0 and V_i, then the function Ψ^ε may turn out not to be an eigenfunction of this operator. An explanatory example is given below.

Equations (231)–(233) are represented in $\mathcal{R}^6$ by the following rules. Each Faddeev component Ψ_i^ε and the equation itself containing it on the left-hand side are written in their proper representation $\langle r_i|$. Therefore, in $\mathcal{R}^6$ the components of Ψ_i^ε are given by

$$\Psi_i^\varepsilon(\mathbf{r}_i) \equiv G_0(\mathbf{r}_i; Z)V_i(\mathbf{r}_i)\Psi W^\varepsilon(\mathbf{r}_i), \tag{234}$$

and are summed in (232) according to the rule

$$\Psi^\varepsilon(\mathbf{r}_i) = \Psi_i^\varepsilon(\mathbf{r}_i) + \sum_{k \neq i} K(\gamma_{ki})\Psi_k^\varepsilon(\mathbf{r}_i) \tag{235}$$

and satisfy three ($i = 1,1,2$) differential equations:

$$(H_O(\mathbf{r}_i) - E)\Psi_i^\varepsilon(\mathbf{r}_i) = -V_i(\mathbf{r}_i) \times \left(\Psi_i^\varepsilon(\mathbf{r}_i) + \sum_{k \neq i} K(\gamma_{ik})\psi_k^\varepsilon(\mathbf{r}_i) \right). \tag{236}$$

The physical solutions of this system are sought in the C^2 class of functions satisfying definite boundary conditions

$$\Psi_i^\varepsilon(\mathbf{r}_i) \to (x_i y_i)^{-1} U_i^{as}(\mathbf{r}_i), \quad r \to \infty, \tag{237}$$

where U_i^{as} is a known function corresponding to the studied three-particle configuration: a bound state or scattering states.

Following Ref. 21 but using the operators $K(\gamma_{ki})$, let us recall how and why the system of three equations (236) reduces to a single equation in the case of identical particles.

In addition to the required conditions (229) that the particles be identical and the Pauli postulate (230), the equations

$$\Psi_i(\mathbf{r}_i) = S_{jk}^{\pm}\Psi_i(\mathbf{r}_i), \quad i = 1,2,3, \tag{238}$$

are satisfied. They imply that by construction, for three (fermions) bosons each Faddeev component is a function (anti) symmetric under reflection $\mathbf{x}_i \to -\mathbf{x}_i$ (permutation of particles labeled j and k). We act with the operator $K(\gamma_{ki})$ on Eq. (234) defining the component Ψ_i^ε. Then, owing to the symmetry conditions (230) on the wave function and the identical nature of the interactions (229), this equation becomes (234) defining the component Ψ_k. This means that identical particles generate simple relations between the Faddeev components:

$$\Psi_i(\mathbf{r}_k) = K(\gamma_{ki})\Psi_k(\mathbf{r}_i). \tag{239}$$

Owing to these, the (anti)symmetric wave function (235) is expressed in terms of one of its components and two kinematical operators:

$$\Psi^\varepsilon(\mathbf{r}_i) = S_i\Psi_i^\varepsilon(\mathbf{r}_i) = \left(I + \sum_{k \neq 1} K(\gamma_{ki}) \right) \Psi_i^\varepsilon(\mathbf{r}_i). \tag{240}$$

Therefore, in the case of identical fermions or bosons, all three equations of the system (236) take a functionally identical form:

$$(H_0(\mathbf{r}) - E)\Psi_i^\varepsilon(\mathbf{r}) = -V(\mathbf{r})S_i(\mathbf{r})\Psi_i^\varepsilon(\mathbf{r}), \tag{241}$$

where $\mathbf{r} \equiv \mathbf{r}_i$ and the omitted index i can take any of three values.

Let us discuss the essential difference between the structures of the Schrödinger and Faddeev equations. In the Schrödinger equation (223), two components of the total interaction undergo the kinematical transformation, while in each Faddeev equation (236) only two components of the desired wave function are transformed. This structure of the Faddeev equations in $\mathcal{R}^6$ significantly simplifies their reduction to systems of integro-differential (differential) equations in $\mathcal{R}_+^2$ ($\mathcal{R}_+^1$).

Let us turn to the analysis of this reduction in the cases of central and S-wave interactions. We shall consider the cases of nonidentical and identical particles separately for

these two interactions. We shall pay special attention to the equivalence of the two methods of reducing the original Faddeev equations to equations in $\mathcal{R}_+^1$. The pair interactions of the two types considered [Eqs. (217)–(220)] commute with the operators $\mathbf{l}^2$, l_3, and P. Therefore, (225) is satisfied, and the Faddeev components (231) and the wave function (232) are eigenfunctions of these operators. Then for these components we have exact representations in the form of series of the type (84)–(87) with a multiple index $\varepsilon=(\ell,m,\sigma)$. Everywhere below, we shall use such an ansatz for the initial representation, and where possible we shall supplement the set ε by other conserved quantum numbers.

3.4.1. Angular analysis in the case of central interactions

Suppose that the particles are nonidentical. We represent the components Ψ_i^ε by the series

$$\Psi_i^\varepsilon(\mathbf{r}_i)=2r^{-2}\operatorname{cosec}2\varphi_i U_i^\ell(\mathbf{r}_i),$$

$$U_i^\ell(\mathbf{r}_i)=\sum_{ab} U_{iab}^\ell(r,\varphi_i)\mathcal{Y}_{ab}^{\ell m}(\hat{x}_i,\hat{y}_i), \tag{242}$$

$$U_{iab}^\ell(r,\varphi_i)=\sum_L \tilde{W}_{Lab}(\sigma_i)U_{iLab}^\ell(r); \tag{243}$$

$$\Psi_i^\varepsilon(\mathbf{r}_i)=r^{-2}\sum_L\sum_{ab} Y_{Lab}^{\ell m}(\Omega_i)U_{iLab}^\ell(r). \tag{244}$$

The kinematical images of these series can be expressed according to the rules (130)–(132) and (145). Then for the wave function (235) we obtain the expansions

$$\Psi^\varepsilon(\mathbf{r}_i)=2r^{-2}\operatorname{cosec}2\varphi_i\sum_{ab}\mathcal{Y}_{ab}^{\ell m}(\hat{x}_i,\hat{y}_i)\times\Bigg(U_{iab}^\ell(r,\varphi_i)+\sum_{k\neq 1}\sum_{a'b'}\langle r,\varphi_i|H_{aba'b'}^\ell|U_{ka'b'}^\ell\rangle\Bigg), \tag{245}$$

$$\langle r,\varphi_i|h_{aba'b'}^\ell(\gamma_{ki})|U_{ka'b'}^\ell\rangle\equiv\int_{C_-(\varphi_i;\gamma_{ki})}^{C_+(\varphi_i,\gamma_{ki})}d\varphi_k h_{aba'b'}^\ell\times(\varphi_i,\varphi_k;\gamma_{ki})U_{ka'b'}^\ell(r,\varphi_k);$$

$$\Psi^\varepsilon(\mathbf{r}_i)=r^{-2}\sum_{Lab}Y_{Lab}^{\ell m}(\Omega_i)\Bigg(U_{Lab}^\ell(r,\varphi_i)+\sum_{k\neq i}\sum_{a'b'}\langle ab|K(\gamma_{ki})|a'b'\rangle_{L\ell}U_{kla'b'}^\ell(r)\Bigg). \tag{246}$$

In (236) we replace the interactions, the Faddeev components, and the wave function by their bispherical series (219), (242), and (245). Using (206), we project each of the three resulting equations onto its proper bispherical basis. Then we obtain a system of integro-differential equations in $\mathcal{R}_+^2$ which is infinite in the indices a and b:

$$(\partial_r^2+r^{-1}\partial_r-r^{-2}\tilde{L}_{ab}^2(\varphi_i)+E-V_i(r\cos\varphi_i))U_{iab}^\ell(r,\varphi_i)=V_i(r\cos\varphi_i)\sum_{k\neq i}\sum_{a'b'}\langle r,\varphi_i|h_{aba'b'}^\ell(\gamma_{ik})|U_{ka'b'}^\ell\rangle. \tag{247}$$

We make this system into a boundary-value problem by imposing the boundary conditions

$$U_{iab}^\ell(r,\varphi_i)=0,\quad \varphi_i=0,\ \pi/2,\quad r\in\mathcal{R}_+^1;\quad r=0,\forall\varphi_i; \tag{248}$$

$$U_{ab}^\ell(r,\varphi_i)\to U_{iab}^{as}(r,\varphi_i)\equiv\langle\ell mab|U_i^{as}\rangle,\quad r\to\infty. \tag{249}$$

Then the Faddeev components (242) will be regular functions with the required asymptotic form (237).

