Reduction in systems with local symmetry*)

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This review is devoted to problems associated with the study of dynamical systems with a finite number of degrees of freedom possessing local symmetry. The procedure of reduction of the system of dynamical equations to the normal form, where the Cauchy problem has a unique solution, is discussed within the framework of the classical Lagrangian and Hamiltonian theory. Special attention is given to the geometrical reduction scheme, which allows the physical subspace in the phase space of a degenerate dynamical system to be distinguished, and makes it possible to find the explicit form of the corresponding canonical variables without introducing additional gauge-fixing conditicns (gauges) into the theory. The two reduction procedures, the geometrical method and the gauge-fixing method, are compared in order to understand what conditions on the gauges guarantee the correctness of the reduction procedure.
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1. INTRODUCTION

The Lagrangians of systems with functional action invariant under local transformations of the generalized coordinates, i.e., transformations with parameters which are arbitrary functions of the space-time variables, belong to the class of so-called *degenerate* Lagrangian systems. The history of the study of degenerate Lagrangian systems, despite the venerable age of classical mechanics itself,[1)] extends for no more than half a century. The systematic analysis of these systems, initiated by the need to use the Hamiltonian formulation for problems in electrodynamics and gravitation, was begun in the late 1940s in studies by Dirac[2] and Bergmann.[3] They formulated the principles of the new theory, referred to as generalized Hamiltonian dynamics or the Dirac–Bergmann formalism. The further development of this science showed that this formalism forms an essential part of the quantum field description of all the fundamental interactions. The theory of gravitation, the unified theory of the electroweak and strong interactions, and models of grand unification are all field-theoretic models with generalized Hamiltonian dynamics, the degeneracy of which is associated with a local invariance: *reparameter or gauge invariance.*

The modern literature devoted to the various aspects of generalized Hamiltonian theory includes a large number of very well known books and reviews (see the list in Ref. 4),[2)] acquaintance with which persuades one of the refreshing lack of similarity to the theory of nondegenerate dynamical systems. The most important and clear manifestation of the uniqueness of this subject is the procedure of the *reduction* of a degenerate theory. Reduction is a systematic method of constructing a nondegenerate theory which is "equivalent" to the original degenerate theory, owing to the elimination of the "inessential" variables. In spite of its special features, the reduction of degenerate theories represents a certain generalization of the operation of eliminating *cyclic* or *ignorable* coordinates, well known since the end of the last century, which occurs in systems possessing a continuous symmetry associated with a Lie group. As in the case of reduction of the number of degrees of freedom in classical mechanics, reduction in degenerate dynamical systems also arises for purely geometrical, symmetry reasons lying at the heart of the theory. The present review is devoted to the discussion of these geometrical aspects of the reduction operation and demonstration of its immense importance in determining the physical content of gauge and reparametrization-invariant theories.

• **The geometrization of dynamics and observables.** The first step in the geometrization of all forms of interaction after the Einstein formulation of the general theory of relativity was taken in 1918 by Weyl,[5] who, in developing a unified theory of electromagnetism and gravity, formulated the principle of local scale invariance of a theory as a geometrical principle explaining the existence of the electromagnetic field. However, the key idea which initiated the modern geometrical treatment of an interaction in terms of connections in a principal fiber bundle[3)] was that of replacing the group of scale transformations in gravity by the group of local phase transformations in the charge space of electrodynamics. Its generalization, after the formulation of the classical theory of non-Abelian fields by Yang and Mills[8] in 1954, definitively transformed the principle of local gauge invariance into a geometrical foundation for constructing quantum field models of the fundamental interactions. Such a formulation of the dynamics of fundamental fields exclusively in geometrical terms necessarily leads to a degenerate theory, and, along with the elegant mathematics and clarity of the main ideas of the theory, introduces an extremely complicated problem which has no analog in the nondegenerate case. On the one hand, the principle of local invariance assigns a physical meaning only to invariant quantities, while requiring that the theory contain "extra" degrees of freedom which are not manifested in observable effects.[4)] The latter requirement implies that the correct statement of a physical problem presupposes the possibility of the construction, on the basis of the original geometrical variables, of new but observable (physical) variables, in terms of which physical reality will be described unambiguously, without "superfluous" elements. However, it is clear that without a unique and effective scheme for this identification of observable physical quantities with the original fundamental geometrical variables, the theory will not be complete and constructive. The reduction procedure is designed to effect this identification of physical quantities with bare geometrical ones by separating the so-called physical and unphysical sectors in a theory with local symmetry. Two reduction schemes will be discussed in this review. After a brief description of the standard approach based on the *gauge-fixing* technique, we analyze in detail the alternative *geometrical* or *gaugeless* method. There are two reasons for this: to present the geometrical reduction scheme itself, and to analyze the limitations following from it on the traditional method of gauge-fixing in generalized Hamiltonian dynamics.

• **Reduction in the gauge-fixing method.** Historically, the first experience in dealing with unphysical degrees of freedom arose within classical electrodynamics when, in the early twentieth century in connection with the desire to derive the Maxwell equations from a variational principle, the concept of the vector potential of the electromagnetic field was introduced.[5)] The price paid for this was the absence of a one-to-one relation between the electric and magnetic field strengths and the vector potential, which led to the appearance of unobservable variables in the theory. In classical electrodynamics, a functional arbitrariness of this type due to the introduction of unobservable degrees of freedom into the theory did not present any particular difficulty, because a simple relation between the vector potential of the electromagnetic field and the observables—the electric and magnetic field strengths—was known. After supplementing the equations of motion by an auxiliary condition, for example, the Coulomb condition, the unphysical components of the vector potential can be eliminated completely. As a result, a unique correspondence between the observables and the original fundamental variables arises in the theory. The need to generalize this simple construction from electrodynamics to the case of the canonical formulation of gravity led Dirac to develop a general scheme for reducing the phase space of degenerate systems, called the gauge-fixing method (Ref. 11).[6)] This method of identifying the physical degrees of freedom based on the introduction of additional conditions—gauges—into the theory which eliminate the unphysical degrees of freedom became the traditional method used in both classical and quantum theories of degenerate systems with local symmetries.

The representation of the scattering matrix of non-Abelian fields as a path integral[12] obtained by the gauge-fixing method has been used successfully to solve a number of perturbative problems in the theories of the electroweak and strong interactions. However, it later turned out that outside the framework of perturbation theory the gauge-fixing method led to a fundamental difficulty, the problem of Gribov ambiguities.[13] The Gribov analysis of the auxiliary Coulomb condition and, furthermore, the Singer topological no-go theorem for a global gauge in a non-Abelian theory[14,15] showed that rigorous definition of the class of admissible gauges was necessary. Although now the necessary condition for gauge functions to be admissible—nonzero Faddeev–Popov determinant—is well known, the question of the sufficient conditions on gauges which ensure the correct isolation of the physical space remains unanswered. Clearly, it is impossible to solve the problem of determining admissible gauges in generalized Hamiltonian dynamics without detailed knowledge of the actual structure of the physical and unphysical sectors of the theory. Therefore, the alternative reduction scheme based on the explicit separation of these sectors without the introduction of any auxiliary gauge conditions acquires special importance. We shall refer to this scheme as the *geometrical scheme*, having in mind the geometrical approach in the group analysis of differential equations, or the *gaugeless scheme*, thereby emphasizing it as an alternative to the traditional method.

• **Reduction in geometrical terms.** The scheme of reduction without resorting to gauge conditions described in this review has its roots in the well known operation of reducing or lowering the order of a differential equation admitting a symmetry under the action of a Lie group.[16–19] The reduction operation based on the corresponding integrals of the motion was used, beginning with the studies by Jacobi, Lie, and Poincaré, either with the goal of simplifying the original system of equations, or in order to prove their square-integrability. The introduction of the concept of noncommutative integrable systems[20] led to generalization of the method of reducing the order of equations by means of integrals of the motion in involution to the case of integrals forming a Lie algebra. The method of geometrical reduction is a generalization of these ideas to the case of a symmetry of a Hamiltonian system associated with the action of infinite-dimensional Lie groups and pseudogroups.[21] The problem of the reduction of degenerate Lagrangian systems corresponds to the problem posed by Lie: "...for a given equation and the symmetry group it admits, finding the orbit equations of any of its solutions and the equations whose solutions determine the set of all orbits." [18] This splitting of the equations into a *solvable* system determining a family of inequivalent solutions and an *automorphic system* specifying for each solution of the solvable system a set of solutions equivalent to it, is referred to as group decomposition of a system of differential equations or the Lie–Wess bundle.[18,19] In the terminology used in physics applications, this splitting corresponds to splitting the equations of motion into gauge-invariant and purely gauge sectors. The Cauchy problem in degenerate Lagrangian and Hamiltonian systems was formulated from a similar point of view most clearly in Ref. 22, where the Levi-Civita method was used to solve the problem of reducing a system of differential equations with invariant equations in involution.[23] The generalization of this method to the case of noninvolutive invariant relations forming a Lie pseudoalgebra is impossible without the development of an effective method of going to the equivalent set of involutive constraints or, in physics terminology, *Abelianization of the constraints*.[7)] In this review we shall discuss two methods of Abelianization of constraints. The first is based on the procedure of constraint resolution (Ref. 41), whose constructiveness and well-definedness is not guaranteed, while the second[26,27] is associated with the use of generalized canonical transformations admissable in degenerate systems.

• **Outline of the review.** To give a systematic exposition of the entire scheme of geometrical reduction, including the procedure of Abelianization of constraints, we must give some information from the theory of degenerate dynamical systems. While trying not to repeat the explanation of well known facts, we constructively review the general definitions and then begin directly with the problem of formulating the Cauchy problem in the Lagrangian and Hamiltonian approaches for degenerate dynamical systems. Next we concentrate on describing the method of gaugeless reduction in phase space, and deal with the question of the admissible gauge conditions. Our entire discussion will be carried out within the framework of mechanical systems with a finite number of degrees of freedom and without discussion of possible topological obstacles. In this review we shall restrict ourselves to the coordinate description of the reduction procedure. The geometrical, coordinate-free description of nondegenerate mechanical systems is discussed in Refs. 28–31, and the corresponding aspects of degenerate theories are studied in Refs. 32–36.

2. THE LAGRANGIAN FORMALISM

Let us begin with a summary of those points of the Lagrangian formulation of the theory of degenerate systems which will be particularly important from the viewpoint of the reduction problem and the statement of the Cauchy problem.

• **Generalized coordinate Lagrangians and evolution.** The configuration of a classical Lagrangian system with N degrees of freedom at some time $t=T$ is specified by a set of numbers $q_i(T)$, $i=1,\ldots,N$. The set of all possible configurations M is called the configuration space, the quantities q_i are the generalized Lagrangian coordinates, and the evolution of the system is the change of the system configuration with time, where the point q_i describes a curve on the manifold M: the classical trajectory of the system. The problem of classical mechanics is to describe the evolution of the system, i.e., to determine its classical trajectory.

• **The principle of classical determinism.** A classical trajectory in configuration space is defined as a solution of the differential equations of motion. The basic requirement imposed on the dynamical equations is that the Newton–Laplace principle of "classical determinism" be satisfied. The mathematical statement of this principle is that it is possible to determine uniquely the configuration of a system at an arbitrary time t on the basis of knowledge of the generalized coordinates $q_i(t)$ together with the time derivatives up to some order (velocities, accelerations) at any fixed instant of time $t=T$. In other words, the differential equations of motion must admit the correct statement of the Cauchy problem.

• **The principle of least action and the equations of motion.** The most important discovery of classical mechanics is the possibility of deriving such differential equations of motion from a variational problem. A large variety of variational principles are known. Their history and formulations are discussed systematically in the excellent texts by Levi-Civita and Amaldi[23] and by Whittaker.[37] Modern field-theoretic formulations usually start from the Hamilton–Ostrogradskiĭ principle of least action, according to which the equations of motion for a classical trajectory, the Euler–Lagrange equations, follow from the condition for the existence of an extremum of the so-called action functional.[8)] More precisely, it is assumed that the integral

$$\mathcal{S}\left[q,\frac{dq}{dt},\ldots,\frac{d^{k-1}q}{dt^{k-1}}\right]=\int dt\mathcal{L}\left(q,\frac{dq}{dt},\frac{d^2q}{dt^2},\ldots,\frac{d^kq}{dt^k},t\right) \tag{2.1}$$

exists. It is specified by the system Lagrangian $\mathcal{L}$, a function of the generalized Lagrangian coordinates and their time derivatives up to some order k. The Lagrangian determines the dynamics of the system from the condition that the classical trajectory is an extremum of the action functional $\mathcal{S}$ for certain boundary conditions on the variation. The large class of dynamical systems encountered in nature is described by second-order equations, and so it is common to restrict oneself to the study of Lagrangians which are functions of the coordinates and their first time derivatives.[9)] In this case the necessary condition for an extremum of the action

$$\mathcal{S}[q]=\int_{t_1}^{t_2}dt\mathcal{L}(q,\dot q,t) \tag{2.2}$$

with boundary conditions on the variation $\delta q(t_1)=\delta q(t_2)=0$ is that the Euler–Lagrange equations for the functions $q_i(t)$ be satisfied:[10)]

$$L_i[q]:=\frac{d}{dt}\left(\frac{\partial\mathcal{L}(a,\dot q,t)}{\partial\dot q_i}\right)-\frac{\partial\mathcal{L}(q,\dot q,t)}{\partial q_i}=0,\quad i=1,..,N, \tag{2.3}$$

which can be rewritten as

$$W_{ij}\ddot q_j-l_i=0, \tag{2.4}$$

with the notation

$$W_{ij}(q,\dot q,t)=\frac{\partial^2\mathcal{L}}{\partial\dot q_i\partial\dot q_j}, \tag{2.5}$$

$$l_i(q,\dot q,t)=-\frac{\partial^2\mathcal{L}}{\partial q_j\partial\dot q_i}\dot q_j-\frac{\partial^2\mathcal{L}}{\partial t\partial\dot q_i}+\frac{\partial\mathcal{L}}{\partial q_i}. \tag{2.6}$$

The matrix W_{ij} is called the Hessian matrix for the system of equations of motion (2.3). Its determinant, the Hessian, is one of the fundamental characteristics of a mechanical system. Depending on whether or not the Hessian matrix is degenerate, i.e., $\det\|W_{ij}\|=0$ or $\det\|W_{ij}\|\neq0$, mechanical systems are classified as *degenerate* or *nondegenerate*, respectively.[11)] The terms *singular* and *nonsingular* are sometimes preferred. The introduction of this classification is fully justified because these two types of theory possess fundamentally different properties, beginning already with the statement of the evolution problem.

• **The Cauchy problem in nondegenerate theories.** If a theory is nondegenerate,

$$\det\|W_{ij}\| \neq 0, \tag{2.7}$$

then, assuming that q_i and $\dot{q}_i$ are independent quantities, we can treat the Euler–Lagrange equations (2.3) as an algebraic system of equations for the unknown accelerations $\ddot{q}_i$ and solve them for the second time derivatives:

$$\ddot{q}_i = W_{ij}^{-1} l_j, \quad i,j=1,...,N. \tag{2.8}$$

This manner of writing the equations of motion, the so-called normal form of representing differential equations, implies that the "local" principle of "classical determinism" is satisfied. In fact, for the Euler–Lagrange equations written in the form (2.8) there always exists a unique solution in the neighborhood of arbitrary initial data, specified as a set of initial coordinates and velocities, because the Cauchy–Kowalewski theorem about the existence and uniqueness of the solutions is valid for a system of differential equations solved for the highest derivative.

