Elastodynamical properties of nuclear matter from the observed activity of neutron stars

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The results of studies devoted to describing the large-scale motions of the nuclear matter of neutron stars are reviewed. The approach is based on the idea that stellar nuclear matter is an elastic Fermi continuum possessing the properties of a compensated magneto-active plasma. The fundamental dynamical equations of motion of self-gravitating nuclear matter are taken to be the equations of nuclear elastodynamics, formulated in the macroscopic theory of nuclear collective processes. A variational method is presented for calculating the frequencies of gravitational-elastic spheroidal (s-mode) and torsional (t-mode) vibrations of a neutron star, and the star's stability with respect to linear deformations is studied. The effectiveness of the elastodynamical method is illustrated by calculations of the periods of global gravitational elastic vibrations within the standard homogeneous model, and also for a family of realistic models of neutron stars, constructed on the basis of the relativistic equation of gravitational equilibrium and the nuclear-matter equations of state taking into account the heterophase nature of the nuclear statistical equilibrium. The motions of the magnetized Ae phase of the neutron-star matter are described using the magneto-hydrodynamical approach, which is used to calculate the frequencies of toroidal and poloidal Alfvén oscillations. It is emphasized that magneto-plasma vibrations can significantly affect the electromagnetic activity of pulsars. [S1063-7796(99)00504-5]

1. INTRODUCTION

The main value of the discovery of neutron stars[1] for nuclear physics is that they provide a unique astrophysical laboratory for studying the equilibrium and dynamical properties of macroscopic nuclear matter. At present, more than 500 neutron stars are known to exist in the Galaxy.[2] They are seen as isolated sources of pulsating radio emission (radiopulsars), and also as sources of pulsating x rays (x-ray pulsars) of binary systems, where the radiation is produced owing to the accretion of matter from the companion at the magnetic poles of the rotating neutron star. A fairly complete idea of the development of the physics of neutron stars over the last 30 years since the discovery of pulsars can be obtained from Refs. 3–13.

In recent theoretical studies concerning the nuclear aspects of the physics of neutron stars, the central position is occupied by studies devoted to the equilibrium properties of these massive compact objects from the final stage of stellar evolution (see, for example, Refs. 14–22 and references therein). These studies have significantly deepened the earlier ideas about the equation of state of nuclear matter and have fundamentally enhanced our understanding of thermodynamical phase transitions in stellar nuclear matter in the density, temperature, chemical composition, and magnetic field strength.[21,22] The methods of evolutionary analysis developed by the present time allow us to judge with some degree of reliability the details of the stratified structure of neutron stars and to calculate accurately the density, pressure, temperature, and other profiles determining the global equilibrium parameters of neutron stars such as the mass M, the radius R, the moment of inertia J, the critical (Kepler) frequency of gravitationally stable rotation Ω_K, the magnetic field strength B, and others.[1)]

Meanwhile, the features of the dynamical behavior of nuclear matter in the cores of these compact objects have been less well studied. This is mainly related to the absence of any clear ideas about the laws governing the continuum mechanics of self-gravitating nuclear matter. The macroscopic electrodynamics of wave processes capable of supporting a strongly magnetized nuclear medium remains poorly studied.

In this review we present arguments that macroscopic nuclear matter possesses the properties of an elastic matter continuum and a compensated, strongly magnetized plasma. In particular, we show that the fundamental equations of the continuum mechanics of a continuous nuclear medium can be taken to be the equations of nuclear elastodynamics, established in laboratory nuclear physics in the study of strongly collectivized processes such as giant resonances and fission, while the electrodynamics of wave processes which can occur in the Ae phase of neutron-star matter can be studied using the equations of magnetohydrodynamics.

Let us begin by listing a number of observations which clearly demonstrate the incorrectness of identifying continuous nuclear matter with an incompressible fluid. The knowledge accumulated so far about the evolution of massive stars leaves no doubt that it is only at the early and mature stages of evolution that stellar matter can be treated as a high-temperature plasma[30,31] whose large-scale motions obey the laws of gas dynamics. The idea that normal stars are spherical, self-gravitating gaseous masses has been convincingly confirmed by numerous studies of variable main-sequence stars (primarily, the Cepheids), in which the periodic variations of the brightness (luminosity) are well described by the hydrodyamical theory of radial pulsations.[32–37] However, the gas-dynamical concept of the behavior of the stellar medium ceases to be justified for stars in the final stage of evolution—white dwarfs and neutron stars. Theoretical calculations and observations show that as the nuclear fuel burns, the matter becomes significantly more dense owing to gravitational compression, as a result of which in the final products of stellar evolution it acquires the properties of an extremely stiff, rigid-body-like material. It has now been rigorously established that the variability of the luminosity of white dwarfs is due to shear spheroidal and torsional, essentially nonradial, vibrations,[37–41] which are accompanied by significant anisotropic (shear) distortions of the internal stresses in the star. It is known from condensed-matter physics that such vibrations can be supported only by rigid-body elastic forces, and not by gas–liquid forces. Convincing evidence that the nuclear matter of neutron stars possesses the properties of a solid (practically incompressible) medium is provided by the phenomenon of stellar shaking, detected as a sudden disturbance of the regular pulsations of pulsar radio emission (see p. 71 of Ref. 3). In the two-component model of a rotating, magnetized neutron star, the observed disturbance is related to the appearance of critical elastic stresses in the stiff peripheral core, as a result of which its stable coupling to the inner, denser core is lost (Ref. 42; see also Refs. 8 and 13). The high degree of incompressibility of nuclear matter, which makes it difficult to excite radial pulsations, indicates that large-scale (seismic) matter fluctuations in the interior of a neutron star, as in white dwarfs, must have a nonradial, elastodynamical nature (Refs. 111–115).[2] In this review we present one of the possible approaches to this problem.

Special attention is paid to the conclusions following from relatively recent studies carried out in laboratory nuclear physics with the goal of understanding the regular, empirically established features of the systematics of the data on giant resonances and fission. In describing the resonance response of the nucleus, modeled as a small particle of continuous matter, it has been found that it manifests the properties of an elastic sphere, rather than a drop of incompressible fluid, as suggested earlier. In the modern macroscopic theory of nuclear collective motion, giant electric and magnetic resonances are treated as fast (diabatic) excitations of, respectively, elastic spheroidal and torsional quasistatic waves (or, in other words, long-wavelength, nonradial elastodynamical vibrations).[46–56] An important achievement of this branch of laboratory nuclear theory is the rigorous formulation of fundamental equations modeling the elastic-like behavior of a continuous nuclear medium: the equations of nuclear elastodynamics. The expression obtained in this theory for the potential energy of the elastodynamical vibrations of a macroscopic nuclear particle has the form of the macroscopic energy of elastic deformations subject to the classical Hooke's law, although the microscopic nature of this energy has an essentially quantum origin due to anisotropic distortions of the Fermi sphere (see, for example, Refs. 54 and 56). One of the main results of the elastodynamical approach is the transparent physical treatment and exact quantitative description of the scaling rules,[3)] which can be clearly traced in the experimental data on the integrated characteristic parameters of giant resonances. Indirect indications of elastic-like behavior of macroscopic nuclear matter have been obtained in studies of adiabatic (slow) collective processes such as nuclear fusion[57–59] and the fission of nuclei of heavy and superheavy elements.[60]

Since a neutron star is a large-scale distribution of nuclear matter, it appears completely natural to study the continuum mechanics of nuclear matter in its intrinsic gravitational field, using the equations of nuclear elastodynamics, thereby assuming that the laws of motion of a dense nuclear medium established in laboratory experiments on nuclei remain in force for their giant cosmic twin. Such studies have been performed in Refs. 61–66, and the first half of this review is devoted to systematic discussion of them.

There is no doubt that the electromagnetic activity of a pulsar is related to anomalously high magnetization of the matter in its core.[67,68] A direct, observed consequence of the presence of a magnetic field in a neutron star is the strong linear polarization of radio pulsars. The physical reason for the superstrong magnetization of neutron stars can be understood by assuming[69,70] that even weakly magnetized stellar matter remains in an ionized (plasma) state during its evolution, and stellar collapse occurs with conservation of the magnetic flux. A characteristic dynamical feature of a magnetized and compensated plasma is the ability to support magneto-plasma (Alfvén) oscillations, the possible propagation of which in a neutron star was apparently first suggested in Refs. 71 and 72. In Ref. 71 it was shown that the energetics of a Crab-like nebula can be understood by assuming that at its center is a neutron star releasing the magnetic energy stored during the contraction stage of its creation by transforming the energy of the residual (after the supernova explosion) hydromagnetic vibrations into the energy of electromagnetic radiation. However, it seems to us that this idea has not been developed constructively. In this review we present a systematic exposition of the theory of nonradial Alfvén vibrations in a neutron star formulated in Refs. 73–76, and we describe a detailed calculation of the periods of nonradial magneto-hydrodynamical (MHD) vibrations of an essentially elastodynamical nature.

The review is organized as follows. In the second part of the Introduction we briefly discuss neutron stars. In Sec. 2 we present the elastodynamical model of global gravitational-elastic nonradial vibrations of neutron stars, which can be induced by a massive companion in a binary system, or preserved as the residual effect of the implosive creation of a pulsar. In Sec. 3 we present a constructive comparison of the hydrodynamical and elastodynamical approaches to describing self-gravitating nuclear matter and discuss the appearance of nonradial gravitational vibrations of neutron stars in the electromagnetic activity of pulsars. The theory of magneto-plasma quasistatic waves in a neutron star is presented in Secs. 4 and 5. Emphasizing the elastodynamical nature of Alfvén vibrations, we present detailed calculations of the periods of magneto-plasma oscillations. Section 6 contains the conclusions of our analysis.

1.1. Neutron stars

A neutron star (pulsar) is a spherical, magnetized, compact object of radius $R \sim 10-15$ km (for comparison, $R_\odot = 695\,980$ km) and mass $(0.3-2.5)M_\odot$ ($M_\odot = 1.989 \times 10^{33}$ g), in the core of which the matter is compressed by intrinsic gravitational forces to densities close to the normal nuclear density $\rho \sim 2.8 \times 10^{14}$ g/cm^3. The moment of inertia of a neutron star is $J = (2/5)MR^2 \sim 10^{44}-10^{45}$ g·cm^3. The most characteristic features of the nuclear medium of a neutron star are superhigh magnetization and high conductivity. The magnetic field strength at the stellar surface reaches $B \sim 10^{11}-10^{13}$ G (Ref. 67). The average magnetic moment is $\mu_{NS} \sim 10^{30}$ G·cm^3, and the average electrical conductivity is $\sigma \sim 6 \times 10^{22}$ sec^{-1}. The spatial distribution of pulsars displays a clearly expressed clustering near the Galactic plane in a disk about 500 pc thick, and the average age of the activity in the radio range is estimated to be $\tau \sim 10^6 - 10^8$ yr (Refs. 8 and 13). According to current estimates, a neutron star is created every 15–20 years.[9] The characteristic periods of pulsar radio emission lie in the range from 1.6 msec (PSR 1937+21 is presently the fastest pulsar) to 4.3 sec(PSR 1845-19 is presently the slowest). The discovery of the Crab pulsar, in the vicinity of which are clear signs of matter disintegrated and dispersed by an explosion, confirmed the hypothesis of Baade and Zwicky[77] about the birth of neutron stars in supernova explosions.[5] The qualitative picture of the creation of this pulsar is explained by the magneto-rotational scenario of implosive creation in the supernova of 1054.[31,78,79] At the critical moment when the reserves of nuclear fuel are exhausted, the gravitational instability arising in the weakly magnetized, slowly rotating predecessor star[4)] develops in such a way that the tendency for matter to fall into the center (implosion—explosion toward the inside) is accompanied by increasing density until the forces of gravitational compression are no longer in equilibrium with the pressure of the degenerate neutron Fermi continuum. The strongly magnetized and rapidly rotating compact object produced at the center ultimately forms a neutron star—a pulsar,[81] and the rest of the mass [rather larger, about $(2-6)M_\odot$] of the original star is ejected by magnetic pressure into the surrounding space in the form of a rapidly cooling, radio-wave emitting nebula. In the formation process the neutron star is heated to a temperature of 10^{11} K (10 MeV), and then rapidly cools to a temperature $T \sim 10^7 - 10^8$ K (10–100 keV; Ref. 82).

In Fig. 1 we systematically present the model of a neutron star constructed on the basis of the relativistic Tolman–Oppenheimer–Volkoff equation of gravitationalequilibrium with the typical parameters of the nuclear-matter equation of state, given in Table I, calculated by taking into account the heterophase nature of the nuclear statistical equilibrium. Equations of state are discussed in detail in Refs. 8, 21, and 22. In Figs. 2 and 3 we show the density and pressure profiles, calculated by the Hartree–Fock method, for the equations of state listed above.[22]

2. THE ELASTODYNAMICS OF SELF-GRAVITATING NUCLEAR MATTER

The elastic-like behavior of the matter continuum of neutron stars is constructively expressed by the fact that its motions are described by the equations of nuclear elastodynamics:[56,64]

$$\frac{d\rho}{dt} + \rho \frac{\partial V_i}{\partial x_i} = 0, \tag{2.1}$$

$$\rho \frac{dV_i}{dt} + \frac{\partial P_{ik}}{\partial x_k} - \rho \frac{\partial U}{\partial x_i} = 0, \tag{2.2}$$

$$\frac{dP_{ij}}{dt} + P_{ik}\frac{\partial V_j}{\partial x_k} + P_{jk}\frac{\partial V_i}{\partial x_k} + P_{ij}\frac{\partial V_k}{\partial x_k} = 0, \tag{2.3}$$

where ρ is the density of the nuclear medium, V_i are the components of the velocity field of the elastic displacements, and P_{ij} is the elastic stress tensor (summation over repeated indices is understood). The first equation, (2.1), is the familiar continuity equation. Equation (2.2) describes the motion of the nuclear-matter flow. Equation (2.3) controls the dynamics of the internal stresses. The above definition of the mass density ρ, the three components of the flow velocity V_i, and the nine components of the elastic stress tensor P_{ij} are, respectively, the zeroth, first, and second moments of the one-particle distribution function in phase space. We are studying the motion of the nuclear medium on scales where the dominant role is played by the bulk forces of intrinsic (Newtonian) gravity. Therefore, U in Eq. (2.2) is understood as the gravitational potential satisfying the Poisson equation:

$$\Delta U = -4\pi G \rho, \tag{2.4}$$

where G is the gravitational constant. As a result, we arrive at the closed system of equations (2.1)–(2.4) describing the dynamics of an ideally elastic continuous medium (elastoplasticity effects are ignored) in its intrinsic gravitational field. In the rest of our discussion these equations are treated as the fundamental equations of the elastodynamics of self-gravitating nuclear matter. The introduction of Eq. (2.3) ensures the possibility of describing both the hydrodynamical and the elastodynamical properties of the nuclear medium within a unified scheme, the manifestation of these properties being related to the nature of the local distortions of the equilibrium Fermi distribution. As a constructive demonstration of this statement, let us consider the propagation of plane-wave perturbations in a uniform, isotropic Fermi continuum.

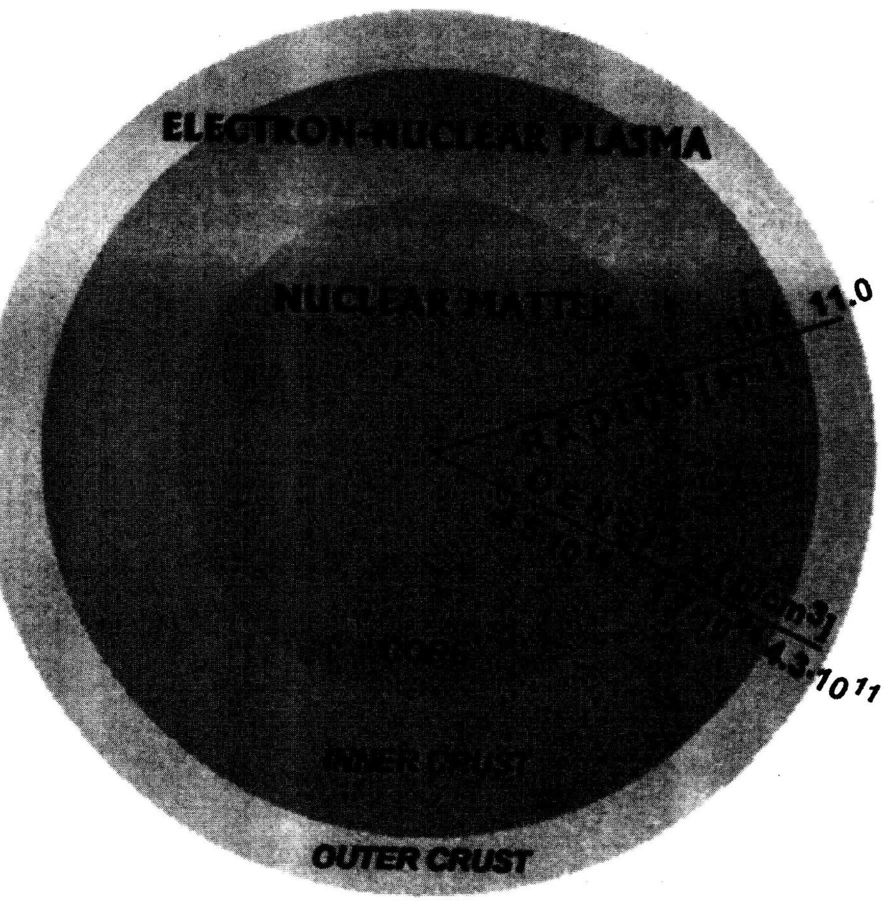

FIG. 1. Schematic picture of the cross section of a neutron star of radius 11 km and mass $1.4M_{\odot}$. The degree of shading reflects the stratified nature of the matter distribution in the cores of the neutron star. The inner part is a massive core of radius 9 km, composed of nuclear (mainly neutron) matter with average density $1.5\rho_N$ (ρ_N is the normal nuclear-matter density). In the peripheral region of the star the matter is a strongly degenerate electron–nuclear plasma (the Ae phase), and the inner core of thickness ~ 1.5 km is formed of neutron-enriched atomic nuclei (with an insignificant admixture of degenerate electrons). The outer core of depth 0.5 km consists of free electrons and iron nuclei forming a quasicrystalline lattice.

