Dynamical effects in (2+1)-dimensional theories with four-fermion interaction

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The critical behavior of some of the simplest (2+1)-dimensional field-theoretic models with four-fermion interaction at nonzero temperature, chemical potential, and external gauge fields is studied. It is shown that an external magnetic field catalyzes spontaneous symmetry breaking. The chromomagnetic gluon condensate in QCD_3 can also spontaneously break chiral symmetry. The thermodynamics of these effects is studied in detail. The conditions under which superconducting phase transitions and also dynamical generation of a Chern–Simons term are possible in three-dimensional models of the Gross–Neveu type are derived. © *1998 American Institute of Physics.* [S1063-7796(98)00505-1]

INTRODUCTION

It is well known that many phenomena in nature arise from phase transformations. Therefore, study of the vacuum has long been decisive in choosing the most important areas of development of modern physics, including elementary-particle physics. The existence of elementary-particle physics without the phenomenon of spontaneous symmetry breaking leading to rearrangement of the ground state is now unthinkable.

The progress in modern elementary-particle physics is to a large degree due to the recognition of symmetry principles and, perhaps even more, to symmetry-breaking mechanisms in models of elementary particles. One of the most important steps in the construction of gauge field theory was the understanding of the role of the Higgs mechanism, spontaneous symmetry breakdown accompanied by the appearance of mass for the gauge bosons. This effect forms the basis of the unified theory of electroweak interactions, grand unification models, and so on. However, this symmetry-breaking mechanism, which requires the introduction of fundamental scalar fields into the theory, is clearly insufficient at a deeper level. There is another, dynamical mechanism of spontaneous symmetry breaking, first studied in Ref. 1. The fundamental fields in this case are spinor fields whose bound states are bosons. As a rule, dynamical symmetry breaking is realized in models with four-fermion interaction, and one special feature of such models is their nonrenormalizability in (3+1)-dimensional spacetime. (In spite of their nonrenormalizability, attempts have been made to use such models for meaningful calculations of dynamical quantities outside the framework of ordinary perturbation theory.[2]) They are therefore convenient for constructing effective models of hadrons.[3] Moreover, dynamical breaking of chiral invariance using four-fermion Lagrangians forms the basis of the description of spin effects in strong-interaction physics[4] and elsewhere.

We would like to make particular mention of a series of studies (see Ref. 5) on gauge field theories in the presence of external supercritical Coulomb-like forces. This physical situation can arise in collisions of heavy relativistic ions. It turns out that here also there is dynamical symmetry breaking, and fermions acquire a mass.

From the viewpoint of constructing a systematic chiral field theory, it is the (1+1)- and (2+1)-dimensional (rather than 4-dimensional) versions of four-fermion models which are of greatest interest. The Lagrangian of one of these, the Gross–Neveu (GN) model, has the form[6]

$$L_\psi=\sum_{k=1}^{N}\bar{\psi}_k i\hat{\partial}\psi_k+\frac{g_0}{2N}\left(\sum_{k=1}^{N}\bar{\psi}_k\psi_k\right)^2. \tag{1}$$

In two dimensions the model (1) is obviously perturbatively renormalizable, but its renormalizability in (2+1) (3) dimensions has been proven only relatively recently,[7] using the nonperturbative $1/N$ expansion. Owing to its simplicity, the two-dimensional GN model can be used to illustrate asymptotic freedom, spontaneous breakdown of chiral invariance, and other fundamental features of quantum chromodynamics. However, the model is also interesting for practical reasons, because it can be used to describe quasi-one-dimensional conductors, called Peierls dielectrics.[8] The vacuum properties of the (1+1)-dimensional GN model have already been studied at nonzero temperature and chemical potential[9–11] and in external electric and gravitational fields.[12] A recent series of studies[13] has been devoted to the phase transitions in this field theory in two-dimensional spacetime with topology $R^1\times S^1$.

The three-dimensional GN model has become an object of study relatively recently in connection with the discovery of high-temperature superconductivity (HTSC; see Ref. 14), and also with attempts to understand the quantum Hall effect.[15] The Hall effect is observed in planar samples located in a strong magnetic field, and HTSC occurs in materials like La_2CuO_4 in which the conduction electrons are concentrated in the planes formed by the Cu and O atoms. This is why in the last decade many theoretical physicists have intensively studied $(2+1)$-dimensional field theories, using techniques from quantum field theory in condensed-matter physics. Among the most fashionable models at present are 3-dimensional quantum electrodynamics, the nonlinear σ model, and, in particular, the $(2+1)$- dimensional GN model (1).

The point is that in 3-dimensional theories with four-fermion interaction the facts of spontaneous symmetry breaking[7] and also dynamical mass generation are well established, and so attempts have already been made using models of this type to explain HTSC.[16] Moreover, such models can be used to describe the properties of planar antiferromagnets, because in the continuum limit some of the lattice solid-state models of these materials have Lagrangian of the form (1) (Ref. 17). Like any other 3-dimensional theory, models like the GN model can serve as a good foundation for understanding the physical processes occurring in thin films. Finally, the field theory (1) is a special sort of laboratory allowing the prediction of new effects in the four-dimensional world.

The present review is devoted to the systematic study of the vacuum properties of some $(2+1)$-dimensional theories with four-fermion interaction as a function of T, μ, and external gauge fields, and also to the related new dynamical effects.

A key feature common to the models studied here can be demonstrated by the following simple example. Let us consider a system of free fermions in R^{2+1} spacetime interacting with an external uniform magnetic field. Its Lagrangian has the form

$$L=\bar{\psi}(i\hat{\partial}-e\hat{A}-m)\psi.$$

In the massless limit, the condensate $\langle\bar{\psi}\psi\rangle$ is found to be

$$\lim_{m\to 0}\langle\bar{\psi}\psi\rangle=-eH/\pi.$$

(This result is obtained by using the effective-potential method and the relation between the absolute minimum of the potential and Bogolyubov quasi-averages,[18] because the fermion mass term introduced into this Lagrangian explicitly breaks chiral symmetry, while the condensate is a quasi-average which in our case does not vanish for $m\to 0$.) From this we see that in the presence of an external uniform magnetic field, the ground state of the quantum-field system of massless fermions in R^{2+1} is degenerate, and chiral symmetry is spontaneously broken. This phenomena originates in the modification of the infrared regime owing to the interaction of the fermion spin with the external magnetic field.

In Sec. 1 the nonperturbative $1/N$ expansion is used for a detailed study of the critical properties of the three-dimensional Gross–Neveu theory in the presence of an external magnetic field, temperature T, and chemical potential μ. We describe the heretofore unknown phenomenon of dynamical breaking of the chiral and flavor symmetries by an arbitrarily small external magnetic field H. We construct phase portraits of the model in the variables (T,μ), (H,T), and (H,μ), and also find the critical values of the parameters at which the original symmetry is restored. In addition, we show that in the (T,μ) plane there exists a tricritical point at which the curve of first-order phase transitions is transformed into a second-order critical curve.

In Sec. 2 we show that the GN model with $SU(3)$ fermionic fields is exactly solvable in the leading order of the $1/N$ expansion in an external chromomagnetic field. This model can be viewed as a three-dimensional analog of the theory effectively describing the low-energy region of quantum chromodynamics. Here the external non-Abelian field enters as the field of the gluon condensate, in the background of which processes involving quarks occur. We have shown that external constant chromomagnetic fields, both Abelian-like and non-Abelian, act as catalysts of the spontaneous breakdown of chiral invariance. (We note that the choice of fields modeling the ground state of QCD is not limited to the configurations that we consider; this is demonstrated in Refs. 19 and 20, where (anti-)self-dual fields and also other possible field configurations[21] which are solutions of the Yang–Mills equations have been studied in detail.) Therefore, one of the reasons for chiral symmetry breaking in QCD is the nonzero chromomagnetic gluon condensate. We have also found the critical value of the temperature at which chiral symmetry is restored.

In Sec. 3 we study in detail two 3-dimensional models with more complicated fermion interaction. We show that both chiral phase transitions and transitions of the superconducting type with spontaneous breakdown of the $U(1)$ group are possible in those models. In addition, we pay special attention to the possibility of dynamical generation of a Chern–Simons term (topological mass of the gauge field), which causes particles to acquire fractional spin and statistics. For the example of these models, we show that violation of P parity is necessary but not sufficient for the spontaneous generation of a Chern–Simons term. We also discover the effect where the order parameter of the theory (i.e., the quantity in one-to-one correspondence with the vacuum structure) is the topological mass of the gauge field.