We use the method proposed in Ref. 67 to reduce the resulting boundary-value problem (247)–(249) to a one-dimensional problem. First we label the equations of the system (247) by the multiple index $\tau\equiv(i,a,b)$. The equation containing the desired function U_{iab}^ℓ on the left-hand side will be called the τ equation. In this equation we replace all the bispherical components $U_{ka'b'}^\ell(r,\varphi_i)$ by series of the type (243) in their proper basis functions $\tilde{W}_{La'b'}(\varphi_k)$. Let us see how these functions are mapped by the operators $\tilde{L}^2$, $h_{aba'b'}^\ell$, and V_i. According to (82), the function $\tilde{W}_{Lab}(\varphi_i)$ is the eigenfunction for the operator $\tilde{L}^2(r,\varphi_i)$. Owing to the property (183), the function $\tilde{W}_{La'b'}(\varphi_k)$ is mapped by the operator $h_{aba'b'}^\ell(\gamma_{ki})$ into the function $\tilde{W}_{Lab}(\varphi_i)$. Finally, from (217)–(220) it follows that the image $V_i W_{Lab}$ is a linear combination of the functions $\tilde{W}_{Lab}(\varphi_i)$ in the index L. Therefore, the τ equation in question reduces to a sum over the index L which is equal to zero. Each term in the sum is a product of the function $\tilde{W}_{Lab}(\varphi_i)$ and a differential relation in the variable r for the functions $U_{kLa'b'}(r)$. The functions $\tilde{W}_{Lab}$ with different L but identical a and b are linearly independent. Therefore, this sum is zero in $\mathcal{R}_2^+$ if and only if each differential relation is zero. This implies that the τ equation reduces to a system of differential equations for the functions $U_{kLa'b'}(r)$.

Let us formulate these statements. First of all, the countable ($L=a+b,\ a+b+2,\ldots$) set of functions $\tilde{W}_{Lab}(\varphi_i)$ is a complete basis for the τ equation of the system (247) containing the function U_{iab}^ℓ with selected indices i, a, and b on the left-hand side. Second, in this and only this equation the following spectral representations are valid for the operators $\tilde{L}^2$, $h_{aba'b'}^\ell(\gamma_{ki})$, and V_i:

$$\tilde{L}_{ab}^2(\varphi_i)=\sum_L|\tilde{W}_{Lab}(\varphi_i)\rangle(L+2)^2\langle\tilde{W}_{Lab}(\varphi_i)|, \tag{250}$$

$$h_{aba'b'}^\ell(\gamma_{ki})=\sum_L|\tilde{W}_{Lab}(\varphi_i\rangle\times\langle ab|K(\gamma_{ki})|a'b'\rangle_{L\ell}\langle\tilde{W}_{Lab}(\varphi_k)|,\quad k\neq i, \tag{251}$$

$$V_i(r\cos\varphi_i)=\sum_{L',L=a+b}|\tilde{W}_{L'ab}(\varphi_i)\rangle V^{LL'}_{iab}\langle\tilde{W}_{Lab}(\varphi_i)|. \quad (252)$$

Using the representation (243) and (250)–(252), we project each τ equation of the system (247) and its boundary conditions (248) and (249) onto the corresponding proper basis of the functions $\tilde{W}_{Lab}(\varphi_i)$. Therefore, in $\mathcal{R}^1_+$ we have a boundary-value problem for the unknown functions $U^{\ell}_{iLab}(r)$:

$$(\partial_r^2+r^{-1}\partial_r-r^{-2}(L+2)^2+E)U^{\ell}_{iLab}(r)$$
$$=\sum_{L'}V^{LL'}_{iab}(r)\Bigg(U^{\ell}_{iL'ab}(l)$$
$$+\sum_{k\neq i}\sum_{a'b'}\langle a'b'|K(\gamma_{ki})|ab\rangle_{L'l}U^{\ell}_{kL'a'b'}(r)\Bigg), \quad (253)$$

$$U^{\ell}_{iLab}(0)=0;\quad U^{\ell}_{iLab}(r)\to\langle\tilde{W}_{Lab}(\varphi_i)|U^{as}_{ab}(r,\varphi_i)\rangle,$$
$$r\to\infty. \quad (254)$$

The hyperharmonics (78) are products of bispherical harmonics and the functions (79). Therefore, expansion of the Faddeev components first in bispherical series (242) and then in series (243) in the functions $|L\bar{a}b\rangle$ is equivalent to the resulting expansion of these components in hyperspherical series (244). The same boundary-value problem (253), (254) can thus be derived in another way: by substituting the ansatz (244) into (236) and the boundary conditions (237) and then projecting the resulting equations onto their proper hyperspherical bases.

The passage from the Faddeev equations in the bispherical basis (247) to the Faddeev equations in the hyperspherical basis (253) is thus effected by projecting the former equations onto the corresponding proper basis functions $\tilde{W}_{Lab}(\varphi_i)$. This statement was proved in the most general case in Ref. 61 and is mentioned in an abbreviated published form in Ref. 67. The inverse transformation, i.e., derivation of the Faddeev equations in the bispherical basis from the Faddeev equations in the hyperspherical basis, was realized in Ref. 112 only in a special case, namely, for a system of three identical bosons with S-wave interactions and zero total angular momentum.

Let us consider the case of identical particles. Interchange of the particles labeled j and k is equivalent to reflection of the vector $\mathbf{x}_i$. In this operation the bispherical harmonics $\mathcal{Y}^{\ell m}_{ab}(\hat{x}_i,\hat{y}_i)$ acquire the phase factor $(-1)^b$. Therefore, the additional symmetry condition (238) will be automatically satisfied if in the ansatz (242) and (244) for the desired Faddeev component of the three (fermion) boson wave functions we define all the bispherical and hyperspherical components with (even) odd indices b to be zero:

$$u^{\ell}_{iab}(r,\varphi_i),\quad U^{\ell}_{iLab}(r)\equiv 0,\quad \begin{cases}\text{fermions} & b=0,2,\ldots,\\ \text{bosons} & b=1,2,\ldots.\end{cases}$$

Substituting the modified expansions of the Faddeev components into (240) in this manner, we obtain the expansions of the wave function in the case of identical particles in the representation $\langle\mathbf{r}_i|$:

$$\Psi^{\varepsilon}(\mathbf{r}_i)=2r^{-2}\operatorname{cosec}2\varphi_i\sum_{ab}\mathcal{Y}^{\ell m}_{ab}(\hat{x}_i,\hat{y}_i)$$
$$\times\sum_{a'b'}\langle r,\varphi_i|S^{\ell}_{aba'b'}|U^{\ell}_{ia'b'}\rangle,$$
$$\langle\ell mab|S_i|\ell mabU^{\ell}_{ia'b'}\rangle\equiv\langle r,\varphi_i|S^{\ell}_{iaba'b'}|U^{\ell}_{ia'b'}\rangle$$
$$\equiv\delta_{aa'}\delta_{bb'}U^{\ell}_{iab}(r,\varphi)$$
$$+\sum_{k\neq i}\langle r,\varphi_i|h^{\ell}_{aba'b'}(\gamma_{ki}|U^{\ell}_{ia'b'}\rangle, \quad (255)$$

$$\Psi^{\varepsilon}(\mathbf{r}_i)=r^{-2}\sum_{Lab}Y^{\ell m}_{Lab}(\Omega_i)\sum_{a'b'}\langle ab|S_i|a'b'\rangle_{L\ell}U^{\ell}_{Lab}(r),$$
$$\langle L\ell mab|S_i|L\ell ma'b'\rangle$$
$$\equiv\langle ab|S_i|a'b'\rangle_{L\ell}$$
$$=\delta_{aa'}\delta_{bb'}+\sum_{k\neq i}\langle ab|K(\gamma_{ki})|a'b'\rangle_{L\ell}. \quad (256)$$

3.4.2. Angular analysis in the case of S-wave interactions

Let us turn to the angular analysis of Eq. (241) for three bosons with S-wave interactions specified by (218) with $b=0$. In this equation we set $V_i=V^0_i$ and represent the functions Ψ^{ε}_i and Ψ^{ε} by bispherical series (242) and (255) with even index b. We project the resulting equation onto its proper bispherical basis. Owing to the projection property (221) of the interaction V^0_i, for each bispherical component U^{ℓ}_{iab} with index $b\neq 0$ we obtain a homogeneous equation of the type (209) with known solution (213), and the equation for the component $U^{\ell}_{i\ell 0}$ with $b=\ell$ will be inhomogeneous. We write this equation as

$$(\partial_r^2+r^{-1}\partial_r-r^{-2}\tilde{L}^2_{l0}+E)U^{\ell}(r,\varphi)$$
$$=V(r\cos\varphi)\langle r,\varphi|S^{\ell}_i|U^{\ell}\rangle, \quad (257)$$

where we have dropped the indices i, $a=\ell$, and $b=0$,

$$U^{\ell}\equiv U^{\ell}_{i\ell 0},\quad \langle r,\varphi|S^{\ell}|U^{\ell}\rangle\equiv\langle r,\varphi|S^{\ell}_{i\ell 0\ell 0}|U^{il0}\rangle, \quad (258)$$

and, according to Eqs. (115), (117), (123), (127), (132), and (255),

$$\langle r,\varphi|S^{\ell}_i|U^{\ell}\rangle=U^{\ell}(r,\varphi)+(4/\sqrt{3})\int_{C_-(\varphi,\pi/3)}^{C_+(\varphi,\pi/3)}d\varphi' P_{\ell}$$
$$\times(u_{yy'}(\varphi,\varphi';\pi/3))U^{\ell}(r,\varphi'),$$
$$u_{yy'}(\varphi,\varphi';\pi/3)=(2(\cos 2\varphi+\cos 2\varphi')$$
$$-1)/(4\sin\varphi\sin\varphi'). \quad (259)$$

Let us describe the construction of the wave function Ψ^{ε}. If $E<0$, then $U^{\ell}_{iab}\equiv 0$ for $b\neq 0$, and the bispherical series (255) of this function degenerates into the equation

$$\Psi^{\varepsilon}(\mathbf{r})=2r^{-2}\operatorname{cosec}2\varphi\langle r,\varphi|S^{\ell}|U^{\ell}\rangle\mathcal{Y}^{\ell m}_{l0}(\hat{x},\hat{y}). \quad (260)$$

This function is an eigenfunction of the operators $\mathbf{l}^2_x$ and $\mathbf{l}^2_y$:

$$\mathbf{l}_x^2\Psi^\varepsilon=0, \quad (\mathbf{l}_y^2-l(l+1))\Psi^\varepsilon=0.$$

This means that for $E<0$ five quantum numbers are conserved: $\varepsilon=(\ell,m,\sigma,a,b)$, where $\sigma=1$, $a=\ell$, and $b=0$. In the case $E\geqslant 0$ the bispherical series (255) of the wave function Ψ^ε contains only one interaction-dependent term, and the other terms are *a priori* known explicitly. Therefore, everywhere below for either sign of the energy E we shall study only the parts of the Faddeev components

$$\Psi_i^\varepsilon(\mathbf{r})=2r^{-2}\operatorname{cosec}2\varphi U_{\ell 0}^{\ell}(r,\varphi)\mathcal{Y}_{\ell 0}^{\ell m}(\hat{x},\hat{y}), \quad \mathbf{r}\equiv\mathbf{r}_i, \tag{261}$$

depending on the potential, and the function (260) corresponding to them.