2.1. Zero-temperature waves in isotropic nuclear matter

By uniform, isotropic neutron matter we mean a continuous medium in which the distribution of internal stresses P_{ij} at zero temperature is specified by the pressure P of a Fermi gas, completely degenerate in spin and isospin, compressed to the normal nuclear density ρ. In the Thomas–Fermi approximation, the equilibrium parameters of this medium are specified by the following equivalent expressions:

$$P_{ij}=\delta_{ij}P, \quad P=\frac{\rho v_F^2}{5}, \quad v_F=\frac{\hbar k_F}{m^*}, \quad \rho=nm^*, \quad n=\frac{k_F^3}{3\pi^2}, \tag{2.5}$$

TABLE I. Parameters of the nuclear-matter equations of state (ES) used in constructing (on the basis of the equations of general relativity) realistic models of neutron stars whose integral characteristics (mass and radius) are given in Table II. E/A is the binding energy per nucleon at average particle density n, K is the compressibility, m^*/m is the ratio of the effective nucleon mass m^* in nuclear matter at a given saturation density and the free nucleon mass m, and a_{sym} is the symmetry energy. The abbreviations for the equations of state G_{200}^{π}, HV, and UVII are taken from Ref. 22.

ES	E/A [MeV]	n [F^{-3}]	K [MeV]	m^*/m	a_{sym} [MeV]
G_{200}^{π}	−15.95	0.145	200	0.80	36.8
HV	−11.5	0.175	202	0.79	29.3
UVII	−15.98	0.145	285	0.77	36.8

$$P=\frac{2}{3}\mathcal{E}_{\mathcal{N}}, \quad \mathcal{E}_{\mathcal{N}}=\frac{3}{5}n\varepsilon_F, \quad \epsilon_F=\frac{1}{2}m^*v_F^2. \tag{2.6}$$

A more complete description of the equilibrium properties of nucleon matter taking into account the nonuniformity of the density distribution is given in Ref. 51.

In momentum space, an isotropic stress distribution corresponds to spherically symmetric filling of single-particle orbitals by nucleons [Eq. (2.5), rewritten as $p_F^2/m^{*2}=5P/\rho$ $=$const is the equation of the Fermi sphere]. Spherically symmetric compression and expansion of the Fermi sphere in momentum space corresponds to isotropic changes of the stresses in the bulk of the medium:

$$P_{ij}\rightarrow(P+\delta P)\delta_{ij}. \tag{2.7}$$

In an ideal liquid or gas the perturbations propagate without spoiling the isotropy of the equilibrium isotropic stress distribution, i.e., without the appearance of shearing stresses. Mathematically, this property is reflected by the condition that the stress tensor (2.7) be spherical (all the normal stresses in the medium are expressed in terms of a single scalar quantity, the pressure). In contrast to a fluid, the perturbation of an isotropic rigid-body elastic medium is accompanied by the appearance of anisotropic stresses. Such perturbations are described by the tensor

$$P_{ij}\rightarrow P\delta_{ij}+\delta P_{ij}, \tag{2.8}$$

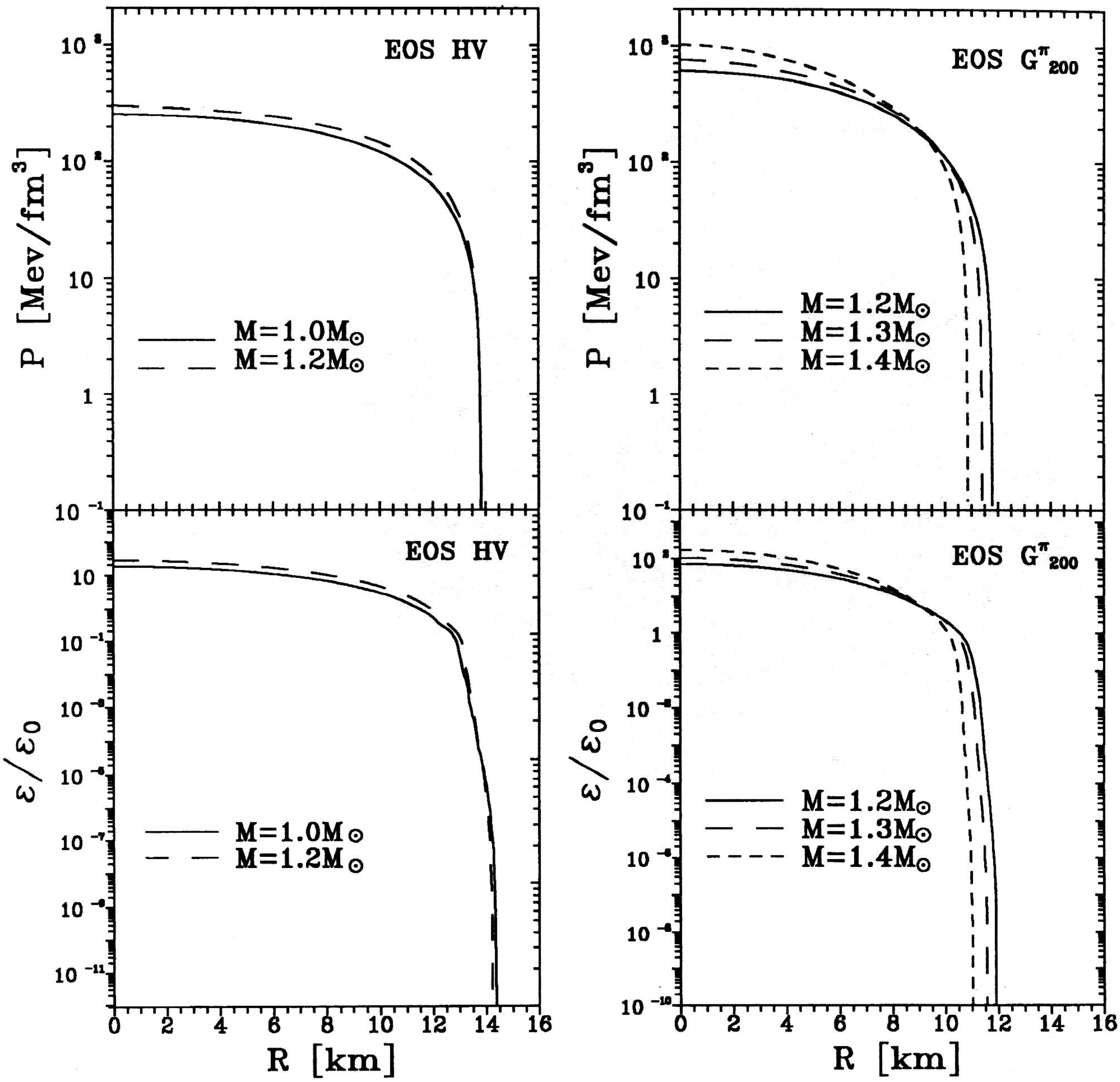

FIG. 2. Density and pressure profiles (in units of the energy density of nuclear matter, $\varepsilon_0 = 140$ MeV/F^3), calculated using the Tolman–Oppenheimer–Volkoff equilibrium equation and realistic nuclear-matter equations of state (Table III) for models of neutron stars of the given masses.

The anisotropy is expressed by the tensor δP_{ij}, which contains nonzero off-diagonal elements. In momentum space, the substitution (2.8) corresponds to anisotropic quadrupole deformation of the equilibrium Fermi distribution or, stated differently, quadrupole deformation of the Fermi system.

According to the modern theory of continuous media, the rheology of a matter continuum is determined by the dynamics of the internal stresses, which are manifested in the ability of the medium to support the propagation of perturbations in the form of waves specific to the type of medium. We shall show that the properties of an ideal Fermi liquid are manifested under linear deformations of Thomas–Fermi neutron matter which are accompanied by exclusively isotropic (compression) distortions of the Fermi distribution in momentum space, i.e., the medium can support the propagation of essentially longitudinal vibrations with phase velocity $c_L = v_F/\sqrt{3}$. Meanwhile, under perturbations leading to deformation of the equilibrium Fermi distribution of the neutrons in momentum space, neutron matter manifests the properties of an elastic Fermi continuum, i.e., a medium capable of supporting elastodynamical zero-temperature waves, namely, a wave propagating longitudinally with phase velocity $c_l = (3/5)^{1/2} v_F$ and a transverse wave with phase velocity $c_t = v_F/\sqrt{5}$.

Hydrodynamical zero sound in degenerate nuclear matter. Let us consider linear perturbations of compressible neutron matter at rest under which the Fermi sphere undergoes isotropic distortions. Here we shall consider only perturbations which do not lead to rearrangement of the original Fermi distribution, the structure of which is determined by the average field U. This actually implies that under such

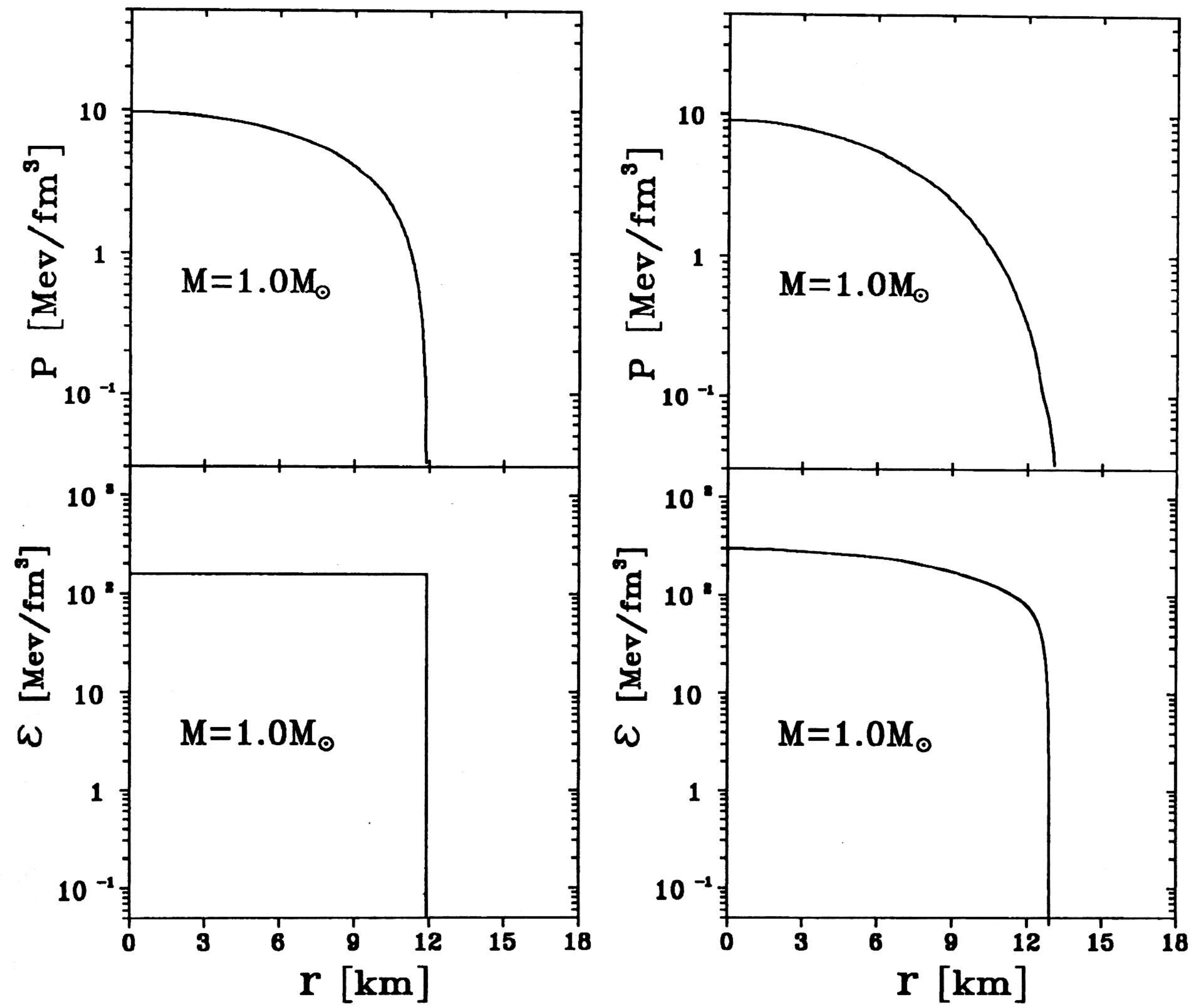

FIG. 3. Comparison of the density and pressure profiles for a neutron star of mass equal to one solar mass in the homogeneous model (right-hand figure) with average density equal to twice the nuclear density, and in the inhomogeneous model constructed on the basis of a realistic nuclear-matter equation of state.

motions the mean-field potential is not changed, i.e., $\delta U = 0$. Under these conditions, Eqs. (2.1)–(2.3) are linearized by a substitution of the form

$$\rho\to\rho+\delta\rho,\quad V_i\to V_i(=0)+\delta V_i,$$

$$P_{ij}\to\left[P\left(=\frac{\rho v_F}{5}\right)+\delta P\right]\delta_{ij}. \tag{2.9}$$

As a result, we arrive at the closed system of equations of the linear hydrodynamics of a viscous Fermi liquid:

$$\frac{\partial\delta\rho}{\partial t}+\rho\frac{\partial\delta V_i}{\partial x_i}=0, \tag{2.10}$$

$$\rho\frac{\partial\delta V_i}{\partial t}+\frac{\partial\delta P}{\partial x_i}=0, \tag{2.11}$$

$$\frac{\partial\delta P}{\partial t}+\frac{5}{3}P\frac{\partial\delta V_k}{\partial x_k}=0. \tag{2.12}$$

If (2.11) is differentiated with respect to time, then by using (2.12) we immediately obtain the equation of a longitudinal hydrodynamical wave:

$$\delta\ddot{\mathbf{V}}-c_L^2\Delta\,\delta\mathbf{V}=0\to\omega_L=c_Lk,\quad c_L=\sqrt{\frac{5P}{3\rho}}=\frac{v_F}{\sqrt{3}}, \tag{2.13}$$

expressing one of the main conclusions of the theory of a Fermi liquid, namely, that an ideal Fermi liquid can support the propagation of longitudinal sound waves (k is the wave vector) at zero temperature. As seen from (2.13), a longitudinal zero-sound wave is characterized by a dispersionless propagation law with phase velocity c_L. The essentially quantum nature of Landau zero sound means that it can be excited at zero temperature, i.e., in a completely degenerate Fermi continuum. Sound-wave propagation is impossible in a classical ideal gas, because the equilibrium pressure is proportional to T (the equation of state of a classical ideal gas is $P=nkT$, where k_B is the Boltzmann constant).

Zero-temperature elastodynamical waves in degenerate neutron matter. Let us now consider linear perturbations in a compressible neutron medium under which the Fermi sphere undergoes anisotropic distortions. In this case Eqs. (2.1)–(2.3) are linearized by the substitution

$$\rho \to \rho + \delta\rho, \quad V_i \to V_i(=0) + \delta V_i,$$

$$P_{ij} \to P\left(=\frac{\rho v_F}{5}\right)\delta_{ij} + \delta P_{ij}, \tag{2.14}$$

which leads to the following system of equations for dissipationless linearized fluctuations:

$$\frac{\partial \delta\rho}{\partial t} + \rho\frac{\partial \delta V_i}{\partial x_i} = 0, \tag{2.15}$$

$$\rho\frac{\partial \delta V_i}{\partial t} + \frac{\partial \delta P_{ij}}{\partial x_j} = 0, \tag{2.16}$$

$$\frac{\partial \delta P_{ij}}{\partial t} + P\left[\frac{\partial \delta V_i}{\partial x_j} + \frac{\partial \delta V_j}{\partial x_i} + \delta_{ij}\frac{\partial \delta V_k}{\partial x_k}\right] = 0. \tag{2.17}$$

In this case it is convenient to transform to the displacement field $\mathbf{D}(\mathbf{r},t)$, related to the velocity field $\delta\mathbf{V}(\mathbf{r},t)$ as

$$\delta V_i(\mathbf{r},t) = -\dot{D}_i(\mathbf{r},t). \tag{2.18}$$

Substituting (2.18) into (2.17), we obtain

$$\delta P_{ij} = P\left(\frac{\partial D_i}{\partial x_j} + \frac{\partial D_j}{\partial x_i} + \delta_{ij}\frac{\partial D_k}{\partial x_k}\right). \tag{2.19}$$

Comparing this expression for the stresses in neutron matter (induced by distortions of the Fermi distribution in momentum space) with the stresses obeying Hooke's law for an isotropic, ideally elastic material,[85]

$$\delta P_{ij} = \mu\left(\frac{\partial D_i}{\partial x_j} + \frac{\partial D_j}{\partial x_i}\right) + \lambda\delta_{ij}\frac{\partial D_k}{\partial x_k}, \tag{2.20}$$

we conclude that the neutron Fermi continuum behaves as an ideal elastic medium in which the elasticity modulus λ and the shear modulus μ (the Lamé parameters) are equal to each other and coincide in absolute value with the pressure, $\lambda = \mu = P$. Moreover, substituting (2.18) and (2.19) into the equation of motion for the velocity field of elastic deformations δV_i (2.17), we find that this equation takes the form

$$\rho\ddot{\mathbf{D}} = 2P \,\mathrm{grad}\,\mathrm{div}\,\mathbf{D} + P\Delta\mathbf{D}, \tag{2.21}$$

coinciding with the Lamé equation

$$\rho\ddot{\mathbf{D}} = (\lambda + \mu)\mathrm{grad}\,\mathrm{div}\,\mathbf{D} + \mu\Delta\mathbf{D} \tag{2.22}$$

describing the linear elastodynamics of an ideally elastic, isotropic, continuous medium. The analogy established between the equations of motion of the neutron Fermi continuum, perturbations of which are accompanied by anisotropic distortions of the Fermi distribution, and the Lamé equation allows us to use the classification of wave motions from the linear theory of elasticity. According to the latter, both longitudinal and transverse elastic displacement waves can propagate in an ideally elastic medium.[85] The procedure for splitting elastodynamical waves into longitudinal and transverse waves is described in detail in the literature, and so here we shall only present the main results, which follow from representation of the vector field of elastic displacements as the sum

$$\mathbf{D} = \mathbf{D}_l + \mathbf{D}_t, \tag{2.23}$$

where $\mathbf{D}_l$ obeys the equation

$$\ddot{\mathbf{D}}_l - c_l^2\Delta\mathbf{D}_l = 0, \quad \mathrm{curl}\;\mathbf{D}_l = 0 \tag{2.24}$$

for a longitudinal wave propagating with phase velocity

$$c_l = \sqrt{\frac{\lambda + 2\mu}{\rho}} = \sqrt{\frac{3}{5}}v_F. \tag{2.25}$$

The equation for transverse wave motions has the form

$$\ddot{\mathbf{D}}_t - c_t^2\Delta\mathbf{D}_t = 0, \quad \mathrm{div}\,\mathbf{D}_t = 0. \tag{2.26}$$

This wave propagates with velocity

$$c_t = \sqrt{\frac{\mu}{\rho}} = \frac{v_F}{\sqrt{5}}. \tag{2.27}$$

A transverse wave does not create compressions and rarefactions, and so it can propagate in an incompressible neutron (nuclear) medium.

