Variational expansions in quantum chromodynamics[1)]

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The results obtained in the last few years using nonperturbative variational expansions in quantum chromodynamics are reviewed. The review begins with examples which explain the method of constructing variational series in quantum field theory, along with how the properties of their convergence can be controlled by special parameters. The variational perturbation theory for quantum chromodynamics is then formulated, and a nonperturbative expansion in a new small parameter is constructed. Various phenomenological applications of this approach are discussed. © *1999 American Institute of Physics.* [S1063-7796(99)00105-9]

1. INTRODUCTION

The theoretical foundation of quantum field theory for performing calculations involving only Lagrangian parameters is perturbation theory. Its use along with the renormalization procedure allows important results to be obtained in quantum electrodynamics, in the theory of electroweak interactions, and in the description of the perturbative region of quantum chromodynamics. However, the specific features of quantum field theory are such that a sufficiently complete study of the structure of a quantum field model within the framework of perturbation theory is not possible, even in theories with a small coupling constant. This is the case, in particular, for the modern theory of strong interactions, quantum chromodynamics. There, nonperturbative effects play the decisive role, both in understanding fundamental phenomena such as quark and gluon confinement, and in describing hadron phenomenology and the correspondence between theoretical results and experimental data.

The development of nonperturbative methods in quantum field theory has received a great deal of attention. The spectrum of such studies is very broad, and in the literature one can find quite a variety of approaches to the problem of going beyond perturbation theory. The goal of the present study is to review the results obtained in recent years using so-called variational or "floating" expansions in quantum chromodynamics. The basic idea behind variational series is quite simple. In order to explain it, let us first recall how a perturbative expansion is constructed.

In the usual version of perturbation theory, the total action corresponding to a physical system is split into a free part and a part describing the interaction. The latter is treated as a perturbation, and the coupling constant entering into it is viewed as the small expansion parameter. As a rule, this treatment leads to asymptotic series which, albeit not "well behaved," are nevertheless widely used in physics and allow useful information about the system in question to be extracted for weak coupling. As the interaction constant grows, the perturbation series becomes worse and worse. The reason for this is understood: now the treatment of the interaction term as a perturbation of the free system is no longer adequate, since the physical system in question has properties far from those of a free system. In this case it is necessary to split the total action in a different way, such that the new "interaction term" can be treated as a perturbation not only when the coupling constant is small, but for a wider range of its values. Of course, here one must also worry about whether this procedure, which is similar to ordinary perturbation theory, allows the possibility of calculating corrections.

How is it possible to seek a functional which can be used as a perturbation with more justification than the usual interaction term? One possibility, realized in variational perturbation theory (VPT), is to probe the system by using a variational type of functional to study the system's response to a change of the trial parameters. Here it turns out to be convenient to apply the functional-integration formalism, which in field theory is widely used both to study general questions[1,2] and to find various approximations.[3–6] In spite of the term "perturbation" appearing in the name of this approach, the VPT method is nonperturbative, since it is not essentially based on the use of the coupling constant as the small expansion parameter. In the VPT method, a given quantity can be approximated by constructing series, different from those of ordinary perturbation theory, which allow quantum systems to be studied not only in the weak-coupling region, but also far beyond it (see the reviews devoted to this subject,[7,8] in which mainly quantum-mechanical systems and scalar models of field theory have been studied). The possibility of constructing expansions different from perturbative ones and having different convergence properties has been pointed out in Ref. 9. The case of the quantum-mechanical anharmonic oscillator has been studied in Refs. 10 and 11. The use of the functional-integration method to describe quantum systems on the basis of expansions of this type was proposed in Refs. 12–14. The method of variational probing of a system, which is based on the functional-integration formalism and admits a natural generalization to the case of quantum field theory, was proposed in Ref. 15. The development of this

approach in Refs. 16 and 17, and also in later studies,[18–26] has demonstrated the effectiveness of the VPT method for studying quantum field models.

At the present time there are different versions of the variational method (see, for example, the early studies[27,28] and also Refs. 29–32) which can be used to perform nonperturbative calculations. One of them is the method of Gaussian effective potential,[33–36] which allows the evaluation of the effective potential, an important energy characteristic of a field-theoretical model,[37] without using the usual loop expansion. The use of the VPT method allows the Gaussian effective potential to be obtained in the first nontrivial order of the variational expansion for various ways of choosing the trial functional.[21] The possibility of performing a nonperturbative calculation of the effective potential allows the study of questions such as the existence of phase transitions in a theory,[38–40], renormalization outside perturbation theory, and the triviality of the φ^4 model in four dimensions.[41–44]

One serious problem of many variational methods is the difficulty of evaluating the accuracy and stability of the results obtained by the variational procedure. The reason is that the formulation of the method often does not contain an algorithm for calculating the needed corrections. As a result, it becomes difficult to determine the degree to which the so-called main contribution found by the variational method corresponds to the object under study, in particular, whether or not the object is directly related to some energy characteristic, and in what region the results are applicable. The VPT method allows *ab initio* the determination of an algorithm for calculating corrections, so that the effect of corrections on the main contribution can be studied. Moreover, the VPT series is not a strict construction specified once and for all. Special parameters characterizing the variational probe allow the convergence properties of the VPT expansion to be controlled. Series of this type whose convergence properties can be influenced by varying special parameters are referred to as variational or floating series. In contrast to the asymptotic expansions characteristic of perturbation theory, the VPT approach allows the construction in some cases of approximating series with a finite region of convergence.[16,17,19,22] There is also the interesting possibility of constructing Leibniz series, which allow upper and lower bounds on the quantity of interest to be estimated, using only the first few terms of the series. The control parameters can be used to optimize such estimates.[7]

Modernizations of the ordinary perturbative expansion in which the original interaction potential is taken as the perturbation are receiving a great deal of attention. For example, in studying systems with singular potentials, the usual perturbation theory proves to be poorly adapted for an adequate description.[45] This is the case because the asymptotic behavior of the free wave functions differs essentially from that of the exact solutions. The situation can be improved if the singular part of the potential is taken into account exactly and the perturbative expansion is constructed on the basis of its regular part.[45] Among the approaches allowing one to go beyond the perturbative region, mention should also be made of the method of the linear δ expansion,[35,46,47] the method of self-similar interpolations,[48] and the nonperturbative approach studied in Ref. 49. An important feature of the VPT method discussed in the present review is that it naturally combines an optimization procedure and a regular method of calculating corrections.

In this review we shall study the method of variational expansions and its application to problems in quantum chromodynamics. In this case the VPT method can be used as the basis for constructing an expansion parameter which is smaller than unity for any value of the original coupling constant.[50] First we shall explain the method of constructing this small expansion parameter, using a simple model which allows us to describe the idea behind the method more clearly. We shall show that the new expansion not only allows a considerable extension of the "lifetime" of perturbation theory and makes it possible to obtain a good-quality approximation for larger values of the coupling constant, but also offers the possibility of analyzing the strong-coupling limit. In the case of quantum chromodynamics this method allows a unified treatment of both the traditional perturbative region and the region beyond it.[51,52]

The renormalization-group resummation of the perturbative expansion gives rise to an invariant charge which possesses unphysical features like a ghost pole in the one-loop approximation. The inclusion of higher-order corrections does not eliminate this difficulty, but leads only to additional unphysical cuts in the complex Q^2 plane. A possible solution of this problem within the perturbative approach consisting of the imposition of an additional analyticity requirement following from the Källén–Lehmann representation was proposed in Ref. 53, and within the context of the renormalization-group method in Ref. 54. The analytic approach was developed for the case of quantum chromodynamics in Refs. 55 and 56, where new, interesting features of this treatment were discovered.

An important feature of the VPT approach is the fact that its use can ensure the correct analytic properties of the running expansion parameter, which reflect the general principles of local quantum field theory.[1,57,58] In particular, the preservation of these analytic properties allows the self-consistent determination of the running parameter in the time-like region[59] and a consistent description of inclusive τ-lepton decay[60] (see also Refs. 61–64). The presence of an infrared fixed point corresponding to the VPT expansion parameter is quite consistent with the low-energy, smeared experimental data on e^+e^- annihilation into hadrons,[65] which can be obtained by a special procedure for smoothing resonances. A generalization to the massive case using the renormalization scheme with subtraction at some Euclidean point has been studied in Refs. 66–68. Along with these questions, we shall also review some other applications of the method. In particular, we shall study the possibility of calculating the renormalon contribution and its role in describing semileptonic inclusive τ decay[69] (see also Refs. 70 and 71), and also the use of the variational approach to describe the mass spectrum of heavy quarkonia on the basis of the sum-rule method of quantum chromodynamics.[72]

This review is organized as follows. In the next section we shall study the main ideas on which the method of variational perturbation theory is based. Using a simple example,

we shall show how to construct a new small expansion parameter, and how the expansion in this parameter works in the nonperturbative region of large coupling constant. The method of the a expansion will then be formulated for quantum chromodynamics, and its correspondence to the representations arising in the potential model of quark confinement will be studied, along with the problem of the stability of the results. A number of phenomenological applications of the method such as low-energy e^+e^- annihilation into hadrons, inclusive τ-lepton decay, and the description of the mass spectrum of heavy quarkonia will be studied in the following sections of the review. The results are discussed in the Conclusion, and in the appendices we give further explanatory material.

2. VARIATIONAL SERIES: BASIC IDEAS

It is convenient to use the functional-integral formalism for constructing variational expansions in quantum field theory. This formalism was developed in the course of generalizing the path-integral formulation of quantum mechanics,[2)] first given by Feynman in 1948.[73] The development of the functional formulation of quantum field theory began with the studies of Bogolyubov,[76] Matthews and Salam,[77] Gel'fand and Minlos,[78] Khalatnikov,[79] and Fradkin.[80] The functional-integration technique is now one of the most widely used methods in quantum field theory. It has proved effective not only for solving such problems as the quantization of gauge theories,[81] but also in developing various approximation methods allowing the nonperturbative analysis of field-theory models. For example, it can be used as the basis for studying various asymptotic regimes in quantum field models, such as the behavior of the Green functions in the infrared region,[3] and for the development of the eikonal approximation[5,6,82] and the method of Gaussian effective potential.[33,34,36]

In the functional-integral formalism, calculations in quantum field theory are performed using Gaussian functional quadratures of the form

$$\int D\varphi \exp\left\{-\left[\frac{1}{2}\langle\varphi\hat{K}\varphi\rangle+\langle\varphi J\rangle\right]\right\} = \left[\det\frac{\hat{K}}{-\partial^2+m^2}\right]^{-1/2}\exp\left[\frac{1}{2}\langle J\hat{K}^{-1}J\rangle\right]. \tag{2.1}$$

Such Gaussian integrals are used to construct perturbation series, in semiclassical analyses, and for estimating functional integrals by the method of steepest descents. The Gaussian quadrature (2.1) can be used as the basis of the functional-integral formalism in quantum field theory. The method of variational perturbation theory is also based on Gaussian functional quadratures of the type (2.1).

The construction of variational expansions. To be specific, let us consider the φ^4 model in Euclidean space. We construct the VPT series for the 2ν-point Green function:

$$G_{2\nu}=\int D\varphi\{\varphi^{2\nu}\}\exp(-S[\varphi]), \tag{2.2}$$

where

$$\{\varphi^{2\nu}\}=\varphi(x_1)\cdots\varphi(x_{2\nu}),$$

and the action functional has the form

$$S[\varphi]=S_0[\varphi]+\frac{m^2}{2}S_2[\varphi]+\lambda S_4[\varphi],$$

$$S_0[\varphi]=\frac{1}{2}\int dx(\partial\varphi)^2,\quad S_p[\varphi]=\int dx\varphi^p. \tag{2.3}$$

The integration measure in (2.2) is normalized so that

$$\int D\varphi\exp\left(-S_0[\varphi]-\frac{m^2}{2}S_2[\varphi]\right)=1. \tag{2.4}$$

We write down the VPT expansion of the Green function (2.2), introducing the variational functional $\tilde{S}[\varphi]$, in the form

$$G_{2\nu}=\sum_{n=0}^{\infty}G_{2\nu,n}, \tag{2.5}$$

where the elements of the VPT series (2.5) are represented as a functional integral:

$$G_{2\nu,n}=\frac{(-1)^n}{n!}\int D\varphi\{\varphi^{2\nu}\}(\lambda S_4[\varphi]-\tilde{S}[\varphi])^n \times\exp\left(-S_0[\varphi]-\frac{m^2}{2}S_2[\varphi]-\tilde{S}[\varphi]\right). \tag{2.6}$$

The original Green function (2.2) and, consequently, the full sum of the series in (2.5) do not depend on the type of variational probe $\tilde{S}[\varphi]$. This functional can be fairly arbitrary, its form being restricted only by the requirement that the action in the exponent in (2.6) be positive-definite. However, the "principle of calculability" significantly restricts the arbitrariness in the choice of the variational addition $\tilde{S}[\varphi]$. We should actually require that the functional integral (2.6) be a Gaussian, or reduce to one by means of some transformation, so that in the end it is possible to use the definition (2.1). Therefore, not only variational additions with functionals quadratic in the fields are allowed, but also, for example, ones with functionals admitting the use of Gaussian quadrature after Fourier transformation:

$$F(A[\varphi])=\int_{-\infty}^{\infty}dx\int_{-\infty}^{\infty}\frac{dp}{2\pi}F(p)\exp[\pm i(A[\varphi]-p)x], \tag{2.7}$$

where $A[\varphi]$ is a functional quadratic in the fields.

Let us choose the VPT functional $\tilde{S}[\varphi]$ in the form of a sum of a harmonic term quadratic in the fields and a functional of the anharmonic type allowing the degree to be lowered by means of the transformation (2.7):

$$\tilde{S}[\varphi]=\frac{M^2}{2}S_2[\varphi]+\theta^2S_2^2[\varphi]. \tag{2.8}$$

The variational parameters M and θ in (2.8) are then fixed on the basis of some optimization procedure. The passage to a Gaussian quadrature is effected by the transformation

$$\exp(-\theta^2 S_2^2[\varphi]) = \int_{-\infty}^{\infty} \frac{du}{2\sqrt{\pi}} \exp\left\{-\frac{u^2}{4} \pm iu\theta S_2[\varphi]\right\}. \tag{2.9}$$

As a result, the VPT series (2.6) for the Green function in question is written as

$$G_{2\nu,n} = \sum_{k=0}^{n} \sum_{l=0}^{n-k} \frac{1}{l!k!(n-k-l)!} \int D\varphi\{\varphi^{2\nu}\} \times (-\lambda S_4[\varphi])^k \theta^{2l} (M^2-m^2)^{n-k-l} \left(\frac{S_2[\varphi]}{2}\right)^{n+l-k} \times \exp\left\{-\left(S_0[\varphi] + \frac{m^2}{2} S_2[\varphi] + \theta^2 S_2^2[\varphi]\right)\right\}. \tag{2.10}$$

It is convenient to represent the expression $(S_2[\varphi]/2)^{n+l-k}$ as a differential operator in the parameter M^2, making the replacement

$$\left(\frac{S_2[\varphi]}{2}\right)^{n+l-k} \to \left(-\frac{\partial}{\partial M^2}\right)^{n+l-k} \tag{2.11}$$

Then the pre-exponential factor in (2.10) contains not only $\{\varphi^{2\nu}\}$, but also the usual factor $(-\lambda S_4[\varphi])^k$ arising also in perturbation theory and generating ordinary vertices in the graphs. Owing to the change of the quadratic form in the exponent, which amounts to a change of the mass parameter $\chi^2 = M^2 + iu\theta$, the propagator is modified and (2.10) takes the form

$$G_{s2\nu,n} = \sum_{k=0}^{n} \sum_{l=0}^{n-k} \frac{1}{l!(n-k-1)!} \int_{-\infty}^{\infty} \frac{du}{2\sqrt{\pi}} \exp\left(-\frac{u^2}{4}\right) \times \theta^{2l} (M^2-m^2)^{n-k-l} \left(-\frac{\partial}{\partial M^2}\right)^{n+l-k} \tilde{g}_{2\nu}^{(k)}(\chi^2), \tag{2.12}$$

where

$$\tilde{g}_{2\nu,n}^{(k)}(\chi^2) = \frac{1}{k!} \int D\varphi\{\varphi^{2\nu}\}(-\lambda S_4[\varphi])^k \times \exp\left\{-\left[S_0[\varphi] + \frac{\chi^2}{2} S_2[\varphi]\right]\right\}. \tag{2.13}$$

Using (2.1), Eq. (2.13) can be rewritten as

$$\tilde{g}_{2\nu,n}^{(k)}(\chi^2) = \det\left[-\frac{\partial^2+\chi^2}{-\partial^2+m^2}\right]^{-1/2} \cdot g_{2\nu,n}^{(k)}(\chi^2), \tag{2.14}$$

where $g_{2\nu,n}^{(k)}(\chi^2)$ is represented as a set of kth-order graphs with the modified propagator

$$\Delta(p,\chi^2) = \frac{1}{p^2+\chi^2}. \tag{2.15}$$

Thus, to construct the Nth order of the VPT expansion we can use the graphs of perturbation theory up to order N with modified propagators and, in general, vertices. Here the structure of the VPT series is considerably different from that of perturbation theory. The possibility of using the standard diagrammatic technique, which is ensured by the method of constructing the VPT expansions, is important from the technical point of view, since it means that the results obtained in perturbation theory can be used.