1. CRITICAL PROPERTIES OF THE THREE-DIMENSIONAL GN MODEL FOR $T,\mu,H\neq 0$

In this section we shall study the phase structure of the three-dimensional Gross–Neveu model for $\mu,T,H\neq 0$. The Lagrangian has the form (1) with $\hat{\partial}=\partial_\mu\Gamma^\mu$, where Γ^μ is a 4×4 matrix whose algebra is described in Appendix A, and each of the N fermion fields is a 4-component Dirac spinor:

$$\psi_k=\begin{pmatrix}\psi_{1k}\\ \psi_{2k}\end{pmatrix}. \tag{2}$$

Here $k=1,\dots,N$, and ψ_{ik} $(i=1,2)$ are ordinary two-component Dirac spinors. In this case the Lagrangian (1) is invariant under two discrete transformations. The first is

$$\psi_k\to\Gamma^3\psi_k;\quad k=1,\dots,N \tag{3}$$

[the matrix Γ^3 is given by Eq. (A8) in Appendix A] and represents the locations of the spinor generations ψ_{1k} and ψ_{2k} from (2). [Symmetry under (3) is called flavor symmetry.] The second transformation has the form

$$\psi_k \to \Gamma^5 \psi_k; \quad k=1,...,N, \tag{4}$$

with Γ^5 given by (A9). We shall refer to symmetry under (4) as discrete chiral symmetry, because $\Gamma^5 \sim \Gamma^0\Gamma^1\Gamma^2\Gamma^3$ and formally has the same form as the generator of chiral transformations for spinor fields in four-dimensional spacetime.

The Lagrangian (1) is also invariant under continuous transformations of the form

$$\psi_k \to \exp(i\alpha)\psi_k; \quad k=1,...,N. \tag{5}$$

We shall study the phase properties of this model in the leading order of the $1/N$ expansion.

1.1. The case $T,\mu \neq 0$

At zero T and μ the Lagrangian (1) is equivalent, in terms of the equations of motion for the field σ, to the following auxiliary Lagrangian:

$$L_\sigma = \sum_{k=1}^{N} [\bar\psi_k i\hat\partial\psi_k + \sigma\bar\psi_k\psi_k] - \frac{N\sigma^2}{2g_0}. \tag{6}$$

In terms of this Lagrangian, the discrete symmetries (3) and (4) become

$$\psi_k \to \Gamma\psi_k, \quad \sigma \to -\sigma; \quad k=1,...,N, \tag{7}$$

where Γ is one of the matrices Γ^3 or Γ^5. It is easy to show that in the leading order of the $1/N$ expansion the effective potential in the equivalent auxiliary model (6) has the form[7,22]

$$\frac{1}{N} V_0(\sigma) = \frac{\sigma^2}{2g_0} - 2\int \frac{d^3p}{(2\pi)^3} \ln(p^2+\sigma^2). \tag{8}$$

Here the integration runs over three-dimensional Euclidean space. Equation (8) contains ultraviolet (UV) divergences which are eliminated by renormalization. For this we assume that $|p|<\Lambda$ in (8), and we easily find

$$\frac{1}{N} V_0(\sigma) = \frac{\sigma^2}{2}\left(\frac{1}{g_0} - \frac{2\Lambda}{\pi^2}\right) + \frac{|\sigma|^3}{3\pi}. \tag{9}$$

From this, using the normalization condition

$$\left.\frac{d^2V_{\mathrm{eff}}}{d\sigma^2}\right|_{\sigma=m} = \frac{N}{g(m)},$$

it can be shown that

$$\frac{1}{g_0} - \frac{2\Lambda}{\pi^2} = \frac{1}{g(m)} - \frac{2m}{\pi} \equiv \frac{1}{g}, \tag{10}$$

[m is the normalization point, and $g(m)$ is the renormalized coupling constant], where we have introduced the new parameter g, which is independent of both the normalization point m (because g_0 is independent of m) and the UV-cutoff parameter Λ [because $g(m)$ is independent of Λ]. Substituting (10) into (9), we obtain the final renormalization-invariant expression for $V_0(\sigma)$:

$$\frac{1}{N} V_0(\sigma) = \frac{\sigma^2}{2g} + \frac{|\sigma|^3}{3\pi}. \tag{11}$$

It is easily seen that for $g>0$ the absolute minimum of the function (11) is located at zero, and so the discrete symmetries (3) and (4) of the model are not broken. If $g<0$, the point corresponding to the global minimum of the potential is

$$\sigma_0 \equiv \langle\sigma\rangle = -\pi/g \equiv M. \tag{12}$$

The discrete symmetries of the model are spontaneously broken, and the fermions dynamically acquire a mass M (12).

Let us now assume that our system described by the Lagrangian (1) [or, equivalently, the Lagrangian (6)] is located in a heat bath. In this case we must use the methods of quantum field theory at finite temperature developed in Ref. 23. Then the properties of the system in thermodynamical equilibrium are determined by the thermodynamical potential $\Omega(T,\mu)$:

$$\beta V\Omega(T,\mu) = -\ln \mathrm{Tr}\, \exp^{-\beta(\hat H - \mu\hat N)}, \tag{13}$$

where $\beta = 1/T$, V is the two-dimensional volume of the system, $\hat H$ is the Hamiltonian, and $\hat N$ is a generator of the group of transformations (5):

$$\hat N \sim \int d^2x \bar\psi(\vec x,t)\Gamma^0\psi(\vec x,t).$$

The quantity $\Omega(T,\mu)$ is the value of the effective potential $V_{T\mu}(\sigma)$ at the global minimum. To find $V_{T\mu}(\sigma)$, it is sufficient to transform the integration measure for the Euclidean energy variable in (8) according to the rule[24]

$$\int \frac{dp_0}{2\pi} f(p_0) \to T\sum_{n=-\infty}^{\infty} f(p_{0n});$$

$$p_{0n} = (2n+1)\pi T - i\mu. \tag{14}$$

Now we have

$$\frac{1}{N} V_{T\mu}(\sigma) = \frac{\sigma^2}{2g_0} - 2T\sum_{n=-\infty}^{\infty}\int \frac{d^2p}{(2\pi)^2} \ln[p_1^2+p_2^2+\sigma^2 + \pi^2T^2(2n+1)^2 - \mu^2 - 2i\mu\pi T(2n+1)]. \tag{15}$$

Summing over n in this expression, we obtain

$$V_{T\mu}(\sigma) = V_0(\sigma) - \frac{NT}{2\pi}\int_0^\infty dx \ln[(1+\exp[-\beta\sqrt{x+\sigma^2} - \beta\mu])(1+\exp[-\beta\sqrt{x+\sigma^2}+\beta\mu])], \tag{16}$$

where $V_0(\sigma)$ is the effective potential at zero T and μ [(8) or (11)].

Without loss of generality, here we shall study this function only for positive σ. The stationarity equation for (16) is

$$\partial V_{T\mu}(\sigma)/\partial\sigma = N\sigma f_{T\mu}(\sigma) = 0, \tag{17}$$

where

$$f_{T\mu}(\sigma) = \frac{1}{g} + \frac{\sigma}{\pi} + \frac{T}{\pi}\ln[(1+e^{-\beta\sigma-\beta\mu})(1 + e^{-\beta\sigma+\beta\mu})]. \tag{18}$$

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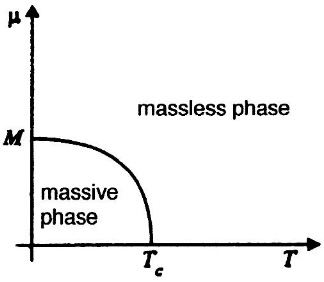

FIG. 1. Phase portrait of the Gross–Neveu model for $g<0$ in the variables μ,T.

Let $T\neq 0$. The function $f_{T\mu}(\sigma)$ increases monotonically on the interval $(0,\infty)$, and $f_{T\mu}(\infty)=\infty$. Moreover,

$$f_{T\mu}(0)=\frac{1}{g}+\frac{T}{\pi}\ln[(1+e^{-\beta\mu})(1+e^{\beta\mu})]. \tag{19}$$

We see from (19) that for $g>0$, $f_{T\mu}(0)>0$, and so $f_{T\mu}(\sigma)>0$. In this case the only solution of the stationarity equation (17) is the point $\sigma=0$, which is the statistical average of the field σ. This point is invariant under the discrete transformations (7), and so the vacuum of the model (here and below by vacuum for $T,\mu\neq 0$ we mean the state of thermodynamical equilibrium of the system) is also symmetric under these transformations for any μ and T.

Now let $g<0$ and, in addition, let T and μ take values such that $f_{T\mu}(0)<0$. Since $f_{T\mu}(\sigma)$ is a monotonically growing function, in this case there must exist a single nonzero point σ_0 depending on T and μ such that $f_{T\mu}(\sigma_0)=0$. Moreover, $f_{T\mu}(\sigma)\leq 0$ for $\sigma\leq\sigma_0$. It then follows from (17) that the first derivative of the function $V_{T\mu}(\sigma)$ with respect to σ is negative on the interval $(0,\sigma_0)$, i.e., $V_{T\mu}(\sigma)$ decreases on this interval, and so $V_{T\mu}(0)>V_{T\mu}(\sigma_0)$. We have thus shown that for $g<0$ and $f_{T\mu}(0)<0$ the point $\sigma_0\neq 0$ is a global minimum of the potential (16), and the symmetry (7) of the model is spontaneously broken. It is easily seen that $\sigma_0\to 0$ for $f_{T\mu}(0)\to 0$. Obviously, for $g<0$ and $f_{T\mu}(0)>0$ the stationarity equation (17) will have only a single root $\sigma=0$. This implies that at values of T and μ for which $f_{T\mu}(0)$ becomes positive, the symmetry (7) of the model is restored. A systematic study of phase transitions in condensed-matter physics, and also in elementary-particle physics and cosmology, was made in Ref. 25.