2.2. The standard model of a neutron star

The heuristic value of standard stellar models is that they can be used to obtain analytic estimates of the equilibrium and dynamical parameters of stars.[34] In the standard model, a neutron star is idealized as a spherical, homogeneous mass of incompressible matter possessing the properties of a nonrelativistic, degenerate, neutron Fermi continuum, compressed by intrinsic gravitational forces to densities close to the normal nuclear density.

The equilibrium distribution of the intrinsic gravitational field is given by the familiar solution of Eq. (2.4):

$$U_0^{in} = -\frac{2\pi}{3}G\rho_0(r^2 - 3R^2), \quad r \le R, \tag{2.28}$$

$$U_0^{ex} = \frac{4\pi R^3}{3r}G\rho_0, \quad r > R, \tag{2.29}$$

where the subscript zero denotes the equilibrium characteristics.

A consequence of the spherical symmetry of the gravitational interaction is the isotropic equilibrium distribution of stresses (pressure) in the stellar volume:

$$P_{ij}^0(r) = P_0(r)\delta_{ij}. \tag{2.30}$$

The radial dependence of the pressure is determined from the equilibrium equation with the boundary condition that the pressure at the center of the star is determined by the internal pressure of the nuclear matter, $P_N(\rho_0)$:

$$\nabla P_0(r) = \rho_0\nabla U_0^{in}(r), \quad P_0(r=0) = P_N(\rho_0). \tag{2.31}$$

The solution of this equation has the form

$$P_0(r) = P_N(\rho_0) - (2\pi/3)G\rho_0^2 r^2. \tag{2.32}$$

We note that $P_N(\rho_0)$ is the main characteristic carrying information about the nuclear-matter equation of state, and therefore relating nuclear physics to neutron-star physics. In the standard model, $P_N(\rho_0)$ is understood as the pressure of degenerate neutron matter:[8]

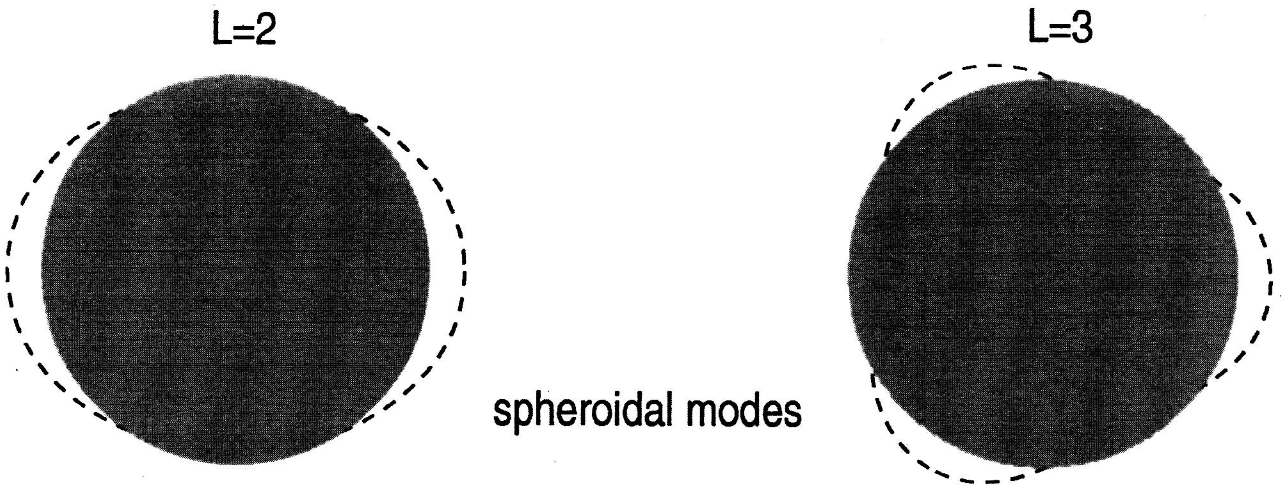

FIG. 4. Schematic depiction of the motions of nuclear matter in a neutron star undergoing spheroidal quadrupole ($L=2$) and octupole ($L=3$) nonradial gravitational-elastic vibrations.

$$P_N(\rho_0)=(2/3)\mathcal{E}_N(\rho_0)=K\rho_0^{5/3}, \quad \rho_0=m^*\frac{2}{3\pi^2}k_F^3,$$
$$K=\frac{\hbar^2(3\pi^2)^{2/3}}{5m^{*8/3}}. \tag{2.33}$$

Comparing Fig. 4 with Figs. 2 and 3, we get an idea of the degree of reliability of the sharp-boundary approximation for the density profile on which the standard neutron-star model is based. For a given density, from (2.32) we easily obtain an estimate of the neutron-star radius, defined as the radius of the spherical surface free from stresses $P_0(r=R)=0$, i.e., the surface where the elastic stresses due to the intrinsic gravitation are balanced by the internal pressure of the neutron Fermi continuum. From the condition

$$P_0(R)=P_N(\rho_0)-(2\pi/3)G\rho_0^2R^2=0 \quad \rightarrow \quad P_N=\frac{2\pi}{3}G\rho_0^2R^2, \tag{2.34}$$

we find that the equilibrium radius and mass of a neutron star in the standard model are given by

$$R=\sqrt{\frac{3P_N}{2\pi G\rho_0^2}}, \quad M=(4\pi/3)\rho_0R^3. \tag{2.35}$$

In Table II we give the numerical values for M and R calculated from these expressions, and in Fig. 2 we show the density and pressure profiles. These estimates agree satisfactorily with the results of the mass and radius calculations obtained using relativistic neutron-star models and given in Table III.

It has been repeatedly pointed out in the literature[13] that since the radius R of a neutron star is comparable to its gravitational radius $R_G=2GM/c^2\sim3$ km, reliable results can be obtained only taking into account effects from the general theory of relativity. Nevertheless, the above estimates convincingly show that the standard model based on Newtonian gravity and using the equation of purely neutron matter as the equation of state of nuclear matter leads to values of the integrated equilibrium parameters (the mass, radius, and moment of inertia) in good agreement with the predictions of realistic models of neutron stars.[21,22] This suggests that the use of the nonrelativistic gravitational equation will not result in serious errors even in calculating the periods of nonradial gravitational-elastic modes.

2.3. Nonradial gravitational-elastic vibrations of a neutron star

In an incompressible elastic medium, nonradial long-wavelength vibrations (quasistatic elastic waves) are the only

TABLE II. Periods P_L (in msec) of nonradial gravitational spheroidal (s-mode) and torsional (t-mode) elastodynamical vibrations for realistic (inhomogeneous) models of neutron stars constructed on the basis of the Tolman–Oppenheimer–Volkoff equation of hydrostatic equilibrium, and three versions of the nuclear-matter equation of state with the parameters given in Table I.

Inhomogeneous model			Gravitational s-mode			Gravitational t-mode		
ES	$M/M_\odot$	R [km]	P_2 [msec]	P_3 [msec]	P_4 [msec]	P_2 [msec]	P_3 [msec]	P_4 [msec]
HV	1.0	14.38	0.41	0.30	0.25	0.60	0.44	0.36
HV	1.2	14.23	0.37	0.27	0.22	0.54	0.39	0.33
HV	1.9	11.28	0.14	0.11	0.09	0.21	0.15	0.13
G_{200}^{π}	1.2	11.95	0.26	0.19	0.17	0.39	0.29	0.24
G_{200}^{π}	1.3	11.59	0.23	0.17	0.15	0.35	0.26	0.22
G_{200}^{π}	1.4	11.03	0.20	0.15	0.13	0.30	0.22	0.18
UVII	2.1	9.20	0.09	0.07	0.06	0.14	0.10	0.08

TABLE III. Periods P_L (in msec) of nonradial gravitational-elastic spheroidal (s-mode) and torsional (t-mode) pulsations, calculated using the standard homogeneous model of a neutron star (a self-gravitating spherical mass of degenerate neutron matter).

Homogeneous model			Spheroidal mode			Torsional mode	
$M/M_\odot$	R [km]	P_2 [msec]	P_3 [msec]	P_4 [msec]	P_2 [msec]	P_3 [msec]	P_4 [msec]
0.89	12.34	0.14	0.11	0.08	0.20	0.14	0.12
0.95	12.10	0.13	0.10	0.08	0.19	0.13	0.11
1.00	11.89	0.13	0.09	0.07	0.18	0.13	0.10
1.04	11.70	0.12	0.09	0.07	0.17	0.12	0.10
1.09	11.54	0.12	0.08	0.07	0.16	0.12	0.09
1.14	11.38	0.11	0.08	0.06	0.16	0.11	0.09
1.18	11.24	0.10	0.08	0.06	0.15	0.11	0.09
1.22	11.12	0.10	0.07	0.06	0.15	0.10	0.08
1.26	11.00	0.10	0.07	0.06	0.14	0.10	0.08
1.30	10.89	0.09	0.07	0.06	0.14	0.09	0.08
1.34	10.78	0.09	0.07	0.05	0.13	0.09	0.08
1.37	10.69	0.09	0.06	0.05	0.13	0.09	0.07
1.41	10.59	0.09	0.06	0.05	0.13	0.09	0.07
1.44	10.51	0.09	0.06	0.05	0.12	0.09	0.07
1.48	10.43	0.09	0.06	0.05	0.12	0.09	0.07

possible type of dynamical activity. The fundamental frequencies of nonradial vibrations can be calculated analytically, using the energy variational principle.[84] Assuming that mass flow is absent in the equilibrium state and using the standard linearization procedure,

$$\rho\to\rho_0+\delta\rho(=0),\quad V_i\to V_i^0(=0)+\delta V_i,$$

$$P_{ij}\to\delta_{ij}P_0+\delta P_{ij},\quad U\to U_0+\delta U,$$

Eqs. (2.1)–(2.4) can be transformed to

$$\frac{\partial\delta V_i}{\partial x_i}=0, \tag{2.36}$$

$$\rho_0\frac{\partial\delta V_i}{\partial t}+\frac{\partial\delta P_{ij}}{\partial x_j}-\rho_0\frac{\partial\delta U}{\partial x_i}=0, \tag{2.37}$$

$$\frac{\partial\delta P_{ij}}{\partial t}+P_0\left(\frac{\partial\delta V_i}{\partial x_j}+\frac{\partial\delta V_j}{\partial x_i}\right)+\delta_{ij}\left(\delta V_k\frac{\partial P_0}{\partial x_k}\right)=0, \tag{2.38}$$

$$\Delta\,\delta U=0. \tag{2.39}$$

Then, performing scalar multiplication of Eq. (2.37) by δV_i and integrating over the stellar volume, we obtain the energy-balance equation:

$$\frac{\partial}{\partial t}\int_V\frac{1}{2}\rho_0\delta V^2d\tau-\int_V\delta P_{ij}\frac{\partial\delta V_i}{\partial x_j}d\tau-\oint_S[\rho_0\delta U\delta V_i-\delta P_{ij}\delta V_j]d\sigma_i=0, \tag{2.40}$$

which controls the energy conservation in the vibration process. We can write the velocity fluctuations of the perturbed flow δV_i and the self-gravitational potential δU as

$$\delta V_i(\mathbf{r},t)=-\xi_i^L(\mathbf{r})\dot\alpha_L(t),\quad \delta U(\mathbf{r},t)=\phi^L(\mathbf{r})\alpha_L(t), \tag{2.41}$$

where L is the multipole order of the vibrations. The normal coordinate $\alpha_L(t)$ determines the time dependence of the fluctuating variables. The instantaneous elastic displacement field is denoted by $\boldsymbol{\xi}^L(\mathbf{r})$. Substituting (2.41) into (2.38), we find that the stress fluctuations are given by the tensor

$$\delta P_{ij}(\mathbf{r},t)=\left[P_0(\mathbf{r})\left(\frac{\partial\xi_i^L(\mathbf{r})}{\partial x_j}+\frac{\partial\xi_j^L(\mathbf{r})}{\partial x_i}\right)+\delta_{ij}\left(\xi_k^L(\mathbf{r})\frac{\partial P_0(\mathbf{r})}{\partial x_k}\right)\right]\alpha_L(t). \tag{2.42}$$

The linear part of the elastic stress tensor δP_{ij} (the dynamical characteristic of the elasticity of the material) along with the elastic-deformation tensor U_{ij} (the kinematical characteristic of elastodynamical displacements),

$$\delta P_{ij}\sim 2P(r)U_{ij},\quad U_{ij}=\frac{1}{2}\left(\frac{\partial\xi_i^L}{\partial x_j}+\frac{\partial\xi_j^L}{\partial x_i}\right)\alpha_L(t), \tag{2.43}$$

indicate that the distribution of elastic distortions in a nuclear medium obeys Hooke's law.[85] Owing to the separation of the spatial and time dependences of the fluctuating variables, the substitution of (2.41) and (2.42) into the energy-balance equation (2.40) transforms it to

$$\frac{d\mathcal{H}}{dt}=0,\quad \mathcal{H}=\frac{\mathcal{M}_L\dot\alpha_L^2}{2}+\frac{\mathcal{K}_L\alpha_L^2}{2}, \tag{2.44}$$

where the inertial parameter $\mathcal{M}_L$ and the stiffness parameter $\mathcal{K}_L$ are defined as

$$\mathcal{M}_L=\int_V\rho_0\xi_i^L\xi_i^Ld\tau,\quad \mathcal{K}_L=\frac{1}{2}\int_V P_0\left(\frac{\partial\xi_i^L}{\partial x_j}+\frac{\partial\xi_j^L}{\partial x_i}\right)^2d\tau. \tag{2.45}$$

In deriving the equation for the stiffness $\mathcal{K}_L$ we neglected surface effects. This approximation can be treated as the elastodynamical analog of the Cowling hydrodynamical approximation.[36] The Cowling approximation can be expressed constructively as

$$\left[\rho_0\Phi-\xi_k\frac{\partial P_0}{\partial x_k}\right]_R=0, \tag{2.46}$$

which will be used below as the boundary condition for finding the arbitrary integration constants in the calculation of the elastic-displacement field.

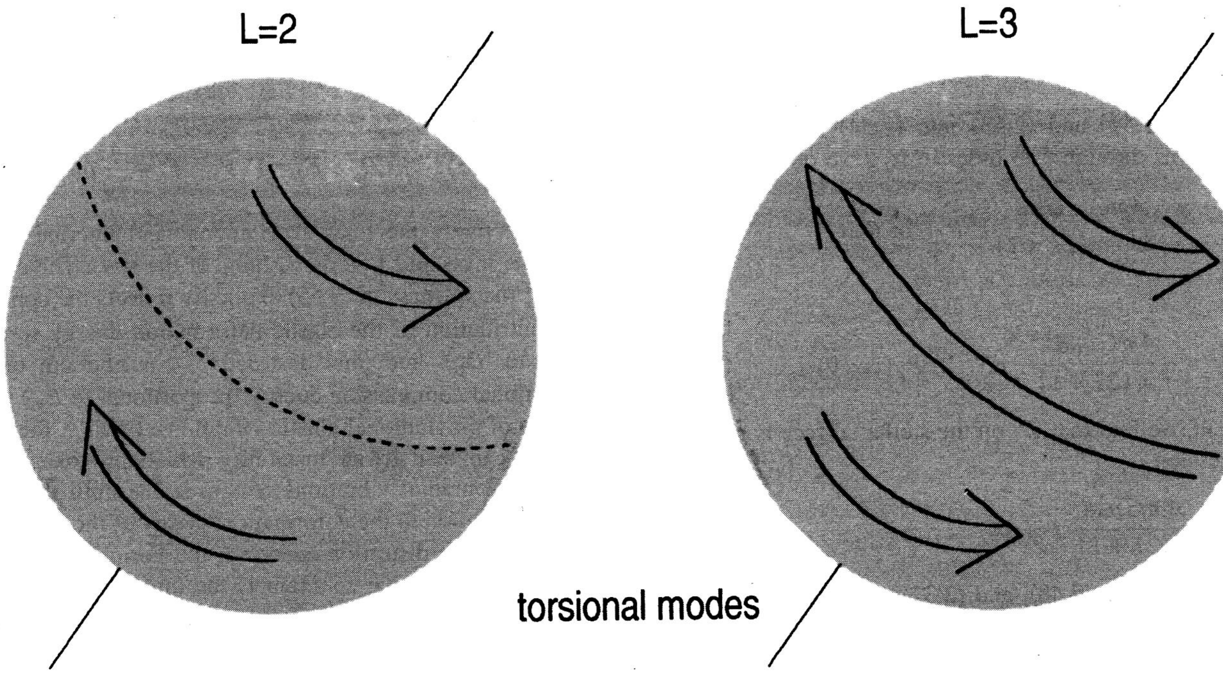

FIG. 5. Schematic depiction of torsional quadrupole ($L=2$) and octupole ($L=3$) nonradial gravitational-elastic vibrations of a neutron star.