Nonperturbative expansion in a small parameter. In order to demonstrate the key features of the introduction of a small nonperturbative expansion parameter, let us consider a simple example. We define the "vacuum functional" [3] of the zero-dimensional φ^4 theory,

$$W(g) = \int_{-\infty}^{\infty} dx \exp(-S[x]), \tag{2.16}$$

with the "action functional" $S[x]$ written as the sum of the free action $S_0[x]$ and the interaction part $S_I[x]$:

$$S[x] = S_0[x] + S_I[x] = x^2 + gx^4. \tag{2.17}$$

As before, we shall be interested in Gaussian quadratures of the form

$$\int dx P(x) \exp(-ax^2), \tag{2.18}$$

where $P(x)$ is a polynomial.

The standard method is to expand $\exp(-S[x])$ in powers of the coupling constant g. Here it is natural to use Gaussian integrals (2.18), and we obtain the usual perturbation series

$$W(g) = \sum_{k=0}^{\infty} \omega_k \tag{2.19}$$

with the coefficients

$$\omega_k = \frac{1}{k!} \int_{-\infty}^{\infty} dx(-gx^4)^k \exp(-S_0[x]). \tag{2.20}$$

Owing to the asymptotic nature of the expansion (2.19), the function $W(g)$ cannot be uniquely reconstructed from the series (2.19) if we deal only with the series (2.19) and do not take into account any additional information about the function $W(g)$. For example, the function $W(g)+\exp(-1/g)$ will have the same series (2.19), but it behaves completely differently in the nonperturbative region. Of course, in this very simple example it is not difficult, using the integral representation (2.16), to find additional conditions which allow the correct formulation of the problem of summing the perturbative expansion. However, we recall that serious difficulties arise when we attempt to do this in the case of field theory, where the additional information needed to sum uniquely the perturbation series can be found only for several simple one- and two-dimensional models. In some sense the problem of obtaining such information is in many respects equivalent to solving the strong-coupling problem. However, the use of Gaussian integrals (2.18) offers greater possibilities than simply obtaining the perturbation series. Here we shall show how a nonperturbative small expansion parameter can be constructed by using, as for perturbation theory, Gaussian quadratures.

We rewrite the original action (2.17) in the form

$$S[x] = S_0'[x] + S_I'[x], \tag{2.21}$$

where

$$S_0'[x] = \zeta^{-1} x^2 \tag{2.22}$$

and

$$S_I'[x]=gx^4-(\zeta^{-1}-1)x^2. \tag{2.23}$$

The total action and the original quantity $W(g)$ are independent of the introduced parameter ζ, but such dependence does arise when $W(g)$ is approximated by a finite number of terms of the VPT series. However, it is possible to use the freedom in choosing ζ to construct a new expansion parameter. We shall also study the question of the degree to which the new expansion allows us to move into the nonperturbative region.

It is clear from the start that if it turns out that the optimal parameter ζ is smaller than unity, then in writing the action in the form (2.21) with the components (2.22) and (2.23) we at least have the possibility of improving the perturbative expansion, since the new interaction part (2.23) can be viewed as a perturbation for a larger set of field configurations. In addition, it is known that for fixed variational parameter the VPT series remains asymptotic, although, in contrast to perturbation theory, it allows $W(g)$ to be approximated for large values of the coupling constant. We have already mentioned that an important role is played here by the principle of induced convergence, which allows a significant extension of the range of possible values of the coupling constant.

Expansion in the interaction term (2.23) leads to a VPT series for the original quantity (2.16):

$$W(g)=\sum_{n=0}^{\infty} W_n, \tag{2.24}$$

where

$$W_n=\frac{1}{n!}\int dx(-S_I'[x])^N\exp(-S_0'[x]) \tag{2.25}$$

$$=\sum_{k=0}^{n}\frac{1}{(n-k)!k!}\int dx(-gx^4)^k$$

$$\times[(\zeta^{-1}-1)x^2]^{n-k}\exp(-S_0'[x]).$$

As before, it is convenient to introduce the auxiliary parameter κ, writing the free action as

$$S_0'=\zeta^{-1}x^2\Rightarrow[1+\kappa(\zeta^{-1}-1)]x^2 \tag{2.26}$$

and setting $\kappa=1$ at the end of all the calculations. In this case any power of the expression $[(\zeta^{-1}-1)x^2]$ in (2.25) can be obtained by means of the corresponding number of differentiations with respect to the parameter κ. After this, the expression $(-gx^4)^k$ remaining in the pre-exponential factor in (2.25) leads to the standard graphs, but with modified propagator:

$$\Delta=\frac{1}{1+\kappa(\zeta^{-1}-1)}. \tag{2.27}$$

For $\kappa=1$ the propagator is $\Delta=\zeta$.

As a result, the terms of the VPT series can be written as

$$W_n=\sum_{k=0}^{n}\frac{1}{(n-k)!}\left(-\frac{\partial}{\partial\kappa}\right)^{n-k}\omega_k, \tag{2.28}$$

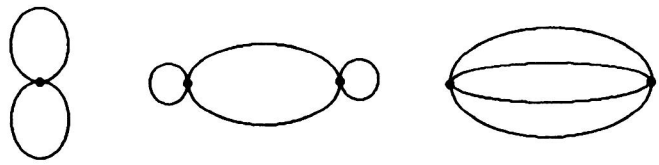

FIG. 1. Lower-order graphs illustrating the relation (2.31) between the numbers of internal lines and vertices.

where the quantities

$$\omega_k=\frac{1}{k!}\int dx(-gx^4)^k\exp(-x\Delta^{-1}x) \tag{2.29}$$

are calculated by using the ordinary diagrammatic technique with the propagator (2.27).

Let us consider the structure of (2.28). First of all, we note that differentiation with respect to the parameter κ leads to the appearance of an additional factor $(1-\zeta)$. In fact, it follows from (2.27) that

$$\frac{1}{m!}\left(-\frac{\partial}{\partial\kappa}\right)^m\Delta(\kappa=1)=(1-\zeta)^m\Delta(\kappa=1). \tag{2.30}$$

We also note that in this case for any graph with I internal lines and V vertices we have[4)]

$$I=2V. \tag{2.31}$$

The lowest orders of this equation are illustrated in Fig. 1.

An internal line of a graph corresponds to a propagator and therefore leads to a factor ζ. A vertex gives a factor g, and one differentiation with respect to κ gives a factor $(1-\zeta)$. Therefore, we can schematically write

$$W_n\sim(g\zeta^2)^n+(1-\zeta)(g\zeta^2)^{n-1}+\cdots$$

$$+(1-\zeta)^{n-1}(g\zeta^2)+(1-\zeta)^n. \tag{2.32}$$

It is clear from this expression that if $(1-\zeta)$ is chosen to be proportional to $g\zeta^2$, the expression for W_n will contain an overall factor $a^n\equiv(1-\zeta)^n$, and $a\equiv(1-\zeta)$ will serve as the expansion parameter in the series of variational perturbation theory. Therefore, the requirement that a single expansion parameter exist dictates the following equation for ζ:

$$1-\zeta=Cg\zeta^2 \tag{2.33}$$

with some positive constant C. From this we obtain the equation relating the parameter a of the VPT expansion to the original coupling constant:

$$g=\frac{1}{C}\frac{a}{(1-a)^2}. \tag{2.34}$$

It is easily seen from (2.34) that for any positive value of the original coupling constant g, the new expansion parameter a satisfies the condition

$$0\leq a<1. \tag{2.35}$$

The remaining arbitrariness is concentrated in the parameter C, which can be determined by some optimization procedure; such procedures are described in Ref. 7.

Here we shall consider another optimization method, in which it is assumed that certain information of an "experi-

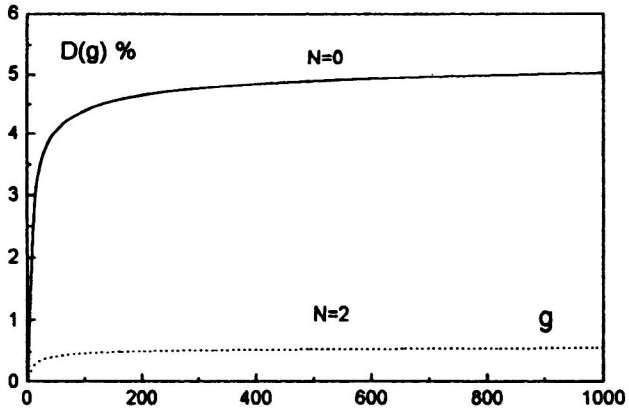

FIG. 2. Relative error (in percent) of the approximation of (2.16) in the lowest orders of variational perturbation theory.

mental'' nature can be used. This method is convenient in quantum chromodynamics, where ''normalization'' can be done by using experimental information. We assume that we know the ''experimental'' value of the function $W(g)$ at some normalization point g_0:

$$W(g_0)=W_{\exp}. \tag{2.36}$$

Equation (2.36) or, more generally, minimization of the modulus of the corresponding difference, can be used to determine the variational parameter and to calculate $W(g)$ for all other values of the coupling constant g. It turns out that the error in this approximation is only a few percent for the entire range of variation of the coupling constant g, even in the lowest orders of the a expansion. In Fig. 2 we present graphs of the relative error of the approximation

$$D(g)=\left|\frac{W^{(N)}(g)-W_{\text{exact}}(g)}{W_{\text{exact}}(g)}\right|$$

for small values $N=0$, and $N=2$ of the order of the approximation. As the ''experimental value'' we used the value of $W(g)$ at the point $g_0=1$.

We stress that this method has not only allowed us to extend the range of small values of the coupling constant for which the approximations give reasonable results, thereby improving perturbation theory, but has also taken us to the essentially nonperturbative region of strong coupling, where $g\to\infty$. The reason for this important result is the induced convergence of the series. In this case the variational parameters are fine-tuned order by order in accordance with some variational principle which, in spite of the harmonic variational procedure that we have used, ensures the convergence of the series. Table I demonstrates the induced convergence of the VPT series for the case considered above. Such empirical convergence was first noticed in Refs. 83 and 84. Further study of this topic and additional references can be found in Ref. 85. It is also possible to prove rigorously the induced convergence of the variational expansion for the case of the anharmonic oscillator.[86,87]

In Table I we give the results of calculations of the relative error of the approximation

$$D(g)=\left|\frac{W_{\text{theor}}(g)}{W_{\exp}(g)}-1\right|$$

TABLE I. Illustration of the induced convergence of the VPT series. The relative error of the VPT approximation $D(g)$ is given for various orders N at the coupling-constant values $g=10$ and $g=1000$.

N	0	1	2	3	4	6	8
C	1.14	2.64	3.56	5.46	6.12	8.71	11.33
$D(g=10)$ (%)	2.76	4.83	0.26	0.73	0.038	0.006	0.0012
$D(g=1000)$ (%)	5.01	6.52	0.56	1.13	0.089	0.017	0.0033

for the coupling constants $g=10$ and $g=1000$, obtained by using the ''experimental'' information for $g_0=1$. The parameter $C(N)$ for odd N is found from the condition for the minimum:

$$\min\left|W^{(N)}(g_0)-W_{\exp}\right|.$$

For even N the equation $W^{(N)}(g_0)=W_{\exp}$ has a root.

Before turning to chromodynamics, we note that the idea of variational perturbation theory can be applied not only in the context described above—to approximate quantities which can be represented as functional integrals—but also for other purposes, such as to optimize the approximations obtained by solving equations iteratively (see Appendix A).

3. THE VARIATIONAL EXPANSION IN QUANTUM CHROMODYNAMICS

Many problems in quantum chromodynamics require the use of nonperturbative methods. In the present section we discuss the application of the VPT method for constructing variational expansions in quantum chromodynamics and perform some nonperturbative calculations using them. A VPT expansion in quantum chromodynamics leading to a new small parameter was proposed in Ref. 50. This parameter turns out to be smaller than unity for any value of the coupling constant. Using this approach, it is possible not only to extend the range of applicability of the variational expansion in relation to perturbation theory, thus stabilizing its properties at scales of the order of a few GeV, but also to study essentially nonperturbative effects. Here we shall discuss the possible interrelationship between the results obtained in the VPT approach and the potential model of quark confinement.[51,52] Then, following Ref. 65, we shall study the problem of describing e^+e^- annihilation at low energies. In this review we shall use massless renormalization schemes belonging to the class of minimal schemes. The use of momentum renormalization schemes, in which the subtraction is done at some Euclidean point, was studied in Refs. 66–68.

Construction of the variational series. Let us consider the construction of the VPT series for quantum chromodynamics on the basis of a variational procedure of the harmonic type. The action functional of quantum chromodynamics is written as

$$S(A,q,\varphi)=S_2(A)+S_2(q)+S_2(\varphi)+gS_3(A,q,\varphi)+g^2S_4(A), \tag{3.1}$$

where $S_2(A)$, $S_2(q)$, and $S_2(\varphi)$ are the free action functionals of the gluon, quark, and ghost fields, respectively. Here

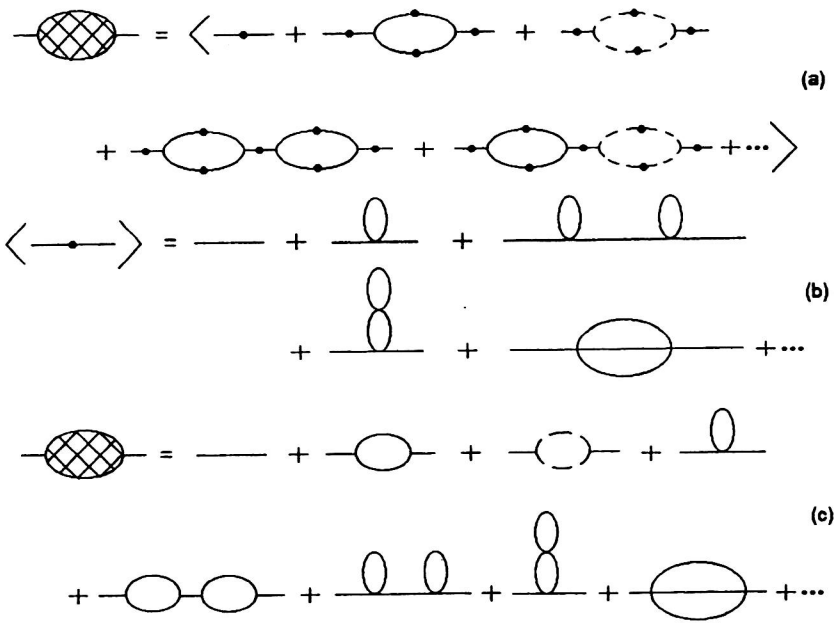

FIG. 3. Diagrammatic representation of the full gluon propagator in perturbation theory using the χ transformation. Solid lines correspond to the gluon propagator, dashed lines correspond to the ghost-field propagator, and lines with a point correspond to the function $D(\chi)$.

$S_2(A)$ also includes a gauge-fixing term, which we shall take to be the expression defining the covariant α_G gauge. The functional $S_3(A,q,\varphi)$ in (3.1) specifies the Yukawa interaction of the gluons with each other, with quarks, and with ghosts:

$$S_3(A,q,\varphi)=S_3(A)+S_3(A,q)+S_3(A,\varphi). \tag{3.2}$$

The functionals $S_3(A)$, $S_3(A,q)$, and $S_3(A,\varphi)$ generate three-gluon vertices of the type (AAA), $(\bar{q}Aq)$, and $(\varphi A\varphi)$, respectively. The term $S_4(A)$ generates four-gluon vertices $(AAAA)$. Let us transform this term, introducing the auxiliary fields $\chi^a_{\mu\nu}$ via

$$\exp[ig^2S_4(A)]=\int D\chi\exp\left\{\frac{i}{2}\int dxdy\chi^a_{\mu\nu}(x)\times[\Delta^{-1}(x,y)]^{ab}_{\mu\nu;\mu_1\nu_1}\chi^b_{\mu_1\nu_1}(y)+i\frac{g}{\sqrt{2}}\int dx\chi^a_{\mu\nu}(x)f^{abc}A^b_\mu(x)A^c_\nu(x)\right\}, \tag{3.3}$$

where $\Delta(x,y)$ is the gluon propagator in the χ field:

$$[\Delta(x,y)]^{ab}_{\mu\nu;\mu_1\nu_1}=\delta(x-y)\delta^{ab}\delta_{\mu\mu_1}\delta_{\nu\nu_1}. \tag{3.4}$$

After the χ transformation, the diagrammatic representation of the Green functions will contain only graphs of the Yukawa type. In addition to the ordinary three-point vertices, vertices of the type $A\chi A$ appear. Therefore, a Green function of QCD can be written in the form of a functional integral as

$$G(\cdots)=\int D\chi D_{\rm QCD}(\cdots)\exp\{i[S(A,\chi)+S_2(q)+S_2(\varphi)+S_2(\chi)+gS_3(A,q,\varphi)]\}, \tag{3.5}$$

where

$$S(A,\chi)=\frac{1}{2}\int dxdyA^a_\mu(x)[D^{-1}(x,y|\chi)]^{ab}_{\mu\nu}A^b_\nu(y), \tag{3.6}$$

with gluon propagator $D(x,y|\chi)$ in the χ field

$$[D^{-1}(x,y|\chi)]^{ab}_{\mu\nu}=[(-g_{\mu\nu}\partial^2+\partial_\mu\partial_\nu)\delta^{ab}+g\sqrt{2}f^{abc}\chi^c_{\mu\nu}+\text{gauge terms}]\delta(x-y), \tag{3.7}$$

in which "gauge terms" denotes terms associated with gauge-fixing. The integration measure $D_{\rm QCD}$ in (3.5) specifies the standard integrations over gluon, quark, and ghost fields.