Summary. In the case of nondegenerate systems, the Cauchy problem for the Euler–Lagrange equations of motion (2.3) has a unique solution in the neighborhood of arbitrary initial values of the coordinates and velocities.

• **The Cauchy problem in degenerate theories.** In theories with degenerate Lagrangians, the situation is fundamentally different. Here, since the inverse Hessian matrix does not exist, it is impossible to reduce the equations of motion to the normal form, which makes it impossible to formulate the classical Cauchy problem. However, the possibility of some modification of it is not excluded, and this is used explicitly or implicitly in all dynamical problems associated with degenerate systems, for example, the Cauchy problem in gravity and the problem of determining the evolution operator in the theory of non-Abelian gauge fields.

The correct statement of the Cauchy problem presupposes that the following conditions are satisfied:

(a) Existence of a solution.

(b) Uniqueness of a solution.

(c) Arbitrariness of the initial data.

The only possible modification is obviously to relax the requirements (b) and (c). Therefore, everywhere in what follows we assume that the degenerate theory is consistent in the sense of the existence of a solution, and only (either) nonuniqueness of the solution and (or) existence of the solution only for certain initial data are possible. In addition, from the viewpoint of physical applications, the most important point is the generalized statement of the problem where in the original degenerate theory it is possible to choose a set of generalized Lagrangian coordinates such that the Cauchy problem is correct in the classical sense for describing the evolution of some of these variables. The construction of such a set of variables forms the essence of the procedure of the *Lagrangian reduction* of degenerate systems. The problem of Lagrangian reduction, which requires separate, careful analysis, is not the subject of the present review, and so we shall only mention its main aspects in connection with the statement of the Cauchy problem.

Owing to the degeneracy of the theory, the rank of the Hessian matrix is smaller than the number of degrees of freedom N:[12)]

$$\operatorname{rank}\|W_{ij}(q,\dot{q},t)\| = R < N. \tag{2.9}$$

In contrast to the nondegenerate case, now the Euler–Lagrange equations (2.3), viewed as a system of N linear, algebraic, inhomogeneous equations with unknown $\ddot{q}_j$, $j=1,\dots,N$, are solvable only for R accelerations, because the rank of the fundamental matrix of the system is R:

$$\ddot{q}_\alpha = Q_\alpha(q_1,\dots,q_N,\dot{q}_1,\dots,\dot{q}_N,\ddot{q}_{R+1},\dots,\ddot{q}_N), \quad \alpha=1,...,R. \tag{2.10}$$

In fact, (2.9) implies the existence for the matrix $\|W_{ij}(q,\dot{q},t)\|$ of R linearly independent vectors $\mu_i^a(q,\dot{q},t)$ with nonzero eigenvalues $\xi^a(q,\dot{q})$,

$$W_{ij}(q,\dot{q},t)\mu_i^a(q,\dot{q},t) = \xi^a(q,\dot{q},t)\mu_j^a(q,\dot{q},t), \tag{2.11}$$

and $N-R$ linearly independent null vectors $\mu^a(q,\dot{q},t)$ with zero eigenvalues,

$$W_{ij}(q,\dot{q},t)\mu_i^a(q,\dot{q},t) = 0, \quad i,j=1,...,N, \quad a=1,...,N-R. \tag{2.12}$$

Contracting the eigenvectors with nonzero eigenvalues with (2.3), we obtain (2.10), while contraction of the null vectors with (2.3) gives a system of $N-R$ equations which do not contain accelerations:

$$\chi_a(q,\dot{q},t) = l_i(q,\dot{q},t)\mu_i^a(q,\dot{q},t) = 0. \tag{2.13}$$

The following situations can arise when analyzing Eqs. (2.10) and (2.13):

(i) The equations are inconsistent.

(ii) Eq. (2.13) is satisfied identically.

(iii) Of the $N-R$ equations (2.13), r (r_1) equations are functionally independent (dependent), and r_2 equations are satisfied identically.

Situation (i) indicates that the action functional does not have an extremum, and so it is excluded from further analysis as a case of no physical interest. Situation (ii), where all the $N-R$ equations (2.13) are satisfied identically, implies that (2.10) represents a system of differential equations for R coordinates $(q_1,\dots,q_R)$ in normal form, the right-hand sides of which depend on arbitrary functions of $(q_{R+1},\dots,q_N)$. Therefore, here we have the situation where the Cauchy problem has no unique solution; after the initial conditions on the coordinates $(q_1,\dots,q_R)$ and the corresponding velocities $(\dot{q}_1,\dots,\dot{q}_R)$ are fixed, the solution of (2.10) contains $N-R$ arbitrary functions.

Summary. If all the $N-R$ relations (2.13) are satisfied identically, the solution of the Cauchy problem for the Euler–Lagrange equations (2.3), after fixing the initial conditions on the R coordinates and the corresponding velocities, contains $N-R$ arbitrary functions.

The most complicated situation to analyze is (iii), where, in general, not all the expressions in (2.13) are satisfied identically. In this case the $N-R$ functionally independent equations give constraints on the possible values of the generalized coordinates and velocities, the so-called *Lagrangian constraints.*[13)] The appearance of Lagrangian constraints immediately presents the problem of whether or not the entire scheme is self-consistent, since the original variational problem presupposed independent variations of all the coordinates. For the further analysis, without loss of generality we can assume that

$$\operatorname{rank}\left\|\frac{\partial\chi_a(q,\dot q,t)}{\partial q_i},\frac{\partial\chi_a(q,\dot q,t)}{\partial\dot q_i}\right\|=r_1+r_2,\quad r_1+r_2\leqslant N-R, \tag{2.14}$$

$$\operatorname{rank}\left\|\frac{\partial\chi_a(q,\dot q,t)}{\partial\dot q_i}\right\|=r_2. \tag{2.15}$$

Then (2.13) can be replaced by r_1+r_2 equivalent relations of the form

$$\chi_a^1(q,t)=0,\quad a=1,\dots,r_1, \tag{2.16}$$

$$\chi_a^2(q,\dot q,t)=0,\quad a=1,\dots,r_2. \tag{2.17}$$

The functions χ_a^1 and χ_a^2 are traditionally called *Lagrangian constraints* of type A and B, respectively (Ref. 4c). Therefore, in the singular case the original system of equations of motion (2.3) is reduced to the system of equations (2.10), (2.16), and (2.17), which must be investigated for consistency. It is clear that this system will be consistent when the constraints χ_a^1 and χ_a^2 are conserved in time, i.e., when their total time derivative is zero. We begin our analysis of the consistency conditions with the constraints of type A:

$$\frac{d\chi_a^1(q,t)}{dt}=\frac{\partial\chi_a^1(q,t)}{\partial q_i}\dot q_i+\frac{\partial\chi_a^1(q,t)}{\partial t}=0. \tag{2.18}$$

If these equations are not satisfied identically when the Lagrangian constraints A and B are taken into account, then, in general, (2.18) gives rise to new Lagrangian constraints of both types, which in principle can lower the rank of the Hessian matrix, which in turn decreases the number of second-order equations (2.10). The resulting system of equations taking into account the new Lagrangian constraints of type A must again be checked for consistency, and so on. For consistent theories, the process of generating new constraints of type A stops after a finite number of steps, and the total number of constraints will be smaller than the number of degrees of freedom N. After this, it is necessary to perform the analogous procedure for the constraints of type B, which leads to the equations

$$\frac{d\chi_a^2(q,\dot q,t)}{dt}=\frac{\partial\chi_a^2(q,\dot q,t)}{\partial\dot q_i}\ddot q_i+\frac{\partial\chi_a^2(q,\dot q,t)}{\partial q_i}\dot q_i+\frac{\partial\chi_a^2(q,\dot q,t)}{\partial t}=0. \tag{2.19}$$

Again, if these equations are not satisfied identically when all the equations of motion existing at this stage and the Lagrangian constraints of types A and B are taken into account, then (2.19) leads to new Lagrangian constraints of both types, which can again lower the rank of the Hessian matrix and thereby decrease the number of independent equations of second order in the time. For consistent theories, the repetition of this procedure ultimately leads to a complete system of the form

$$\ddot q_\alpha=Q_\alpha(q_1,\dots,q_N,\dot q_1,\dots,\dot q_N,\ddot q_{r'+1},\dots,\ddot q_N),\quad \alpha=1,\dots,r', \tag{2.20}$$

$$\chi_{a'}^1(q,t)=0,\quad a'=1,\dots,r_1', \tag{2.21}$$

$$\chi_{a''}^2(q,\dot q,t)=0,\quad a''=1,\dots,r_2'', \tag{2.22}$$

where $r_1+r_2<r_1'+r_2''<N$, $r'<R$, $r_1'+r_2''+r'<N$, and the completeness condition for the system of constraints implies that no new constraints appear in the evolution process, i.e., the equations

$$\frac{d\chi_{a'}^1(q,t)}{dt}=0,\quad a'=1,\dots,r_1', \tag{2.23}$$

$$\frac{d\chi_{a''}^2(q,\dot q,t)}{dt}=0,\quad a''=1,\dots,r_2'', \tag{2.24}$$

are satisfied identically, taking into account all the expressions (2.20)–(2.22).

Summary. For systems with degenerate Lagrangian, the definitive system of equations is given by (2.20)–(2.22), among which only the equations containing second derivatives with respect to the time are true equations of motion, while the rest are restrictions on the initial conditions on the coordinates and velocities, i.e., Lagrangian constraints.

The general problem of studying the integrability of degenerate Lagrange equations of motion and determining the nature of the general solutions in the sense of determining the degree to which they are nonunique is quite complicated. Its analysis requires the use of techniques from the modern geometrical theory of nonlinear differential equations[21] based on the ideas of formal expansion of solutions in Taylor series and extension of a system of differential equations to an integrable system. Without delving deeply into this problem, let us mention just a few of the main features of the Cauchy problem for Lagrange equations of motion in degenerate theories:

• Whereas the arbitrariness in the general solution of the equations of motion following from nondegenerate Lagrangians reduces to the existence of $2N$ arbitrary constants whose values are uniquely determined by the initial data on the generalized coordinates and velocities, in the degenerate case this is not so. The degree and nature of a given arbitrariness can be established by studying two limiting cases:

1. Maximal arbitrariness occurs when all the Lagrangian constraints are satisfied identically. In this case the general solution of the equations of motion depends on $N-R$ arbitrary functions of the time and $2R$ arbitrary constants.
2. Minimal arbitrariness occurs when all the Lagrangian constraints are constraints of type A. Then the general solution depends on $2(N-R)$ arbitrary constants.

In contrast to the nondegenerate case, the initial values of the generalized coordinates and velocities cannot be chosen arbitrarily, but must satisfy the Lagrangian constraints (2.21) and (2.22).

• **The Noether identities and the Cauchy problem.** The completeness condition mentioned in obtaining the system of equations (2.20)–(2.22) is related to the consistency of the statement of the Cauchy problem, when the Lagrangian constraints (2.21) and (2.22) are treated as conditions on the initial data for a second-order system in normal form (2.20). For the exceptionally important singular theories, the degeneracy of which is associated with invariance of the action under transformations containing arbitrary functions of time as group parameters, the integrability or, equivalently, consistency of the Cauchy problem is guaranteed by the well known Noether identities,[39,40] which ensure that the constraint conditions are satisfied at any instant of time if they are satisfied at one instant of time (Ref. 4h). In addition, the Noether identities determine the functional arbitrariness of the general solutions of the equations of motion associated with the rank of the symmetry group of the action functional.

3. THE HAMILTONIAN FORMALISM

The analogy between mechanical and optical phenomena discovered by Hamilton and presented in his report to the Irish Academy of Sciences in 1824 initiated new formulations of dynamical principles of motion. As the history of science shows, it was this analogy and the Hamiltonian form of the equations of motion which a century later played an exceptionally important role in the creation and establishment of quantum theory.

Hamiltonian mechanics, which began with the studies of Poisson, Hamilton, Ostrogradskiĭ, and Liouville, was first based exclusively on the canonical coordinates of the symplectic structure in Euclidean space.[14)] A more general treatment based on the Poisson structure first appeared in connection with the theory of functional groups and the theory of the integration of first-order partial differential equations in the studies by Lie.[25] However, the Lie approach was forgotten by both mathematicians and physicists, and we are indebted to Dirac[2] (1950) for reviving the general view of the Poisson structure of Hamiltonian mechanics. Dirac was the first to introduce the general symplectic structure of a manifold, defining the Poisson bracket purely axiomatically.[15)]

• **The Hamiltonian form of nondegenerate Lagrangian systems.** The system of Lagrange equations (2.8) consisting of N second-order differential equations in normal form can be replaced by the infinite set of procedures of the first-order system equivalent to it, also in normal form, with $2N$ unknowns, if for the new unknown functions we take, along with the $q_i(t)$, the N first derivatives $v_i(t)=\dot{q}_i(t)$, or, altogether, N independent functions $v_i(t)=v_i(q(t),\dot{q}(t),t)$. To obtain the first-order equations, Hamilton used a special set of such functions. These variables, auxiliary to the coordinates, are the generalized or canonical momenta, related to the velocities of the Legendre transformation. The generating function of the direct transformation is the Lagrangian of the system $\mathcal{L}(q,\dot{q},t)$ itself:

$$p_i=\frac{\partial\mathcal{L}(q,\dot{q},t)}{\partial\dot{q}_i},\quad i=1,\dots,N, \tag{3.1}$$

and the generating function of the inverse transformation

$$\dot{q}_i=\frac{\partial H(q,p,t)}{\partial p_i},\quad i=1,\dots,N, \tag{3.2}$$

is the Hamilton function or the Hamiltonian $H(p,q,t)$, which is related to the Lagrangian as

$$H(q,p,t)=\sum_{i=1}^{N}\left(p_i\dot{q}_i-\mathcal{L}(q,\dot{q},t)\right)\Big|_{\dot{q}\to q,p,t}, \tag{3.3}$$

where it is assumed that all the velocities are expressed in terms of the momenta, using the direct transformation (3.1). The Legendre transformation takes the Euler–Lagrange equations of motion (2.8) into the canonical Hamilton equations[16)]

$$\dot{q}_i=\frac{\partial H(q,p)}{\partial p_i}, \tag{3.4}$$

$$\dot{p}_i=-\frac{\partial H(q,p)}{\partial q_i}. \tag{3.5}$$

Following Gibbs, the space of the canonical variables p,q is called the $2N$-dimensional phase space Γ. After introducing the Poisson bracket of two functions $A(q,p,t)$ and $B(q,p,t)$ defined on Γ,

$$\{A,B\}=\sum_{i=1}^{N}\left(\frac{\partial A}{\partial q_i}\frac{\partial B}{\partial p_i}-\frac{\partial A}{\partial p_i}\frac{\partial B}{\partial q_i}\right), \tag{3.6}$$

the equations of motion (3.4) acquire the symmetric form

$$\dot{q}_i=\{q_i,H\},\quad \dot{p}_i=\{p_i,H\}. \tag{3.7}$$

The equivalence of these canonical equations to the Euler–Lagrange equations can also be established by using the variational problem for a conditional extremum in phase space:

$$S[q,p]=\int_{t_1}^{t_2}dt[p_i\dot{q}_i-H(q,p,t)] \tag{3.8}$$

with the boundary conditions

$$\delta q(t_1)=\delta q(t_2)=0. \tag{3.9}$$

Here nondegeneracy of the Hessian matrix, $\det\|W_{ij}\|\neq 0$, is the necessary condition for this equivalence.