Let us separate the variables r and φ in (257). For this we write the desired function U_{iab}^{ℓ} and the matrix representations of all the operators (250)–(252) contained in this equation in the abbreviated notation of (258) and (259):

$$U^{\ell}(r,\varphi)=\sum_L U_L^{\ell}(r)\tilde{W}_{L\ell 0}(\varphi), \tag{262}$$

$$\tilde{L}_{\ell 0}^2(r,\varphi)=\sum_L |\tilde{W}_{L\ell 0}(\varphi)\rangle(L+2)^2\langle\tilde{W}_{L\ell 0}(\varphi)|, \tag{263}$$

$$S^{\ell}=\sum_{L=\ell}^{\infty}|\tilde{W}_{L\ell 0}(\varphi)\rangle S_L^{\ell}\langle\tilde{W}_{L\ell 0}(\varphi)|, \tag{264}$$

$$S_L^{\ell}\equiv S_{\ell 0\ell 0}^{\ell}=1+2\langle\ell 0|K(\pi/3)|\ell 0\rangle_{L\ell}. \tag{265}$$

$$V(r\cos\varphi)=\sum_{LL'}|\tilde{W}_{L\ell 0}(\varphi)\rangle V_{\ell 0}^{LL'}(r)\langle\tilde{W}_{L\ell 0}(\varphi)|. \tag{266}$$

Now, using the representations (262)–(266), we project Eq. (257) onto its proper basis of the functions $\tilde{W}_{L\ell 0}$, $L=\ell,\ell+2,\ldots$. In the end we obtain the desired system of one-dimensional equations:

$$(\partial_r^2+r^{-1}\partial_r-r^{-2}(L+2)^2+E)U_L^{\ell}(r)$$
$$=R_L^{\ell}(r)\equiv\sum_{L'}V_{\ell 0}^{LL'}(r)S_{L'}^{\ell}U_{L'}^{\ell}(r). \tag{267}$$

We conclude by using (262) and (264) to express the wave function (260) in terms of hyperspherical components obeying the system of equations (267):

$$\Psi^\varepsilon(\mathbf{r})=r^{-2}\sum_L Y_{\ell 0}^{\ell m}(\Omega)S_L^{\ell}U_L^{\ell}(r),$$
$$\varepsilon=(\ell,m,\sigma,a,b), \quad \sigma=1, \quad a=\ell, \quad b=0. \tag{268}$$

4. EXACT SOLUTIONS OF THE FADDEEV EQUATIONS

In Sec. 4.1 we discuss the conditions for the existence of, and the methods of constructing, spurious solutions of the Faddeev equations (236), and we give examples of spurious solutions in the case of central and S- and P-wave interactions. In Sec. 4.2 we consider two special cases in which both spurious solutions and solutions with a physical meaning can be found exactly.

4.1. Spurious solutions

Let us begin by explaining why spurious solutions can exist, and why their study is interesting. In the Schrödinger equation (223), the entire Faddeev sum (232) is the unknown function Ψ^ε. In the system of Faddeev equations (233) the terms of this sum are the unknown functions. In general, the sum of three nonzero functions can be identically zero. Therefore, the system (233) without any auxiliary conditions can have a nontrivial and regular solution $(\Psi_1^\varepsilon,\Psi_2^\varepsilon,\Psi_3^\varepsilon)$, with the sum (232) of its components identically equal to zero, $\Psi^\varepsilon\equiv 0$. This solution [which we write as the set (S_1,S_2,S_3)] simultaneously makes the right- and left-hand sides of the system (236) vanish:

$$S_i(\mathbf{r}_i)+\sum_{k\neq i}K(\gamma_{ik})S_i(\mathbf{r}_i)=0, \quad i=1,2,3, \tag{269}$$

$$(H_0(\mathbf{r}_i)-E)S_i(\mathbf{r}_i)=0, \quad i=1,2,3. \tag{270}$$

The set (S_1,S_2,S_3) of twice continuously differentiable functions satisfying only the system of equations (269), (270) is commonly[100] called a spurious solution of the Faddeev equations (236). Since a spurious solution satisfies the free Schrödinger equation (270), it does not contain any information about pair potentials. According to (269), a spurious solution corresponds to a trivial solution ($\Psi^\varepsilon\equiv 0$) of the Schrödinger equation (222). Therefore, a spurious solution does not correspond to any physical state of a three-particle system. As was noted in Ref. 108, the system of equations (269), (270) always has a nontrivial two-parameter solution. We shall construct it.

Let Ψ_0 and E be eigenfunctions and eigenvalues of the free Hamiltonian H_0. For any numbers c_i the functions

$$S_i(\mathbf{r}_i)\equiv c_i\Psi_0(\mathbf{r}_i), \quad i=1,2,3, \tag{271}$$

satisfy (270). We set $c_1=-c_2-c_3$. Then the coefficients c_2 and c_3 remain independent, and the functions (271) satisfy Eq. (269). Therefore, any solution Ψ_0 of the free Schrödinger equation corresponds to a two-parameter spurious solution (271) of the Faddeev equations (236). The components S_i of this solution will not always be the eigenfunctions of a given set of operators commuting with the full Hamiltonian H. Let us give an example. We replace the function Ψ_0 in (271) by the eigenfunction (215) of six momentum operators. We impose the condition $c_1=-c_2-c_3$. The components of the resulting spurious solution will be eigenfunctions of the same momentum operators, but, owing to (215), they will not be eigenfunctions of the operators $\mathbf{l}^2$, l_3, and P. Do there always exist spurious solutions corresponding to an *a priori* specified set $\varepsilon=(\ell,m,\sigma)$ of eigenvalues of such operators? If such solutions exist, how can they be constructed explicitly, and how can they be taken into account in solving the Faddeev equations for a system of three particles with arbitrary masses and central or b-wave interactions? These questions were answered in Ref. 71. We shall give the main results: the proof of the existence criterion for spurious solutions in the class of functions $\mathcal{A}^\varepsilon$ and the method of eliminating such solutions.

First we shall study the condition (269), and then the condition (270). According to the condition (269), the sum of

all the components of a spurious solution must be zero in *each* of three ($i=1,2,3$) sets of Jacobi coordinates. Let us consider these conditions as three equations for the unknown components. We shall seek the components in the form of hyperspherical series in their proper bases:

$$S_i(\mathbf{r}_i)=r^{-2}\sum_L \sum_{ab} S_{iLab}(r)Y_{Lab}(\Omega_i). \tag{272}$$

We substitute these series into (269) and take into account (142). Then for each value of L we obtain a linear, homogeneous system of equations

$$S_{iLa'b'}(r)+\sum_{k\neq i}\sum_{a'b'}\langle ab|K(\gamma_{ki})|a'b'\rangle_{L\ell}S_{kLa'b'}(r)=0, \tag{273}$$

in which $i=1,2,3$ and the indices a,b,a',b' take all the values allowed for given L and ℓ. The number of such values is unbounded. Therefore, the matrix $\mathbf{M}^L$ of this system (273) has finite dimension equal to three times the number of hyperharmonics (55) with given quantum numbers L, ℓ, and m. It is known from matrix theory[2] that the system (273) has

$$N^L=\dim\mathbf{M}^L-\operatorname{rank}\mathbf{M}^L$$

nontrivial and linearly independent solutions if and only if the determinant of this matrix is zero:

$$\det\mathbf{M}^L=0. \tag{274}$$

Let $\mathcal{D}$ be the set of values of L for which the conditions (274) are satisfied. If this set is empty ($\mathcal{D}=\varnothing$), i.e., $\det\mathbf{M}^L\neq 0$ for any L, there are no nonzero functions (272) satisfying the equations (269). In this trivial case (233) has no spurious solutions in the class $\mathcal{A}^\varepsilon$. Since (269) reduces to an uncoupled system of equations (273), the general solution (272) is the linear combination

$$S_i(\mathbf{r}_i)=\sum_{L\in\mathcal{D}} C^L S_i^L(r,\Omega_i) \tag{275}$$

with numerical coefficients C^L and particular solutions

$$S_i^L(\mathbf{r}_i)=\sum_{ab} S_{iLab}(r)Y_{Lab}^{\ell m}(\Omega_i), \tag{276}$$

possessing the quantum number L in addition to the set (ℓ,m,σ). Let us formulate the statements proved above as lemmas.