Inside the χ average, all interactions are Yukawa-type interactions. Four-gluon vertices appear in the expansion of the functional integral in the field χ. Let us illustrate this situation by the example of the full vector-field propagator in the case of gluodynamics. For an arbitrary Green function we can write

$$G(\cdots)=\langle G_{\rm Yuk}(\cdots|\chi)\rangle, \tag{3.8}$$

where $\langle\ldots\rangle$ denotes the χ-functional average

$$\langle\cdots\rangle=\int D\chi[\cdots]\exp[iS_0(\chi)], \tag{3.9}$$

and the Green function in the χ field,

$$G_{\rm Yuk}(\cdots|\chi)=\int DA[\cdots]\exp\{i[S(A,\chi)+S^{\rm Yuk}_{\rm YM}(A)]\}, \tag{3.10}$$

is determined only by graphs of the Yukawa type with gluon propagator $D(x,y|\chi)$.

In Fig. 3(a) we show the full gluon propagator $\langle D_{\rm full}(x,y|\chi)\rangle$. Graphs with four-gluon vertices arise in expanding $\langle D(x,y|\chi)\rangle$ in a perturbation series [Fig. 3(b)]. They are added to the Yukawa graphs, and we obtain the standard diagrammatic representation of the perturbation series [Fig. 3(c)].

Let us now turn to the construction of the VPT expansion and introduce the auxiliary parameters ζ and ξ, rewriting the action functional in (3.5) as

$$S(A,q,\varphi,\chi)=S_0'(A,q,\varphi,\chi)+S_I'(A,q,\varphi,\chi), \tag{3.11}$$

where

$$S_0'(A,q,\varphi,\chi)=\zeta^{-1}[S(A,\chi)+S_2(q)+S_2(\varphi)]+\xi^{-1}S_2(\chi) \tag{3.12}$$

and

$$S_I'(A,q,\varphi,\chi)=gS_3(A,q,\varphi)-(\zeta^{-1}-1)[S(A,\chi)+S_2(q)+S_2(\varphi)]-(\xi^{-1}-1)S_2(\chi). \tag{3.13}$$

The exact value of the quantity for which the VPT expansion is used, for example, a Green function, is, of course, independent of the auxiliary parameters ζ and ξ. However, the approximation of the quantity by a finite number of terms of the VPT series, obtained in the expansion in powers of the action $S_I'(A,q,\varphi,\chi)$, will depend on these parameters. We shall use the freedom in choosing ζ and ξ for our goal of constructing a new small expansion parameter.

Technically, it is convenient to rewrite $S_0'(A,q,\varphi,\chi)$ by replacing ζ^{-1} by $[1+\kappa(\zeta^{-1}-1)]$ and ξ^{-1} by $[1+\kappa(\xi^{-1}-1)]$ in (3.12) and setting $\kappa=1$ at the end of all the calculations. In this case, any power of the expression $(\zeta^{-1}-1)\times[S(A,\chi)+S_2(q)+S_2(\varphi)]+(\xi^{-1}-1)S_2(\chi)$, which appears in the pre-exponential factor after expansion of (3.13) in powers, can be obtained by differentiating the corresponding number of times with respect to the parameter κ. Then the pre-exponential factor inside the functional integral will contain only powers of the action $gS_3(A,q,\varphi)$, which generate Yukawa graphs of QCD with modified propagators, determined by the corresponding quadratic forms in the new "free" action S_0'. The VPT series for the Green functions is written as

$$G(\cdots)=\sum_n\sum_{k=0}^{n}\frac{1}{(n-k)!}\left(-\frac{\partial}{\partial k}\right)^{n-k}\frac{i^k}{k!}\times\int D\chi D_{\mathrm{QCD}}(\cdots)\times[gS_3(A,q,\varphi]^k\exp[iS_0'(A,q,\varphi,\chi)], \tag{3.14}$$

where the above replacement has been made in $S_0'(A,q,\varphi,\chi)$. Next, it is convenient to rescale the fields:

$$(A,q,\varphi)\Rightarrow\frac{(A,q,\varphi)}{\sqrt{\kappa(\zeta^{-1}-1)}},\qquad \chi\Rightarrow\frac{x}{\sqrt{1+\kappa(\xi^{-1}-1)}}, \tag{3.15}$$

which brings the propagators to the standard form and modifies only the vertices of the graphs. Then, integrating over the field χ, for the Green function of ν fields we find

$$G(\cdots)=\sum_n\sum_{k=0}^{n}\frac{1}{(n-k)!}\left(-\frac{\partial}{\partial\kappa}\right)^{n-l}\frac{1}{1+\kappa(\zeta^{-1}-1)^{\nu/2}}\times\frac{i^k}{k!}\int D_{\mathrm{QCD}}(\cdots)[q_3S_3(A,q,\varphi)]^k\times\exp\{i[S_0(A,q,\varphi)+g_4^2S_4(A)]\}. \tag{3.16}$$

Here $S_0(A,q,\varphi)$ no longer contains the term corresponding to the field χ and is now the ordinary free action functional of QCD. The factors g_3 and g_4 at the Yukawa and four-gluon vertices are defined as

$$g_3=\frac{g}{[1+\kappa(\zeta^{-1}-1)]^{\frac{3}{2}}},\qquad g_4=\frac{g}{[1+\kappa(\xi^{-1}-1)]^{\frac{1}{2}}}. \tag{3.17}$$

The structure of the resulting VPT expansion has been analyzed in Refs. 50 and 51, where it was shown that for a special relation between the variational parameters and the coupling constant it is possible to construct a new nonperturbative expansion parameter. This parameter is always smaller than unity for any value of the original coupling constant. Here we shall show how the nonperturbative small parameter arises in the case of gluodynamics.

A small expansion parameter in quantum chromodynamics. As in the example studied above, differentiation with respect to the parameter κ leads to several additional factors. Let us see which factors arise when the original formulation is used before applying the rescaling of the fields (3.15). In this case the differentiation operator $(-\partial/\partial\kappa)^l$ leads to the factor $(1-\zeta)^l$, which acts on the gluon propagator, and to the factor $(1-\xi)^l$ in acting on the propagator of the field χ. Defining for convenience the parameter η, the power of which specifies the order of the expansion, we can symbolically write the structure of the VPT series as

$$1+\eta(1-\zeta)+\eta^2[(1-\zeta)^2+g^2\zeta^3+g^2\xi]+\eta^3[(1-\zeta)^3+g^2\zeta^3\times(1-\zeta)+g^2\xi(1-\zeta)+g^2\xi(1-\xi)]+\cdots. \tag{3.18}$$

As an illustration, in Fig. 4 we show the graphs for the full gluon propagator corresponding to the new expansion. A slash through a line denotes differentiation with respect to the parameter κ and corresponds to a factor $(1-\zeta)$ for a gluon line and $(1-\xi)$ when the gluon line arose via a χ field.

We see from (3.18) that by setting $\xi=\zeta^3$ and $(1-\zeta)^2\sim g^2\zeta^3$ we obtain the nth term of the VPT series containing the overall factor $(1-\zeta)^n$, while the second condition guarantees that the inequality $(1-\zeta)<1$ is satisfied for all positive values of the coupling constant.

A similar result is also obtained in the more general case where fermions are included. Here the formulation with rescaling of the fields (3.15) is more convenient. We note that in the context of gauge invariance, the choice $\xi=\zeta^3$ also serves to make the constants in Yukawa-type interactions and the

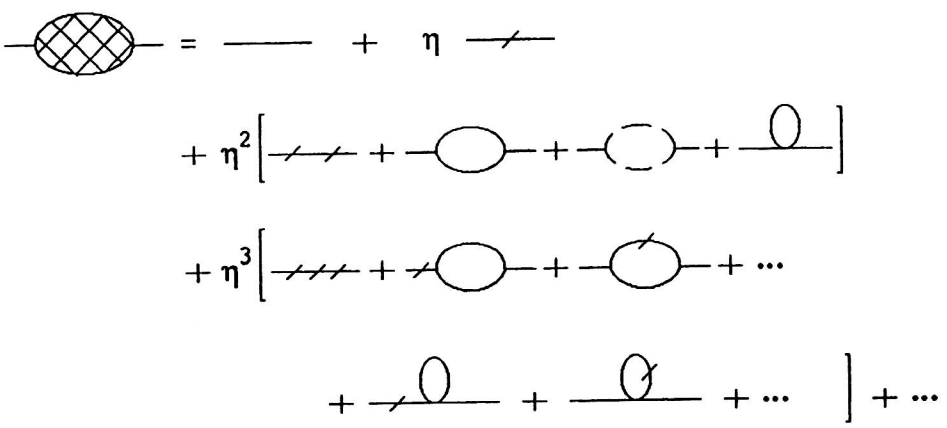

FIG. 4. Graphs corresponding to the VPT expansion of the full gluon propagator.

four-gluon interaction consistent. This choice ensures that the Slavnov–Taylor identities are satisfied for the renormalization constants obtained in this approach.

Thus, we obtain a new expansion parameter $a=1-\zeta$ related to the coupling constant g as

$$\lambda=\frac{g^2}{(4\pi)^2}=\frac{1}{C}\frac{a^2}{(1-a)^3},\quad a=1-\zeta, \tag{3.19}$$

where C is a positive constant. It follows from (3.19) that for any value of the coupling constant g the new expansion parameter a satisfies the inequality $0\leqslant a<1$.

Let us present the result for the VPT expansion for Green functions up to order a^7, which can be used to perform calculations at the two-loop level in this approach. We rewrite the expression for the Green functions as

$$G(\cdots)=\int D_{\mathrm{QCD}}(\cdots)V(A,q,\varphi)\exp(iS_0), \tag{3.20}$$

and then, using (3.16)–(3.19), we find

$$\begin{aligned}V=1&+aA_3+a^2\left[\frac{1}{2}A_3^2+\frac{3}{2}A_3\right]+a^3\left[\frac{1}{6}A_3^3+\frac{3}{2}A_3^2+A_3A_4\right.\\&\left.+3A_4+\frac{15}{8}A_3\right]+a^4\left[\frac{1}{24}A_3^4+\frac{1}{2}A_4^2+\frac{1}{2}A_3^2A_4+\frac{3}{4}A_3^3\right.\\&\left.+\frac{9}{2}A_3A_4+3A_3^2+6A_4+\frac{35}{16}A_3\right]+a^5\left[\frac{1}{120}A_3^5\right.\\&\left.+\frac{1}{6}A_3^3A_4+\frac{1}{2}A_3A_4^2+\frac{1}{4}A_3^4+3A_3^2A_4+3A_4^2+\frac{33}{16}A_3^3\right.\\&\left.+\frac{99}{8}A_3A_4+5A_3^2+10A_4+\frac{315}{128}A_3\right]+a^6\left[\frac{1}{720}A_3^6\right.\\&\left.+\frac{1}{24}A_3^4A_4+\frac{1}{4}A_3^2A_4^2+\frac{1}{6}A_4^3+\frac{1}{16}A_3^5+\frac{5}{4}A_3^3A_4\right.\\&\left.+\frac{15}{4}A_3A_4^2+\frac{7}{8}A_4^4+\frac{21}{2}A_3^2A_4+\frac{21}{2}A_4^2+\frac{143}{32}A_3^3\right.\\&\left.+\frac{129}{16}A_3A_4\frac{15}{2}A_3^3+15A_4+\frac{693}{256}A_3\right]+O(a^7),\end{aligned} \tag{3.21}$$

where $A_3=4\pi(iS_3)/\sqrt{C}$ and $A_4=(4\pi)^2(iS_4)/C$.

It is easily shown that the Nth order of the VPT series coincides with the Nth order of perturbation theory with accuracy $O(g^{N+1})$:

$$G_{\mathrm{VPT}}^{(N)}=\sum_{n=0}^{N}G_n=G_{\mathrm{PT}}^{(N)}+O(g^{N+1}). \tag{3.22}$$

Therefore, at small values of the coupling constants, the VPT expansion leads to the same results as standard perturbation theory. However, as we shall see later, in the nonperturbative region, when the coupling constant becomes large and direct use of perturbation theory is impossible, the constructed a expansion, as in the simple example considered above, remains well defined and allows a consistent treatment outside the framework of weak coupling.

Renormalization. Using dimensional regularization with $d=4-2\varepsilon$, for the renormalization constants Z_1 and Z_3 in the leading order of the VPT expansion we find

$$\begin{aligned}Z_1&=1+\lambda\zeta^3\left[N\left(\frac{17}{6}-\frac{3}{2}\alpha G\right)-\frac{4}{3}N_f\right]\frac{1}{2\varepsilon},\\Z_3&=1+\lambda\zeta^3\left[N\left(\frac{13}{3}-\alpha G\right)-\frac{4}{3}N_f\right]\frac{1}{2\varepsilon}.\end{aligned} \tag{3.23}$$

We recall that the coupling constant $\lambda\equiv\alpha_s/4\pi$ and the parameter ζ are related to each other as $(1-\zeta)^2=C\lambda\zeta^3$.

From (3.23) we find

$$\lambda_0=\mu^{2\varepsilon}Z_1^2Z_3^{-3}\lambda=\lambda\mu^{2\varepsilon}\left[1-\lambda\zeta^3\frac{\beta_0}{\varepsilon}\right], \tag{3.24}$$

where $\beta_0=11-\frac{2}{3}N_f$. From this, for the β function we find the expression

$$\begin{aligned}\beta(\lambda)&=\lim_{\varepsilon\to 0}\mu^2\frac{\partial\lambda}{\partial\mu^2}=-\beta_0\left(\lambda\frac{\partial}{\partial\lambda}-1\right)(\lambda^2\zeta^3)\\&=-2\beta_0\lambda^2\frac{\zeta^4}{3-\zeta}.\end{aligned} \tag{3.25}$$

Solving the corresponding renormalization-group equation, we obtain

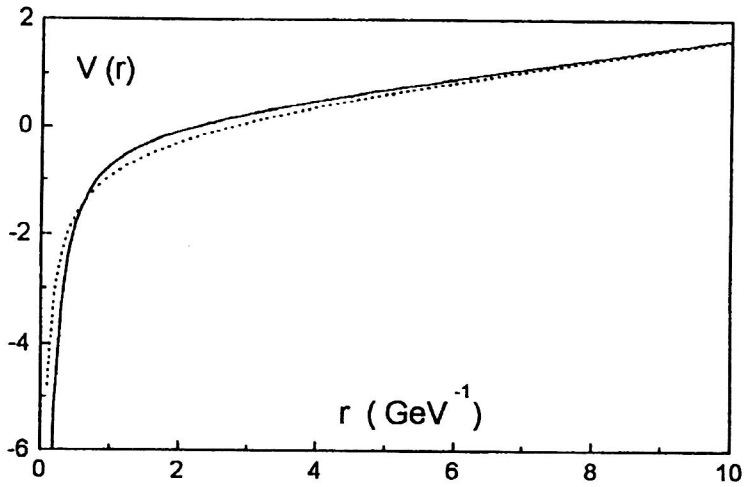

FIG. 5. Static quark interaction potential. The solid line corresponds to the VPT potential, and the dotted line to the phenomenological potential,[89] which reproduces the meson-spectroscopy data well.

$$\ln\frac{Q^2}{\Lambda^2}=\frac{C}{2\beta_0}f(\zeta), \tag{3.26}$$

where the function $f(\zeta)$ has the form

$$f(\zeta)=\frac{2}{(1-\zeta)^2}+\frac{12}{1-\zeta}+21\ln\frac{\zeta}{1-\zeta}-\frac{9}{\zeta}. \tag{3.27}$$

It is easily seen that in the perturbative region, when $Q^2\gg\Lambda^2$ ($\zeta\sim 1$), the expressions given above readily give the well known one-loop result:

$$\lambda(Q^2)=\frac{1}{\beta_0\ln(Q^2/\Lambda^2)}.$$

For decreasing Q^2 and correspondingly increasing coupling constant, the logarithmic growth is replaced by power-law growth, $\lambda(Q^2)\sim 1/Q^2$, as will be shown below. In the potential quark model, this behavior in the infrared region is consistent with the phenomenology of meson spectroscopy.

Relation to the potential quark model. In order to fix the parameters of our expansion, we shall use nonperturbative information associated with large-distance physics and obtained from the phenomenology of meson spectroscopy. This will allow the evolution law of the running coupling constant $\alpha_s(Q^2)$ to be completely determined without the use of any high-energy experimental data, in contrast to the usual case in determining the scale parameter Λ of quantum chromodynamics. In other words, we shall find the relation between the universal tension σ in the linear part of the static quark–antiquark potential,

$$V_{\rm lin}(r)=\sigma r, \tag{3.28}$$

which can be determined from meson spectroscopy (see, for example, Refs. 88 and 89 and the review of Ref. 90), and the behavior of the invariant charge in the perturbative ultraviolet region.

If we assume that the $q\bar{q}$ potential in momentum space can be written as

$$V(q^2)=-\frac{16\pi}{3}\frac{\bar{\alpha}_s(q^2)}{q^2}, \tag{3.29}$$

where $\bar{\alpha}_s(q^2)$ describes the regions of both large and small q^2 and has singular infrared behavior $\bar{\alpha}_s(q^2)\sim q^{-2}$, we obtain the potential (3.28) growing linearly at large distances. The singular behavior of the invariant charge in the infrared region corresponds to the asymptotic behavior of the β function

$$\beta(\lambda)\simeq-\lambda \tag{3.30}$$

at large values of the coupling constant.

