PHYSICS OF PARTICLES AND NUCLEI VOLUME 30, NUMBER 5 SEPTEMBER–OCTOBER 1999

Spaces with contravariant and covariant affine connections and metrics

Sawa Manoff*
Bogoliubov Laboratory of Theoretical Physics, Joint Institute for Nuclear Research, 141980 Dubna, Russia

Fiz. Élem. Chastits At. Yadra **30**, 1211–1269 (September–October 1999)

The theory of spaces with contravariant and covariant affine connections, whose components differ not only in sign, and metrics [$(\bar{L}_n, g)$ spaces] is worked out within the framework of tensor analysis over differentiable manifolds and in a volume necessary for further consideration of the kinematics of vector fields and the Lagrangian theory of tensor fields over $(\bar{L}_n, g)$ spaces. The possibility of introducing affine connections for contravariant and covariant tensor fields, whose components differ not only in sign, over differentiable manifolds with finite dimensions is discussed. The action of the deviation operator, having an important role for deviation equations in gravitational physics, is considered for the case of contravariant and covariant vector fields over differentiable manifolds with different affine connections (called $\bar{L}_n$ spaces). A deviation identity for contravariant vector fields is obtained. The notions of covariant, contravariant, covariant projective, and contravariant projective metric are introduced in $(\bar{L}_n, g)$ spaces. The action of the covariant and the Lie differential operator on the different types of metric is found. The concepts of symmetric covariant and contravariant (Riemannian) connection are defined and presented by means of the covariant and contravariant metric and the corresponding torsion tensors. The different types of relative tensor fields (tensor densities) as well as the invariant differential operators acting on them are considered. The invariant volume element and its properties under the action of different differential operators are investigated. © *1999 American Institute of Physics.* [S1063-7796(99)00305-8]

1. INTRODUCTION

In the present review, differentiable manifolds with contravariant and covariant affine connections, whose components differ not only in sign, and metrics [spaces with contravariant and covariant affine connections and metrics, $(\bar{L}_n, g)$ spaces] are considered as models of space-time. On the basis of the differential-geometric structures of the $(\bar{L}_n, g)$ spaces the kinematics of vector fields and the dynamics of tensor fields are worked out as useful tools in mathematical models for the description of physical interactions and, in particular, the gravitational interaction in modern gravitational physics. The general results found for differentiable manifolds with different contravariant and covariant affine connections and metrics can be specialized for spaces with one affine connection and metrics [the so-called (L_n, g) spaces] as well as for (pseudo-)Riemannian spaces with or without torsion [the so-called U_n and V_n spaces]. Most of the results are given either in index-free form or in a coordinate or a noncoordinate basis. The main objects in such investigations can be arranged in the following scheme:

SPACES WITH CONTRAVARIANT AND COVARIANT AFFINE CONNECTIONS AND METRICS
differential operators
covariant differential operator,
Lie differential operator,
operator of curvature,
deviation operator,
extension operator,
affine connections, metrics,
special tensor fields, tensor densities,
invariant volume element.
KINEMATIC CHARACTERISTICS
OF CONTRAVARIANT TENSOR FIELDS
relative velocity (shear, rotation and expansion velocities),
relative acceleration (shear, rotation and expansion accelerations),
deviation equations,
geodesic and auto-parallel equations,
Fermi–Walker transports,
conformal transports.
LAGRANGIAN THEORY OF TENSOR FIELDS
Lagrangian density,
variational principles,
Euler–Lagrange equations,
energy–momentum tensors.

In this review we consider only the elements of the first part of the above scheme related to spaces with contravariant and covariant affine connections and metrics. The review is an introduction to the theory of $(\bar{L}_n, g)$ spaces. It contains formulas necessary for the development of the mechanics of tensor fields and for constructing mathematical models of different dynamical systems described by the use of the main objects under consideration. The general results found for differentiable manifolds with different (not only in sign) contravariant and covariant affine connections and metrics can be specialized for spaces with one affine connection and a metric [so-called (L_n, g) spaces] as well as for (pseudo-)Riemannian spaces with or without torsion [the so-called U_n and V_n spaces]. Most of the results are given either

1063-7796/99/30(5)/23/$15.00

in index-free form or in a coordinate or a noncoordinate basis. This has been done to help the reader to choose the right form of the results for his own further considerations. The main conclusions are summarized in the last section.

The $(\bar{L}_n, g)$ spaces have interesting properties which could be of use in theoretical physics, especially in theoretical gravitational physics. In these types of spaces the introduction of a contravariant nonsymmetric affine connection for contravariant tensor fields and the introduction of a symmetric (Riemannian, Levi-Civita connection) for covariant tensor fields is possible. On this basis we can consider flat spaces $[(\bar{M}_n, g)$ spaces$]$ with predetermined torsion for contravariant vector fields and with torsion-free connection for covariant vector fields. In an analogous way, such structures could be induced in (pseudo-)Riemannian spaces $[(\bar{V}_n, g)$ spaces$]$.

1.1. Space-time geometry and differential geometry

In the last few years new attempts have been made[1–5] to revive the ideas of Weyl[6,7] for using manifolds with independent affine connection and metric (spaces with affine connection and metric) as a model of space-time in the theory of gravitation.[3] In such spaces the connection for cotangent vector fields (as dual to the tangent vector fields) differs from the connection for the tangent vector fields only in sign. This fact is due to the definition of dual vector bases in dual vector spaces over points of a manifold, which is a trivial generalization of the definition of dual bases of algebraic dual vector spaces from multilinear algebra.[8–11] On the one hand, the whole modern differential geometry is built as a rigorous logical structure having as one of its main assumptions the canonical definition for dual bases of algebraic dual vector spaces (with equal dimensions).[12] On the other hand, the possibility of introducing a noncanonical definition for dual bases of algebraic dual vector spaces (with equal dimensions) has been pointed out by many mathematicians[13] who have not exploited this possibility for further evolution of the differential-geometric structures and its applications. The canonical definition of dual bases of dual spaces is so naturally embedded in the basis of differential geometry that there has been no need to change it.[14–17] But the recent evolution of mathematical models for describing the gravitational interaction on a classical level shows a tendency to generalizations, using spaces with affine connection and metric, which can also be generalized using the freedom of the differential-geometric preconditions. It has been proved that an affine connection, which at a point or over a curve in Riemannian spaces can vanish [a fact leading to the principle of equivalence in the Einstein theory of gravitation (ETG)], can also vanish for a special choice of the basis system in a space with affine connection and metric.[18–20] This fact shows that the equivalence principle in the ETG could be considered only as a physical interpretation of a corollary of the mathematical formalism used in this theory. Therefore, every *differentiable manifold with affine connection and metric can be used as a model for space-time in which the equivalence principle holds.* But if the manifold has two different (not only in sign) connections for tangent and cotangent vector fields, the situation changes and is worth investigating.

The basic concepts in differential geometry related to the concepts considered in this review are defined, for the most part, in textbooks and monographs on differential geometry (see, for example, Refs. 21–27).

2. ALGEBRAIC DUAL VECTOR SPACES. CONTRACTION OPERATOR

The concept of an algebraic dual vector space can be introduced in such a way[9] that the two vector spaces (the considered one and its dual vector space) are two independent (finite) vector spaces with equal dimensions.

Let X and X^* be two vector spaces with equal dimensions: $\dim X = \dim X^* = n$. Let S be an operator (mapping) such that for every pair of elements $u \in X$ and $p \in X^*$ it gives an element of the field K (R or C), i.e.,

$$S:(u,p) \to z \in K, \quad u \in X, \quad p \in X^*. \tag{1}$$

Definition. The operator (mapping) S is called a *contraction operator* if it is a bilinear symmetric mapping, i.e., if it fulfills the following conditions:

(a) $S(u,p_1+p_2) = S(u,p_1) + S(u,p_2)$, $\forall u \in X$, $\forall p_i \in X^*$, $i=1,2$.

(b) $S(u_1+u_2,p) = S(u_1,p) + S(u_2,p)$, $\forall u_i \in X$, $i=1,2$, $\forall p \in X^*$.

(c) $S(\alpha u,p) = S(u,\alpha p) = \alpha S(u,p)$, $\alpha \in K$.

(d) Nondegeneracy: if $u_1,\ldots,u_n$ are linearly independent in X and $S(u_1,p)=0,\ldots, S(u_n,p)=0$, then p is the null element in X^*. In an analogous way, if $p_1,\ldots,p_n$ are linearly independent in X^* and $S(u,p_1)=0,\ldots, S(u,p_n)=0$, then u is the null element in X.

(e) Symmetry: $S(u,p) = S(p,u)$, $\forall u \in X$, $\forall p \in X^*$.

Let $e_1,\ldots,e_n$ be an arbitrary basis in X, and let $e^1,\ldots,e^n$ be an arbitrary basis in X^*. Let $u = u^i e_i \in X$ and $p = p_k e^k \in X^*$. From properties (a)–(c) it follows that

$$S(u,p) = f^k{}_i \cdot u^i \cdot p_k, \tag{2}$$

where

$$f^k{}_i = S(e_i, e^k) = S(e^k, e_i) \in K. \tag{3}$$

Thus, the result of the action of the contraction operator S is expressed in terms of a bilinear form. The nondegeneracy property (d) means nondegeneracy of the bilinear form. The result $S(u,p)$ can be defined in different ways by giving arbitrary numbers $f^k{}_i \in K$ for which the condition $\det(f^k{}_i) \neq 0$ and, at the same time, the conditions (a)–(d) are fulfilled.

Definition. *(Mutually) algebraic dual vector spaces.* The spaces X and X^* are called (mutually) dual spaces if a contraction operator acting on them is given and they are considered together with this operator [i.e., (X, X^*, S) with $\dim X = n = \dim X^*$ defines the two (mutually) dual spaces X and X^*].

The definition for (mutually) algebraic dual spaces allows, for a given vector space X, an infinite number of vector spaces X^* (dual to X in different ways) to be constructed. In order to avoid this nonuniqueness, Efimov and Rosendorn[9] introduced the concept of equivalence between dual vector spaces [which is an additional condition for the definition of (mutually) dual spaces].

Definition. *Vector spaces equivalently dual to X.* Let X_1^* and X_2^* be two n-dimensional vector spaces dual to X. If a linear isomorphism exists between them, such that

$$S(u,p)=S(u,p'), \quad \forall u\in X, \quad \forall p\in X_1^*, \quad p'\in X_2^*, \tag{4}$$

where p' is the element of X_2^* corresponding to p of X_1^* by means of the linear isomorphism, then X_1^* and X_2^* are called vector spaces equivalently dual to X.

Proposition. All linear (vector) spaces dual to a given vector space X are equivalent to each other.

To prove this proposition, it is sufficient to show that if for X and X^* we are given an arbitrary S, then for an arbitrary basis $e_1,\dots,e_n\in X$ one can find a unique basis $e^1,\dots,e^n$ dual to it in the space X^*, i.e., $e^1,\dots,e^n\in X^*$ can be found in a unique way, so that $S(e_i,e^k)=f^k{}_i$, where $f^k{}_i\in K$ are prescribed numbers.[28] The proof is analogous to the proof given by Efimov and Rosendorn[9] for the case $S=C:C(e_k,e^i)=g^i_k$, with $g^i_k=1$ for $k=i$ and $g^i_k=0$ for $k\neq i$. $C(e_k,e^i)=g^i_k$ means that the basis vector field e^i dual to $\{e_k\}$ is orthogonal to all basis vectors e_k for which $k\neq i$. The contraction operator C corresponds to the canonical approach:

$$C(u,p)=C(p,u)=p(u)=p_i\cdot u^i. \tag{5}$$

The new definition of algebraic dual spaces actually corresponds to that in the common approach. Only the dual basis vector e^i is not orthogonal to the basis vectors $e_k:S(e_k,e^i)=f^i{}_k\neq g^i_k$. It is sufficient to note that for an arbitrary element $p\in X^*$ the corresponding linear form

$$S(u,p)=p_i\cdot u^{\bar{i}}=p_i\cdot f^i{}_k\cdot u^k=p_{\bar{i}}\cdot u^i \tag{6}$$

is given, where $p_1,\dots,p_n$ are the constant components of the given vector $p\in X^*$. The last equality can also be written in the form

$$S(u,p)=S(p,u)=p(u)=p_i\cdot u^{\bar{i}}. \tag{7}$$

Remark. The generalization of the concept of algebraic dual spaces for the case of vector fields over a differentiable manifold is a trivial one. The vector fields are considered as sections of vector bundles over a manifold. The vector bases become dependent on the points of the manifold, and the numbers $f^i{}_j$ are considered as functions over the manifold.

Remark. If the basis vectors in the tangent space $T_x(M)$ at a point x of a manifold M ($\dim M=n$) are the coordinate vector fields ∂_i and in the dual vector space (the cotangent space) $T_x^*(M)$ the basis $\{dx^k\}$ is defined as a dual to the basis $\{\partial_i\}$, where dx^k are the differentials of the coordinates x^k of the point x in a given chart, then $S(\partial_i,dx^k)=f^k{}_i$ [$f^k{}_i\in C^r(M)$]. After multiplying the last equality by $f_k{}^l$ and taking into account the relation $f^k{}_i\cdot f_k{}^l=g^l_i$ we obtain the condition $S(\partial_i,f_k{}^l\cdot dx^k)=g^l_i$, which is equivalent to the result of the action of the contraction operator C on the vectors ∂_i and e^l, where $e^l=f^l{}_k\cdot dx^k$. The new vectors e^l are not in general coordinate differentials of the coordinates x^l at $x\in M$. They would be differentials of new coordinates $x^{l'}=x^{l'}(x^k)$ if the relation $dx^{l'}=A_k{}^{l'}\cdot dx^k$ were connected with the condition $e^l=dx^{l'}$ and $x^{l'}=\int dx^{l'}$. In an analogous way, in the case when $S(f_l{}^i\cdot\partial_i,dx^k)=g^k_l$ the new vectors $e_l=f_l{}^i\cdot\partial_i$ in general are again not coordinate vector fields $\partial_{i'}$; e_l would again be coordinate vector fields if by changing the charts (the coordinates) at a point $x\in M$ the condition $f_l{}^i=\partial x^i/\partial x^{l'}$ were fulfilled.

Thus, the definition of algebraic dual vector fields over manifolds by means of the contraction operator S as a generalization of the contraction operator C allows considerations including functions $f^i{}_j(x^k)$ instead of the Kronecker symbol g^i_j.

The contraction operator S can be easily generalized to a multilinear contraction operator S.

3. CONTRAVARIANT AND COVARIANT AFFINE CONNECTIONS. COVARIANT DIFFERENTIAL OPERATOR

3.1. Affine connection. Covariant differential operator

The concept of an affine connection can be defined in different ways, but in all definitions a linear mapping is given, which to a given vector of a vector space over a point x of a manifold M juxtaposes a corresponding vector from the same vector space at this point. The corresponding vector is identified as a vector of the vector space over another point of the manifold M. The method of identification is called a transport from one point to another point of the manifold.

Vector and tensor fields over a differentiable manifold are endowed with the structure of a linear (vector) space by defining the corresponding operations at every point of the manifold.

Definition. *Affine connection* over a differentiable manifold M. Let $V(M)$ ($\dim M=n$) be the set of all (smooth) vector fields over the manifold M. The mapping $\nabla:V(M)\times V(M)\to V(M)$, by means of $\nabla(u,w)\to\nabla_u w$, u, $w\in V(M)$, with ∇_u as a covariant differential operator along the vector field u (see the definition below), is called an affine connection over the manifold M.

Definition. A covariant differential operator along the vector field u is a linear differential operator (mapping) ∇_u with the following properties:

(a) $\nabla_u(v+w)=\nabla_u v+\nabla_u w$, $u,v,w\in V(M)$;

(b) $\nabla_u(f\cdot v)=(uf)\cdot v+f\cdot\nabla_u v$, $f\in C^r(M)$, $r\geq 1$;

(c) $\nabla_{u+v}w=\nabla_u w+\nabla_v w$;

(d) $\nabla_{fu}v=f\cdot\nabla_u v$;

(e) $\nabla_u f=uf$, $f\in C^r(M)$, $r\geq 1$;

(f) $\nabla_u(v\otimes w)=\nabla_u v\otimes w+v\otimes\nabla_u w$ (Leibniz rule), where $\otimes$ denotes the tensor product.

The result of the action of the covariant differential operator $\nabla_u v$ is often called the covariant derivative of the vector field v along the vector field u.

In a given chart (coordinate system), the determination of $\nabla_{e_\alpha}e_\beta$ in the basis $\{e_\alpha\}$ defines the components $\nabla^\alpha_{\beta\gamma}$ of the affine connection ∇:

$$\nabla_{e\alpha}e_\beta=\nabla^\gamma_{\alpha\beta}\cdot e_\gamma, \quad \alpha,\beta,\gamma=1,\dots,n. \tag{8}$$

The $\{\nabla^{\gamma}_{\alpha\beta}\}$ have the transformation properties of a linear differential geometric object.[21,29]

Definition. *Space with affine connection.* A differentiable manifold M endowed with an affine connection ∇, i.e., the pair (M,∇), is called a space with affine connection.

3.2. Contravariant and covariant affine connections

The action of the covariant differential operator on a contravariant (tangential) coordinate basis vector field ∂_i over M along another contravariant coordinate basis vector field ∂_j is determined by the affine connection $\nabla=\Gamma$ with components Γ^k_{ij} which in a given chart (coordinate system) are defined by the expression

$$\nabla_{\partial_j}\partial_i=\Gamma^k_{ij}\cdot\partial_k. \tag{9}$$

For a noncoordinate contravariant basis $e_\alpha\in T(M)$, with $T(M)=\cup_{x\in M}T_x(M)$,

$$\nabla_{e_\beta}e_\alpha=\Gamma^\gamma_{\alpha\beta}\cdot e_\gamma. \tag{10}$$

Definition. *Contravariant affine connection.* The affine connection $\nabla=\Gamma$ induced by the action of the covariant differential operator on contravariant vector fields is called a contravariant affine connection.

The action of the covariant differential operator on a covariant (dual to contravariant) basis vector field e^α [$e^\alpha\in T^*(M)$, with $T^*(M)=\cup_{x\in M}T^*_x(M)$] along a contravariant basis (noncoordinate) vector field e_β is determined by the affine connection $\nabla=P$ with components $P^\alpha_{\beta\gamma}$ defined by the expression

$$\nabla_{e_\beta}e^\alpha=P^\alpha_{\gamma\beta}\cdot e^\gamma. \tag{11}$$

For a coordinate covariant basis dx^i,

$$\nabla_{\partial_j}dx^i=P^i_{kj}\cdot dx^k. \tag{12}$$

Definition. *Covariant affine connection.* The affine connection $\nabla=P$ induced by the action of the covariant differential operator on covariant vector fields is called a covariant affine connection.

Definition. *Space with contravariant and covariant affine connections* ($\bar{L}_n$ space). A differentiable manifold with a contravariant affine connection Γ and a covariant affine connection P is called a space with contravariant and covariant affine connections.

The connection between the two connections Γ and P is based on the connection between the two dual spaces $T(M)$ and $T^*(M)$, which in turn is based on the existence of the contraction operator S. Usually, commutation relations are required between the contraction operator and the covariant differential operator, in the form

$$S\circ\nabla_u=\nabla_u\circ S. \tag{13}$$

If the last operator equality in the form $\nabla_{\partial_k}\circ S=S\circ\nabla_{\partial_k}$ is used for acting on the tensor product $dx^i\otimes\partial_j$ of two basis vector fields $dx^i\in T^*(M)$ and $\partial_j\in T(M)$, then

$$f^i{}_{j,k}=\Gamma^l_{jk}\cdot f^i{}_l+P^i_{lk}\cdot f^l{}_j,$$

$$f^i{}_{j,k}:=\partial_k f^i{}_j \text{ (in a coordinate basis).} \tag{14}$$

The last equality can be considered from two different points of view:

1. If $P^i_{jk}(x^l)$ and $\Gamma^i_{jk}(x^l)$ are given as functions of the coordinates in M, then the equality appears as a system of equations for the unknown functions $f^i{}_j(x^l)$. The solutions of these equations determine the action of the contraction operator S on the basis vector fields for given components of both connections. The integrability conditions for the equations can be written in the form

$$R^m{}_{jkl}\cdot f^i{}_m+P^i{}_{mkl}\cdot f^m{}_j=0, \tag{15}$$

where $R^m{}_{jkl}$ are the components of the contravariant curvature tensor, constructed by means of the contravariant affine connection Γ, and $P^i{}_{mkl}$ are the components of the covariant curvature tensor, constructed by means of the covariant affine connection P, where $[R(\partial_i,\partial_j)]dx^k=P^k{}_{lij}\cdot dx^l$, $[R(\partial_i,\partial_j)]\partial_k=R^l{}_{kij}\cdot\partial_l$, $R(\partial_i,\partial_j)=\nabla_{\partial_i}\nabla_{\partial_j}-\nabla_{\partial_j}\nabla_{\partial_i}$.

2. If $f^i{}_j(x^l)$ are given as functions of the coordinates in M, then the conditions for $f^i{}_j$ determine the connection between the components of the contravariant affine connection Γ and the components of the covariant affine connection P on the basis of the predetermined action of the contraction operator S on the basis vector fields.

If $S=C$, i.e., $f^i{}_j=g^i_j$, then the conditions for $f^i{}_j$ are fulfilled for every $P=-\Gamma$, i.e.,

$$P^i_{jk}=-\Gamma^i_{jk}. \tag{16}$$

This fact can be formulated as the following proposition:

Proposition. $S=C$ is a sufficient condition for $P=-\Gamma$ ($P^i_{jk}=-\Gamma^i_{jk}$).

Corollary. If $P\neq-\Gamma$, then $S\neq C$, i.e., if the covariant affine connection P differs from the contravariant affine connection Γ not only in sign, then the contraction operator S must differ from the canonical contraction operator C (if S commutes with the covariant differential operator).

This corollary allows the introduction of different (not only in sign) contravariant and covariant connections by using a contraction operator S that is different from the canonical contraction operator C.

Example. If $f^i{}_j=e^\varphi\cdot g^i_j$, where $\varphi\in C^r(M)$, $\varphi\neq0$, then $P^i_{jk}=-\Gamma^i_{jk}+\varphi,k\cdot g^i_j$.

3.3. Covariant derivatives of contravariant tensor fields

The *action* of a covariant differential operator along a contravariant vector field u is called the *transport along the contravariant vector field u* (or *transport along u*).

The result of the action of a covariant differential operator on a tensor field is called the *covariant derivative* of this tensor field.

The result ∇_uV of the action of ∇_u on a contravariant tensor field V is called the *covariant derivative of the contravariant tensor field V along the contravariant vector field u* (*or covariant derivative of V* along u).