Summary. The classical theorem about the equivalence of the description of the evolution of nondegenerate systems in terms of variables defined on the phase space and in terms of configuration space runs as follows: if the functions $q_i(t),\dot{q}_i(t)$ satisfy the Euler–Lagrange equations, then the functions $q_i(t),p_i(t)$ related to them by a direct Legendre transformation, with the Lagrangian as the generating function, satisfy the Hamilton equations and vice versa, and every solution of the canonical equations is transformed into a solution of the Lagrange equations by the inverse Legendre transform.

• **Hamiltonian dynamics in degenerate systems.** Let us now discuss the Hamiltonian dynamics for systems with degenerate Lagrangians. Let the rank of the Hessian matrix be nonmaximal:

$$\operatorname{rank}\|W_{ij}(q,\dot{q},t)\|=R<N. \tag{3.10}$$

Then, according to the definition of the canonical momenta (3.1), only R of the N canonical momenta p_i are independent functions of the velocities. In other words, we have relations of the form

$$\varphi_\alpha(q,p)=0, \quad \alpha=1,\dots,N-R, \tag{3.11}$$

between the canonical variables of phase space, referred to by Bergmann as *primary constraints*. We note that none of the $N-R$ primary constraints (3.11) are functionally dependent:

$$\operatorname{rank}\left\|\frac{\partial\varphi_\alpha(q,p)}{\partial p_i},\frac{\partial\varphi_\alpha(q,p)}{\partial q_i}\right\|=N-R, \tag{3.12}$$

and they can be resolved for the $N-R$ momenta:[17]

$$\operatorname{rank}\left\|\frac{\partial\varphi_\alpha(q,p)}{\partial p_i}\right\|=N-R. \tag{3.13}$$

Therefore, in contrast to nondegenerate systems, where the Legendre transform determines a one-to-one relationship between the space of states M' and the phase space Γ, for degenerate systems there is only projection.

Summary. **In the case of systems with degenerate Lagrangians, the Legendre transform maps the entire space of states M' onto the $(2N-R)$-dimensional surface of primary constraints Γ_1.**

This fact implies that the canonical Hamiltonian as the generating function of the inverse Legendre transform:[18]

$$H_c(q_i,p_a,\dot{q}_{R+\alpha})=p_i\dot{q}_i-\mathcal{L}(q,\dot{q})|_{\dot{q}_a=f_a},$$

$$i=1,\dots,n, \quad a=1,\dots,R, \quad \alpha=1,\dots,N-R, \tag{3.14}$$

for a degenerate system is defined on the surface of primary constraints $\Gamma_1\subset\Gamma$, and not in the entire phase space. This implies that now, instead of $2N$ equations on Γ, there are only $2N-R$ Hamilton equations of motion defined on Γ_1. In order to obtain a complete set of $2N$ Hamilton equations, Dirac took a fundamentally new step: he introduced the concept of *weak* and *strong* equalities, which establish the equivalence relation on a set of functions defined in the entire phase space of the system. Without analyzing these concepts in detail, we shall assume that two functions $f(q,p)$ and $g(q,p)$ are equal in the weak sense,[19] $f(q,p)\simeq g(q,p)$, if they coincide with each other on the constraint surface. According to this equivalence relation, it can be shown that there exists an infinite set of functions $H_c(q,p)$, defined in the entire phase space, differing from each other by a linear combination of primary constraints, and equivalent to $H_c(q_i,p_a)$ in the sense that

$$\{q_i,H_c(q_i,p_a)\}\simeq\{q_i,H_c(q,p)\},$$

$$\{p_i,H_c(q_i,p_a)\}\simeq\{p_i,H_c(q,p)\}.$$

In order to encompass the entire class of equivalent Hamiltonians $H_c(q_i,p_a)$, Dirac defined the *full* Hamiltonian as

$$H_T=H_c(q,p)+u_\alpha(t)\varphi_\alpha(q,p), \quad \alpha=1,\dots,N-R, \tag{3.15}$$

introducing $N-R$ arbitrary functions $u_\alpha(t)$ and fixing one function $H_c(q,p)$ defined in the entire phase space and satisfying the condition $H_c(q,p)\simeq H_c(q_i,p_a)$. This full Hamiltonian H_T for systems with degenerate Lagrangian can be used to write the equations of motion in Hamilton–Dirac form:

$$\dot{q}_i\simeq\{q_i,H_T(q,p)\}, \tag{3.16}$$

$$\dot{p}_i\simeq\{p_i,H_T(q,p)\}, \quad i=1,\dots,N, \tag{3.17}$$

$$\varphi_\alpha(q,p)\simeq0, \quad \alpha=1,\dots,N-R, \tag{3.18}$$

which is equivalent to the system of Euler–Lagrange equations (2.20)–(2.22). Here equivalence, as in the nondegenerate case, is understood in the following sense. If the functions $q_i(t)$, $i=1,\dots,N$, are solutions of the Euler–Lagrange equations, then the functions $q_i(t)$ and $p_i(t)=\partial\mathcal{L}(q(t),\dot{q}_i(t))/\partial\dot{q}_i(t)$ will be solutions of the Hamilton equations (3.16)–(3.18) for some choice of functions $u_\alpha(t)$, $\alpha=1,\dots,N-R$. Conversely, if for some choice of functions $u_\alpha(t)$ the functions $q_i(t)$ and $p_i(t)$ are solutions of the Hamilton–Dirac equations, then the functions $q_i(t)$ satisfy the Euler–Lagrange equations. Owing to this equivalence, the analysis of the consistency of the Lagrange equations of motion described in the preceding section carries over to the system of Hamilton–Dirac equations. As in analyzing the Lagrange equations of motion, where the consistency of the theory was checked according to consistency of the dynamics in the presence of Lagrangian constraints (2.13), in the present case it is necessary to check that the primary constraints (3.11) are stationary in time:

$$0\equiv\frac{d\varphi_\alpha(q,p)}{dt}:=\{\varphi_\alpha(q,p),H_T(q,p)\}$$

$$=\{\varphi_\alpha(q,p),H_c(q,p)\}+u_\beta(t)\{\varphi_\alpha(q,p),\varphi_\beta(q,p)\}\simeq0. \tag{3.19}$$

The following situations can arise in analyzing Eqs. (3.16)–(3.18) and (3.19):

(i) The equations are inconsistent.

(ii) Equations (3.19) are satisfied identically.

(iii) Some of the $N-R$ equations (3.19) form a system of functionally independent equations which are used to determine the u_α, $\alpha=1,\dots,\alpha_0$.

Situation (i) implies the absence of extrema in the action functional, where it is impossible to satisfy (3.19) by any choice of u_α. This case is thus eliminated from consideration, as being of no physical interest. Situation (ii) requires no comment; the theory is consistent. In situation (iii) we must distinguish the special case where the system of constraints is closed relative to the Poisson-bracket operation:

$$\{\varphi_\alpha(q,p),\varphi_\beta(q,p)\}=f_{\alpha\beta\gamma}(q,p)\varphi_\gamma(q,p),$$

$$\alpha,\beta,\gamma=1,\dots,N-R. \tag{3.20}$$

In connection with this, we note that the set of constraints and the Poisson bracket establish an equivalence relation between dynamical quantities. Owing to the importance of this relation, Dirac introduced a corresponding terminology. He called any function $A(q,p)$ whose Poisson brackets with all the constraints in the theory are weakly equal to zero,

$$\{A(q,p),\varphi_\alpha(q,p)\}\simeq 0,$$

a dynamical quantity of the *first class*, and all quantities not belonging to the first class *second-class* quantities. The constraints themselves are classified according to this terminology.

After the primary constraints are taken into account, the dependence on all the u_α drops out of the stationarity conditions (3.19), and the latter become additional constraints on the generalized coordinates and momenta:

$$\chi(p,q)=0, \tag{3.21}$$

which Dirac called *secondary* constraints, thereby emphasizing the fact that they originate in the dynamical equations of motion and not only in the Legendre transformations, as occurs for primary constraints. Obviously, when secondary constraints are present the consistency of the theory must be analyzed in the same way as in the case of primary constraints. The consistency analysis ends when at some step of the process the appearance of new constraints terminates. To analyze the general case (iii), where not all the Poisson brackets $\{\varphi_\alpha(q,p),\varphi_\beta(q,p)\}$ are equal to zero, taking into account all the constraints of the preceding steps, it is convenient to introduce a unified notation for all the constraints of the second, third, etc., steps using φ_j, $j=N-R+1,\ldots,J$. Since in this case the consistency conditions do not give new constraints, they can be treated as a system of inhomogeneous linear equations for the u_α:

$$\{\varphi_j(q,p),H_c(q,p)\}+u_\alpha(t)\{\varphi_j(q,p),\varphi_\alpha(q,p)\}\simeq 0,$$

$$\alpha=1,\ldots,N-R, \tag{3.22}$$

and the coefficients $u(t)$ can be fixed as functions of (q,p): $u_\alpha(t):=U(q,p)$. If $\operatorname{rank}\|\{\varphi_i(q,p),\varphi_\alpha(q,p)\}\|=A$, then the general solution for $u_\alpha(t)$ contains a term in the form of a linear combination of A arbitrary functions $v_a(t)$ and solutions of the corresponding homogeneous system $V_{a\alpha}$:

$$u_\alpha(t)\{\varphi_i(q,p),\varphi_\alpha(q,p)\}\simeq 0, \tag{3.23}$$

$$u_\alpha(t)=U_\alpha(p,q)+v_a(t)V_{a\alpha}, \quad a=1,\ldots,A. \tag{3.24}$$

Accordingly, two structures of the full Hamiltonian are distinguished:

$$H_T=H_c'(p,q)+v_a(t)\phi_a(p,q), \tag{3.25}$$

where

$$H_c'(p,q):=H_c(p,q)+U_\alpha(p,q)\varphi_\alpha(p,q),$$

$$\phi_a(p,q)=V_{a\alpha}(p,q)\varphi_\alpha(p,q). \tag{3.26}$$

Summary. The generalized Hamiltonian dynamics in the theory with degenerate Lagrangian is determined by the full Hamiltonian of the system H_T, which is equal to the sum of the two first-class quantities H' and $v_a(t)\phi_a(p,q)$ with arbitrary functions of time $v_a(t)$ equal in number to the number of first-class primary constraints. The presence of arbitrary functions in the Hamilton–Dirac equation implies that it is impossible to uniquely determine the dynamics of the generalized coordinates and momenta in terms of given initial data satisfying the complete set of constraints of the system.

This ambiguity, which is completely consistent with the description of evolution in the Lagrangian approach, has the same origin: the invariance of the equations of motion under a group of local gauge transformations, the rank of which is determined by the number of first-class primary constraints. Bypassing discussion of the well known hypothesis of Dirac about the structure of the generating functions of local gauge symmetry transformations and about the role of first-class constraints (see Refs. 4, 44, and 45), here we shall concentrate on the main problem originating in this feature of the dynamics of degenerate systems with local symmetries: the reduction problem, i.e., the determination of the nondegenerate Hamiltonian system which is "equivalent" to the original system. The rest of our discussion will be devoted to explaining the meaning of the word "equivalent."

4. REDUCTION IN SYSTEMS WITH FIRST-CLASS CONSTRAINTS

In our discussion of the problem of reduction, we shall limit ourselves to the case where the theory contains only first-class constraints. We shall begin by answering the question of the sense in which the original degenerate theory with local symmetry can be compared to the nondegenerate theory equivalent to it.

• **Determination of the reduced phase space.** To make all concepts precise, we shall take the case of a mechanical system defined in a $2n$-dimensional Euclidean phase space Γ with canonical coordinates q_i and their conjugate momenta p_i, and provided with a canonical symplectic structure $\{q_i,p^j\}=\delta_i^j$. According to the generalized Hamiltonian formulation of systems with local symmetry, the dynamics unfolds on the $(2n-m)$-dimensional submanifold Γ_c of the phase space specified by a complete set of m functionally independent relations

$$\varphi_\alpha(p,q)=0, \tag{4.1}$$

which form a closed system under the Poisson-bracket operation:

$$\{\varphi_\alpha(p,q),\varphi_\beta(p,q)\}=f_{\alpha\beta\gamma}(p,q)\varphi_\gamma(p,q), \tag{4.2}$$

i.e., which are first-class constraints. Completeness is understood in the sense of satisfaction of the relations

$$\{\varphi_\alpha(p,q),H_C(p,q)\}=g_{\alpha\gamma}\varphi_\gamma(p,q), \tag{4.3}$$

with the canonical Hamiltonian of the system $H_C(p,q)$. Owing to the presence of the constraints, the dynamics of the system is described by the generalized Poincaré–Cartan form

$$\Theta=\sum_{i=1}^{n} p_i dq_i-H_E(p,q)dt \tag{4.4}$$

with the *generalized Hamiltonian* $H_E(p,q)$ differing from the canonical Hamiltonian $H_C(p,q)$ by the addition of a linear combination of all the first-class constraints with undetermined coefficients $u_\alpha(t)$:

$$H_E(p,q)=H_C(p,q)+u_\alpha(t)\varphi_\alpha(p,q). \tag{4.5}$$

From the completeness condition (4.3) with H_C replaced by H_E it follows that when only first-class constraints are present, the functions $u_\alpha(t)$ cannot be determined within the internal terms of the theory. This is a manifestation of the local symmetry present in the system and causes the dynamics of some of the coordinates to be undefined. The existence of the arbitrary functions $u_\alpha(t)$ implies that there is not a one-to-one correspondence between the space of physical states of the system and the subspace Γ_c. In other words, each point in Γ_c corresponds to one physical state, while each physical state corresponds to several points in Γ_c. *The subspace of the full phase space* Γ *whose points are in one-to-one correspondence with the physical states of the system is called the reduced phase space of the degenerate theory and denoted as* Γ^*. Clearly, $\Gamma^*\subset\Gamma_c$, because a trajectory of the system beginning at a point in the phase space belonging to the constraint surface Γ_c remains in this space at all subsequent times, owing to the stationarity condition for the constraints. To verify that each subspace Γ^* does actually exist and that its dimension is $2n-2m$, where n is the number of degrees of freedom of the system and m is the number of all first-class constraints, we must refine the definition of a physical state of a classical system with local symmetry. We shall assume that the space of physical states of the system is specified by a finite set of *physical variables* O^A, each of which in a theory with first-class constraints is defined, following Dirac, as a dynamical quantity satisfying the relations[20)]

$$\{O^A(p,q),\varphi_\alpha(p,q)\}=d^A_{\alpha\gamma}(p,q)\varphi_\gamma(p,q). \tag{4.6}$$

Then the reduced space is defined as follows. If we treat (4.6) as a system of m linear first-order differential equations on O^A, then, owing to the integrability conditions (4.2), each function can be defined completely and uniquely in terms of its initial values on the $2(n-m)$-dimensional submanifold of the full phase space Γ (Ref. 32). It is this subspace of the constraint surface which defines the desired *reduced phase space* Γ^*.

Summary. The physical state of a system described by a degenerate 2*n*-dimensional Hamiltonian system with *m* first-class constraints is defined by a set of 2(*n*−*m*) invariant Dirac observables (4.6) defined on the reduced phase space Γ^*.

We have thus explained the meaning of the reduction operation for a system with first-class constraints as the problem of constructing a nondegenerate Hamiltonian system equivalent to the original system. Here the word "equivalence" implies that, first, the nondegenerate system must have a $2(n-m)$-dimensional phase space **isomorphic** to the reduced space of the degenerate theory, and, second, its Hamiltonian dynamics must be **canonically equivalent** to the dynamics of the Dirac observables in the degenerate case. To solve this problem, we must find a set of $2(n-m)$ "physical coordinates" Q_i^*, P_i^* specifying the reduced phase space and choose m auxiliary pairs of coordinates determining the gauge degrees of freedom of the system.