The representation of confinement described above has been used in several approaches. One of the first attempts to unify the description of small and large distances by means of a single expression for the invariant charge was made in Ref. 91, where a simple prescription was proposed, according to which the original asymptotically free expression for the running coupling constant is modified as follows:

$$\bar{\alpha}_s(Q^2)=\frac{4\pi}{\beta_0}\frac{1}{\ln(Q^2/\Lambda^2)}\Rightarrow\frac{4\pi}{\beta_0}\frac{1}{\ln(1+Q^2/\Lambda^2)}. \tag{3.31}$$

The ultraviolet asymptotically free behavior of the running coupling constant is preserved in this model, and the singularity of the coupling constant required to obtain the linear potential appears in the infrared region. Thus, in Ref. 91 the phenomenological quark–antiquark potential was obtained on the basis of a smooth model matching of the perturbative behavior at large momenta and the assumed infrared asymptotic behavior. Of course, this approach is completely model-dependent. A more systematic treatment of this problem is given in Refs. 92–94, where the analysis is performed using an approximate solution of the Schwinger–Dyson equations explicitly including the Slavnov–Taylor identities. This question has also been studied in lattice calculations (see, for example, the review of Ref. 95).

The static quark interaction potential obtained in the first order of the VPT expansion was found in Ref. 50 and is shown by the solid line in Fig. 5. For comparison, the dotted line in Fig. 5 shows the phenomenological potential taken from Ref. 89, which agrees well with the data from meson spectroscopy and is close to the VPT potential.

Thus, in the VPT approach it is possible to determine the behavior of the invariant charge in the ultraviolet region by using information extracted from large-distance physics. Before turning to this topic, it is important to study the question of the stability of our results.

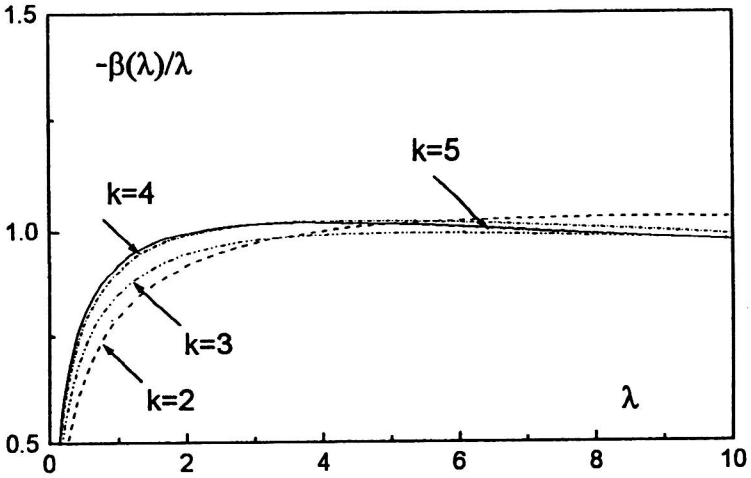

FIG. 6. Behavior of $-\beta^{(k)}(\lambda)/\lambda$ for $k=2,3,4,5$.

Corrections and stability. An arbitrary Green function $G(\cdots)$ corresponding to an even number of fields can be written through $O(a^5)$ as

$$G(\cdots)=g_0(\cdots)+\frac{a^2}{C}g_2(\cdots)+3\frac{a^3}{C}g_2(\cdots)$$
$$+\frac{a^4}{C^2}[6Cg_2(\cdots)+g_4(\cdots)]$$
$$+\frac{a^5}{C^2}[10Cg_2(\cdots)+6g_4(\cdots)], \qquad (3.32)$$

where $(\cdots)$ denotes the set of arguments of the corresponding fields, and $g_k(\cdots)$ are the coefficients of the perturbative expansion with the parameter λ, and are written in functional-integral form as

$$g_0(\cdots)=\int DA(\cdots)\exp[iS_0],$$
$$g_2(\cdots)=\int DA(\cdots)\left[\frac{(iS_3)^2}{2!}+\frac{(iS_4)}{1!}\right]\exp[iS_0],$$
$$g_4(\cdots)=\int DA(\cdots)\left[\frac{(iS_3)^4}{4!}+\frac{(iS_4)^2}{2!}\right.$$
$$\left.+\frac{(iS_3)^2}{2!}\frac{(iS_4)}{1!}\right]\exp[iS_0]. \qquad (3.33)$$

The VPT expansion of Green functions with an odd number of fields can be written down similarly.

In accordance with the mechanism of induced convergence (Refs. 83, 84, 86, 96, and 97), the variational parameter C is fine-tuned in each order of the approximation, using some optimization principle. Here we shall perform the calculations at the four-loop level, corresponding to the approximations $O(a^2)$, $O(a^3)$, $O(a^4)$, and $O(a^5)$, and compare the resulting β functions.

Using dimensional regularization and the results of Ref. 98 (see also the review of Ref. 99) and performing the calculations in the covariant gauge with arbitrary gauge parameter α_G, we find the charge renormalization constant[51] Z_λ ($\lambda_0=\mu^{2\varepsilon}Z_\lambda\lambda$):

$$Z_\lambda^{-1}=\frac{Z_3\tilde{Z}_3^2}{\bar{Z}_1^2}=1+\frac{1}{\varepsilon}\left[\frac{a^2}{3C}(11N-2N_f)+\frac{a^3}{C}(11N-2N_f)\right.$$
$$+\frac{a^4}{6NC^2}(34N^3-13N^2N_f+3N_f+132CN^2$$
$$-24CNN_f)+\frac{a^5}{3NC^2}(102N^3-39N^2N_f+9N_f$$
$$\left.+110CN^2-20CNN_f)\right], \qquad (3.34)$$

where Z_3 and $\tilde{Z}_3$ are the renormalization constants of the gluon and ghost fields, and $\bar{Z}_1$ is the renormalization constant of the ghost–gluon–ghost vertex. We shall verify directly that, as expected, the charge renormalization constant Z_λ is independent of the gauge parameter α_G; this serves as an additional test of the correctness of the calculations.

Knowing the charge renormalization constant Z_λ, we can calculate the β function corresponding to (3.34):

$$\beta(\lambda)=-\frac{1}{C^2}\frac{a^2}{(2+a)(1-a)^2}\left[2\beta_0a^2+9\beta_0a^3+4\left(6\beta_0\right.\right.$$
$$\left.\left.+\frac{\beta_1}{2C}\right)a^4+5\left(10\beta_0+6\frac{\beta_1}{2C}\right)a^5\right], \qquad (3.35)$$

where β_0 and $\beta_1=102-38N_f/3$ are the perturbative coefficients of the β function.

Keeping only terms of order $O(a^2)$, $O(a^3)$, $O(a^4)$, and $O(a^5)$ in the expression for the renormalization constant (3.34), we obtain the four β functions corresponding to these approximations: $\beta^{(2)}$, $\beta^{(3)}$, $\beta^{(4)}$, and $\beta^{(5)}$. In Fig. 6 we show the functions $-\beta^{(k)}(\lambda)/\lambda$ for the parameters $C_2=0.977$, $C_3=4.1$, $C_4=10.4$, and $C_5=21.5$. The approach of the ratio $-\beta^{(k)}(\lambda)/\lambda$ to unity at large coupling constants corresponds to an infrared singularity of the running charge $\bar{\alpha}_s(Q^2)\sim Q^{-2}$ at small Q^2. The increase of the variational parameter C_k with increasing order of the approximation occurs for the same reasons as in the simple example of the induced-convergence mechanism studied above.

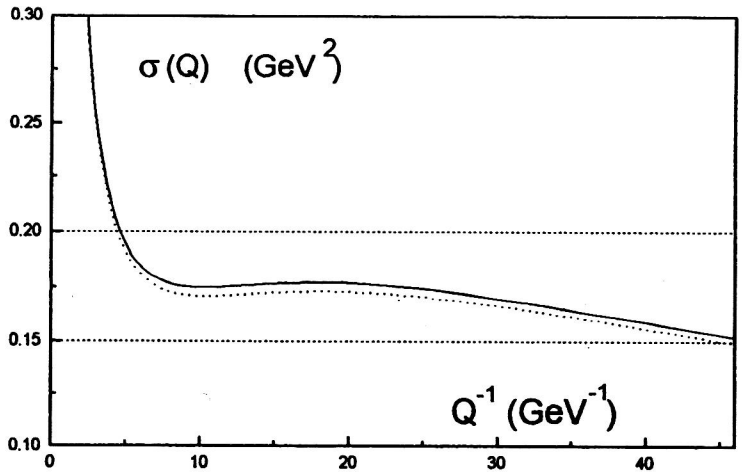

FIG. 7. The function $\sigma(Q^2)$ as a function of the inverse momentum Q^{-1} for the normalization points $Q_0=50$ MeV (solid line) and $Q_0=150$ MeV (dotted line). The horizontal lines indicate the range of σ compatible with the available experimental data.

To determine completely all the parameters from large-distance physics, let us consider the parameter σ in the linear part of the quark potential, $V_{\text{lin}}(r)=\sigma r$. Its phenomenological value is $\sigma\simeq 0.15-0.20\,\text{GeV}^2$ (Refs. 88–91). The invariant charge $\bar{\alpha}_s(Q^2)$ in the infrared region behaves as

$$\bar{\alpha}_s(Q^2)\simeq\frac{3}{2}\frac{\sigma}{Q^2}. \tag{3.36}$$

For normalization at some Q_0 we shall use the value from Ref. 89: $\frac{3}{2}\sigma=0.2652\,\text{GeV}^2$ ($\sigma=0.1768\,\text{GeV}^2$).

The renormalization-group evolution of the expansion parameter a is described by the equation

$$Q=Q_0\exp[\phi(a,N_f)-\phi(a_0,N_f^0)], \tag{3.37}$$

where

$$\phi(a,N_f)=\frac{1}{2}\int^{\lambda}\frac{d\lambda}{\beta(\lambda)} \tag{3.38}$$

and the β function is defined as in (3.35).

The behavior of $\sigma(Q^2)=\frac{2}{3}Q^2\alpha_s(Q^2)$ as a function of the inverse momentum Q^{-1} (in order to have some analog of the distance) is shown in Fig. 7 for two normalization points: $Q_0=50$ MeV (solid line) and $Q_0=150$ MeV (dotted line). The result at small momenta lies inside the range of the phenomenological estimates $\sigma\simeq 0.15-0.20\,\text{GeV}^2$ (in Fig. 7 the corresponding corridor is indicated by the two horizontal lines) and, as expected, depends weakly on the value of Q_0 and on the corresponding number of active quarks, N_f^0. We have made the calculation for $N_f^0=3$, which seems reasonable for our purposes, since a significant fraction of the information about the nonrelativistic quark interaction potential comes from data on the charmonium family. Nevertheless, we note that there is no strong sensitivity to the number of active quarks, N_f^0, and, for example, for $N_f^0=2$ the results are similar.

Thus, all the parameters have been found on the basis of information of a nonperturbative nature extracted from meson spectroscopy. The law describing the evolution of the expansion parameter $a=a(Q^2)$, given by (3.37), makes it possible to find the invariant charge in the ultraviolet region. For example, for the parameters found above on the Z-boson mass scale we obtain $\bar{\alpha}_s(M_Z=91.2\,\text{GeV})=0.126$. Taking into account the fact that all the parameters were fixed on the basis of essentially nonperturbative information corresponding to the infrared region, the value found for the coupling constant in the ultraviolet region appears quite reasonable. Thus, the region of applicability of the VPT method is considerably larger than that of ordinary perturbation theory. With this approach, it is possible to study on a unified basis not only the ultraviolet region of small coupling constant, but also the essentially infrared region.

4. e^+e^- ANNIHILATION INTO HADRONS AT LOW ENERGIES

In approximating some physical quantity, such as the known ratio $R(s)$ for e^+e^- annihilation into hadrons, by a partial sum of the perturbation series, the important problem of the dependence of the results on the selected renormalization scheme arises, owing to the truncation of this series. In particular, such a dependence is manifested outside the asymptotic ultraviolet region. On the one hand, the dependence of a physical quantity on the renormalization scheme can be viewed as an annoying fact. On the other hand, an additional degree of freedom appears, which can be used to construct an optimal expansion having a wider range of applicability than ordinary perturbation theory. In particular, as was shown in Refs. 100–102, in this way it is possible to enlarge the region typical of ordinary perturbation theory and move to low energies. The result obtained in those studies was based on optimization of the scheme dependence appearing in the third-order perturbative approximation. Improved perturbative coefficients were used,[103,104] and the principle of minimal sensitivity[96,97] was chosen to optimize the scheme dependence.[101,102] The e^+e^- annihilation process that we shall consider here was analyzed in Ref. 100, using another method of optimization of the scheme dependence, based on the effective-charge method.[105] We note that the dependence on the renormalization scheme can be substantially decreased in analytic perturbation theory.[106]

We shall use the results obtained for the ratio $R(s)$ in e^+e^- annihilation into hadrons. To make the comparison more convenient, as in Ref. 102 we shall study the range of $Q=\sqrt{s}$ from 0 to 6 GeV. It is impossible to use perturbative

expressions for the direct description of the experimentally observed quantity $R(s)$ at small s, owing to the presence of threshold singularities of the form $(\alpha_s/v)^n$ in the perturbative expansion. A smearing method was proposed in Ref. 107, which allows comparison with experiment. According to that study,[107] the problem of resonances and threshold singularities can be solved as follows. Instead of the original quantity $R(s)$ defined in terms of the discontinuity of the correlation function $\Pi(s)$ on the cut,

$$R(s)=\frac{1}{2i}[\Pi(s+i\epsilon)-\Pi(s-i\epsilon)], \tag{4.1}$$

one introduces the smeared quantity

$$R_\Delta(s)=\frac{1}{2i}[\Pi(s+i\Delta)-\Pi(s-i\Delta)], \tag{4.2}$$

with a finite value of Δ. For values of s near threshold, the quantity (4.1) is very sensitive to threshold singularities, in approaching which the perturbation series ceases to work. By going a finite distance Δ from the real axis in the complex q^2 plane, as in (4.2), we can be sure that when the correct perturbative approximation is used it is possible to describe the quantity in (4.2). The parameter Δ which must be used for effective comparison with the experimental data was estimated to be $1-3$ GeV^2 in Ref. 107.

The "experimental" curve corresponding to (4.2) arises if, using the dispersion relation

$$\Pi(q^2)=\text{const}+\frac{1}{\pi}\int_0^\infty ds\frac{R(s)}{s-q^2-i\epsilon}, \tag{4.3}$$

we rewrite (4.2) as

$$R_\Delta(s)=\frac{\Delta}{\pi}\int_0^\infty ds'\frac{R(s')}{(s-s')+\Delta^2}. \tag{4.4}$$

Then, substituting into this the corresponding fit to the experimental data on e^+e^- annihilation $R_{\text{exp}}(s)$, we can find the experimental curve corresponding to (4.2). Such curves have been found in Refs. 101 and 102 for several values of the parameter Δ estimated in Ref. 107. We shall use them for comparison with the results obtained in our approach.

No difficulty with small v arises when studying smeared quantities. Nevertheless, it is still impossible to use perturbation theory directly to describe $R_\Delta(s)$. Actually, the ratio R in (4.4), parametrized by using the invariant charge with unphysical singularities, causes the integral in (4.4) to diverge. Therefore, despite the fact that the use of the smeared quantity (4.2) allows the difficulty with threshold singularities to be avoided, there is a problem associated with the behavior of the running coupling constant in the infrared region. The use of the VPT method makes it possible to avoid this difficulty.

We shall restrict ourselves to the first nontrivial order determined by the VPT expansion (3.21). In this case, for $R(s)$ we obtain

$$R(s)=3\sum_f Q_f^2 T(v_f)\,\theta(s-4m_f^2)\left[1+g(v_f)\frac{\alpha_{\text{eff}}(Q)}{\pi}\right]. \tag{4.5}$$

The functions $v_f(s)$, $T(v)$, and $g(v)$ are defined as[5)]

$$v_f=\sqrt{1-4m_f^2/s},$$

$$T(v)=\frac{v(3-v^2)}{2},$$

$$g(v)=\frac{4\pi}{3}\left[\frac{\pi}{2v}-\frac{3+v}{4}\left(\frac{\pi}{2}-\frac{3}{4\pi}\right)\right]. \tag{4.6}$$

We shall consider two expressions for $\alpha_{\text{eff}}(Q)/\pi$ corresponding to different approximations [respectively, $O(a^2)$ and $O(a^3)$]:

$$\frac{\alpha_{\text{eff}}^{(2)}}{\pi}=\frac{4}{C_2}a^2,\quad \frac{\alpha_{\text{eff}}^{(3)}}{\pi}=\frac{4}{C_3}a^2(1+3a). \tag{4.7}$$

The values of the parameters C_2 and C_3 are not related to the fit to the e^+e^- data; they were found by us earlier from the condition that the renormalization-group β function behaves as $\beta(\lambda)\simeq-\lambda$ for sufficiently large coupling constant. This behavior corresponds to singular infrared behavior of the invariant charge $\alpha_s(Q^2)\sim Q^{-2}$ and ensures linear growth of the static quark–antiquark potential at large distances. In this way we fix the parameters C_i entering into (4.7) by using the data on hadron spectroscopy. As noted above, this gives $C_2=0.977$ and $C_3=4.1$. For the quark masses, as in Ref. 102, we take the values $m_u=5.6$ MeV, $m_d=9.9$ MeV, $m_s=199$ MeV, $m_c=1.350$ GeV, and $m_b=4.75$ GeV.

The running expansion parameter as a function of the momentum Q is determined by the equation

$$Q=Q_0\exp\left\{\frac{C_i}{4b_0}[f_i(a)-f_i(a_0)]\right\}, \tag{4.8}$$

where

$$f_2(a)=\frac{2}{a^2}+\frac{12}{a}+21\ln\frac{1-a}{a}-\frac{9}{1-a},$$

$$f_3(a)=\frac{2}{a^2}-\frac{6}{a}-48\ln a-\frac{18}{11}\frac{1}{1-a}+\frac{624}{121}\ln(1-a)+\frac{5184}{121}\ln\left(1+\frac{9}{2}a\right). \tag{4.9}$$

To determine all the parameters we use the normalization $\alpha_{\text{eff}}(Q_0)=\alpha_0$, and for Q_0 we choose the value of the Z-boson mass, $Q_0=M_Z=91.2$ GeV, and $\alpha_0=0.12$. Now there are no longer any free parameters. It should be noted that we have not used any information extracted from the data on low-energy e^+e^- annihilation.