It follows from the expressions for the inertia $\mathcal{M}_L$ and stiffness $\mathcal{K}_L$ that the instantaneous displacement field $\boldsymbol{\xi}(\mathbf{r})$ is the only quantity to be determined which is needed for calculating the frequencies of intrinsic gravitational-elastic nonradial vibrations. It was shown in Ref. 83 that the instantaneous displacement field arising in nonradial vibrations of a spherical mass of elastic matter is found as solutions of the vector Laplace equation:

$$\Delta \boldsymbol{\xi}(\mathbf{r})=0, \quad \operatorname{div} \boldsymbol{\xi}(\mathbf{r})=0. \tag{2.47}$$

According to the Lamb treatment[86] of the eigenmodes of an ideally elastic sphere, they can be classified as spheroidal modes described by the poloidal solution of (2.47):

$$\boldsymbol{\xi}(\mathbf{r})=\frac{N_p}{L+1}\operatorname{curl}\ \operatorname{curl}\ \mathbf{r}\, r^L P_L(\mu)=N_p \operatorname{grad} r^L P_L(\mu),$$

$$\mu=\cos\theta, \tag{2.48}$$

and torsional modes corresponding to toroidal solutions of the form

$$\boldsymbol{\xi}(\mathbf{r})=N_t \operatorname{curl}\mathbf{r}\, r^L P_L(\mu). \tag{2.49}$$

Here $P_L(\mu)$ denotes the Legendre polynomial of multipole order L. The most important feature of this variational method is that the frequencies $\omega^2=\mathcal{K}_L/\mathcal{M}_L$ of the two branches of spheroidal (s-mode) and torsional (t-mode) vibrations can be calculated in a unified way as the eigenmodes of the oscillator Hamiltonian (2.44). We note that in the Cowling approximation the specific form of the arbitrary constants N_p and N_t is not fundamental, since they enter into both $\mathcal{M}_L$ and $\mathcal{K}_L$ quadratically.

2.4. The periods of spheroidal gravitational-elastic nonradial vibrations: the *s* mode

In spheroidal multipole vibrations, an arbitrary spherical surface inside the stellar volume takes the shapes of harmonic spheroids, given in the coordinate frame with fixed polar axis by the equation

$$r'(t)=r[1+\alpha_L(t)P_L(\cos\theta)]. \tag{2.50}$$

Here r is the radius of the unperturbed spherical surface. Figure 5 illustrates the spheroidal quadrupole and octupole vibrations of the star shape. To determine the arbitrary constant N_p in the expression for the poloidal instantaneous displacement field (2.48), we impose the following (dynamical) boundary condition:

$$\left[\rho\phi^L-\xi_r^L\frac{\partial P_0}{\partial r}\right]_{r=R_0}=0, \tag{2.51}$$

in which the only unknown quantity is the function ϕ^L determining the surface fluctuations of the gravitational potential (2.41). The variation δU satisfying the Laplace equation is determined by the following solutions of the latter:

$$\delta U^{in}=B_L r^L P_L(\cos\theta)\alpha_L, \quad r\leqslant R, \tag{2.52}$$

$$\delta U^{ex}=C_L r^{-(L+1)}P_L(\cos\theta)\alpha_L, \quad r>R. \tag{2.53}$$

The arbitrary constants B_L and C_L are fixed by the standard boundary conditions:

$$U_0^{in}(r')+\delta U^{in}(r')=U_0^{ex}(r')+\delta U^{ex}(r')\big|_{r'=R',(r=R)}, \tag{2.54}$$

$$\left.\frac{\partial U_0^{in}(r')}{\partial r'}+\frac{\partial \delta U^{in}(r')}{\partial r'}=\frac{\partial U_0^{ex}(r')}{\partial r'}+\frac{\partial \delta U^{ex}(r')}{\partial r'}\right|_{r'=R',(r=R)} \tag{2.55}$$

Substituting (2.52) and (2.53) into (2.54) and (2.55), and keeping terms through first order in α_L, we obtain

$$\delta U^{in}=-\frac{4\pi}{R^{L-2}}\frac{G\rho_0}{(2L+1)}r^L P_L(\cos\theta)\alpha_L, \tag{2.56}$$

and

$$\delta U^{ex}=-\frac{4\pi G\rho_0 R^{L+3}}{(2L+1)}r^{-(L+1)}P_L(\cos\theta)\alpha_L. \tag{2.57}$$

As a result, we find that ϕ^L on the stellar surface is given by the expression

$$\phi^L=-\frac{4\pi G\rho_0 R^2}{(2L+1)}P_L(\mu). \tag{2.58}$$

Substituting (2.58), (2.48), and (2.32) into (2.51), we find

$$N_p=\frac{3}{L(2L+1)R^{L-2}}. \tag{2.59}$$

The components of the poloidal field in spherical coordinates have the form

$$\xi_r=N_p r^{L-1}P_L(\mu),$$

$$\xi_\theta=-N_p(1-\mu^2)^{1/2}r^{L-1}\frac{dP_L(\mu)}{d\mu},\quad \xi_\phi=0. \tag{2.60}$$

Substituting (2.60) into (2.45) and integrating over the complete solid angle, we obtain the following expressions for the inertia and stiffness:

$$\mathcal{M}_L=4\pi L N_p^2\int_0^{R_0}\rho_0(r)r^{2L}dr,$$

$$\mathcal{K}_L=8\pi N_p^2 L(L-1)(2L-1)\int_0^{R_0}P_0(r)r^{2L-2}dr. \tag{2.61}$$

These will prove extremely useful for what follows. The details of how to calculate these integrals are given in Appendix A. The final expression for calculating the frequency of the gravitational-elastic s mode has the form

$$\omega_s=\left[\frac{2(2L-1)(L-1)\int_0^{R_0}P_0(r)r^{2L-2}dr}{\int_0^{R_0}\rho_0(r)r^{2L}dr}\right]^{1/2}. \tag{2.62}$$

It is easily seen that the monopole $(L=0)$ and dipole $(L=1)$ modes in the homogeneous model are excluded. The excitation of monopole (purely radial) vibrations is impossible, owing to the incompressibility of the stellar nuclear medium. Perturbation of the dipole velocity field of elastic displacements can lead only to motion of the star center of mass without any change of the internal state, since the stiffness coefficient vanishes for $L=1$.[5)]

The standard homogeneous model leads to the expressions

$$\mathcal{M}_L=\frac{27MR^2}{L(2L+1)^3},$$

$$\mathcal{K}_L=36E_N\frac{L-1}{L(2L+1)^2}-45E_G\frac{(L-1)(2L-1)}{L(2L+1)^3}, \tag{2.65}$$

where

$$E_G=(3/5)GM^2/R,\quad E_N=\mathcal{E}_N V,\quad \mathcal{E}_N=\frac{3}{10}\rho v_F^2 \tag{2.66}$$

are respectively the total gravitational and internal energy (M is the mass and V is the volume of the star). The expression for the stiffness in (2.65) explicitly reflects the constructive contribution of the elastic-deformation energy (proportional to E_N) and the destructive contribution of the gravitational compression energy (proportional to E_G) to the stiffness of gravitational-elastic vibrations. Clearly, this combination can lead to an instability which provokes stellar shaking. Nonradial vibrations remain stable until the dominant contribution to the deformation energy of the star comes from the elastic-distortion energy of the Fermi sphere of the neutron matter, which counteracts the intrinsic gravitational energy. The final expression for the s-mode frequency can be written as

$$\omega_s^2=\omega_0^2(2L+1)(L-1)\left[1-\Gamma\frac{5(2L-1)}{4(2L+1)}\right],$$

$$\omega_0^2=\frac{3E_N}{4MR^2},\quad \Gamma=\frac{E_G}{E_N}. \tag{2.67}$$

The parameter Γ is a measure of the vibrational stability of the neutron star. The star becomes unstable with respect to spheroidal gravitational-elastic deformations when $\mathcal{K}_{L=2}=0$. This can occur only when the ratio of the gravitational and internal energies reaches the critical value $\Gamma_{cr}=4/3$. As a result, we arrive at the following stability condition for the spheroidal vibrations of a neutron star:

$$\omega_s(L=2)\geqslant 0\ \rightarrow\ \Gamma\leqslant\frac{4}{3}. \tag{2.68}$$

This condition is satisfied for all the neutron-star models listed in Table II.

The eigenfrequencies of spheroidal modes in the standard model (where $P_N=\rho_0 v_F^2/5$) can be written in a different form [equivalent to (2.67)]:

$$\omega_s^2=\omega_F^2\frac{2}{5}(2L+1)(L-1)-\omega_G^2(2L-1)(L-1)$$

$$=\omega_F^2\frac{2}{5}(2L+1)(L-1)\left[1-\beta\frac{5(2L-1)}{2(2L+1)}\right],$$

$$\beta=\left(\frac{\omega_F}{\omega_G}\right)^2. \tag{2.69}$$

This representation emphasizes the characteristic difference between the frequency of quantum elastic oscillations $\omega_F^2=v_F^2/R^2$ and the gravitational frequency $\omega_G^2=4\pi G\rho_0/3$, which is that ω_F depends on the stellar radius and density, while ω_G depends only on the stellar density. In this representation β is the parameter of vibrational stability of gravitational-elastic vibrations. We note that the first term in (2.69) exactly coincides with the expression for the frequency (related to the energy $E=\hbar\omega$) of nuclear giant electric resonances of multipole order $L\geqslant 2$ obtained in the elastodynamical collective nuclear model.[47,54,56] As already noted, in this model electric giant resonances are treated as a manifestation of the spheroidal nonradial vibrations of a macroscopic nuclear particle, in which the restoring force of the elastic deformations (which exactly coincides with the Hooke force of linear elasticity theory) is due to the reaction of the nucleus to anisotropic distortions of the Fermi sphere. The similarity between a neutron star and a giant nucleus with mass number $A\sim 10^{57}$ is missed in this observation, but, strictly speaking, the analogy is not completely justified.

Let us conclude this section by making one important comment concerning the potential possibilities of this variational method. In the basic expression for the frequency of spheroidal gravitational-elastic nonradial vibrations (2.62), the density and pressure profiles $\rho_0(r)$ and $P_0(r)$ are present as input parameters of the equilibrium configuration. With this in mind, the constraint related to the use of the Newtonian approximation and also the assumption that the mass distribution is homogeneous can be weakened. In order to obtain reliable estimates of the s-mode frequencies, these local equilibrium characteristics can be taken from realistic models of neutron stars. The results of calculating the periods of spheroidal modes for several realistic models of neutron stars are given in Table II. Comparison of these calculations with the predictions of the standard model (see Table III) shows that the estimates of the latter determine the lower limit on the periods of spheroidal vibrations.

2.5. The periods of torsional gravitational-elastic nonradial vibrations: the *t* mode

The property of dynamical elasticity of a neutron Fermi continuum indicates that a neutron star can undergo transverse-shear, torsional vibrations. The geometrical picture of torsional vibrations is given by the following representation of the toroidal velocity field of the elastic displacements:

$$\delta\mathbf{V}=N_t\,\mathrm{curl}\,\mathbf{r}r^L P_L(\cos\theta)\dot{\alpha}_L(t)=[\mathbf{\Omega}(\mathbf{r},t)\times\mathbf{r}], \quad (2.70)$$

where

$$\mathbf{\Omega}(\mathbf{r},t)=N_t\,\mathrm{grad}\,r^L P_L(\cos\theta)\dot{\alpha}_L(t),$$

is the angular velocity field of differentially rotational vibrations. The normal coordinate $\alpha_L(t)$ in this case is interpreted as the infinitesimal angle of twist of the elastic displacement field about the polar axis. In spherical coordinates the components of the toroidal elastic displacement field have the form

$$\xi_r=0, \quad \xi_\theta=0, \quad \xi_\phi=-N_t r^L(1-\mu^2)^{1/2}\frac{dP_L(\mu)}{d\mu}. \quad (2.71)$$

The boundary condition fixing the arbitrary constant N_t has the form

$$\delta\mathbf{V}(\mathbf{r},t)|_{r=R}=[\mathbf{\Omega}_0\times\mathbf{r}] \quad \text{with } \mathbf{\Omega}_0=\dot{\alpha}_L(t)\,\mathrm{grad}\,P_L(\mu) \rightarrow N_t=\frac{1}{R^{L-1}}. \quad (2.72)$$

Let us discuss in more detail the case of quadrupole torsional vibrations. The components of the velocity field of these motions are given by $\delta V_x=-\Omega_z y$, $\delta V_y=\Omega_z x$, and $\delta V_z=0$. Formally, this field has the same form as the velocity field of rigid-body rotation. However, in quadrupole torsion the angular velocity $\mathbf{\Omega}$ is not a constant vector, but a vector field with components $\Omega_x=0$, $\Omega_y=0$, and $\Omega_z=\dot{\alpha}z$. The Cartesian components of the toroidal field of quadrupole torsion have the form $\xi_x=-yz$, $\xi_y=xz$, and $\xi_z=0$. The vibrational motions corresponding to such shifts are axially symmetric, antiphase oscillations of the north and south hemispheres of the star about the z axis. A schematic picture of the elastic displacements in differential-rotational quadrupole and octupole nonradial vibrations of a neutron star is given in Fig. 5.

Substituting (2.71) into (2.45) and integrating over the total solid angle, we find

$$\mathcal{M}_L=4\pi N_t^2\frac{L(L+1)}{(2L+1)}\int_0^{R_0}\rho_0(r)r^{2L+2}dr,$$

$$\mathcal{K}_L=4\pi N_t^2 L(L^2-1)\int_0^{R_0}P_0(r)r^{2L}dr. \quad (2.73)$$

We again stress that this representation can be used to calculate the t-mode frequency

$$\omega_t=\left[\frac{(2L+1)(L-1)\int_0^{R_0}P_0(r)r^{2L}dr}{\int_0^{R_0}\rho_0(r)r^{2L+2}dr}\right]^{1/2} \quad (2.74)$$

on the basis of the density and pressure profiles taken from realistic models of neutron stars.[6)]

Calculations of the mass and the stiffness parameters in the standard model of a neutron star give

$$\mathcal{M}_L=3MR_0^2\frac{L(L+1)}{(2L+1)(2L+3)},$$

$$\mathcal{K}_L=2E_N\frac{L(L^2-1)}{(2L+1)}-\frac{5}{2}E_G\frac{L(L^2-1)}{(2L+3)}. \quad (2.77)$$

For $L=1$ the mass parameter coincides with the moment of inertia of an absolutely rigid sphere, $J_0=\frac{2}{5}MR_0^2$. It clearly follows from the expression for the stiffness that a dipole torsional excitation is not an eigenmode of the torsional vibrations of a neutron star (since in this case the stiffness coefficient vanishes), but corresponds to rigid-body rotation. We therefore conclude that the multipole order of the lowest stable torsional mode is $L=2$. In the standard model the t-mode frequency of global gravitational-elastic nonradial vibrations of a neutron star is given by

$$\omega_t^2=\frac{1}{2}\omega_0^2(2L+3)(L-1)\left[1-\Gamma\frac{5(2L+1)}{4(2L+3)}\right], \quad (2.78)$$

where ω_0 and η were defined above [see Eq. (2.67) of the preceding subsection]. A neutron star remains stable with respect to a quadrupole torsional deformation if and only if

$$\omega_t(L=2)\geqslant 0 \quad \text{or} \quad \Gamma\leqslant 1.15. \tag{2.79}$$

Let us give another equivalent representation of Eq. (2.78):

$$\omega_t^2=\omega_F^2\frac{1}{5}(2L+3)(L-1)-\omega_G^2(2L+1)(L-1)$$

$$=\omega_F^2\frac{1}{5}(2L+3)(L-1)\left[1-\beta\frac{5(2L+1)}{2(2L+3)}\right], \tag{2.80}$$

where the Fermi frequency ω_F, the gravitational frequency ω_G, and the vibrational stability parameter β were defined above. The first term in (2.80) exactly coincides with the expression for the frequency of nuclear magnetic giant resonances, which in nuclear elastodynamics[56] are described in terms of quasistatic torsional waves excited in the volume of the nucleus modeled as a macroscopic particle of nuclear matter. This analogy again displays the resemblance between a neutron star and its microscopic twin—an atomic nucleus. We again stress that torsional (differential-rotational) vibrations of a neutron star are exclusively due to the dynamical elasticity of the degenerate Fermi continuum. Such modes do not arise in the gaseous medium of a main-sequence star governed by the equations of hydrodynamics.