Lemma. *In the class $\mathcal{A}^\varepsilon$ the system (269) has nontrivial solutions if and only if $\mathcal{D}\neq\varnothing$. All such solutions are linear combinations (275) of the functions (276) with hyperspherical components obeying the conditions (273).*

Now from all the solutions (275) of (269) let us choose those satisfying the condition (270). These will be spurious solutions. We substitute the desired solutions S_i in their general form (275), (276) into (270). Using (69), we project the resulting equations onto their proper hyperspherical bases. Then by the substitution $z=\sqrt{E}r$, $Z_\nu=S_{iLab}$, $\nu=L+2$ we reduce the equation for each desired component $S_{iLab}(z)$ with $L\in\mathcal{D}$ to the Bessel equation (211). It has nontrivial regular solutions only for real $z>0$, i.e., for $E>0$. All such solutions are proportional to Bessel functions:

$$S_{iLab}=D_{iab}^L J_{L+2}(\sqrt{E}r),\quad E>0. \tag{277}$$

For each value $L\in\mathcal{D}$ the functions (277) depend on r in the same way, and so they satisfy the system (273) if and only if the coefficients D_{iab}^L satisfy the system of equations

$$D_{iab}^L+\sum_{k\neq i}\sum_{a'b'}\langle ab|K(\gamma_{ki})|a'b'\rangle_{L\ell}D_{ka'b'}^L=0, \tag{278}$$

which has the same matrix $\mathbf{M}^L$ as the system (273).

Substituting the ansatz (277) into the condition (276), we obtain the particular spurious solution with quantum numbers ℓ, m, σ and $L\in\mathcal{D}$:

$$S_i^L(\mathbf{r}_i)=r^{-2}J_{L+2}(\sqrt{E}r)\sum_{ab} D_{iab}^L Y_{Lab}^{\ell m}(\Omega_i). \tag{279}$$

If all such solutions are substituted into the sum (275), we obtain the general spurious solution with quantum numbers $\varepsilon=(\ell,m,\sigma)$:

$$S_i^\varepsilon=r^{-2}\sum_{L\in\mathcal{D}} C^L J_{L+2}(\sqrt{E}r)\sum_{ab} D_{iab}^L Y_{Lab}^{\ell m}(\Omega_i). \tag{280}$$

Owing to the lemma and Eqs. (277)–(280), the following criterion is valid.

Theorem 2. *In the class $\mathcal{A}^\varepsilon$ the Faddeev equations (236) have spurious solutions $(S_1^\varepsilon,S_2^\varepsilon,S_3^\varepsilon)$ if and only if $E>0$ and $\mathcal{D}\neq\varnothing$. All such spurious solutions are sums (280) in which the coefficients C^L are arbitrary numbers, and the coefficients D_{iab}^L obey the conditions (278).*

Now we shall prove that all spurious solutions possess the same quantum number in addition to the set $\varepsilon=(\ell,m,\sigma)$. Let $\mathcal{S}^\varepsilon$ and $\mathcal{U}^\varepsilon$ be linear subspaces of the space $\mathcal{A}^\varepsilon$ whose bases in $\mathcal{S}^5$ and $\mathcal{R}^6$ form hyperharmonics (55) with the corresponding indices $L\in\mathcal{D}$ and $L\notin\mathcal{D}$. Then the operators

$$P_s\equiv\sum_{L\in\mathcal{D}}\sum_{ab}|Y_{Lab}^{\ell m}(\Omega_i)\rangle\langle Y_{Lab}^{\ell m}(\Omega_i)|,$$

$$P_u\equiv\sum_{L\notin\mathcal{D}}\sum_{ab}|Y_{Lab}^{\ell m}(\Omega_i)\rangle\langle Y_{Lab}^{\ell m}(\Omega_i)| \tag{281}$$

are projectors onto the corresponding subspaces $\mathcal{S}^\varepsilon$ and $\mathcal{U}^\varepsilon$ orthogonal to each other with respect to integration over the hypersphere $\mathcal{S}^5$ in the set of hyperangles Ω_i. Since the kinematical transformation of the hyperharmonics (142) is unitary, the representations (281) of the projectors P_s and P_u are invariant with respect to the index i. Taking this into account, we operate with the projectors on the spurious solutions (280). Then we obtain the equations

$$P_sS_i^\varepsilon=p_sS_i^\varepsilon,\quad p_s=1;\quad P_uS_i^\varepsilon=0, \tag{282}$$

implying that the components S_i^ε of all the spurious solutions $(S_1^\varepsilon,S_2^\varepsilon,S_3^\varepsilon)$ are eigenfunctions of the operator P_s corresponding to the same eigenvalue $p_s=1$ and belonging to the set $\mathcal{S}^\varepsilon$. The subspace $\mathcal{U}^\varepsilon$ of the Faddeev equation (233) therefore cannot have any spurious solutions.

Using this fact, from the Faddeev equations we derive modified equations which have no spurious solutions. Since $\mathcal{U}^\varepsilon$ is the orthogonal complement of $\mathcal{S}^\varepsilon$ with respect to $\mathcal{A}^\varepsilon$,

the components of any nontrivial solution of (233) can be represented as sums of their projections onto these subspaces:

$$\Psi_i^\varepsilon = P_u\Psi_i^\varepsilon + P_s\Psi_i^\varepsilon . \tag{283}$$

Using this representation, it follows by *reductio ad absurdum* that

$$\sum_{k=1}^{3} P_u\Psi_k^\varepsilon \neq 0, \quad \sum_{k=1}^{3} P_s\Psi_k^\varepsilon = 0. \tag{284}$$

Suppose that for any $i=1,2,3$ and nonzero functions $R_i \in \mathcal{S}^\varepsilon$ we have

$$\sum_{k=1}^{3} \langle \mathbf{r}_i | P_u\Psi_k^\varepsilon(\mathbf{r}_k)\rangle = 0, \quad \sum_{k=1}^{3} \langle \mathbf{r}_i | P_s\Psi_k^\varepsilon(\mathbf{r}_k)\rangle = R_i .$$

In these proposed equations we replace the functions Ψ_i^ε and R_i by their hyperspherical series of the type (65),

$$\Psi_i^\varepsilon(\mathbf{r}_i) = \sum_L \sum_{ab} \Psi_{iLab}^{\ell}(r) Y_{Lab}^{\ell m}(\Omega_i),$$

$$R_i(\mathbf{r}_i) = \sum_L \sum_{ab} R_{iLab}^{\ell}(r) Y_{Lab}^{\ell m}(\Omega_i),$$

and we represent the operators P_s and P_u in the form (281). We project the resulting equations onto the hyperspherical basis by means of (145). For each fixed L we thus obtain the system of equations

$$\Psi_{iLab}^{\ell} + \sum_{k\neq i} \sum_{a'b'} \langle ab|K(\gamma_{ki})|a'b'\rangle_{L\ell} \Psi_{kLa'b'}^{\ell} = \begin{cases} 0, & L \notin \mathcal{D}, \\ R_{iLab}^{\ell}, & L \in \mathcal{D}, \end{cases} \tag{285}$$

where a and b take all the values possible for given ε and L. If $L \notin \mathcal{D}$, the homogeneous system (285) has only a trivial solution, since its matrix $\mathbf{M}^L$ is not degenerate. If $L \in \mathcal{D}$, the inhomogeneous system (285) has a degenerate matrix $\mathbf{M}^L$ and is inconsistent according to the Fredholm alternative.[2] The resulting contradictions prove the validity of (284).

Let us write $U_i^\varepsilon \equiv P_u\Psi_i^\varepsilon$. Using (281)–(284), we project (233) onto the subspace $\mathcal{U}^\varepsilon$. Any nontrivial solution of the resulting modified Faddeev equations

$$(H_0 - E)U_i^\varepsilon = -P_uV_i\Psi^\varepsilon = -P_uV_i\sum_{k=1}^{3} U_k^\varepsilon \tag{286}$$

belongs to the space $\mathcal{U}^\varepsilon$, and the corresponding wave function $\Psi^\varepsilon = U_1^\varepsilon + U_2^\varepsilon + U_3^\varepsilon$ is always nonzero.

We conclude our study of the general properties of spurious solutions by formulating a theorem which summarizes the statements (283), (284), and (286) that we have proved, and we give some explanatory remarks.

Theorem 3. *In the class $\mathcal{A}^\varepsilon$ the Faddeev components Ψ_i^ε can be represented as a sum (283) of orthogonal terms $P_u\Psi_i^\varepsilon$ and $P_s\Psi_i^\varepsilon$ satisfying (284). The terms $U_i^\varepsilon \equiv P_u\Psi_i^\varepsilon$ satisfy the system of equations (286), which has no spurious solutions.*

We note that the necessary condition ($E>0$) for the existence of spurious solutions has been stated in a recent study,[100] and the representation (283) has been proved in Ref. 103 by another, less transparent method. According to the terminology of that study, the spaces $\mathcal{U}^\varepsilon$ and $\mathcal{S}^\varepsilon$ are referred to as the spaces of physical and spurious solutions of the Faddeev equations. The main statement of Theorem 3 can therefore be formulated as follows: the Faddeev equations in the physical space (286) do not have any spurious solutions.