In Fig. 8 we show the graphs of the functions $\alpha_{\text{eff}}^{(2)}/\pi$ and $\alpha_{\text{eff}}^{(3)}/\pi$ defined as in (4.7), which practically coincide. For comparison, we also show the behavior of the one-loop coupling constant corresponding to the perturbative case. The closeness of the VPT curves in Fig. 8 shows that the results obtained in different orders of the VPT approximation are stable. As we saw above, a similar picture is also observed in a wide range of coupling constant for the β functions corresponding to various approximation levels. The same stability with respect to the inclusion of higher-loop corrections also occurs for the smeared quantity (4.4).

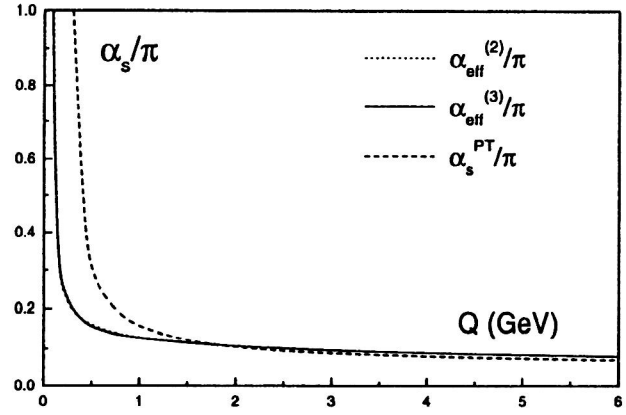

FIG. 8. Behavior of the functions $\alpha_{\text{eff}}^{(2)}/\pi$ and $\alpha_{\text{eff}}^{(3)}/\pi$ in the VPT approach and the corresponding one-loop perturbative coupling constant α_s.

The behavior of the function R_Δ for $\Delta=3\ \text{GeV}^2$ is shown in Fig. 9 [the two curves corresponding to the functions in (4.7) practically coincide, and we give only the graph for $\alpha_{\text{eff}}^{(3)}$]. The experimental smeared curve is taken from Ref. 102. We have also given the theoretically calculated result from this study, obtained by optimizing the third order of the perturbative expansion on the basis of the principle of minimal sensitivity. Thus, the result found even in the first order of the VPT reproduces the experimental curve quite well and is close to the result obtained on the basis of optimization of the third order of ordinary perturbation theory. Interestingly, a similar situation also occurs in analytic perturbation theory,[106] where an infrared-stable point exists, as in the VPT approach.

Recently, there has been a great deal of discussion about the "freezing" of the QCD coupling constant at low energies. Many of the model approaches based on QCD require such freezing (see the detailed discussion of this question in Ref. 102 and in the references cited therein). Unfortunately, at present there is no direct experimental information about the behavior of the QCD coupling constant in the deep infrared region. Only some integrated characteristics of this behavior are known. One of them, the smeared quantity R_Δ, was studied above. Another experimental quantity convenient for comparison is the integral

$$I=\int_0^{1\ \text{GeV}} dQ\,\frac{\alpha_s^{\text{eff}}(Q)}{\pi}\simeq 0.2\ \text{GeV}, \tag{4.10}$$

which is independent of the fit to the data.[109] Its value has cleverly been extracted from jet physics. In our case, for $\alpha_{\text{eff}}^{(2)}$ the value of this integral is 0.239 GeV, and for $\alpha_{\text{eff}}^{(3)}$ it is 0.237 GeV.

Here we have studied a method of constructing variational series in quantum chromodynamics, using renormalization prescriptions belonging to the class of MS schemes. The generalization to the case of MM renormalization schemes was discussed in Refs. 64 and 66–68. It turned out that the result obtained for the function $R_\Delta(s)$ is very close to that for the $\overline{\text{MS}}$ case discussed above. This indicates that the VPT approach is stable with respect to the choice of renormalization scheme.

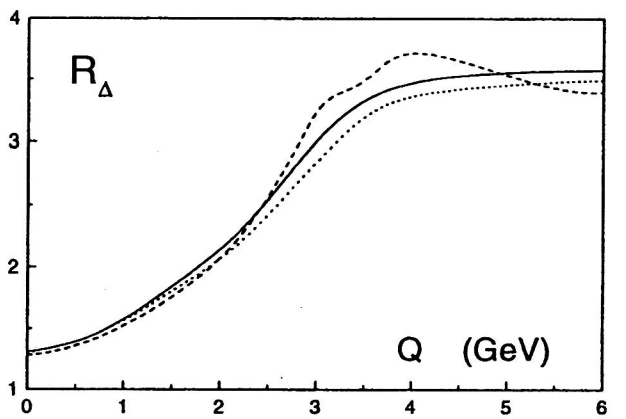

FIG. 9. Graph of the smeared quantity R_Δ for $\Delta=3\ \text{GeV}^2$. The solid line corresponds to the VPT result. The experimental smeared curve, shown by the dashed line, is taken from Ref. 102. The result of optimization of the third order of the perturbative expansion, performed in Ref. 102 on the basis of the principle of minimal sensitivity, is shown by the dotted line.

5. INCLUSIVE τ-LEPTON DECAY

In accordance with the renormalization-group method,[1] the invariant charge is defined in the spacelike, Euclidean region. To parametrize processes in quantum chromodynamics, for which timelike momenta are typical, for example, e^+e^- annihilation, it is necessary to use a special analytic continuation procedure. This has been studied in Refs. 103 and 110–113 within the perturbative approach. As will become clear below, to perform the analytic continuation self-consistently, it is essential that the running coupling constant have analytic properties consistent with the Källén–Lehmann representation, which are obviously violated by the perturbative approximation leading to unphysical singularities like a ghost pole. Here we shall study the problem of defining the invariant charge in the timelike region, following Ref. 59. In this section we shall also discuss the use of our method for describing inclusive τ-meson decay (Refs. 60, 61, and 64)[6] along with the effect of the renormalon contribution on the value of the coupling constant extracted from τ decay,[69,71] and, following Ref. 72, we shall touch upon the question of using the VPT together with the method of QCD sum rules to describe the mass spectrum of heavy quarkonia.

The invariant charge in the timelike region. Let us consider the D function,[119] related to the vector-current correlation function $\Pi(q^2)$ in the relation

$$i\int d^4x\,\exp(iq\cdot x)\langle 0|T\{J_\mu(x)J_\nu(0)\}|0\rangle$$
$$\propto(q_{\mu\nu}-g_{\mu\nu}q^2)\Pi(q^2) \tag{5.1}$$

as[7]

$$D(q^2)=q^2\left(-\frac{d}{dq^2}\right)\Pi(q^2). \tag{5.2}$$

The relation between the D function and the ratio $R(s)$ is determined by the dispersion integral

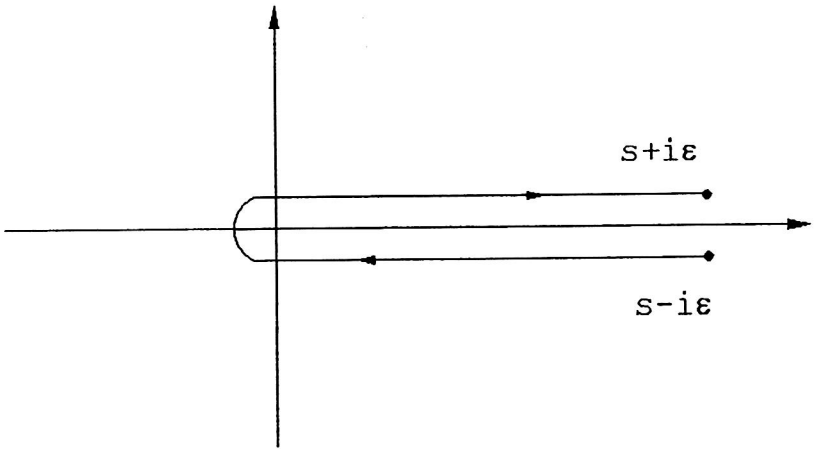

FIG. 10. Integration contour in the transform (5.4).

$$D(q^2)=-q^2\int_0^\infty \frac{ds}{(s-q^2)^2}R(s). \tag{5.3}$$

Therefore, the D function is analytic in the complex q^2 plane, cut along the positive real axis. Equation (5.3) can be solved for $R(s)$, and the inverse relation can be written as

$$R(s)=-\frac{1}{2\pi i}\int_{s-i\epsilon}^{s+i\epsilon}\frac{dz}{z}D(z). \tag{5.4}$$

The integration contour in (5.4) connecting the points $s-i\epsilon$ and $s+i\epsilon$ is shown in Fig. 10.

We write $D(q^2)$ and $R(s)$ as

$$D(q^2)\propto\sum_f Q_f^2[1+d_0\lambda(q^2)+d_1\lambda^2(q^2)+\cdots]$$

$$\equiv\sum_f Q_f^2[1+d_0\lambda^{\rm eff}(q^2)], \tag{5.5}$$

and

$$R(s)\propto\sum_f Q_f^2[1+r_0\lambda_s(s)+r_1\lambda_s^{(2)}(s)+\cdots]$$

$$\equiv\sum_f Q_f^2[1+r_0\lambda_s^{\rm eff}(s)]. \tag{5.6}$$

Here the coefficients d_k and r_k and the function λ depend on the number of fermions f, and the subscript s on the coupling constant in (5.6) denotes the s channel. Thus, for the observable $R(s)$ defined for timelike arguments we can write down a representation analogous to that for the Euclidean (t-channel) quantity (5.5).[8] It is therefore natural to take the function $\lambda_s^{\rm eff}(s)$ in (5.6) as the definition of the effective coupling constant in the timelike region.

Taking into account the fact that the coefficients d_0 and r_0 in (5.5) and (5.6) coincide, we obtain the following relation between the t- and s-channel constants:

$$\lambda^{\rm eff}(q^2)=-q^2\int_0^\infty \frac{ds}{(s-q^2)^2}\lambda_s^{\rm eff}(s) \tag{5.7}$$

and

$$\lambda_s^{\rm eff}(s)=-\frac{1}{2\pi i}\int_{s-i\epsilon}^{s+i\epsilon}\frac{dz}{z}\lambda_t^{\rm eff}(z). \tag{5.8}$$

We can attempt to use perturbation theory to find the s-channel coupling constant from (5.8), ignoring the fact that the perturbative approximation violates the analytic properties needed for (5.7) and (5.8) to be consistent. This treatment leads to known π^2 contributions to the perturbative s-channel coefficients. These contributions are very important from the viewpoint of phenomenology. For example, at the τ-lepton mass scale, where $\bar{\alpha}_s(M_\tau^2)\simeq 0.35$, the difference between the t- and s-channel constants due to the π^2 contributions reaches 20%. However, the violation of the analytic properties in the perturbative treatment means that the s-channel charge found in this manner does not reproduce the original function when substituted into (5.7). This indicates that in the usual perturbative treatment it is not possible to determine self-consistently the coupling constant in the timelike region.

In the VPT approach the s-channel charge corresponding to (5.8) is found from the relation[59]

$$\lambda_s^{(i)}(s)=\frac{1}{2\pi i}\frac{1}{2\beta_0}[\phi^{(i)}(a_+)-\phi^{(i)}(a_-)], \tag{5.9}$$

where the $a_\pm$ obey the equation

$$f(a_\pm)=f(a_0)+\frac{2\beta_0}{C}\left[\ln\frac{s}{Q_0^2}\pm i\pi\right], \tag{5.10}$$

in which the functions $f(a)$ are determined in the lowest orders by (4.9), and the corresponding functions $\phi(a)$ have the form

$$\phi^{(2)}(a)=\frac{1}{1-a}[2-11a-4(1-a)\ln a+3(1-a)\ln(1-a)] \tag{5.11}$$

and

$$\phi^{(3)}(a)=-4\ln a-\frac{72}{11}\frac{1}{1-a}+\frac{318}{121}\ln(1-a)$$

$$+\frac{256}{363}\ln\left(1+\frac{9}{2}a\right). \tag{5.12}$$

The coupling constants $\lambda(q^2)$ and $\lambda_s(s)$ obviously have identical ultraviolet tails determined by asymptotic freedom and the same infrared limits. However, these functions differ from each other for finite values of the arguments. For

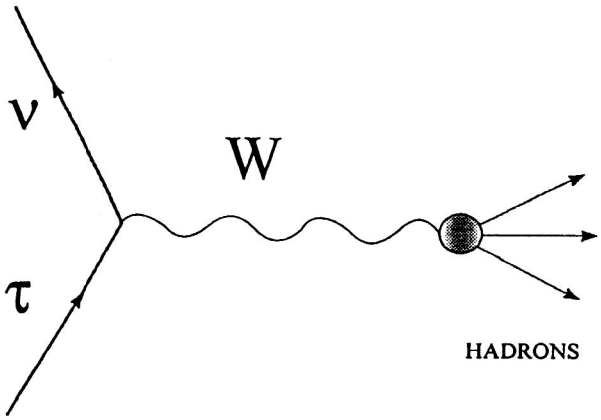

FIG. 11. Graph of inclusive τ-lepton decay.

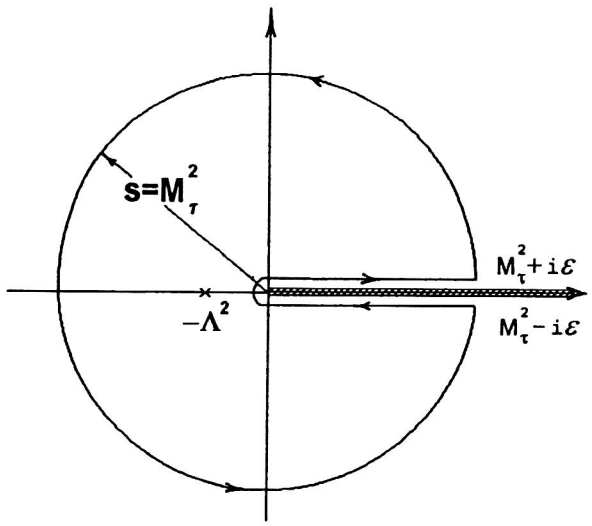

FIG. 12. Passage to the contour representation for R_τ.

example, on the τ-lepton mass scale this difference is of order 7%, which is considerably smaller than in the perturbative treatment and is therefore important from the viewpoint of the phenomenology of low-energy processes. A similar situation occurs for the t and s coupling constants in the analytic approach in quantum chromodynamics.[114]

τ decay. The only presently known lepton which has a hadronic decay mode is the τ lepton. Inclusive τ decay (the corresponding graph is shown in Fig. 11) offers unique possibilities for low-energy tests of quantum chromodynamics. On the one hand, the τ-lepton mass $M_\tau = 1777.05^{+0.29}_{-0.26}$ MeV (Ref. 121) is large enough that hadronic decay modes are possible, while, on the other, it is small on the chromodynamic scale and lies in the low-energy region.

It is inclusive τ-lepton decay which is the most interesting, as it can in principle be described without any important model assumptions,[122–125] and it allows the value of the coupling constant $\bar{\alpha}_s(M_\tau^2)$ to be found with fairly high accuracy. A quantity convenient for study is the experimentally measured value of R_τ, which is determined by the ratio of the hadronic and leptonic decay widths:

$$R_\tau = \frac{\Gamma[\tau^- \to \nu_\tau \text{ hadrons}(\gamma)]}{\Gamma[\tau^- \to \nu_\tau e^- \bar{\nu}_e(\gamma)]}. \tag{5.13}$$

The starting point for the theoretical analysis is the expression

$$R_\tau = 2\int_0^{M_\tau^2} \frac{ds}{M_\tau^2}\left(1-\frac{s}{M_\tau^2}\right)^2\left(1+\frac{2s}{M_\tau^2}\right)\tilde{R}(s), \tag{5.14}$$

where

$$\tilde{R}(s) = \frac{N}{2\pi i}[\Pi(s+i\epsilon)-\Pi(s-i\epsilon)],$$

$$\Pi(s) = \sum_{q=d,s} |V_{uq}|^2(\Pi_{uq,V}(s)+\Pi_{uq,A}(s)). \tag{5.15}$$

Here the V_{uq} are the elements of the Kobayashi–Maskawa matrix, and the normalization factor N is defined in such a way that at the parton level $\tilde{R}^{(0)}_{\text{parton}} = 3$. In the case of massless quarks considered below, the vector and axial-vector hadron correlation functions in (5.15), $\Pi_{uq,V}$ and $\Pi_{uq,A}$, coincide.