In Tables II and III we present numerical values of the periods of global nonradial torsional pulsations of a neutron star for the three lowest multipole orders, calculated in both the standard and realistic inhomogeneous models. We see from comparison of the numerical values given in these tables that the estimates of the standard model determine the lower bound on the periods of torsional gravitational-elastic vibrations. Comparison of the numerical values of the periods of the lowest torsional modes with the periods of spheroidal modes shows that torsional nonradial vibrations are slow compared with spheroidal ones. The measured period of the Crab pulsar is $P=33.3$ msec, that of the Vela pulsar is $P=89.3$ msec, and that of the Heming pulsar is $P=273.1$ msec. The periods of all the pulsars presently known lie in the range $P\sim 1.6$ msec–4.3 sec, whereas our estimates of the periods of gravitational-elastic nonradial modes give $P\sim 0.01-0.1$ sec. It follows from this comparison that nonradial gravitational vibrations, like the radial vibrations studied earlier, are unrelated to the pulsating radio emission of neutron stars. However, these vibrations may be related to the fine structure of the spectra of complex or C-pulsars.[7] A characteristic feature of the latter is the fact that the average pulse profile of these pulsars reveals substructures whose nature remains a mystery. In particular, along with a clearly distinguishable pulse localized approximately between the peaks of the main pulses, there are clear structures called subpulses with a periodicity of order 10^{-4} sec.[7,13] The calculated results given in Tables II and III show that the periods of nonradial oscillations are of the same order. Therefore, as first pointed out in Ref. 111, the variations in the intensity of radio emission recorded over the time interval of the micropulses can be attributed to residual gravitational vibrations of neutron stars.

3. NONRADIAL VIBRATIONS IN THE HYDRODYNAMICAL MODEL: THE KELVIN *f* MODE

In order to trace the differences between the elastodynamical and hydrodynamical treatments of the motions of the nuclear matter of neutron stars, let us derive the spectrum of eigenvibrations of a spherical, homogeneous mass of incompressible, inviscid fluid, using the variational method presented above. The motions of an inviscid fluid in its intrinsic gravitational field are described by the equations

$$\frac{d\rho}{dt}+\rho\frac{\partial V_i}{\partial x_i}=0, \tag{3.1}$$

$$\rho\frac{dV_i}{dt}+\frac{\partial P_{ik}}{\partial x_k}+\rho\frac{\partial U}{\partial x_i}=0, \quad P_{ik}=P\delta_{ik}, \tag{3.2}$$

$$\Delta U=4\pi G\rho. \tag{3.3}$$

The main feature distinguishing the hydrodynamical description from the elastodynamical one is that the distribution of internal stresses in a fluid is isotropic, and so it is described by a scalar pressure function P rather than by a tensor, as in the case of an elastic continuum (i.e., in a fluid always $P_{ij}=P\delta_{ij}$). A characteristic feature of the dynamical behavior of a fluid is the conservation of this property as perturbations propagate. In other words, external perturbations do not spoil the isotropy of the equilibrium internal stresses. Therefore, the hydrodynamical equations are linearized by the replacement $P_{ij}\rightarrow(P+\delta P)\delta_{ij}$, and for small perturbations the fluid motions obey linear equations of the form

$$\frac{\partial\delta V_i}{\partial x_i}=0, \tag{3.4}$$

$$\rho_0\frac{\partial\delta V_i}{\partial t}+\frac{\partial\delta P}{\partial x_i}+\rho_0\frac{\partial\delta U}{\partial x_i}=0, \tag{3.5}$$

$$\Delta\delta U=0. \tag{3.6}$$

Following the variational principle described above, we perform scalar multiplication of the linearized Euler equation (3.5) by δV_i and integrate the result over the stellar volume:

$$\frac{\partial}{\partial t}\int_V\rho_0\frac{\delta V^2}{2}d\tau+\oint_S(\delta P+\rho_0\delta U)\delta V_i d\sigma_i=0. \tag{3.7}$$

Then, using the Rayleigh factorization procedure, we obtain the standard equation for normal vibrations:

$$\mathcal{M}_L\ddot{\alpha}_L^2+\mathcal{K}_L\alpha_L^2=0, \tag{3.8}$$

where the mass parameter $\mathcal{M}_L$ and the stiffness parameter $\mathcal{K}_L$ are given by

$$\mathcal{M}_L=\int_V\rho_0\xi_i^L\xi_i^L d\tau, \quad \mathcal{K}_L=\oint_S(p^L+\rho_0\phi^L)\xi_i^L d\sigma_i. \tag{3.9}$$

The velocity fluctuations δV_i of the excited flow and the self-gravitational potential δU are calculated using the scheme described in the preceding section. However, in the hydrodynamical model under consideration the displacement field is specified by the potential function $\xi_i^L=\mathrm{grad}_i\,\psi^L$. This representation is a consequence of the fact that undamped

TABLE IV. Frequencies of quadrupole ω_2, octupole ω_3, and hexadecapole ω_4 nonradial spheroidal gravitational modes in the homogeneous model of a neutron star. The masses M (in units of the solar mass $M_\odot$) and radii R were calculated using the expressions of the standard homogeneous model with the equation of state for purely neutron matter of average density ρ_0, written in fractions of the normal nuclear density ρ_N.

Model parameters			Nuclear elastodynamics			Classical hydrodynamics		
ρ_0/ρ_N	$M/M_\odot$	R [km]	ω_2 [10^4 sec^{-1}]	ω_3 [10^4 sec^{-1}]	ω_4 [10^4 sec^{-1}]	ω_2 [10^4 sec^{-1}]	ω_3 [10^4 sec^{-1}]	ω_4 [10^4 sec^{-1}]
1.0	1.00	11.90	1.48	2.11	2.60	0.79	1.16	1.44
1.2	1.09	11.54	1.62	2.32	2.85	0.87	1.27	1.58
1.4	1.18	11.25	1.75	2.50	3.08	0.94	1.37	1.71
1.6	1.26	11.00	1.87	2.68	3.29	1.00	1.47	1.83
1.8	1.34	10.79	1.99	2.84	3.49	1.06	1.55	1.94
2.0	1.41	10.60	2.09	2.99	3.68	1.12	1.64	2.04
2.2	1.48	10.43	2.20	3.14	3.86	1.17	1.72	2.14
2.4	1.54	10.28	2.29	3.28	4.03	1.23	1.79	2.24
2.6	1.60	10.15	2.39	3.41	4.20	1.28	1.87	2.33
2.8	1.67	10.02	2.48	3.54	4.36	1.32	1.94	2.42
3.0	1.72	9.91	2.56	3.66	4.51	1.37	2.01	2.50

oscillations in a fluid propagate essentially as longitudinal sound waves. Substituting this field into the incompressibility equation, we obtain

$$\Delta\psi^L=0, \quad \psi^L=Nr^LP_L(\mu). \tag{3.10}$$

The constant N_L is fixed by the well known Neumann boundary condition:

$$\delta V_r(r')=\dot{r}'|_{r'=R',(r=R)} \quad\rightarrow\quad N=\frac{1}{LR^{L-2}}. \tag{3.11}$$

The final expression for the displacement field in spheroidal hydrodynamical vibrations takes the form

$$\xi^L=\frac{1}{LR^{L-2}}\nabla r^LP_L(\mu). \tag{3.12}$$

The only unknown variable is the pressure variation δP. Acting with the divergence operator on (3.5) and using (3.4) and (3.6), we find that δP obeys the Laplace equation, which is supplemented by the condition that stresses be absent on the vibrating surface:

$$\Delta\delta P=0, \quad P_0(r')+\delta P(r')|_{r'=R',(r=R)}=0. \tag{3.13}$$

The solution of (3.13) has the form

$$\delta P=p^L(\mathbf{r})\alpha_L(t), \quad p^L(\mathbf{r})=\frac{4\pi}{3R^{L-2}}G\rho_0^2r^LP_L. \tag{3.14}$$

For the inertia and stiffness we find[62]

$$\mathcal{M}_L=\frac{4\pi\rho_0R^5}{L(2L+1)}, \quad \mathcal{K}_L=\frac{32}{3}\pi^2G\rho_0^2R^5\frac{(L-1)}{(2L+1)^2}. \tag{3.15}$$

As a result, we obtain the well known Kelvin formula (Refs. 62, 84, 86, and 89):

$$(\omega_L^K)^2=\omega_G^2\frac{2L(L-1)}{2L+1}, \quad \omega_G^2=GM/R^3, \tag{3.16}$$

characterizing the eigenfrequencies of the f mode of the nonradial, gravitational vibrations of a star modeled as a spherical, homogeneous, self-gravitating mass of incompressible, inviscid fluid. The physical content of our expression (2.64) for the spheroidal eigenmodes of a self-gravitating elastic sphere is in many respects analogous to the Kelvin formula (3.16). In particular, the lowest-order vibration in both models has multipole order $L=2$. We stress that the ratio of the s-mode frequencies of the nonradial vibrations of a self-gravitating elastic sphere (2.64) and the Kelvin f-mode frequencies (3.16) of the nonradial vibrations of a spherical mass of incompressible, inviscid fluid satisfies the inequality

$$\frac{\omega_s^2}{(\omega_L^K)^2}=\frac{(2L+1)}{L}>1 \quad \text{for } L\geqslant 2, \tag{3.17}$$

from which it follows that $\omega_s\rightarrow\sqrt{2}\omega_L^K$ for $L\rightarrow\infty$. Therefore, for identical L and ρ_0 the frequencies of the spheroidal, nonradial vibrations of a self-gravitating elastic sphere are always higher than those of the nonradial gravitational vibrations of a spherical fluid mass. For comparison, in Table IV we give the numerical values of the frequencies obtained in nuclear elastodynamics (2.67) and in classical hydrodynamics (3.16). We see that the two approaches give the same growth of the frequency with increasing multipole order of the vibrations (the periods vary approximately as $P_L\sim 1/L$): the more massive the star, the higher the frequencies of the gravitational vibrations (and, accordingly, the shorter the periods).

3.1. Tidal vibrations

Modern evolution calculations based on realistic equations of state clearly indicate the stratified nature of the hadronic matter distribution in the cores of neutron stars. The density of the inner region is about three orders of magnitude larger than that of the periphery. Therefore, perturbations induced, for example, by residual matter fluctuations after a supernova explosion will most likely be preserved only in the peripheral layer of the star. Accordingly, the question arises of how much the vibrational frequencies of the surface layer differ from the frequencies of nonradial vibrations of the entire volume of the neutron star. Such a model may prove useful for the further study of tidal vibrations in a neutron star which forms part of a binary system, where tides are induced by the orbital motion of the massive companion.

Below we present calculations of the eigenmodes of the spheroidal vibrations of the surface layer, using the hydrodynamical theory.

The equilibrium parameters of the model. In the dynamical-layer model, the stratified structure of a neutron star is treated in a simplified manner as a rigid, inert core of density ρ_c surrounded by a dynamical layer of density $\rho<\rho_c$. The radii of the core and of the entire star will be denoted by R_c and R, respectively, and $\Delta R=R-R_c$ is the depth of the outer shell.

In the nonrelativistic model, the distribution of Newtonian gravity in the star is given by

$$\Delta U_1=4\pi G\rho_c, \quad r<R_c, \tag{3.18}$$

$$\Delta U=4\pi G\rho, \quad R_c<r<R, \tag{3.19}$$

$$\Delta U_2=0, \quad r>R. \tag{3.20}$$

Using the standard boundary conditions

$$U_1=U|_{r=R_c}, \quad U=U_2|_{r=R}, \tag{3.21}$$

$$\left.\frac{dU_1}{dr}=\frac{dU}{dr}\right|_{r=R_c}, \quad \left.\frac{dU}{dr}=\frac{dU_2}{dr}\right|_{r=R}, \tag{3.22}$$

we find

$$U=\frac{2\pi}{3}G\rho\left(r^2-3R^2+\frac{2R_c^3}{r}\right)-\frac{4\pi}{3}G\rho_c\frac{R_c^3}{r}. \tag{3.23}$$

It is easily checked that in the limit $R_c\to 0$ (along with $\rho_c\to 0$) this model reproduces the potential of the homogeneous model. The locally equilibrium pressure P in the peripheral layer can be calculated from the equation of hydrostatic equilibrium with the boundary condition of zero stress at the surface:

$$\nabla P=-\rho\nabla U, \quad P(R)=0, \tag{3.24}$$

where $U=U_2$ is the potential inside the external layer of the star. The solution of (3.24) has the form

$$P(r)=\frac{2\pi}{3}G\rho^2(R^2-r^2)+\frac{4\pi}{3}G\frac{R_c^3}{R}\rho(\rho_c-\rho)\frac{R-r}{r}. \tag{3.25}$$

We again note that for $R_c=0$ we obtain the pressure in the homogeneous model.

The periods of tidal vibrations. To find the eigenmodes of the normal vibrations, we write the fluctuating variables in separable form:

$$\delta V_i(\mathbf{r},t)=\xi_i^L(\mathbf{r})\dot{\alpha}_L(t), \quad \delta P(\mathbf{r},t)=p^L(\mathbf{r})\alpha_L(t),$$

$$\delta U(\mathbf{r},t)=\phi^L(\mathbf{r})\alpha_L(t). \tag{3.26}$$

The instantaneous displacement field $\xi_L(\mathbf{r})$ obeys the equation

$$\operatorname{div}\xi_L=0, \tag{3.27}$$

the solution of which we shall seek in the form of a poloidal vector field:

$$\xi_L=\operatorname{curl}\,\operatorname{curl}\,\mathbf{r}\chi_L, \quad \chi_L=[A_L^1 r^L+A_L^2 r^{-L-1}]P_n(\mu). \tag{3.28}$$

To find the constants entering into the expressions for the flow velocity in nonradial vibrations of the surface layer, we use the condition that the core is impenetrable:

$$\delta V_r|_{r=R_c}=0, \quad \dot{R}_c=0, \tag{3.29}$$

which also reflects its inertia. At the stellar surface we impose the standard Neumann boundary condition:

$$\delta V_r|_{r=R}=\dot{R}(t)=R\dot{\alpha}_L(t)P_L(\mu),$$

$$R(t)=R(1+\alpha_L(t)P_L(\mu)). \tag{3.30}$$

From (3.29) and (3.30) we find the explicit form of the constants A_L^1 and A_L^2:

$$A_L^1=\frac{A_L}{L(L+1)}, \quad A_L^2=-\frac{A_L}{L(L+1)}R_c^{2L+1},$$

$$A_L=\frac{R^{L+3}}{R^{2L+1}-R_c^{2L+1}}. \tag{3.31}$$

The spherical components of the instantaneous displacement field ξ_L are written as

$$\xi_r=A_L\frac{r^{2L+1}-R_c^{2L+1}}{r^{L+2}}P_L(\mu), \tag{3.32}$$

$$\xi_\theta=-\frac{A_L}{L(L+1)}\frac{(L+1)r^{2L+1}+LR_c^{2L+1}}{r^{L+2}}$$

$$\times(1-\mu^2)^{1/2}\frac{dP_L(\mu)}{d\mu}, \tag{3.33}$$

$$\xi_\phi=0. \tag{3.34}$$

In the case under study, the variation of the potential ϕ^L at the stellar surface is given by the general solution of the Laplace equation:[65]

$$\phi^L=-\frac{4\pi}{2L+1}\rho GR^2\left[1+\frac{\rho_c-\rho}{\rho}\left(\frac{R_c}{R}\right)^{L+3}\right]P_L(\mu). \tag{3.35}$$

For the fluctuating surface pressure we have[65]

$$p^L=\frac{4\pi}{3}\rho^2GR^2\left[1+\frac{\rho_c-\rho}{\rho}\left(\frac{R_c}{R}\right)^3\right]P_L(\mu). \tag{3.36}$$

The inertia and stiffness parameters of small vibrations of the shell of the neutron star are given by

$$M_L=\frac{4\pi\rho R^5}{L(2L+1)}\left[1+\frac{2L+1}{L+1}\frac{R_c^{2L+1}}{R^{2L+1}-R_c^{2L+1}}\right], \tag{3.37}$$

$$K_L=\frac{16\pi^2G}{3(2L+1)}\rho^2R^5\left[\frac{2(L-1)}{2L+1}+\frac{\rho_c-\rho}{\rho}\left(\frac{R_c}{R}\right)^3\right.$$

$$\left.\times\left(1-\frac{3}{2L+1}\left(\frac{R_c}{R}\right)^L\right)\right]. \tag{3.38}$$

Let us give numerical estimates of the eigenfrequencies of vibrations of the outer shell for a typical neutron star of radius $R\approx 10$ km. We assume that the depth of the surface core participating in the vibrations is $\Delta R\approx 0.5$ km, the matter density of the inner core is $\rho_c\approx 2\times 10^{14}\,\mathrm{g\cdot cm^{-3}}$, and that of the outer layer is $\rho \approx 4.3 \times 10^{11}\,\mathrm{g \cdot cm^{-3}}$. From (3.37) and (3.38) we obtain an estimate of the periods (in seconds) of the tidal vibrations:

$$P_1 = 7.9 \times 10^{-3}, \quad P_2 = 1.9 \times 10^{-3},$$
$$P_3 = 1.1 \times 10^{-3}, \quad P_4 = 0.9 \times 10^{-3}. \tag{3.39}$$

A noteworthy feature of the inhomogeneous two-component model (with the condition that the massive core remain unperturbed) is the conclusion that the lowest mode is a dipole mode. It should be noted that the absolute values of the frequencies of nonradial vibrations of the peripheral layer are lower than those of the frequencies in the homogeneous stellar model. The closeness of the periods of the tidal vibrations to those of pulsar electromagnetic radiation shows that tidal gravitational fluctuations of the matter in the peripheral layer of the neutron star can affect the star's electromagnetic activity.

The analytic results that we have obtained suggest several conclusions.