The action of a covariant differential operator on contravariant tensor fields with rank >1 can be determined in a trivial manner on the basis of the Leibniz rule which the operator obeys. Then the action of the operator ∇_{∂_j} on a tensor basis $\partial_A = \partial_{j_1} \otimes \ldots \otimes \partial_{j_l}$ can be written in the form

$$\nabla_{\partial_j}\partial_A = \nabla_{\partial_j}[\partial_{j_1} \otimes \ldots \otimes \partial_{j_l}] = (\nabla_{\partial_j}\partial_{j_1} \otimes \partial_{j_2} \ldots \otimes \partial_{j_l}) + (\partial_{j_1} \otimes \nabla_{\partial_j}\partial_{j_2} \otimes \ldots \otimes \partial_{j_l}) + \ldots + (\partial_{j_1} \otimes \ldots \otimes \nabla_{\partial_i}\partial_{j_l})$$
$$= \Gamma^{i_1}_{j_1 j} \cdot \partial_{i_1} \otimes \ldots \otimes \partial_{j_l} + \ldots + \Gamma^{i_l}_{j_l j} \cdot \partial_{j_1} \otimes \ldots \otimes \partial_{i_l}$$
$$= \left(\sum_{k=1}^{l} g^{i}_{j_k} \cdot g^{i_k}_{m} \cdot g^{i_1}_{j_1} \cdot g^{i_2}_{j_2} \ldots g^{i_{k-1}}_{j_{k-1}} \cdot g^{i_{k+1}}_{j_{k+1}} \ldots g^{i_l}_{j_l} \cdot \right) \cdot \Gamma^{m}_{ij} \cdot (\partial_{i_1} \otimes \ldots \otimes \partial_{i_l}).$$

If we introduce the abbreviations

$$S_{Am}{}^{Bi} = -\sum_{k=1}^{l} g^{i}_{j_k} \cdot g^{i_k}_{m} \cdot g^{i_1}_{j_1} \cdot g^{i_2}_{j_2} \ldots g^{i_{k-1}}_{j_{k-1}} \cdot g^{i_k}_{j_{k+1}} \ldots g^{i_l}_{j_l}, \quad (17)$$

$$\Gamma^{B}_{A_j} = -S_{Am}{}^{Bi} \cdot \Gamma^{m}_{ij}, \quad A = j_1 \ldots j_l, \quad B = i_1 \ldots i_l, \quad (18)$$

then $\nabla_{\partial_j}\partial_A$ can be written in the form

$$\nabla_{\partial_j}\partial_A = \Gamma^{B}_{A_j} \cdot \partial_B = -S_{Am}{}^{Bi} \cdot \Gamma^{m}_{ij} \cdot \partial_B. \quad (19)$$

The quantities $S_{Am}{}^{Bi}$ obey the following relations:
(a) $S_{Bi}{}^{Aj} \cdot S_{Ak}{}^{Cl} = -g^{l}_{i} \cdot S_{Bk}{}^{Cj}$, $\dim M = n$, $l = 1, \ldots, N$;
(b) $S_{Bi}{}^{Bj} = -N \cdot n^{N-1} \cdot g^{j}_{i}$;
(c) $S_{Bi}{}^{Ai} = -N \cdot g^{A}_{B}$.

Here

$$g^{A}_{B} = g^{j_1}_{i_1} \ldots g^{j_{m-1}}_{i_{m-1}} \cdot g^{j_m}_{i_m} \cdot g^{j_{m+1}}_{i_{m+1}} \ldots g^{j_l}_{i_l} \quad (20)$$

is defined as the *multi-Kronecker symbol* of rank l:

$$g^{A}_{B} = 1, \quad i_k = j_k \text{ (for all } k \text{ simultaneously)}$$
$$= 0, \quad i_k \neq j_k, \quad k = 1, \ldots, l. \quad (21)$$

The covariant derivative along a contravariant vector field u of a contravariant tensor field $V = V^A \cdot \partial_A$ can be written in a coordinate basis as

$$\nabla_u V = (V^A{}_{,i} + \Gamma^{A}_{Bi} \cdot V^B) \cdot u^i \cdot \partial_A = V^A{}_{;i} \cdot u^i \cdot \partial_A, \quad (22)$$

where

$$V^A{}_{;i} = V^A{}_{,i} + \Gamma^{A}_{Bi} \cdot V^B \quad (23)$$

is called the first covariant derivative of the components V^A of the contravariant tensor field V along the contravariant coordinate basis vector field ∂_i:

$$\nabla_{\partial_i} V = V^A{}_{;i} \cdot \partial_A. \quad (24)$$

In an analogous way, for the second covariant derivative $\nabla_\xi \nabla_u V$ we find

$$\nabla_\xi \nabla_u V = (V^A{}_{;j;i} \cdot u^j + V^A{}_{;j} \cdot u^j{}_{;i}) \cdot \xi^i \cdot \partial_A = (V^A{}_{;j} \cdot u^j)_{;i} \cdot \xi^i \cdot \partial_A,$$

where

$$V^A{}_{;j;i} = (V^A{}_{;j})_{,i} + \Gamma^{A}_{Bi} \cdot V^B{}_{;j} - \Gamma^{k}_{ji} \cdot V^A{}_{;k} \quad (25)$$

is the second covariant derivative of the components V^A of the contravariant vector field V. Here

$$\nabla_\xi \nabla_u V - \nabla_u \nabla_\xi V = [(V^A{}_{;i;j} - V^A{}_{;j;i}) \cdot u^i \cdot \xi^j + V^A{}_{;j} \cdot (u^j{}_{;i} \cdot \xi^i - \xi^j{}_{;i} \cdot u^i)] \cdot \partial_A. \quad (26)$$

3.4. Covariant derivatives of covariant tensor fields

In an analogous way, the covariant derivative of a covariant vector field can be written in the form

$$\nabla_u p = (p_{i,j} + P^{k}_{ij} \cdot p_k) \cdot u^i \cdot dx^i = p_{i;j} \cdot u^j dx^i,$$
$$p \in T^*(M) \text{ (in a coordinate basis)}. \quad (27)$$

The action of the covariant differential operator on covariant tensor fields with rank >1 is generalized in a trivial manner on the basis of the Leibniz rule, which holds for this operator. Then the action of the operator ∇_{∂_j} on the basis $dx^A = dx^{j_1} \otimes \ldots \otimes dx^{j_l}$ can be written in the form

$$\nabla_{\partial_j} dx^B = P^{B}_{Aj} \cdot dx^A = -S_{Am}{}^{Bi} \cdot P^{m}_{ij} \cdot dx^A, \quad (28)$$

where $P^{B}_{Aj} = -S_{Am}{}^{Bi} \cdot P^{m}_{ij}$.

The covariant derivative of a covariant tensor field $W = W_A \cdot dx^A = W_B \cdot e^B$ can be written in the form

$$\nabla_u W = (W_{A,j} + P^{B}_{Aj} \cdot W_B) \cdot u^j \cdot dx^A = W_{A;j} \cdot u^j \cdot dx^A \text{ (in a coordinate basis)}. \quad (29)$$

The form of the covariant derivative of a mixed tensor field follows from the form of the derivative of contravariant and covariant basis tensor fields and the Leibniz rule:

$$\nabla_u K = \nabla_u (K^A{}_B \cdot \partial_A \otimes dx^B) = K^A{}_{B;j} \cdot u^j \cdot \partial_A \otimes dx^B$$
$$= (K^A{}_{B,j} + \Gamma^{A}_{Cj} \cdot K^C{}_B + P^{D}_{Bj} \cdot K^A{}_D) \cdot u^j \cdot \partial_A \otimes dx^B \text{ (in a coordinate basis)}. \quad (30)$$

If the *Kronecker tensor* is defined in the form

$$\mathrm{Kr} = g^{i}_{j} \cdot \partial_i \otimes dx^j = g^{\alpha}_{\beta} \cdot e_\alpha \otimes e^\beta, \quad (31)$$

then the components of the contravariant and the covariant affine connection differ from each other by the components of the covariant derivative of the Kronecker tensor, i.e.,

$$\Gamma^{i}_{jk} + P^{i}_{jk} = g^{i}_{j;k}, \quad \Gamma^{\alpha}_{\beta\gamma} + P^{\alpha}_{\beta\gamma} = g^{\alpha}_{\beta/\gamma}. \quad (32)$$

Remark. In the special case when $S = C$ and in the canonical approach, $g^{i}_{j;k} = 0$ $(g^{\alpha}_{\beta/\gamma} = 0)$.

4. LIE DIFFERENTIAL OPERATOR

The Lie differential operator $\mathcal{L}_\xi$ along the contravariant vector field ξ is another operator which can be constructed by means of a contravariant vector field. Its definition can be considered as a generalization of the concept of the Lie derivative of tensor fields (Refs. 13, 21, 30, and 31).

Definition. $\mathcal{L}_\xi$:= *Lie differential operator along the contravariant vector field* ξ with the following properties:
(a) $\mathcal{L}_\xi$: $V \to \bar{V} = \mathcal{L}_\xi V$, $V, \bar{V} \in \otimes^l(M)$;
(b) $\mathcal{L}_\xi$: $W \to \bar{W} = \mathcal{L}_\xi W$, $W, \bar{W} \in \otimes_k(M)$;
(c) $\mathcal{L}_\xi$: $K \to \bar{K} = \mathcal{L}_\xi K$, $K, \bar{K} \in \otimes^l{}_k(M)$;
(d) linear operator with respect to tensor fields, $\mathcal{L}_\xi(\alpha \cdot V_1 + \beta \cdot V_2) = \alpha \cdot \mathcal{L}_\xi V_1 + \beta \cdot \mathcal{L}_\xi V_2$, $\alpha, \beta \in F$ (R or C), $V_i \in \otimes^l(M)$, $i = 1, 2$,

$\mathcal{L}_\xi(\alpha\cdot W_1+\beta\cdot W_2)=\alpha\cdot\mathcal{L}_\xi W_1+\beta\cdot\mathcal{L}_\xi W_2$, $W_i\in\otimes_k(M)$, $i=1,2$,

$\mathcal{L}_\xi(\alpha\cdot K_1+\beta\cdot K_2)=\alpha\cdot\mathcal{L}_\xi K_1+\beta\cdot\mathcal{L}_\xi K_2$, $K_i\in\otimes^l{}_k(M)$, $i=1,2$;

(e) linear operator with respect to the contravariant field ξ,

$\mathcal{L}_{\alpha\cdot\xi+\beta\cdot u}=\alpha\cdot\mathcal{L}_\xi+\beta\cdot\mathcal{L}_u$, $\alpha,\beta\in F$ (R or C), ξ, $u\in T(M)$;

(f) differential operator obeying the Leibniz rule,

$\mathcal{L}_\xi(S\otimes U)=\mathcal{L}_\xi S\otimes U+S\otimes\mathcal{L}_\xi U$, $S\in\otimes^m{}_q(M)$, $U\in\otimes^k{}_l(M)$;

(g) action on a function $f\in C^r(M)$, $r\geqslant 1$, $\mathcal{L}_\xi f=\xi f$, $\xi\in T(M)$;

(h) action on a contravariant vector field,

$\mathcal{L}_\xi u=[\xi,u]$, $\xi,u\in T(M)$, $[\xi,u]=\xi\circ u-u\circ\xi$,

$\mathcal{L}_\xi e_\alpha=[\xi,e_\alpha]=-(e_\alpha\xi^\beta-\xi^\gamma\cdot C^\beta_{\gamma\alpha})\cdot e_\beta$,

$\mathcal{L}_{e_\alpha}e_\beta=[e_\alpha,e_\beta]=C_{\alpha\beta}{}^\gamma\cdot e_\gamma$, $C_{\alpha\beta}{}^\gamma\in C^r(M)$,

$\mathcal{L}_\xi\partial_i=-\xi^j{}_{,i}\cdot\partial_j$, $\mathcal{L}_{\partial_i}\partial_j=[\partial_i,\partial_j]=0$;

(i) action on a covariant basis vector field,

$\mathcal{L}_\xi e^\alpha=k^\alpha{}_\beta(\xi)\cdot e^\beta$, $\mathcal{L}_{e_\gamma}e^\alpha=k^\alpha{}_{\beta\gamma}\cdot e^\beta$,

$\mathcal{L}_\xi dx^i=k^i{}_j(\xi)\cdot dx^j$, $\mathcal{L}_{\partial_k}dx^i=k^i{}_{jk}\cdot dx^j$.

The action of the Lie differential operator on a covariant basis vector field is determined by its action on a contravariant basis vector field and the commutation relations between the Lie differential operator and the contraction operator S.

4.1. Lie derivatives of contravariant tensor fields

The Lie differential operator $\mathcal{L}_\xi$ along a contravariant vector field ξ is another operator which can be constructed by means of a contravariant vector field. It is an operator mapping a contravariant tensor field V into a contravariant tensor field $\tilde{V}=\mathcal{L}_\xi V$.

The *action of the Lie differential operator* along a contravariant vector field ξ is called *dragging-along of the contravariant vector field* ξ (or *dragging-along of* ξ).

The *result of the action* ($\mathcal{L}_\xi V$) *of the Lie differential operator* $\mathcal{L}_\xi$ on V is called the Lie *derivative of the contravariant tensor field V along the contravariant vector field* ξ (or the *Lie derivative of V along* ξ).

The commutator of two Lie differential operators

$$[\mathcal{L}_\xi,\mathcal{L}_u]=\mathcal{L}_\xi\circ\mathcal{L}_u-\mathcal{L}_u\circ\mathcal{L}_\xi \tag{33}$$

has the following properties:

(a) Action on a function,

$$[\mathcal{L}_\xi,\mathcal{L}_u]f=(\mathcal{L}_\xi\circ\mathcal{L}_u-\mathcal{L}_u\circ\mathcal{L}_\xi)f=[\xi,u]f$$
$$=(\mathcal{L}_\xi u)f=[\nabla_\xi,\nabla_u]f,$$

$f\in C^r(M)$, $r\geqslant 2$.

(b) Action on a contravariant vector field,

$$[\mathcal{L}_\xi,\mathcal{L}_u]v=(\mathcal{L}_\xi\circ\mathcal{L}_u-\mathcal{L}_u\circ\mathcal{L}_\xi)v=\mathcal{L}_\xi\mathcal{L}_u v-\mathcal{L}_u\mathcal{L}_\xi v$$
$$=\mathcal{L}_v\mathcal{L}_u\xi=-\mathcal{L}_{\mathcal{L}_u\xi}v=\mathcal{L}_{\mathcal{L}_\xi u}v.$$

(c) The Jacobi identity,

$$\langle[\mathcal{L}_\xi,[\mathcal{L}_u,\mathcal{L}_v]]\rangle\equiv[\mathcal{L}_\xi,[\mathcal{L}_u,\mathcal{L}_v]]+[\mathcal{L}_v,[\mathcal{L}_\xi,\mathcal{L}_u]]+[\mathcal{L}_u,[\mathcal{L}_v,\mathcal{L}_\xi]]\equiv 0. \tag{34}$$

The Lie derivative of a contravariant vector field,

$$\mathcal{L}_\xi u=[\xi,u]=(\mathcal{L}_\xi u^i)\cdot\partial_i=(\xi^k\cdot u^i{}_{,k}-u^k\cdot\xi^i{}_{,k})\cdot\partial_i, \tag{35}$$

where

$$\mathcal{L}_\xi u^i=\xi^k\cdot u^i{}_{,k}-u^k\cdot\xi^i{}_{,k} \tag{36}$$

is called the *Lie derivative of the components* u^i *of the vector field u along the contravariant vector field* ξ (or the *Lie derivative of the components* u^i *along* ξ) *in a coordinate basis.*

In a noncoordinate basis the Lie derivative can be written in a manner analogous to what was done in a coordinate basis:

$$\mathcal{L}_\xi u=[\xi,u]=(\mathcal{L}_\xi u^\alpha)\cdot e_\alpha$$
$$=(\xi^\beta\cdot e_\beta u^\alpha-u^\beta\cdot e_\beta\xi^\alpha+C_{\beta\gamma}{}^\alpha\cdot\xi^\beta\cdot u^\gamma)\cdot e_\alpha, \tag{37}$$

where

$$\mathcal{L}_\xi u^\alpha=\xi^\beta\cdot e_\beta u^\alpha-u^\beta\cdot e_\beta\xi^\alpha+C_{\beta\gamma}{}^\alpha\cdot\xi^\beta\cdot u^\gamma \tag{38}$$

is called the *Lie derivative of the components* u^α *of the contravariant vector field u along the contravariant vector field* ξ *in a noncoordinate basis* (or the *Lie derivative of the components* u^α *along* ξ). $\mathcal{L}_{e_\beta}u$ can be written in the form

$$\mathcal{L}_{e_\beta}u=(e_\beta u^\alpha-C_{\gamma\beta}{}^\alpha\cdot u^\gamma)\cdot e_\alpha$$
$$=u^\alpha{}_{//\beta}\cdot e_\alpha=-\mathcal{L}_u e_\beta=(\mathcal{L}_{e_\beta}u^\alpha)\cdot e_\alpha, \tag{39}$$

where

$$\mathcal{L}_{e_\beta}u^\alpha=u^\alpha{}_{//\beta}=e_\beta u^\alpha-C_{\gamma\beta}{}^\alpha\cdot u^\gamma. \tag{40}$$

The second Lie derivative $\mathcal{L}_\xi\mathcal{L}_u v$ in a noncoordinate basis will have the form

$$\mathcal{L}_\xi\mathcal{L}_u v=[\xi^\beta\cdot e_\beta(\mathcal{L}_u v^\alpha)-(\mathcal{L}_u v^\beta)\cdot e_\beta\xi^\alpha-C^\alpha_{\gamma\beta}$$
$$\cdot(\mathcal{L}_u v^\gamma)\cdot\xi^\beta]\cdot e_\alpha=(\mathcal{L}_\xi\mathcal{L}_u v^\alpha)\cdot e_\alpha, \tag{41}$$

where $\mathcal{L}_\xi\mathcal{L}_u v^\alpha=\xi^\beta\cdot e_\beta(\mathcal{L}_u v^\alpha)-(\mathcal{L}_u v^\beta)\cdot e_\beta\xi^\alpha-C_{\gamma\beta}{}^\alpha\cdot(\mathcal{L}_u v^\gamma)\cdot\xi^\beta$ is called the *second Lie derivative of the components* v^α *along u and* ξ *in a noncoordinate basis.*

The action of the Lie differential operator on a contravariant tensor field with rank $k>1$ can be generalized by virtue of the validity of the Leibniz rule under the action of this operator on the bases of the tensor fields.

The result of the action of the operator $\mathcal{L}_\xi$ on a basis $\partial_A=\partial_{j_1}\otimes\ldots\otimes\partial_l$ can be found by the use of the already known relation $\mathcal{L}_\xi\partial_{j_k}=-\mathcal{L}_{\partial_{j_k}}\xi=-\xi^m{}_{ij_k}\cdot\partial_m$. Then $\mathcal{L}_\xi\partial_A=S_{Am}{}^{Bn}\cdot\xi^m{}_{,n}\cdot\partial_B$, and

$$\mathcal{L}_\xi V=\mathcal{L}_\xi(V^A\cdot\partial_A)=(\mathcal{L}_\xi V^A)\cdot\partial_A$$
$$=(\xi^k\cdot V^A{}_{,k}+S_{Bk}{}^{Al}\cdot V^B\cdot\xi^k{}_{,l})\cdot\partial_A, \tag{42}$$

where $\mathcal{L}_\xi V^A=\xi^k\cdot V^A{}_{,k}+S_{Bk}{}^{Al}\cdot V^B\cdot\xi^k{}_{,l}$ is the *Lie derivative of the components* V^A *of the contravariant tensor field V along the contravariant vector field* ξ *in a coordinate basis* (or *the Lie derivative of the components* V^A *along* ξ *in a coordinate basis*).

For $\mathcal{L}_\xi\mathcal{L}_u V$ we obtain

$$\mathcal{L}_\xi\mathcal{L}_u V=(\mathcal{L}_\xi\mathcal{L}_u V^A)\cdot\partial_A$$
$$=[\xi^k(\mathcal{L}_u V^A)_{,k}+S_{Bk}{}^{Al}\cdot(\mathcal{L}_u V^B)\cdot\xi^k{}_{,l}]\cdot\partial_A, \tag{43}$$

where $\mathcal{L}_\xi \mathcal{L}_u V^A = \xi^k(\mathcal{L}_u V^A)_{,k} + S_{Bk}{}^{Al}\cdot(\mathcal{L}_u V^B)\cdot \xi^k{}_{,l}$ is called the *second Lie derivative of the components* V^A *along* u *and* ξ *in a coordinate basis.*

The result of the action of the Lie differential operator $\mathcal{L}_\xi$ on a noncoordinate basis e_A can be found in a manner analogous to what was done for a coordinate basis. Since

$$\mathcal{L}_\xi e_\beta = -\xi^\alpha{}_{//\beta}\cdot e_\alpha, \quad \xi^\alpha{}_{//\beta} = e_\beta \xi^\alpha - C^\alpha_{\gamma\beta}\cdot \xi^\gamma, \tag{44}$$

$$\mathcal{L}_\xi e_A = \mathcal{L}_\xi[e_{\alpha_1}\otimes\ldots\otimes e_{\alpha_l}] = (\mathcal{L}_\xi e_{\alpha_l}\otimes e_{\alpha_2}\ldots\otimes e_{\alpha_l})$$
$$+(e_{\alpha_1}\otimes \mathcal{L}_\xi e_{\alpha_2}\otimes\ldots\otimes e_{\alpha_l}) + \ldots + (e_{\alpha_1}\otimes\ldots\otimes \mathcal{L}_\xi e_{\alpha_l})$$
$$= S_{A\alpha}{}^{B\beta}\cdot \xi^\alpha{}_{//\beta}\cdot e_B, \tag{45}$$

$$A = \alpha_1\ldots\alpha_l, \quad B = \beta_1\ldots\beta_l, \quad e_B = e_{\beta_1}\otimes\ldots\otimes e_{\beta_l},$$

we have

$$\mathcal{L}_\xi e_A = S_{A\alpha}{}^{B\beta}\cdot \xi^\alpha{}_{//\beta}\cdot e_B \text{ and } \mathcal{L}_\xi V = \mathcal{L}_\xi(V^A\cdot e_A)$$
$$= (\mathcal{L}_\xi V^A)\cdot e_A = (\xi^\alpha\cdot e_\alpha V^A + S_{B\alpha}{}^{A\beta}\cdot V^B\cdot \xi^\alpha{}_{//\beta})\cdot e_A. \tag{46}$$

The explicit form of the expression for $S_{B\alpha}{}^{A\beta}\cdot \xi^\alpha{}_{//\beta}$ can be given as

$$S_{B\alpha}{}^{A\beta}\cdot \xi^\alpha{}_{//\beta} = S_{B\alpha}{}^{A\beta}\cdot e_\beta \xi^\alpha - S_{B\alpha}{}^{A\beta}\cdot C_{\gamma\beta}{}^\alpha\cdot \xi^\gamma, \tag{47}$$

and if we introduce the abbreviations

$$C_{B\gamma}{}^A = S_{B\alpha}{}^{A\beta}\cdot C_{\gamma\beta}{}^\alpha = -S_{B\alpha}{}^{A\beta}\cdot C_{\beta\gamma}{}^\alpha, \tag{48}$$

$$S_{B\alpha}{}^{A\beta}\cdot \xi^\alpha{}_{//\beta} = S_{B\alpha}{}^{A\beta}\cdot e_\beta \xi^\alpha - C_{B\alpha}{}^A\cdot \xi^\alpha, \tag{49}$$

then $\mathcal{L}_\xi V^A$ can be written in the form

$$\mathcal{L}_\xi V^A = \xi^\alpha\cdot e_\alpha V^A + S_{B\alpha}{}^{A\beta}\cdot V^B\cdot \xi^\alpha{}_{//\beta} = \xi^\alpha\cdot e_\alpha V^A + S_{B\alpha}{}^{A\beta}$$
$$\cdot V^B\cdot(e_\beta \xi^\alpha - C_{\gamma\beta}{}^\alpha\cdot \xi^\gamma) = \xi^\alpha\cdot(e_\alpha V^A - S_{B\beta}{}^{A\gamma}\cdot V^B$$
$$\cdot C_{\alpha\gamma}{}^\beta) + S_{B\beta}{}^{A\gamma}\cdot V^B\cdot e_\gamma \xi^\beta = \xi^\alpha\cdot V^A{}_{//\alpha} + S_{B\alpha}{}^{A\beta}$$
$$\cdot V^B\cdot e_\beta \xi^\alpha. \tag{50}$$

$\mathcal{L}_\xi V^A$ is called *the Lie derivative of the components* V^A *of the contravariant tensor field* V *along* ξ *in a noncoordinate basis.* Here

$$V^A{}_{//\alpha} = e_\alpha V^A - S_{B\beta}{}^{A\gamma}\cdot V^B\cdot C_{\alpha\gamma}{}^\beta = e_\alpha V^A - C_{B\alpha}{}^A\cdot V^B. \tag{51}$$

In a noncoordinate basis the following relations are valid:

$$\mathcal{L}_{e_\alpha} V = V^A{}_{//\alpha}\cdot e_A, \quad \mathcal{L}_{e_\alpha} e_A = -C_{A\alpha}{}^B\cdot e_B. \tag{52}$$

The quantity

$$S_{B\alpha}{}^{A\beta} = -\sum_{k=1}^{l} g^{i_1}_{j_1}\ldots g^{i_{k-1}}_{j_{k-1}}\cdot g^{i_k}_\alpha\cdot g^\beta_{j_k}\cdot g^{i_{k+1}}_{j_{k+1}}\ldots g^{i_l}_{j_l}, \tag{53}$$

where $l = 1,\ldots,N$, $B = j_1\ldots j_l$, $A = i_1\ldots i_l$, is the *multicontraction symbol with rank N.*

4.2. Connections between the covariant and the Lie differentiations

The action of the covariant differential operator and the action of the Lie differential operator on functions are identified with the action of the contravariant vector field in the construction of both operators. The contravariant vector field acts as a differential operator on functions over a differentiable manifold M:

$$\nabla_\xi f = \xi f = \mathcal{L}_\xi f = \xi^i\cdot \partial_i f = \xi^\alpha\cdot e_\alpha f, \quad f\in C^r(M), \quad \xi\in T(M).$$

If we compare the Lie derivative with the covariant derivative of a contravariant vector field in a noncoordinate (or coordinate) basis,

$$\mathcal{L}_\xi u = (\mathcal{L}_\xi u^\alpha)\cdot e_\alpha = (\xi^\beta\cdot e_\beta u^\alpha - u^\beta\cdot e_\beta \xi^\alpha$$
$$+ C_{\beta\gamma}{}^\alpha\cdot \xi^\beta\cdot u^\gamma)\cdot e_\alpha,$$
$$\nabla_\xi u = (u^\alpha{}_{/\beta}\cdot \xi^\beta)\cdot e_\alpha = (\xi^\beta\cdot e_\beta u^\alpha + \Gamma^\alpha_{\gamma\beta}\cdot u^\gamma\cdot \xi^\beta)\cdot e_\alpha, \tag{54}$$

we see that both expressions have a common term of the type $\xi u^\alpha = \xi^\beta\cdot e_\beta u^\alpha$, allowing a relation between the two derivatives.