In the perturbative analysis of semileptonic τ decay we immediately encounter a difficulty in using the initial expression (5.14). The parametrization of the function $\tilde{R}(s)$, which in our case of massless quarks simply coincides with the ratio $R(s)$ for e^+e^- annihilation into hadrons, using the perturbative coupling constant possessing unphysical singularities, leads to singularities of the integrand in (5.14). A way out of this difficulty was proposed in Ref. 122. The integral (5.14) can be represented as a combination of integrals along the lips of the cut in the complex s plane (see Fig. 12). Then the Cauchy theorem can be used to transform this integral into an integral over the contour $|s| = M_\tau^2$. Finally, after integrating by parts, we arrive at the contour representation for R_τ in terms of the D function (5.2):

$$R_\tau = \frac{1}{2\pi i}\oint_{|z|=1}\frac{dz}{z}(1-z)^3(1+z)D(M_\tau^2 z). \tag{5.16}$$

We write the D function as

$$D(M_\tau^2 z) = d_0[1+d_1\lambda^{\text{eff}}(M_\tau^2 z)] \tag{5.17}$$

and isolate from R_τ the contribution of strong interactions Δ_τ:

$$R_\tau = R_\tau^{(0)}(1+\Delta_\tau), \tag{5.18}$$

where the quantity $R_\tau^{(0)}$ corresponding to the parton level is defined as

$$R_\tau^{(0)} = 3(|V_{ud}|^2+|V_{us}|^2)S_{\text{EW}}. \tag{5.19}$$

Here $S_{\text{EW}} = 1.0194$ is the known electroweak factor, and the elements of the Kobayashi–Maskawa matrix have the values $|V_{ud}| = 0.9753$ and $|V_{us}| = 0.221$ (Refs. 121 and 123).

For Δ_τ we find the expression

$$\Delta_\tau = \frac{1}{2\pi i}d_1\oint_{|z|=1}\frac{dz}{z}(1-z)^3(1+z)\lambda^{\text{eff}}(M_\tau^2 z), \tag{5.20}$$

which, after making the substitution $z = -M_\tau^2\exp(i\theta)$ and taking into account the fact that $d_1 = 4$, can be rewritten in the following form convenient for numerical calculations:

$$\Delta_\tau=\frac{2}{\pi}\int_{-\pi}^{\pi}d\theta(1+2e^{i\theta}-2e^{3i\theta}-e^{4i\theta})\lambda^{\rm eff}(M_\tau^2e^{i\theta}). \tag{5.21}$$

The question of how many active quarks there are must be studied specially for massless renormalization schemes. This problem does not arise in the mass-dependent renormalization-group formalism which originated with Bogolyubov and was later developed by Shirkov.[128,129] An algorithm of the latter type for smooth matching was used in Ref. 130 to analyze the behavior of the invariant charge in a wide range of momentum transfers. However, in MS-type schemes an additional ansatz is needed in order that different numbers of quarks can be treated consistently. It is common to use a matching of the running coupling constant corresponding to different numbers of fermions in the Euclidean region for $Q=\xi m_q$ with some matching parameter $1\leqslant\xi\leqslant 2$ (Ref. 126; see also Ref. 127). Such a procedure obviously leads to a discontinuity of the derivative and spoils the analytic properties of the invariant charge. In the VPT approach this difficulty can be avoided by making use of the possibility of the self-consistent determination of the effective charge in the timelike region, in which the number of active quarks is directly related to the energy of quark-pair creation. The passage between regions with different numbers of fermions is determined by the system of equations[9)]

$$\frac{1}{\beta_0(f-1)}\,{\rm Im}\,\phi(a_+^{(f-1)})=\frac{1}{\beta_0(f)}\,{\rm Im}\,\phi(a_+^{(f)}),$$

$$\frac{1}{C^{(f-1)}}{\rm Im}[(a_+^{(f-1)})^2(1+3a_+^{(f-1)})]$$

$$=\frac{1}{C^{(f)}}{\rm Im}[(a_+^{(f)})^2(1+3a_+^{(f)})], \tag{5.22}$$

which allows us to determine the relation between the parameters $C^{(f)}$ and $a_0^{(f)}$ corresponding to regions with different numbers of active quarks. After performing this procedure, we arrive at the following attractive physical picture. The running expansion parameter in the Euclidean region, reconstructed by using the dispersion integral (5.7), as in mass-dependent schemes does not correspond to any definite number of fermions, and it will "know" about all the thresholds in the physical region.

The analysis of the experimental data on inclusive τ-lepton decay performed in the order a^3 in Ref. 60 led to the following values of the s- and t-channel coupling constants: $\alpha_s(M_\tau^2)=0.37$ and $\alpha(M_\tau^2)=0.40$. Thus, the value of the t-channel coupling constant is somewhat larger than the value of the invariant charge determined in the timelike region. A similar situation also occurred in the analysis of Refs. 116 and 117 based on analytic perturbation theory. The $O(a^5)$ calculation performed in Ref. 61 leads to a somewhat smaller value of the extracted coupling constants. The renormalon contribution significantly affects the value of the invariant charge found from the same experimental data. We shall discuss these questions in the following subsection.

The renormalon contribution. Let us now consider the method by which the contribution of renormalon chains is effectively included.[69] In the lowest order, the perturbative approximation for the D function has the form $D(t,\lambda)=1+4\lambda(\mu^2)$, where $t=Q^2/\mu^2$. The standard renormalization-group summation of the leading logarithms leads to the replacement $\lambda(\mu^2)\to\bar\lambda(t,\lambda)$. However, owing to the ghost pole in the running coupling constant for $Q^2=\Lambda^2_{\rm QCD}$, this replacement spoils the analytic properties of the D function in the complex $q^2=-Q^2$ plane, in which the D function is analytic with a cut along the positive real axis. This situation can be rectified by noting that the solution of the renormalization-group equation that we have used, which is equivalent to the replacement $\lambda(\mu^2)\to\bar\lambda(t,\lambda)$, is not unique. The general solution is a function of the running coupling constant with asymptotic behavior $1+4\lambda$ at small λ. In order to preserve the analytic properties of the D function, we write it as a dispersion integral of the function $R(s)=(1/\pi){\rm Im}\,\Pi(s+i\epsilon)$. If we now use the renormalization-group method for $R(s)$, we find $D(t,\lambda)=1+4\lambda_{\rm eff}(t,\lambda)$ with the following Borel representation for the effective coupling constant:

$$\lambda_{\rm eff}(t,\lambda)=\int_0^\infty db\,\exp\left[-\frac{b}{\bar\lambda(t,\lambda)}\right]B(b), \tag{5.23}$$

where $B(b)=\Gamma(1+b\beta_0)\Gamma(1-b\beta_0)$.

Thus, in the Borel b plane there are singularities at the points $b\beta_0=-1,-2,\ldots$ and $b\beta_0=1,2,\ldots$, which correspond to ultraviolet and infrared renormalons. The first infrared renormalon singularity at $b\beta_0=1$ corresponds to power corrections $1/Q^2$, which, as is well known, are absent in the operator expansion. Although the absence of a singularity at $b=1/\beta_0$ is not a rigorously established fact (see, for example, the discussion in Ref. 125), it is reasonable from the viewpoint of consistency with the structure of the operator expansion to assume that the infrared renormalon singularities begin at $b=2/\beta_0$. This leads to an additional constraint on the choice of renormalization-group solution. Integrating by parts in the original dispersion representation of the D function and then choosing $R(s)$ to have the standard renormalization-group form, we find the following expression for the effective constant:

$$\lambda_{\rm eff}(t,\lambda)=\int_0^\infty d\tau\omega(\tau)\frac{\bar\lambda(kt,\lambda)}{1+\bar\lambda(kt,\lambda)\beta_0\ln\tau}. \tag{5.24}$$

Here the factor k depends on the choice of renormalization prescription. The resulting function

$$\omega(\tau)=\frac{2\tau}{(1+\tau)^3} \tag{5.25}$$

describes the distribution of virtual momenta in the quark loop. This function coincides with that used in Ref. 131 and found on the basis of other considerations and, as shown in Fig. 13, is close to the exact one.[132]

The function $B(b)$ arising in the Borel transform of the effective constant (5.24) has the form

$$B(b)=\Gamma(1+b\beta_0)\Gamma(2-b\beta_0). \tag{5.26}$$

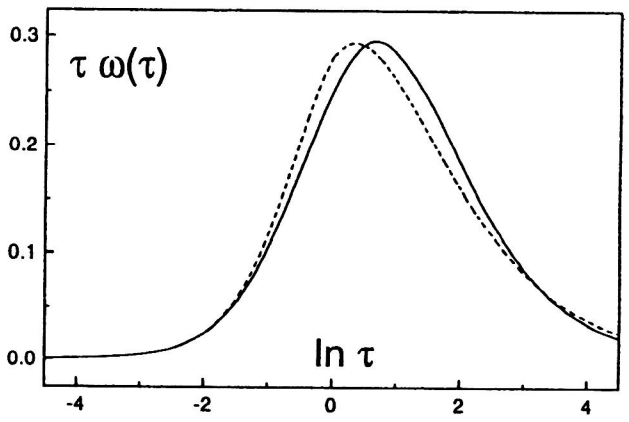

FIG. 13. Virtual momentum distribution $\tau\omega(\tau)$ in a quark loop. The solid line corresponds to the model function (5.25), and the dashed line to the exact calculation.[132]

Thus, the procedure described above does not change the location of the ultraviolet singularities, while the infrared singularities now begin at $b=2/\beta_0$. In contrast to the perturbative approach leading to singularities of the integrand in (5.24), in the method of the a expansion the integral (5.24) turns out to be well defined.

Let us return to the study of inclusive τ-lepton decay and use the trick described above to estimate the effect of the renormalon contribution on the value of the coupling constant extracted from τ decay.

As before, we write $R_\tau=R_\tau^0(1+\Delta_\tau)$, where the contribution of the strong interaction can be written as

$$\Delta_\tau=12d_1\int_0^{M_\tau^2}\frac{ds}{M_\tau^2}\left(\frac{s}{M_\tau^2}\right)^2\left(1-\frac{s}{M_\tau^2}\right)\tilde{\lambda}(ks). \tag{5.27}$$

In the $\overline{\text{MS}}$ scheme $k=\exp(-5/3)$ (Ref. 133), and the effective constant $\tilde{\lambda}$ is related to the expansion parameter as

$$\tilde{\lambda}(\sigma)=\frac{a^2}{C}+3\frac{a^3}{C}+\frac{a^4}{C}\left(6+\frac{1}{C}\frac{d_2}{d_1}\right)+\frac{a^5}{C}\left(10+\frac{6}{C}\frac{d_2}{d_1}\right), \tag{5.28}$$

where d_1 and d_2 are the coefficients of the perturbative expansion of the D function:[134]

$$d_1=4,\quad d_2=\frac{2}{3}[365-22N_f-8\zeta(3)(33-2N_f)]. \tag{5.29}$$

Here $\zeta(n)$ is the Riemann ζ function, and $\zeta(3)\simeq 1.202$.

The dependence of the running parameter a on the momentum variable σ is determined by the renormalization-group method and can be found as the solution of the equation

$$\ln\frac{\sigma}{\sigma_0}=\frac{C}{2\beta_0}[\bar{f}(a)-\bar{f}(a_0)], \tag{5.30}$$

where σ_0 is the normalization point and a_0 is the corresponding value of the parameter a. The function $\bar{f}$ has the form

$$\bar{f}(a)=\frac{1}{5(5+3B)}[x_1J(a,a_1)+x_2J(a,a_2)+x_3J(a,a_3)], \tag{5.31}$$

where

$$J(a,b)=-\frac{2}{a^2b}-\frac{4}{ab^2}-\frac{12}{ab}-\frac{9}{(1-a)(1-b)}+\frac{4+12b+21b^2}{b^3}\ln a+\frac{30-21b}{(1-b)^2}\ln(1-a)-\frac{(2+b)^2}{b^3(1-b)^2}\ln(a-b). \tag{5.32}$$

In (5.31) the parameters a_i are the roots of the polynomial

$$\phi(a)=1+\frac{9}{2}a+2(6+a)a^2+5\left(5+3\frac{\beta_1}{2C\beta_0}\right)a^3 \tag{5.33}$$

and

$$x_i=\frac{1}{(a_i-a_j)(a_i-a_k)}, \tag{5.34}$$

where the subscripts $\{i,j,k\}$ are written cyclically. The solution of (5.30) determining the running parameter $a=a(\sigma)$ is unique, owing to the monotonicity of the function $\bar{f}(a)$.

For the experimental value[10] $R_\tau^{\text{exp}}=3.56\pm0.03$ (Refs. 135 and 136), in the case of three active quarks we find the corresponding value of the strong coupling constant at the τ-lepton mass scale: $\alpha_s(M_\tau^2)=0.326\pm0.015$. This can be compared with the value $\alpha_s(M_\tau^2)=0.40$ found in Ref. 60 in $O(a^3)$ calculations and without the renormalon contribution. Thus, the contribution of renormalon chains and also of the next terms of the a expansion turns out to be important for extracting the strong-interaction coupling constant from the experimental data on inclusive τ-lepton decay. For the evolution of the value of $\alpha_s(M_\tau^2)$ found at the Z-boson scale it is necessary to take into account the thresholds due to the heavy c and b quarks. As noted above, in this approach it is possible to use a matching procedure for which the coupling constant determined in the timelike region and its derivative remain continuous at the threshold $s=4m_q^2$.[11] Using the values of the heavy quark masses $m_c=1.6$ GeV and $m_b=4.5$ GeV, for the ratio R_Z we obtain $R_Z=20.85\pm0.03$, which agrees well with the available experimental data.[121]

6. THE QCD SUM RULES AND THE *a* EXPANSION

In this section we shall study the problem of describing $q\bar{q}$ bound states by the method of QCD sum rules (Ref. 139; see also the review of Ref. 140), applying the VPT approach to describe not only the perturbative, but also, following a recent study,[72] the essentially nonperturbative contributions. The VPT method gives a unified description of the system; the explicit separation (from the viewpoint of the usual treatment based on the operator expansion) into the perturbative part containing the logarithmic dependence and the nonperturbative part determined by power corrections is not always natural and is possible only in very simple cases, such as the example studied in Ref. 72.

We shall begin our analysis with the hadron correlation function corresponding to a current with matrix structure Γ, whose imaginary part at the two-loop level can be written as

$$\mathrm{Im}\,\Pi^{\Gamma}(s)=\frac{1}{4\pi}[\Pi_{\Gamma}^{(0)}(s)+4\lambda\Pi_{\Gamma}^{(1)}(s)]. \tag{6.1}$$

The expressions for the components $\Pi^{(0)}$ and $\Pi^{(1)}$ which we shall need later are given in Appendix B.

Let us consider the first moment associated with the D function:

$$M_{1+N_\Gamma}^{(\Gamma)}(Q^2)\equiv-\frac{d\Pi^{\Gamma}(Q^2)}{dQ^2}$$

$$=\frac{1}{\pi}\int_{4m^2}^{\infty}\frac{ds}{(s+Q^2)^2}\mathrm{Im}\,\Pi^{\Gamma}(s). \tag{6.2}$$

Defining the variable $\sigma=s-4m^2$, which is related to the virtual momentum in the renormalon representation, and $u^2=\sigma/(\sigma+4m^2)$, we find

$$M_{1+N_\Gamma}^{(\Gamma)}(Q^2)=\frac{1}{4\pi^2}\int_0^{\infty}d\sigma\frac{(\sigma+4m^2)^{N_\Gamma}}{(Q^2+\sigma+4m^2)^{2+N_\Gamma}}$$

$$\times[\Pi_{\Gamma}^{(0)}(u)+4\lambda\Pi_{\Gamma}^{(1)}(u)]. \tag{6.3}$$

An additional problem which must be discussed specially is that of the so-called Coulomb singularities. The perturbative expansion is made in powers of the effective parameter λ/u. In the perturbative treatment of the higher-order moments, the region of small u dominates and the perturbative approximation becomes inapplicable (Ref. 142; see also Refs. 143 and 144). Therefore, for a consistent treatment of the higher-order moments it is necessary to perform some sort of summation of the Coulomb singularities. Passage to the renormalon representation studied above allows such a summation to be made.[72] This can be demonstrated for a simple example, which we do in Appendix C. There we also study the relation to the well known Sommerfeld–Sakharov factor.[145,146]

We rewrite (6.3) as

$$M_{1+N_\Gamma}^{(\Gamma)}(Q^2)=\frac{1}{4\pi^2}\int_0^{\infty}d\sigma\frac{(\sigma+4m^2)^{N_\Gamma}\Pi_{\Gamma}^{(0)}(u)}{(Q^2+W(\sigma))^{2+N_\Gamma}}, \tag{6.4}$$

where

$$W(\sigma)=(\sigma+4m^2)\left[1-4\lambda\frac{\psi_\Gamma(u)}{\Pi_{\Gamma}^{(0)}(u)}\right] \tag{6.5}$$

and

$$\psi_\Gamma(u)=(1-u^2)^{1+N_\Gamma}\int_0^{u}du\frac{2u}{(1-u^2)^{2+N_\Gamma}}\Pi_{\Gamma}^{(1)}(u). \tag{6.6}$$

The application of the renormalization-group method to the integrand in (6.4), where the current has zero anomalous dimension, leads to replacement of the coupling constant λ and the mass m by the running parameters $\bar{\lambda}(\sigma)$ and $\bar{m}(\sigma)$, where in the $\overline{MS}$ scheme that we are using the argument σ must be scaled by the factor $k_{\overline{MS}}=\exp(-5/3)$ (Ref. 133). When the anomalous dimension of the current is nonzero, an additional overall factor related to it arises; however, as will be clear from what follows, it proves unimportant for the conclusions based on analysis of moments with large n, and so this factor will not play any role in our treatment.

We shall use the running parameters of the VPT approach. As noted above, this does not lead to any difficulties associated with determining the integral when these parameters enter into the integrand. Thus, the expression for the nth-order moment has the form

$$M_n^{(\Gamma)}(Q^2)=\frac{1}{4\pi^2}\int_0^{\infty}d\sigma\frac{[\sigma+4m^2(k\sigma)]^{N_\Gamma}}{[Q^2+W(\sigma)]^{1+n}}\Pi_{\Gamma}^{(0)}(u). \tag{6.7}$$

Here

$$W(\sigma)=[\sigma+4m^2(k\sigma)]\left[1-4\lambda_{\mathrm{eff}}(k\sigma)\frac{\psi_\Gamma(u)}{\Pi_{\Gamma}^{(0)}(u)}\right], \tag{6.8}$$

and now $u^2=\sigma/[\sigma+4m^2(k\sigma)]$. In the sum-rule method the mass of the first resonance can be defined in terms of the moment ratio

$$R_n^{\Gamma}=\frac{M_{n-1}^{\Gamma}}{M_n^{\Gamma}} \tag{6.9}$$

for large n.