The results of this subsection can be displayed as a phase portrait of the model in the plane of the variables T and μ. In Fig. 1 we show the critical curve given by the equation $f_{T\mu}(0)=0$. It divides the set of points (T,μ) corresponding to the phase with massive fermions and spontaneous breaking of the discrete symmetries from the set corresponding to the massless, symmetric phase of the theory. On the critical curve the temperature and chemical potential are related as

$$\mu(T)=T\ln K(T), \tag{20}$$

where

$$K(T)=-1+\frac{1}{2}\exp(\beta M)+\sqrt{\left[-1+\frac{1}{2}\exp(\beta M)\right]^2-1},$$

and M is the fermion mass for $T,\mu=0$. The function (20) vanishes at the point $T_c=M/(2\ln 2)$. Moreover, $\mu(0)=M$, $\mu'(T_c)=-\infty$, and $\mu'(0)=0$. In crossing the critical curve a second-order phase transition occurs in the theory, because the order parameter—the point at which the potential reaches a global minimum—on this curve is a continuous function of the external parameters T and μ.

Up to now we have assumed that $T\neq 0$. For $T=0$ the picture is qualitatively different. Here the effective potential can be obtained from (16) for $T\to 0$:

$$V_\mu(\sigma)=V_0(\sigma)-\frac{N}{6\pi}\Theta(\mu-\sigma)(\mu-\sigma)^2(\mu+2\sigma), \tag{21}$$

where $\Theta(x)$ is the Heaviside step function, and the stationarity equation takes the form

$$\sigma\left[\frac{1}{g}+\frac{\sigma}{\pi}+\frac{1}{\pi}\Theta(\mu-\sigma)(\mu-\sigma)\right]=0.$$

Study of the potential (21) shows that for $g>0$ the symmetries (7) are not broken for any value of the chemical potential. However, if $g<0$ there exists a critical value of the chemical potential $\mu_c=M$ below which the global minimum of the function $V_\mu(\sigma)$ is $\sigma_0=M$, and for $\mu>\mu_c$ the absolute minimum is located at zero. Since the order parameter undergoes a discontinuity at the point μ_c, we have a first-order phase transition here. Accordingly, on the phase diagram in Fig. 1 the point $(0,M)$ is a tricritical point,[26] because at it the phase-transition curve goes from being a line of first-order phase transitions to a line of second-order ones and vice versa.

1.2. The case $H\neq 0$. The catalysis effect

Let us now study the critical properties of the three-dimensional GN model in an external constant magnetic field H for $T,\mu=0$. The Lagrangian of the model in terms of the auxiliary scalar field $\sigma(x)$ in this case takes the form

$$L_\sigma=\sum_{k=1}^{N}[\bar\psi_k(i\hat\partial-e\hat A)\psi_k+\sigma\bar\psi_k\psi_k]-\frac{N\sigma^2}{2g_0}, \tag{22}$$

where $A\equiv A_\mu\Gamma^\mu$, e is the fermion charge, and the vector potential corresponding to constant external magnetic field H has the form $A_{0,1}=0$, $A_2=x_1H$. The generating functional for the Green functions of the scalar field σ is

$$\exp\{iW(J)\}=\int D\bar\psi D\psi D\sigma\exp\left\{i\int d^3x[L_\sigma+J(x)\sigma(x)]\right\}.$$

Integrating this over the spinor fields, we obtain

$$\exp\{iW(J)\}=\int D\sigma \exp\left\{i\int d^3x[L_{\rm eff}(\sigma)+J(x)\sigma(x)]\right\},$$

where

$$\int d^3xL_{\rm eff}(\sigma)=\int d^3x(-N\sigma^2/(2g_0))-iN \operatorname{Tr}\ln\Delta. \tag{23}$$

In this expression $\Delta=i\hat{\partial}-e\hat{A}+\sigma$. Now we assume that the field σ is independent of the spacetime point. Then in the leading order of the $1/N$ expansion the effective potential of the model in an external magnetic field takes the form

$$V_{\rm eff}\equiv V_H(\sigma)=-L_{\rm eff}(\sigma),$$

i.e.,

$$V_H(\sigma)=\frac{N\sigma^2}{2g_0}+\frac{iN}{v}\operatorname{Tr}\ln\Delta, \tag{24}$$

where $v=\int d^3x$. It has been shown[27] that the function (24) is symmetric under the transformation $\sigma\to-\sigma$, and so it is sufficient to study its properties for $\sigma\geq 0$.

Let us first turn to the causal Green function of the operator Δ, which can be written as[29]

$$\Delta^{-1}_{\alpha\beta}(x,t;x',t')=-i\Theta(t-t')\sum_{\{n\}}\psi^{(+)}_{\{n\}\alpha}(x,t)\times\bar{\psi}^{(+)}_{\{n\}\beta}(x',t')+i\Theta(t'-t)\times\sum_{\{n\}}\psi^{(-)}_{\{n\}\alpha}(x,t)\bar{\psi}^{(-)}_{\{n\}\beta}(x',t'). \tag{25}$$

Here $\{n\}=(i,n,k)$, where $i=1,2$; $n=0,1,2,\dots$; and k is a real number, $-\infty<k<\infty$. In addition, $\psi^{(\pm)}_{\{n\}}$ are the positive- and negative-frequency orthonormal solutions of the Dirac equation $\Delta\psi=0$ and have the form (T denotes the transpose)

$$\psi^{(\pm)T}_{1nk}(x,t)=\exp(\mp i\varepsilon_n t+ikx_2)\left(\sqrt{\frac{\varepsilon_n\mp\sigma}{4\pi\varepsilon_n}}h_{n,k}(x_1),\pm\sqrt{\frac{\varepsilon_n\pm\sigma}{4\pi\varepsilon_n}}h_{n-1,k}(x_1),0,0\right),$$

$$\psi^{(\pm)T}_{2nk}(x,t)=\exp(\mp i\varepsilon_n t+ikx_2)\times\left(0,0,\sqrt{\frac{\varepsilon_n\pm\sigma}{4\pi\varepsilon_n}}h_{n,k}(x_1),\pm\sqrt{\frac{\varepsilon_n\mp\sigma}{4\pi\varepsilon_n}}h_{n-1,k}(x_1)\right), \tag{26}$$

where

$$h_{n,k}(x_1)=\frac{(eH)^{1/4}}{(2^n n!\sqrt{\pi})^{1/2}}\exp(-\xi^2/2)H_n(\xi), \tag{27}$$

$H_n(\xi)$ are the Hermite polynomials, $\varepsilon_n=\sqrt{\sigma^2+2eHn}$, and $\xi=\sqrt{eH}(x_1-k/eH)$. The functions (27) satisfy the conditions

$$\int dx_1 h^2_{n,k}(x_1)=\frac{1}{eH}\int dk\, h^2_{n,k}(x_1)=1. \tag{28}$$

In addition, in (26) it is assumed that $h_{-1,k}(x_1)\equiv 0$.

We now have the information needed to calculate the quantity

$$\frac{\partial}{\partial\sigma}V_H(\sigma)=\frac{N\sigma}{g_0}+\frac{iN}{v}\operatorname{Tr}(\Delta^{-1}). \tag{29}$$

First we need to get rid of the Θ functions in (25), using the rule

$$\mp\Theta(\pm t)\exp(\mp i\varepsilon_n t)f(\varepsilon_n)=\int\frac{d\omega}{2\pi i}\frac{f(\pm\omega)\exp(-i\omega t)}{\omega\mp(\varepsilon_n-i0)}. \tag{30}$$

Then, substituting the expression for the Green function into (29), we find

$$\frac{\partial}{\partial\sigma}V_H(\sigma)=\frac{N\sigma}{g_0}-\frac{iN\sigma}{2\pi^2 v}\int d^3x\int d\omega\int dk\times\sum_{n=0}^{\infty}\frac{h^2_{n-1,k}(x_1)+h^2_{n,k}(x_1)}{\omega^2-\varepsilon_n^2+i0}.$$

We can perform the integration over the variable k [see (28)]:

$$\frac{\partial}{\partial\sigma}V_H(\sigma)=\frac{N\sigma}{g_0}-\frac{iN\sigma eH}{2\pi^2}\int d\omega\sum_{n=0}^{\infty}\frac{s_n}{\omega^2-\varepsilon_n^2+i0}, \tag{31}$$

where $s_n=2-\delta_{0n}$. We change to the Euclidean metric in (31), i.e., we make the replacement $\omega\to i\omega$ and use the α representation.[28] After integrating over α and summing over n, we find

$$\frac{\partial}{\partial\sigma}V_H(\sigma)=\frac{N\sigma}{g_0}-\frac{N\sigma eH}{2\pi^{3/2}}\int_0^\infty\frac{d\alpha}{\sqrt{\alpha}}\times\exp(-\alpha\sigma^2)\coth(eH\alpha). \tag{32}$$