Let us give some examples of spurious solutions from Ref. 71. We shall begin with the most general case of central interactions. Then we turn to the special cases of S- and P-wave interactions. Owing to (217)–(221), to go from central interactions to $\tilde{b}$-wave ones we need to restrict the indices b and b' from taking values different from $\tilde{b}$ in (276)–(280). This has been proved for S-wave interactions in Sec. 3.4.2, and a similar proof exists for P-wave interactions.

Spurious solutions in the case of central interactions. We note that only for $L,\ell=0$ or $L=2$, $\ell=1$ can the indices a,b of the hyperharmonics (55) take only the values $a,b=0$ and and $a,b=1$, respectively. In these cases there is only one nonzero Raynal–Revai coefficient, and it is equal to unity: $\langle aa|K(\gamma_{ki})|aa\rangle_{L\ell}=1$. Equations (142) therefore degenerate into the equations

$$Y_{Laa}^{\ell m}(\Omega_i) = Y_{Laa}^{\ell m}(\Omega_k), \quad \ell,L=0; \quad l=1, \quad L=2, \tag{287}$$

and all the elements of the matrices $\mathbf{M}^0$ and $\mathbf{M}^2$ of the systems (278) are unity. Then

$$\dim \mathbf{M}^L = 3, \quad \operatorname{rank} \mathbf{M}^L = 1, \quad L,\ell=0; \quad L=2, \quad \ell=1.$$

In both cases the general solution of the system (278) is such that D_{2ab}^L and D_{3ab}^L are arbitrary numbers not simultaneously equal to zero, and

$$D_{1ab}^L = -D_{2ab}^L - D_{3ab}^L .$$

From (279) we find the Faddeev components

$$S_i^L(\mathbf{r}_i) = r^{-2}J_{L+2}(\sqrt{E}r)(D_{2ab}^L(\delta_{i2}-\delta_{i1}) + D_{3ab}^L(\delta_{i3}-\delta_{i1}))Y_{Lab}^{\ell m}(g\mathcal{Q}_i) \tag{288}$$

of the particular spurious solutions possessing the conserved quantum numbers $L,\ell,m,a,b=0$, $\sigma=1$ or $L=2$, $\ell,a,b=1$, $m=0,\pm1$, $\sigma=1$. These solutions (288) are independent of the particle mass and degenerate into trivial ones if all three particles are identical.

Let us prove this statement. We recall that for any state $|\varepsilon\rangle$ of a system of three identical particles, the components of a physical solution of the Faddeev equations (233) satisfy the auxiliary conditions (238) and (239). Let us impose analogous conditions on the components of the spurious solution. Let

$$S_i^\varepsilon(\mathbf{x}_i,\mathbf{y}_i) = \pm S_i^\varepsilon(-\mathbf{x}_i,\mathbf{y}_i);$$

$$S_i^\varepsilon(\mathbf{r}_i) = S_k^\varepsilon(\mathbf{r}_k(\mathbf{r}_i)), \quad i,k=1,2,3; \quad k\neq i. \tag{289}$$

Using (287), we can show that the symmetry conditions (289) are satisfied only by trivial spurious solutions (288) with coefficients $D_{iab}^L=0$, $i=1,2,3$, as required.

Spurious solutions in the case of S-wave interactions. In the case of S-wave interactions, the set of quantum numbers $\varepsilon=(\ell,m,\sigma,a,b)$, $a=\ell$, $b=0$ is conserved. For any L and ℓ the indices a,b and a',b' in (276)–(280) can take only the following values: $a,a'=\ell$ and $b,b'=0$. Therefore, the matrix $\mathbf{M}^L$ of the system (278) always has dimension three. All the off-diagonal elements of this matrix can be found from (170), according to which

$$M^L_{ki}\equiv\langle\ell 0|K(\gamma_{ki})|\ell 0\rangle_{L\ell}$$
$$=(-\sin\gamma_{ki})^{\ell}\frac{P_n^{(\ell+1/2,1/2)}(-\cos 2\gamma_{ki})}{P_n^{(\ell+1/2,1/2)}(-1)},\quad n=(L-\ell)/2. \tag{290}$$

We reduce the condition (274) to the equations

$$\det M^L=1-(M^L_{21})^2-(M^L_{13})^2-(M^L_{32})^2+2M^L_{21}M^L_{13}M^L_{32}=0. \tag{291}$$

Let $\ell=0$, $L=2$. Then $a,b,a',b'=0$. Using (290) and (291), we find that for any particle masses

$$\operatorname{rank}\mathbf{M}^2=2;\quad \det\mathbf{M}^2=0;\quad M^2_{ii}=1,$$

$$M^2_{ki}=\cos 2\gamma_{ki},\quad i,k=1,2,3;\quad k\neq i.$$

Then, using the two properties of the kinematical angles (4),

$$\gamma_{21},\gamma_{13},\gamma_{32}>0,\quad \gamma_{32}=\pi-\gamma_{21}-\gamma_{13},$$

we can show that all the solutions of the system (278) are proportional to the solution

$$D^L_{iab}=\sin(L(\gamma_{21}+\gamma_{13})\delta_{i1}-(-1)^L L(\gamma_{13}\delta_{i2}+\gamma_{21}\delta_{i3})),$$
$$i=1,2,3. \tag{292}$$

Substituting this into (279), we find the three ($i=1,2,3$) components of the spurious solution possessing the quantum numbers $\ell,a,b=0$, $\sigma=1$, and $L=2$ and depending on the particle masses through the kinematical angles:

$$S^L_i=r^{-2}J_{L+2}(\sqrt{E}r)Y^{\ell m}_{Lab}(\Omega_i)\times\sin(L(\gamma_{21}+\gamma_{13})\delta_{i1}$$
$$-(-1)^L L(\gamma_{13}\delta_{i2}+\gamma_{21}\delta_{i3})). \tag{293}$$

Let $L,\ell=1$. Then everywhere $a,a'=1$ and $b,b'=1$. Owing to (290) and (291), we have

$$\operatorname{rank}\mathbf{M}^1=2;\quad \det\mathbf{M}^1=0;\quad M^1_{ii}=1,$$

$$M^1_{ki}=-\cos\gamma_{ki},\quad k,i=1,2,3;\quad k\neq i.$$

The solution of (278) and the corresponding spurious solution (279) are given by (292) and (293) with indices $L,\ell,a=1$, $\sigma=-1$, and $b=0$.

Let L and ℓ be arbitrary, but the particles have equal masses. Then $a,a'=\ell$ and $b,b'=0$, and all $|\gamma_{ki}|=\pi/3$. Therefore, all the off-diagonal elements (290) of the matrix $\mathbf{M}^L$ are equal to each other, and the condition (291) becomes simpler: for each fixed ℓ it reduces to the Diophantine equation

$$(-2)^{\ell-1}P^{(\ell+1/2,1/2)}_{(L-\ell)/2}(-1)=P^{(\ell+1/2,1/2)}_{(L-\ell)/2}(1/2) \tag{294}$$

in L. For $\ell=0$ or $\ell=1$ there is one solution: $L=2$ or $L=1$, and the corresponding components (293) of the spurious solution satisfy the symmetry conditions (289) and have the form

$$S^L_i(\mathbf{r}_i)=(-1)^{L+1}(\sqrt{3}/2)r^{-2}J_{L+2}(\sqrt{E}r)Y^{\ell m}_{Lab}(\Omega_i). \tag{295}$$

We note that in the case of S-wave interactions, the spatial part of the wave function which is a spin-quartet ($s=3/2$) for neutron scattering on the deuteron in the state with $\ell=0$ is completely symmetric.[21] The Faddeev components of this part of the wave function satisfy the system (233), which has the spurious solution (295) with $L=2$.

Spurious solutions in the case of P-wave interactions. Let the interactions be P-wave ones and the particle masses be arbitrary. Now in addition to the numbers ℓ, m, and σ the number $b=1$ is conserved, and the index a can take no more than three values $a=\ell,\ell\pm1$ and, in general, is not conserved. Therefore, the indices a,b and a',b' in (276)–(280) of these solutions can take only the values $b,b'=1$ and $a,a'=\ell,\ell\pm1$. Then the dimension of the matrix $\mathbf{M}^L$ of the system (278) cannot be more than nine. The exceptions are the case $L=2$, $\ell=1$ studied above and the cases $L=2,\ell=0$ and $L,\ell=1$. Let us discuss these.

Let $\ell=0$ and $L=2$. Then $a,b,a',b'=1$. Using the expressions given in Ref. 25, we find all the off-diagonal elements of the matrix $\mathbf{M}^2$ of the system (278). We obtain $M^2_{ii}=1$ and $M^2_{ki}=\cos 2\gamma_{ki}$. This matrix coincides with the matrix $\mathbf{M}^2$ studied earlier in the case of S-wave potentials. Therefore, (292) and (293) are valid also in this case, when $\ell=0$, $a,b=1$, and $L=2$.

Let $\ell,L=1$. Then $a,a'=1$ and $b,b'=0$. Now the matrix $\mathbf{M}^1$ coincides with the matrix $\mathbf{M}^1$ studied above in the case of S-wave potentials. This means that (292) and (293) are valid also in this case, i.e., for $a,b,\ell,L=1$ and $a=0$.