Let us consider the mass spectrum of the charmonium and bottomonium families ($c\bar{c}$ and $b\bar{b}$ bound states). In this approach the parameters are the value of the coupling constant α_0 at some normalization point, which we shall take to be the τ-lepton mass scale, and also the value of the heavy-quark mass (in this case, m_c or m_b) specified at some scale (we shall use 1 GeV). Here, of course, the values corresponding to the best fit should be consistent with the other data. As far as the parameter C is concerned, we shall not vary it, but take for it the value $C=4.1$ found earlier, corresponding to the $O(a^3)$ approximation for the renormalization constants.[51] The value of the coupling constant $\alpha_0(M_\tau^2)=0.379$ was found in Ref. 72. The masses of the c and b quarks were found to be $m_c=1.51\,\mathrm{GeV}$ and $m_b=4.72\,\mathrm{GeV}$, in good agreement with other results.[121]

At large n the main contribution to the moments is determined by the neighborhood of the minimum of the function $W(\sigma)$: $\sigma=\tilde{\sigma}$. The analysis of the various channels performed in Ref. 72 showed that for $c\bar{c}$ systems this minimum is reached at $\tilde{\sigma}\sim4-6\,\mathrm{GeV}^2$, and for $b\bar{b}$ states at $\tilde{\sigma}\sim20-30\,\mathrm{GeV}^2$. The main contribution to the moments (6.7) at large n is determined by the stationary point, and for the moment ratio we can obviously write

$$R_n(Q^2)\xrightarrow{n\to\infty}Q^2+W(\tilde{\sigma}). \tag{6.10}$$

This quantity should be compared with the corresponding hadron ratio $R_n^{\mathrm{had}}(Q^2)$, for which in the narrow-resonance approximation we easily find

$$R_n^{\mathrm{had}}(Q^2)\xrightarrow{n\to\infty}Q^2+M_R^2. \tag{6.11}$$

Therefore, at any value of Q for the resonance mass we obtain the expression

$$M_R=\sqrt{W(\tilde{\sigma})}. \tag{6.12}$$

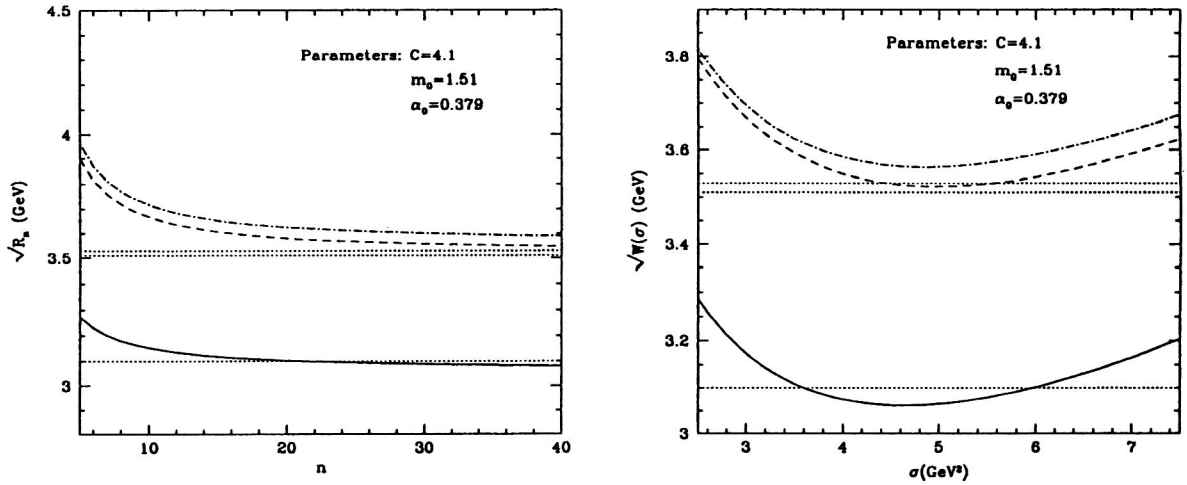

FIG. 14. Behavior of the functions $\sqrt{R_n}$ as a function of the number n and $\sqrt{W(\sigma)}$ as a function of σ for $c\bar{c}$ bound states. The solid line corresponds to the vector current, the dashed line to the axial-vector current, and the dot–dash line to the A' meson. The horizontal lines indicate the masses of the corresponding mesons, whose values are taken from Ref. 137.

We note that the fact that the Q dependence drops out is nontrivial. This dependence is present in the usual treatment and requires special analysis (see the discussion in Ref. 140).

In Figs. 14 and 15 we show the functions $\sqrt{R_n}$ and $\sqrt{W(\sigma)}$ for $c\bar{c}$ and $b\bar{b}$ systems, respectively. The solid lines correspond to the vector current, the dashed lines to the axial-vector current, and the dot–dash lines to the A' meson. The horizontal lines show the masses of the corresponding mesons, whose values are taken from Ref. 137. The results for $c\bar{c}$ systems with c-quark mass $m_c = 1.51$ GeV are summarized in Table II, and those for $b\bar{b}$ systems with b-quark mass $m_b = 4.72$ GeV are given in Table III. The c- and b-quark masses used here are in good agreement with estimates given by other approaches.[121]

To summarize briefly the analysis of this section, we remark that in this approach the stability region of the moments in the variable n is considerably larger than in the usual treatment, where the stability interval lies, as a rule, near $n = 6-8$. Another positive feature is the fact that at large n the Q^2 dependence for the ratio of moments calculated in this way is exactly the same as in the hadronic sector. The splitting between the vector, axial-vector, and A' states is reproduced with good accuracy. However, the approach to describing quarkonium spectroscopy which we have outlined in this section is only in the rudimentary stages of its development. A number of questions need to be answered, and a great deal of work remains to be done. For example, the question of describing scalar and pseudoscalar states needs to be studied; it is apparently necessary to include several other additional effects for them.[72]

7. CONCLUSION

As is well known, in constructing a perturbative expansion, the perturbation is usually taken to be the part of the full Lagrangian referred to as the interaction Lagrangian, which contains the higher powers of the fields. A free system is described by the harmonic part quadratic in the fields. As a result, for models with a single coupling constant one obtains a power-series expansion, which for most cases of

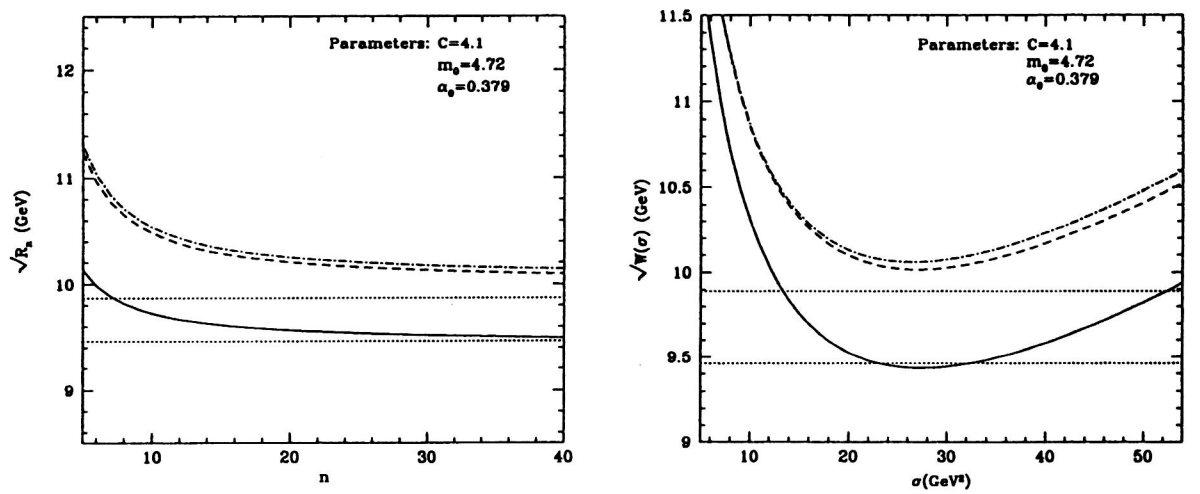

FIG. 15. The same as in Fig. 14, but for $b\bar{b}$ systems.

TABLE II. Mass spectrum of $c\bar{c}$ systems.

Current	M_{exp} (GeV)	$M_{theor}=\sqrt{W(\bar{\sigma})}$ (GeV)	$\bar{\sigma}$ (GeV^2)
j_V	3.10	3.06	4.6
j_A	3.51	3.52	4.8
$j_{A'}$	3.53	3.56	4.8
j_S	3.42	3.27	6.0
j_P	2.98	3.19	3.9

interest is asymptotic. At small values of the coupling constant the perturbation series, despite its asymptotic nature, can be used to analyze a fairly large class of problems of practical importance. Nevertheless, the solution of many problems makes it necessary to go beyond perturbation theory.

In this review we have discussed the results obtained on the basis of nonperturbative variational expansions in quantum field theory. The idea behind this approach is fairly simple. The usual splitting of the total action functional corresponding to a quantum system into the free part quadratic in the fields and the part describing the interaction and containing higher powers of the fields is to some extent purely conventional. The system is described by the full Lagrangian, and the traditional splitting of it into a free part and an interaction part is largely dictated by the need to find approximate solutions by using perturbation theory, when an exact integration procedure is unknown. Calculations can be performed in the functional-integral formalism, owing to Gaussian functional quadratures. Their obvious use, when an expansion in powers of the original interaction part of the Lagrangian is made inside the functional integral, leads to the traditional perturbation theory. However, as we have shown, perturbation theory is not the only possible consequence of Gaussian integration. There is another method of approximating functional integrals and performing calculations in quantum field theory. This method is based on probing the system by means of special functionals containing trial parameters of the variational type. By studying the response of the system to the change of the parameters of the variational probe, it is possible to choose them in an optimal manner such that the variational approximation is good for the system not only in the weak-coupling regime.

This approach, which we refer to as variational perturbation theory, combines a variational principle with a regular method of calculating corrections. In the VPT approach the original action functional is rewritten with an addition of the variational type, and an expansion is made in the effective interaction part of the action. Special parameters entering into the variational functional allow the properties of the resulting floating series to be controlled. Thus, in contrast to many other nonperturbative approaches, already from the very beginning in the VPT the quantity of interest is represented as a series which allows the necessary corrections to be calculated and the stability of the results to be evaluated. In other words, the method of variational expansions studied here offers the possibility of ascertaining the degree to which the main contribution determined by the variational method on the basis of some optimization principle correctly describes the system, as well as the region of applicability of the results. As in standard perturbation theory, only Gaussian functional quadratures are used to construct the VPT. Of course, the resulting series has a different structure and, moreover, some of the Feynman rules are modified at the level of the propagators and vertices.

TABLE III. Mass spectrum of $b\bar{b}$ systems.

Current	M_{exp} (GeV)	$M_{theor}=\sqrt{W(\bar{\sigma})}$ (GeV)	$\bar{\sigma}$ (GeV^2)
j_V	9.46	9.43	27
j_A	9.89	10.02	26
$j_{A'}$	-	10.07	26
j_S	9.86	9.80	30
j_P	-	9.56	24

In the case of quantum chromodynamics the application of the idea of constructing variational series leads to a new small expansion parameter. This parameter obeys an equation whose solutions are always smaller than unity for any value of the original coupling constant. While remaining within the region of applicability of this expansion it is possible to go to significantly lower energies than in the case of ordinary perturbation theory. The method used here ensures the induced convergence of the VPT series. At present, this type of convergence has been proved only for simple models. In the case of quantum chromodynamics, it is not yet possible to give a rigorous proof of the induced convergence of the variational expansion described here. However, our analysis leads to several arguments of an empirical nature in favor of a possible convergence of this type. For example, the behavior of the nonperturbative β function found by the method of the a expansion in various orders of the VPT approximation has turned out to be surprisingly stable with respect to the effect of corrections in the sense of induced convergence in a wide range of coupling constants. An important feature of the VPT approach is the fact that for sufficiently small $\bar{\alpha}_s$ the variational expansion reproduces the standard perturbation theory, and the perturbative high-energy physics is preserved. In moving to low energies, where ordinary perturbation theory breaks down [$\bar{\alpha}_s(Q^2) \sim 1$], the parameter of the VPT expansion remains small and we stay within the region of applicability of the method.

In this study we have considered a number of phenomenological applications of the VPT method. One is the description of e^+e^- annihilation into hadrons at low energies (on the QCD scale). To allow comparison with the experimental data we have used a special method of smoothing resonances, the so-called smearing procedure. The use of the smearing method allows the difficulties associated with threshold singularities in the perturbative expansion to be avoided. However, direct application of perturbation theory for describing the corresponding experimental data is still impossible. The reason for this is the unphysical singularities which occur in the perturbative invariant charge in the infrared region. Such singularities are absent in the method of the a expansion; the smearing integrals are well defined, and already in the first order of the VPT expansion good agreement with the experimental data is obtained.

The application of the usual renormalization-group resummation in perturbation theory produces unphysical singularities like a ghost pole in the behavior of the invariant charge. The presence of such singularities leads to a conflict with the fundamental principles of the theory. An important feature of the VPT approach is the possibility of preserving the analytic properties of the effective expansion constant dictated by the general principles of quantum field theory. It is this fact which allows the self-consistent introduction of the definition of the invariant charge in the timelike region. This concept is convenient for constructing a matching procedure in the timelike region, according to which the number of active quarks is directly related to the quark–antiquark pair-production thresholds. In contrast to the usual matching procedure in the Euclidean region, this approach does not violate analyticity. The absence of unphysical singularities when the VPT method is used is essential for obtaining the renormalon representation. The VPT approach, which is fundamentally a nonperturbative approach, allows a new view of the method of QCD sum rules, in particular, the application of the latter to describing the meson mass spectra. The correct analytic properties turn out to be of fundamental importance for describing inclusive τ-meson decay. This process offers the unique possibility of performing low-energy tests of quantum chromodynamics. The accuracy of the experimental data on it is considerably higher than for many other low-energy processes, and the method of constructing the theoretical description in principle allows assumptions of a model nature to be avoided. However, the use of perturbation theory leads to the above-mentioned unphysical singularities, and the analyticity needed for a consistent description is spoiled. Unphysical singularities can be avoided in the a-expansion method considered here. In this respect the VPT approach is similar to analytic perturbation theory, in which renormalization-group resummation is performed in such a way that there is no conflict with the general principles of the theory. Consequently, the correct analytic properties are restored, owing to the nonperturbative contributions which appear automatically. Despite the difference between these two methods, one based on the variational expansion and the other on renormalization-group summation with imposition of the analyticity condition, they have a number of features in common and often lead to similar results.

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APPENDIX A. SOLUTION OF EQUATIONS BY THE METHOD OF VARIATIONAL ITERATIONS

In this appendix we shall briefly discuss another possibility offered by variational perturbation theory: finding approximate solutions of equations by using the method of variational iterations. This technique, as in the case of functional integrals, leads to floating expansions whose convergence properties can be controlled by means of special variational parameters. It is not our intention here to go into any detailed discussion of this question. We shall only give an illustration of the possibilities offered by the VPT idea for finding solutions of equations.

To be specific, let us consider the nonrelativistic Lipmann–Schwinger equation for the Green function $G(E;\mathbf{p},\mathbf{q})$:

$$\left(E-\frac{\mathbf{p}^2}{2m}\right)G(E;\mathbf{p},\mathbf{q})-\int\frac{d\mathbf{k}}{(2\pi)^3}V(\mathbf{p},\mathbf{q})G(E;\mathbf{p},\mathbf{q})$$
$$=(2\pi)^3\delta(\mathbf{p}-\mathbf{k}). \tag{A1}$$

Iterating (A1), we obtain the well known series of standard perturbation theory, which can be interpreted as the series for the rescattering of a free particle with propagator

$$G_0(E;\mathbf{p},\mathbf{q})=\frac{(2\pi)^3\delta(\mathbf{p}-\mathbf{k})}{E-\mathbf{p}^2/2m} \tag{A2}$$

by the potential $V(\mathbf{p},\mathbf{q})$. This representation is possible in the weak-coupling limit. A different approach is required for strong coupling.

Let us consider the application of the method of variational perturbation theory to this problem. We write the Lippmann–Schwinger equation (A1) in symbolic form:

$$G-G_0VG=G_0. \tag{A3}$$

In accordance with the VPT, we change the zeroth-order approximation, writing (A3) in the equivalent form

$$G-\tilde{G}_0\tilde{V}G=\tilde{G}_0, \tag{A4}$$

with

$$\tilde{V}=V+[\tilde{G}_0^{-1}-G_0^{-1}]. \tag{A5}$$

Thus, iterations of this equation can be interpreted as the series for the rescattering of an effective particle with propagator $\tilde{G}_0$ by a new effective potential $\tilde{V}$ (see Fig. 16).