After substituting $e_\beta u^\alpha$ and $e_\beta \xi^\alpha$ from the equalities $e_\beta u^\alpha = u^\alpha{}_{/\beta} - \Gamma^\alpha_{\gamma\beta}\cdot u^\gamma$ and $e_\beta \xi^\alpha = \xi^\alpha{}_{/\beta} - \Gamma^\alpha_{\gamma\beta}\cdot \xi^\gamma$ into the expression for $\mathcal{L}_\xi u$ we obtain

$$\mathcal{L}_\xi u^\alpha = u^\alpha{}_{/\beta}\cdot \xi^\beta - \xi^\alpha{}_{/\beta}\cdot u^\beta - T_{\beta\gamma}{}^\alpha\cdot \xi^\beta\cdot u^\gamma$$
$$= u^\alpha{}_{/\beta}\cdot \xi^\beta - (\xi^\alpha{}_{/\beta} - T_{\beta\gamma}{}^\alpha\cdot \xi^\gamma)\cdot u^\beta, \tag{55}$$

where

$$T_{\beta\gamma}{}^\alpha = \Gamma^\alpha_{\gamma\beta} - \Gamma^\alpha_{\beta\gamma} - C_{\beta\gamma}{}^\alpha = -T_{\gamma\beta}{}^\alpha, \tag{56}$$

$$\mathcal{L}_\xi u = (\mathcal{L}_\xi u^\alpha)\cdot e_\alpha = (u^\alpha{}_{/\beta}\cdot \xi^\beta - \xi^\alpha{}_{/\beta}\cdot u^\beta - T_{\beta\gamma}{}^\alpha\cdot \xi^\beta\cdot u^\gamma)\cdot e_\alpha$$
$$= \nabla_\xi u - \nabla_u \xi - T(\xi,u), \tag{57}$$

with

$$T(\xi,u) = T_{\beta\gamma}{}^\alpha\cdot \xi^\beta\cdot u^\gamma\cdot e_\alpha = -T(u,\xi),$$
$$T(e_\beta,e_\gamma) = T_{\beta\gamma}{}^\alpha\cdot e_\alpha. \tag{58}$$

The contravariant vector field $T(\xi,u)$ is called the *(contravariant) torsion vector field* (or *the torsion vector field*).

If we use the equality following from the expression for $\mathcal{L}_u v^\beta$,

$$v^\beta{}_{/\gamma}\cdot u^\gamma - u^\beta{}_{/\gamma}\cdot v^\gamma = \mathcal{L}_u v^\beta + T_{\alpha\gamma}{}^\beta\cdot u^\alpha\cdot v^\gamma, \tag{59}$$

in the expression

$$\nabla_u\nabla_v \xi - \nabla_v\nabla_u \xi = [(\xi^\alpha{}_{/\beta/\gamma} - \xi^\alpha{}_{/\gamma/\beta})\cdot v^\beta\cdot u^\gamma + \xi^\alpha{}_{/\beta}$$
$$\cdot(v^\beta{}_{/\gamma}\cdot u^\gamma - u^\beta{}_{/\gamma}\cdot v^\gamma)]\cdot e_\alpha,$$

then

$$\nabla_u\nabla_v \xi - \nabla_v\nabla_u \xi = [(\xi^\alpha{}_{/\beta/\gamma} - \xi^\alpha{}_{/\gamma/\beta})\cdot v^\beta\cdot u^\gamma + \xi^\alpha{}_{/\beta}$$
$$\cdot(\mathcal{L}_u v^\beta + T_{\gamma\delta}{}^\beta\cdot u^\gamma\cdot u^\gamma\cdot v^\delta)]\cdot e_\alpha$$
$$= [(\xi^\alpha{}_{/\beta/\gamma} - \xi^\alpha{}_{/\gamma/\beta})\cdot v^\beta\cdot u^\gamma + \xi^\alpha{}_{/\beta}$$
$$\cdot(\mathcal{L}_u v^\beta + T^\beta(u,v)]\cdot e_\alpha,$$

$$T^\beta(u,v) = T_{\delta\gamma}{}^\beta\cdot u^\delta\cdot v^\gamma,$$

$$\nabla_{T(u,v)}\xi = \xi^\alpha{}_{/\beta}\cdot T_{\gamma\delta}{}^\beta\cdot u^\gamma\cdot v^\delta\cdot e_\alpha,$$

$$\nabla_u\nabla_v\xi-\nabla_v\nabla_u\xi-\nabla_{\mathcal{L}uv}\xi$$
$$=(\xi^\alpha{}_{/\beta/\gamma}-\xi^\alpha{}_{/\gamma/\beta})\cdot v^\beta\cdot u^\gamma\cdot e_\alpha+\nabla_{T(u,v)}\xi, \quad (60)$$

or

$$\nabla_u\nabla_v\xi-\nabla_v\nabla_u\xi-\nabla_{\mathcal{L}_{uv}}\xi-\nabla_{T(u,v)}\xi,$$
$$=(\xi^\alpha{}_{/\beta/\gamma}-\xi^\alpha{}_{/\gamma/\beta})\cdot v^\beta\cdot u^\gamma\cdot e_\alpha,$$
$$\nabla_{e_\gamma}\nabla_{e_\beta}\xi-\nabla_{e_\beta}\nabla_{e_\gamma}\xi-\nabla_{\mathcal{L}_{e_\gamma}e_\beta}\xi-\nabla_{T(e_\gamma,e_\beta)}\xi$$
$$=(\xi^\alpha{}_{/\beta/\gamma}-\xi^\alpha{}_{/\gamma/\beta})\cdot e_\alpha. \quad (61)$$

In a coordinate basis the contravariant torsion vector will have the form

$$T(\xi,u)=T^i_{kl}\cdot\xi^k\cdot u^l\cdot\partial_i=(\Gamma^i_{lk}-\Gamma^i_{kl})\cdot\xi^k\cdot u^l\cdot\partial_i, \quad (62)$$

$$T_{kl}{}^i=\Gamma^i_{lk}+\Gamma^i_{kl},\quad T(\partial_k,\partial_l)=T_{kl}{}^i\cdot\partial_i. \quad (63)$$

The Lie derivative $\mathcal{L}_\xi u$ can now be written as

$$\mathcal{L}_\xi u=(\mathcal{L}_\xi u^i)\cdot\partial_i=(u^i{}_{;k}\cdot\xi^k-u^k\cdot\xi^i{}_{;k}-T_{kl}{}^i\cdot\xi^k\cdot u^l)\cdot\partial_i,$$
$$\mathcal{L}_\xi u^i=u^i{}_{;k}\cdot\xi^k-u^k\cdot\xi^i{}_{;k}-T_{kl}{}^i\cdot\xi^k\cdot u^l. \quad (64)$$

The connection between the covariant derivative and the Lie derivative of a contravariant tensor field can be found in the same way as in the case of a contravariant vector field.

4.3. Lie derivative of covariant basis vector fields

Lie derivative of covariant coordinate basis vector fields. The commutation relations between the Lie differential operator $\mathcal{L}_\xi$ and the contraction operator S in the case of basis coordinate vector fields can be written in the form

$$\mathcal{L}_\xi\circ S(dx^i\otimes\partial_j)=S\circ\mathcal{L}_\xi(dx^i\otimes\partial_j),$$
$$\mathcal{L}_\xi\circ S(e^\alpha\otimes e_\beta)=S\circ\mathcal{L}_\xi(e^\alpha\otimes e_\beta), \quad (65)$$

where

$$\mathcal{L}_\xi\circ S(dx^i\otimes\partial_j)=\xi f^i{}_j=f^i{}_{j,k}\cdot\xi^k,$$
$$S\circ\mathcal{L}_\xi(dx^i\otimes\partial_j)=S(\mathcal{L}_\xi dx^i\otimes\partial_j)+S(dx^i\otimes\mathcal{L}_\xi\partial_j). \quad (66)$$

By means of the nondegenerate inverse matrix $(f^i{}_j)^{-1}=(f_j{}^i)$ and the connections $f^i{}_k\cdot f_j{}^k=g^i_j$, $f^k{}_i\cdot f_k{}^j=g^j_i$, after multiplication of the equality for $k^i{}_l(\xi)$ by $f_m{}^j$ and summation over j, an explicit expression for $k^i{}_j(\xi)$ is obtained in the form

$$k^i{}_j(\xi)=f_j{}^l\cdot\xi^k{}_{,l}\cdot f^i{}_k+f_j{}^l\cdot f^i{}_{l,k}\cdot\xi^k. \quad (67)$$

For $\mathcal{L}_{\partial_k}dx^i=k^i{}_j(\partial_k)\cdot dx^j=k^i{}_{jk}\cdot dx^j$ we obtain the corresponding form

$$\mathcal{L}_{\partial_k}dx^i=k^i{}_{jk}\cdot dx^j=f_j{}^l\cdot f^i{}_{l,k}\cdot dx^j,$$

$$k^i{}_{jk}=f_j{}^l\cdot f^i{}_{l,k}. \quad (68)$$

On the other hand, from the commutation relations between S and the covariant differential operator ∇_ξ, the connection between the partial derivatives of $f^i{}_j$ and the components of the contravariant and covariant connections Γ and P follows in the form

$$f^i{}_{l,k}=P^i_{mk}\cdot f^m{}_l+\Gamma^m_{lk}\cdot f^i{}_m. \quad (69)$$

After substituting the last expression into the expressions for $k^i{}_j(\xi)$ and $k^i{}_{jk}$, the corresponding quantities are obtained in the form

$$k^i{}_j(\xi)=f_j{}^l\cdot\xi^k{}_{,l}\cdot f^i{}_k+(P^i_{jk}+f_j{}^l\cdot\Gamma^m_{lk}\cdot f^i{}_m)\cdot\xi^k, \quad (70)$$

$$k^i{}_j(\partial_k)=k^i{}_{jk}=P^i_{jk}+f_j{}^l\cdot\Gamma^m_{lk}\cdot f^i{}_m,$$

$$\mathcal{L}_\xi dx^i=[f_j{}^l\cdot\xi^k{}_{,l}\cdot f^i{}_k+(P^i_{jk}+f_j{}^l\cdot\Gamma^m_{lk}\cdot f^i{}_m)\cdot\xi^k]\cdot dx^j, \quad (71)$$

$$\mathcal{L}_{\partial_k}dx^i=k^i{}_{jk}\cdot dx^j=(P^i_{jk}+f_j{}^l\cdot\Gamma^m_{lk}\cdot f^i{}_m)\cdot dx^j. \quad (72)$$

If we introduce the abbreviations

$$\xi^{\bar{i}}{}_{,\underline{j}}=f^i{}_k\cdot\xi^k{}_{,l}\cdot f_j{}^l,\quad \Gamma^{\bar{i}}_{\underline{j}k}=f_j{}^l\cdot\Gamma^m_{lk}\cdot f^i{}_m, \quad (73)$$

then the Lie derivatives of covariant coordinate basis vector fields dx^i along the contravariant vector fields ξ and ∂_k can be written in the form

$$\mathcal{L}_\xi dx^i=[\xi^{\bar{i}}{}_{,\underline{j}}+(P^i_{jk}+\Gamma^{\bar{i}}_{\underline{j}k})\cdot\xi^k]\cdot dx^j,$$

$$\mathcal{L}_{\partial_k}dx^i=(P^i_{jk}+\Gamma^{\bar{i}}_{\underline{j}k})\cdot dx^j. \quad (74)$$

Lie derivative of covariant noncoordinate basis vector fields. By analogy with the case of covariant coordinate basis vector fields, the Lie derivatives of covariant noncoordinate basis vector fields can be obtained in the form

$$\mathcal{L}_\xi e^\alpha=[\xi^{\bar{\alpha}}{}_{//\underline{\beta}}+(P^\alpha_{\beta\gamma}+\Gamma^{\bar{\alpha}}_{\underline{\beta}\gamma})\cdot\xi^\gamma]\cdot e^\beta$$
$$=[e_{\underline{\beta}}\xi^{\bar{\alpha}}+(P^\alpha_{\beta\gamma}+\Gamma^{\bar{\alpha}}_{\underline{\beta}\gamma}+C_{\underline{\beta}\gamma}{}^{\bar{\alpha}})\cdot\xi^\gamma]\cdot e^\beta, \quad (75)$$

$$\mathcal{L}_{e_\gamma}e^\alpha=(P^\alpha_{\beta\gamma}+\Gamma^{\bar{\alpha}}_{\underline{\beta}\gamma}+C_{\underline{\beta}\gamma}{}^{\bar{\alpha}})\cdot e^\beta, \quad (76)$$

where

$$\xi^{\bar{\alpha}}{}_{//\underline{\beta}}=f^\alpha{}_\gamma\cdot\xi^\gamma{}_{//\delta}\cdot f_\beta{}^\delta=f^\alpha{}_\gamma\cdot(e_\delta\xi^\gamma)\cdot f_\beta{}^\delta+f^\alpha{}_\gamma\cdot C_{\delta\sigma}{}^\gamma\cdot f_\beta{}^\delta\cdot\xi^\sigma$$
$$=e_{\underline{\beta}}\xi^{\bar{\alpha}}+C_{\underline{\beta}\sigma}{}^{\bar{\alpha}}\cdot\xi^\sigma,$$

$$e_{\underline{\beta}}\xi^{\bar{\alpha}}=f^\alpha{}_\gamma\cdot(e_\delta\xi^\gamma)\cdot f_\beta{}^\delta,\quad C_{\underline{\beta}\sigma}{}^{\bar{\alpha}}=f^\alpha{}_\gamma\cdot C_{\delta\sigma}{}^\gamma\cdot f_\beta{}^\delta, \quad (77)$$

$$\Gamma^{\bar{\alpha}}_{\underline{\beta}\gamma}=f_\beta{}^\delta\cdot\Gamma^\sigma_{\delta\gamma}\cdot f^\alpha{}_\sigma.$$

4.4. Lie derivatives of covariant tensor fields

The action of the Lie differential operator on covariant vector and tensor fields is determined by its action on covariant basis vector fields and on the functions over M.

In a coordinate basis the Lie derivative of a covariant vector field p along a contravariant vector field ξ can be written in the form

$$\mathcal{L}_\xi p=\mathcal{L}_\xi(p_i\cdot dx^i)=(\mathcal{L}_\xi p_i)\cdot dx^i$$
$$=[p_{i,k}\cdot\xi^k+p_j\cdot\xi^{\bar{j}}{}_{,\underline{i}}+p_j\cdot(P^j_{ik}+\Gamma^{\bar{j}}_{\underline{i}k})\cdot\xi^k]\cdot dx^i$$
$$=[p_{i;k}\cdot\xi^k+\xi^{\bar{k}}{}_{;\underline{i}}\cdot p_k+T_{k\underline{i}}{}^{\bar{j}}\cdot p_j\cdot\xi^k]\cdot dx^i, \quad (78)$$

where

$$\xi^{\bar{j}}{}_{;\underline{i}}=f^j{}_k\cdot\xi^k{}_{;l}\cdot f_i{}^l,\quad T_{k\underline{i}}{}^{\bar{j}}=f^j{}_l\cdot T_{km}{}^l\cdot f_i{}^m,$$

$$T_{ki}{}^{j}=\Gamma^{j}_{ik}-\Gamma^{j}_{ki} \text{ (in a coordinate basis).} \tag{79}$$

In a noncoordinate basis the Lie derivative $\mathcal{L}_{\xi}p$ has the form

$$\begin{aligned}\mathcal{L}_{\xi}p&=\mathcal{L}_{\xi}(p_{\alpha}\cdot e^{\alpha})=(\mathcal{L}_{\xi}p_{\alpha})\cdot e^{\alpha}\\&=\{(e_{\gamma}p_{\alpha}+P^{\beta}_{\alpha\gamma}\cdot p_{\beta})\cdot\xi^{\gamma}+p_{\beta}\\&\quad\cdot[e_{\underline{\alpha}}\xi^{\bar{\beta}}+(\Gamma^{\bar{\beta}}_{\underline{\alpha}\gamma}+C_{\underline{\alpha}\gamma}{}^{\bar{\beta}})\cdot\xi^{\gamma}]\}\cdot e^{\alpha}\\&=(p_{\alpha/\beta}\cdot\xi^{\beta}+\xi^{\bar{\beta}}{}_{/\underline{\alpha}}\cdot p_{\beta}+T_{\gamma\underline{\alpha}}{}^{\bar{\beta}}\cdot p_{\beta}\cdot\xi^{\gamma})\cdot e^{\alpha},\end{aligned} \tag{80}$$

where

$$\xi^{\bar{\beta}}{}_{/\underline{\alpha}}=f^{\beta}{}_{\delta}\cdot\xi^{\delta}{}_{/\gamma}\cdot f_{\alpha}{}^{\gamma},\quad T_{\gamma\underline{\alpha}}{}^{\bar{\beta}}=f_{\alpha}{}^{\delta}\cdot T_{\gamma\delta}{}^{\sigma}\cdot f^{\beta}{}_{\sigma},$$

$$T_{\beta\gamma}{}^{\alpha}=\Gamma^{\alpha}_{\gamma\beta}-\Gamma^{\alpha}_{\beta\gamma}-C_{\beta\gamma}{}^{\alpha} \text{ (in a noncoordinate basis).} \tag{81}$$

The action of the Lie differential operator on covariant tensor fields is determined by its action on basis tensor fields. In a coordinate basis,

$$\begin{aligned}\mathcal{L}_{\xi}W&=\mathcal{L}_{\xi}(W_{A}\cdot dx^{A})=(\xi W_{A})\cdot dx^{A}+W_{A}\cdot\mathcal{L}_{\xi}dx^{A}\\&=(\mathcal{L}_{\xi}W_{A})\cdot dx^{A},\quad W\in\otimes_{k}(M),\end{aligned}$$

$$\mathcal{L}_{\xi}dx^{B}=-k^{m}{}_{n}(\xi)\cdot S_{Am}{}^{Bn}\cdot dx^{A},\quad \mathcal{L}_{\xi}dx^{m}=k^{m}{}_{n}(\xi)\cdot dx^{n}, \tag{82}$$

$$\mathcal{L}_{\xi}dx^{B}=[-\xi^{\bar{k}}{}_{,\underline{l}}\cdot S_{Ak}{}^{Bl}-S_{Am}{}^{Bn}\cdot(P^{m}_{nl}+\Gamma^{\bar{m}}_{\underline{nl}})\cdot\xi^{l}]\cdot dx^{A}.$$

After introducing the abbreviations

$$\bar{\Gamma}^{B}_{Ak}=-S_{Ai}{}^{Bj}\cdot\Gamma^{\bar{i}}_{\underline{j}k},\quad P^{B}_{Ak}=-S_{Ai}{}^{Bj}\cdot P^{i}_{jk}, \tag{83}$$

$\mathcal{L}_{\xi}W$ can be written in the form

$$\begin{aligned}\mathcal{L}_{\xi}W&=(\mathcal{L}_{\xi}W_{A})\cdot dx^{A}=[\xi^{k}\cdot W_{A,k}-\xi^{\bar{k}}{}_{,\underline{l}}\cdot S_{Ak}{}^{Bl}\\&\quad\cdot W_{B}+(P^{B}_{Al}+\bar{\Gamma}^{B}_{Al})\cdot W_{B}\cdot\xi^{l}]\cdot dx^{A},\end{aligned} \tag{84}$$

where

$$\begin{aligned}\mathcal{L}_{\xi}W_{A}&=\xi^{k}\cdot W_{A,k}-\xi^{\bar{k}}{}_{,\underline{l}}\cdot S_{Ak}{}^{Bl}\cdot W_{B}+(P^{B}_{Al}+\bar{\Gamma}^{B}_{Al})\cdot W_{B}\cdot\xi^{l}\\&=\xi^{k}\cdot W_{A;k}-S_{A\bar{k}}{}^{B\underline{l}}\cdot W_{B}\cdot(\xi^{k}{}_{;l}-T_{lj}{}^{k}\cdot\xi^{j})\\&=\xi^{k}\cdot W_{A;k}-S_{Ak}{}^{Bl}\cdot W_{B}\cdot(\xi^{\bar{k}}{}_{;\underline{l}}-T^{\bar{k}}_{\underline{l}j}\cdot\xi^{j}),\end{aligned} \tag{85}$$

$$\mathcal{L}_{\partial_{j}}W_{A}=W_{Aj}+(P^{B}_{Aj}+\bar{\Gamma}^{B}_{Aj})\cdot W_{B},$$

$$\mathcal{L}_{\partial_{j}}dx^{B}=-S_{Ai}{}^{Bl}\cdot(P^{i}_{lj}+\Gamma^{\bar{i}}_{\underline{l}j})\cdot dx^{A}=(P^{B}_{Aj}+\bar{\Gamma}^{B}_{Aj})\cdot dx^{A}. \tag{86}$$

The second Lie derivative of the components W_A of the covariant tensor field W can be written in the form

$$\begin{aligned}\mathcal{L}_{\xi}\mathcal{L}_{u}W_{A}&=\xi^{k}\cdot(\mathcal{L}_{u}W_{A})_{,k}-\xi_{,\underline{l}}{}^{\bar{k}}\cdot S_{Ak}{}^{Bl}\cdot\mathcal{L}_{u}W_{B}\\&\quad+(P^{B}_{Al}+\bar{\Gamma}^{B}_{Al})\cdot\xi^{l}\cdot\mathcal{L}_{u}W_{B}.\end{aligned} \tag{87}$$

In a noncoordinate basis, $\mathcal{L}_{\xi}W$ has the form

$$\mathcal{L}_{\xi}W=(\xi W_{A})\cdot e^{A}+W_{B}\cdot(\mathcal{L}_{\xi}e^{B})=(\mathcal{L}_{\xi}W_{A})\cdot e^{A}, \tag{88}$$

where

TABLE I. Relations between transport conditions and types of dragging-along.