(i) The limiting case $R_c \to 0$ corresponds to vibrations of the full mass of the star:

$$M_L = \frac{4\pi\rho R^5}{L(2L+1)}, \quad K_L = \frac{32}{3}\pi^2\rho^2 G R^5 \frac{(L-1)}{(2L+1)^2}, \tag{3.40}$$

and we again arrive at the Kelvin spectrum:

$$(\omega_L^K)^2 = \omega_G^2 \frac{2L(L-1)}{2L+1}, \quad \omega_G^2 = \frac{4\pi}{3} G\rho. \tag{3.41}$$

(ii) In the limit of small depth $\Delta R = R - R_c \ll R_c$ we have

$$\omega_L^2 = \omega_G^2 \frac{2L(L^2-1)}{2L+1} \frac{\Delta R}{R}. \tag{3.42}$$

(iii) When $\rho_c = \rho$,

$$\omega_L^2 = \omega_G^2 \frac{2L(L-1)}{2L+1} \left[1 + \frac{2L+1}{L+1} \frac{R_c^{2L+1}}{R^{2L+1} - R_c^{2L+1}} \right]^{-1}. \tag{3.43}$$

It follows from (3.43) that for homogeneous stellar density the vibrational frequencies of the peripheral layer of finite depth are lower than those of the entire mass. This comparison of the homogeneous and inhomogeneous models of a neutron star show that a characteristic dynamical manifestation of the inhomogeneity in the radial distribution of the mass is the presence of the dipole f mode in the spectrum of stellar gravitational vibrations. This mode appears only owing to the nonhomogeneousity of the density profile and is the lowest stable mode. As shown in Ref. 66, this conclusion is independent of the specific type of inhomogeneity of the density profile. We can therefore conclude that the presence of the dipole mode is the main feature distinguishing the inhomogeneous two-component model from the homogeneous Kelvin model, in which the lowest stable mode is the quadrupole mode.

In summarizing this comparison of the hydrodynamical and elastodynamical approaches, we note the following. The frequencies predicted by the elastodynamical model are about 1.5–2 times higher than those obtained in the hydrodynamical approach. This difference has an exclusively dynamical origin and is due to the fact that the restoring force of the vibrations of a spherical fluid mass in its intrinsic gravitational field is determined only by the surface fluctuations of the gravitational field and the gravitational pressure, whereas in the elastodynamical model the stability of the equilibrium shape of the neutron star and the restoring force of the vibrations are determined by the balance of the elastic deformation forces in the degenerate Fermi continuum and the intrinsic gravitational forces. This adequately reflects the firmly established fact that the stability of a neutron star with respect to small-amplitude deformations (like the absolute stability of the equilibrium configuration) is determined by the competition between the destructive pressure of the gravitational compression of the star and the constructive pressure of the degenerate nuclear matter preventing the star from collapsing. This is one of the reasons why the elastodynamical model of continuum mechanics is more justified and suitable for describing the eigenmotions of self-gravitating nuclear matter than is the hydrodynamical model. However, the above hydrodynamical estimates may prove useful for analyzing the seismology of compact objects.

In conclusion, we make a final observation directly related to the elastodynamical treatment of the motion of matter in neutron stars. The two-component model of a neutron star (a stiff peripheral shell undergoing elastic vibrations relative to the denser core) allows a disturbance in the radio emission of a pulsar to be viewed as stellar shaking caused when the companion passes through the periastron of the binary system. In this situation a tidal perturbation of the peripheral shell of the neutron star due to the approach of the companion should, most likely, lead to rapidly damped, nonradial, torsional vibrations of the outer core relative to the inner core. It is not impossible that repeated disturbances of the pulsating radio emission of neutron stars is a direct indication that these pulsars are components of binary systems.

4. MAGNETO-PLASMA VIBRATIONS OF NEUTRON STARS

Before the discovery of pulsars, Ginzburg[69] and Woltjer[70] showed that the strong magnetization of a neutron star can be understood physically by assuming that the collapse of weakly magnetized, massive, main-sequence stars (with $B \sim 1 - 10^3$ G) with dimensions of the order of the solar size ($R \sim 10^5 - 10^6$ km) occurs with conservation of the magnetic flux. In this case, if the magnetic lines of force in the stellar matter are completely frozen, the catastrophic decrease of the size to tens of kilometers must be accompanied by an increase of the magnetic field strength to $10^{11} - 10^{13}$ G (Ref. 67). Making this hypothesis actually implies that the nuclear matter of the formed neutron star must remain ionized (albeit partially), i.e., it must possess the properties of a magnetized compensated plasma. One of the characteristic signs of the latter is its ability to support undamped magnetohydrodyanamical (MHD) Alfvén waves.[84,90] This fact in neutron-star physics was first pointed out by Hoyle, Narlikar, and Wheeler.[71] They showed that the magnetic energy stored during the collapse stage must, after the neutron star is formed, be released via transformation of the energy of residual magneto-plasma oscillations inside the star into the energy of electromagnetic radiation into the surrounding space. The possibility of releasing magnetic energy via radiation was also suggested by Pacini.[91] However, as noted in Ref. 92, this idea was subsequently undeservedly forgotten. In this section we shall perform a variational calculation and make numerical estimates of the periods of MHD modes, on the basis of which it can be concluded that the hypothesis of the magneto-plasma mechanism of electromagnetic activity of neutron stars is consistent.

However, before discussing the physics of magneto-plasma, nonradial vibrations in neutron stars, we should make special mention of the observations made in Refs. 93 and 94. There it was argued that the neutron fraction of matter in a pulsar is located in the ferromagnetic phase. In particular, it was shown[93] that the spontaneous orientation of the neutron magnetic moments can form a stable magnetization of a spherical mass of neutron matter with a value of the total magnetic moment sufficient for explaining the phenomenon of pulsating radiation as the magnetic-dipole radio emission of a rotating neutron star in the lighthouse model (Refs. 7, 11, 12, and 95–97). In a recent study,[94] additional arguments were presented in favor of a magnetically ordered state. The main one is the self-consistent estimate of the average magnetic field strength and the neutron-star density. One of the remarkable consequences of this hypothesis is the prediction of electromagnetic activity of a neutron star due to spin vibrations. It was estimated[94] that the periods of these vibrations lie within the millisecond range on the pulsar time scale.

It should be added that according to recent calculations of the structure of a neutron star, only the peripheral layer can be associated with the Ae phase, whereas the structural content of the deeper regions is apparently neutron matter possessing the properties of the B phase[7)] of superfluid ^{3}He. The characteristic feature of superfluid rotation is that it can be accompanied by the formation of quantum vortices.[98] This idea is supported by the conclusions of the quantum-macroscopic theory of disturbances in the radio emission of pulsars proposed in Ref. 99, according to which sharp spikes in the frequency of radio emission can be attributed to the loss of stable coupling of the Tkachenko vortex lattice (formed by magnetized vortex filaments oriented along the rotation axis) or, in other words, the breaking off of magnetized vortex filaments from the peripheral core.[100]

4.1. Nonradial MHD vibrations of a neutron star in the homogeneous model

In this section we shall study the motion of the ionized matter of a neutron star due to the presence of a strong magnetic field in its core. This model is based on the assumption that the Ae phase possesses the properties of a compensated, magnetically active plasma. As is well known, the dynamics of matter in the plasma aggregate state are controlled by the equations of magneto-hydrodynamics:[84]

$$\rho\frac{d\mathbf{V}}{dt}=-\nabla W+\frac{1}{4\pi}(\mathbf{B}\nabla)\mathbf{B},\quad W=P+\frac{B^2}{8\pi},\tag{4.1}$$

$$\operatorname{div}\mathbf{B}=0,\quad \frac{\partial\mathbf{B}}{\partial t}=\operatorname{curl}[\mathbf{V}\times\mathbf{B}],\tag{4.2}$$

where ρ is the density, $\mathbf{V}$ is the velocity of the medium, $\mathbf{B}$ is the magnetic field strength, and W is the magneto-hydrostatic pressure (d/dt is the substantive derivative).

When the matter is incompressible, the linearized MHD equations containing the solution corresponding to the propagation of an ordinary magneto-hydrodynamical wave can be written as

$$\frac{\partial\delta V_i}{\partial x_i}=0,\quad \frac{\partial\delta B_k}{\partial x_k}=0,\tag{4.3}$$

$$\rho\frac{\partial\delta V_i}{\partial t}-\frac{B_k}{4\pi}\frac{\partial\delta B_i}{\partial x_k}=0,\tag{4.4}$$

$$\frac{\partial\delta B_i}{\partial t}-B_k\frac{\partial\delta V_i}{\partial x_k}=0,\tag{4.5}$$

where δV_k and δB_k are the components of the fluctuations of the velocity and the magnetic field strength. In obtaining Eqs. (4.3)–(4.5) we used the trivial solution of the Laplace equation $\Delta\,\delta W=0$ for the pressure fluctuations: $\delta W=0$. This corresponds to the case where gravitational vibrations are not excited, and magneto-plasma oscillations are the only degree of activity of the plasma medium.

Chandrasekhar and Fermi showed[101] that the presence of a homogeneous field in a star leads to the same effect as rotation, i.e., it tends to squash the star at the poles. For a homogeneous, self-gravitating spherical mass of radius R and mass M with uniform field strength B inside the star, the degree of flattening of the star is estimated as $\varepsilon\sim E_{\mathrm{mag}}/E_{\mathrm{gr}}$, where $E_{\mathrm{mag}}\sim B^2R^3$ is the magnetic energy and $E_{\mathrm{gr}}\sim GM^2/R$ is the gravitational energy. For neutron stars $\varepsilon\sim10^{-5}$, and so we can neglect the asphericity due to the presence of the magnetic field, and calculate the frequencies of MHD vibrations for the spherical equilibrium configuration.

We are interested in the eigenfrequency spectrum of nonradial Alfvén vibrations.[8)] These eigenfrequencies can be calculated by using the energy variational principle as follows. Scalar multiplication of (4.4) by δV_i and integration over the stellar volume (on the surface of the star it is assumed that $\delta\mathbf{B}|_{r=R}=0$) leads to the energy-balance equation:

$$\frac{\partial}{\partial t}\int_V\frac{\rho\,\delta V^2}{2}d\tau-\frac{\mu}{4\pi}\int_V\delta V_iB_k\frac{\partial\delta B_i}{\partial x_k}d\tau=0.\tag{4.6}$$

Small deviations of the flow velocity and of the magnetic field strength are conveniently written as

$$\delta V_i=\xi_i(\mathbf{r})\dot\alpha(t),\quad \delta B_i=h_i(\mathbf{r})\alpha(t).\tag{4.7}$$

Substituting (4.7) into (4.5), we find

$$h_i=B_k\frac{\partial\xi_i}{\partial x_k}.\tag{4.8}$$

Substitution of (4.7) into (4.6) gives

$$M_L\ddot\alpha+K_L\alpha=0,\tag{4.9}$$

where M_L is the inertia and K_L is the stiffness of hydromagnetic vibrations:[73]

$$M_L=\int_V \rho\xi_i\xi_i d\tau, \quad K_L=\frac{\mu}{4\pi}\int_V h_i h_i d\tau. \tag{4.10}$$

It follows from these expressions that only the displacement field is needed to determine the frequency $\omega^2=K_L/M_L$. Below, we shall consider the case where the magnetic field **B**, uniform inside the star, is directed along the z axis:

$$B_r=\mu B, \quad B_\theta=-(1-\mu^2)^{1/2}B, \quad B_\phi=0, \quad \mu=\cos\theta. \tag{4.11}$$

We note that it is this field configuration inside the star that was studied in Ref. 101.

The elastodynamical nature of MHD vibrations. When studying the propagation of MHD waves in a magnetically active plasma, it is common to use the mechanical analogy developed by Alfvén[90] between a magneto-plasma (essentially transverse) wave propagating along magnetic field lines frozen in a perfectly conducting medium, and a transverse elastic stress wave propagating along a string under tension. This analogy emphasizes the fact that the presence of a uniform magnetic field inside an infinitely conducting, compensated plasma gives it the features of an elastic continuum in the sense that the propagation of a transverse wave serves as the main dynamical sign of an elastic continuous medium.[84,90] Adhering to this analogy, we shall assume that hydromagnetic vibrations in the spherical volume of a star with constant magnetic field inside develop like the elastodynamical vibrations of a spherical mass of an elastic continuum. In other words, we assume that the displacement fields in a star undergoing MHD vibrations are described by poloidal and toroidal displacement fields, just as in the case of spheroidal and torsional gravitational vibrations.[9)]

The poloidal MHD mode. Let us consider perturbations associated with the excitation of the poloidal velocity field:

$$\delta\mathbf{V}_p=\frac{N_p}{L+1}\operatorname{curl}\operatorname{curl}\mathbf{r}\,r^L P_L(\mu)\dot\alpha_L(t)$$

$$=N_p\operatorname{grad} r^L P_L(\mu)\dot\alpha_L(t), \quad N_p=\frac{1}{LR^{L-2}}. \tag{4.12}$$

The spherical components of the instantaneous displacement field have the form

$$\xi_r=\frac{r^{L-1}}{R^{L-2}}P_L(\mu), \quad \xi_\theta=-\frac{r^{L-1}}{LR^{L-2}}(1-\mu^2)^{1/2}\frac{\partial P_L(\mu)}{\partial\mu},$$

$$\xi_\phi=0. \tag{4.13}$$

The spatial dependence of the magnetic field fluctuations is given by (see Appendix B)

$$h_r=(L-1)\frac{Br^{L-2}}{R^{L-2}}P_{L-1}(\mu),$$

$$h_\theta=\frac{-Br^{L-2}}{R^{L-2}}(1-\mu^2)^{1/2}\frac{\partial P_{L-1}(\mu)}{\partial\mu}, \quad h_\phi=0. \tag{4.14}$$

Substituting (4.13) and (4.14) into (4.10), we find

$$M_L=\frac{4\pi\rho R^5}{L(2L+1)}, \quad K_L=B^2R^3\frac{L-1}{2L-1}. \tag{4.15}$$

As a result, for the frequency of the poloidal MHD mode we obtain

$$\omega^2=\Omega_A^2 L(L-1)\frac{2L+1}{2L-1}, \quad \Omega_A^2=\frac{V_A^2}{R^2}=\frac{B^2}{4\pi\rho R^2}, \tag{4.16}$$

where Ω_A is the natural unit of frequency of Alfvén magneto-plasma oscillations.

The toroidal MHD mode. The above-noted physical analogy between the behavior of a magnetized plasma and an elastic continuum suggests the possibility of exciting the toroidal hydromagnetic mode. We recall that the toroidal velocity field of elastic displacements in a system with fixed polar axis is written as

$$\delta\mathbf{V}_t=N_t\operatorname{curl}\mathbf{r}r^L P_L(\mu)\dot\alpha_L(t)=[\mathbf{\Omega}(\mathbf{r},t)\times\mathbf{r}], \tag{4.17}$$

where

$$\mathbf{\Omega}(\mathbf{r},t)=N_t\operatorname{grad} r^L P_L(\mu)\dot\alpha_L(t), \quad N_t=\frac{1}{R^{L-1}}, \tag{4.18}$$

is the angular-frequency field of local torsional vibrations. The spherical components of the toroidal displacement field are written as

$$\xi_r=0, \quad \xi_\theta=0, \quad \xi_\phi=-\frac{r^L}{R^{L-1}}(1-\mu^2)^{1/2}\frac{\partial P_L(\mu)}{\partial\mu}, \tag{4.19}$$

and the corresponding fluctuations in the magnetic field intensity are

$$h_r=0, \quad h_\theta=0,$$

$$h_\phi=B(L+1)\frac{r^{L-1}}{R^{L-1}}(1-\mu^2)^{1/2}\frac{\partial P_{L-1}(\mu)}{\partial\mu}. \tag{4.20}$$

Calculations of the mass parameter and of the stiffness parameter of toroidal MHD vibrations give

$$M_L=4\pi\rho R^5\frac{L(L+1)}{(2L+1)(2L+3)},$$

$$K_L=B^2R^3\frac{L(L-1)(L+1)^2}{(2L+1)(2L-1)}. \tag{4.21}$$

As a result, the discrete frequency spectrum of toroidal MHD vibrations can be written as

$$\omega_t^2=\Omega_A^2(L^2-1)\frac{2L+3}{2L-1}, \tag{4.22}$$

where the Alfvén frequency Ω_A was defined above.

The spectral equations (4.16) and (4.22) are the main result of the theory. It follows from these expressions that the eigenfrequencies of the hydromagnetic modes of a neutron star are proportional to the magnetic field intensity inside the star B and inversely proportional to its radius R. The periods of MHD vibrations $P_{\rm hm}$ (both poloidal and toroidal) fall off monotonically with increasing multipole order L as $P_{\rm hm}\sim 1/L$.

TABLE V. Periods P_L (in sec) of global nonradial poloidal and toroidal magneto-hydrodynamical (MHD) vibrations, calculated in the constant-field approximation for a family of homogeneous neutron-star models with the parameters given in the first four columns of the table.