There are various approaches to solving the reduction problem in systems with first-class constraints. Below, we shall briefly describe alternative methods of constructing the physical and gauge degrees of freedom: the standard approach with the introduction of additional gauge conditions (gauges), and the purely geometrical method of Hamiltonian reduction without the use of any gauges. Regarding the last method, we only note that the idea of performing the reduction exclusively within the internal terms of the theory stems primarily from the desire to fully preserve the global properties of the original theory with the "extra" degrees of freedom.

• **The Dirac gauge-fixing method.** The general principles of the introduction of gauge conditions in the Hamiltonian approach as auxiliary constraints imposed on the canonical variables were proposed by Dirac in connection with the construction of the canonical formalism of the theory of gravitation.[11] According to Dirac's idea, to specify a $2(n-m)$-dimensional reduced space Γ^* as a surface in the full phase space Γ, along with the m constraint equations it is possible to introduce into the theory m auxiliary constraints on the coordinates, the *gauges*

$$\chi_\alpha(p,q)=0. \tag{4.7}$$

Here it is assumed that the gauge conditions satisfy the following requirements:

(1) When they are used, the undetermined Lagrange multipliers must be fixed uniquely as functions of the generalized coordinates and momenta.

(2) Together with the constraints, the gauge conditions must define a $2(n-m)$-dimensional space Σ.

(3) The surface Σ must be "canonically equivalent" to the surface Γ^*.

We immediately note that the main problem is to give a rigorous formulation and to satisfy the last condition on the gauges. Requirements (1) and (2) can be satisfied if the condition

$$\det\|\{\chi_\alpha(p,q),\varphi_\beta(p,q)\}\|\,|_\Sigma\neq 0 \tag{4.8}$$

is satisfied. The satisfaction of condition (2) follows directly from the implicit-function theorem. We shall show that the condition (4.8) makes it possible to find the unknown Lagrange multipliers $u_\alpha(t)$ from the condition for the gauges (4.7) to be stationary in time,

$$\dot\chi_\alpha=\{\chi_\alpha,H_C\}+\sum_\beta\{\chi_\alpha,\varphi_\beta\}u_\beta=0, \tag{4.9}$$

and thereby to fix the dynamics of the system uniquely. In fact, the equations in (4.9) form a compatible system of inhomogeneous algebraic equations for the unknowns $u_\beta(t)$, with (4.8) serving as the compatibility condition. After determining the Lagrange multipliers as functions of the coordinates and momenta, we have the following picture: the dynamics of the system is specified in the form of $2n$ Hamiltonian equations of motion

$$\dot{q}_i \simeq \{q_i, H_E^*\}, \tag{4.10}$$

$$\dot{p}_i \simeq \{p_i, H_E^*\}, \tag{4.11}$$

and $2m$ constraint equations

$$\varphi_\alpha(q,p) \simeq 0, \tag{4.12}$$

$$\chi_\alpha(q,p) \simeq 0, \tag{4.13}$$

which actually represent auxiliary conditions on the initial data in the Cauchy problem for the system of Hamilton–Dirac equations (4.10) and (4.11). The symbol $\simeq$ in (4.10) and (4.11) denotes weak equality taking into account all the constraints and gauges, and H_E^* denotes the generalized Hamiltonian H_E, in which the Lagrange multipliers are fixed by means of (4.9). The Hamilton–Dirac equations (4.10) and (4.11) therefore represent a system of weak equations, i.e., the constraints and gauges must be taken into account in them only after taking all the Poisson brackets. However, it turns out that by using a modification of the Poisson bracket, more precisely, by replacing the Poisson brackets by Dirac brackets, it is possible to eliminate the weak equalities from the theory. The Dirac brackets are determined by the Poisson- bracket operation and the system of first-class constraints, augmented by the gauges:

$$\{F,G\}_D := \{F,G\} - \{F,\xi_s\} C^{-1}_{ss'} \{\xi_{s'}, G\}, \tag{4.14}$$

where ξ denotes the set of all constraints and gauges, and C^{-1} is the matrix inverse of $C_{\alpha\beta} := \{\xi_\alpha, \xi_\beta\}$. The remarkable result of Dirac was to observe that the equations of motion can be written with the Poisson brackets replaced by the new brackets (4.14):

$$\dot{q}_i \simeq \{q_i, H_c\}_D, \tag{4.15}$$

$$\dot{p}_i \simeq \{p_i, H_c\}_D, \tag{4.16}$$

where all the constraints can be set equal to zero before taking the Dirac brackets. This fact is a consequence of the following important property of the Dirac bracket:

$$\{F, \xi_s\}_D = 0, \tag{4.17}$$

where $F(q,p)$ is an arbitrary function. The theory thus now contains only strong equalities, and the problem of reducing the phase space has been solved, abeit implicitly, since the number of Hamilton equations of motion remains unchanged, even though the presence of constraints and gauges indicates that they are not independent.

Summary. The correct addition of gauges to the set of first-class constraints makes it possible to take into account the fact that the original canonical coordinates are not independent by replacing the original canonical symplectic structure by a new structure determined by the Dirac brackets, and it allows the number of degrees of freedom to be effectively reduced from $2n$ to $2(n-m)$:

$$\sum_{i=1}^{n} \{q_i, p_i,\}_{\text{P.B.}} = n, \quad \sum_{i=1}^{n} \{q_i, p_i,\}_{\text{D.B.}} = n - m.$$

The new symplectic structure, which depends on the choice of gauge conditions, is in general complex, and so it leads to serious difficulties in quantization. However, there is a special case in which the Dirac bracket coincides with the canonical Poisson bracket for a regular system defined on Γ^*:

$$\{F,G\}_D|_{\varphi=0,\ \chi=0} = \sum_{i=1}^{n-m} \left\{ \frac{\partial \bar{F}}{\partial Q_i^*} \frac{\partial \bar{G}}{P_i^*} - \frac{\partial \bar{F}}{\partial P_i^*} \frac{\partial \bar{G}}{Q_i^*} \right\}. \tag{4.18}$$

It follows from this representation of the Dirac bracket that in terms of the conjugate coordinates Q_i^*, P_i^* $(i=1,\dots,n-m)$, the reduced phase space is parametrized in such a way that the constraints vanish identically, and an arbitrary function $F(p,q)$ specified on the reduced phase space can be written as

$$F(p,q)|_{\varphi=0,\ \chi=0} = \bar{F}(P^*, Q^*).$$

Therefore, if in the Dirac gauge-fixing method the requirement (3) on the gauge is satisfied, i.e., if the surface Σ is canonically equivalent to the surface Γ^* and the gauges are chosen "luckily" in the sense that (4.18) holds, then the problem of determining the "true dynamical degrees of freedom" is thereby solved explicitly.

• **The Faddeev gauge-fixing method.** Another method of reduction by gauge fixing was suggested in the well known study of Faddeev[32] devoted to the quantization of systems with constraints using functional integration. The basic idea of the Faddeev method, in contrast to the Dirac method, was to introduce an explicit parametrization of the reduced phase space. As in the Dirac method, gauge conditions $\chi_\alpha(p,q)=0$ are introduced which, in addition to the conditions (4.8), satisfy the additional requirement of being Abelian:

$$\{\chi_\alpha(p,q), \chi_\beta(p,q)\} = 0. \tag{4.19}$$

The Abelian nature of the gauge conditions (4.19) makes it possible, by a canonical transformation

$$q_i \mapsto Q_i = Q_i(q,p)$$

$$p_i \mapsto P_i = P_i(q,p) \tag{4.20}$$

to go to new variable coordinates such that the first m momenta coincide with the gauge conditions χ_α:

$$P_\alpha = \chi_\alpha(q,p). \tag{4.21}$$

The condition (4.8) allows the constraints (4.1) to be resolved for the coordinates Q_α, expressing them as functions of the $(n-m)$ pairs of canonically conjugate coordinates and momenta (Q_i^*, P_i^*), which are the internal coordinates of the $2(n-m)$-dimensional surface Σ defined by the relations

$$P_\alpha = 0,$$

$$Q_\alpha = Q_\alpha(Q^*, P^*). \tag{4.22}$$

This completes the reduction of the phase space, in the sense of distinguishing the $2(n-m)$ independent variables which evolve according to the usual Hamilton equations with the Hamiltonian

$$H^*(Q^*, P^*) := H_E(q,p)|_{\bar{Q}_\alpha = \bar{Q}_\alpha(P^*, Q^*), \bar{P}_\alpha = 0}. \tag{4.23}$$

Summary. The variables (Q_i^*, P_i^*) in the Faddeev method actually represent a set of physical degrees of

freedom if it is possible to choose auxiliary gauge conditions χ_α such that the surface Σ is canonically equivalent to the reduced phase space Γ^* of the system.

Therefore, both in the Dirac method and in this approach the problem of defining admissible gauge conditions must also be solved.

• **The gaugeless method of Hamiltonian reduction.** The invariance of the theory under local gauge transformations is due to the presence of unphysical degrees of freedom in the theory, which in the Hamiltonian approach is reflected in the existence of Lagrange multipliers which cannot be defined within the internal terms of the theory. The method of phase-space reduction by introducing gauges eliminates this arbitrariness and fixes the Lagrange multipliers as functions of the generalized coordinates and momenta by the introduction of additional gauge conditions, but leaves open the question of the separation of the original degrees of freedom into physical and unphysical ones. In contrast, the method of gaugeless reduction, to which we now turn, is based on the idea of explicit separation of the set of generalized phase-space coordinates into physical (gauge-invariant) and unphysical (gauge-noninvariant) ones. Let us begin our discussion by studying theories of a particular form, namely, theories which contain only Abelian constraints.

Let us take a degenerate theory with a complete set of Abelian constraints $\varphi_\alpha(q,p)=0$, $\alpha=1,\ldots,m<n$:

$$\{\varphi_\alpha(q,p),\varphi_\beta(q,p)\}=0,\quad \alpha,\beta=1,\ldots,m. \tag{4.24}$$

In this case an explicit parametrization of the reduced phase space can be specified as follows. According to a well known theorem (see, for example, Refs. 37 and 50), it is always possible to construct a canonical transformation to new variables

$$q_i\mapsto Q_i=Q_i(q,p),$$
$$p_i\mapsto P_i=P_i(q,p), \tag{4.25}$$

such that m of the new momenta $(\bar{P}_1,\ldots,\bar{P}_m)$ become equal to the Abelian constraints φ_α:

$$\bar{P}_\alpha=\varphi_\alpha(q,p), \tag{4.26}$$

while the remaining $(n-m)$ pairs of new canonical variables $(Q_1^*,P_1^*,\ldots,Q_{n-m}^*,P_{n-m}^*)$ will form a basis for the gauge-invariant observables O. To verify this, let us explain the structure of the canonical Hamiltonian in these variables. Owing to the completeness condition for the system of constraints, we have

$$\{\varphi_\alpha(q,p),H_c(q,p)\}=g_{\alpha\beta}(q,p)\varphi_\beta(q,p), \tag{4.27}$$

with some fixed functions $g_{\alpha\beta}$. It follows from (4.27) in the new coordinates P,Q,

$$\frac{\partial\bar{H}_C(P,Q)}{\partial\bar{Q}_\alpha}=\bar{g}_{\alpha\beta}(P,Q)\bar{P}_\beta,$$
$$\bar{H}_C(P,Q)=H_C(p(P,Q),q(P,Q)), \tag{4.28}$$

that the canonical Hamiltonian can be written as

$$\bar{H}_C(P,Q)=\bar{H}_0(Q^*,P^*,\bar{P})+\bar{\Psi}_\alpha(Q,P)\bar{P}_\alpha, \tag{4.29}$$

with some function $H_0(Q^*,P^*,\bar{P})$ independent of the coordinate $\bar{Q}$:[21)]

$$\{\bar{P}_\alpha,\bar{H}_0(P,Q)\}=0, \tag{4.30}$$

and determining the gauge-invariant part of the canonical Hamiltonian. As far as the function $\bar{\Psi}_\alpha(Q,P)$ is concerned, it is defined in terms of the structure functions $\bar{g}_{\alpha\beta}(P,Q)$ by the equation

$$\frac{\partial\bar{\Psi}_\gamma(P,Q)}{\partial\bar{Q}_\alpha}=\bar{g}_{\alpha\gamma}(P,Q). \tag{4.31}$$

This implies that in terms of the original variables p,q, the canonical Hamiltonian has the form

$$H_C(p,q)=H_0(q,p)+\Psi_\alpha(p,q)\varphi_\alpha(p,q), \tag{4.32}$$

where $H_0(p,q)$ is a gauge-invariant function:

$$\{H_0(p,q),\varphi_\alpha(p,q)\}=0, \tag{4.33}$$

and $\Psi_\alpha(p,q)$ is related to $\bar{\Psi}_\alpha(Q,P)$ as

$$\Psi_\alpha(p,q)=\bar{\Psi}_\alpha(P(p,q),Q(p,q)). \tag{4.34}$$

Now Eq. (4.31) can be rewritten in canonically invariant form using the Poisson bracket:

$$\{\varphi_\alpha(p,q),\Psi_\gamma(p,q)\}=g_{\alpha\gamma}(p,q). \tag{4.35}$$

However, it should be noted that there is a fundamental advantage in using the adapted coordinates (P,Q). In these coordinates the invariant part of the Hamiltonian is determined by the simple decomposition

$$H_0(P,Q)=\bar{H}_C(P,Q)-\frac{\partial\bar{H}_C}{\partial\bar{P}_\alpha}\bar{P}_\alpha, \tag{4.36}$$

while in the original coordinates p,q it can be represented by only a variational derivative:

$$H_0(p,q)=\left[H_C(p,q)-\frac{\delta H_C}{\delta\varphi_\alpha}\varphi_\alpha\right]. \tag{4.37}$$

Using the representation (4.29) for the canonical Hamiltonian, the generalized Hamilton–Dirac equations (taking the generalized Hamiltonian as the generator)

$$\dot{q}_i=\{q_i,H_E\},$$
$$\dot{p}_i=\{p_i,H_E\} \tag{4.38}$$

in the new coordinates (Q,P) have the form

$$\dot{\bar{Q}}_A\simeq\{\bar{Q}_A,\bar{H}_E\}=\left[\frac{\partial\bar{H}_0(Q^*,P^*,\bar{P})}{\partial\bar{P}_A}+\psi_A(Q,P)\right.$$
$$\left.\left.+u_\alpha(t)\delta_{\alpha A}\right]\right|_{\bar{P}_A=0}, \tag{4.39}$$

$$\dot{\bar{P}}_A\simeq0,\quad A=1,\ldots,m, \tag{4.40}$$

$$\dot{Q}^*_\alpha \simeq \{Q^*,\bar{H}_E\} = \left.\frac{\partial \bar{H}_0(Q^*,P^*,\bar{P})}{\partial P^*_\alpha}\right|_{\bar{P}_\alpha=0} = \frac{\partial \bar{H}^*(Q^*,P^*)}{\partial P^*_\alpha}, \tag{4.41}$$

$$\dot{P}^*_\alpha \simeq \{P^*,\bar{H}_E\} = -\left.\frac{\partial \bar{H}_0(Q^*,P^*,\bar{P})}{\partial Q^*_\alpha}\right|_{\bar{P}_\alpha=0} = -\frac{\partial \bar{H}^*(Q^*,P^*)}{\partial Q^*_\alpha}, \tag{4.42}$$

$$\bar{P}_A \simeq 0, \tag{4.43}$$

with the physical Hamiltonian

$$H^*(P^*,Q^*) \equiv H_C(P,Q)|_{\bar{P}_\alpha=0}. \tag{4.44}$$

Summary. In the special canonical coordinates $(\bar{Q},\bar{P})$ and (Q^*,P^*), the canonical equations of motion take the form

$$\dot{Q}^* = \{Q^*,H^*\}, \quad \dot{\bar{Q}} = u(t),$$
$$\dot{P}^* = \{P^*,H^*\}, \quad \dot{\bar{P}} = 0, \tag{4.45}$$

where H^* depends only on $(n-m)$ pairs of new gauge-invariant canonical variables (Q^*,P^*), and the form of the canonical equations (4.45) displays the explicit factorization of the variables:

$$2n\left\{\begin{pmatrix} q_1 \\ p_1 \\ \vdots \\ q_n \\ p_n \end{pmatrix}\right. \mapsto \begin{matrix} 2(n-m)\left\{\begin{pmatrix} Q^* \\ P^* \end{pmatrix}\right. & \text{physical variables} \\[2ex] 2m\left\{\begin{pmatrix} \bar{Q} \\ \bar{P} \end{pmatrix}\right. & \text{unphysical variables.} \end{matrix} \tag{4.46}$$