Transport condition	Type of dragging-along and transports
$P^{\alpha}_{\beta\gamma}+\Gamma^{\bar{\alpha}}_{\underline{\beta}\gamma}+C_{\underline{\beta}\gamma}{}^{\bar{\alpha}}=\bar{F}^{\alpha}_{\beta\gamma}$, $P^{i}_{jk}+\Gamma^{\bar{i}}_{\underline{j}jk}=\bar{F}^{i}_{jk}$.	$\mathcal{L}_{e_{\gamma}}e^{\alpha}=\bar{F}^{\alpha}_{\beta\gamma}\cdot e^{\beta}$, $\mathcal{L}_{\partial_{k}}dx^{i}=\bar{F}^{i}_{jk}\cdot dx^{j}$. Transport with arbitrary dragging-along
$P^{\alpha}_{\beta\gamma}+\Gamma^{\bar{\alpha}}_{\underline{\beta}\gamma}=\bar{A}_{\gamma}\cdot g^{\alpha}_{\beta}$, $P^{i}_{jk}+\Gamma^{\bar{i}}_{\underline{j}k}=\bar{A}_{k}\cdot g^{i}_{j}$.	$\mathcal{L}_{e_{\gamma}}e^{\alpha}=\bar{A}_{\gamma}\cdot e^{\alpha}+C_{\underline{\beta}\gamma}{}^{\bar{\alpha}}\cdot e^{\beta}$, $\mathcal{L}_{\partial_{k}}dx^{i}=\bar{A}_{k}\cdot dx^{i}$. Transport with collinear dragging-along
$P^{\alpha}_{\beta\gamma}+\Gamma^{\bar{\alpha}}_{\underline{\beta}\gamma}=0$, $P^{i}_{jk}+\Gamma^{\bar{i}}_{\underline{j}k}=0$.	$\mathcal{L}_{e_{\gamma}}e^{\alpha}=C_{\beta\gamma}{}^{\bar{\alpha}}\cdot e^{\beta}$, $\mathcal{L}_{\partial_{k}}dx^{i}=0$. Transport with invariant dragging-along

$$\begin{aligned}\mathcal{L}_{\xi}W_{A}&=\xi^{\beta}\cdot e_{\beta}W_{A}-S_{A\alpha}{}^{B\beta}\cdot W_{B}\cdot e_{\underline{\beta}}\xi^{\bar{\alpha}}\\&\quad+(P^{B}_{A\gamma}+\bar{\Gamma}^{B}_{A\gamma}+\bar{C}_{A\gamma}{}^{B})\cdot W_{B}\cdot\xi^{\gamma},\end{aligned}$$

$$\bar{\Gamma}^{B}_{A_{\gamma}}=-S_{A\alpha}{}^{B\beta}\cdot\Gamma^{\bar{\alpha}}_{\underline{\beta}\gamma},\quad \bar{C}_{A\gamma}{}^{B}=-S_{A\alpha}{}^{B\beta}\cdot C_{\underline{\beta}\gamma}{}^{\bar{\alpha}},$$

$$\mathcal{L}_{e_{\beta}}e^{B}=(P^{B}_{A\beta}+\bar{\Gamma}^{B}_{A\beta}+\bar{C}_{A\beta}{}^{B})\cdot e^{A},$$

$$\mathcal{L}_{e_{\beta}}W_{A}=e_{\beta}W_{A}+(P^{B}_{A\beta}+\bar{\Gamma}^{B}_{A\beta}+\bar{C}_{A\beta}{}^{B})\cdot W_{B}. \tag{89}$$

The second Lie derivative of W_A in a noncoordinate basis has the form

$$\begin{aligned}\mathcal{L}_{\xi}\mathcal{L}_{u}W_{A}&=\xi^{\beta}\cdot e_{\beta}(\mathcal{L}_{u}W_{A})-S_{A\alpha}{}^{B\beta}\cdot(\mathcal{L}_{u}W_{B})\cdot e_{\underline{\beta}}\xi^{\bar{\alpha}}\\&\quad+(P^{B}_{A\gamma}+\bar{\Gamma}^{B}_{A\gamma}+\bar{C}^{B}_{A\gamma})\cdot\xi^{\gamma}\cdot\mathcal{L}_{u}W_{B}.\end{aligned} \tag{90}$$

The Lie derivatives of covariant basis tensor fields can be given in terms of the covariant derivatives of the components of the contravariant vector field ξ and the torsion tensor:

$$\mathcal{L}_{\xi}e^{B}=[-S_{A\alpha}{}^{B\beta}\cdot\xi^{\bar{\alpha}}{}_{/\underline{\beta}}+P^{B}_{A\gamma}\cdot\xi^{\gamma}+\bar{T}_{A\gamma}{}^{B}\cdot\xi^{\gamma}]\cdot e^{A}, \tag{91}$$

where $\bar{T}_{A\gamma}{}^{B}=S_{A\alpha}{}^{B\beta}\cdot T_{\underline{\beta}\gamma}{}^{\bar{\alpha}}=S_{A\bar{\alpha}}{}^{B\underline{\beta}}\cdot T_{\beta\gamma}{}^{\alpha}$.

$\mathcal{L}_{\xi}W_{A}$ will then have the form

$$\begin{aligned}\mathcal{L}_{\xi}W_{A}&=\xi^{\beta}\cdot W_{A/\beta}-S_{A\bar{\alpha}}{}^{B\underline{\beta}}\cdot W_{B}\cdot(\xi^{\alpha}{}_{/\beta}-T_{\beta\gamma}{}^{\alpha}\cdot\xi^{\gamma})\\&=\xi^{\beta}\cdot W_{A/\beta}-S_{A\alpha}{}^{B\beta}\cdot W_{B}\cdot(\xi^{\bar{\alpha}}{}_{/\underline{\beta}}-T_{\underline{\beta}\gamma}{}^{\bar{\alpha}}\cdot\xi^{\gamma}).\end{aligned} \tag{92}$$

The generalization of the Lie derivatives for mixed tensor fields is analogous to that for covariant derivatives of mixed tensor fields.

4.5. Classification of linear transports with respect to the connections between contravariant and covariant affine connections

By means of the Lie derivatives of covariant basis vector fields, a classification can be proposed for the connections between the components Γ^{i}_{jk} ($\Gamma^{\alpha}_{\beta\gamma}$) of the contravariant affine connection Γ and the components P^{i}_{jk} ($P^{\alpha}_{\beta\gamma}$) of the covariant affine connection P. On this basis, linear transports (induced by the covariant differential operator or by connections) and draggings-along (induced by the Lie differential operator) can be considered as connected with each other through commutation relations of both operators with the contraction operator.

The classification of the relations between the affine connections is analogous to the classification proposed by Schouten[29] and considered by Schmutzer.[32]

5. CURVATURE OPERATOR. BIANCHI IDENTITIES

5.1. Curvature operator

One of the well known operators constructed by means of the covariant and the Lie differential operators, which has been used in the differential geometry of differentiable manifolds, is the curvature operator.

Definition. *Curvature operator.* The operator

$$R(\xi,u)=\nabla_\xi\nabla_u-\nabla_u\nabla_\xi-\nabla_{\pounds_\xi u}=[\nabla_\xi,\nabla_u]-\nabla_{[\xi,u]},$$
$$\xi,u\in T(M), \tag{93}$$

is called *the curvature operator* (or the operator of the curvature).

1. Action of the curvature operator on a function of the class $C^r(M)$, $r\geqslant 2$, over a manifold M:

$$[R(\xi,u)]f=0,\quad f\in C^r(M),\quad r\geqslant 2.$$

2. $[R(\xi,u)]fv=f\cdot[R(\xi,u)]v$, $f\in C^r(M)$, $r\geqslant 2$, $v\in T(M)$.

3. Action of the curvature operator on a contravariant vector field:

$$[R(\xi,u)]v=\nabla_\xi\nabla_u v-\nabla_u\nabla_\xi v-\nabla_{\pounds_\xi u}v=[(v^\delta{}_{/\beta/\gamma}-v^\delta{}_{/\gamma/\beta})$$
$$\cdot u^\beta\cdot\xi^\gamma+v^\delta{}_{/\alpha}\cdot T_{\beta\gamma}{}^\alpha\cdot\xi^\beta\cdot u^\gamma]\cdot e_\delta$$
$$=[(v^i{}_{;j;k}-v^i{}_{;k;j})\cdot u^j\cdot\xi^k+v^i{}_{;j}\cdot T^j_{kl}\cdot\xi^k\cdot u^l]\cdot\partial_i. \tag{94}$$

Thus, we can find for $\forall\xi\in T(M)$ and $\forall u\in T(M)$ the following relation in a coordinate basis:

$$v^i{}_{;k;l}-v^i{}_{;l;k}=-v^j\cdot R^i{}_{jkl}+v^i{}_{;j}\cdot T^j{}_{kl}, \tag{95}$$

where the quantities

$$R^i{}_{jkl}=\Gamma^i_{jl,k}-\Gamma^i_{jk,l}+\Gamma^m_{jl}\cdot\Gamma^i_{mk}-\Gamma^m_{jk}\cdot\Gamma^i_{ml} \tag{96}$$

are called the *components of the (contravariant) curvature tensor (Riemannian tensor) in a coordinate basis.*

4. Action of the curvature operator on contravariant tensor fields.

For $V=V^A\cdot e_A=V^B\cdot\partial_B$, $V\in\otimes^l(M)$, and the bases e_A and ∂_B the following relations can be proved, using the properties of $S_{Ak}{}^{Bl}$ and Γ^B_{Ai}:

$$[R(\xi,u)](f\cdot V)=f\cdot[R(\xi,u)]V, \tag{97}$$

$$[R(\xi,u)]V=V^A\cdot[R(\xi,u)]e_A=V^B\cdot[R(\xi,u)]\partial_B, \tag{98}$$

$$[R(\partial_j,\partial_i)]\partial_A=R^B{}_{Aji}\cdot\partial_B=-S_{Ak}{}^{Bl}\cdot R^k{}_{lji}\cdot\partial_B, \tag{99}$$

where

$$R^B{}_{Aji}=-S_{Ak}{}^{Bl}\cdot R^k{}_{lji},\quad S_{Ak}{}^{Bl}{}_{,i}=0, \tag{100}$$

$$R^B{}_{Aji}=\Gamma^B_{Ai,j}-\Gamma^B_{Aj,i}+\Gamma^C_{Ai}\cdot\Gamma^B_{Cj}-\Gamma^C_{Aj}\cdot\Gamma^B_{Ci}, \tag{101}$$

$$[R(\xi,u)]V=-S_{Bk}{}^{Al}\cdot V^B\cdot R^k{}_{lij}\cdot\xi^i\cdot u^j\cdot\partial_A. \tag{102}$$

On the other hand, it follows from the explicit construction of $[R(\xi,u)]V$ that

$$[R(\xi,u)]V=(V^A{}_{;i;j}-V^A{}_{;j;i}+V^A{}_{;k}\cdot T^k{}_{ji}\cdot u^i\cdot\xi^j\cdot\partial_A,$$
$$V^A{}_{;i;j}-V^A{}_{;j;i}=-S_{Bk}{}^{Al}\cdot V^B\cdot R^k{}_{lji}-T^k{}_{ji}\cdot V^A{}_{;k},$$
$$[R(\partial_j,\partial_i)]-\nabla_{T(\partial_j,\partial_i)}]V=(V^A{}_{;i;j}-V^A{}_{;j;i})\cdot\partial_A. \tag{103}$$

5. The action of the curvature operator on covariant vector fields is determined by its structure and by the action of the covariant differential operator on a covariant tensor field.

In a coordinate basis,

$$[R(\xi,u)]p=(\nabla_\xi\nabla_u-\nabla_u\nabla_\xi-\nabla_{\pounds_\xi u})p=p_l\cdot P^l{}_{ikj}\cdot\xi^k\cdot u^j\cdot dx^i$$
$$=(p_{i;j;k}-p_{i;k;j}+T_{kj}{}^l\cdot p_{i;l})\cdot u^j\cdot\xi^k\cdot dx^i, \tag{104}$$

$$[R(\partial_k,\partial_l)]dx^i=P^i{}_{jkl}\cdot dx^j, \tag{105}$$

where the quantities

$$P^i{}_{jkl}=P^i_{jl,k}-P^i_{jk,l}+P^m_{jk}\cdot P^i_{ml}-P^m_{jl}\cdot P^i_{mk}=-P^i{}_{jlk} \tag{106}$$

are called the components of the *covariant curvature tensor* in a coordinate basis.

Special case: $S=C:f^i{}_j=g^i_j:P^i_{jk}+\Gamma^i_{jk}=0$:

$$P^i{}_{jkl}=-R^i{}_{jkl}. \tag{107}$$

In a noncoordinate basis,

$$[R(\xi,u)]p=p_\delta\cdot P^\delta{}_{\alpha\beta\gamma}\cdot\xi^\beta\cdot u^\gamma\cdot e^\alpha$$
$$=(p_{\alpha/\gamma/\beta}-p_{\alpha/\beta/\gamma}+T_{\beta\gamma}{}^\delta\cdot p_{\alpha/\delta})\cdot\xi^\beta\cdot u^\gamma\cdot e^\alpha, \tag{108}$$

$$P^\alpha{}_{\delta\beta\gamma}=e_\beta P^\alpha_{\delta\gamma}-e_\gamma P^\alpha_{\delta\beta}+P^\sigma_{\delta\beta}\cdot P^\alpha_{\sigma\gamma}$$
$$-P^\sigma_{\delta\gamma}\cdot P^\alpha_{\sigma\beta}-C_{\beta\gamma}{}^\sigma\cdot P^\alpha_{\delta\sigma}. \tag{109}$$

The quantities $P^\alpha{}_{\delta\beta\gamma}=-P^\alpha{}_{\delta\gamma\beta}$ are called the components of the covariant curvature tensor in a noncoordinate basis.

For a covariant tensor field $W=W_A\cdot dx^A=W_C\cdot e^C\in\otimes_k(M)$ we have, in a coordinate basis,

$$W_{A;i;j}-W_{A;j;i}=S_{Am}{}^{Bn}\cdot W_B\cdot P^m{}_{nij}+W_{A;l}\cdot T_{ij}{}^l, \tag{110}$$

and in a noncoordinate basis,

$$W_{A/\beta/\gamma}-W_{A/\gamma/\beta}=S_{A\alpha}{}^{B\delta}\cdot W_B\cdot P^\alpha{}_{\delta\beta\gamma}+W_{A/\delta}\cdot T_{\beta\gamma}{}^\delta. \tag{111}$$

5.2. Bianchi identities

If we write down the cycle of the action of the curvature operator on contravariant vector fields, i.e., if we write

$$\langle[R(\xi,u)]v\rangle=[R(\xi,u)]v+[R(v,\xi)]u+[R(u,v)]\xi \tag{112}$$

and give the explicit form of every term in the cycle, then by the use of the covariant and the Lie differential operator, after some (not so difficult) calculations, we can find identities in the form

$$[R(\xi,u)]v+[R(v,\xi)]u+[R(u,v)]\xi$$
$$\equiv T(T(\xi,u),v)+T(T(v,\xi),u)+T(T(u,v),\xi)$$
$$+(\nabla_\xi T)(u,v)+(\nabla_v T)(\xi,u)+(\nabla_u T)(v,\xi), \quad (113)$$

or in the form

$$\langle[R(\xi,u)]v\rangle\equiv\langle T(T(\xi,u),v)\rangle+\langle(\nabla_\xi T)(u,v)\rangle. \quad (114)$$

These identities are called *Bianchi identities of the first type* (or *of type 1*), where

$$\langle T(T(\xi,u),v)\rangle\equiv T(T(\xi,u),v)+T(T(v,\xi),u)$$
$$+T(T(u,v),\xi),$$
$$\langle(\nabla_\xi T)(u,v)\rangle\equiv(\nabla_\xi T)(u,v)+(\nabla_v,T)(\xi,u)$$
$$+(\nabla_u T)(v,\xi),$$
$$\nabla_\xi[T(u,v)]=(\nabla_\xi T)(u,v)+T(\nabla_\xi u,v)+T(u,\nabla_\xi v).$$

By the use of the curvature operator and the covariant differential operator, a new operator $(\nabla_w R)(\xi,u)$ can be constructed in the form

$$(\nabla_w R)(\xi,u)=[\nabla_w,R(\xi,u)]-R(\nabla_w\xi,u)-R(\xi,\nabla_w u), \quad (115)$$

where

$$[\nabla_w R(\xi,u)]=\nabla_w\circ R(\xi,u)-R(\xi,u)\circ\nabla_w,\quad w,\xi,u\in T(M).$$

The operator $(\nabla_w R)(\xi,u)$ has the structure

$$(\nabla_w R)(\xi,u)=\nabla_w\nabla_\xi\nabla_u-\nabla_w\nabla_u\nabla_\xi+\nabla_u\nabla_\xi\nabla_w-\nabla_\xi\nabla_u\nabla_w$$
$$+\nabla_u\nabla_{\nabla_w\xi}-\nabla_{\nabla_w\xi}\nabla_u+\nabla_{\nabla_w u}\nabla_\xi-\nabla_\xi\nabla_{\nabla_w u}$$
$$+\nabla_{\mathcal{L}_\xi u}\nabla_w-\nabla_w\nabla_{\mathcal{L}_\xi u}+\nabla_{\mathcal{L}_\xi\nabla_w u}-\nabla_{\mathcal{L}_u\nabla_w\xi}. \quad (116)$$

This operator obeys the so-called *Bianchi identity of the second type* (or *of type 2*):

$$\langle(\nabla_w R)(\xi,u)\rangle\equiv\langle R(w,T(\xi,u))\rangle, \quad (117)$$

where

$$\langle(\nabla_w R)(\xi,u)\rangle\equiv(\nabla_w R)(\xi,u)+(\nabla_u R)(w,\xi)$$
$$+(\nabla_\xi R)(u,w),$$
$$\langle R(w,T(\xi,u))\rangle\equiv R(w,T(\xi,u))+R(u,T(w,\xi))$$
$$+R(\xi,T(u,w)).$$

The Bianchi identity of type 2 can be written in a coordinate or in a noncoordinate basis as an identity for the components of the contravariant curvature tensor:

$$R^i{}_{j\langle kl;m\rangle}\equiv R^i{}_{j\langle kn}\cdot T_{lm\rangle}{}^n\equiv -R^i{}_{jn\langle k}\cdot T_{lm\rangle}{}^n, \quad (118)$$

where

$$R^i{}_{j\langle kl;m\rangle}\equiv R^i{}_{jkl;m}+R^i{}_{jmk;l}+R^i{}_{jlm;k},$$
$$R^i{}_{j\langle kn}\cdot T_{lm\rangle}{}^n\equiv R^i{}_{jkn}\cdot T_{lm}{}^n+R^i{}_{jmn}\cdot T_{kl}{}^n+R^i{}_{jlr}\cdot T_{mk}{}^r. \quad (119)$$

For the commutator

$$[\nabla_w,R(\xi,u)]=\nabla_w\circ R(\xi,u)-R(\xi,u)\circ\nabla_w$$

the following commutation identity is valid:

$$\langle[\nabla_w,R(\xi,u)]\rangle\equiv-\langle R(w,\mathcal{L}_\xi u)\rangle, \quad (120)$$

where

$$\langle[\nabla_w,R(\xi,u)]\rangle\equiv[\nabla_w,R(\xi,u)]+[\nabla_u,R(w,\xi)]$$
$$+[\nabla_\xi,R(u,w)],$$
$$\langle R(w,\mathcal{L}_\xi u)\rangle\equiv R(w,\mathcal{L}_\xi u)+R(u,\mathcal{L}_w\xi)+R(\xi,\mathcal{L}_u w). \quad (121)$$

The curvature operator and the Bianchi identities have been applied in differentiable manifolds with one affine connection. They can also find applications in considerations concerning the characteristics of differentiable manifolds with affine connections and metrics. The structure of the curvature operator leads to the construction of another operator called the deviation operator.

6. DEVIATION OPERATOR

By means of the structure of the *curvature operator*,

$$R(\xi,u)=\nabla_\xi\nabla_u-\nabla_u\nabla_\xi-\nabla_{\mathcal{L}_\xi u}=[\nabla_\xi,\nabla_u]-\nabla_{[\xi,u]}, \quad (122)$$

the commutator $[\nabla_w,R(\xi,u)]$ $[w,\xi,u\in T(M)]$ can be represented in the form

$$[\nabla_w,R(\xi,u)]=[\nabla_w,\mathcal{L}\Gamma(\xi,u)]+[\nabla_w,[\nabla_\xi,\nabla_u]]$$
$$-[\nabla_w,[\mathcal{L}_\xi,\nabla_u]], \quad (123)$$

where

$$\mathcal{L}\Gamma(\xi,u)=\mathcal{L}_\xi\nabla_u-\nabla_u\mathcal{L}_\xi-\nabla_{\mathcal{L}_\xi u}=[\mathcal{L}_\xi,\nabla_u]-\nabla_{[\xi,u]}. \quad (124)$$

The operator $\mathcal{L}\Gamma(\xi,u)$ appears as a new operator, constructed by means of the Lie differential operator and the covariant differential operator.[33–36]

Definition. The operator $\mathcal{L}\Gamma(\xi,u)$ is called the *deviation operator*. Its properties include the following relations:

1. Action of the deviation operator on a function f: $[\mathcal{L}\Gamma(\xi,u)]f=0$, $f\in C^r(M)$, $r\geqslant 2$.

2. Action of the deviation operator on a contravariant vector field:

$$[\mathcal{L}\Gamma(\xi,u)](fv)=f[\mathcal{L}\Gamma(\xi,u)]v,\quad \xi,u,v\in T(M),$$
$$[\mathcal{L}\Gamma(\xi,u)]v=v^\beta[\mathcal{L}\Gamma(\xi,u)]e_\beta=v^j[\mathcal{L}\Gamma(\xi,u)]\partial_j$$
$$=u^\gamma v^\beta[\mathcal{L}\Gamma(\xi,e_\gamma)]e_\beta=u^j v^i[\mathcal{L}\Gamma(\xi,\partial_j)]\partial_i.$$

The connections between the action of the deviation operator and that of the curvature operator on a contravariant vector field can be given in the form

$$[\mathcal{L}\Gamma(\xi,u)]v=[R(\xi,u)]v+[\nabla_u\nabla_v-\nabla_{\nabla_u v}]\xi$$
$$-T(\xi,\nabla_u v)+\nabla_u[T(\xi,v)]. \quad (125)$$

In a coordinate basis, $[\mathcal{L}\Gamma(\xi,\partial_l)]\partial_k$ has the form

$$[\mathcal{L}\Gamma(\xi,\partial_l)]\partial_k=[\xi^i{}_{;k;l}-R^i{}_{klj}\cdot\xi^j+(T_{jk}{}^i\cdot\xi^j)_{;l}]\cdot\partial_i$$
$$=(\mathcal{L}_\xi\Gamma^i_{kl})\cdot\partial_i, \qquad (126)$$

where

$$\nabla_{\partial_j}[T(\xi,\partial_i)]-T(\xi,\nabla_{\partial_j}\partial_i)=(T_{li}{}^k\cdot\xi^l)_{;j}\cdot\partial_k.$$

The quantity $\mathcal{L}_\xi\Gamma^i_{kl}$ is called the *Lie derivative of contravariant affine connection* along the contravariant vector field ξ. It can also be written in the form

$$\mathcal{L}_\xi\Gamma^i_{kl}=\xi^i{}_{,k,l}+\xi^j\cdot\Gamma^i_{kl,j}-\xi^i{}_{,j}\cdot\Gamma^j_{kl}+\xi^j{}_{,k}\cdot\Gamma^i_{jl}+\xi^j{}_{,l}\cdot\Gamma^i_{kj}. \qquad (127)$$

By means of $\mathcal{L}_\xi\Gamma^i_{kl}$ the expression for $[\mathcal{L}\Gamma(\xi,u)]v$ can be represented in the form

$$[\mathcal{L}\Gamma(\xi,u)]v=v^k\cdot u^l\cdot(\mathcal{L}_\xi\Gamma^i_{kl})\cdot\partial_i=[\xi^i{}_{;k;l}\cdot v^k\cdot u^l-R^i{}_{klj}$$
$$\cdot v^k\cdot u^l\cdot\xi^j+(T_{jk}{}^i\cdot\xi^j)_{;l}\cdot v^k\cdot u^l]\cdot\partial_i. \qquad (128)$$

In this way, the second covariant derivative $\nabla_u\nabla_v\xi$ of the contravariant vector field ξ can be represented by means of the deviation operator in the form

$$\nabla_u\nabla_v\xi=([R(u,\xi)]v)+\nabla_\xi\nabla_u v-\mathcal{L}_\xi(\nabla_u v)$$
$$-\nabla_u[T(\xi,v)]+[\mathcal{L}\Gamma(\xi,u)]v$$
$$=([R(u,\xi)]v)+\nabla_\xi\nabla_u v$$
$$-\nabla_u\mathcal{L}_\xi v-\nabla_{\mathcal{L}_\xi u}v-\nabla_u[T(\xi,v)]. \qquad (129)$$

For $v=u$ the last identity is called the *generalized deviation identity*.[33] It is used for analysis of deviation equations in spaces with affine connection and metric (L_n spaces, U_n spaces, and V_n spaces), where deviation equations are considered with respect to their structure and solutions,[37–41] and as a theoretical basis for gravitational-wave detectors in (pseudo-)Riemannian spaces without torsion (V_n spaces).[42–48] Deviation equations of Synge and Schild and their generalization for $(\bar{L}_n,g)$ spaces are considered in Ref. 28.