As in the case of the harmonic variational procedure, we choose

$$\tilde{G}_0=\zeta G_0. \tag{A6}$$

As a result, the VPT series can be written as

$$G=\sum_{n=0}^{\infty}G_n, \tag{A7}$$

$$G_n=\sum_{k=0}^{n}\frac{1}{(n-k)!}\left(-\frac{\partial}{\partial\kappa}\right)^{n-k}g_k.$$

The functions g_k in (A7) can be represented by the usual set of graphs with the modified propagator

$$\tilde{G}_0'=\frac{1}{1+\kappa(\zeta^{-1}-1)}G_0, \tag{A8}$$

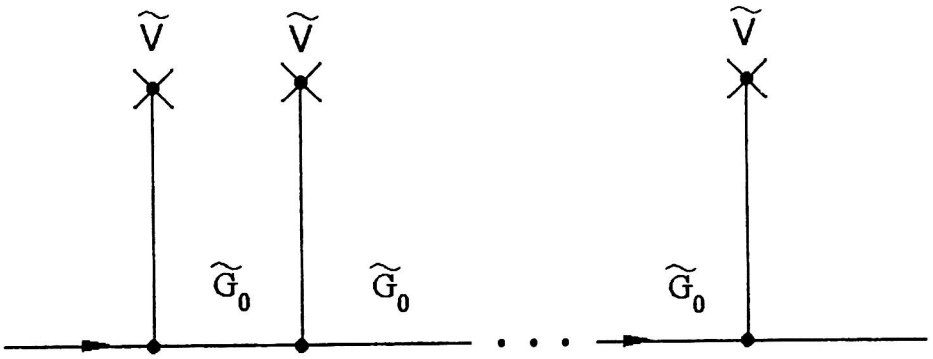

FIG. 16. Diagrammatic interpretation of the iterative solution in the method of variational perturbation theory.

where for convenience we have again introduced the parameter κ, which we set equal to unity at the end of all the calculations.

Analysis of the structure of the VPT expansion shows that the elements of the VPT series will be proportional to a new expansion parameter a when the original constant g and a are related as[12)]

$$g=\frac{1}{C}\frac{a}{1-a}. \tag{A9}$$

Here C is a parameter of the variational type, which must be determined on the basis of some optimization procedure.

In order to be able to compare the VPT result, let us consider the case of a separable interaction:

$$V(\mathbf{p},\mathbf{q})=gf(\mathbf{p})f(\mathbf{q}), \tag{A10}$$

In this case there exists the exact solution

$$G(E;\mathbf{p},\mathbf{q})=\frac{(2\pi)^3\delta(\mathbf{p}-\mathbf{k})}{E-\mathbf{p}^2/2m}+\frac{gf(\mathbf{p})f(\mathbf{q})}{(E-\mathbf{p}^2/2m)(E-\mathbf{q}^2/2m)}\times\left[1-g\int\frac{d\mathbf{k}}{(2\pi)^3}\frac{f^2(\mathbf{k}^2)}{E-\mathbf{k}^2/2m}\right]^{-1}. \tag{A11}$$

We restrict ourselves to the first nontrivial order of the VPT approximation, which allows us to obtain an equation for the variational parameter C and in the present case corresponds to the expansion through order a^2. In this order $(N=2)$ we have

$$G^{(2)}(E;\mathbf{p},\mathbf{q})=g_0(E;\mathbf{p},\mathbf{q})+a\frac{g_1(E;\mathbf{p},\mathbf{q})}{C}+a^2\left[\frac{g_2(E;\mathbf{p},\mathbf{q})}{C^2}+\frac{g_1(E;\mathbf{p},\mathbf{q})}{C}\right], \tag{A12}$$

where the perturbative coefficients $g_k(E;\mathbf{p},\mathbf{q})$ have the form

$$g_0(E;\mathbf{p},\mathbf{q})=\frac{(2\pi)^3\delta(\mathbf{p}-\mathbf{k})}{E-\mathbf{p}^2/2m},$$

$$g_1(E;\mathbf{p},\mathbf{q})=\frac{gf(\mathbf{p})f(\mathbf{q})}{(E-\mathbf{p}^2/2m)(E-\mathbf{q}^2/2m)}, \tag{A13}$$

$$g_2(E;\mathbf{p},\mathbf{q})=\frac{g^2f(\mathbf{p})f(\mathbf{q})}{(E-\mathbf{p}^2/2m)(E-\mathbf{q}^2/2m)}\times\int\frac{d\mathbf{k}}{(2\pi)^3}\frac{f^2(\mathbf{k}^2)}{E-\mathbf{k}^2/2m}.$$

The principle of minimal sensitivity

$$\frac{\partial G^{(2)}(E;\mathbf{p},\mathbf{q})}{\partial C}=0$$

determines the variational parameter C as

$$C=-\int\frac{d\mathbf{k}}{(2\pi)^3}\frac{f^2(\mathbf{k}^2)}{E-\mathbf{k}^2/2m}. \tag{A14}$$

As a result, it is easily verified by comparison with (A11) that

$$G^{(2)}(E;\mathbf{p},\mathbf{q})=G_{\text{exact}}(E;\mathbf{p},\mathbf{q}), \tag{A15}$$

while all the following terms of the VPT series are equal to zero.

APPENDIX B. CORRELATION FUNCTIONS OF MASSIVE-QUARK CURRENTS

In this appendix we give the two-loop expressions for the correlation functions of massive-quark currents. The currents given below correspond to the format $j_\Gamma=\ldots(J^{PC})$.

- *The scalar current*: $j_S=\bar\psi_i\psi_j(0^{++})$
- *The pseudoscalar current*: $j_P=i\bar\psi_i\gamma_5\psi_j(0^{-+})$
- *The vector current*: $j_V=\bar\psi_i\gamma_\mu\psi_j(1^{--})$
- *The axial-vector current*: $j_A=(q_\mu q_\nu/q^2-g_{\mu\nu})\bar\psi_i\gamma_\nu\gamma_5\psi_j(1^{++})$
- *The A′ current*: $j_{A'}=\bar\psi_i\partial_\mu\gamma_5\psi_j(1^{+-})$

To parametrize the common components of the correlation functions it is convenient to define the functions[140]

$$A(u)=(1+u^2)\left[\frac{\pi^2}{6}+\ln\frac{1+u}{1-u}\ln\frac{1+u}{2}+2l\left(\frac{1-u}{1+u}\right)+2l\left(\frac{1+u}{2}\right)-2l\left(\frac{1-u}{2}\right)-4l(u)+l(u^2)\right]+3u\ln\frac{1-u^2}{4u}-u\ln u, \tag{B1}$$

$$A'(u)=(1+u^2)\left[2l\left[\left(\frac{1-u}{1+u}\right)^2\right]-2l\left(\frac{u-1}{u+1}\right)-3\ln\frac{1-u}{1+u}\ln\frac{1+u}{2}+2\ln\frac{1-u}{1+u}\ln u\right], \tag{B2}$$

where

$$l(x)=-\int_0^x dt\frac{1}{t}\ln(1-t) \tag{B3}$$

is the Spence function.

The required functions Π^0 and Π^1 have the following form.

The vector current (Ref. 140) ($\Gamma=\gamma_\mu$):

$$N_V=0, \tag{B4}$$

$$\Pi_V^0=\frac{1}{2}u(3-u^2), \tag{B5}$$

$$\Pi_V^1=2\left[\left(1-\frac{u^2}{3}\right)A(u)+P_V(u)\ln\frac{1+u}{1-u}+Q_V(u)\right], \tag{B6}$$

$$P_V(u)=\frac{1}{24}(33+22u^2-7u^4), \tag{B7}$$

$$Q_V(u)=\frac{1}{4}(5u-3u^3). \tag{B8}$$

The axial-vector current (Ref. 140) ($\Gamma=\gamma_5\gamma_\nu(q_\mu q_\nu/q^2-g_{\mu\nu})$):

$$N_A=1, \tag{B9}$$

$$\Pi_A^0=u^3, \tag{B10}$$

$$\Pi_A^1=\frac{4}{3}\left[u^2A(u)+P_A(u)\ln\frac{1+u}{1-u}+Q_A(u)\right], \tag{B11}$$

$$P_A(u)=\frac{1}{32}(21+59u^2-19u^4-3u^6), \tag{B12}$$

$$Q_A(u)=\frac{1}{16}(-21u+30u^3+3u^5). \tag{B13}$$

The A′ current (Ref. 140) ($\Gamma=\partial_\mu\gamma_5$):

$$N_{A'}=2, \tag{B14}$$

$$\Pi_{A'}^0=\frac{1}{2}u^3, \tag{B15}$$

$$\Pi_{A'}^1=\frac{2}{3}\left[u^2A(u)+P_{A'}(u)\ln\frac{1+u}{1-u}+Q_{A'}(u)\right], \tag{B16}$$

$$P_{A'}(u)=\frac{1}{16}(13+28u^2+17u^4-2u^6), \tag{B17}$$

$$Q_{A'}(u)=\frac{1}{24}(-39y+47y^3+6u^5). \tag{B18}$$

The scalar current (Ref. 141) ($\Gamma=1$):

$$N_S=1, \tag{B19}$$

$$\Pi_S^0=\frac{3}{2}u^3, \tag{B20}$$

$$\Pi_S^1=2\left[u^2A'(u)+P_S(u)\ln\frac{1+u}{1-u}+Q_S(u)\right], \tag{B21}$$

$$P_S(u)=\frac{1}{16}(3+34u^2-13u^4), \tag{B22}$$

$$Q_S(u)=\frac{1}{8}(21u-3u^3). \tag{B23}$$

The pseudoscalar current (Ref. 141) ($\Gamma=\gamma_5$):

$$N_P=1, \tag{B24}$$

$$\Pi_P^0=\frac{3}{2}u, \tag{B25}$$

$$\Pi_P^1=2\left[A'(u)+P_P(u)\ln\frac{1+u}{1-u}+Q_P(u)\right], \tag{B26}$$

$$P_P(u)=\frac{1}{16}(19-48u+2u^2+3u^4), \tag{B27}$$

$$Q_P(u)=\frac{1}{8}(21u-3u^3). \tag{B28}$$

APPENDIX C. COULOMB SINGULARITIES

Here we shall use a simple model to study the question of how the representation (6.4) allows us to avoid the difficulty associated with Coulomb singularities mentioned in the text and to give a consistent treatment of the moments with large n.

As is well known (Ref. 142; see also Refs. 143 and 144), the perturbative expansion for $\mathrm{Im}\,\Pi(s)$ even for small coupling constant is poorly defined for small velocities

$$u=\sqrt{1-\frac{4m^2}{s}}, \tag{C1}$$

because the effective expansion parameter is λ/u. Owing to this fact, the direct use of the original perturbation series is not justified for moments with large n, for which the dominant contribution is determined by the region $u\sim 1/\sqrt{n}$, and the corresponding effective expansion parameter is $\sqrt{n}\lambda$.

Such singular expressions for nonrelativistic Coulomb systems are summed by using the Sommerfeld–Sakharov factor:[145,146]

$$S=\frac{X}{1-\exp(-X)}, \quad X=\frac{16\pi^2}{3}\frac{\lambda}{u}. \tag{C2}$$

Let us adopt a simple model to study this question, using the method of describing quarkonium given here. We write the nth moment as

$$M_n(Y)=\int_0^\infty d\sigma\frac{\rho(\sigma)}{(\sigma+Y)^{n+1}}, \tag{C3}$$

where the virtual momentum is $\sigma=s-4m^2$, and $Y=Q^2+4m^2$. Then we choose the spectral function $\rho(\sigma)$ normalized by the condition $\rho(\sigma,\lambda=0)=1$ as

$$\rho(\sigma)=\left(\frac{\sqrt{\sigma}}{\lambda+\sqrt{\sigma}}\right)^p, \tag{C4}$$

where p is a positive integer.

The perturbative expansion (C4), which has the form

$$\rho_{\rm pert}(\sigma)\sim 1-p\frac{\lambda}{\sqrt{\sigma}}+O\left(\frac{\lambda}{\sqrt{\sigma}}\right)^2\sim 1-\frac{p}{2m}\frac{\lambda}{u}+O\left(\frac{\lambda}{u}\right)^2, \tag{C5}$$

shows that Coulomb singularities are present at small u.

Assuming that the function $\rho(\sigma)$ possesses the smoothness properties,[144] the moments for large n can be written as

$$M_n(Y)\sim Y^{-(n+1)}\int_0^\infty d\sigma\rho(\sigma)\exp\left(-n\frac{\sigma}{Y}\right)\left(1+O\left(\frac{1}{n}\right)\right). \tag{C6}$$

The dominant contribution is determined by the region $\sigma\sim Y/n$, which in terms of the variable u corresponds to $u\sim 1/\sqrt{n}$. Meanwhile, without using the perturbative approximation for the original moments $M_n^{\rm ex}$ defined according to (C3) with the spectral function (C4), the analogous asymptotic expression at large n, where small velocities again dominate in the integral, is

$$\rho_{\rm ex}\to\left(1+\sqrt{n}\frac{\lambda}{\sqrt{Y}}\right)^{-p}. \tag{C7}$$

If we start from the perturbative expression in $O(\lambda)$, we obtain

$$\rho_{\rm pert}\to 1-p\sqrt{n}\frac{\lambda}{\sqrt{Y}}. \tag{C8}$$

We can now see explicitly that for sufficiently large n the use of the perturbative expression (C8) is not justified.

Let us now see what happens in our approach. For correspondence with the operator expansion, we must first rewrite the expression for the moments (C3) in the renormalon representation, which in the present case has the form

$$M_n^{\rm re}(Y)=\int_0^\infty d\sigma'\frac{1}{(f(\sigma')+Y)^{n+1}}, \tag{C9}$$

where the function $f(\sigma')$ is defined implicitly by the equation

$$\sigma'=\int_0^{f(\sigma')}d\sigma\rho(\sigma). \tag{C10}$$

This expression is obtained by making the change of variable $\sigma=f(\sigma')$ and assuming that the spectral function $\rho(\sigma)$ is such that $f(\infty)=\infty$ and $f(0)=0$.

Although the transformation defined by (C9) and (C10) is exact, we are interested in the order λ. In this approximation, for the function $f(\sigma')$ we obtain the explicit expression

$$f(\sigma')=\sigma'+2p\lambda\sqrt{\sigma'}. \tag{C11}$$

In this approach no problem with Coulomb singularities arises, and the first approximation is already sufficient for reproducing the n dependence of the moments for large n. As in the sum-rule method, we have the ratio $R_n=M_n/M_{n-1}$, and we can calculate three functions: the exact one $R_n^{\rm ex}$, the perturbative one $R_n^{\rm pert}$ in the order λ, and the sum based on the renormalon representation $R_n^{\rm re}$ in the same order λ. These functions are shown in Fig. 17 for the parameters $Y=1$, $\lambda=0.1$, and $p=1$. A different choice of parameters leads to a similar result.

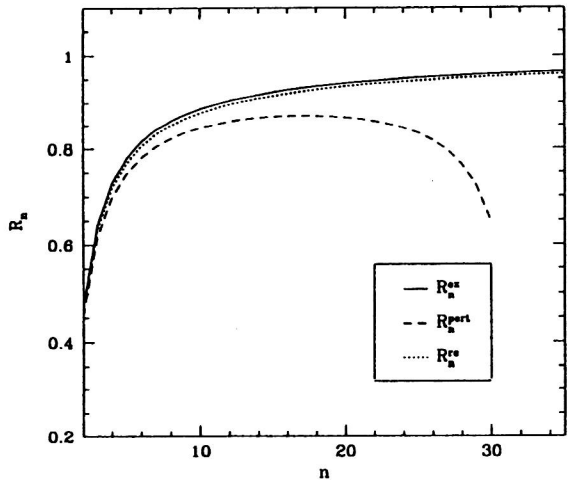

FIG. 17. Moment ratios $R_n^{\rm ex}$, $R_n^{\rm pert}$, and $R_n^{\rm re}$ for the parameters $Y=1$, $\lambda=0.1$, and $p=1$.

We note that when the renormalon representation is used, the region $u\sim 1/\sqrt{n}$ no longer defines the leading contribution to the integral for the moments at large n. Now the stationary point is determined by some $\tilde{\sigma}$ independent of n, which completely corresponds to our analysis of the QCD sum rules.

[1)]Dedicated to the memory of Iosif Noraĭrovich Sisakyan, a remarkable physicist and a true friend.

[2)]A detailed description of the path-integral method can be found in Refs. 74 and 75.

[3)]In considering this simple example, we shall always have in mind a field model, and so we find it convenient to use the corresponding terminology.

[4)]We are considering the vacuum functional, and so all the graphs of interest are vacuum graphs.

[5)]The function $g(v)$ in (4.6) corresponds to the Schwinger interpolation formula.[108] The exact expressions for two-loop massive correlation functions which we shall use below to study the QCD sum rules are given in Appendix B.

[6)]In the analytic perturbation theory mentioned above, the question of determining the invariant charge in the timelike region has been studied in Refs. 114 and 115, and semileptonic τ decay has been studied in Refs. 116–118.

[7)]Here we use the common convention $Q^2=-q^2$, so that the Euclidean region corresponds to positive Q^2.

[8)]We note that $\lambda_s^{(2)}$ in (5.6) does not coincide with the square of λ_s, and the expansion in (5.6) is not a power series. A similar situation also occurs in analytic perturbation theory. The non-power expansions corresponding to the analytic approach have been studied in Ref. 120.

[9)]We give the system of equations (5.22) determining the matching procedure in the order a^3. The function $\phi(a)$ in it is given by (5.12), and a_+ satisfies Eq. (5.10).

[10)]Another experimental value of R_τ, corresponding to the world average given in Ref. 137, was analyzed in Ref. 69.

[11)]Earlier, we gave the system of equations (5.22) corresponding to the $O(a^3)$ case. For the $O(a^5)$ case considered here, the system of equations determining the matching procedure can be found along with some other details in Ref. 138.

[12)]The problem of the motion of a particle in an external field corresponds to the simplest relation between the new expansion parameter a and the original coupling constant g.

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