Model parameters				Poloidal MHD mode			Toroidal MHD mode		
$M/M_\odot$	R [km]	ρ/ρ_N,	$B\,[10^{13}\,\mathrm{G}]$	P_2 [sec]	P_3 [sec]	P_4 [sec]	P_2 [sec]	P_3 [sec]	P_4 [sec]
			0.1	31.70	19.40	14.20	20.80	14.70	11.50
			2.0	1.51	0.93	0.68	0.99	0.70	0.55
0.5	9.8	0.9	4.0	0.77	0.47	0.35	0.51	0.36	0.28
			6.0	0.52	0.32	0.23	0.34	0.24	0.19
			8.0	0.39	0.24	0.18	0.26	0.18	0.14
			0.1	35.60	21.80	15.90	23.30	16.50	12.90
			2.0	1.70	1.04	0.76	1.11	0.79	0.61
0.7	11.0	0.9	4.0	0.87	0.53	0.39	0.57	0.40	0.31
			6.0	0.58	0.36	0.26	0.38	0.27	0.21
			8.0	0.44	0.27	0.20	0.29	0.20	0.16
			0.1	40.60	24.90	18.20	26.60	18.80	14.70
			2.0	1.93	1.18	0.86	1.27	0.90	0.70
0.9	10.6	1.3	4.0	0.99	0.61	0.44	0.65	0.46	0.36
			6.0	0.67	0.41	0.30	0.44	0.31	0.24
			8.0	0.50	0.31	0.22	0.33	0.23	0.18
			0.1	46.00	28.20	20.60	30.10	21.30	16.60
			2.0	2.19	1.34	0.98	1.43	1.01	0.79
1.1	10.6	1.6	4.0	1.12	0.69	0.50	0.74	0.52	0.41
			6.0	0.75	0.46	0.34	0.49	0.35	0.27
			8.0	0.57	0.35	0.25	0.37	0.26	0.21
			0.1	53.20	32.60	23.80	34.80	24.60	19.20
			2.0	2.53	1.55	1.13	1.66	1.17	0.92
1.4	10.2	2.3	4.0	1.30	0.80	0.58	0.85	0.60	0.47
			6.0	0.87	0.53	0.39	0.57	0.40	0.32
			8.0	0.66	0.40	0.29	0.43	0.30	0.24

In Table V we give numerical estimates of the periods $P_L=2\pi/\omega_L$ of magneto-plasma poloidal and toroidal oscillations of the lowest multipole order, calculated for a family of homogeneous models of neutron stars with the simplest configuration of constant magnetic field inside the star and dipole field outside it. The periods of poloidal magneto-plasma oscillations are always slightly larger (on the average, $P_\mathrm{p}/P_\mathrm{t}\sim 1.2-1.6$) than the periods of toroidal Alfvén modes. Using the data of Table V, we can also trace the general trends in the variation of the periods as a function of the parameters of the stellar model. For both modes the period grows as the stellar matter becomes denser. In other words, the heavier the star, the larger the periods of its hydromagnetic oscillations. The principal result worth noting is that when the field strength is above 10^{13} G, the periods of the Alfvén oscillations almost exactly coincide with the periods of pulsar radio emission: $0.016<P<5$ sec.[67] As B decreases the periods of MHD oscillations become longer (and, accordingly, the frequencies decrease). In other words, the periods of the Alfvén oscillations must grow as the stellar magnetic field is depressed. These conclusions follow from the assumption that the magnetic field strength reaches the values in Table V. For smaller values $B\sim 10^{12}$ G (such fields are assumed in the inclined-rotator model), the periods of Alfvén oscillations lie in the range $5<P<50$ sec. The radiation due to magneto-plasma vibrations with such periods, superimposed on the magnetic-dipole radiation due to the rotation, can be manifested as modulations of the amplitude of the latter. It also should not be forgotten that these conclusions were obtained in the model with uniform distribution of the compensated plasma throughout the spherical volume of the neutron star. However, it follows from calculations of the structure of this compact object that the Ae phase is mainly localized in the peripheral core of the star, where the matter density is lower than in deeper regions (see Fig. 1). In connection with this it is appropriate to reestimate the frequencies of MHD vibrations using the model taking this into account.

5. ALFVÉN VIBRATIONS IN THE PERIPHERAL CORE OF A NEUTRON STAR

In this section we shall perform a variational calculation and numerically estimate the frequencies of MHD eigenvibrations localized in the outer core of a neutron star, i.e., in the region where an electron–nuclear plasma is most likely to exist. The neutron star is idealized as a two-component object, in full analogy with the Baym–Pethick–Pines–Ruderman model[42] mentioned above (see also Refs. 8 and 13), which attributes disturbances of pulsars to shear seismic vibrations of the outer (less dense) core relative to the denser inner core. We shall also use the arguments of Ref. 94, assuming that the neutron-enriched nuclear matter of the massive core is located in the ferromagnetic phase. This at least clarifies the physical origin of the strong magnetic field at the surface of a compensated electron–nuclear plasma, on the background of which Alfvén vibrations can develop.

The poloidal mode. In the case of poloidal MHD vibrations, the velocity field of elastic displacements at the surface of the inner core of radius R_c can be found by using the impenetrability condition:

$$\delta V_r|_{r=R_c}=0 \quad \text{for } R_c=0. \tag{5.1}$$

At the stellar surface we impose the standard boundary condition

$$\delta V_r|_{r=R}=\dot R(t)=RP_L(\mu)\dot\alpha_L(t), \tag{5.2}$$

where $R(t)=R[1+\alpha_L(t)P_L(\mu)]$, L being the multipole order of the spheroidal distortions of the surface. For the poloidal vector field we have

$$\boldsymbol{\xi}_L=\text{curl curl}\,\mathbf{r}\chi_L,\quad \chi_L=[A_L^1r^L+A_L^2r^{-L-1}]P_L(\mu). \tag{5.3}$$

From (5.1) and (5.2) we find the explicit form of the arbitrary constants A_L^1 and A_L^2:

$$A_L^1=\frac{A_L}{L(L+1)},\quad A_L^2=-\frac{A_L}{L(L+1)}R_c^{2L+1},$$

$$A_L=\frac{R^{L+3}}{R^{2L+1}-R_c^{2L+1}}. \tag{5.4}$$

The components of the instantaneous displacement field ξ_L in spherical coordinates are written as

$$\xi_r=A_L\frac{r^{2L+1}-R_c^{2L+1}}{r^{L+2}}P_L(\mu), \tag{5.5}$$

$$\xi_\theta=\frac{-A_L}{L(L+1)}\frac{(L+1)r^{2L+1}+LR_c^{2L+1}}{r^{L+2}}P_L^1(\mu), \tag{5.6}$$

$$\xi_\phi=0, \tag{5.7}$$

where $P_L^1(\mu)=(1-\mu^2)^{1/2}dP_L(\mu)/d\mu$ is the first-order associated Legendre polynomial. The inertia parameter M calculated using this field is[65]

$$M=\frac{4\pi\rho}{L(2L+1)}A_L^2R^{2L+1}\left[1+\frac{L}{L+1}X^{2L+1}\right](1-X^{2L+1}),$$

$$X=R_c/R, \tag{5.8}$$

where X varies within the range $0<X<1$. We stress that here ρ is the density of the electron–nuclear plasma (the Ae phase) localized in the peripheral core of the star.

Then, substituting (4.11) and (5.3) into (4.8), we find that the components of the magnetic field-strength fluctuations take the form

$$h_r=\frac{A_LB}{r^{L+3}}[(L-1)r^{2L+1}P_{L-1}(\mu)+(L+2)R_c^{2L+1}P_{L+1}(\mu)], \tag{5.9}$$

$$h_\theta=\frac{A_LB}{r^{L+3}}[r^{2L+1}P_{L-1}^1(\mu)-R_c^{2L+1}P_{L+1}^1(\mu)],\quad h_\phi=0. \tag{5.10}$$

Then for the stiffness of hydromagnetic poloidal vibrations we find

$$K=A_L^2B^2R^{2L-1}\left[\frac{L-1}{2L-1}+\frac{2L+1}{(2L+3)(2L-1)}X^{2L-1}\right.$$
$$\left.-\frac{L+2}{2L+3}X^{2(2L+1)}\right]. \tag{5.11}$$

It is easily seen that in the limit $X\to0$ we reproduce the result of the homogeneous model:

$$\omega_{\mathbf{p}}^2=\Omega_A^2L(L-1)\frac{2L+1}{2L-1}. \tag{5.12}$$

The toroidal mode. Let us now consider nonradial toroidal MHD vibrations. In the frame with fixed polar axis z, the toroidal velocity field has the form

$$\delta\mathbf{V}=\text{curl }\,\mathbf{r}\chi_L\dot\alpha_L(t),\quad \chi_L=[A_L^1r^L+A_L^2r^{-L-1}]P_L(\mu). \tag{5.13}$$

The arbitrary constants A_L^1 and A_L^2 are fixed by boundary conditions analogous to those used above to study spheroidal vibrations. For differential rotational vibrations, the distortions of the stellar surface are given by $R(t)=R[1+\alpha_L(t)P_L^1(\mu)]$, and so for $r=R$ we must take

$$\delta V_\phi|_{r=R}=\dot R(t)=RP_L^1(\mu)\dot\alpha_L(t). \tag{5.14}$$

We assume that the inner boundary remains at rest:

$$\delta V_\phi|_{r=R_c}=0,\quad \dot R_c=0. \tag{5.15}$$

As a result, we obtain

$$A_L^1=A_L,\quad A_L^2=-A_LR_c^{2L=1},\quad A_L=\frac{R^L}{R^{2L+1}-R_c^{2L+1}}. \tag{5.16}$$

Using the separable representation (4.7) for the velocity field of torsional vibrations (5.14), we find the components of the toroidal instantaneous displacement field:

$$\xi_r=0,\quad \xi_\theta=0,\quad \xi_\phi=A_L\left[r^L-\frac{R_c^{2L+1}}{r^{L+1}}\right]P_L^1(\mu). \tag{5.17}$$

Substitution of (4.11) and (5.17) into (4.8) leads to the following expressions for the components of the fluctuating magnetic field strength (see also Appendix B):

$$h_r=0,\quad h_\theta=0,$$

$$h_\phi=A_LB\left[(L+1)r^{L-1}P_{L-1}^1(\mu)+L\frac{R_c^{2L+1}}{r^{L+2}}P_{L+1}^1(\mu)\right]. \tag{5.18}$$

Calculations of the inertia and stiffness coefficients of toroidal MHD vibrations give

$$M=A_L^2\frac{4\pi\rho L(L+1)R^{2L+3}}{(2L+1)(2L+3)}\left[1-(2L+3)X^{2L+1}\right.$$
$$\left.+\frac{(2L+1)^2X^{2L+3}}{2L-1}-\frac{(2L+3)}{2L-1}X^{2(2L+1)}\right],$$

$$K = A_L^2 B^2 \frac{L(L+1)R^{2L+1}}{(2L+1)} \left[\frac{L^2-1}{2L-1} + \frac{3X^{2L+1}}{(2L-1)(2L+3)} - \frac{L(L+2)}{2L+3} X^{2(2L+1)} \right]. \quad (5.19)$$

As expected, for $X(=R_c/R) \to 0$ we arrive at the result of the homogeneous model:[73]

$$\omega_t^2 = \Omega_A^2 (L^2-1) \frac{2L+3}{2L-1}, \quad (5.20)$$

where the basic (Alfvén) frequency Ω_A is defined above.

The two-component model can be used to obtain lower and upper limits on the frequencies of the Alfvén MHD eigenvibrations of a neutron star. The neutron-star parameters given in the literature, obtained using various nuclear-matter equations of state, lie within the following ranges:

(i) depth of the peripheral core $\Delta R = R - R_c = R(1-X)$: $0.3 < \Delta R < 0.8$ km;

(ii) average density of the surface core $10^8 < \rho < 10^{11}$ g/cm^3;

(iii) surface magnetic field strength $10^{10} < B < 10^{13}$ G.

The results of numerical analysis of the model are given in Figs. 6 and 7. In Fig. 6 we show the graph of period versus magnetic field strength, often given in the literature.[67,92,104] The calculated periods of nonradial poloidal and toroidal MHD oscillations in a surface core of depth $\Delta R = 0.5$ km are shown by the lines in Fig. 7, which are numbered according to the values of the matter density in the region where the Ae phase is localized. We see that the model predictions fall fairly closely into the square denoting the pulsar region on this diagram. Figure 7 illustrates the dependence of the calculated periods on the depth of the layer in which Alfvén vibrations of the lowest multipole orders are excited. The estimates show that the calculated periods of MHD oscillations are close to the periods of electromagnetic radiation of radio pulsars. We are inclined to believe that the coincidence between the periods of MHD vibrations and the principal periods of radio pulsations of neutron stars is not accidental and supports the above-mentioned Hoyle–Narlikar–Wheeler hypothesis[71] that the low-frequency hydromagnetic oscillations arising as a residual effect of a supernova burst of the second type may be an effective source of the electromagnetic activity of neutron stars. Analysis of the evolution of pulsars shows that the original magnetic field must be destroyed over a time of order $\tau_m \sim 2 \times 10^6$ yr.[104] Since the periods of hydromagnetic oscillations are inversely proportional to the magnetic field strength ($P_{\rm hm} \sim 1/B$), the adiabatic depression of the latter must tend to increase the periods of Alfvém vibrations. Therefore, the coherent nature of magneto-plasma oscillations inside a neutron star must be manifested outside the star as pulses propagating along the magnetic field lines and generating pulsating electromagnetic radiation by bunches of charged particles ejected from the stellar surface. It is known[105,106] that magneto-hydrodynamical waves in interstellar space can accelerate charged particles along field lines and thereby generate radiation (synchrotron or bending radiation).

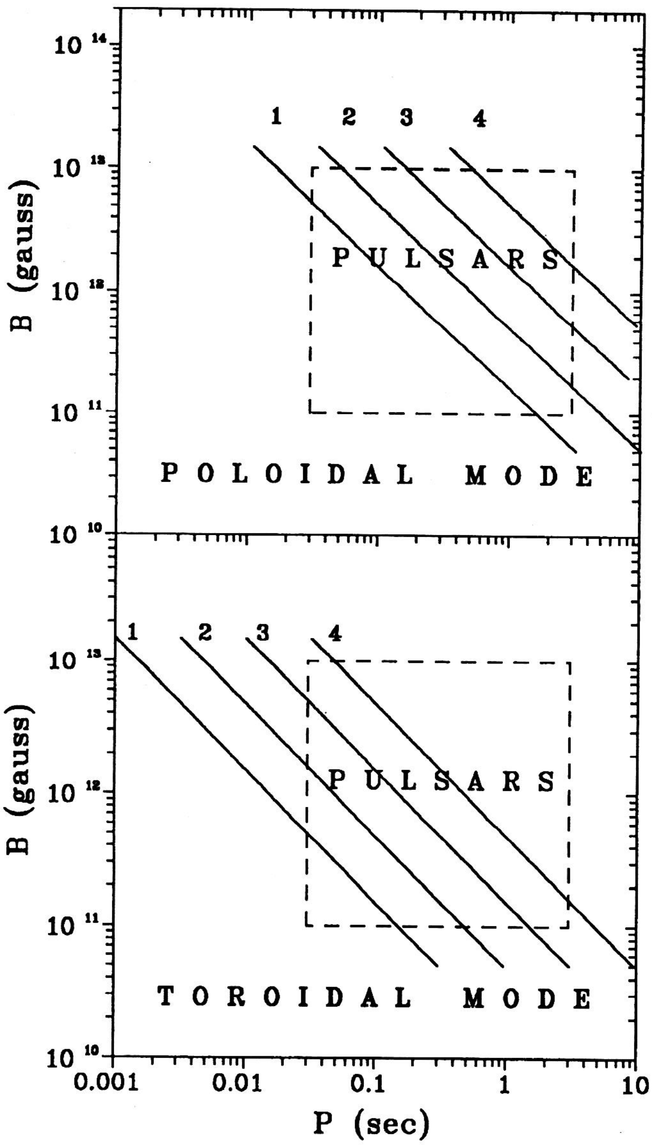

FIG. 6. Graph of the period versus magnetic field strength, $P-B$. The calculated periods of quadrupole poloidal and toroidal MHD vibrations are shown by the lines, which correspond to different values of the electron–nuclear plasma density (the Ae phase), localized in the outer core of depth $\Delta R = 0.5$ km: (1) $\rho = 10^8$, (2) $\rho = 10^9$, (3) $\rho = 10^{10}$, (4) $\rho = 10^{11}$ (in g/cm^3). The square denotes the region of P and B of actually observed pulsars.