2. Action of the deviation operator on a contravariant tensor field:

$$[\mathcal{L}\Gamma(\xi,u)]V=u^\gamma\cdot V^A\cdot[\mathcal{L}\Gamma(\xi,e_\gamma)]e_A=u^\gamma\cdot V^B\cdot(\mathcal{L}_\xi\Gamma^A_{B\gamma})\cdot e_A$$
$$=-(S_{B\alpha}{}^{A\beta}\cdot V^B\cdot\mathcal{L}_\xi\Gamma^\alpha_{\beta\gamma}\cdot u^\gamma)e_A, \quad V\in\otimes^k(M), \qquad (130)$$

where

$$\mathcal{L}_\xi\Gamma^A_{B\gamma}=-S_{B\alpha}{}^{A\beta}\cdot\mathcal{L}_\xi\Gamma^\alpha_{\beta\gamma},$$
$$([\mathcal{L}\Gamma(\xi,u)]e_B)=(\mathcal{L}_\xi\Gamma^A_{B\gamma})\cdot u^\gamma\cdot e_A$$
$$=-S_{B\alpha}{}^{A\beta}\cdot[\xi^\alpha{}_{/\beta/\gamma}-R^\alpha{}_{\beta\gamma\delta}\cdot\xi^\delta+(T_{\delta\beta}{}^\alpha$$
$$\cdot\xi^\delta)_{/\gamma}]\cdot u^\gamma\cdot e_A. \qquad (131)$$

3. The deviation operator obeys an identity analogous to the Bianchi identity of the first type for the curvature operator:

$$\langle[\mathcal{L}\Gamma(\xi,u)]v\rangle\equiv\langle(\nabla_\xi\nabla_u-\nabla_{\nabla_\xi u})v\rangle+\langle T(T(\xi,u),v)\rangle$$
$$-\langle T(u,\nabla_\xi v)\rangle, \quad \xi,u,v\in T(M), \qquad (132)$$

where

$$\langle[\mathcal{L}\Gamma(\xi,u)]v\rangle=[\mathcal{L}\Gamma(\xi,u)]v+[\mathcal{L}\Gamma(v,\xi)]u$$
$$+[\mathcal{L}\Gamma(u,v)]\xi,$$
$$\langle(\nabla_\xi\nabla_u-\nabla_{\nabla_\xi u})v\rangle=(\nabla_\xi\nabla_u-\nabla_{\nabla_\xi u})v+(\nabla_v\nabla_\xi-\nabla_{\nabla_v\xi})u$$
$$+(\nabla_u\nabla_v-\nabla_{\nabla_u v})\xi,$$
$$\langle T(u,\nabla_\xi v)\rangle=T(u,\nabla_\xi v)+T(v,\nabla_u\xi)+T(\xi,\nabla_v u). \qquad (133)$$

In a noncoordinate basis, this identity acquires the form

$$(\mathcal{L}_\xi\Gamma^\gamma_{\alpha\beta})\cdot v^\alpha\cdot u^\beta+(\mathcal{L}_u\Gamma^\gamma_{\alpha\beta})\cdot\xi^\alpha\cdot v^\beta+(\mathcal{L}_v\Gamma^\gamma_{\alpha\beta})\cdot u^\alpha\cdot\xi^\beta$$
$$\equiv\xi^\gamma{}_{/\alpha/\beta}\cdot v^\alpha\cdot u^\beta+u^\gamma{}_{/\alpha/\beta}\cdot\xi^\alpha\cdot v^\beta+v^\gamma{}_{/\alpha/\beta}\cdot u^\alpha\cdot\xi^\beta$$
$$+T_{\langle\alpha\beta}{}^\kappa T_{\kappa\delta>}{}^\gamma\cdot v^\alpha\cdot\xi^\beta\cdot u^\delta-T_{\alpha\beta}{}^\gamma\cdot(u^\alpha\cdot v^\beta{}_{/\delta}\cdot\xi^\delta$$
$$+v^\alpha\cdot\xi^\beta{}_{/\delta}\cdot u^\delta+\xi^\alpha\cdot u^\beta{}_{/\delta}\cdot v^\delta). \qquad (134)$$

The commutator of the covariant differential operator and the deviation operator obeys the identity

$$\langle[\nabla_w,\mathcal{L}\Gamma(\xi,u)]\rangle\equiv\langle[\nabla_w,[\mathcal{L}_\xi,\nabla_u]]\rangle-\langle R(w,\mathcal{L}_\xi u)\rangle, \qquad (135)$$

where

$$\langle[\nabla_w,\mathcal{L}\Gamma(\xi,u)]\rangle=[\nabla_w,\mathcal{L}\Gamma(\xi,u)]+[\nabla_u,\mathcal{L}\Gamma(v,\xi)]$$
$$+[\nabla_\xi,\mathcal{L}\Gamma(u,v)],$$
$$\langle[\nabla_w,[\mathcal{L}_\xi,\nabla_u]]\rangle=[\nabla_w,[\mathcal{L}_\xi,\nabla_u]]+[\nabla_u,[\mathcal{L}_w,\nabla_\xi]]$$
$$+[\nabla_\xi,[\mathcal{L}_u,\nabla_w]],$$
$$\langle R(w,\mathcal{L}_\xi u)\rangle=R(w,\mathcal{L}_\xi u)+R(u,\mathcal{L}_w\xi)$$
$$+R(\xi,\mathcal{L}_u w), \quad \xi,u,w\in T(M).$$

4. The action of the deviation operator on covariant vector fields is determined by its structure and especially by the Lie differential operator.

In a noncoordinate basis,

$$[\mathcal{L}\Gamma(\xi,e_\gamma)]e^\alpha=\mathcal{L}_\xi\nabla_{e_\gamma}e^\alpha-\nabla_{e_\gamma}\mathcal{L}_\xi e^\alpha-\nabla_{\mathcal{L}\xi e_\gamma}e^\alpha$$
$$=(\mathcal{L}_\xi P^\alpha_{\beta\gamma})\cdot e^\beta, \qquad (136)$$

where

$$\mathcal{L}_\xi P^\alpha_{\beta\gamma}=\xi^\delta\cdot e_\delta P^\alpha_{\beta\gamma}+P^\alpha_{\delta\gamma}\cdot e_{\underline{\beta}}\xi^{\bar{\delta}}+P^\alpha_{\delta\gamma}\cdot(P^\delta_{\beta\rho}+\Gamma^{\bar{\delta}}_{\underline{\beta}\rho}$$
$$+C_{\underline{\beta}\rho}{}^{\bar{\delta}})\cdot\xi^\rho-e_\gamma(e_{\underline{\beta}}\xi^{\bar{\alpha}})-e_\gamma[(P^\alpha_{\beta\rho}+\Gamma^{\bar{\alpha}}_{\underline{\beta}\rho}$$
$$+C_{\underline{\beta}\rho}{}^{\bar{\alpha}})\cdot\xi^\rho]-P^\delta_{\beta\gamma}\cdot[e_{\underline{\delta}}\xi^{\bar{\alpha}}+(P^\alpha_{\delta\rho}+\Gamma^{\bar{\alpha}}_{\underline{\delta}\rho}$$
$$+C_{\underline{\delta}\rho}{}^{\bar{\alpha}})\cdot\xi^\rho]+P^\alpha_{\beta\delta}\cdot(e_\gamma\xi^\delta-C_{\rho\gamma}{}^\delta\cdot\xi^\rho), \qquad (137)$$

$$e_{\underline{\beta}}\xi^{\bar{\delta}}=f_\beta{}^\sigma\cdot e_\sigma\xi^\kappa\cdot f^\delta{}_\kappa, \quad \Gamma^{\bar{\delta}}_{\underline{\beta}\rho}=f^\delta{}_\kappa\cdot f_\beta{}^\sigma\cdot\Gamma^\kappa_{\sigma\rho}. \qquad (138)$$

The expression for $\mathcal{L}_\xi P^\alpha_{\beta\gamma}$ can also be written in the form

$$\mathcal{L}_\xi P^\alpha_{\beta\gamma}=-P^\alpha{}_{\beta\gamma\delta}\cdot\xi^\delta-\xi^{\bar{\alpha}}{}_{/\underline{\beta}/\gamma}+T_{\underline{\beta}\delta}{}^{\bar{\alpha}}\cdot\xi^\delta{}_{/\gamma}+T_{\underline{\beta}\delta}{}^{\bar{\alpha}}{}_{/\gamma}\cdot\xi^\delta. \qquad (139)$$

The quantity $\mathcal{L}_\xi P^\alpha_{\beta\gamma}$ is called the *Lie derivative of the components $P^\alpha_{\beta\gamma}$ of the covariant affine connection P in a noncoordinate basis.*

Special case: $S=e^\varphi\cdot C$: $f^\alpha{}_\beta=e^\varphi\cdot g^\alpha_\beta$, $f^i{}_j=e^\varphi\cdot g^i_j$, $f_i^j=e^{-\varphi}\cdot g^j_i$:

$$\mathcal{L}_\xi P^i_{jk}=-P^i{}_{jkl}\cdot\xi^l-\xi^i{}_{;j;k}+T_{jl}{}^i\cdot\xi^l{}_{;k}+T_{jl}{}^i{}_{;k}\cdot\xi^l. \qquad (140)$$

The Lie derivative of the components of the covariant affine connection P can be used in considerations related to deviation equations for covariant vector fields.

7. EXTENDED COVARIANT DIFFERENTIAL OPERATOR. EXTENDED DERIVATIVE

If Γ^i_{jk} are components of a contravariant affine connection Γ and P^i_{jk} are components of a covariant affine connection P in a given (here, coordinate) basis in an $(\bar{L}_n,g)$ space, then $\bar{\Gamma}^i_{jk}$ and $\bar{P}^i_{jk}$ with

$$\bar{\Gamma}^i_{jk}=\Gamma^i_{jk}-\bar{A}^i{}_{jk},\quad \bar{P}^i_{jk}=P^i_{jk}-\bar{B}^i{}_{jk},$$

$$\bar{A}=\bar{A}^i{}_{jk}\cdot\partial_i\otimes dx^j\otimes dx^k,\quad \bar{B}=\bar{B}^i{}_{jk}\cdot\partial_i\otimes dx^j\otimes dx^k,$$

$$\bar{A},\bar{B}\in\otimes^1{}_2(M),$$

are components (in the same basis) of a new contravariant affine connection $\bar{\Gamma}$ and a new covariant affine connection $\bar{P}$, respectively.

$\bar{\Gamma}$ and $\bar{P}$ correspond to a new [extended with respect to ∇_u, $u\in T(M)$] covariant differential operator ${}^e\nabla_u$,

$${}^e\nabla_{\partial k}\partial_j=\bar{\Gamma}^i_{jk}\cdot\partial_i,\quad {}^e\nabla_{e\beta}e_\alpha=\bar{\Gamma}^\gamma_{\alpha\beta}\cdot e_\gamma,$$

$${}^e\nabla_{\partial k}dx^i=\bar{P}^i_{jk}\cdot dx^j,\quad {}^e\nabla_{e\gamma}e^\alpha=\bar{P}^\alpha_{\beta\gamma}\cdot e^\beta,$$

with the same properties as the covariant differential operator ∇_u.

If we choose the tensors $\bar{A}$ and $\bar{B}$ with certain predefined properties, then we can find $\bar{\Gamma}$ and $\bar{P}$ with predetermined characteristics. For instance, we can find $\bar{P}$ for which ${}^e\nabla_u g=0$, $\forall u\in T(M)$, $g\in\otimes_2(M)$, although $\nabla_u g\neq 0$ for the covariant affine connection P. On the other hand, $\bar{A}^i{}_{jk}$ and $\bar{B}^i{}_{jk}$ are related to each other on the basis of the commutation relations for ${}^e\nabla_u$ and ∇_u with the contraction operator S. From

$$\nabla_u\circ S=S\circ\nabla_u,\quad {}^e\nabla_u\circ S=S\circ{}^e\nabla_u,$$

we have

$$(S\circ{}^e\nabla_{\partial_k})(\partial_j\otimes dx^i)=\Gamma^l_{jk}\cdot f^i{}_l-\bar{A}^l{}_{jk}\cdot f^i{}_l+P^i_{lk}\cdot f^l{}_j-\bar{B}^i{}_{lk}\cdot f^l{}_j,$$

$$({}^e\nabla_{\partial_k}\circ S)(\partial_j\otimes dx^i)={}^e\nabla_{\partial_k}(S(\partial_j\otimes dx^i))$$

$$={}^e\nabla_{\partial_k}(f^i{}_j)=\partial_k(f^i{}_j)=f^i{}_{j,k}.$$

Therefore,

$$f^i{}_{j,k}=\Gamma^l_{jk}\cdot f^i{}_l-\bar{A}^l{}_{jk}\cdot f^i{}_l+P^i_{lk}\cdot f^l{}_j-\bar{B}^i{}_{lk}\cdot f^l{}_j.$$

Since the equality $(\nabla_{\partial_k}\circ S)(\partial_j\otimes dx^i)=(S\circ\nabla_{\partial_k})(\partial_j\otimes dx^i)$ leads to the relation $f^i{}_{j,k}=\Gamma^l_{jk}\cdot f^i{}_l+P^i_{lk}\cdot f^l{}_j$, we obtain the connection between $\bar{A}^i{}_{jk}$ and $\bar{B}^i{}_{jk}$ in the form $\bar{A}^l{}_{jk}\cdot f^i{}_l+\bar{B}^i{}_{lk}\cdot f^l{}_j=0$.

Therefore, $\bar{B}^i{}_{jk}=-\bar{A}^l{}_{mk}\cdot f^i{}_l\cdot f_j{}^m=-\bar{A}^{\bar{i}}{}_{jk}$ and $\bar{A}^i{}_{jk}=-\bar{B}^m{}_{lk}\cdot f^l{}_j\cdot f_m{}^i=-\bar{B}^i{}_{\bar{j}k}$.

We can write ${}^e\nabla_{\partial_k}$ in the form ${}^e\nabla_{\partial_k}=\nabla_{\partial_k}-\bar{A}_{\partial_k}$ with $\bar{A}_{\partial_k}=\bar{A}^i{}_{jk}\cdot\partial_i\otimes dx^j$.

The operator ${}^e\nabla_u$ can also be written in the form ${}^e\nabla_u=\nabla_u-\bar{A}_u$ with $\bar{A}_u=\bar{A}^i{}_{jk}\cdot u^k\cdot\partial_i\otimes dx^j$.

Here $\bar{A}_u$ appears as a mixed tensor field of second rank, but acting on tensor fields as a covariant differential operator, because ${}^e\nabla_u$ is defined as a covariant differential operator with the same properties as the covariant differential operator ∇_u.

Definition. *Extended (to ∇_u) covariant differential operator.* The linear differential operator ${}^e\nabla_u:v\to{}^e\nabla_u v=\bar{v}$, $v,\bar{v}\in\otimes^k{}_l(M)$, with the properties of ∇_u.

From the properties of ${}^e\nabla_u$ and ∇_u we obtain the properties of the operator $\bar{A}_u$:

$$\bar{A}_u:v\to\bar{A}_u v,\quad u\in T(M),\quad v,\bar{A}_u v\in\otimes^k{}_l(M).$$

(a) $\bar{A}_u(v+w)=\bar{A}_u v+\bar{A}_u w$, $v,w\in\otimes^k{}_l(M)$.

(b) $\bar{A}_u(f\cdot v)=f\cdot\bar{A}_u v$, $f\in C^r(M)$.

(c) $\bar{A}_{u+v}w=\bar{A}_u w+\bar{A}_v w$.

(d) $\bar{A}_{f\cdot u}v=f\cdot\bar{A}_u v$.

(e) $\bar{A}_u f=0$.

(f) $\bar{A}_u(v\otimes w)=\bar{A}_u v\otimes w+v\otimes\bar{A}_u w$, $v\in\otimes^k{}_l(M)$, $w\in\otimes^m{}_r(M)$.

(g) $\bar{A}_u\circ S=S\circ\bar{A}_u$ (commutation relation with the contraction operator S).

All the properties of $\bar{A}_u$ correspond to the properties of ${}^e\nabla_u$ and ∇_u as well defined covariant differential operators. In fact, $\bar{A}_u$ can be defined as $\bar{A}_u=\nabla_u-{}^e\nabla_u$. If $\bar{A}_u$ is a given mixed tensor field, then ${}^e\nabla_u$ can be constructed in a unique way.

On the basis of the above considerations we can formulate the following proposition:

Proposition. To every covariant differential operator ∇_u and a given tensor field $\bar{A}_u\in\otimes^1{}_1(M)$ acting as a covariant differential operator on a tensor field in an $(\bar{L}_n,g)$ space there corresponds an extended covariant differential operator ${}^e\nabla_u=\nabla_u-\bar{A}_u$.

In accordance with property (c), $\bar{A}_{u+v}=\bar{A}_u+\bar{A}_v$, the operator $\bar{A}_u$ must be linear in u. On the other hand, $\bar{A}_u$ as a mixed tensor field of second rank can be represented by the use of the existing [in $(\bar{L}_n,g)$ space] contravariant and covariant metrics $\bar{g}$ and g, respectively, in the form $\bar{A}_u=\bar{g}(A_u)$, where A_u is a covariant tensor field of second rank constructed by the use of a tensor field C and a contravariant vector field u in such a way that A_u is linear in u. There are at least three possibilities for construction of a covariant tensor field A_u of second rank in such a way that A_u is linear in u, i.e., $A_u=C(u)=C_{ij}(u)\cdot dx^i\otimes dx^j$ with:

1. $A_u=C(u)=A(u)=A_{ij\bar{k}}\cdot u^k\cdot dx^i\otimes dx^j$, $A=A_{ijk}\cdot dx^i\otimes dx^j\otimes dx^k\in\otimes_3(M)$, $u\in T(M)$.
2. $A_u=C(u)=\nabla_u B=B_{ij;k}\cdot u^k\cdot dx^i\otimes dx^j$, $B=B_{ij}\cdot dx^i\otimes dx^j\in\otimes_2(M)$, $u\in T(M)$.
3. $A_u=C(u)=A(u)+\nabla_u B$, $A(u)=A_{ij\bar{k}}\cdot u^k\cdot dx^i\otimes dx^j$, $A=A_{ijk}\cdot dx^i\otimes dx^j\otimes dx^k\in\otimes_3(M)$, $u\in T(M)$; $\nabla_u B=B_{ij;k}\cdot u^k\cdot dx^i\otimes dx^j$, $B=B_{ij}\cdot dx^i\otimes dx^j\in\otimes_2(M)$, $u\in T(M)$.

An extended covariant differential operator ${}^e\nabla_u=\nabla_u-\bar{A}_u$ can obey additional conditions determining the structure of the mixed tensor field $\bar{A}_u$ (acting on tensor fields as a covariant differential operator). One can impose given conditions on ${}^e\nabla_u$ leading to definite properties of $\bar{A}_u$ and vice versa: one can impose conditions on the tensor field $\bar{A}_u$ leading to definite properties of ${}^e\nabla_u$.

Every extended covariant differential operator and every covariant differential operator has its special type of transport of covariant vector fields.

8. METRICS

We have introduced the concept of a contraction operator acting on two vectors belonging to two different vector spaces with equal dimensions at a point of a differentiable manifold M and associating with them a function over M. If a contraction operator acts on two vectors belonging to one and the same vector space, then this operator is connected with the concept of a metric.

Definition. *Metric.* Contraction operator S acting on two vectors of one and the same vector space and mapping them to an element of the field F (R or C).

Definition. *Metric over a differentiable manifold M.* Contraction operator S acting on two vector fields whose vectors at every fixed point $x\in M$ belong to one and the same vector space, i.e., $S:(u,v)\rightarrow S(u,v)\in C^r(M)$, $u_x,v_x\in N_x(M)$.

8.1. Covariant metric

Definition. *Covariant metric.* Contraction operator S acting on two contravariant vector fields over a manifold M, whose action is identified with the action of a covariant symmetric tensor field of rank two on the two vector fields, i.e.,

$$S(u,v)\equiv g(u,v):=S(g,q)=S(g,u\otimes v)$$
$$=S(g\otimes(u\otimes v)),\quad q=u\otimes v. \tag{141}$$

The tensor $g=g_{\alpha\beta}\cdot e^\alpha\cdot e^\beta=g_{ij}\cdot dx^i\cdot dx^j$ is called the *covariant metric tensor field* (*covariant metric*), and $g(x)=g_x\in\otimes_{2/x}(M)$ is called *the covariant metric tensor* (*covariant metric*) at a point $x\in M$.