Integrating both sides of this equation over σ from σ to ∞ and dropping the unimportant σ-independent constants, we find

$$V_H(\sigma)=\frac{N\sigma^2}{2g_0}+\frac{NeH}{4\pi^{3/2}}\int_0^\infty\frac{d\alpha}{\alpha^{3/2}}\times\exp(-\alpha\sigma^2)\coth(eH\alpha). \tag{33}$$

The integral in this expression diverges at the lower limit. Making identity transformations in (33), we can localize this divergence in the effective potential for $H=0$:

$$V_H(\sigma)=V_0(\sigma)+\frac{NeH}{4\pi^{3/2}}\int_0^\infty\frac{d\alpha}{\alpha^{3/2}}\exp(-\alpha\sigma^2)\times\left[\coth(eH\alpha)-\frac{1}{eH\alpha}\right]. \tag{34}$$

Here

$$V_0(\sigma)\equiv\frac{N\sigma^2}{2g_0}+\frac{N}{4\pi^{3/2}}\int_0^\infty\frac{d\alpha}{\alpha^{5/2}}\exp(-\alpha\sigma^2)$$
$$=\frac{\sigma^2}{2g}+\frac{\sigma^3}{3\pi}. \quad (35)$$

We note that Eq. (34) for $V_H(\sigma)$ can be derived by the Schwinger proper-time method.[30]

The effective potential (34) can be written more compactly as[32]

$$V_H(\sigma)=\frac{N\sigma^2}{2g}+\frac{NeH\sigma}{2\pi}-\frac{N(2eH)^{3/2}}{2\pi}\zeta\left(-\frac{1}{2},\frac{\sigma^2}{2eH}\right), \quad (36)$$

where $\zeta(s,v)$ is the generalized Riemann zeta function,[33] and the stationarity equation for it becomes

$$\frac{\partial V_H(\sigma)}{\partial\sigma}=\frac{N\sigma}{g}+\frac{NeH}{2\pi}-\frac{N\sigma\sqrt{2eH}}{2\pi}\zeta\left(\frac{1}{2},\frac{\sigma^2}{2eH}\right)=0. \quad (37)$$

For $\sigma\to 0$ the ζ function has the expansion[34]

$$\zeta\left(\frac{1}{2},\frac{\sigma^2}{2eH}\right)=\frac{\sqrt{2eH}}{\sigma}+\text{const}+o(\sigma/\sqrt{2eH}). \quad (38)$$

Substituting this into (37), we easily see that the stationarity equation does not have a solution $\sigma=0$ for $H\neq 0$. *Therefore, in an external magnetic field both the flavor symmetry (3) and the chiral symmetry (4) of the three-dimensional Gross–Neveu model are spontaneously broken independently of the sign of the coupling constant g for arbitrarily small H.* This phenomenon is called the catalysis of spontaneous symmetry breaking by an external magnetic field.

It was shown in Refs. 22 and 27 that (37) has the single solution $\sigma_0(H)$, and its properties depend significantly on H and g. Let us assume that $g>0$ and that the external field is weak, i.e., $eHg^2\ll 1$. Substituting (38) into (37), in this case we find

$$\sigma_0(H)=egH/2\pi+\dots . \quad (39)$$

(This fermion-mass asymptote is justified in more detail in Ref. 27.)

If $g<0$, then $\sigma_0(H)\to M$ for $H\to 0$, and so here we need to use the following expansion[33] ($x\equiv 2eH/\sigma^2$):

$$\zeta\left(\frac{1}{2},\frac{1}{x}\right)=-2x^{-1/2}+\frac{1}{2}x^{1/2}+\frac{1}{\sqrt{\pi}}$$
$$\times\sum_{n=1}^{m}B_{2n}\frac{\Gamma(2n-1/2)}{(2n)!}x^{2n-1/2}$$
$$+O(x^{2m+3/2}). \quad (40)$$

Here B_{2n} are the Bernoulli numbers, $B_2=\frac{1}{6},\dots$. Using (40), from (37) we find in this case

$$\sigma_0(H)=M\{1+(eH)^2/(12M^4)+o((eH)^2/M^4)\}. \quad (41)$$

Let us now consider large values of the external magnetic field. Following Refs. 27 and 35, it can be shown that in this case

$$\sigma_0(H)\approx 0.45\sqrt{eH}. \quad (42)$$

1.3. The case $H, T\neq 0$

Now we shall study the combined effect of temperature and an external constant magnetic field on the phase structure of the three-dimensional GN model. To obtain the effective potential in this case it is sufficient to make the Euclidean rotation $\omega\to ip_0$ in (31), and then apply the operator (14). After simple algebra we arrive at the following expression for $V_{HT}(\sigma)$:

$$V_{HT}(\sigma)=\frac{N\sigma^2}{2g_0}+\frac{NeHT}{2\pi}\sum_n\int_0^\infty\frac{d\alpha}{\alpha}\exp(-\alpha\sigma^2$$
$$-\alpha(2n+1)^2\pi^2T^2)\coth(eH\alpha). \quad (43)$$

We split (43) into two terms:

$$V_{HT}(\sigma)\equiv V_T(\sigma)+\tilde V_{TH}(\sigma), \quad (44)$$

where

$$V_T(\sigma)=\frac{N\sigma^2}{2g_0}+\frac{NT}{2\pi}\sum_n\int_0^\infty\frac{d\alpha}{\alpha^2}\exp(-\alpha\sigma^2-\alpha(2n$$
$$+1)^2\pi^2T^2),$$
$$\tilde V_{HT}(\sigma)=\frac{NT}{2\pi}\sum_n\int_0^\infty\frac{d\alpha}{\alpha^2}\exp(-\alpha\sigma^2-\alpha(2n$$
$$+1)^2\pi^2T^2)[eH\alpha\coth(eH\alpha)-1]. \quad (45)$$

Let us now study the function (44) at the absolute minimum. The stationarity equation for it has the form

$$\partial V_{TH}(\sigma)/\partial\sigma=N\sigma[f_T(\sigma)-F(\sigma)]=0, \quad (46)$$

where $f_T(\sigma)$ coincides with $f_{T\mu}(\sigma)$ (18) at $\mu=0$, and $NF(\sigma)=\partial\tilde V_{HT}(\sigma)/\partial\sigma$:

$$F(\sigma)=\frac{T}{\pi}\sum_n\int_0^\infty\frac{d\alpha}{\alpha}\exp(-\alpha\sigma^2-\alpha(2n+1)^2\pi^2T^2)$$
$$\times[eH\alpha\coth(eH\alpha)-1]. \quad (47)$$

We see from (47) that $F(\sigma)>0$ for $\sigma\geqslant 0$. Moreover, $F'(\sigma)<0$, i.e., $F(\sigma)$ is a monotonically decreasing function on the semiaxis $\sigma\geqslant 0$. In Appendix B it is shown that $F(0)=\text{const}<\infty$. There we obtain upper and lower bounds on the value of $F(0)$, from which it is seen that $F(0)\to\infty$ for $T\to 0$ and $F(0)\to 0$ for $T\to\infty$. We find from (46) that for any nonzero temperature there always exists a solution $\sigma=0$ of this equation. Depending on the relation between $f_T(0)$ and $F(0)$, the stationarity equation may or may not have nonzero solutions which, obviously, satisfy the equation

$$f_T(\sigma)=F(\sigma). \quad (48)$$

For example, at sufficiently small T, when $f_T(0)<F(0)$, Eq. (48) has a single solution $\sigma_0\neq 0$ [the uniqueness is a consequence of the fact that $f_T(\sigma)$ is a monotonically increasing function and $F(\sigma)$ a monotonically decreasing function]. Obviously, $f_T(\sigma)<F(\sigma)$ if $0<\sigma<\sigma_0$. It therefore follows from (46) that $V'_{HT}<0$ for $0<\sigma<\sigma_0$. Then on the interval

$(0,\sigma_0)$ the potential $V_{HT}(\sigma)$ is a monotonically decreasing function, i.e., $V_{HT}(0)>V_{HT}(\sigma_0)$, and its global minimum is located at the point σ_0. In this case the discrete symmetries (3) and (4) of the model are spontaneously broken, and the fermions acquire a nonzero mass σ_0.

At sufficiently large T, when $f_T(0)>F(0)$, Eq. (48) again has no solutions because $f_T(\sigma)$ is a monotonically increasing function and $F(\sigma)$ a monotonicaly decreasing function. In this case the vacuum of the model is symmetric, and the fermions are massless. We therefore see that in the (T,H) plane the equation

$$f_T(0)=F(0) \qquad (49)$$

determines the critical curve l_c. It separates the region of parameters (T,H) corresponding to the symmetric vacuum from the region where the discrete symmetries (3) and (4) are spontaneously broken. If as T and H vary we intersect the critical curve, a second-order phase transition occurs in the model. This is easily understood when we remember that for $f_T(0)\to F(0)$ the solution of the stationarity equation σ_0 tends to zero. This means that the effective fermion mass on l_c is a continuous function of the parameters T and H, which is the necessary and sufficient condition for a second-order phase transition.