A fact of fundamental importance here is that the arbitrary functions $u(t)$ are present only in the equations of the system which contain the velocities of ignorable coordinates $\bar{Q}_\alpha$, while the equations of motion for the coordinates (Q^*,P^*) form a separate, closed system which does not contain the variables $(\bar{Q},\bar{P})$. Since there is no arbitrariness in these equations, it is they which are the true equations of motion, while the coordinates (Q^*,P^*) are true dynamical objects, since they satisfy the requirements imposed on physical quantities:[22] they are gauge-invariant, and their evolution is determined by the system of $2(n-m)$ Hamiltonian equations of motion without any restrictions on the initial data. The corresponding solutions of the equations fill the entire manifold of dimension $2(n-m)$, the points of which are in one-to-one correspondence with the points of the space of the physical states of the system. It is thereby possible, without introducing additional gauge conditions into a theory containing only Abelian constraints, to effect the desired reduction simply by choosing special variables, which are related to the original ones by a canonical transformation guaranteeing the required "equivalence" between the nondegenerate reduced system and the original singular one.

The direct generalization of this reduction method to the non-Abelian case is not possible, because momenta cannot be identified with constraints when they are non-Abelian. However, there is an important observation which actually allows the analysis of reduction for a system with arbitrary first-class constraints to be reduced to the case of Abelian constraints discussed above. The essential point is that there is a great deal of freedom in the description of systems with constraints;[23] in addition to the arbitrariness in the choice of canonical variables, there is an additional freedom associated with the description of the constraint surface Γ_c. In connection with this nonuniqueness of the description, the question arises of whether or not it is possible to represent a given constraint surface using other functions in involution. The answer to this question is yes.[24]

Summary. The statement about the *Abelianization* of constraints is that it is possible to replace the constraints φ_α by an equivalent set of constraints Φ_α:

$$\Phi_\alpha = D_{\alpha\beta}\varphi_\beta, \quad \det\|D\||_{\varphi=0} \neq 0, \tag{4.48}$$

which specify the same surface Γ_c but form an Abelian algebra.

The proof of this statement is briefly discussed in Appendix A. Here we only note that it is based on the explicit resolution of the constraints (see Refs. 4f, 4j, and 41), on the use of gauge-fixing conditions,[43] or on the direct construction of the Abelianization matrix using the solutions of a defined system of first-order linear partial differential equations (see Refs. 26 and 27 and Appendix A).

Taking this into account, let us finally formulate the general scheme of gaugeless reduction of degenerate theories with a general form of the first-class constraint algebra. Let us take a degenerate system with canonical Hamiltonian $H_c(q,p)$ and a full irreducible set of first-class constraints $\varphi_A(q,p)$:

$$\{\varphi_A(q,p),\varphi_B(q,p)\} = f_{ABC}(q,p)\varphi_C(q,p). \tag{4.49}$$

Using the procedure of local Abelianization, we replace the non-Abelian constraints by the equivalent set of Abelian constraints $\Phi_A(q,p) = D_{AB}(q,p)\varphi_A(q,p)$ and transform to special adapted variables:

$$q_i \mapsto Q_i = Q_i(q,p),$$
$$p_i \mapsto P_i = P_i(q,p), \tag{4.50}$$

such that m of the new momenta $(\bar{P}_1,\ldots,\bar{P}_m)$ become equal to the new Abelian constraints Φ_A:

$$\bar{P}_A = \Phi_A(q,p). \tag{4.51}$$

The generalized Hamiltonian in the new variables and in terms of the Abelianized constraints has the form

$$\bar{H}_E(Q,P) = \bar{H}_0(Q^*,P^*,\bar{P}) + \bar{\psi}_B(Q,P)\bar{P}_B + u_A(t)\bar{D}^{-1}_{AB}(Q,P)\bar{P}_B, \tag{4.52}$$

which demonstrates the existence of the ignorable coordinates $\bar{Q}_A$ and the canonical variables (Q^*,P^*), the evolution of which is uniquely determined from the Hamilton equations of motion with the reduced Hamiltonian

$$\bar{H}^*(Q^*,P^*)=\bar{H}_0(Q^*,P^*,\bar{P})|_{\bar{P}=0}. \tag{4.53}$$

Summary. In degenerate theories with non-Abelian first-class constraints, it is possible to perform the phase-space reduction using only generalized canonical transformations, without the use of auxiliary gauge conditions.

Carrying out this reduction scheme requires knowledge of the explicit form of both the Abelianization matrix and the canonical transformations (4.50), which in turn is related to the solution of complete systems of first-order linear partial differential equations. It is here that the gaugeless reduction scheme can encounter technical difficulties, the severity of which depends on the functional form of the system Lagrangian. In spite of this, the fact that a gaugeless reduction scheme exists is of fundamental importance, since such a scheme is based on the validity, from the viewpoint of generalized Hamiltonian dynamics, of applying generalized canonical transformations to "special" coordinates, which guarantees the correctness of this reduction. This can serve as the starting point for analyzing the above-mentioned problems of determining the class of admissible gauges in both the Dirac method and the Faddeev method.

• **Analysis of admissible gauges.** In this section we compare the two reduction procedures, the gauge-fixing method and the gaugeless method, and give some simple conditions on gauges which guarantee the correctness of the gauge-fixing method itself.[25]

Since all forms of representation of degenerate theories must necessarily be related to each other by generalized canonical transformations, we can take the following as the definition of an admissible gauge: *A gauge is admissible if and only if there exists a canonical transformation relating the Hamilton–Dirac equations (4.39) for the canonical variables Q^*,P^* to the equations of motion obtained using it in the gauge-fixing method (4.15).*

In "special" coordinates, gauge conditions of general form look like

$$\chi_A(\bar{Q},\bar{P},Q^*,P^*)=0. \tag{4.54}$$

It follows from this definition that the gauge conditions (4.54) belong to the class of admissible gauges if they do not impose any restrictions on the dynamical variables (Q^*,P^*) and lead to the physical Hamiltonian (4.53). The requirement of nondegeneracy of the Faddeev–Popov determinant (4.8) in the coordinates (Q,P),

$$\det\|\{\chi_A(Q,P),\Phi_B(Q,P)\}\|=\det\left\|\frac{\partial\bar{\chi}_A(Q,P)}{\partial\bar{Q}_B}\right\|_{\Gamma^*}\neq 0, \tag{4.55}$$

is the condition for resolvability of the constraints and gauges in terms of pairs of canonically conjugate variables in the form

$$\bar{P}_A=0,$$

$$\bar{Q}_A=f_A(Q^*,P^*). \tag{4.56}$$

We shall show that if the functions $f_A(Q^*,P^*)$ are defined for arbitrary values of (Q^*,P^*) from their domain of definition, then the gauges

$$\bar{\chi}'=\bar{Q}_A-f_A(Q^*,P^*)=0 \tag{4.57}$$

belong to the class of allowed gauges. For this it is sufficient to show that the reduced Hamiltonian corresponding to the gauges (4.57) coincides with the physical Hamiltonian (4.53). Defining the Lagrange multipliers from the stationarity conditions for the gauge conditions (4.57),

$$\dot{\bar{\chi}}_A=\{\bar{\chi}_A,\bar{H}_E\}=0, \tag{4.58}$$

we obtain the following representation for the generalized Hamiltonian $\bar{H}_E$:

$$\bar{H}_E=\bar{H}_0+\left[-\frac{\partial\bar{H}_0}{\partial\bar{P}_B}-\frac{\partial\bar{\psi}_C}{\partial\bar{P}_B}\bar{P}_C+\{\bar{f}_B,\bar{H}_c\}\right]\bar{P}_B. \tag{4.59}$$

Expanding the function $\bar{H}_0$ in a Taylor series with a remainder,

$$\begin{aligned}\bar{H}_0(Q^*,P^*,\bar{P})&=\bar{H}_0(Q^*,P^*,\bar{P})|_{\bar{P}=0}\\&+\left(\bar{P}_A\frac{\partial}{\partial\bar{P}_A}\right)\bar{H}_0(Q^*,P^*,\bar{P})\Big|_{\bar{P}=0}\\&+\frac{1}{2}\left(\bar{P}_A\frac{\partial}{\partial\bar{P}_A}\right)^2\bar{H}_0(Q^*,P^*,\Theta\bar{P}),\end{aligned} \tag{4.60}$$

where $0\leqslant\Theta_A\leqslant 1$, the generalized Hamiltonian is written as

$$\bar{H}_E=\bar{H}^*+\bar{F}_B(Q^*,P^*,\bar{P})\bar{P}_B, \tag{4.61}$$

where $\bar{H}^*$ coincides with the reduced Hamiltonian (4.53) in the gaugeless scheme, and

$$\bar{F}_B(Q^*,P^*,\bar{P})=\left[\frac{1}{2}\left(\bar{P}_A\frac{\partial}{\partial\bar{P}_A}\frac{\partial}{\partial\bar{P}_B}\right)\bar{H}_0(Q^*,P^*,\Theta\bar{P})-\frac{\partial\bar{\psi}_A}{\partial\bar{P}_B}\bar{P}_A+\{\bar{f}_B,\bar{H}_c\}\right].$$

This representation shows that the gauge is actually admissible, because the Hamilton equations of motion obtained using $\bar{H}_E$ coincide with the equations of motion for Q^*,P^* in the gaugeless reduction scheme.

Gauge conditions depending only on ignorable coordinates,

$$\bar{\chi}_A(\bar{Q})=0,$$

$$\det\left\|\left\|\frac{\partial\bar{\chi}_A(\bar{Q})}{\partial\bar{Q}_B}\right\|\right\|_\Sigma \neq 0, \tag{4.62}$$

or, written as a decomposition

$$\bar{\chi}_A = \bar{Q}_A - C_A = 0, \tag{4.63}$$

where C_A are arbitrary constants, occupy a special place among admissible gauges. Gauges of the type (4.62), which we shall term *canonical*, are the most natural from the viewpoint of gaugeless reduction. The Lagrange multipliers determined from the stationarity conditions for a canonical gauge,

$$\bar{u}_A = -\frac{\partial\bar{H}_c}{\partial\bar{P}_A} - \bar{\psi}_A, \tag{4.64}$$

lead to the generalized Hamiltonian

$$\bar{H}_E = \bar{H}_c - \frac{\partial\bar{H}_c}{\partial\bar{P}_A}\bar{P}_A = \bar{H}^* - \bar{F}_{AB}\bar{P}_A\bar{P}_B, \tag{4.65}$$

coinciding with the corresponding expression obtained by the gaugeless reduction scheme. Since this definition of canonical gauge is constructive only for special coordinates (Q,P), the problem arises of formulating a condition on canonical gauges which would allow invariant, independently of the choice of coordinates, verification that the gauge is canonical. Below, we shall discuss such a criterion for a gauge to belong to the class of canonical gauges. To obtain this criterion we note that, according to the definition of the reduced Hamiltonian (4.53), the requirement that a gauge be independent of the physical variables (Q^*,P^*) can be expressed in the following canonically invariant form:[26)]

$$\{\chi_\beta(p,q),H^*(p,q)\}|_{\Gamma^*}=0. \tag{4.66}$$

The criterion in this form is far from useful for practical applications, but it can be rewritten in a more convenient form using the definition of the Dirac brackets. In the special coordinates (Q,P) we have the following representation for the canonical Hamiltonian:

$$\begin{aligned}\bar{H}_C(P,Q) &= \bar{H}_0(Q^*,P^*,\bar{P}) + \bar{\Psi}_\alpha(Q,P)\bar{P}_\alpha \\ &= \bar{H}^*(P^*,Q^*) + F_\alpha(Q,P)\bar{P}_\alpha.\end{aligned} \tag{4.67}$$

Using the fact that neither the gauge itself nor the matrix $\bar{\Delta}_{\alpha\beta} = \{\bar{\chi}_\alpha,\bar{P}_\beta\}$ depends on the physical variables, we have the expressions

$$\begin{aligned}\{\bar{\chi}_\alpha(\bar{Q}),\bar{H}_C(P,Q)\} &= \bar{\Delta}_{\alpha\beta}(\bar{Q})F_\beta(P,Q) \\ &\quad + \{\bar{\chi}_\alpha(\bar{Q}),F_\beta(P,Q)\}\bar{P}_\beta,\end{aligned} \tag{4.68}$$

$$\begin{aligned}\{\bar{\Delta}_{\alpha\beta}(\bar{Q}),\bar{H}_C(P,Q)\} &= \{\bar{\Delta}_{\alpha\beta}(\bar{Q})F_\gamma(P,Q)\}\bar{P}_\gamma \\ &\quad + \{\bar{\Delta}_{\alpha\beta}(\bar{Q}),\bar{P}_\gamma\}F_\gamma(P,Q).\end{aligned}$$

from which, eliminating the functions $F_\gamma(P,Q)$,[27)] we obtain

$$\begin{aligned}\{\bar{\Delta}_{\alpha\beta}(\bar{Q}),\bar{H}_C(P,Q)\}|_{\Gamma^*} &= \{\bar{\Delta}_{\alpha\beta}(\bar{Q}),\bar{P}_\gamma\}\bar{\Delta}^{-1}_{\gamma\sigma}(\bar{Q}) \\ &\quad \times\{\bar{\chi}_\sigma(\bar{Q}),\bar{H}_C(P,Q)\}|_{\Gamma^*}.\end{aligned}$$

Using the definition of the Dirac brackets, this condition takes the more attractive form

$$\{\bar{\Delta}_{\alpha\beta}(\bar{Q}),\bar{H}_C(P,Q)\}_{D(\bar{P},\bar{\chi})}|_{\bar{P}=0,\bar{\chi}=0}=0. \tag{4.69}$$