In both cases ($\ell=0$, $L=2$ and $\ell,L=1$) the particular spurious solutions (293) for equal particle masses reduce to the corresponding spurious solutions (295), which satisfy (289) ensuring the complete antisymmetry of the wave function Ψ^{ε}. Accordingly, let us consider a system of three neutrons in the state with $\ell=1$, $s=3/2$, and total angular momentum ($\mathbf{j}=\mathbf{l}+\mathbf{s}$) equal to 5/2. In this state the neutrons interact only via P-wave potentials, and the spatial part of their wave function is completely antisymmetric. Its Faddeev components satisfy the system (233), which has the spurious solution (295) with $L=1$.

4.2. Examples of exact physical and spurious solutions

Here we shall consider a system of three identical bosons with S-wave interactions of the oscillator, confining, and centrifugal types. Such interactions are described by (217)–(219) with $b=0$, and accordingly

$$V(x)=(\omega x)^2/6=(\omega r\cos\varphi)^2/6,\quad \omega,x\in/\mathcal{R}^1_+; \tag{296}$$

$$V(x)\to\infty,\quad x\to\infty, \tag{297}$$

$$V(x)=\alpha x^{-2}=\alpha(r\cos\varphi)^{-2},\quad |\alpha|,x\in\mathcal{R}^1_+. \tag{298}$$

Oscillator potentials. In Ref. 106, Friar and Gibson studied the structure of the Faddeev component Ψ_1^ε of the wave function

$$\Psi^\varepsilon(\mathbf{r})=(1+P^+ +P^-)\Psi_1^\varepsilon(\mathbf{r}), \quad \mathbf{r}\equiv\mathbf{r}_1, \quad P^\pm=K(\pm\pi/3) \tag{299}$$

of the ground state $\varepsilon=(\ell,m,\sigma,a,b)$, $\ell,m,a,b=0$, $\sigma=1$, of three identical bosons with S-wave potentials (296). In this case the Schrödinger equation (223) for the wave function (299) reduces to an equation with separable variables x and y:

$$(x^{-2}\partial_x(x^2\partial_x)+y^{-2}\partial_y(y^2\partial_y)-\omega^2(x^2+y^2)/4+E)\Psi^\varepsilon(r)=0, \tag{300}$$

and the Faddeev equation (236) for the component Ψ_1^ε has the form

$$6(\Delta_x+\Delta_y+E)\Psi_1^\varepsilon(x,y)=(\omega,x)^2\Psi^\varepsilon(r). \tag{301}$$

Substituting the known exact solution[113] of the Schrödinger equation (300),

$$\Psi^\varepsilon(r)=3\exp(-\omega r^2/2), \quad E=3\omega, \tag{302}$$

into the Faddeev equation (302), the authors derived the equation

$$2(\Delta_x+\Delta_y+3\omega)\Psi_1^\varepsilon(x,y)=(\omega x)^2\exp(-\omega r^2/4). \tag{303}$$

By definition, the image (299) of the solution of this equation must coincide with the exact wave function (302). Therefore, in the ansatz

$$\Psi_1^\varepsilon(x^2,y^2)=\exp(-\omega r^2/4)+S(x^2,y^2) \tag{304}$$

used for the desired function S the auxiliary condition

$$(1+P^+ +P^-)S(x^2,y^2)\equiv 0, \quad \forall x,y\in\mathcal{R}_+^1 \tag{305}$$

was imposed. Further, Friar and Gibson noted that, owing to the identities

$$P^\pm(x^2+y^2)\equiv x^2+y^2,$$

$$(1+P^+ +P^-)(x^2-y^2)\equiv 0,$$

$$\forall x,y\in\mathcal{R}_+^1,$$

this condition is satisfied by all functions of the type

$$S(x^2,y^2)=(\omega/4)(x^2-y^2)Z(s), \quad s\equiv\sqrt{\omega/2}r. \tag{306}$$

Using the substitution (304) and (306), the authors reduced (303) to an inhomogeneous Bessel equation for the function $Z(s)$.

As is well known,[8] the general regular solution of this inhomogeneous equation can be represented as a sum of the general solution of the corresponding homogeneous equation and any particular solution of the original equation. The general regular solution of the homogeneous Bessel equation is the product of the Bessel function J_ν of a given order and an arbitrary constant λ, and a particular solution of the inhomogeneous Bessel equation is given by a convolution of a Green function with its right-hand side.

Using the above-described representation of the function Z and the ansatz (306), Friar and Gibson first found the explicit expression

$$S(x^2,y^2)=((x^2-y^2)/\omega r^4)\Big(\lambda J_4(t)+\pi\int_0^t ds s^3\exp(-s^2)$$

$$\times(Y_4(t)J_4(s)-J_4(t)Y_4(s))\Big), \quad t\equiv\sqrt{3\omega}r, \tag{307}$$

for the unusual term in the S-wave Faddeev component (304) and referred to it as a spurious term. Although the spurious term (307) is not determined uniquely, for any value of the parameter λ it does not give any contribution to the wave function (299), owing to (305). Moreover, the spurious term falls off slowly and oscillates asymptotically for $r\to\infty$. In other words, the Faddeev equation (301) or (303) satisfies a one-parameter $[\Psi_1^\varepsilon=\Psi_1^\varepsilon(x,y;\lambda)]$ family of solutions (304), (307) with oscillating asymptotic behavior. However, any member of this family corresponds to only a single solution (302) of the Schrödinger equation (300) with exponentially falling asymptotic behavior. As was noted by Friar and Gibson, this situation occurs only in the specific case that they considered, and the possible existence of spurious solutions of the Faddeev components in general requires special study.

Confining potentials. The conclusions of Friar and Gibson were generalized by the present author in Ref. 57, where it was proved that spurious terms of the Faddeev components exist for bound states with $\ell=0$ of three identical bosons with arbitrary S-wave confining potentials (297). This generalization appeared necessary for the following reasons.

As is well known,[19] the three-particle Schrödinger equation with a confining potential (297) has only positive eigenvalues ($E>0$), and the eigenfunctions corresponding to them belong to the class $\mathcal{L}^2(\mathcal{R}^6)$. However, the existence and uniqueness theorem for the solutions of the Faddeev equations in the case of pair potentials growing at large distances has not been proved[21] for any class of functions. Moreover, the asymptotic behavior of possible solutions of these equations is unknown.

For these reasons, in the case of confining potentials it is necessary to study the structure of the Faddeev components in a class of functions $\mathcal{A}^\varepsilon$ broader than the class of square-integrable functions $\mathcal{L}^2$. Here it is sufficient to limit ourselves to seeking all the Faddeev components which correspond to a square-integrable wave function and have positive total energy. An example of how this can be done is given in Ref. 57. Let us describe the main results.

We recall that in the case of S-wave interactions, the bispherical component U^ℓ and the hyperspherical components U_L^ℓ of the wave function (260), (268) of three identical bosons obey Eqs. (257) and (267). Let us study the structure of the general regular solutions of these equations in the case $E>\ell=0$, for the time being not assuming a specific form of the confining potential. First, we note that for $\ell=0$ the definition (79) gives

$$\tilde{W}_{L00}(\varphi)=(2/\sqrt{\pi})\sin(L+2)\varphi, \quad L=0,2,..., \tag{308}$$

and the expansion (262) becomes a discrete Fourier transform:[3]

$$U^0(r,\varphi)=(2\sqrt{\pi})\sum_{L=0}^{\infty}\sin(L+2)\varphi U_L^0(r). \tag{309}$$

Using (264) and (265), we find

$$S_L^0=1+4\frac{\sin(L+2)\pi/3}{(L+2)\sin 2\pi/3},\quad L=0,2,...,$$

$$S_0^0=1,\quad S_2^0=0,\quad S_L^0\neq 0,\quad \forall L>2. \tag{310}$$

Since $S_2^0=0$, for any potential V the right-hand side of (267) for the function U_2^0 vanishes identically, and the system (267) breaks up into the system ($L=0,4,6,\dots$) of equations

$$(\partial_r^2+r^{-1}\partial_r-(L+2)^2r^{-2}+E)U_L^0(r)$$

$$=R_L^0(r)\equiv\sum_{L'\neq 2}V_{00}^{LL'}(r)S_{L'}^0U_{L'}^0(r), \tag{311}$$

not containing the function U_2^0, and a differential relation

$$(\partial_r^2+r^{-1}\partial_r-16r^{-2}+E)U_2^0(r)$$

$$=R_2^0(r)\equiv\sum_{L'\neq 2}V_{00}^{2L'}(r)S_{L'}^0U_{L'}^0(r), \tag{312}$$

determining the function U_2^0 in terms of all the other functions U_L^0 with indices $L\neq 2$. By replacing the argument $t\equiv r\sqrt{E}$ we reduce this relation to the canonical inhomogeneous Bessel equation. The general solution $U_2^0(t)$ of this equation regular at the origin contains the general regular solution $\lambda J_4(t)$ of the corresponding homogeneous equation and the particular solution of the original equation. Therefore, for $E>0$, Eq. (312) has a one-parameter family of solutions with parameter λ and oscillatory asymptotic behavior at $r\to\infty$:

$$U_2^0(r;\lambda)=\lambda J_4(t)+(\pi/2)\int_0^r dss(Y_4(t)J_4(s)$$

$$-J_4(t)Y_4(s))R_2^0(s/\sqrt{E}),\quad t=\sqrt{E}r. \tag{313}$$

The general solution (309) of (257) thus also depends on λ and can be written as a sum of the physical (in brackets) and spurious terms:

$$U^0(r,\varphi)=(2/\sqrt{\pi})\left(\sum_{L\neq 2}U_L^0(r)\sin(L+2)\varphi\right)$$

$$+(2/\sqrt{\pi})U_2^0(r;\lambda)\sin 4\varphi. \tag{314}$$

This representation generates the following splitting of each ($i=1,2,3$) Faddeev component (261):

$$\Psi_i^\varepsilon(\mathbf{r}_i;\lambda)=\frac{2}{\pi r^2}\sum_{L\neq 2}^{\infty}U_L^0(r_i)\frac{\sin(L+2)\varphi}{\sin 2\varphi}$$

$$+\frac{4}{\pi r^2}U_2^0(r_i;\lambda)\cos 2\varphi. \tag{315}$$

The second terms of these sums are spurious. In fact, owing to the equation $S_2^0=0$, they do not contribute at all to the hyperspherical expansion (268) of the wave function:

$$\Psi^\varepsilon(\mathbf{r}_i)=\frac{2}{\pi r^2}\left(3U_0^0(r_i)+\sum_{L=4}^{\infty}S_L^0U_L^0(r_i)\frac{\sin(L+2)\varphi}{\sin 2\varphi}\right). \tag{316}$$