(a) Action of the covariant metric on two contravariant vector fields in a coordinate basis:

$$g(u,v)=g_{kl}\cdot f^k{}_i\cdot f^l{}_j\cdot u^i\cdot{}^j=g_{\bar{i}\bar{j}}u^i\cdot v^j=g_{kl}\cdot u^{\bar{k}}\cdot v^{\bar{l}}=u_l\cdot v^{\bar{l}}=u_{\bar{j}}\cdot v^j,\quad g_{\bar{i}\bar{j}}=f^k{}_i\cdot f^l{}_j\cdot g_{kl},\quad u^{\bar{k}}=f^k{}_i\cdot u^i,$$
$$u_{\bar{j}}=g_{\bar{i}\bar{j}}\cdot u^i,\quad u_l=g_{kl}\cdot u^{\bar{k}}=g_{\bar{k}l}\cdot u^k. \tag{142}$$

Remark. $g(u,v)$ is also called the *scalar product of the contravariant vector fields u* and *v over the manifold M.* When $v=u$, we have

$$g(u,u)=g_{\bar{\alpha}\bar{\beta}}\cdot u^\alpha\cdot u^\beta=g_{\alpha\beta}\cdot u^{\bar{\alpha}}\cdot u^{\bar{\beta}}=u_{\bar{\alpha}}\cdot u^\alpha=u_\alpha\cdot u^{\bar{\alpha}} \tag{143}$$
$$:=u^2=\pm|u|^2:=\pm l_u^2, \tag{144}$$

and $g(u,u)=u^2=\pm l_u^2$ is called the *square of the length* of the contravariant vector field u.

(b) The action of the covariant metric on a contravariant vector field u can be introduced by means of the contraction operator S in a coordinate basis as

$$g(u):=S^j{}_k(g,u)=S^j{}_k(g_{ij}\cdot dx^i,dx^j,u^k\cdot\partial_k)=g_{ij}\cdot u^k\cdot S^j{}_k(dx^i\cdot dx^j,\partial_k)=g_{ij}\cdot u^k\cdot f^j{}_k\cdot dx^i=g_{i\bar{k}}\cdot u^k\cdot dx^i$$
$$=g_{ij}\cdot u^{\bar{j}}\cdot dx^i=u(g),\quad g_{i\bar{k}}=g_{ij}\cdot f^j{}_k. \tag{145}$$

Remark. The abbreviation $u(g)$ is equivalent to the abbreviation $(u)(g):=S(u,g)$. It should not be considered as the result of the action of the contravariant vector field u on g. Such action of u on g is (until now) not defined.

The action of the covariant metric g on a contravariant vector field u considered in index form (in a given basis) is called *lowering indices* by means of g. The result of the action of g on $u\in T(M)$ is a covariant vector field $g(u)\in T^*(M)$. On this basis, g can be defined as a linear mapping (operator) which maps every element of $T(M)$ onto a corresponding element of $T^*(M)$, i.e., $g:u\rightarrow g(u)\in T^*(M)$, $u\in T(M)$.

Covariant symmetric affine connection. In a noncoordinate basis, the covariant affine connection P will have the form

$$P^\gamma_{\alpha\beta}=\bar{P}^\gamma_{\alpha\beta}+\frac{1}{2}\cdot U^\gamma_{\alpha\beta}, \tag{146}$$

where

$$\bar{P}^\gamma_{\alpha\beta}=\tfrac{1}{2}(P^\gamma_{\alpha\beta}+P^\gamma_{\beta\alpha}+C_{\alpha\beta}{}^\gamma),$$
$$U_{\alpha\beta}{}^\gamma=P^\gamma_{\alpha\beta}-P^\gamma_{\beta\alpha}-C_{\alpha\beta}{}^\gamma=-U_{\beta\alpha}{}^\gamma. \tag{147}$$

The components of the covariant derivative of a covariant metric tensor field g can be represented by means of the covariant symmetric affine connection. If

$$g_{\alpha\beta;\gamma}=e_\gamma g_{\alpha\beta}+\bar{P}^\delta_{\alpha\gamma}\cdot g_{\delta\beta}+\bar{P}^\delta_{\beta}\cdot g_{\alpha\delta} \tag{148}$$

is the covariant derivative of the components $g_{\alpha\beta}$ of the covariant metric tensor g with respect to the covariant symmetric affine connection $\bar{P}$ in a noncoordinate basis, then

$$g_{\alpha\beta/\gamma}=g_{\alpha\beta;\gamma}+\tfrac{1}{2}(U_{\alpha\gamma}{}^\delta\cdot g_{\delta\beta}+U_{\beta\gamma}{}^\delta\cdot g_{\alpha\delta}). \tag{149}$$

On the other hand, the components of the covariant symmetric affine connection $\bar{P}^\delta_{\alpha\beta}$ can be written in the form

$$g_{\delta\gamma}\cdot\bar{P}^\delta_{\alpha\beta}=-\{\alpha\beta,\gamma\}+K_{\alpha\beta\gamma}+C_{\alpha\beta\gamma}+\tfrac{1}{2}(g_{\delta\alpha}\cdot U_{\beta\gamma}{}^\delta+g_{\delta\beta}\cdot U_{\alpha\gamma}{}^\delta)=-\{\alpha\beta,\gamma\}+C_{\alpha\beta\gamma}$$
$$+\tfrac{1}{2}(g_{\alpha\gamma;\beta}+g_{\beta\gamma;\alpha}-g_{\alpha\beta;\gamma}), \tag{150}$$

where

$$\{\alpha\beta,\gamma\}=\frac{1}{2}(e_\alpha g_{\beta\gamma}+e_\beta g_{\alpha\gamma}-e_\gamma g_{\alpha\beta}),$$

$$K_{\alpha\beta\gamma}=\frac{1}{2}(g_{\alpha\gamma/\beta}+g_{\beta\gamma/\alpha}-g_{\alpha\beta/\gamma}),$$

$$C_{\alpha\beta\gamma}=\frac{1}{2}(g_{\delta\alpha}\cdot C_{\beta\gamma}{}^{\delta}+g_{\delta\beta}\cdot C_{\alpha\gamma}{}^{\delta}+g_{\delta\gamma}\cdot C_{\alpha\beta}{}^{\delta}). \quad (151)$$

By means of the last expressions, $P^{\gamma}_{\alpha\beta}$ can be represented in the form

$$g_{\delta\gamma}\cdot P^{\delta}_{\alpha\beta}=-\{\alpha\beta,\gamma\}+K_{\alpha\beta\gamma}+U_{\alpha\beta\gamma}+C_{\alpha\beta\gamma}, \quad (152)$$

where

$$U_{\alpha\beta\gamma}=\frac{1}{2}(g_{\delta\alpha}\cdot U^{\delta}_{\beta\gamma}+g_{\delta\beta}\cdot U^{\delta}_{\alpha\gamma}+g_{\delta\gamma}\cdot U^{\delta}_{\alpha\beta}). \quad (153)$$

In the special case when the condition $g_{\alpha\beta;\gamma}=0$ is imposed, the following proposition can be proved:

Proposition. A necessary and sufficient condition for $g_{\alpha\beta;\gamma}=0$ is the condition

$$g_{\delta\gamma}\cdot \bar{P}^{\delta}_{\alpha\beta}=-\{\alpha\beta,\gamma\}+C_{\alpha\beta\gamma}. \quad (154)$$

The proof follows imediately from (150).

In a coordinate basis, the covariant derivative of the components g_{ij} of g can be represented in an analogous way by means of the components P^{i}_{jk} of the covariant symmetric affine connection:

$$g_{ij;k}=g_{ij,k}+\bar{P}^{l}_{ik}\cdot g_{lj}+\bar{P}^{l}_{jk}\cdot g_{il}+\frac{1}{2}(U^{l}_{ik}\cdot g_{lj}+U^{l}_{jk}\cdot g_{il})$$

$$=g_{ij/k}+\frac{1}{2}(U^{l}_{ik}\cdot g_{lj}+U^{l}_{jk}\cdot g_{il}), \quad (155)$$

where

$$g_{ij/k}=g_{ij,k}+\bar{P}^{l}_{ik}\cdot g_{lj}+\bar{P}^{l}_{jk}\cdot g_{il}. \quad (156)$$

Action of the Lie differential operator on the covariant metric. In a coordinate basis, $\mathcal{L}_\xi g$ will take the form

$$\mathcal{L}_\xi g=(\mathcal{L}_\xi g_{ij})\cdot dx^i\cdot dx^j=[g_{ij;k}\cdot \xi^k+g_{kj}\cdot \xi^{\bar{k}}{}_{;\underline{i}}+g_{ik}\cdot \xi^{\bar{k}}{}_{;\underline{j}}$$

$$+(g_{kj}\cdot T_{l\underline{i}}{}^{\bar{k}}+g_{ik}\cdot T_{l\underline{j}}{}^{\bar{k}})\cdot \xi^l]\cdot dx^i\cdot dx^j. \quad (157)$$

The following relations are also fulfilled:

$$\mathcal{L}_\xi[g(u,v)]=\xi[g(u,v)]$$

$$=(\mathcal{L}_\xi g)(u,v)+g(\mathcal{L}_\xi u,v)+g(u,\mathcal{L}_\xi v),$$

$$\mathcal{L}_\xi[g(u)]=(\mathcal{L}_\xi g)(u)+g(\mathcal{L}_\xi u), \quad \xi,u,v\in T(M). \quad (158)$$

The action of the Lie differential operator is called *dragging-along of a contravariant vector field.* On the basis of draggings-along of the metric tensor field g, concepts such as arbitrary (nonmetric) draggings-along, quasi-projective draggings-along, conformal motions, and motions can be defined and considered, as in (L_n,g) spaces. Here we will only define different types of draggings-along.

1. *Arbitrary (nonmetric) draggings-along:*

$$\mathcal{L}_\xi g=q_\xi, \quad \forall \xi\in T(M), \quad q_\xi\in\otimes_{\text{sym}2}(M).$$

2. *Quasi-projective draggings-along:*

$$\mathcal{L}_\xi g=\frac{1}{2}[p\otimes g(\xi)+g(\xi)\otimes p],$$

$$\xi\in T(M), \quad p\in T^*(M).$$

3. *Conformal-invariant draggings-along (conformal motions):*

$$\mathcal{L}_\xi g=\lambda\cdot g, \quad \lambda\in C^r(M), \quad \xi\in T(M).$$

4. *Isometric draggings-along (motions):*

$$\mathcal{L}_\xi g=0, \quad \xi\in T(M).$$

For all types of draggings-along the changes of the scalar product of two contravariant vector fields and the changes of the length of these fields can be found and used in the same way as in (L_n,g) spaces.

8.2. Covariant projective metric

If a covariant metric field g is given and there exists a contravariant vector field u whose squared length is $g(u,u)=e\neq 0$, then we can construct a new covariant tensor field which is orthogonal to the vector field u. It possesses properties analogous to those of the covariant tensor field g acting on contravariant vector fields in every subspace $T_x^{\perp u}(M)$ in $T_x(M)$ orthogonal to $u(x)=u_x\in T_x(M)$, where $(T_x^{\perp u}(M)=\{\xi_x\}:g_x(\xi_x,u_x)=0)$, $g_x\in\otimes_{\text{sym}2/x}(M)$.

Definition. *Covariant projective metric.* Covariant metric orthogonal to a given nonisotropic (non-null) vector field $u\,[e=g(u,u)\neq 0]$, i.e., covariant metric h_u satisfying the condition $h_u(u)=u(h_u)=0$ and constructed by means of the covariant metric g and u in the form

$$h_u=g-\frac{1}{g(u,u)}\cdot g(u)\otimes g(u)=g-\frac{1}{e}\cdot g(u)\otimes(u). \quad (159)$$

The properties of the covariant projective metric follow from its construction and from the properties of the covariant metric g:

(a) $h_u(u)=u(h_u)=0, \quad [g(u)](u)=g(u,u)=e.$

(b) $h_u(u,u)=0.$

(c) $h_u(u,v)=h_u(v,u)=0, \quad \forall v\in T(M).$

8.3. Contravariant metric

Definition. *Contravariant metric.* Contraction operator S acting on two covariant vector fields over a manifold M whose action is identified with the action of a contravariant symmetric tensor field of rank two on the two vector fields, i.e.,

$$S(p,q)\equiv\bar{g}(p,q):=S(\bar{g},w):=S(\bar{g},p\otimes q)$$

$$=S(\bar{g}\otimes(p\otimes q)), \quad w=p\otimes q.$$

The tensor field $\bar{g}=g^{\alpha\beta}\cdot e_\alpha\cdot e_\beta=g^{ij}\cdot\partial_i\cdot\partial_j$ is called the *contravariant metric tensor field* (*contravariant metric*); $\bar{g}(x)=\bar{g}_x\in\otimes^2_x(M)$ is called the *contravariant metric tensor* in $x\in M$.

The properties of the contravariant metric are determined by the properties of the contraction operator and its identification with the contravariant symmetric tensor field of rank 2. On this basis the following properties can be proved:

(a) Action of the contravariant metric on two covariant vector fields in a coordinate basis,

$$\bar{g}(p,q)=g^{kl}\cdot f^i{}_k\cdot f^j{}_l\cdot p_i\cdot q_j=g^{\bar{i}\bar{j}}\cdot p_{\bar{i}}\cdot q_{\bar{j}}=q^{kl}\cdot p_{\bar{k}}\cdot q_{\bar{l}}=p^{\bar{j}}\cdot q_j$$
$$=p_k\cdot q^{\bar{k}},\quad p_{\bar{k}}=f^i{}_k\cdot p_i,\quad q_{\bar{l}}=f^j{}_l\cdot q_j,\quad p^{\bar{j}}=g^{\bar{j}\bar{i}}\cdot p_i.$$

[When $q=p$, the quantity $\bar{g}(p,p)=p^2=\pm|p|^2$ is called the *square of the length* of the covariant vector field p.]

(b) Action of the contravariant metric $\bar{g}$ on a covariant vector field,

$$\bar{g}(p)=p(\bar{g})=g^{ij}\cdot p_k\cdot f^k{}_j\cdot \partial_i=g^{ij}\cdot p_{\bar{j}}\cdot \partial_i=g^{i\bar{k}}\cdot p_k\cdot \partial_i$$
$$=p^i\cdot \partial_i,\quad g^{i\bar{k}}=g^{il}\cdot f^k{}_l,\quad p^i=g^{ij}\cdot p_{\bar{j}}=g^{i\bar{j}}\cdot p_j.$$

The action of the contravariant metric $\bar{g}$ on a covariant vector field p in a given basis is called raising of indices by means of the contravariant metric. The result of this action is a contravariant vector field $\bar{g}(p)$. On this basis, $\bar{g}$ can be defined as a linear mapping (operator) which maps an element of $T^*(M)$ onto an element of $T(M)$:

$$\bar{g}:p\to\bar{g}(p)\in T(M),\quad p\in T^*(M).$$

The connection between the contravariant and covariant metric can be determined by the conditions

$$g[\bar{g}(u)]=u,\quad u\in T(M),\quad g[\bar{g}(p)]=p,\quad p\in T^*(M). \tag{160}$$

In a coordinate basis, these conditions take the form

$$g^{ij}\cdot g_{\bar{j}\bar{k}}=g^i_k,\quad g_{ij}\cdot g^{\bar{j}\bar{k}}=g^k_i. \tag{161}$$

From the last expressions we obtain the relations

$$g[\bar{g}]=g_{ij}g^{\bar{i}\bar{j}}=n,\quad \bar{g}[g]=g^{\bar{i}\bar{j}}\cdot g_{ij}=n,\quad \dim M=n. \tag{162}$$

Contravariant symmetric affine connection. From the transformation properties of the components of the contravariant affine connection, it follows that the quantity

$$\tfrac{1}{2}(\Gamma^\gamma_{\alpha\beta}+\Gamma^\gamma_{\beta\alpha}-C_{\alpha\beta}{}^\gamma)\ \text{ or }\ \tfrac{1}{2}(\Gamma^k_{ij}+\Gamma^k_{ji})$$

has the same transformation properties as the contravariant affine connection itself. This fact can be used as usual for representing the contravariant affine connection by means of its symmetric and antisymmetric part in the form

$$\Gamma^k_{ij}=\bar{\Gamma}^k_{ij}-\tfrac{1}{2}T_{ij}{}^k,\quad \bar{\Gamma}^k_{ij}=\tfrac{1}{2}(\Gamma^k_{ij}+\Gamma^k_{ji}),$$
$$T_{ij}{}^k=\Gamma^k_{ji}-\Gamma^k_{ij}\ \text{(in a coordinate basis)},$$
$$\Gamma^\gamma_{\alpha\beta}=\bar{\Gamma}^\gamma_{\alpha\beta}-\tfrac{1}{2}T_{\alpha\beta}{}^\gamma,\quad \bar{\Gamma}^\gamma_{\alpha\beta}=\tfrac{1}{2}(\Gamma^\gamma_{\alpha\beta}+\Gamma^\gamma_{\beta\alpha}-C_{\alpha\beta}{}^\gamma),$$
$$T_{\alpha\beta}{}^\gamma=\Gamma^\gamma_{\beta\alpha}-\Gamma^\gamma_{\alpha\beta}-C_{\alpha\beta}{}^\gamma\ \text{(in a noncoordinate basis)}. \tag{163}$$

The quantities $\bar{\Gamma}^k_{ij}$ ($\bar{\Gamma}^\gamma_{\alpha\beta}$) are called the components of the *contravariant symmetric affine connection* in a coordinate (respectively, in a noncoordinate) basis.

The components of the covariant derivative of the contravariant metric tensor field $\bar{g}$ can be represented by means of the contravariant symmetric affine connection. If we introduce the abbreviations

$$g^{\alpha\beta}{}_{;\gamma}=e_\gamma g^{\alpha\beta}+\bar{\Gamma}^\alpha_{\delta\gamma}\cdot g^{\delta\beta}+\bar{\Gamma}^\beta_{\delta\gamma}\cdot g^{\alpha\delta}\ \text{(in a noncoordinate basis)},$$

$$g^{ij}{}_{/k}=g^{ij}{}_{,k}+\bar{\Gamma}^i{}_{lk}\cdot g^{lj}+\bar{\Gamma}^j_{lk}\cdot g^{il}\ \text{(in a coordinate basis)}, \tag{164}$$

where $g^{\alpha\beta}{}_{;\gamma}$ is the covariant derivative of the components of the contravariant metric tensor $\bar{g}$ with respect to the contravariant symmetric affine connection $\bar{\Gamma}$ in a noncoordinate basis, then

$$g^{\alpha\beta}{}_{/\gamma}=g^{\alpha\beta}{}_{;\gamma}-\tfrac{1}{2}(T^\alpha_{\delta\gamma}\cdot g^{\delta\beta}+T_{\delta\gamma}{}^\beta\cdot g^{\alpha\delta}). \tag{165}$$

By means of the explicit expression for $g^{\alpha\beta}{}_{/\kappa}\cdot g^{\kappa\gamma}$ and the usual method for expressing the components of the symmetric affine connection, the components of the contravariant symmetric affine connection can be represented in the form

$$g^{\alpha\delta}\cdot g^{\beta\kappa}\cdot\bar{\Gamma}^\gamma_{\delta\kappa}=\bar{K}^{\alpha\beta\gamma}-\{^{\alpha\beta,\gamma}\}-\bar{C}^{\alpha\beta\gamma}$$
$$-\tfrac{1}{2}g^{\gamma\delta}(g^{\beta\kappa}T_{\kappa\delta}{}^\alpha+g^{\alpha\kappa}T_{\kappa\delta}{}^\beta), \tag{166}$$

where

$$\{^{\alpha\beta,\gamma}\}=\tfrac{1}{2}(g^{\alpha\kappa}\cdot e_\kappa g^{\beta\gamma}+g^{\beta\kappa}\cdot e_\kappa g^{\alpha\gamma}-g^{\gamma\kappa}\cdot e_\kappa g^{\alpha\beta}),$$
$$\bar{K}^{\alpha\beta\gamma}=\tfrac{1}{2}(g^{\alpha\gamma}{}_{/\kappa}\cdot g^{\beta\kappa}+g^{\beta\gamma}{}_{/\kappa}\cdot g^{\alpha\kappa}-g^{\alpha\beta}{}_{/\kappa}\cdot g^{\gamma\kappa}),$$
$$\bar{C}^{\alpha\beta\gamma}=\tfrac{1}{2}(g^{\gamma\delta}\cdot g^{\beta\kappa}\cdot C_{\kappa\delta}{}^\alpha+g^{\gamma\delta}\cdot g^{\alpha\kappa}\cdot C_{\kappa\delta}{}^\beta+g^{\alpha\delta}\cdot g^{\beta\kappa}\cdot C_{\delta\kappa}{}^\gamma). \tag{167}$$

The brackets $\{^{\alpha\beta,\gamma}\}$ are called *Christoffel symbols of the first kind* for the contravariant symmetric affine connection in a noncoordinate basis.

The components $\Gamma^\gamma_{\alpha\beta}$ of the contravariant affine connection Γ can be written by means of the last abbreviations in the form

$$g^{\alpha\delta}\cdot g^{\beta\kappa}\cdot\Gamma^\gamma_{\delta\kappa}=-\{^{\alpha\beta,\gamma}\}+\bar{K}^{\alpha\beta\gamma}-\bar{T}^{\alpha\beta\gamma}-\bar{C}^{\alpha\beta\gamma}, \tag{168}$$

where

$$\bar{T}^{\alpha\beta\gamma}=\tfrac{1}{2}(g^{\gamma\delta}\cdot g^{\beta\kappa}\cdot T_{\kappa\delta}{}^\alpha+g^{\gamma\delta}\cdot g^{\alpha\kappa}$$
$$\cdot T_{\kappa\delta}{}^\beta+g^{\alpha\delta}\cdot g^{\beta\kappa}\cdot T_{\delta\kappa}{}^\gamma). \tag{169}$$

By using the connections between the components of the covariant metric, the components of the contravariant metric, and their derivatives,

$$g_{\bar{\alpha}\bar{\beta}}\cdot g^{\beta\gamma}=g^\gamma_\alpha,\quad g_{\bar{\alpha}\bar{\delta}}\cdot e_\kappa g^{\delta\gamma}=-g^{\delta\gamma}\cdot e_\kappa(g_{\bar{\alpha}\bar{\delta}}),$$
$$g_{\bar{\alpha}\bar{\delta}}\cdot g^{\gamma\delta}{}_{/\kappa}=-g^{\gamma\delta}\cdot g_{\bar{\alpha}\bar{\delta}/\kappa}, \tag{170}$$
$$g_{\bar{\alpha}\bar{\delta}/\kappa}=f^\gamma{}_\alpha\cdot f^\beta_\delta\cdot g_{\gamma\beta/\kappa},$$

the components of the contravariant affine connection Γ can be represented in a noncoordinate basis in the form

$$\Gamma^\gamma_{\alpha\beta}=\left\{{\gamma \atop \alpha\beta}\right\}-\bar{K}_{\alpha\beta}{}^\gamma-\bar{S}_{\alpha\beta}{}^\gamma-\bar{C}_{\alpha\beta}{}^\gamma, \tag{171}$$

where

$$\left\{{\gamma \atop \alpha\beta}\right\}=\tfrac{1}{2}g^{\gamma\delta}[e_\beta(g_{\bar{\alpha}\bar{\delta}})+e_\alpha(g_{\bar{\beta}\bar{\delta}})-e_\delta(g_{\bar{\alpha}\bar{\beta}})]=-g_{\bar{\alpha}\bar{\rho}}$$
$$\cdot g_{\bar{\beta}\bar{\sigma}}\cdot\{^{\rho\sigma,\gamma}\},\quad \bar{K}_{\alpha\beta}{}^\gamma=-g_{\bar{\alpha}\bar{\rho}}\cdot g_{\bar{\beta}\bar{\sigma}}\cdot\bar{K}^{\rho\sigma\gamma},$$
$$\bar{S}_{\alpha\beta}{}^\gamma=-g_{\bar{\alpha}\bar{\rho}}\cdot g_{\bar{\beta}\bar{\sigma}}\cdot\bar{T}^{\rho\sigma\gamma},\quad \bar{C}_{\alpha\beta}{}^\gamma=-g_{\bar{\alpha}\bar{\rho}}\cdot g_{\bar{\beta}\bar{\sigma}}\cdot\bar{C}^{\rho\sigma\gamma}. \tag{172}$$

The brackets $\{^{\gamma}_{\alpha\beta}\}$ are called *generalized Christoffel symbols of the second kind* for the contravariant symmetric affine connection in a noncoordinate basis.