However, a defect of this criterion is the fact that it is written in terms of Abelianized constraints $\bar{P}$. In order to have a criterion for the gauge to be canonical in terms of non-Abelian constraints, we use an important property of the Abelianization matrix:

$$\varphi_\alpha = \mathcal{D}_{\alpha\beta}\bar{P}_\beta. \tag{4.70}$$

First, we note that the Dirac bracket is invariant under generalized canonical transformations:[3]

$$\{\bar{F}(P,Q),\bar{G}(P,Q)\}_{D(\bar{P},\bar{\chi})} = \{F(p,q),G(p,q)\}_{D(\varphi,\chi)},$$

and so, instead of (4.69), we have

$$\begin{aligned}&\mathcal{D}_{\alpha\gamma}\{\Delta_{\gamma\beta}(p,q),H_C(p,q)\}_{D(\varphi,\chi)} + \Delta_{\gamma\beta}(p,q) \\ &\quad\times\{\mathcal{D}_{\alpha\gamma},H_C(p,q)\}_{D(\varphi,\chi)}|_{\varphi=0,\chi=0}=0.\end{aligned} \tag{4.71}$$

Now to obtain the desired result it is necessary to check that the Abelianization matrix depends only on $\bar{P}$ and $\bar{Q}$, and, as a result,

$$\{\mathcal{D}_{\alpha\gamma},H_C(p,q)\}_{D(\varphi,\chi)}|_{\varphi=0,\chi=0}=0. \tag{4.72}$$

This can be verified by studying the structure of the generator of gauge transformations. This generator can be written either as a linear combination of non-Abelian first-class constraints,[4] so that[44,45]

$$G = \varepsilon_\alpha(q,p,t)\varphi_\alpha(q,p),$$

or in terms of the Abelian constraints

$$G = \bar{\varepsilon}_\alpha(\bar{Q},\bar{P},t)\bar{P}_\alpha. \tag{4.73}$$

In (4.73) the functions—parameters of the gauge transformations—$\varepsilon_A(\bar{Q})$ depend only on the ignorable coordinates $(\bar{Q},\bar{P})$, owing to the factorized form of the equations of motion. According to (4.73), an arbitrary gauge-invariant function I depends only on Q^* and P^*:

$$\{I,G\}=0. \tag{4.74}$$

Therefore, using (4.74) with the generator of gauge transformations G, in terms of the non-Abelian constraints and the Abelianization matrix (4.70),

$$G = \bar{\varepsilon}_\alpha(\bar{Q},\bar{P},t)\mathcal{D}^{-1}_{\alpha\beta}\varphi_\beta,$$

for $I \equiv Q^*,P^*$ we obtain

$$\{Q^*_i,G\}=0 \Rightarrow \frac{\partial\mathcal{D}^{-1}_{\alpha\beta}}{\partial Q^\prime_i}=0,$$

$$\{P_i^*,G\}=0 \Rightarrow \frac{\partial \mathcal{D}^{-1}_{\alpha\beta}}{\partial P_i^*}=0, \tag{4.75}$$

taking into account the nondegeneracy of the matrix $\mathcal{D}$ and the functional independence of the constraints. This completes the proof that the Abelianization matrix is independent of the variables P^* and Q^*. Then from (4.71) we obtain the condition

$$\{\Delta_{\alpha\beta}(p,q),H_C(p,q)\}_D|_{\varphi=0,\chi=0}=0, \tag{4.76}$$

where now the matrix $\Delta_{\alpha\beta}=\{\chi_\alpha,\varphi_\beta\}$ is calculated in terms of non-Abelian constraints and for an arbitrary choice of canonical coordinates.

Summary. The requirement that the Dirac bracket of the matrix $\Delta_{\alpha\beta}=\{\chi_\alpha,\varphi_\beta\}$ with the canonical Hamiltonian vanish on the constraint and gauge surface,

$$\{\Delta_{\alpha\beta}(p,q),H_C(p,q)\}_D|_{\varphi=0,\chi=0}=0, \tag{4.77}$$

is an invariant criterion for the gauge χ_α to belong to the class of canonical gauges.

5. CONCLUDING REMARKS

In writing this article, it was not our goal to discuss all the aspects of, or even to give a complete bibliography on, the reduction procedure in degenerate dynamical systems. The range of problems associated with the subject of this review is extremely broad, and includes both fundamental mathematical problems and complicated problems of realizing the procedure of Hamiltonian reduction in field-theoretic models of immediate physical interest.

Let us mention a few of the topics which we have not discussed here and plan to return to in future publications:

• Development of the method of Hamiltonian reduction for systems possessing reparametrization invariance.

• Generalization of the method of Hamiltonian reduction to field-theoretic models.

• Comparison with the Batalin–Fradkin–Vilkovisky covariant method of describing degenerate systems with the introduction of additional ghost variables.

• Study of the global properties of the Hamiltonian reduction procedure.

Owing to the infinite number of dynamical variables, the operation of reduction of the number of degrees of freedom in gauge field theories is incomparably more complex than in mechanical systems, and careful analysis of the fundamental problems is necessary. Aside from the obvious technical difficulties, there is, foremost, the problem of global analysis of the reduction phenomenon. Obviously, without the correct first step, understanding of the local properties of the theory, it is not possible to obtain any results of a global nature. The results presented in this review are based on a local analysis in the sense that in all the manifolds considered we have restricted ourselves to a region which can be covered by a single map. However, guided by the traditional principle stated by Tatarinov[50] as the principle of *"the presumption of analyticity,"* i.e., **relations which are established locally give a result which is also true globally**, we can attempt to arrive at some conclusions about the properties of the system as a whole. And then, every time we find that this principle is violated, we can resort to the injunction of Newton, hidden in the anagram: *"Data aequatione quotcunque fluentes quantitae involvente fluxiones invenire et vice versa."*

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APPENDICES

A. Algorithm for the Abelianization of first-class constraints

An important ingredient in the gaugeless reduction of degenerate systems with local symmetry is the procedure of Abelianization of first-class constraints. The possibility of transforming to Abelian constraints follows from the well known fact[28)] that, if the first-class constraints are resolved for the canonical momenta, they are necessarily Abelian. It should be noted that, although the implicit-function theorem, whose applicability in the present case is guaranteed by (3.13), implies the possibility of such a resolution, the question of the global equivalence of the original constraint surface to the surface now described by Abelian constraints remains open. Obviously, if the Abelianized constraints $\Phi_\alpha(p,q)$ are related to the original non-Abelian constraints as

$$\Phi_\alpha(p,q)=\mathcal{D}_{\alpha\gamma}\varphi_\gamma(p,q), \tag{A1}$$

then the condition of global equivalence implies the existence of a nondegenerate matrix $\mathcal{D}(p,q)$ (with $\det\|\mathcal{D}_{\alpha\gamma}\|\,|_{\Gamma_c} \neq 0$ on the entire constraint surface) which is a global solution of a system of first-order nonlinear partial differential equations:

$$\{\mathcal{D}_{\alpha\gamma},\mathcal{D}_{\beta\lambda}\}\varphi_\gamma+\{\mathcal{D}_{\alpha\lambda},\varphi_\gamma\}\mathcal{D}_{\beta\gamma}+\{\varphi_\gamma,\mathcal{D}_{\beta\lambda}\}\mathcal{D}_{\alpha\gamma} + f_{\gamma\sigma\lambda}\mathcal{D}_{\alpha\gamma}\mathcal{D}_{\beta\sigma}=0. \tag{A2}$$

The analysis of the existence of such a solution is an extremely complex problem. However, it turns out to be possible to reduce the problem of finding the solutions of (A2) to the problem of finding the particular solution of a complete set of first-order linear partial differential equations.[29)] More precisely, in this appendix we shall describe a recursion procedure for Abelianizing constraints satisfying an algebra of the general form

$$\{\varphi_\alpha(p,q),\varphi_\beta(p,q)\}=f_{\alpha\beta\gamma}(p,q)\varphi_\gamma(p,q), \quad \alpha=1,2,\dots,m, \tag{A3}$$

by transformation of the Dirac equivalence (A1) to new Abelian constraints $\Phi_\alpha(p,q)$ with matrix having the structure

$$\mathcal{D} := \underbrace{\mathcal{D}^1(p,q)\cdots\mathcal{D}^m(p,q)}_{m}, \tag{A4}$$

where each matrix $\mathcal{D}^k$ is a product of k $m\times m$ matrices:

$$\mathcal{D}^k:=\mathcal{R}^{a_k+k}(p,q)\prod_{i=k-1}^{0}\mathcal{S}^{a_k+i}(p,q),\quad a_k:=\frac{k(k+1)}{2},\tag{A5}$$

$$\mathcal{R}^{a_k+k}=\begin{array}{c}\begin{array}{cc} \overbrace{\hphantom{\boxed{\mathrm{I}}\ }}^{k} & \overbrace{\hphantom{\boxed{\mathrm{B}^{a_k+k}}}}^{m-k}\end{array}\\ \left(\begin{array}{cc} \boxed{\mathrm{I}} & 0 \\ 0 & \boxed{\mathrm{B}^{a_k+k}} \end{array}\right)\end{array},\tag{A6}$$

$$\mathcal{S}^{a_k+i}$$

$$=\begin{array}{c} \overbrace{\hphantom{1\ 0\ \cdots\ 0\ 0}}^{k}\ \overbrace{\hphantom{0\ 0\ \cdots\ 0\ 0}}^{m-k} \\ \left(\begin{array}{ccccc|ccccc}
1 & 0 & \cdots & 0 & 0 & 0 & 0 & \cdots & 0 & 0\\
0 & 1 & \cdots & 1 & 0 & 0 & 0 & \cdots & 0 & 0\\
\vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots\\
0 & 0 & \cdots & 1 & 0 & 0 & 0 & \cdots & 0 & 0\\
0 & 0 & \cdots & 0 & 1 & 0 & 0 & \cdots & 0 & 0\\
\hline
0 & \cdots & C^{a_k+i}_{k} & \cdots & 0 & 1 & 0 & \cdots & 0 & 0\\
0 & \cdots & C^{a_k+i}_{k+1} & \cdots & 0 & 0 & 1 & \cdots & 0 & 0\\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots\\
0 & \vdots & C^{a_k+i}_{m-1} & \cdots & 0 & 0 & 0 & \cdots & 1 & 0\\
0 & \cdots & C^{a_k+i}_{m} & \cdots & 0 & 0 & 0 & \cdots & 0 & 1
\end{array}\right) \\ \underbrace{\hphantom{0\ \cdots}}_{k-i}\hphantom{C\ \cdots\ 0\ 0\ 0\ \cdots\ 0\ 0\ 0\ 0} \end{array}.\tag{A7}$$

The constraints obtained by acting with k matrices of this type on the original constraints Φ^0_β, i.e., the constraints of the (a_k+k)th step of the Abelianization procedure,

$$\Phi^{a_k+k}_{\alpha}:=(\mathcal{D}^k\cdot\mathcal{D}^{k-1}\cdots\mathcal{D}^1)_{\alpha\beta}\Phi^0_\beta,\tag{A8}$$

form a set in which k constraints have zero Poisson bracket with any other constraint because the functions C and B satisfy a complete system of first-order linear partial differential equations:

$$\{\Phi^{a_k+i-1}_{\bar\alpha_k},C^{a_k+i}_{\alpha_k}\}=0,\tag{A9}$$

$$\{\Phi^{a_k+i-1}_{k},C^{a_k+i}_{\alpha_k}\}=f^{a_k+i-1}_{k\alpha_k\gamma_k}C^{a_k+i}_{\gamma_k}-f^{a_k+i-1}_{k\alpha_k i+1},\tag{A10}$$

$$\{\Phi^{a_k+k-1}_{\bar\alpha_k},B^{a_k+k}_{\alpha_k\beta_k}\}=0,\tag{A11}$$

$$\{\Phi^{a_k+k-1}_{k},B^{a_k+k}_{\alpha_k\beta_k}\}=-f^{a_k+k-1}_{k\gamma_k\beta_k}B^{a_k+k}_{\alpha_k\gamma_k},\tag{A12}$$

where $\alpha_k=k+1,\dots,m$, $\bar\alpha_k=1,2,\dots,k-1$, and the $f^{a_k+i}_{\alpha\gamma\beta}$ are the structure functions of the (a_k+i)th step of the constraint algebra.

The validity of the representation (A4) and the completeness of the system of differential equations for the matrices S and $\mathcal{R}$ were proved in Ref. 26 by the method of induction. From the algebraic point of view, the Abelianization procedure is an iteration procedure in which "equivalent" algebras A^{a_i} formed by the constraints $\Phi^{a_i}_\alpha$ are constructed:

$$\mathcal{A}^0\underbrace{\stackrel{\mathcal{S}^1}{\rightarrow}\mathcal{A}^1\stackrel{\mathcal{R}^2}{\rightarrow}}_{\mathcal{D}^1}\mathcal{A}^2\underbrace{\stackrel{\mathcal{S}^3}{\rightarrow}\mathcal{A}^3\stackrel{\mathcal{S}^4}{\rightarrow}\mathcal{A}^4\stackrel{\mathcal{R}^5}{\rightarrow}}_{\mathcal{D}^2}\mathcal{A}^5\dots$$

$$\underbrace{\stackrel{\mathcal{S}^{a_k}}{\rightarrow}\mathcal{A}^{a_k}\dots\stackrel{\mathcal{R}^{a_k+k}}{\rightarrow}}_{\mathcal{D}^k}\mathcal{A}^{a_k+k}\dots\ .\tag{A13}$$

The Abelianization procedure consists of a_m successive steps forming an Abelian algebra of dimension m, equivalent to the original non-Abelian algebra, so that at the a_kth step the algebra $\mathcal{A}^{a_k}$ possesses a center of k elements $\mathcal{Z}_k[A]=(\Phi^{a_k}_1,\Phi^{a_k}_2,\dots,\Phi^{a_k}_k)$. In other words, the matrix $\mathcal{D}^k$ transforms the algebra $\mathcal{A}^k$ into the algebra $\mathcal{A}^{k+1}$, with a center of $k+1$ elements.

B. The Christ–Lee–Prokhorov Abelian model

Here we shall use the example of a very simple mechanical system to compare the procedure of gaugeless reduction with the gauge-fixing scheme. This example demonstrates the existence of gauges which, in spite of the fact that the condition (4.8) is satisfied, still lead to distortion of the true dynamics of the reduced system.

• **Formulation of the model and gaugeless reduction.** Let us consider the exactly solvable mechanical system corresponding to the Lagrangian[47,48]

$$\mathcal{L}=\tfrac{1}{2}(\dot x_1^2+\dot x_2^2+y^2(x_1^2+x_2^2))-y(\dot x_1x_2-x_1\dot x_2)-V(x_1^2+x_2^2),\tag{B1}$$

where x_1, x_2, and y are independent variables. The Hessian matrix has rank two, and so there is one primary constraint

$$p_y=0,\tag{B2}$$

which follows directly from the definition of the canonical momentum p_y. Applying the standard procedure for transforming to the Hamiltonian formalism, we obtain the full Hamiltonian

$$H_T=\tfrac{1}{2}(p_1^2+p_2^2)-y(x_1p_2-x_2p_1)+V(x_1^2+x_2^2)+u(t)p_y\tag{B3}$$

and the secondary constraint from the stationarity condition for the primary one:

$$\varphi:=x_1p_2-x_2p_1=0.\tag{B4}$$

It is easily verified that there are no other constraints in the theory, and the primary and secondary constraints form a complete set of first-class constraints:

$$\{\varphi,p_y\}=0.$$

This implies that there exists a group of local gauge transformations under which the Lagrangian of the system is (quasi)invariant. The generating function of the corresponding infinitesimal local gauge transformations is constructed from the constraints and has the form

$$G=-\dot{\varepsilon}(t)p_y+\varepsilon(t)(x_1p_2-x_2p_1). \tag{B5}$$

The transformations of the generalized coordinates generated by G,

$$x_1'=x_1+\{x_1,G\}=x_1-\varepsilon(t)x_2,$$

$$x_2'=x_2+\{x_2,G\}=x_2+\varepsilon(t)x_1,$$

$$y'=y+\{y,G\}=y-\dot{\varepsilon}(t), \tag{B6}$$

are rotations by the angle $\varepsilon(t)$ about the axis perpendicular to the (x_1,x_2) plane.