Let us study the features of the critical curve in greater detail. Equation (49) determines the temperature as a function of the external magnetic field. Therefore, in the (T,H) plane the critical curve can be specified as

$$l_c=\{(H,T):T=T_c(H)\}. \qquad (50)$$

The function $T_c(H)$ will be referred to as the critical temperature. Let us evaluate its behavior at large and small H.

We assume that $g>0$. Using the estimates for $F(0)$ obtained in Appendix B, the critical temperature must satisfy the following inequalities:

$$\Phi_1(T_c(H),H)>f_T(0)=F(0)>\Phi_2(T_c(H),H), \qquad (51)$$

where Φ_1 and Φ_2 are respectively the upper (B3) and lower (B5) limits on $F(0)$. Replacing $f_T(0)$ by (18) for $\mu=0$ and solving the resulting inequality for $H\to\infty$, we find

$$C_1\sqrt{eH}<T_c(H)<C_2\sqrt{eH}, \qquad (52)$$

where $C_{1,2}$ are known constants (in order not to encumber the text with inessential equations, we do not give $C_{1,2}$ explicitly here). It follows from (52) that for $H\to\infty$

$$T_c(H)\sim C_3\sqrt{eH}, \qquad (53)$$

where C_3 is an unknown constant. Solving (51) for small H, we find

$$C_4geH<T_c(H)<C_5geH. \qquad (54)$$

From this it follows that for $H\to 0$

$$T_c(H)\sim C_6geH. \qquad (55)$$

In the last expressions $C_{4,5}$ are known constants, and C_6 is unknown. The phase portrait of the model for $g>0$ is given in Fig. 2.

Let us now study the critical temperature at negative g. Here for $H\to\infty$, $T_c(H)$ is proportional to $\sqrt{eH}$ with unknown proportionality factor (the arguments are the analog of those for the case $g>0$). In the region $H\to 0$ we obtain the asymptote of the function $F(\sigma)$ (Ref. 22):

$$F(\sigma)=-\frac{(eH)^2}{12\pi\sigma}\frac{\partial}{\partial\sigma}\left[\frac{1}{\sigma}\tanh\frac{\sigma}{2T}\right]+o((eH)^2). \qquad (56)$$

From this we find the value of $F(0)$ and substitute it into (49). The solution of the latter for small parameter eH has the form

$$T_c(H)=T_c+(eH)^2/[T_c^3 288\ln 2]+o(eH)^2), \qquad (57)$$

where $T_c=M/(2\ln 2)$ is the critical temperature at $H=0$ (see Sec. 1). The results of the analysis are shown in Fig. 3 as the phase portrait of the three-dimensional GN model for $g<0$.

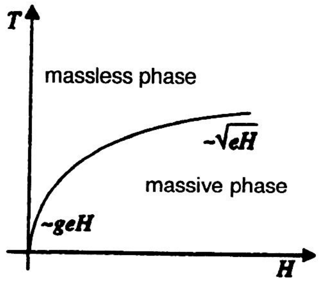

FIG. 2. Phase portrait of the Gross–Neveu model for $g>0$ in the variables T,H.

1.4. The case $H,\mu\neq 0$

As in the preceding cases, the phase structure of the Gross–Neveu model in an external constant magnetic field at nonzero chemical potential is determined by using the effective potential. To find it, we can start from Eq. (31). Assuming for now that in addition to H and μ the temperature is also nonzero, we must make a Euclidean rotation in (31) and replace the integration over ω by summation over the Matsubara frequencies [see (14)]. Here we shall slightly modify this procedure. Instead of summing over the eigenvalues of the Dirac operator (as in calculating the effective potentials in the two preceding cases), in the resulting expression we sum over Matsubara frequencies:[25]

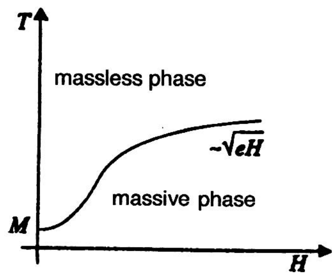

FIG. 3. Phase portrait of the Gross–Neveu model for $g<0$ in the variables T,H.

$$\frac{\partial}{N\partial\sigma}V_{H\mu T}(\sigma)=\left\{\frac{\sigma}{g_0}-\frac{eH\sigma}{2\pi}\sum_n\frac{s_n}{\varepsilon_n}\right\}+\frac{eH\sigma}{2\pi}\sum_n\frac{s_n}{\varepsilon_n}[[1+\exp^{-\beta(\varepsilon_n+\mu)}]^{-1}+[1+\exp^{-\beta(\varepsilon_n-\mu)}]^{-1}]. \tag{58}$$

Here $\varepsilon_n=\sqrt{\sigma^2+2eHn}$ and $s_n=2-\delta_{0n}$. It is easily seen that the expression in curly brackets in (58) is, up to the coefficient N, just the right-hand side of (31). Therefore, after integrating both sides of (58) from σ to ∞, we obtain the spectral representation for the effective potential at H,T, $\mu\neq 0$:

$$V_{H\mu T}(\sigma)=V_H(\sigma)-\frac{NTeH}{2\pi}\sum_{n=0}^{\infty}s_n\ln\{[1+\exp^{-\beta(\varepsilon_n+\mu)}][1+\exp^{-\beta(\varepsilon_n-\mu)}]\}, \tag{59}$$

where the function $V_H(\sigma)$ is given in (34) or (36). Finally, after taking the limit $\beta\to\infty$ in (59), we obtain the effective potential for $H,\mu\neq 0$, $T=0$:

$$V_{H\mu}(\sigma)=V_H(\sigma)-\frac{eHN}{2\pi}\sum_{n=0}^{\infty}s_n\Theta(\mu-\varepsilon_n)(\mu-\varepsilon_n), \tag{60}$$

the stationarity equation for which has the form

$$0=\frac{\sigma}{g}+\frac{eH}{2\pi}-\frac{\sigma\sqrt{2eH}}{2\pi}\zeta\left(\frac{1}{2},\frac{\sigma^2}{2eH}\right)+\frac{eH}{2\pi}\times\sum_{n=0}^{\infty}s_n\Theta(\mu-\epsilon_n)\frac{\sigma}{\epsilon_n}. \tag{61}$$

Let us first study the function (60) at the absolute minimum for $g>0$. First, we show that the symmetric phase is realized in the theory for $\mu>\sigma_0(H)$, where $\sigma_0(H)$ is the single solution of the stationarity equation (37) in the case $\mu=0$, $H\neq 0$. For this we split the parameter plane (μ,H) (for $\mu,H\geqslant 0$) into regions Ω_n:

$$(\mu,H)=\bigcup_{n=0}^{\infty}\Omega_n;$$

$$\Omega_n=\{(\mu,H):2eHn\leqslant\mu^2\leqslant 2eH(n+1)\}. \tag{62}$$

Obviously, in the region Ω_0 only the first term inside the sum will contribute to (61), while in Ω_1 the first and second terms inside the sum are nonzero, and so on. In what follows we shall need the very important expansion of the Riemann ζ function[34] ($\vartheta=\sigma^2/eH$):

$$\zeta\left(\frac{1}{2},\vartheta\right)=\sum_{i=0}^{k}(\vartheta+i)^{-1/2}-2\sqrt{k+\vartheta}-\sum_{i=k}^{\infty}f_i(\vartheta), \tag{63}$$

where

$$f_i(\vartheta)=\frac{1}{2}\int_i^{i+1}\frac{(u-i)du}{(u+\vartheta)^{3/2}}>0. \tag{64}$$

We assume that $(\mu,H)\in\Omega_0$. Taking into account the asymptotes (39) and (42), it is obvious that for sufficiently small and large H the curve $\mu=\sigma_0(H)$ passes through the region Ω_0. Therefore, inside Ω_0 there are points satisfying the condition $\mu>\sigma_0(H)$. In this case the stationarity equation (61) can be written as follows, using (63) with $k=0$:

$$\frac{\sigma}{f}+\frac{\sigma^2}{\pi}+\frac{\sigma\sqrt{2eH}}{2\pi}\sum_{i=0}^{\infty}f_i\left(\frac{\sigma^2}{2eH}\right)-\frac{eH}{2\pi}[1-\Theta(\mu-\sigma)]=0. \tag{65}$$

For $\mu>\sigma$ this equation has only one solution $\sigma_1=0$. If $\mu\leqslant\sigma$, (65) formally coincides with (37), as is easily seen from the expansion (63), and will have a solution $\sigma_0(H)$ only for $\mu<\sigma_0(H)$. If, as stipulated initially, $\mu>\sigma_0(H)$, Eq. (65) will not have solutions located in the region $\mu\leqslant\sigma$. Therefore, for points from Ω_0 with the condition $\mu>\sigma_0(H)$ the corresponding effective potential (60) has a single stationary point $\sigma_1=0$. Similar calculations can be performed for any region Ω_n, and so the following statement is valid. Points of the (μ,H) plane lying above the curve $\mu=\sigma_0(H)$ correspond to an effective potential whose global minimum occurs at $\sigma_1=0$, i.e., for $\mu>\sigma_0(H)$ the massless phase of the theory symmetric under the discrete transformations (3) and (4) is obtained.