In an analogous way, when contravariant and covariant metric fields are given, the covariant affine connection can be represented by means of the both types of tensor metric fields in the form

$$P^{\gamma}_{\alpha\beta}=-\underline{\begin{Bmatrix}\gamma\\ \alpha\beta\end{Bmatrix}}+\underline{K}_{\alpha\beta}{}^{\gamma}+\underline{U}_{\alpha\beta}{}^{\gamma}+\underline{C}_{\alpha\beta}{}^{\gamma}, \quad (173)$$

where

$$\underline{\begin{Bmatrix}\gamma\\ \alpha\beta\end{Bmatrix}}=g^{\overline{\gamma\sigma}}\cdot\{\alpha\beta,\gamma\},\quad \underline{K}_{\alpha\beta}{}^{\gamma}=g^{\overline{\gamma\sigma}}\cdot K_{\alpha\beta\sigma},$$

$$\underline{U}_{\alpha\beta}{}^{\gamma}=g^{\overline{\gamma\sigma}}\cdot U_{\alpha\beta\sigma},\quad \underline{C}_{\alpha\beta}{}^{\gamma}=g^{\overline{\gamma\sigma}}\cdot C_{\alpha\beta\sigma}. \quad (174)$$

The brackets $\underline{\{^{\gamma}_{\alpha\beta}\}}$ are called *generalized Christoffel symbols of the second kind* for the covariant symmetric affine connection in a noncoordinate basis.

The same expressions can also be obtained in a coordinate basis.

In the special case when the condition of vanishing of the covariant derivatives of the contravariant metric with respect to the contravariant symmetric affine connection is imposed, i.e., $g^{\alpha\beta}{}_{;\gamma}=0$, the components of the contravariant symmetric affine connection can be written in the form $\bar{\Gamma}^{\gamma}_{\alpha\beta}=\{^{\gamma}_{\alpha\beta}\}-\bar{C}_{\alpha\beta}{}^{\gamma}$.

The last expression is a necessary and sufficient condition for $g^{\alpha\beta}{}_{;\gamma}=0$. In a coordinate basis, the necessary and sufficient condition for $g^{ij}{}_{/k}=0$ takes the form $\bar{\Gamma}^{k}_{ij}=\{^{k}_{ij}\}$.

On the basis of the connection between the covariant derivative of the contravariant tensor metric field and the covariant derivative of the covariant tensor metric field,

$$\nabla_{\xi}g=-g(\nabla_{\xi}\bar{g})g,\quad (\nabla_{\xi}\bar{g})[g(u)]=-\bar{g}[(\nabla_{\xi}g)(u)],$$

$$\forall\xi,\forall u\in T(M),\quad \nabla_{\xi}\bar{g}=-\bar{g}(\nabla_{\xi}g)\bar{g},$$

$$(\nabla_{\xi}g)[\bar{g}(p)]=-g[(\nabla_{\xi}\bar{g})(p)],$$

$$\forall\xi\in T(M),\quad \forall_{p}\in T^{*}(M),$$

one can prove that there is a one-to-one correspondence between the transports of g and $\bar{g}$. *Every transport of the covariant tensor metric field g induces a corresponding transport of the contravariant tensor metric field* $\bar{g}$ and *vice versa.*

8.4. Contravariant projective metric

The concept of a contravariant projective metric with respect to a nonisotropic (non-null) contravariant vector field u can be introduced in two different ways:

(a) by the definition

$$h^{u}=\bar{g}-\frac{1}{g(u,u)}\cdot u\otimes u=\bar{g}-\frac{1}{e}u\otimes u,\quad e=g(u,u)\neq 0; \quad (175)$$

(b) by inducing it from the covariant projective metric, using the relations between the covariant and contravariant metric,

$$h^{u}=\bar{g}(h_{u})\bar{g}=\bar{g}-\frac{1}{e}\cdot u\otimes u,\quad \bar{g}(g)\bar{g}=\bar{g},$$

$$\bar{g}(g(u)\otimes g(u))\bar{g}=u\otimes u. \quad (176)$$

h^{u} is called the *contravariant projective metric* with respect to the nonisotropic contravariant vector field u.

The properties of the contravariant projective metric are determined by its structure.

9. BIANCHI IDENTITIES FOR THE COVARIANT CURVATURE TENSOR

9.1. Bianchi identity of the first type for the covariant curvature tensor

The existence of contravariant and covariant metrics allows us to consider the action of the curvature operator on a covariant vector field $g(v)=g_{\alpha\bar{\beta}}\cdot v^{\beta}\cdot e^{\alpha}=g_{i\bar{j}}\cdot v^{j}\cdot dx^{i}$, constructed by the use of the covariant metric g and a contravariant vector field v.

The identity

$$\langle\bar{g}\{([R(\xi,u)]g)(v)\}\rangle\equiv\langle\bar{g}([R(\xi,u)][g(v)])\rangle$$
$$-\langle[R(\xi,u)]v\rangle\equiv\langle\bar{g}([R(\xi,u)]$$
$$\times[g(v)])\rangle-\langle T(T(\xi,u),v)\rangle$$
$$-\langle(\nabla 0_{\xi}T)(u,v)\rangle \quad (177)$$

is called the *Bianchi identity of the first type* (*of type 1*) *for the covariant curvature tensor.*

In a coordinate basis, the Bianchi identity of first type will have the form

$$P^{l}{}_{\langle ijk\rangle}\equiv-g^{m\bar{n}}\cdot R^{\bar{l}}{}_{m\langle ij}\cdot g_{k>n}, \quad (178)$$

$$R^{l}{}_{\langle ijk\rangle}\equiv-g^{l\bar{m}}\cdot g_{mn}\cdot P^{n}{}_{\langle\bar{i}jk\rangle}\equiv T^{l}_{\langle ij;k\rangle}+T_{\langle ij}{}^{m}\cdot T_{mk>}{}^{l}. \quad (179)$$

It is obvious that the form of the Bianchi identity of the first type for the components of the covariant curvature tensor is not so simple as the form of the Bianchi identity for the components of the contravariant curvature tensor.

9.2. Bianchi identity of the second type for the covariant curvature tensor

The action of the operator $(\nabla_{w}R)(\xi,u)$ can be extended to an action on covariant vector and tensor fields in the same way as in the case of contravariant vector and tensor fields. By the use of the relation

$$\nabla_{w}\{[R(\xi,u)]p\}=[(\nabla_{w}R)(\xi,u)]p+[R(\nabla_{w}\xi,u)]p$$
$$+[R(\xi,\nabla_{w}u)]p+[R(\xi,u)](\nabla_{w}p),$$
$$w,\xi,u\in T(M),\quad p\in T^{*}(M), \quad (180)$$

we can find the identity

$$\langle(\nabla_{w}R)(\xi,u)\rangle p\equiv\langle R(w,T(\xi,u))\rangle p, \quad (181)$$

where

$$\langle(\nabla_{w}R)(\xi,u)\rangle p=[(\nabla_{w}R)(\xi,u)]p+[(\nabla_{u}R)(w,\xi)]p$$
$$+[(\nabla_{\xi}R)(u,w)]p,$$

$$\langle R(w,T(\xi,u))\rangle p=[R(w,T(\xi,u))]p+[R(u,T(w,\xi))]p+[R(\xi,T(u,w))]p. \qquad (182)$$

The identity (181) is called the *Bianchi identity of the second type (of type 2) for the covariant curvature tensor.*

The Bianchi identity of the second type will have the following form in a coordinate basis:

$$P^i{}_{j\langle kl;m\rangle}=P^i{}_{jkl;m}+P^i{}_{jmk;l}+P^i{}_{jlm;k}\equiv P^i{}_{j\langle kn}\cdot T_{lm>}{}^n$$
$$=P^i{}_{jkn}\cdot T_{lm}{}^n+P^i{}_{jmn}\cdot T_{kl}{}^n+P^i{}_{jlr}\cdot T_{mk}{}^r. \qquad (183)$$

10. INVARIANT VOLUME ELEMENT

10.1. Definition and properties

The concept of volume element of a manifold M can be generalized to that of invariant volume element.[16]

Definition. *The volume element of a manifold* M $(\dim M=n)$:

$$d^{(n)}x=d^{(n)}x=dx^1\wedge\ldots\wedge dx^n \text{ (in a coordinate basis)},$$

$$dV_n=e^1\wedge\ldots\wedge e^n \text{ (in a noncoordinate basis)}.$$

The properties of the volume element can be represented as follows:

$$d^{(n)}x=\frac{1}{n!}\cdot\varepsilon_A\cdot\omega^A=\frac{1}{n!}\cdot\varepsilon_A\cdot d\hat{x}^A,\quad d^{(n')}x'=J^{-1}\cdot d^{(n)}x,$$

$$dV_n=\frac{1}{n!}\cdot\varepsilon_A\cdot\omega^A=\frac{1}{n!}\cdot\varepsilon_A\cdot\hat{e}^A,\quad dV_n'=J^{-1}\cdot dV_n, \qquad (184)$$

where $J=\det(A_{\alpha'}{}^{\alpha})=\det(\partial x^i/\partial x^{i'})$, $dV_n'=e^{1'}\wedge\ldots\wedge e^{n'}$, $\varepsilon_A=\varepsilon_{i_1\ldots i_n}$, $\omega^A=dx^{i_1}\wedge\ldots\wedge dx^{i_n}$, ε_A is the Levi-Civita symbol,[16] and

$$\varepsilon_{A'}\cdot\omega^{A'}=J^{-1}\cdot\varepsilon_A\cdot\omega^A,\quad \varepsilon_A\cdot\omega^A=J\cdot\varepsilon_{A'}\cdot\omega^{A'},$$

$$\varepsilon_{A'}\cdot d\hat{x}^{A'}=J^{-1}\cdot\varepsilon_A\cdot d\hat{x}^A,\quad \varepsilon_A\cdot d\hat{x}^A=J\cdot\varepsilon_{A'}\cdot d\hat{x}^{A'},$$

$$d^{(n)}x=\frac{1}{n!}\cdot\varepsilon_A\cdot\omega^A=J\cdot\frac{1}{n!}\cdot\varepsilon_{A'}\cdot\omega^{A'}=\frac{1}{n!}\cdot J\cdot\varepsilon_{A'}\cdot d\hat{x}^{A'}.$$

The transformation properties of the volume element correspond to those of a tensor density of weight $\omega=-\frac{1}{2}$. Therefore, for the construction of an invariant volume element (keeping its form, and independent of the choice of a full antisymmetric tensor basis) the volume element must be multiplied by a tensor density with weight $\omega=\frac{1}{2}$ and rank 0. Since the covariant metric tensor field is connected with the basic characteristics of contravariant (and covariant) vector fields and determines, along with them, concepts (such as the length of a contravariant vector, or the cosine of the angle between two contravariant vectors) which in Euclidean geometry are related to the concept of volume element, the covariant metric tensor density $\tilde{Q}_g$ with weight $\omega=\frac{1}{2}$ and rank 0 ($\tilde{Q}_g=|d_g|^{1/2}$) is a suitable multiplier for a volume element.[55]

Definition. The invariant volume element $d\omega$ of a manifold M $(\dim M=n)$:

$$d\omega=\sqrt{-d_g}\cdot d^{(n)}x:=\frac{1}{n!}\cdot\varepsilon_A\cdot\bar{\omega}^A,$$

$$\bar{\omega}^A=\sqrt{-d_g}\cdot\omega^A,\quad d_g<0$$

(invariant volume element in a coordinate basis),

$$d\omega=\sqrt{-d_g}\cdot dV_n,\quad d_g<0$$

(invariant volume element in a noncoordinate basis).

From the transformation properties of $\sqrt{-d_g}$, namely, $\sqrt{-d_g'}=\pm J\cdot\sqrt{-d_g}$, we obtain the invariance of the invariant volume element: $d\omega'=\pm d\omega$, where

$$d\omega'=\sqrt{-d_g'}\cdot d^{(n')}x' \text{ (in a coordinate basis)},$$

$$d\omega'=\sqrt{-d_g'}\cdot dV_n' \text{ (in a noncoordinate basis)}. \qquad (185)$$

Remark. The minus sign in $\pm d\omega$ can be omitted because of the identical configuration (order, orientation) of the basis vector fields in the old and in the new tensor basis.

From the definition of the invariant volume element we obtain the following relations connected with its structure:

$$d\omega'=\frac{1}{n!}\cdot\sqrt{-d_g'}\cdot\varepsilon_{A'}\cdot\omega^{A'}=\frac{1}{n!}\cdot\sqrt{-d_g}\cdot\varepsilon_A\cdot d\omega^A=d\omega. \qquad (186)$$

10.2. Action of the covariant differential operator on an invariant volume element

The action of the covariant differential operator on an invariant volume element is determined by its action on the elements of the construction of the invariant volume element (the Levi-Civita symbol, the full antisymmetric tensor basis, and the metric tensor density). From $d\omega=(1/n!)\cdot\varepsilon_A\cdot\bar{\omega}^A$ and $\nabla_\xi(d\omega)$, it follows that

$$\nabla_\xi(d\omega)=\nabla_\xi\left[\frac{1}{n!}\cdot(\varepsilon_A\cdot\bar{\omega}^A)\right]$$
$$=\frac{1}{n!}[(\xi\varepsilon_A)\cdot\bar{\omega}^A+\varepsilon_A\cdot\nabla_\xi\bar{\omega}^A], \qquad (187)$$

and $\nabla_\xi(d\omega)$ can be written in the form

$$\nabla_\xi(d\omega)=\frac{1}{2}\cdot\bar{g}[\nabla_\xi g]\cdot\frac{1}{n!}\cdot\varepsilon_A\cdot\bar{\omega}^A=\frac{1}{2}\cdot\bar{g}[\nabla_\xi g]\cdot d\omega. \qquad (188)$$

The quantity $\nabla_\xi(d\omega)$ is called the *covariant derivative of the invariant volume element* $d\omega$ along the contravariant vector field ξ.

10.3. Action of the Lie differential operator on an invariant volume element

The action of the Lie differential operator on an invariant volume element is determined in the same way as the action of the covariant differential operator:

$$\mathcal{L}_\xi(d\omega)=\frac{1}{n!}\cdot\mathcal{L}_\xi(\varepsilon_A\cdot\bar{\omega}^A)$$
$$=\frac{1}{n!}[(\xi\varepsilon_A)\cdot\bar{\omega}^A+\varepsilon_A\cdot\mathcal{L}_\xi\bar{\omega}^A]$$
$$=\frac{1}{n!}\cdot\varepsilon_A\cdot\mathcal{L}_\xi\bar{\omega}^A. \quad (189)$$

After some computation, we obtain for $\mathcal{L}_\xi(d\omega)$ the expression

$$\mathcal{L}_\xi(d\omega)=\frac{1}{n!}\cdot\varepsilon_A\cdot\frac{1}{2}\cdot\bar{g}[\mathcal{L}_\xi g]\cdot\bar{\omega}^A$$
$$=\frac{1}{2}\cdot\bar{g}[\mathcal{L}_\xi g]\cdot\frac{1}{n!}\cdot\varepsilon_A\cdot\bar{\omega}^A,$$
$$\mathcal{L}_\xi(d\omega)=\frac{1}{2}\cdot\bar{g}[\mathcal{L}_\xi g]\cdot d\omega. \quad (190)$$

The quantity $\mathcal{L}_\xi(d\omega)$ is called the *Lie derivative of the invariant volume element* $d\omega$ along the contravariant vector field ξ.

Special case: Metric transports ($\nabla_\xi g=0$): $\nabla_\xi(d\omega)=0$.

Special case: Isometric draggings-along (motions) ($\mathcal{L}_\xi g=0$): $\mathcal{L}_\xi(d\omega)=0$.

In some cases, when conservation of the volume is required as an additional condition, one can introduce a new covariant differential operator or a new Lie differential operator which do not change the invariant volume element, i.e., they act on $d\omega$ in the same way that ∇_ξ and $\mathcal{L}_\xi$ act on constant functions.

10.4. Covariant differential operator preserving the invariant volume element

The variation of the invariant volume element $d\omega$ under the action of the covariant differential operator ∇_ξ,

$$\nabla_\xi(d\omega)=\frac{1}{2}\cdot\bar{g}[\nabla_\xi g]\cdot d\omega,$$

allows the introduction of a new covariant differential operator ${}^\omega\nabla_\xi$ preserving the invariant volume element.

Definition. The operator ${}^\omega\nabla_\xi$ is a *covariant differential operator preserving the invariant volume element* $d\omega$ along a contravariant vector field ξ:

$${}^\omega\nabla_\xi=\nabla_\xi-\frac{1}{2}\cdot\bar{g}[\nabla_\xi g].$$

The properties of ${}^\omega\nabla_\xi$ are determined by the properties of the covariant differential operator and the existence of a covariant metric tensor field g connected with its contravariant metric tensor field $\bar{g}$:

(a) Action on an invariant volume element $d\omega$:

$${}^\omega\nabla_\xi(d\omega)=0. \quad (191)$$

This follows from the definition of ${}^\omega\nabla_\xi$ and (188).

(b) Action on a contravariant basis vector field:

$${}^\omega\nabla_{\partial_j}\partial_i=\left(\Gamma^k_{ij}-\frac{1}{2}\cdot g^{\overline{lm}}\cdot g_{lm;j}\cdot g^k_i\right)\cdot\partial_k. \quad (192)$$

(c) Action on a covariant basis vector field:

$${}^\omega\nabla_{\partial_j}dx^i=\left(P^i_{kj}-\frac{1}{2}\cdot g^{\overline{lm}}\cdot g_{lm;j}\cdot g^i_k\right)\cdot dx^k. \quad (193)$$

(d) Action on a function f over M:

$${}^\omega\nabla_\xi f=\xi f-\frac{1}{2}\cdot\bar{g}[\nabla_\xi g]\cdot f,\quad f\in C^r(M),\quad r\geq 1. \quad (194)$$

If we introduce the abbreviations

$$Q_\beta=\bar{g}[\nabla_{e_\beta}g]=g^{\overline{\gamma\delta}}\cdot g_{\gamma\delta/\beta},\quad Q_j=\bar{g}[\nabla_{\partial_j}g]=g^{\overline{kl}}\cdot g_{kl;j}, \quad (195)$$

$$Q=Q_\beta\cdot e^\beta=Q_j\cdot dx^j, \quad (196)$$

$${}^\omega\Gamma^\gamma_{\alpha\beta}=\Gamma^\gamma_{\alpha\beta}-\frac{1}{2}\cdot g^\gamma_\alpha\cdot Q_\beta,\quad {}^\omega P^\alpha_{\gamma\beta}=P^\alpha_{\gamma\beta}-\frac{1}{2}\cdot g^\alpha_\gamma\cdot Q_\beta, \quad (197)$$

$$Q_\xi=\bar{g}[\nabla_\xi g]=Q_\beta\cdot\xi^\beta=Q_j\cdot\xi^j=2\cdot{}_c\theta_\xi, \quad (198)$$

then ${}^\omega\nabla_\xi$, (192), and (193) can be written in the form

$${}^\omega\nabla_\xi=\nabla_\xi-\frac{1}{2}\cdot Q_\xi, \quad (199)$$

$${}^\omega\nabla_{e_\beta}e_\alpha={}^\omega\Gamma^\gamma_{\alpha\beta}\cdot e_\gamma,\quad {}^\omega\nabla_{\partial_j}\partial_i={}^\omega\Gamma^k_{ij}\cdot\partial_k, \quad (200)$$

$${}^\omega\nabla_{e_\beta}e^\alpha={}^\omega P^\alpha_{\gamma\beta}\cdot e^\gamma,\quad {}^\omega\nabla_{\partial_j}dx^i={}^\omega P^i_{kj}\cdot dx^k. \quad (201)$$

The quantities ${}^\omega\Gamma^\gamma_{\alpha\beta}$ are called the components of the *contravariant affine connection* ${}^\omega\Gamma$ preserving the invariant volume element $d\omega$ in a noncoordinate basis, and ${}^\omega P^\gamma_{\alpha\beta}$ are called the components of the covariant affine connection ${}^\omega P$ preserving the invariant volume element $d\omega$ in a noncoordinate basis.

Since ${}^\omega\Gamma^\gamma_{\alpha\beta}$ and ${}^\omega P^\gamma_{\alpha\beta}$ differ from $\Gamma^\gamma_{\alpha\beta}$ and $P^\gamma_{\alpha\beta}$, respectively, by the components of a mixed tensor field $1/2\cdot g^\gamma_\alpha\cdot Q_\beta$ of rank 3, ${}^\omega\Gamma$ and ${}^\omega P$ will have the same transformation properties as the affine connections Γ and P, respectively.

The action of ${}^\omega\nabla_\xi$ on a contravariant vector field u can be written in the form

$${}^\omega\nabla_\xi u=\nabla_\xi u-\frac{1}{2}\cdot Q_\xi\cdot u. \quad (202)$$

If u is considered as a tangent vector to a curve $x^i(\tau)$, i.e.,

$$u=\frac{d}{dr}=u^\alpha\cdot e_\alpha=u^i\cdot\partial_i,\quad u^i=\frac{dx^i}{d\tau}, \quad (203)$$

$$u^\alpha=A_i{}^\alpha\cdot u^i=A_i{}^\alpha\cdot\frac{dx^i}{d\tau},\quad e_\alpha=A_\alpha{}^k\cdot\partial_k,\quad A_i{}^\alpha\cdot A_\alpha{}^k=g^k_i, \quad (204)$$

and the parameter τ is considered as a function of another parameter λ [with a one-to-one (injective) mapping between τ and λ], i.e.,

$$\tau=\tau(\lambda),\quad \lambda=\lambda(\tau), \quad (205)$$

$$u=\frac{d}{d\tau}=\frac{d\lambda}{d\tau}\cdot\frac{d}{d\lambda}=\frac{d\lambda}{d\tau}\cdot v,\quad v=\frac{d}{d\lambda},\tag{206}$$

then ${}^{\omega}\nabla_{\xi}u$ can be represented by means of the vector field v and $\nabla_{\xi}v$ in the form

$$^{\omega}\nabla_{\xi}u=\frac{d\lambda}{d\tau}\cdot\nabla_{\xi}v+\left[\xi\left(\frac{d\lambda}{d\tau}\right)-\frac{1}{2}\cdot Q_{\xi}\cdot\frac{d\lambda}{d\tau}\right]\cdot v.\tag{207}$$

If an additional condition for a relation between λ and τ is given in the form

$$\xi\left(\frac{d\lambda}{d\tau}\right)-\frac{1}{2}\cdot Q_{\xi}\cdot\frac{d\lambda}{d\tau}=0,\tag{208}$$

then for an arbitrary vector field ξ a solution for $\lambda=\lambda(\tau)$ exists in the form

$$\lambda=\lambda_0+\lambda_1\cdot\int\left[\exp\left(\frac{1}{2}\int Q_i\cdot dx^i\right)\right]\cdot d\tau,$$

$$Q_i=Q_i(x^k),\quad \lambda_0,\lambda_1=\text{const},\tag{209}$$

and the connection between ${}^{\omega}\nabla_{\xi}u$ and $\nabla_{\xi}v$ is obtained in the form

$$_{\omega}\nabla_{\xi}u=\frac{d\lambda}{d\tau}\cdot\nabla_{\xi}v=\left[\lambda_1\cdot\exp\left(\frac{1}{2}\int Q_i\cdot dx^i\right)\right]$$

$$\times\nabla_{\xi}v,\quad \lambda_1=\text{const}.\tag{210}$$

It follows from the last expression that there is a possibility of associating the action of ${}^{\omega}\nabla_{\xi}$ on a contravariant vector field u (as a tangential vector field to a given curve) with the action of ∇_{ξ} on the vector field v corresponding to the vector field u (obtained after changing the parameter of the curve). If the vector field v satisfies the condition for parallel transport along ξ induced by the covariant differential operator ∇_{ξ} ($\nabla_{\xi}v=0$), then the vector field u will also satisfy the parallel-transport condition along ξ induced by the covariant differential operator ${}^{\omega}\nabla_{\xi}$ (${}^{\omega}\nabla_{\xi}u=0$).