Let us now make a canonical transformation to the special variables (Y,P_Y), (R,P_R), and $(\bar{\Theta},\bar{P}_{\bar{\Theta}})$ such that the new momentum $\bar{P}_{\bar{\Theta}}$ is equal to the secondary constraint φ:

$$Y=y,\quad P_y=p_Y, \tag{B7}$$

$$R=\sqrt{x_1^2+x_2^2},\quad P_R=\frac{x_1p_1+x_2p_2}{\sqrt{x_1^2+x_2^2}}, \tag{B8}$$

$$\bar{\Theta}=\arctan\left(\frac{x_2}{x_1}\right),\quad \bar{P}_{\bar{\Theta}}=x_1p_2-x_2p_1. \tag{B9}$$

This transformation is canonical and has the inverse

$$y=Y,\quad p_Y=P_y, \tag{B10}$$

$$x_1=R\cos\bar{\Theta},\quad p_1=P_R\cos\bar{\Theta}-\frac{\bar{P}_{\bar{\Theta}}}{R}\sin\bar{\Theta}, \tag{B11}$$

$$x_2=R\sin\bar{\Theta},\quad p_2=P_R\sin\bar{\Theta}+\frac{\bar{P}_{\bar{\Theta}}}{R}\cos\bar{\Theta} \tag{B12}$$

everywhere except for the single point $R=0$ if we assume that $0<\bar{\Theta}<2\pi$. In terms of the new variables, the full Hamiltonian becomes

$$H_T=\frac{1}{2}\left(P_R^2+\frac{\bar{P}_{\bar{\Theta}}^2}{R^2}\right)-Y\bar{P}_{\bar{\Theta}}+V(R^2)+u_YP_Y. \tag{B13}$$

In accordance with the general ideas, the equations of motion

$$\dot{R}=P_R,$$

$$\dot{P}_R=-\frac{\partial V(R^2)}{\partial R},$$

$$\dot{\bar{P}}_{\bar{\Theta}}=0,\quad \dot{\bar{\Theta}}_\alpha=\bar{u}_\Theta(t),$$

$$\dot{P}_Y=0,\quad \dot{Y}=\bar{u}_Y(t) \tag{B14}$$

display the splitting of the phase-space coordinates into two parts, one, (R,P_R), whose dynamics does not contain any arbitrariness and is specified by the physical Hamiltonian

$$H_{\mathrm{Ph}}=\tfrac{1}{2}P_R^2+V(R^2), \tag{B15}$$

and the other, (Y,Θ), with completely arbitrary evolution. Therefore, after the constraints P_Y and $\bar{P}_{\bar{\Theta}}$ are included, reduction to the true dynamical variables (R,P_R) occurs automatically without the introduction of any additional gauge conditions. The gauge invariance can easily be verified using the generating function of infinitesimal local gauge transformations (B5) in the new coordinates:

$$G=-\dot{\varepsilon}(t)P_Y+\varepsilon(t)\bar{P}_{\bar{\Theta}}. \tag{B16}$$

• **The canonical gauge.** Let us now turn to the reduction scheme using gauge-fixing. Every correct reduction must lead to a nondegenerate theory which must be canonically equivalent to the theory obtained in the gaugeless manner. Therefore, let us first ascertain the existence of the canonical gauge for this model. It is easily checked that the gauge choice

$$\chi_1=:y=0,$$

$$\chi_2=:\arctan\left(\frac{x_2}{x_1}\right)=\text{constant}, \tag{B17}$$

$$\det\|\{\chi_\alpha,\varphi_\beta\}\|=1 \tag{B18}$$

allows the Lagrange multipliers u_1 and u_2 in the generalized Hamiltonian

$$H_E=\tfrac{1}{2}(p_1^2+p_2^2)-y(x_1p_2-x_2p_1)+V(x_1^2+x_2^2)$$
$$+u(t)_1p_y+u_2(x_1p_2-x_2p_1),$$

$$u_1=0,$$

$$u_2=y-\frac{x_1p_2-x_2p_1}{\sqrt{x_1^2+x_2^2}}$$

to be fixed and leads to a reduced system which is equivalent to the one obtained earlier in (B14).

• **An inadmissible gauge.** Let us give an example of a gauge which satisfies the condition of nondegeneracy of the Faddeev–Popov matrix, but which is inadmissible. We consider the functions

$$\chi_1\equiv y=0,$$

$$\chi_2\equiv\frac{x_1^2-x_2^2}{x_1^2+x_2^2}-\left(\frac{1}{2}+\frac{A}{\sqrt{x_1^2+x_2^2}}\right)=0,\quad A>0. \tag{B19}$$

It is easily verified that, in spite of the fact that it can also be used to fix the Lagrange multipliers, this gauge is inadmissible because it leads to a restriction on the value of a physical quantity. In the special coordinates (B7) this gauge has the form

$$\chi_1\equiv Y=0,$$

$$\chi_2\equiv\cos2\bar{\Theta}-\left(\frac{1}{2}+\frac{A}{R}\right)=0. \tag{B20}$$

It follows from (B20) that $0<2\bar{\Theta}\leqslant\pi/3$, $5\pi/3\leqslant2\bar{\Theta}<2\pi$, and so the required Faddeev–Popov condition

$$\det\|\{\chi_\alpha,\varphi_\beta\}\|=-2\sin2\bar{\Theta}|_{\Gamma^*}\neq0 \tag{B21}$$

is satisfied on Γ^*, but a restriction on the physical variable R immediately follows:

$$R>2A. \tag{B22}$$

We also note that the proposed criterion (4.76) for gauges to be admissible rejects the gauge (B19).

Let us give another example of a gauge for which the Faddeev–Popov determinant does not vanish anywhere on the gauge and constraint surface, but which is also inadmissible:

$$\chi_1=y=0,$$

$$\chi_2=x_2-\sqrt{x_1\left(\tfrac{2}{3}x_1^2-x_1+2\right)+a}=0. \tag{B23}$$

C. $SU(2)$-invariant Yang–Mills mechanics in $(0+1)$-dimensional space-time

As an example of the procedure of Abelianization of constraints and construction of the reduced system, let us consider a model[48,49] representing the $SU(2)$ gauge-invariant theory of Yang–Mills fields in $(0+1)$-dimensional space-time.

The Lagrangian of the model is given by

$$L=\tfrac{1}{2}(D_tx)_i(D_tx)_i-\tfrac{1}{2}V(x^2), \tag{C1}$$

where x_i and y_i are the components of 3-vectors and D_t is the covariant derivative:

$$(D_tx)_i:=\dot{x}_i+g\epsilon_{ijk}y_jx_k. \tag{C2}$$

The system specified in this manner corresponds to the dimensional reduction of non-Abelian $SU(2)$ theory in $(0+1)$-dimensional space, and the variables x_i are relics of the "matter" fields. Using the Legendre transform

$$p_y^i=\frac{\partial L}{\partial \dot{y}_i}, \tag{C3}$$

$$p^i=\frac{\partial L}{\partial \dot{x}_i}=\dot{x}_i+g\epsilon^{ijk}y_jx_k, \tag{C4}$$

we obtain the canonical Hamiltonian

$$H_C=\tfrac{1}{2}p_ip_i-\epsilon_{ijk}x_jp_ky_i+V(x^2) \tag{C5}$$

and three primary constraints $p_y^i=0$ leading to the secondary constraints

$$\Phi_i=\epsilon_{ijk}x_jp_k=0 \tag{C6}$$

forming an $SO(3)$ algebra

$$\{\Phi_i,\Phi_j\}=\epsilon_{ijk}\Phi_j. \tag{C7}$$

We note that the secondary constraints are dependent in the Dirac sense: $x_i\Phi_i=0$. Turning to the Abelianization procedure, as the independent constraints we choose

$$\Phi_1^{(0)}:=x_2p_3-x_3p_2,\quad \Phi_2^{(0)}:=x_3p_1-x_1p_3. \tag{C8}$$

Now instead of the algebra (C7) we have[30]

$$\{\Phi_1^{(0)},\Phi_2^{(0)}\}=-\frac{x_1}{x_3}\Phi_1^{(0)}-\frac{x_2}{x_3}\Phi_2^{(0)}. \tag{C9}$$

According to the general iteration scheme for constructing the Abelianization matrix (see Ref. 26 or Appendix A above), two steps of equivalence transformations of the constraints are required in this case. First we eliminate $\Phi_1^{(0)}$ from the right-hand side of (C9). This is done by the transformation

$$\Phi_1^{(1)}:=\Phi_1^{(0)},$$

$$\Phi_2^{(1)}:=\Phi_2^{(0)}+C\Phi_1^{(0)}, \tag{C10}$$

with the function C which is a solution of the following partial differential equation:

$$\{\Phi_1^{(0)},C\}=-\frac{x_2}{x_3}C+\frac{x_1}{x_3}. \tag{C11}$$

Using the particular solution of this equation

$$C(x)=\frac{x_1x_2}{x_2^2+x_3^2}, \tag{C12}$$

for the new constraints we obtain an algebra of the form

$$\{\Phi_1^{(1)},\Phi_2^{(1)}\}=-\frac{x_2}{x_3}\Phi_2^{(1)}. \tag{C13}$$

In the second step of the Abelianization procedure we make the transformation

$$\Phi_1^{(2)}:=\Phi_1^{(1)},$$

$$\Phi_2^{(2)}:=B\Phi_2^{(1)}, \tag{C14}$$

with the function B satisfying the equation

$$\{\Phi_1^{(2)},B\}=\frac{x_2}{x_3}B. \tag{C15}$$

Using one of the solutions of this equation, $B(x)=1/x_3$, we arrive at the desired Abelian constraints:

$$\Phi_1^{(2)}=x_2p_3-x_3p_2,$$

$$\Phi_2^{(2)}=\frac{1}{x_3}\left[(x_3p_1-x_1p_3)+\frac{x_1x_2}{x_2^2+x_3^2}(x_2p_3-x_3p_2)\right]. \tag{C16}$$

Now let us write out the canonical transformation to special variables such that the two new momenta coincide with the Abelianized constraints:[31]

$$p_\theta:=\frac{(\mathbf{x}\cdot\mathbf{p})x_1-\mathbf{x}^2p_1}{\sqrt{x_2^2+x_3^2}},\quad p_\phi:=x_2p_3-x_3p_2. \tag{C17}$$

It is easily verified that the point transformation from Cartesian to spherical coordinates

$$x_1=r\cos\theta,\quad r=\sqrt{x_1^2+x_2^2+x_3^2},$$

$$x_2=r\sin\phi\sin\theta,\quad \theta=\arccos\frac{x_1}{\sqrt{x_1^2+x_2^2+x_3^2}},$$

$$x_3=r\cos\phi\sin\theta,\quad \phi=\arctan\left(\frac{x_2}{x_3}\right), \tag{C18}$$

is a transformation of this type. In fact, using the corresponding generating function

$$F[\mathbf{x};p_r,p_\theta,p_\phi]=p_r\sqrt{x_1^2+x_2^2+x_3^2}+p_\theta\arccos\frac{x_1}{\sqrt{x_1^2+x_2^2+x_3^2}}+p_\phi\arctan\left(\frac{x_2}{x_3}\right), \qquad \text{(C19)}$$

we obtain

$$p_1=\frac{\partial F}{\partial x_1}=p_r\cos\theta-p_\theta\frac{\sin\theta}{r}, \qquad \text{(C20)}$$

$$p_2=\frac{\partial F}{\partial x_2}=p_r\sin\theta\sin\phi+p_\theta\frac{\sin\phi\cos\theta}{r}+p_\phi\frac{\cos\phi}{r\sin\theta}, \qquad \text{(C21)}$$

$$p_3=\frac{\partial F}{\partial x_3}=p_r\sin\theta\cos\phi+p_\theta\frac{\cos\phi\cos\theta}{r}-p_\phi\frac{\sin\phi}{r\sin\theta}, \qquad \text{(C22)}$$

and verify that in the new variables the two independent constraints are in fact $p_\theta=0$ and $p_\phi=0$, in accordance with (C17). It is worth noting that by starting with the set of reduced constraints (C6) and making the transformation (C18), we can obtain the representation

$$\Phi_1=-p_\phi, \qquad \text{(C23)}$$

$$\Phi_2=-p_\theta\cos\phi+p_\phi\sin\phi\cot\theta, \qquad \text{(C24)}$$

$$\Phi_3=p_\theta\sin\phi+p_\phi\cos\phi\cot\theta, \qquad \text{(C25)}$$

which is adapted to the Abelianization procedure. The Abelianization matrix for the reduced set of constraints has the form

$$D:=\frac{1}{d}\begin{pmatrix} -d_2\sin\phi-d_3\cos\phi, & d_1\sin\phi, & d_1\cos\phi \\ (d_2\cos\phi-d_3\sin\phi)\cot\theta, & -d_3-d_1\cos\phi\cot\theta, & d_2+d_1\sin\phi\cot\theta \\ \cot\theta, & \sin\phi, & \cos\phi \end{pmatrix}, \qquad \text{(C26)}$$

with arbitrary $\mathbf{d}$ and $d:=d_1\cot\theta+d_2\sin\phi+d_3\cos\phi$. This example illustrates two characteristic features of the Abelianization procedure:

(i) It is not necessary to work with the reduced set of constraints, because the Abelianization procedure automatically leads to an irreducible set of constraints.

(ii) In certain special coordinates, the problem of solving the differential equations becomes an algebraic problem. In the new canonical variables the canonical Hamiltonian (C5) takes the form

$$H_C=\frac{1}{2}p_r^2+\frac{1}{2r^2}\left(p_\theta^2+\frac{p_\phi^2}{\sin^2\theta}\right)-p_\phi y_\phi-p_\theta y_\theta+V(r), \qquad \text{(C27)}$$

with the physical momentum $p_r=(\mathbf{x}\cdot\mathbf{p})/\sqrt{x_1^2+x_2^2+x_3^2}$, and

$$y_\phi:=y_1+y_2\sin\phi+y_3\cos\phi\cot\theta,$$

$$y_\theta:=y_2\cos\phi-y_3\sin\phi.$$

As a result, all the unphysical variables are separated from the physical ones r and p_r, whose time evolution is uniquely determined by the physical Hamiltonian. The latter is obtained from the canonical vanishing of p_ϕ and p_θ in (C27):

$$H_{\text{phys}}=\frac{1}{2}p_r^2+V(r). \qquad \text{(C28)}$$

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1)The *Mécanique Analytique* of J. L. Lagrange,[1] in which practically all the fundamental principles of analytic dynamics were already presented, appeared in 1788.

2)Of course, this list does not pretend to be complete, but only reflects the tastes and biases of the authors.

3)The basics of the theory of fiber bundles are discussed in the classical text of Lichnerowicz[6] and the textbook by Dubrovin *et al.*[7] The geometrical aspects of the formulation of field theories with local symmetry can be found in the monograph by Konoplëva and Popov.[4]

4)Interestingly, the formulation of physical theories on the basis of the principle of local symmetry turned out to be, in some sense, a realization of the hypothesis of Hertz about the force-free nature of interactions. The striving to eliminate the concept of force from mechanics and replace it by the effects of hidden, unobservable connections was the main stimulus for the new formulation of mechanics presented in his remarkable book, ***Die Prinzipien der Mechanik in neuem Zusammenhang dargestellt.***[9]

5)The history of the solution of this problem in the studies of Larmour, Lorentz, Schwarzschild, and Poincaré is traced in Ref. 10.