In our opinion, one of the decisive arguments confirming the hypothesis that magneto-plasma oscillations of a neutron star can be just as effective as rotation in producing pulsating radio emission in the surrounding space would be the observation of long-lived, superfast pulsars with pulsation period $P < 0.5$ msec. For these values of the radio-pulsation periods, the radiation frequency significantly exceeds the Kepler limiting frequency determining the gravitational–rotational stability of a star.[10)] Therefore, the existence of such pulsars is excluded by the model of a unipolar generator, in which a magnetized neutron star generates magnetic-dipole radio emission with period equal to the intrinsic rotational period.[81,107,108] In connection with this, we think that the current observations of the MANIYa program (Multichannel Analysis of Nanosecond Brightness Variability), one goal of which is to seek pulsars with emission varying over time

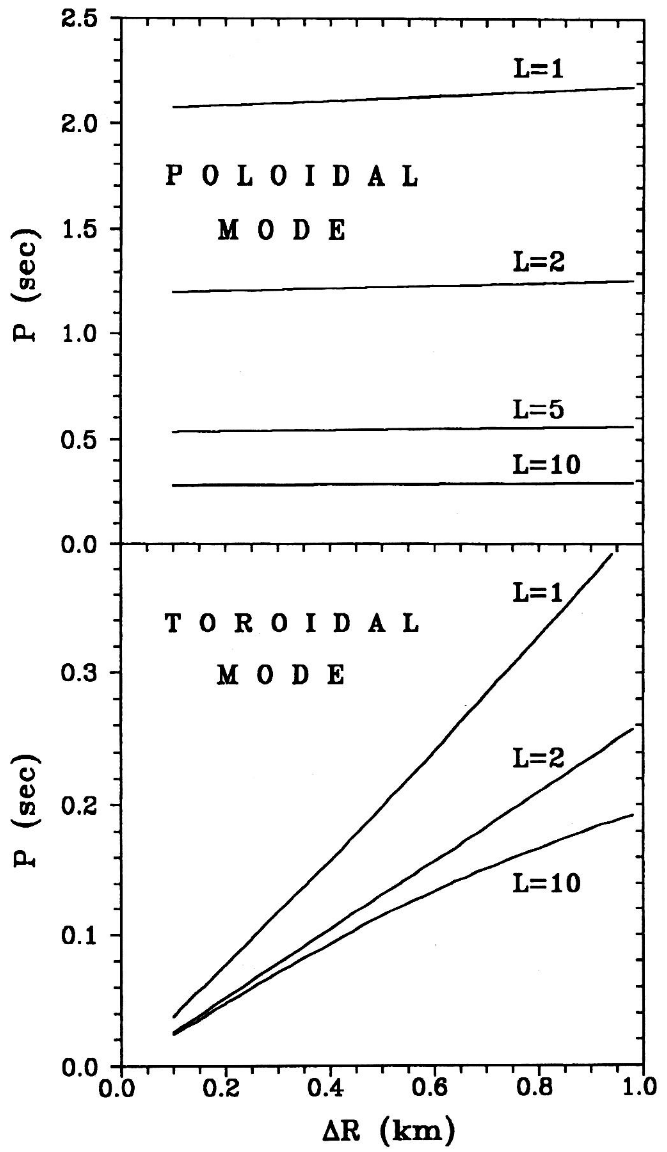

FIG. 7. Periods of poloidal and toroidal multipole MHD oscillations excited in the Ae phase, calculated as a function of the thickness of the outer core, $\Delta R = R - R_c$, for average matter density of the Ae phase $\rho = 4.3 \times 10^{11}$ g/cm^3 and magnetic field strength $B = 0.5 \times 10^{13}$ G.

scales of $10^{-7} - 10^{2}$ sec, are extremely important.[109]

6. CONCLUSION

In this review we have presented the theory of gravitational and magneto-plasma nonradial vibrations of neutron stars based on the idea that nuclear matter is an elastic Fermi continuum possessing the properties of a compensated magneto-active plasma. As the fundamental dynamical equations modeling the motion of the matter in the core of a neutron star we have used the equations of nuclear elastodynamics, proposed in the macroscopic theory of collective processes of laboratory nuclear physics such as fission and giant resonances.

We have constructively compared the predictions of the hydrodynamical and elastodynamical models regarding the behavior of a continuous nuclear medium with the available astrophysical observations. Using the hydrodynamical approach, we have shown that the presence of the dipole mode is a characteristic sign of inhomogeneity in the profile of the stellar density. However, the hydrodynamical model does not allow the description of the principal physical factors governing the eigenvibrations of neutron stars. The vibrational stability of a neutron star is determined by the competition of the constructive forces of elastic deformations of the degenerate Fermi continuum and the destructive forces of gravitational compression. This is quite clearly reflected in the elastodynamical theory of nonradial vibrations and is absolutely absent in the hydrodynamical theory, which indicates that the hydrodynamical paradigm of a continuous nuclear medium is inadequate. One of the main conclusions of our elastodynamical model of nonradial pulsations is that the vibrational dynamics of a neutron star are characterized by two branches of gravitational-elastic nonradial eigenvibrations: the spheroidal (s-mode) and torsional (t-mode) branches. The torsional differential-rotational vibrations of a neutron star are due exclusively to the dynamical elasticity of the degenerate Fermi continuum. Such modes are absent in the gaseous medium of a main-sequence star whose motion is governed by the hydrodynamical equations.

A method of calculating the frequencies (periods) of these vibrations has been developed on the basis of the energy variational principle. The effectiveness of the method was illustrated by analytic calculations of the periods of global nonradial gravitational-elastic modes, using the standard model of a neutron star (modeled as a spherical mass of a homogeneous neutron Fermi continuum, compressed by self-gravity to densities of the order of the normal nuclear density). Realistic estimates of the periods of spheroidal and torsional gravitational modes have been obtained by using models of neutron stars constructed on the basis of the relativistic equilibrium equation, using the nuclear-matter equations of state taking into account the heterophase nature of the nuclear statistical equilibrium. We have analyzed the vibrational (seismic) stability of a neutron star with respect to elastic deformations accompanying global gravitational vibrations, and have shown that for linear deformations obeying Hooke's law, no unstable stresses arise which could provoke stellar shaking. The estimates obtained for the periods of gravitational nonradial modes suggest that these vibrations may be responsible for variations in the intensity of the micropulses observed in the millisecond range of the pulsar spectrum.

We have made a detailed study of the nonradial magneto-plasma oscillations hypothetically induced in the Ae phase by a supernova explosion in the creation of a pulsar or by the companion in a binary system. We have found that the timing of the Alfvén MHD vibrations overlaps with the pulsar time scale. The coincidence between the calculated periods of magneto-plasma vibrations and the observed periods of pulsar radio emission is interpreted as confirmation of the Hoyle–Narlikar–Wheeler hypothesis that weakly damped magneto-plasma vibrations can be a source of the pulsating (linear polarized) radiation which is formed in the magnetosphere surrounding a star. In the magneto-plasma model of the electromagnetic activity of neutron stars, the observed lengthening of the periods of radio pulses can be attributed to the slow depression of the pulsar magnetic field. This conclusion is yet another argument that the magneto-hydrodynamical mechanism of the transformation of the energy of Alfvén vibrations into the energy of electromagnetic radiation can be an effective source of pulsating radio emission of neutron stars, along with unipolar induction generating the geometrical effect of this radiation.

Thus, the astrophysical data obtained to date on the electromagnetic activity of neutron stars, and also the experimental data from laboratory nuclear physics, indicate that a continuous nuclear medium is an elastic Fermi continuum whose motion in the intrinsic gravitational and magnetic fields is adequately described by the equations of nuclear elastodynamics and magneto-hydrodynamics.

This work has been performed under the agreement governing the collaboration between the JINR LVTA, Erevan University (Armenia), Saratov State University (Russia), and the Theoretical Physics Institute of Ludwig Maximilian University (Munich, Germany), and has been partially supported by a grant from the Heisenberg–Landau program of the JINR Laboratory of Theoretical Physics.

APPENDIX A

In this appendix we list the useful expressions which considerably simplify the analytic calculations of the periods of nonradial, gravitational-elastic eigenvibrations. The most awkward calculations can be performed using spherical coordinates with fixed polar axis z. The derivatives of the displacement field in the elastic stress tensor have the form

$$\frac{\partial\xi_1}{\partial x_1}=\frac{\partial\xi_r}{\partial r},\quad \frac{\partial\xi_2}{\partial x_2}=-\frac{(1-\mu^2)^{1/2}}{r}\frac{\partial\xi_\theta}{\partial\mu}+\frac{\xi_r}{r},\quad u=\cos\theta,$$

$$\frac{\partial\xi_3}{\partial x_3}=\frac{1}{r(1-\mu^2)^{1/2}}\frac{\partial\xi_\phi}{\partial\phi}+\frac{\xi_r}{r}+\frac{\xi_\theta}{r}\frac{\mu}{(1-\mu^2)^{1/2}},$$

$$\frac{\partial\xi_1}{\partial x_2}=-\frac{(1-\mu^2)^{1/2}}{r}\frac{\partial\xi_r}{\partial\mu}-\frac{\xi_\theta}{r},\quad \frac{\partial\xi_2}{\partial x_1}=\frac{\partial\xi_\theta}{\partial r},$$

$$\frac{\partial\xi_1}{\partial x_3}=\frac{1}{r(1-\mu^2)^{1/2}}\frac{\partial\xi_r}{\partial\phi}-\frac{\xi_\phi}{r},\quad \frac{\partial\xi_3}{\partial x_1}=\frac{\partial\xi_\phi}{\partial r},$$

$$\frac{\partial\xi_2}{\partial x_3}=\frac{1}{r(1-\mu^2)^{1/2}}\frac{\partial\xi_\theta}{\partial\phi}-\frac{\xi_\phi}{r}\frac{\mu}{(1-\mu^2)^{1/2}},$$

$$\frac{\partial\xi_3}{\partial x_2}=-\frac{(1-\mu^2)^{1/2}}{r}\frac{\partial\xi_\phi}{\partial\mu}. \tag{A1}$$

The expression for the stiffness of elastic nonradial vibrations which can be directly integrated is

$$C=\frac{1}{2}\int_v P(r)\left(\frac{\partial\xi_i}{\partial x_j}+\frac{\partial\xi_j}{\partial x_i}\right)^2 dV=\int_v P(r)\left(\frac{\partial\xi_i}{\partial x_j}+\frac{\partial\xi_j}{\partial x_i}\right)\frac{\partial\xi_j}{\partial x_i}dV$$

$$=\int_v P(r)\left\{2\left[\left(\frac{\partial\xi_1}{\partial x_1}\right)^2+\left(\frac{\partial\xi_2}{\partial x_2}\right)^2+\left(\frac{\partial\xi_3}{\partial x_3}\right)^2\right]+\left[\left(\frac{\partial\xi_1}{\partial x_2}+\frac{\partial\xi_2}{\partial x_1}\right)^2\right.\right.$$

$$\left.\left.+\left(\frac{\partial\xi_1}{\partial x_3}+\frac{\partial\xi_3}{\partial x_1}\right)^2+\left(\frac{\partial\xi_2}{\partial x_3}+\frac{\partial\xi_3}{\partial x_2}\right)^2\right]\right\}dV. \tag{A2}$$

In the calculations we used the following representation of the poloidal elastic displacement field:

$$\xi_{\mathbf{p}}=\frac{N_p}{L+1}\text{curl curl }\mathbf{r}r^L P_L(\mu):\ \xi_r=LN_p r^{L-1}P_L(\mu),$$

$$\xi_\theta=-N_p(1-\mu^2)^{1/2}r^{L-1}\frac{dP_L(\mu)}{d\mu},\quad \xi_\phi=0, \tag{A3}$$

and its derivatives

$$\frac{\partial\xi_1}{\partial x_1}=N_pL(L-1)r^{L-2}P_L(\mu),$$

$$\frac{\partial\xi_2}{\partial x_2}=N_pr^{L-2}\left[\mu\frac{dP_L(\mu)}{d\mu}-L^2P_L(\mu)\right],$$

$$\frac{\partial\xi_3}{\partial x_3}=N_pr^{L-2}\left[LP_L(\mu)-\mu\frac{dP_L(\mu)}{d\mu}\right],$$

$$\frac{\partial\xi_1}{\partial x_2}=-N_p(L-1)r^{L-2}(1-\mu^2)^{1/2}\frac{dP_L(\mu)}{d\mu},$$

$$\frac{\partial\xi_2}{\partial x_1}=-N_p(L-1)r^{L-2}(1-\mu^2)^{1/2}\frac{dP_L(\mu)}{d\mu},$$

$$\frac{\partial\xi_1}{\partial x_3}=0,\quad \frac{\partial\xi_3}{\partial x_1}=0,\quad \frac{\partial\xi_2}{\partial x_3}=0,\quad \frac{\partial\xi_3}{\partial x_2}=0. \tag{A4}$$

The spherical components of the toroidal field of elastic torsional displacements and their derivatives have the form

$$\xi_t=\frac{N_P}{L+1}\text{curl}\,\mathbf{r}\,r^LP_L(\mu):\quad \xi_r=0,\quad \xi_\theta=0,$$

$$\xi_\phi=N_tr^L(1-\mu^2)^{1/2}\frac{dP_L(\mu)}{d\mu}, \tag{A5}$$

$$\frac{\partial\xi_1}{\partial x_1}=0,\quad \frac{\partial\xi_2}{\partial x_2}=0,\quad \frac{\partial\xi_3}{\partial x_3}=0,\quad \frac{\partial\xi_1}{\partial x_2}=0,\quad \frac{\partial\xi_2}{\partial x_1}=0,$$

$$\frac{\partial\xi_1}{\partial x_3}=-N_tr^{L-1}(1-\mu^2)^{1/2}\frac{dP_L(\mu)}{d\mu},$$

$$\frac{\partial\xi_3}{\partial x_1}=LN_tr^{L-1}(1-\mu^2)^{1/2}\frac{dP_L(\mu)}{d\mu},$$

$$\frac{\partial\xi_2}{\partial x_3}=-N_tr^{L-1}\mu\frac{dP_L(\mu)}{d\mu},$$

$$\frac{\partial\xi_3}{\partial x_2}=-N_tr^{L-1}\left[\mu\frac{dP_L(\mu)}{d\mu}-L(L+1)P_L(\mu)\right]. \tag{A6}$$

APPENDIX B

In calculating the variation of the magnetic field in magneto-plasma oscillations, it should be borne in mind that the components of the derivatives of the elastic displacement fields along the direction of constant magnetic field inside the star,

$$h_i=B_k\frac{\partial\xi_i}{\partial x_k}, \tag{B1}$$

have the following explicit form in spherical coordinates:

$$h_r=\left[B_r\frac{\partial}{\partial r}+\frac{B_\theta}{r}\frac{\partial}{\partial\theta}+\frac{B_\phi}{r\sin\theta}\frac{\partial}{\partial\phi}\right]\xi_r-\frac{B_\theta\xi_\theta+B_\phi\xi_\phi}{r},$$

$$h_\theta=\left[B_r\frac{\partial}{\partial r}+\frac{B_\theta}{r}\frac{\partial}{\partial\theta}+\frac{B_\phi}{r\sin\theta}\frac{\partial}{\partial\phi}\right]\xi_\theta+\frac{B_\theta\xi_r-B_\phi\xi_\phi\cot\theta}{r},$$

$$h_\phi=\left[B_r\frac{\partial}{\partial r}+\frac{B_\theta}{r}\frac{\partial}{\partial\theta}+\frac{B_\phi}{r\sin\theta}\frac{\partial}{\partial\phi}\right]\xi_\phi+\frac{B_\phi\xi_r+B_\phi\xi_\theta\cot\theta}{r}. \tag{B2}$$

[1)]At present, the study of the equilibrium properties of collapsed stars is focused on the careful analysis of the hypothesis that massive, gravitationally stable configurations with significant content of strange and quark matter exist.[23–25] One of the most attractive consequences of this hypothesis was the prediction of two new branches in the family of compact astrophysical objects: strange stars and strange dwarfs.[21,22] Moreover, calculations of the equilibrium configurations involving strange matter suggest the possible existence of noncompact planet-like objects[26] (with a small mass like that of Jupiter, $M\sim(10^{-2}-10^{-4})M_\odot$), which are of particular interest for modern astrophysics in light of the well known problem of hidden (dark) matter.[27] Considerable progress in solving this problem has been made owing to the intensive development of the theory and technique of observations by the microlensing method.[28,29]

[2)]The features of radial vibrations of neutron stars have been discussed in Refs. 43–45. However, from the data presently available it can be concluded that neutron stars do not reveal any signs of radial pulsations. It should be stressed that the methods of describing nonradial gravitational vibrations of stars remain less well developed than the theory of radial pulsations.[111–115]

[3)]By scaling rules we mean the smooth systematic dependences of the energy, the total excitation probability, and the width of a giant resonance on the mass number.

[4)]According to the theory of stellar evolution, massive stars of the main sequence with mass $M\sim(4-8)M_\odot$, but no more than ten solar masses, can become neutron stars. The final stage of evolution of stars of mass exceeding $10M_\odot$ is a black hole.[80]

[5)]It is interesting to note that the model of a homogeneous, self-gravitating mass of an ideally elastic continuum with pressure

$$P_0=\frac{2\pi}{3}G\rho_0^2(R^2-r^2) \tag{2.63}$$

leads to a discrete spectrum of s-mode frequencies of the form[61]

$$\omega_s^2=2\omega_G^2(L-1),\quad \omega_G^2=(4\pi/3)G\rho_0, \tag{2.64}$$

where ω_G is the fundamental frequency of the gravitational vibrations in homogeneous models. In this representation, the s mode of nonradial gravitational-elastic vibrations is analogous in physical content to the Kelvin hydrodynamical f mode of nonradial vibrations of a spherical mass of inviscid, incompressible fluid.

[6)]In the model of a homogeneous mass of ideally elastic matter with pressure of the form

$$P_0=\frac{2\pi}{3}G\rho_0^2(R^2-r^2), \tag{2.75}$$

the energy variational principle described above leads to the following expression for the t-mode frequency:[62]

$$\omega_t^2=\omega_G^2(L-1),\quad \omega_G^2=GM/R^3, \tag{2.76}$$

where ω_G is the fundamental unit of frequency of gravitational vibrations in homogeneous models. Torsional gravitational vibrations of stars have been studied earlier in Refs. 87 and 88, where the problem of the discrete frequency spectrum of these oscillations was first formulated. It was first shown in Ref. 62 that the application of the Rayleigh variational principle to the elastodynamical equations allows the discrete spectrum of torsional vibrations to be calculated analytically (see also Refs. 83, 99, and 115).

[7)]The superfluidity of neutron matter would most likely be due to the pairing of neutrons in the $^3P_2$ state. This phase is analogous to the anisotropic ^{3}He B phase.

[8)]Radial magneto-plasma vibrations of a star idealized as a spherical mass of ideally conducting fluid were studied by Schwarzschild[103] (see also Ref. 34) long before the discovery of pulsars.

[9)]These arguments were recently used to construct the magnetic jellium model[102] for describing the electromagnetic response of magnetized spherical particles (clusters) of semimetals and nonmagnetic dielectrics in terms of the nonradial Alfvén vibrations of a compensated electron–hole solid-state plasma.

[10)]According to recent estimates,[110] the maximum value of the frequency of rotation of a formed neutron star about its axis is 1800 rev/sec. According to the rigid-rotor model, the fastest pulsars presently known, PSR 1937 and PSR 1957, must perform about 600 rev/sec.

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