If $\mu<\sigma_0(H)$, it is easy to show that the potential possesses another stationary point $\sigma_2=\sigma_0(H)$, at which, in general, a local minimum occurs. The corresponding ground state is metastable as long as we do not cross the critical curve $\mu=\mu_c(H)$, which is found from the condition

$$V_{H\mu}(0)=V_{H\mu}(\sigma_0(H)). \tag{66}$$

Below this curve the global minimum moves to the point $\sigma_2=\sigma_0(H)\neq 0$, and so here the phase with massive fermions and spontaneously broken symmetries (3) and (4) is stable, while the massless phase is metastable. Since the order parameter undergoes a discontinuity in crossing the critical curve, a first-order phase transition occurs in the model. The more detailed form of the equation for the critical curve $\mu=\mu_c(H)$ can be obtained from (66), using (60):

$$V_H(\sigma_0(H))=V_H(0)-\frac{eHN}{2\pi}\sum_{n=0}^{\infty}s_n\Theta(\mu-\sqrt{2eHn})(\mu-\sqrt{2eHn}). \tag{67}$$

It is clear from these arguments that $\mu_c(H)$ must lie below the curve $\mu=\sigma_0(H)$. However, we have already noted that for both fairly large and fairly small H the line $\mu=\sigma_0(H)$ is located in the region Ω_0. This means that at these values of H the critical curve is also located in Ω_0, and so only the first term need be kept inside the sum in (67). As a result, we obtain the form of the curve $\mu=\mu_c(H)$ at fairly large and small H:

$$\mu_c(H)=\frac{2\pi}{eHN}[V_H(0)-V_H(\sigma_0(H))]. \tag{68}$$

At small H, when $\sigma_0(H)$ is small, (68) gives

$$\mu_c(H)\cong\frac{2\pi}{eHN}\frac{dV_H(0)}{d\sigma}\cdot\sigma_0(H)=\sigma_0(H). \tag{69}$$

For $H\to\infty$ from (42) and (68) we find

$$\mu_c(H) \sim \sqrt{eH}. \quad (70)$$

The situation is less clearly defined at negative values of the coupling g. The point is that here the stationarity equations have some features which allow rigorous results to be obtained only for sufficiently large magnetic fields.

Let us assume that $(\mu,H)\in\Omega_0$ and $g<0$. We consider Eq. (65). For $\mu<\sigma$ it can have a solution $\sigma_0(H)$ [for $\mu<\sigma_0(H)$], or it may not have any solutions at all [for $\mu>\sigma_0(H)$]. For $\mu>\sigma$, Eq. (65) obviously has the trivial solution $\sigma=0$. Moreover, owing to the negative value of g, in this region Eq. (65) may in general have other nontrivial solutions different from $\sigma_0(H)$. In the latter case the process of finding the stationary point at which the potential has a global minimum is considerably more complicated. In order to avoid this situation, we shall single out the values of H at which Eq. (65) can have no more than two solutions $\sigma_1=0$ and $\sigma_2=\sigma_0(H)$.

For this we calculate the integral $f_0(\vartheta)$ in (64) and write (65) as

$$\frac{\sigma}{\pi}\left\{-M+\frac{\sigma^2+eH}{\sqrt{\sigma^2+2eH}}\right\}+\frac{\sigma\sqrt{2eH}}{2\pi}\sum_{i=1}^{\infty}f_i\left(\frac{\sigma^2}{2eH}\right)$$
$$-\frac{eH}{2\pi}[1-\Theta(\mu-\sigma)]=0, \quad (71)$$

where $M=-\pi/g$. The expression in curly brackets in (71) is positive for all $\sigma\geqslant 0$ if

$$2M^2<eH. \quad (72)$$

Now, imposing the constraint (72) on the magnetic field, we arrive at the same conclusions about the phase structure of the model in the region Ω_0 as for the case $g>0$. It can be shown that the stationarity equations for the potential $V_{H\mu}(\sigma)$ for $(\mu,H)\in\Omega_n$ $(n>0)$ also have no more than two solutions $\sigma=0$ and $\sigma_0(H)$ with the constraint (72). Accordingly, it can be stated that for $g<0$ and $eH>2M^2$, in the (μ,H) plane there exists a critical curve $\mu=\mu_c(H)$ specified by Eq. (67). In the region Ω_0 it obviously has the form (68), and therefore $\mu_c(H)\sim\sqrt{eH}$ for sufficiently large H. These conclusions are obtained by analogy with the case $g>0$. Unfortunately, we can say nothing about the behavior of the critical curve at small H for $g<0$ except that $\mu_c(0)=M$ (see Sec. 1.1).

1.5. Discussion of the results

In this section we have followed Refs. 22, 27, 37, and 41 and studied the phase structure of the three-dimensional GN model for nonzero temperature, chemical potential, and external constant magnetic field.

We have obtained an exact expression (20) for the critical curve of this model at $T,\mu\neq 0$ in the leading order of the $1/N$ expansion, and we have shown that a tricritical point exists on this phase diagram. These results were first published in Ref. 37, and were later rediscovered by other authors.[10,38] In addition, in Ref. 10 it was shown that in D-dimensional GN models with $2<D<3$ the tricritical point no longer lies on the boundary, but is shifted to the interior of the region $T,\mu\geqslant 0$. A similar result for the two-dimensional GN model was obtained in Ref. 11.

In this section we have displayed a new feature of an external magnetic field: its ability to effect the spontaneous breakdown of various symmetries.[22,27] (In the contemporary literature this is referred to as the catalysis effect.) In terms of the Gross–Neveu model, this implies that for $g>0$ the flavor and chiral symmetries (3) and (4) are spontaneously broken by any, arbitrarily small, external magnetic field H. Moreover, this serves to stabilize the phase with broken symmetries. For example, for $g<0$ the three-dimensional model (1) is located in the unsymmetric phase with massive fermions, and the external field H only increases the fermion mass. (The effect of an external electric field on the massive phase of the model is the opposite;[36] it decreases the fermion mass until the system crosses over into the massless phase at some E_c).

The effect of a magnetic field on the three-dimensional Gross–Neveu model has also been studied in Refs. 35 and 39, where the relation of the catalyzing role of the magnetic field to the zero modes of the Dirac operator and modification of the infrared regime was studied. This effect was also studied in Ref. 39 and then developed further in several studies,[40] which showed that the catalysis effect is also observed in other field-theoretical models: in three-dimensional quantum electrodynamics, in the Nambu–Jona-Lasinio model, and so on. These studies proposed an explanation of the catalysis effect, which in R^{2+1} arises only owing to the presence of an external magnetic field (with which the fermion spins interact, leading to a fundamental change of the infrared behavior in the system) and is unrelated to the presence of a four-fermion interaction in the original Lagrangian.

It should also be noted that in the present review we are far from mentioning all the external factors on which the vacuum structure of the model (1) can depend. We have not considered the effect of nontrivial topology or spacetime curvature, or finite-volume effects on the phase structure of the three-dimensional GN model. We refer the interested reader to Ref. 42 for more details on these aspects.

2. THE GLUON CONDENSATE AND THREE-DIMENSIONAL $(\bar\psi\psi)^2$ FIELD THEORY

In this section we continue our study of the critical behavior of the three-dimensional Gross–Neveu model, taking into account the effect of external non-Abelian gauge fields, the structure of which has been discussed in Refs. 46–49. The Lagrangian of the model has the form

$$L_\psi=\bar\psi\Gamma^\mu(i\partial_\mu+eA^a_\mu\lambda_a/2)\psi+\frac{g_0}{2N}[\bar\psi\psi]^2, \quad (73)$$

where the field ψ is a four-component Dirac spinor, which can correspond to quarks. In this case e is the quark–gluon interaction constant, and Γ_μ are 4×4 matrices whose algebra is given in Appendix A. The spinor ψ transforms under the fundamental representations of the color group $SU(3)$ and the group $SU(N)$, which is auxiliary in nature and has been introduced in order that the theory contain the parameter $1/N$, which is small for $N\to\infty$. [For simplicity, the group indices, which are summed over, have been omitted from the spinors in (73).] Here λ_a, $a=1,\ldots,8$, are the $SU(3)$ generators, and A^a_μ is an external non-Abelian (color) gauge field. For simplicity, we study a model which is invariant under the discrete chiral transformation (4), although it is also easy to study a model like (73) with continuous chiral invariance.