6)The solution of the reduction problem found by Dirac is expressed in a remarkable way as an effective decrease in the number of degrees of freedom due to the replacement of the Poisson brackets by Dirac brackets.[2]

7)The roots of this operation can be found in the well known Lie–Cartan theorem from the theory of functional groups.[24,25]

8)The limits of applicability of variational principles represent a separate, extremely interesting problem. For example, we note that the standard formulation of the Hamilton–Ostrogradskiĭ integral principle is true only for holonomic systems, while a different principle, the well known Gauss–Hertz principle,[23] is applicable also to nonholonomic systems.[37]

9)It is interesting that, as a rule, the Lagrangians of theories with higher derivatives of the coordinates with respect to time describe objects possessing some internal structure.

10)In physical problems, questions of the existence of an extremum and the sufficient conditions are, as a rule, treated by heuristic arguments. The analysis of such problems is discussed in Ref. 38.

11)The vanishing of the Hessian does not depend on the choice of coordinates, but is an invariant characteristic of the system from the viewpoint of the nonsingular transformations of generalized coordinates allowed in the Lagrangian approach.

12)In what follows the rank of the Hessian matrix will be assumed to be constant in the entire range of variation of the variables $(q,\dot{q})$.

13)Beginning with Dirac's studies, the same term "constraint" is used for both the function itself, and the condition for it to vanish, assuming that this does not cause confusion.

14)The historical details of the development of Hamiltonian theory in the last century can be found in the classical text by Whittaker[37] and the monograph by Polak.[10]

15)To commemorate the enormous contribution of Lie, in the important example of a Poisson structure associated with Lie algebras, the term Lie–Poisson bracket is used to refer to the bracket in the space dual to the algebra.

16)In the classical terminology, the Lagrange function $\mathcal{L}$ and the Hamilton function H are sometimes called characteristic functions, thereby emphasizing the fact that they contain all the information characterizing the system.

17)The last equation implies that the primary constraints necessarily depend on the momentum variables, while dependence on the coordinates may be absent.

18)In reality, the canonical Hamiltonian does not depend on the unresolved velocities $\dot{q}_{R+\alpha}$, owing to a remarkable property of the Legendre transform itself: $H_c(q_i,p_a,\dot{q}_{R+\alpha})=H_c(q_i,p_a)$.

19)Following Dirac, a weak equality is denoted by the symbol $\simeq$, and the usual symbol $=$ is used for a strong one.

20)It follows from this definition that the evolution of a physical variable according to the dynamics specified by the generalized Hamiltonian will be unique and independent of any Lagrange multipliers.

21)In the theory of nondegenerate systems, the coordinates on which the Hamiltonian does not depend are termed *cyclic* or *ignorable*. In the theory of degenerate dynamical systems the same term is used to refer to purely gauge degrees of freedom, despite the fundamental difference; the dynamics of "classical cyclic" coordinates is uniquely determined, whereas the evolution of cyclic coordinates in a degenerate theory contains a functional arbitrariness.

22)We note that the choice of special adapted coordinates P^*, Q^*, and $\bar{Q}$ is not unique. For example, it would have been possible to use the canonical coordinates

$$\overline{P'}_\alpha=\bar{P}_\alpha,$$

$$\overline{Q'}_\alpha=\bar{Q}_\alpha+f_\alpha(Q^*),$$

$$P'^*_i=P^*_i+\bar{P}_\alpha\frac{\partial f_\alpha(Q^*)}{\partial Q^*_i},$$

$$Q'^*_i=Q^*_i. \qquad (4.47)$$

However, in any case, the passage from one set of coordinates to another will be a canonical transformation in the sector of gauge-invariant variables (Q^*,P^*).

23)Bergmann called the full group of transformations which preserve both the form of the Hamilton equations of motion and the constraint surface the *generalized group of canonical transformations*.[3]

24)Since canonical transformations leave the value of the Poisson bracket unchanged, the Abelianization transformation cannot be canonical, but certainly must belong to the class of generalized canonical transformations.

25)The review by Prokhorov[46] is devoted to the discussion of several important features of the gauge-fixing procedure in degenerate theories.

26)Of course, it is possible to have the situation where the Hamiltonian does not depend on any physical variable Q^*. This occurs for cyclic coordinates associated with a global symmetry of the system. It therefore does not pose any danger for the derived criterion.

27)We assume that $\{\bar{\Delta}_{\alpha\beta}(\bar{Q}),\bar{P}_\gamma\}\neq 0$. If this condition is not satisfied, the gauge can depend on the physical coordinates:

$$\chi_\alpha=\bar{Q}_\alpha+f_{g\alpha}(Q^*).$$

However, as pointed out in the analysis of the choice of special coordinates, this dependence can be eliminated by means of a canonical transformation in the physical sector.

28)We note that already Levi-Civita, in finding particular integrals of differential equations, used this procedure to obtain invariant equations in involution.[23]

29)A similar proof has been used in the special case of the algebra

$$\{\varphi_\alpha,\varphi_\beta\}=F_{\alpha\beta}(\varphi_1,\ldots,\varphi_m), \quad \alpha,\beta=1,\ldots,m.$$

Such a set of constraints is called a functional group, following the terminology of Lie.[41,42]

30)In what follows we shall not write out the primary constraints, since they now belong to the center of the algebra, and the procedure of Abelianization of an element of the center is unchanged.

31)Here we introduce compact notation for the 3-vectors **x**, **p** and multiply the constraint $\Phi_2^{(2)}$ by the factor $\sqrt{x_2^2+x_3^2}$, so that the constraints have the same dimension. This multiplication preserves the Abelian nature of the constraints, since $\{\Phi_1^{(2)},\sqrt{x_2^2+x_3^2}\}=0$.

[1] J. L. Lagrange, *Mécanique Analytique*, 4th ed. (Gauthier-Villars, Paris, 1888) [Russ. transl., Gostekhizdat, Moscow, 1950].

[2] P. A. M. Dirac, Can. J. Math. **2**, 129 (1950); **3**, 1 (1951); Proc. R. Soc. London, Ser. A **246**, 326 (1958).

[3] P. G. Bergmann, Phys. Rev. **75**, 680 (1949); J. L. Anderson and P. G. Bergman, Phys. Rev. **83**, 1018 (1951); P. G. Bergmann and J. Goldberg, Phys. Rev. **98**, 531 (1955).

[4] (a) P. A. M. Dirac, *Lectures on Quantum Mechanics* (Yeshiva University, New York, 1964) [Russ. transl., Mir, Moscow, 1968]; (b) N. P. Konopleva and V. N. Popov, *Gauge Fields* (Harwood, Chur, 1981) [Russ. original, Atomizdat, Moscow, 1972]; (c) E. C. G. Sudarshan and N. Mukunda, *Classical Dynamics: A Modern Perspective* (Wiley, New York, 1974); (d) A. J. Hanson, T. Regge, and C. Teitelboim, *Constrained Hamiltonian Systems* (Accademia Nazionale dei Lincei, Rome, 1976); (e) L. D. Faddeev and A. A. Slavnov, *Gauge Fields: Introduction to Quantum Theory* (Addison-Wesley, Redwood City, Calif., 1990) [Russ. original, Nauka, Moscow, 1978]; (f) K. Sundermeyer, *Constrained Dynamics*, Lecture Notes in Physics, Vol. 169 (Springer-Verlag, Berlin, 1982); (g) M. Henneaux, Phys. Rep. **126**, 1 (1985); (h) V. V. Nesterenko and A. M. Chervyakov, *Singular Lagrangians: Lectures for Beginning Students* [in Russian] (JINR, Dubna, 1986); (i) F. I. Razumov and L. D. Solov'ev, Preprints 86-212, 86-213, 86-214, IHEP, Serpukhov (1986) [in Russian]; (j) D. M. Gitman and I. V. Tyutin, *Quantization of Fields with Constraints* (Springer-Verlag, Bonn, 1990) [Russ. original, Nauka, Moscow, 1986]; (k) L. Lusanna, Riv. Nuovo Cimento **14**, No. 3, 1 (1991); (l) M. Henneaux and C. Teitelboim, *Quantization of Gauge Systems* (Princeton University Press, Princeton, N.J., 1992); (m) L. V. Prokhorov and S. V. Shabanov, *Hamiltonian Mechanics of Gauge Systems* [in Russian] (JINR, Dubna, 1993), B1-2-93-312.

[5] H. Weyl, Z. Phys. **56**, 330 (1929).

[6] A. Lichnerowicz, *Global Theory of Connections and Holonomy Groups*, transl. from the French (Noordhoff, Leyden, 1976) [Russ. transl., IL, Moscow, 1960].

[7] B. A. Dubrovin, A. T. Fomenko, and S. P. Novikov, *Modern Geometry—Methods and Applications*, Parts I–III (Springer-Verlag, New York, 1984, 1985, 1990) [Russ. original, Nauka, Moscow, 1986].

[8] C. N. Yang and R. L. Mills, Phys. Rev. **96**, 191 (1954).

[9] H. Hertz, *Die Prinzipien der Mechanik in neuem Zusammenhang dargestellt*, 3 Vol. (Deutsch, Thun, 1996) [Russ. transl., GTTI, Moscow, 1959].

[10] L. S. Polak, *Variational Principles of Mechanics* [in Russian] (Fizmatgiz, Moscow, 1960).

[11] P. A. M. Dirac, Phys. Rev. **114**, 924 (1959).

[12] L. D. Faddeev and V. N. Popov, Phys. Lett. B **25**, 30 (1967).

[13] V. N. Gribov, Nucl. Phys. B **139**, 1 (1978).

[14] I. M. Singer, Commun. Math. Phys. **60**, 7 (1978).

[15] M. A. Solov'ev, Pis'ma Zh. Éksp. Teor. Fiz. **38**, 415 (1983) [JETP Lett. **38**, 504 (1983)]; Teor. Mat. Fiz. **78**, 163 (1989) [Theor. Math. Phys. (USSR)].

[16] V. I. Arnold, *Mathematical Methods of Classical Mechanics*, 2nd ed. (Springer-Verlag, New York, 1989) [Russ. original, Nauka, Moscow, 1979].

[17] P. J. Olver, *Applications of Lie Groups to Differential Equations* (Springer-Verlag, New York, 1986) [Russ. transl., Mir, Moscow, 1989].

[18] L. V. Ovsiannikov, *Group Analysis of Differential Equations* (Academic Press, New York, 1982) [Russ. original, Nauka, Moscow, 1978].

[19] N. H. Ibragimov, *Transformation Groups Applied to Mathematical Physics* (Reidel, Dordrecht, 1985) [Russ. original, Nauka, Moscow, 1983].

[20] A. T. Fomenko, *Symplectic Geometry* (Gordon and Breach, New York, 1988) [Russ. original, Nauka, Moscow, 1988].

[21] J. F. Pommaret, *Systems of Partial Differential Equations and Lie Pseudogroups* (Gordon and Breach, New York, 1978) [Russ. transl., Mir, Moscow, 1983].

[22] S. Shanmugadhasan, J. Math. Phys. **14**, 677 (1973).

[23] T. Levi-Civita and U. Amaldi, *Lezioni di Meccanica razionale* (Nicola Zanichelli, Bologna, 1927) [Russ. transl., IL, Moscow, 1951].

[24] E. Cartan, *Leçons sur les Invariants Intégraux* (Hermann, Paris, 1922) [Russ. transl., Gostekhizdat, Leningrad, 1940].
[25] A. R. Forsyth, *The Theory of Differential Equations* (Cambridge University Press, Cambridge, 1953).
[26] S. A. Gogilidze, A. M. Khvedelidze, and V. N. Pervushin, J. Math. Phys. **37**, 1760 (1996).
[27] S. A. Gogilidze, A. M. Khvedelidze, and V. N. Pervushin, Phys. Rev. D **53**, 2160 (1996).
[28] R. Abraham and J. E. Marsden, *Foundations of Mechanics* (Addison–Wesley, New York, 1978).
[29] W. Thirring, *A Course in Mathematical Physics*, Vols. I, II (Springer-Verlag, New York, 1978).
[30] C. Godbillon, *Géometrie Différentielle et Mécanique Analytique* (Hermann, Paris, 1969) [Russ. transl., Mir, Moscow, 1973].
[31] J. S. Marsden and T. S. Ratiu, *Introduction to Mechanics and Symmetry* (Springer-Verlag, New York, 1994).
[32] L. D. Faddeev, Teor. Mat. Fiz. **1**, 3 (1969) [Theor. Math. Phys. (USSR)].
[33] A. Lichnerowicz, C. R. Acad. Sci. Ser. A **280**, 523 (1975).
[34] M. Gotay, J. Nester, and G. Hinds, J. Math. Phys. **19**, 2388 (1978).
[35] G. Marmo, N. Mukunda, and J. Samuel, Riv. Nuovo Cimento **6**, 2 (1983).
[36] V. P. Pavlov, Teor. Mat. Fiz. **104**, 304 (1995); **105**, 429 (1995) [Theor. Math. Phys. (USSR)].
[37] E. T. Whittaker, *A Treatise on the Analytic Dynamics of Particles and Rigid Bodies*, 4th ed. (Cambridge University Press, Cambridge, 1937) [Russ. transl., ONTI, Moscow, 1937].
[38] H. Rund, *The Hamilton–Jacobi Theory in the Calculus of Variations* (Van Nostrand, 1966).
[39] E. Noether, *Invariante Variationsprobleme*, Nachr. König. Gesell. Wissen. Göttingen, Math.-Phys. Kl. 235 [Engl. transl. in Transport Theory and Stat. Phys. **1**, 186 (1971); Russ. transl., Fizmatgiz, Moscow, 1959].
[40] B. M. Barbashov and V. V. Nesterenko, *Continuous Symmetries in Field Theory: Lectures for Beginning Students* [in Russian] (JINR, Dubna, 1978).
[41] L. P. Eisenhart, *Continuous Groups of Transformations* (Princeton University Press, Princeton, N.J., 1933) [Russ. transl., IL, Moscow, 1947].
[42] J. A. Schouten and W. v. d. Kulk, *Pfaff's Problem and Its Generalizations* (Clarendon Press, Oxford, 1949).
[43] I. A. Batalin and G. A. Vilkovisky, Nucl. Phys. B **234**, 106 (1984).
[44] S. A. Gogilidze, V. V. Sanadze, Yu. S. Surovtsev, and F. G. Tkebuchava, Int. J. Mod. Phys. A **4**, 4165 (1989).
[45] S. A. Gogilidze, V. V. Sanadze, Yu. S. Surovtsev, and F. G. Tkebuchava, Teor. Mat. Fiz. **102**, 56, 66 (1995) [Theor. Math. Phys. (USSR)].
[46] L. V. Prokhorov, Fiz. Élem. Chastits At. Yadra **27**, 1399 (1996) [Phys. Part. Nucl. **27**, 569 (1996)].
[47] N. H. Christ and T. D. Lee, Phys. Rev. D **22**, 939 (1980).
[48] L. V. Prokhorov, Yad. Fiz. **35**, 229 (1982) [Sov. J. Nucl. Phys. **35**, 129 (1982)].
[49] L. V. Prokhorov and S. V. Shabanov, Usp. Fiz. Nauk **161**, 13 (1991) [Sov. Phys. Usp. **34**, 108 (1991)].
[50] Ya. V. Tatarinov, *Lectures on Classical Dynamics* [in Russian] (Moscow State University, Moscow, 1984).

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