As an illustrative example, let us consider the special case of oscillator confining potentials (296). Applying (216), (266), (308), and the identity

$$2\cos^2\varphi\sin(L+2)\sigma\equiv\sin(L+4)\varphi-\sin L\varphi,$$

we can calculate the matrix elements of these potentials:

$$V_{00}^{LL'}(r)=\pm(\omega r^2)\delta_{L',L\pm 2}/12.$$

Since $V_{00}^{LL'}=0$ for $L'\neq L\pm 2$ and $S_2^0=0$, the system (311) splits into a homogeneous equation for the function U_0^0,

$$\partial_r^2+r^{-1}\partial_r-4r^{-2}+E-(\omega r)^2/4)U_0^0(r)=0, \tag{317}$$

and a system ($L=4,6,\dots$) of uncoupled equations not containing the functions U_0^0 and U_2^0:

$$24(\partial_r^2+r^{-1}\partial_r-(L+2)^2r^{-2}+E)U_L^0(r)$$

$$=(\omega r)^2\sum_{L'=L+\pm 2}S_{L'}^0U_{L'}^0(r). \tag{318}$$

For the same reason, Eq. (312) takes the form

$$24(\partial_r^2+r^{-1}\partial_r-16r^{-2}+E)U_2^0(r)$$

$$=(\omega r)^2(3U_0^0(r)+U_6^0(r)). \tag{319}$$

We shall construct the solutions of the system (317)–(319). As is well known,[8] Eq. (317) in the class of functions $\mathcal{L}^2(\mathcal{R}_+^1)$ has a countable ($p=0,1,\dots$) set of solutions, with each eigenvalue $E=E_p$ corresponding to a single eigenfunction containing the Laguerre polynomial L_p^2:

$$E=E_p=(3+2p)\omega,$$

$$U_2^0(r;E_p)=r^2L_p^2(s)\exp(-s/2),\quad s\equiv\omega r^2/2,\quad p=0,1,\dots. \tag{320}$$

For each p we take the regular solution of the system (318) to be the trivial solution

$$U_L^0(r;E_p)\equiv 0,\quad \forall r\in\mathcal{R}_+^1,\quad L=4,6,\dots. \tag{321}$$

Then each ($p=0,1,\dots$) solution (320) of (317) will correspond to its own one-parameter family (313) of solutions of (319):

$$U_2^0(r;E_p,\lambda)=\lambda J_4(t_p)+(\pi\omega^2/16)\int_0^r dss^3(Y_4(t_p)J_4(s)$$

$$-J_4(t_p)Y_4(s))U_0^0(s;E_p), \tag{322}$$

where $t_p\equiv\sqrt{E_p}r$. Equations (314)–(316) are simplified greatly by the choice (321). The series (314) degenerates into a sum of two functions:

$$U^0(r,\varphi;\lambda,E_p)=(2/\sqrt{\pi})(U_0^0(r;E_p)\sin 2\varphi$$

$$+U_2^0(r;\lambda,E_p)\sin 4\varphi).$$

The second of these generates a spurious term of the Faddeev component (315):

$$\Psi_i^{\varepsilon}(\mathbf{r};\lambda)=\frac{2}{\pi r^2}U_0^0(r;E_p)\frac{\sin(2+L)\varphi}{\sin 2\varphi}$$

$$+\frac{4}{\pi r^2}U_2^0(r;\lambda,E_p)\cos 2\varphi.$$

For the ground state ($p=0$) this term is proportional to the spurious term (307) found by Friar and Gibson. The wave function (316) of the p state,

$$\Psi^{\varepsilon}(\mathbf{r})=(2/3\pi)U_0^0(r;E_p),$$

coincides, apart from the normalization, with the corresponding wave function obtained by solving the Schrödinger equation (223) by the hyperharmonics method.[114]

Now let us use the example of (257) with a confining potential (297) and $E>\ell=0$ to explain the methods which make it possible in principle to eliminate spurious terms, and analyze how the numerical solution of this equation with the standard boundary condition leads to a choice of the amplitude λ of the inhomogeneous part of the spurious term.

The most reliable method of eliminating the spurious term is based on Theorem 3. It is realized by adding to (257) the condition that the desired function U^0 be orthogonal to the function $\tilde{W}_{200}$ describing the hyperangular dependence of the spurious term:

$$\int_0^{\pi/2}d\varphi\sin 4\varphi U^0(r,\varphi;\lambda)=0,\quad \forall r\in\mathcal{R}_+^1. \tag{323}$$

This condition is easily incorporated into the known algorithms (Refs. 59, 60, 88, 92, and 93) for numerically solving Eq. (257).

The spurious term is a function which is not integrable over the variable r. Its inhomogeneous part can therefore be eliminated by replacing the condition (323) by the alternative condition

$$\int_0^{\infty}dr(U^0(r,\varphi;\lambda))^2\equiv 0,\quad \forall\varphi\in[0,\pi/2], \tag{324}$$

generating the equation $\lambda=0$. However, this condition, which restricts the class of desired solutions U^0 to the class of $\mathcal{L}^2(\mathcal{R}_+^2)$ functions, is not easily incorporated into the traditional algorithms for numerically solving the Faddeev equations. Such equations for the problem of a three-particle bound state are usually solved for values of the variable r restricted ($r\leqslant R$) by some sufficiently large but finite parameter $R<\infty$, and the condition (324) is replaced by the requirement

$$U^0(r,\varphi;\lambda)\equiv 0,\quad r\geqslant R,\quad \varphi\in[0,\pi/2]. \tag{325}$$

Clearly, for any finite R the solution U^0 of (257) with the boundary condition (325) is square-integrable over the entire region $\mathcal{R}_+^1$. Nevertheless, the calculated function U^0, and thus also its associated Faddeev component (315), contains spurious terms. We shall prove this statement by studying the two possible cases (A and B) where the number $q\equiv\sqrt{E}R$ does or does not coincide with some root of the function J_4:

$$J_4(q)\neq 0\ \text{(A)};\quad J_4(q)=0\ \text{(B)}. \tag{326}$$

Owing to the linear independence of the basis functions (308), the series (309) for the function U^0 vanishes for $r=R$ if and only if all its components U_L^0 are zero at this point. Therefore, the condition (325) generates the equation $U_2^0(R;\lambda)=0$.

In case A, this equation is consistent with the representation (313) only for one, in general nonzero, value of the parameter λ:

$$\lambda=\lambda(R,E)$$

$$=-\frac{\pi}{2J_4(q)}\int_0^R dss(Y_4(t)J_4(s)-J_4(t)Y_4(s))R_2^0(s/\sqrt{E}). \tag{327}$$

Therefore, for all $r<R$ the calculated function U^0 contains a uniquely determined spurious term $\lambda(R,E)J_4(\sqrt{E}r)\sin 4\varphi$. Moreover, if the number q tends to a root of the Bessel function J_4, the function $\lambda(R,E)$ grows without bound, thus generating an instability in the calculated values of the function U^0.

In case B, the first term λJ_4 of the sum (313), proportional to the denominator in (327), vanishes at $r=R$ independently of the value of the parameter λ. Owing to the condition $U_2^0(R;\lambda)=0$ at this point, the second (integral) term of this sum, proportional to the numerator in (327), also vanishes. The fraction (327) therefore becomes meaningless, the parameter λ can take any value, and the function U^0 for all $r\leqslant R$ contains an entire one-parameter family of spurious terms of the type $\lambda J_4(\sqrt{E}r)\sin 4\varphi$. This case (B) is not realized in practice, owing to round-off errors arising, for example, in the calculation of the zeros of the function J_4.

We conclude by formulating the alternative that we have proved.