2.1. The effective potential. Structure of the external fields

It is convenient to perform the phase analysis of the model (73) in terms of the auxiliary scalar field $\sigma(x)$. For this we need to introduce the auxiliary Lagrangian

$$L_\sigma=\bar\psi\left(i\hat\partial+\frac{1}{2}e\lambda^a\hat A^a\right)\psi-\frac{N\sigma^2}{2g_0}+\sigma\bar\psi\psi, \qquad (74)$$

which is equivalent to L_ψ (73) in the equations of motion. By analogy with the preceding section [see the derivation of (24)], it can be shown that in the leading order of the $1/N$ expansion the effective potential of the model (73) in an external non-Abelian field has the form

$$V(\sigma)=\frac{N\sigma^2}{2g_0}+\frac{iN}{v}\,\mathrm{Tr}\,\ln(\hat\Pi+\sigma), \qquad (75)$$

where $\hat\Pi=\Gamma^\mu(i\partial_\mu+eA^a_\mu\lambda_a/2)$ and $v=\int d^3x$. Taking into account the identity (the matrix Γ^5 is given in Appendix A)

$$\hat\Pi+\sigma=\Gamma^5\Gamma^5(\hat\Pi+\sigma)=\Gamma^5(-\hat\Pi+\sigma)\Gamma^5,$$

the effective potential (75) can be written as follows, up to an unimportant, σ-independent constant:

$$V(\sigma)=\frac{N\sigma^2}{2g_0}+\frac{iN}{2v}\,\mathrm{Tr}\,\ln D, \qquad (76)$$

where

$$D=(\hat\Pi+\sigma)(-\hat\Pi+\sigma)=\sigma^2-\Pi_\mu\Pi^\mu-\frac{ie}{4}\Gamma^\mu\Gamma^\nu\lambda^aF^a_{\mu\nu},$$

$$F^a_{\mu\nu}=\partial_\mu A^a_\nu-\partial_\nu A^a_\mu+ef^{abc}A^b_\mu A^c_\nu,\quad \Pi_\mu=i\partial_\mu+\tfrac{1}{2}e\lambda^aA^a_\mu. \qquad (77)$$

In Eqs. (75)–(77) the field σ no longer depends on spacetime points, and f^{abc} are the $SU(3)$ structure constants. We would like to study the three-dimensional model (73) and (74) in external constant (i) chromomagnetic and (ii) chromoelectric fields. In case (i) we let, for example, $F^3_{12}=-F^3_{21}=H\neq 0$, while the other components of the tensor $F^a_{\mu\nu}$ are zero. Two independent choices of the vector field A^a_μ are possible here. In the first case A^a_μ is an Abelian-like vector potential, used in Refs. 44 and 45 to model the vacuum gluon condensate:

$$A^a_\mu=H\delta_{\mu2}x_1\delta^{a3}. \qquad (78)$$

However, we shall not dwell on (78), since here we easily obtain the same results as for the Gross–Neveu model in an external magnetic field (see the preceding section). In the second case the vector potential no longer depends on spacetime points:

$$A^a_\mu=\delta^a_\mu\sqrt{H/e}\quad (a=1,\ldots,8;\ \mu=0,1,2). \qquad (79)$$

This case differs qualitatively from (78) in that the non-Abelian nature of the Yang–Mills fields is explicitly involved. We shall use this gauge field (79) to find and study the effective potential (75) and (76) of the original model. We particularly emphasize the fact that recent theoretical studies indicate that in QCD at sufficiently high temperatures the vacuum is stable, and the gluon condensate in it most likely has the form (78). However, at fairly low temperatures the background gluon field must have non-Abelian components.[47] Therefore, the vector potential (79) is a possible candidate for the role of the gluon condensate in the true QCD vacuum at $T=0$. An external gauge field with this structure and its effect on various physical processes have already been studied.[46]

In case (ii) it is assumed that the only nonzero components of the tensor $F_{\mu\nu}$ are $F^3_{01}=-F^3_{10}=E$. Here again we shall work only with an essentially non-Abelian vector potential, which by analogy with (79) has the form

$$A^1_\mu=(\sqrt{E/e},0,0),\ A^2_\mu=(0,\sqrt{E/e},0);\ A^a_\mu=0;$$
$$a=3,\ldots,8. \qquad (80)$$

The case of an Abelian-like gauge field

$$A^a_\mu=-E\delta_{\mu0}x_1\delta^{a3}$$

is treated similarly to that of the three-dimensional Gross–Neveu model in an external electric field.[36]

Now, going to momentum space in (76), we easily find

$$\frac{1}{N}V(\sigma)=\frac{\sigma^2}{2g_0}+\frac{i}{2(2\pi)^3}\,\mathrm{Tr}\int d^3p\,\ln\tilde D(p), \qquad (81)$$

where Tr denotes the trace over both spinors and $SU(3)$ indices, and $\tilde D(p)$ is the Fourier transform of the operator D from (77). Obviously, $\tilde D(p)$ is a matrix in spinor and color space with three fourfold degenerate eigenvalues:[50,51]

$$d_i(p)=E_i^2(\bar p)-p_0^2;\ i=1,2,3, \qquad (82)$$

where $E_i(\bar p)$ is independent of p_0. [The quantities (82) for each case (i) or (ii) will be given in the corresponding subsections below.] Substituting (82) into (81) and integrating over the variable p_0, we find

$$\frac{1}{N}V(\sigma)=\frac{\sigma^2}{2g_0}-2\int\frac{d^2p}{(2\pi)^2}\sum_{i=1}^{3}E_i(\bar p). \qquad (83)$$

2.2. An external chromomagnetic field

Let us assume that the vacuum gluon condensate is specified by a vector potential of the form (79). In this case the quantities $E_i(\bar p)$ from (82) are easily calculated:[50,52]

$$E_1^2(\bar p)=\bar p^2+\sigma^2;\ \bar p^2=p_1^2+p_2^2;$$

$$E^2_{2,3}(\bar p)=\bar p^2+\sigma^2+\frac{eH}{2}\pm\frac{1}{2}\sqrt{(eH)^2+4eH\bar p^2}. \qquad (84)$$

Substituting (84) into (83), we have ($V\equiv V_H$)

$$\frac{1}{N}V_H(\sigma)=\frac{1}{N}V_0(\sigma)-\frac{|\sigma|^3}{3\pi}+\frac{(\sigma^2+eH)^{3/2}}{3\pi}-\frac{eH}{4\pi}(\sigma^2+eH)^{1/2}-\frac{\sigma^2\sqrt{eH}}{4\pi}\ln[(\sqrt{eH}+\sqrt{\sigma^2+eH})/|\sigma|], \tag{85}$$

where

$$\frac{1}{N}V_0(\sigma)=\frac{\sigma^2}{2g_0}-\frac{3\sigma^2\Lambda}{2\pi}+\frac{|\sigma|^3}{\pi} \tag{86}$$

is the effective potential of the model for $H=0$, and Λ is the ultraviolet-cutoff parameter of the integration region in (83). Owing to the symmetry of the function (85) under reflections $\sigma\to-\sigma$, in what follows we shall consider only the region $\sigma\geqslant 0$. As in Sec. 1, the function (86) can be renormalized and reduced to renormalization-invariant form:

$$\frac{1}{N}V_0(\sigma)=\frac{\sigma^2}{2g}+\frac{\sigma^3}{\pi}, \tag{87}$$

where the finite coupling constant g is independent both of the normalization point and of Λ:

$$\frac{1}{g}=\frac{1}{g_0}-\frac{3\Lambda}{\pi}.$$

The insignificant difference between the potential (87) and the analogous quantity (11) is a consequence of the additional color degrees of freedom of the fermions (quarks) in the model (73). Therefore, in the case $H=0$ it follows from (87) that for $g>0$ the absolute minimum of the function $V_0(\sigma)$ is located at zero, and the symmetry (4) is not broken. For $g<0$ the global minimum occurs at

$$\sigma_0=-\pi/3g\equiv M. \tag{88}$$

In this case the chiral invariance (4) is spontaneously broken, and the quarks acquire a mass M.

Let $H\neq 0$. We therefore need to study the function (85) at the global minimum. Its stationary points satisfy the equation

$$\sigma\{A(H)+4x-G_H(x)\}=0, \tag{89}$$

where

$$A(H)=\frac{2\pi}{f\sqrt{eH}},\quad x=\frac{\sigma}{\sqrt{eH}}$$

$$G_H(x)=\ln[(1+\sqrt{1+x^2})/x]-2\sqrt{1+x^2}. \tag{90}$$

It is easily shown that the function $G_H(x)$ decreases monotonically from $+\infty$ to $-\infty$ on the interval $(0,\infty)$. Therefore, the expression in curly brackets in (89) vanishes at a single point $x_0(H)\neq 0$, and the stationarity equation (89) has two solutions. One of them is $\sigma=0$, and the other is $\sigma_0(H)=\sqrt{eH}x_0(H)$. (If the external field were Abelian-like, then, as shown in Sec. 1, the stationarity equation would not have the solution $\sigma=0$.) Using (89), it is easy to see that the derivative $\partial V_H(\sigma)/\partial\sigma$ is negative on the interval $\sigma\in(0,\sigma_0(H))$. Consequently, the absolute minimum of the function $V_H(\sigma)$ lies at a point $\sigma_0(H)\neq 0$, and the symmetry (4) is spontaneously broken for both $g<0$ and $g>0$ for any value of the external chromomagnetic field $H\neq 0$. Therefore, an external chromomagnetic field, either Abelian-like or non-Abelian, leads to spontaneous breakdown of the chiral symmetry (4), and the fermions thus acquire a nonzero mass $\sigma_0(H)$.