The action of ${}^{\omega}\nabla_{\xi}$ on a metric tensor field g can be represented in the form

$$^{\omega}\nabla_{\xi}g=\nabla_{\xi}g-\frac{1}{2}\cdot Q_{\xi}\cdot g.\tag{211}$$

After contraction of both components of ${}^{\omega}\nabla_{\xi}g$ with $\bar{g}$, i.e., for $\bar{g}[{}^{\omega}\nabla_{\xi}g]=g^{\bar{\alpha}\bar{\beta}}\cdot({}^{\omega}\nabla_{\xi}g)_{\alpha\beta}$, the equality

$$\bar{g}[{}^{\omega}\nabla_{\xi}g]=\left(1-\frac{n}{2}\right)\cdot Q_{\xi}\tag{212}$$

follows.

The trace-free part of ${}^{\omega}\nabla_{\xi}g$,

$$^{\omega}_{-}\nabla_{\xi}g={}^{\omega}\nabla_{\xi}g-\frac{1}{n}\cdot\bar{g}[{}^{\omega}\nabla_{\xi}g]\cdot g,\tag{213}$$

can be written by means of (212) in the form

$$^{\omega}_{-}\nabla_{\xi}g={}^{\omega}\nabla_{\xi}g+\frac{n-2}{2n}\cdot Q_{\xi}\cdot g.\tag{214}$$

Using this form, ${}^{\omega}\nabla_{\xi}g$ can be represented by means of its trace-free part and its trace part in the form

$$^{\omega}\nabla_{\xi}g={}^{\omega}_{-}\nabla_{\xi}g-\frac{n-2}{2n}\cdot Q_{\xi}\cdot g,\tag{215}$$

where $\bar{g}[{}^{\omega}_{-}\nabla_{\xi}g]=0$.

Special case: $\dim M=n=2$: ${}^{\omega}\nabla_{\xi}g=\nabla_{\xi}g$, $\bar{g}[{}^{\omega}\nabla_{\xi}g]=0$.
Special case: $\dim M=n=4$: ${}^{\omega}\nabla_{\xi}g=\nabla_{\xi}g-\frac{1}{4}\cdot Q_{\xi}\cdot g$.

The covariant differential operator preserving the invariant volume element *does not obey the Leibniz rule* when acting on a tensor product $Q\otimes S$ of two tensor fields Q and S:

$$^{\omega}\nabla_{\xi}(Q\otimes S)={}^{\omega}\nabla_{\xi}Q\otimes S+Q\otimes{}^{\omega}\nabla_{\xi}S+\frac{1}{2}\cdot Q_{\xi}\cdot Q\otimes S,$$

$$Q\in\otimes^{k}{}_{l}(M),\quad S\in\otimes^{m}{}_{r}(M).\tag{216}$$

10.5. Trace-free covariant differential operator. Weyl transport. Weyl space

The description of the gravitational interaction and its unification with other types of interactions over differentiable manifolds with affine connections and metric [(L_n,g) spaces] suggests[1] the introduction of an affine connection with a corresponding covariant differential operator ${}^{s}\nabla_{\xi}$ constructed by means of ∇_{ξ} and Q_{ξ} in the form

$$^{s}\nabla_{\xi}=\nabla_{\xi}-\frac{1}{n}\cdot Q_{\xi},\quad \dim M=n\cdot\tag{217}$$

The action of ${}^{s}\nabla_{\xi}$ on a covariant metric tensor field g is determined by

$$^{s}\nabla_{\xi}g=\nabla_{\xi}g-\frac{1}{n}\cdot Q_{\xi}\cdot g,\tag{218}$$

with the condition

$$\bar{g}[{}^{s}\nabla_{\xi}g]=0.\tag{219}$$

On the basis of this relation, the covariant differential operator ${}^{s}\nabla_{\xi}$ is called a *trace-free covariant differential operator.*

If the transport of g by the trace-free covariant differential operator ${}^{s}\nabla_{\xi}$ obeys the condition

$$^{s}\nabla_{\xi}g=0,\tag{220}$$

which is equivalent to the condition for $\nabla_{\xi}g$,

$$\nabla_{\xi}g=\frac{1}{n}\cdot Q_{\xi}\cdot g,\tag{221}$$

then the transport is called *Weyl transport.*

The covariant vector field [see (196)]

$$\bar{Q}=\frac{1}{n}\cdot Q\tag{222}$$

is called the *Weyl vector field.*

A differentiable manifold M ($\dim M=n$) with affine connection and metric, over which for every contravariant vector field $\xi\in T(M)$ the transport of g is a Weyl transport, is called *Weyl space with torsion* (*Weyl–Cartan space*) Y_n (Ref. 1).

The trace-free covariant differential operator ${}^s\nabla_\xi$ is connected with the covariant differential operator ${}^\omega\nabla_\xi$ preserving the invariant volume element $d\omega$ through the relation

$$ {}^\omega\nabla_\xi = {}^s\nabla_\xi - \frac{n-2}{2n}\cdot Q_\xi = \nabla_\xi - \frac{1}{2}\cdot Q_\xi . \tag{223}$$

The actions of the two operators ${}^\omega\nabla_\xi$ and ${}^s\nabla_\xi$ are identical if $\dim M = n = 2$ ($Q_\xi \neq 0$) or if $Q_\xi = 0$.

The components $\Gamma^\alpha_{\beta\gamma}$ of the affine connection Γ can be represented by means of the components of the affine connections corresponding to the operators ${}^\omega\nabla_\xi$ and ${}^s\nabla_\xi$.

The quantity $\nabla_{e_\beta} e_\alpha$ can be written in the form

$$\nabla_{e_\beta} e_\alpha = \frac{1}{2}(\nabla_{e_\alpha} e_\beta + \nabla_{e_\beta} e_\alpha - [e_\alpha, e_\beta]) - \frac{1}{2}\cdot T(e_\alpha, e_\beta), \tag{224}$$

corresponding to the representation of $\Gamma^\gamma_{\alpha\beta}$ in the form

$$\Gamma^\gamma_{\alpha\beta} = \frac{1}{2}(\Gamma^\gamma_{\alpha\beta} + \Gamma^\gamma_{\beta\alpha} - C_{\alpha\beta}{}^\gamma) - \frac{1}{2}\cdot T_{\alpha\beta}{}^\gamma$$
$$= \bar{\Gamma}^\gamma_{\alpha\beta} - \frac{1}{2}\cdot T_{\alpha\beta}{}^\gamma. \tag{225}$$

If we introduce the abbreviations

$$ {}^g\nabla_{e_\beta} e_\alpha = \frac{1}{2}(\nabla_{e_\alpha} e_\beta + \nabla_{e_\beta} e_\alpha - [e_\alpha, e_\beta]) = \bar{\Gamma}^\gamma_{\alpha\beta}\cdot e_\gamma, \tag{226}$$

$$ {}^s\nabla_{e_\beta} e_\alpha = Q_{\alpha\beta}{}^\gamma \cdot e_\gamma = \left(\Gamma^\gamma_{\alpha\beta} - \frac{1}{n}\cdot g^\gamma_\alpha\cdot Q_\beta\right)\cdot e_\gamma, \tag{227}$$

then

$$\nabla_{e_\beta} e_\alpha = {}^s\nabla_{e_\beta} e_\alpha + \frac{1}{n}\cdot Q_\beta \cdot e_\alpha, \tag{228}$$

$$\nabla_{e_\beta} e_\alpha = {}^g\nabla_{e_\beta} e_\alpha + \frac{1}{2}\cdot T(e_\beta, e_\alpha). \tag{229}$$

From (224), (226), and (228), it follows that

$$\nabla_{e_\beta} e_\alpha = \frac{1}{2}\left[{}^g\nabla_{e_\beta} e_\alpha + \frac{1}{2}\cdot T(e_\beta, e_\alpha) + \frac{1}{n}\cdot Q_\beta\cdot e_\alpha + {}^s\nabla_{e_\beta} e_\alpha\right]. \tag{230}$$

The last equality corresponds to the representation of $\Gamma^\gamma_{\alpha\beta}$ in the form

$$\Gamma^\gamma_{\alpha\beta} = \frac{1}{2}\left(\bar{\Gamma}^\gamma_{\alpha\beta} - \frac{1}{2}\cdot T_{\alpha\beta}{}^\gamma + \frac{1}{n}\cdot g^\gamma_\alpha\cdot Q_\beta + Q_{\alpha\beta}{}^\gamma\right). \tag{231}$$

Similarly, using the relations

$$ {}^\omega\nabla_{e_\beta} e_\alpha = {}^\omega\Gamma^\gamma_{\alpha\beta}\cdot e_\gamma = \left(\Gamma^\gamma_{\alpha\beta} - \frac{1}{2}\cdot g^\gamma_\alpha\cdot Q_\beta\right)\cdot e_\gamma$$
$$= \nabla_{e_\beta} e_\alpha - \frac{1}{2}\cdot Q_\beta\cdot e_\alpha, \tag{232}$$

$$\nabla_{e_\beta} e_\alpha = {}^\omega\nabla_{e_\beta} e_\alpha + \frac{1}{2}\cdot Q_\beta\cdot e_\alpha,$$

$$\nabla_{e_\beta} e_\alpha = {}^g\nabla_{e_\beta} e_\alpha + \frac{1}{2}\cdot T(e_\beta, e_\alpha),$$

one can obtain for $\nabla_{e_\beta} e_\alpha$ the expression

$$\nabla_{e_\beta} e_\alpha = \frac{1}{2}\left[{}^g\nabla_{e_\beta} e_\alpha + \frac{1}{2}\cdot T(e_\beta, e_\alpha) + {}^\omega\nabla_{e_\beta} e_\alpha + \frac{1}{2}\cdot Q_\beta\cdot e_\alpha\right]. \tag{233}$$

The last equality is equivalent to the representation of $\Gamma^\gamma_{\alpha\beta}$ in the form

$$\Gamma^\gamma_{\alpha\beta} = \frac{1}{2}\left(\bar{\Gamma}^\gamma_{\alpha\beta} - \frac{1}{2}\cdot T_{\alpha\beta}{}^\gamma + \frac{1}{2}\cdot g^\gamma_\alpha\cdot Q_\beta + {}^\omega\Gamma^\gamma_{\alpha\beta}\right). \tag{234}$$

From (228) and (229), the connection between ${}^g\nabla_{e_\beta} e_\alpha$ and ${}^s\nabla_{e_\beta} e_\alpha$ follows in the form

$$ {}^g\nabla_{e_\beta} e_\alpha = {}^s\nabla_{e_\beta} e_\alpha - \frac{1}{2}\cdot T(e_\beta, e_\alpha) + \frac{1}{n}\cdot Q_\beta\cdot e_\alpha, \tag{235}$$

which is equivalent to the connection between $\bar{\Gamma}^\gamma_{\alpha\beta}$ and $Q_{\alpha\beta}{}^\gamma$:

$$\bar{\Gamma}^\gamma_{\alpha\beta} = Q_{\alpha\beta}{}^\gamma + \frac{1}{2}\cdot T_{\alpha\beta}{}^\gamma + \frac{1}{n}\cdot g^\gamma_\alpha\cdot Q_\beta. \tag{236}$$

On the other hand, there is a connection between ${}^g\nabla_{e_\beta} e_\alpha$ and ${}^\omega\nabla_{e_\beta} e_\alpha$ in the form

$$ {}^g\nabla_{e_\beta} e_\alpha = {}^\omega\nabla_{e_\beta} e_\alpha - \frac{1}{2}\cdot T(e_\beta, e_\alpha) + \frac{1}{2}\cdot Q_\beta\cdot e_\alpha, \tag{237}$$

corresponding to the connection between $\bar{\Gamma}^\gamma_{\alpha\beta}$ and ${}^\omega\Gamma^\gamma_{\alpha\beta}$:

$$\bar{\Gamma}^\gamma_{\alpha\beta} = {}^\omega\Gamma^\gamma_{\alpha\beta} + \frac{1}{2}\cdot T_{\alpha\beta}{}^\gamma + \frac{1}{2}\cdot g^\gamma_\alpha\cdot Q_\beta. \tag{238}$$

10.6. Lie differential operator preserving the invariant volume element

The action of the Lie differential operator $\mathcal{L}_\xi$ on the invariant volume element $d\omega$,

$$\mathcal{L}_\xi(d\omega) = \frac{1}{2}\cdot \bar{g}[\mathcal{L}_\xi g]\cdot d\omega,$$

allows the construction of a new Lie differential operator preserving the invariant volume element $d\omega$.

Definition. ${}^\omega\mathcal{L}_\xi$:=Lie differential operator preserving the invariant volume element $d\omega$ along a contravariant vector field ξ:

$$ {}^\omega\mathcal{L}_\xi = \mathcal{L}_\xi - \frac{1}{2}\cdot\bar{g}[\mathcal{L}_\xi g].$$

The properties of ${}^\omega\mathcal{L}_\xi$ are determined by the properties of the Lie differential operator and the existence of a covariant metric tensor field g connected with a contravariant metric tensor field $\bar{g}$:

(a) Action on the invariant volume element $d\omega$:

$$ {}^\omega\mathcal{L}_\xi(d\omega) = 0. \tag{239}$$

This follows from the definition of ${}^{\omega}\mathcal{L}_\xi$ and (190).

(b) Action on a contravariant basis vector field:

$$
\begin{aligned}
{}^{\omega}\mathcal{L}_{e_\alpha} e_\beta &= \mathcal{L}_{e_\alpha} e_\beta - \frac{1}{2}\cdot \bar{g}[\mathcal{L}_{e_\alpha} g]\cdot e_\beta \\
&= \left(C_{\alpha\beta}{}^{\gamma} - \frac{1}{2}\cdot g^{\overline{\rho\sigma}}\cdot \mathcal{L}_{e_\alpha} g_{\rho\sigma}\cdot g^{\gamma}_{\beta}\right)\cdot e_\gamma, \qquad (240)
\end{aligned}
$$

$$
{}^{\omega}\mathcal{L}_{\partial_i}\partial_j = -\frac{1}{2}\cdot g^{\overline{kl}}\cdot \mathcal{L}_{\partial_i} g_{kl}\cdot \partial_j. \qquad (241)
$$

(c) Action on a covariant basis vector field:

$$
\begin{aligned}
{}^{\omega}\mathcal{L}_{e_\alpha} e^\beta &= \mathcal{L}_{e_\alpha} e^\beta - \frac{1}{2}\cdot \bar{g}[\mathcal{L}_{e_\alpha} g]\cdot e^\beta \\
&= k_{\gamma\alpha}{}^{\beta}\cdot e^\gamma - \frac{1}{2}\cdot \bar{g}[\mathcal{L}_{e_\alpha} g]\cdot e^\beta, \qquad (242)
\end{aligned}
$$

$$
{}^{\omega}\mathcal{L}_{\partial_i} dx^j = k_{mi}{}^{j}\cdot dx^m - \frac{1}{2}\cdot \bar{g}[\mathcal{L}_{\partial_i} g]\cdot dx^j. \qquad (243)
$$

(d) Action on a function f:

$$
{}^{\omega}\mathcal{L}_\xi f = \xi f - \frac{1}{2}\cdot \bar{g}[\mathcal{L}_\xi g]\cdot f, \quad f\in C^r(M), \quad r\geq 1. \qquad (244)
$$

If we introduce the abbreviations

$$
P_\beta = \bar{g}[\mathcal{L}_{e_\beta} g] = g^{\overline{\gamma\delta}}\cdot \mathcal{L}_{e_\beta} g_{\gamma\delta}, \qquad (245)
$$

$$
P_j = \bar{g}[\mathcal{L}_{\partial_j} g] = g^{\overline{kl}}\cdot \mathcal{L}_{\partial_j} g_{kl}, \qquad (246)
$$

$$
P = P_\beta\cdot e^\beta = P_j\cdot dx^j, \qquad (247)
$$

$$
P_\xi = \bar{g}[\mathcal{L}_\xi g] = 2\cdot {}_l\theta_\xi, \qquad (248)
$$

$$
\hat{C}_{\alpha\beta}{}^{\gamma} = C_{\alpha\beta}{}^{\gamma} - \frac{1}{2}\cdot P_\alpha\cdot g^{\gamma}_{\beta}, \quad \hat{C}_{\alpha\beta}{}^{\gamma} \neq -\hat{C}_{\beta\alpha}{}^{\gamma}, \qquad (249)
$$

then ${}^{\omega}\mathcal{L}_\xi$ and (240)–(244) can be written in the form

$$
{}^{\omega}\mathcal{L}_\xi = \mathcal{L}_\xi - \frac{1}{2}\cdot P_\xi, \qquad (250)
$$

$$
\begin{aligned}
{}^{\omega}\mathcal{L}_{e_\alpha} e_\beta &= \mathcal{L}_{e_\alpha} e_\beta - \frac{1}{2}\cdot P_\alpha\cdot e_\beta \\
&= \left(C_{\alpha\beta}{}^{\gamma} - \frac{1}{2}\cdot g^{\gamma}_{\beta}\cdot P_\alpha\right)\cdot e_\gamma = \hat{C}_{\alpha\beta}{}^{\gamma}\cdot e_\gamma, \qquad (251)
\end{aligned}
$$

$$
{}^{\omega}\mathcal{L}_{\partial_i}\partial_j = -\frac{1}{2}\cdot P_i\cdot \partial_j, \qquad (252)
$$

$$
{}^{\omega}\mathcal{L}_{e_\alpha} e^\beta = \mathcal{L}_{e_\alpha} e^\beta - \frac{1}{2}\cdot P_\alpha\cdot e^\beta = k_{\gamma\alpha}{}^{\beta}\cdot e^\gamma - \frac{1}{2}\cdot P_\alpha\cdot e^\beta, \qquad (253)
$$

$$
{}^{\omega}\mathcal{L}_{\partial_i} dx^j = k_{mi}{}^{j}\cdot dx^m - \frac{1}{2}\cdot P_i\cdot dx^j, \qquad (254)
$$

$$
{}^{\omega}\mathcal{L}_\xi f = \xi f - \frac{1}{2}\cdot P_\xi\cdot f, \quad f\in C^r(M), \quad r\geq 1\cdot \qquad (255)
$$

The commutator of two Lie differential operators preserving $d\omega$ has the following properties:

(a) Action on a function f:

$$
\begin{aligned}
[{}^{\omega}\mathcal{L}_\xi, {}^{\omega}\mathcal{L}_u] f &= (\mathcal{L}_\xi u) f + \frac{1}{2}(u P_\xi - \xi P_u) f \\
&= \left[\mathcal{L}_\xi u + \frac{1}{2}(u P_\xi - \xi P_u)\right] f,
\end{aligned}
$$

$$
f\in C^r(M), \quad r\geq 2. \qquad (256)
$$

(b) Action on a contravariant vector field:

$$
\begin{aligned}
[{}^{\omega}\mathcal{L}_\xi, {}^{\omega}\mathcal{L}_u] v &= [\mathcal{L}_\xi, \mathcal{L}_u] v + \frac{1}{2}(u P_\xi - \xi P_u) v \\
&= \left\{ [\mathcal{L}_\xi, \mathcal{L}_u] + \frac{1}{2}(u P_\xi - \xi P_u)\right\} v,
\end{aligned}
$$

$$
\xi, u, v \in T(M). \qquad (257)
$$

(c) The Jacobi identity

$$
\begin{aligned}
\langle [[{}^{\omega}\mathcal{L}_\xi, {}^{\omega}\mathcal{L}_u], {}^{\omega}\mathcal{L}_v]\rangle &= [[{}^{\omega}\mathcal{L}_\xi, {}^{\omega}\mathcal{L}_u], {}^{\omega}\mathcal{L}_v] \\
&+ [[{}^{\omega}\mathcal{L}_v, {}^{\omega}\mathcal{L}_\xi], {}^{\omega}\mathcal{L}_u] \\
&+ [[{}^{\omega}\mathcal{L}_u, {}^{\omega}\mathcal{L}_v], {}^{\omega}\mathcal{L}_\xi] \equiv 0. \qquad (258)
\end{aligned}
$$

The different types of differential operators acting on the invariant volume element can be used to describe different physical systems and interactions over a differentiable manifold with affine connections and metric interpreted as a model of the space-time.

11. CONCLUSIONS

The main conclusions following from our results can be grouped together in the following statements:

A contraction operator S, commuting with the covariant differential operator and with the Lie differential operator, for which the affine connection P determined by the covariant differential operator for covariant tensor fields is different (not only in sign) from the affine connection Γ determined by the covariant differential operator for contravariant tensor fields can be introduced over every differentiable manifold. The components (in a coordinate or in a noncoordinate basis) of the two affine connections P^i_{jk} and Γ^i_{jk} differ from each other by the components $g^i_{j;k}$ of the covariant derivatives of the Kronecker tensor. At least three cases can be distinguished:

(a) $g^i_{j;k} := 0$: $P^i_{jk} + \Gamma^i_{jk} = 0$ [P^i_{jk} differs only in sign from Γ^i_{jk} (canonical case: $S := C$)].

(b) $g^i_{j;k} := \varphi_{,k}\cdot g^i_j$: $P^i_{jk} + \Gamma^i_{jk} = \varphi_{,k}\cdot g^i_j$, $\varphi\in C^r(M)$ [P^i_{jk} differs from Γ^i_{jk} by the derivative of an invariant function $\varphi\in C^r(M)$, $r\geq 2$, along a basis vector field (∂_k or e_k), and the components of the Kronecker tensor in the given basis].

(c) $g^i_{j;k} = g^i_{jk}$: $P^i_{jk} + \Gamma^i_{jk} = q^i_{jk}$, $q\in \otimes^1{}_2(M)$ [P^i_{jk} differs from Γ^i_{jk} by the covariant derivative $g^i_{j;k}$ of the Kronecker tensor along a basis vector field ∂_k (or e_k)].

In cases (b) and (c) the Lie derivatives of covariant tensor fields depend also on structures determined by the affine connections, in contrast to case (a), where the covariant derivative and the Lie derivative of covariant tensor fields are independent of each other's structures (although the Lie derivatives can be expressed by means of the covariant derivatives).

On the basis of our results,[49–51] the kinematics of vector fields has been analyzed.[28,52–55] The Lagrangian theory for tensor fields has been considered[56] and applied to the Einstein theory of gravitation as a special case of a Lagrangian theory of tensor fields[57,58] over V_n spaces ($n=4$), as well as to Einstein's theory of gravitation over $\bar{V}_4$ spaces.[59]

The author is grateful to Prof. Dr. N. A. Chernikov, Dr. N. S. Shavokhina, and Dr. A. B. Pestov from the Joint Institute for Nuclear Research, Dubna (Russia) for helpful discussions; Prof. Dr. St. Dimiev from the Institute for Mathematics and Informatics of the Bulgarian Academy of Sciences and Prof. Dr. K. Sekigawa from the Department of Mathematics of the Faculty of Science at Niigata University (Japan) for their support of the considered topics; and Prof. D. I. Kazakov for his kind hospitality during my stay at the JINR in Dubna.

This work was supported in part by the National Science Foundation of Bulgaria under Grants Nos. F-103, F-498, and F-642.

*Permanent address: Bulgarian Academy of Sciences, Institute for Nuclear Research and Nuclear Energy, Department of Theoretical Physics, Blvd. Tzarigradsko Chaussee 72, 1784 Sofia, Bulgaria. E-mail address: smanov@inrne.bas.bg.

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