Theorem 4. *The boundary condition (325) with selected $R<\infty$ corresponds either to a single spurious term with coefficient λ defined by (327), or an entire one-parameter family of spurious terms. The first case is realized both in theory and in practice for the condition (326) (case A), and the second is theoretically possible for the condition (326) (case B).*

Centrifugal interactions. Let us consider a system of three identical bosons with S-wave interactions (298). We shall study the Faddeev equation (257) with some ℓ by the variable-separation method.[8] We assume that p^2 is the separation constant for the variables r and φ, and that the desired solution has the form

$$U^{\ell}(r,\varphi)=Z_p(\sqrt{E}r)F_p^{\ell}(\varphi). \tag{328}$$

Substituting this ansatz into (257), we derive two uncoupled equations: the Bessel equation (211) with index $\nu=p$ for the function $Z_p(\sqrt{E}r)$ and an integro-differential equation for the function $F_p^{\ell}(\varphi)$:

$$\cos^2\varphi(p^2-\tilde{L}_{\ell 0}^2)F_p^{\ell}(\varphi)=\alpha\langle\varphi|S^{\ell}|F_p^{\ell}\rangle. \tag{329}$$

Let us explain the conditions under which the finite linear combination

$$F_p^\ell(\varphi)=F_{pt}^\ell(\varphi)\equiv\sum_{L=\ell}^{t}B_L^\ell\tilde{W}_{L\ell 0}(\varphi),\quad t<\infty, \tag{330}$$

with as yet unknown constants p and B_L^ℓ, satisfies this equation.

Using the recursion relations for the Jacobi polynomials with subscripts n and $n\pm1$, we prove the key expansion

$$\cos^2\varphi\tilde{W}_{L\ell 0}(\varphi)=\sum_{s=0,\pm1}C_{Ls}^\ell\tilde{W}_{L+2s,\ell 0}(\varphi), \tag{331}$$

in which $C_{Ls}^\ell=0$ for $L+2s<\ell$, and otherwise

$$2C_{L's}^\ell\equiv\delta_{s0}\left(1-\frac{\ell(\ell+1)}{(L+1)(L+3)}\right)+\left(\frac{n(2n+1)(n+\ell+1)(L+\ell+1)}{L(L+2)(L+1)^2}\right)^{1/2},$$

$$s=0,\pm1;\quad L'\equiv L-2\delta_{s1};\quad L=2n+1. \tag{332}$$

Let us substitute the desired function (330) into (329). Using (263), (264), and (331), we derive the relation $p^2=(t+2)^2$ determining the separation constant and a system of linear equations for the desired coefficients B_L^ℓ. For the column $\mathbf{B}^\ell\equiv(B_\ell^\ell,B_{\ell+2}^\ell,\ldots,B_t^\ell)^T$ composed of these coefficients, this system can be written as a homogeneous matrix equation

$$\mathbf{A}^\ell(\alpha)\mathbf{B}^\ell=0 \tag{333}$$

or as a generalized spectral matrix problem

$$\mathbf{A}^\ell(0)\mathbf{B}^\ell=\alpha\mathbf{S}^\ell\mathbf{B}^\ell. \tag{334}$$

The nonzero elements of the diagonal matrix

$$\operatorname{diag}\mathbf{S}^\ell=(S_\ell^\ell,S_{\ell+2}^\ell,\ldots,S_t^\ell)$$

are given by (265), and the elements of the principal ($s=0$), upper ($s=+1$), and lower ($s=-1$) diagonals of the tridiagonal matrix

$$\mathbf{A}^\ell(\alpha)=\mathbf{A}^\ell(0)-\alpha\mathbf{S}^\ell$$

can be expressed in terms of the coefficients (265) and (332) as

$$A_{Ls}^\ell\equiv(p^2-(L+2s+2)^2)C_{L+2s,-s}^\ell-\alpha S_L^\ell\delta_{s0},\quad L=\ell,\ell+2,\ldots,t. \tag{335}$$

Accordingly, the matrix element A_{L0}^ℓ is independent of α if and only if $S_{L'}^\ell=0$. In this special case, according to (264) and (265) the function $W_{L'\ell 0}$ is mapped by the operator S^ℓ identically to zero, and Eqs. (333) become identities if $t=L'$ and only one element of the column $\mathbf{B}$, namely, $B_{L'}^\ell$, is nonzero. According to the ansatz (328), this column corresponds to a nontrivial solution of (257) which is independent of the parameter α:

$$U^\ell(r,\varphi)=B_{L'}^\ell J_{L'+2}(\sqrt{E}r)\tilde{W}_{L'\ell 0}(\varphi).$$

If this spurious solution is substituted into (260), the trivial wave function $\Psi^\varepsilon\equiv0$ is obtained. As was noted in Sec. 4.1, for this three-particle system such a situation is realized only for $L,\ell=1$ or $L=2$, $\ell=0$. In all other cases, when the problem (333) has a nontrivial solution, the Faddeev component defined by (328) and (330) corresponds to the three-particle wave function (260):

$$\Psi^\varepsilon(\mathbf{r})=r^{-2}Z_p(\sqrt{E}r)\sum_{L=\ell}^{t}S_L^\ell B_L^\ell Y_{L\ell 0}^{\ell m}(\Omega). \tag{336}$$

According to the Fredholm alternative,[2] a nontrivial solution of (333) having the form of a column $\mathbf{B}$ with two or more nonzero elements exists if and only if the parameter α is a zero of the determinant of the matrix $A(\alpha)$. It follows from (335) that for any ℓ and t the last element A_{t0}^ℓ of the main diagonal of the matrix $\mathbf{A}$ is $-\alpha S_t^\ell$, and in the special cases $L,\ell=1$ and $L=2,\ell=0$ we have $\partial_\alpha A_{L0}^\ell=0$. Therefore, $\alpha=0$ is a root of the polynomial $\det\mathbf{A}(\alpha)$, and the number of its remaining real zeros must be no greater than

$$N_t^\ell\equiv(t-\ell)/2-\delta_{\ell 0}-\delta_{\ell 1}. \tag{337}$$

If the matrix $\mathbf{A}$ is diagonally dominant, it is nondegenerate.[2] Otherwise, the zeros of its determinant satisfy at least one ($L=\ell,\ell+2,\ldots,t$) of the inequalities

$$|A_{L0}^\ell(0)-\alpha S_L^\ell|<\sum_{s=\pm1}A_{Ls}^\ell(0). \tag{338}$$

We shall give some exact solutions of the problem (333).

Let $\ell=0$. Then for $t=0$ we have only the trivial solution $\mathbf{B}^0=(0)^T$, and for $t=2$ and any α there is the particular solution $B=(0,1)^T$. If $t=4$, the only solution is

$$\alpha=4,\quad \mathbf{B}=(-5,4,5)^T.$$

In the case $t=6$ there are two solutions:

$$\alpha_\pm=9\pm\sqrt{11},$$

$$\mathbf{B}_\pm^0=\left(4,\alpha_\pm-10,\frac{60}{21-\alpha_\pm},\frac{280}{\alpha_\pm(21-\alpha_\pm)}\right)^T.$$

Let $\ell=1$. Then for $t=1$ and any α there is the particular solution $\mathbf{B}^1=(1)^T$, for $t=3$ there is no solution, and for $t=5$ there is one solution

$$\alpha=79/6,\quad \mathbf{B}^1=(79\sqrt{3},-395,120\sqrt{2}/S_5^1)^T.$$

If $\ell>1$ and $t=\ell$ there is only the trivial solution, and for $t=\ell+2$ there is one nontrivial solution

$$\alpha=6/(1+2(-2)^\ell),$$

$$\mathbf{B}^l=(\sqrt{3(\ell+4)}S_{\ell+2}^\ell,\sqrt{2\ell+3}S_\ell^\ell)^T.$$

In other cases the spectral problem (333) or (334) is easily solved by known numerical methods.[2] For completeness, let us discuss the known results[66,109] of the numerical solution of this problem in the case $\ell=0$. Using (332) and (335), we can verify that (338) is not satisfied for any $\alpha\leq0$. This means that the eigenvalues α of the problem (334) with $\ell=0$ are non-negative numbers. The minimum one is $9-\sqrt{11}$, and some others are given in a table in Ref. 66. These data supplement the table of Ref. 109, where Eq. (329) was analyzed numerically in the case $\ell=0$ and $-\alpha=1,2,\ldots,6$. We shall prove that the functions $F_p^0(\varphi)$ corresponding to such negative values of the parameter α cannot be represented in the form (330). Let us assume the contrary:

for $\alpha<\ell=0$ suppose that there is at least one solution of (333) of the type (330). Then the spectral problem (333) has a nontrivial solution **B** corresponding to a selected negative eigenvalue $\alpha<0$. As explained earlier, there are no such eigenvalues. This contradiction proves that all the functions $F_p^0(\varphi)$ calculated in Ref. 109 are *infinite* series in the functions $\tilde{W}_{L\ell 0}$.

We conclude by stating the criterion that we have proved.

Theorem 5. *The Schrödinger equation for a system of three identical bosons with S- wave potentials (298) has exact solutions (336) if and only if the parameter α and the corresponding column $\mathbf{B}^{\ell}(\alpha)$ satisfy the spectral problem (334). For any fixed indices ℓ and t its eigenvalues satisfy the inequality (338), and their number does not exceed the number N_t^{ℓ} given by (337).*

5. CONCLUSION

The main conclusions of this review are briefly stated in the form of Theorems 1–5, or as the conclusions at the end of each subsection. Therefore, here we shall only list some interesting but unsolved problems. The generalization of Theorems 1–5 to the case of three particles with discrete (spin–isospin) degrees of freedom does not lead to any fundamental difficulties and can be obtained after isolating the spin–isospin variables by the methods described above. For the further development of exact methods of solving the three-particle problem with central interactions it is important to analyze the kinematical transformation of the angular basis of Wigner D functions and to study the structure of spurious and physical solutions of the Faddeev equations written in this basis. Just as interesting and important is the problem of constructing the complete and explicit asymptotic expansions of the wave functions near triple and pair collision points. A comparative analysis of the known approaches to solving this problem could not be made in this review and will be presented in a separate study.

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