Let us now study the behavior of the dynamically generated fermion mass $[\equiv\sigma_0(H)]$ at large and small values of H. Let k be a solution of the equation $4k=G_H(k)$. Then for $H\to\infty$ we see that $A(H)\to 0$, i.e., $x_0(H)\to k$. Therefore, for $H\to\infty$

$$\sigma_0(H)\equiv\sqrt{eH}x_0(H)\to k\sqrt{eH}. \tag{91}$$

The asymptote (91) is independent of the sign of the constant g. For comparison, we note that in the case with an external magnetic field (see Sec. 1.2), the fermion mass has the same qualitative behavior for $H\to\infty$.

Now let us consider small values of H. Two cases must be distinguished. Let $g>0$. Clearly, the equation satisfied by $x_0(H)$ for $H\to 0$ has the form

$$A(H)=-\ln x_0(H).$$

From this we easily find $x_0(H)$ and $\sigma_0(H)$ for $H\to 0$:

$$\sigma_0(H)\sim\sqrt{eH}\exp(-2\pi/(g\sqrt{eH})), \tag{92}$$

i.e., the fermion mass falls exponentially with decreasing H. It is interesting that in an external magnetic field the fermion mass decreases linearly, i.e., more slowly than for $H\to 0$ (see Sec. 1.2).

Let $g<0$. in this case at small H the parameter $A(H)$ from (90) tends to $-\infty$, and so the solution $x_0(H)$ of (89) tends to $+\infty$. Using the asymptote of the function $G_H(x)$ at large x in (89), it is easily shown that for $H\to 0$

$$\sigma_0(H)=M\{1+(eH)^2/(72M^4)+o((eH)^2/M^4)\}, \tag{93}$$

where M is the fermion mass for $H=0$ [see (88)].

Thus, both magnetic and chromomagnetic external fields like (78) and (79) induce spontaneous chiral symmetry breaking in four-fermion models.

2.3. An external chromoelectric field

Let us now place the original model in an external constant chromoelectric field whose vector potential has the essentially non-Abelian form (80). Now the quantities $E_{2,3}(\bar p)$ from (82) have the form[51]

$$E_{2,3}^2(\bar p)=\sigma^2+\bar p^2+eE/2\pm\sqrt{eE(\sigma^2+\bar p^2+p_1^2)}, \tag{94}$$

and $E_1(\bar p)$ and $\bar p^2$ are defined in (84). Taking this into account, the effective potential (83) can be written as $(V\equiv V_E)$

$$\frac{1}{N}V_E(\sigma)=\frac{1}{N}V_0(\sigma)+\frac{1}{3\pi}(D_+^3+D_-^3-2|\sigma|^3)$$
$$-\frac{B}{4\pi}(D_+-D_-)-\frac{3eE}{16\pi}(D_++D_-)$$
$$-\frac{1}{4\pi^2}\int_0^{2\pi}d\varphi\,\frac{4(CA-B^2)-C^2}{8\sqrt{C}}$$
$$\times\ln\left[\frac{\sqrt{C}+D_++D_-}{-\sqrt{C}+D_++D_-}\right]. \quad (95)$$

The potential $V_0(\sigma)$ is defined in (86) and (87), and

$$D_\pm=\sqrt{(A\pm B)},\quad A=\sigma^2+eE/2,$$
$$B=\sqrt{\sigma^2 eE},\quad C=eE(1+\cos^2\varphi). \quad (96)$$

We have used polar coordinates in deriving (95).

Let us now study the function (95) at the absolute minimum on the semiaxis $\sigma\geqslant 0$. It is easily shown that

$$\frac{1}{N}\frac{\partial V_E}{\partial\sigma}=\sigma\{H(\sigma)-G_E(\sigma)\}, \quad (97)$$

where

$$H(\sigma)=\frac{1}{g}+\frac{1}{\pi}(\sigma+D_++D_-),$$
$$G_E(\sigma)=\frac{1}{4\pi^2}\int_0^{2\pi}d\varphi\,\frac{C-eE}{\sqrt{C}}\ln\left[\frac{D_++D_-+\sqrt{C}}{D_++D_--\sqrt{C}}\right]. \quad (98)$$

Obviously, $H(\sigma)$ grows monotonically, and $G_E(\sigma)$ falls monotonically for $\sigma\geqslant 0$. Therefore, for $H(0)>G_E(0)$ the stationarity equation $\partial V_E/\partial\sigma=0$ will have a single solution $\sigma=0$, at which the potential $V_E(\sigma)$ reaches its smallest value. However, if $H(0)<G_E(0)$, another stationary point $\sigma_0(E)\neq 0$ appears, at which the function $V_E(\sigma)$ reaches a global minimum. Estimating $G_E(0)$ numerically, we find

$$H(0)-G_E(0)\cong\frac{1}{g}+\sqrt{eE}\cdot 0.2923\ldots. \quad (99)$$

From this we see that for positive g the inequality $H(0)>G_E(0)$ is satisfied for any value of the external chromoelectric field E. In this case the chiral invariance of the model remains unbroken, and the quarks are massless. (We recall that for $g>0$, in contrast to the chromoelectric case, arbitrarily small values of the external chromomagnetic field induce spontaneous chiral symmetry breaking.)

Let us consider the case $g<0$ in more detail. It follows from (99) that at sufficiently low E, $H(0)<G_E(0)$, i.e., the model is located in the phase with spontaneously broken chiral symmetry. However, at sufficiently high E it will be in the phase with unbroken chiral symmetry. The transition from one phase to the other occurs at $E=E_c$:

$$eE_c=(g\cdot 0.2923\ldots)^{-2}\cong M^2\cdot 10.6730\ldots. \quad (100)$$

[The critical chromoelectric field strength is found from the condition that the right-hand side of (99) vanish.] At the critical point we have a second-order phase transition.

Let us now evaluate the fermion mass for $0<E\leqslant E_c$. This can be done by using the stationarity equation. In fact, it is easy to show that for $E\to 0$

$$\sigma_0(E)=M(1-(eE)^2/(72M^4)+\ldots). \quad (101)$$

However, near the critical point $(E\to E_c)$

$$\sigma_0(E)\cong\frac{e(E_c-E)}{\sqrt{eE_c}}\cdot 0.4591\ldots. \quad (102)$$

Finally, we note that the effective potential (95) has no imaginary part. Therefore, the vacuum of the model in an external chromoelectric field (8) is a stable state. In contrast, the vacuum of the Gross–Neveu theory is unstable in an external electric field.[36]

2.4. The case $H,T\neq 0$

Let us now assume that the system described by the Lagrangian (73) is located in a heat bath. Before giving the effective potential for this case, we write the potential (81) including (82) in the more convenient form

$$\frac{1}{N}V(\sigma)=\frac{\sigma^2}{2g_0}+\frac{2i}{(2\pi)^3}\sum_{k=1}^{3}\int d^3p\,\ln d_k(p). \quad (103)$$

Performing calculations analogous to those of Sec. 1.1 for $T,\mu\neq 0$, we obtain

$$\frac{1}{N}V(\sigma)=\frac{\sigma^2}{2g_0}-2\sum_{k=1}^{3}\int\frac{d^2p}{(2\pi)^2}$$
$$\times\{E_k+T\ln[1+\exp[-\beta(E_k+\mu)]]$$
$$+T\ln[1+\exp[-\beta(E_k-\mu)]]\}, \quad (104)$$

where the relation between E_k and $d_k(p)$ is given by (82). Now, assuming that $\mu=0$ and that the field of the gluon condensate is a non-Abelian chromomagnetic field, where the $E_k(\bar{p})$ have the form (84), after trivial algebra from (104) we find the expression for the effective potential at $H,T\neq 0$ $[V(\sigma)\equiv V_{HT}(\sigma)]$:

$$\frac{1}{N}V_{HT}(\sigma)=\frac{1}{N}V_H(\sigma)-\sum_{i=1}^{3}F_i(\sigma), \quad (105)$$

where

$$F_i(\sigma)=\frac{T}{\pi}\int_0^\infty dx\,\ln[1+\exp[-\beta E_i(x)]], \quad (106)$$

$x=\bar{p}^2$, and the function $V_H(\sigma)$ is defined in (85). Since the entire dependence of $E_k(\bar{p})$ on the momentum components reduces to a dependence on the combination $\bar{p}^2$ [see (84)], in (106) we have used the notation $E_k(x)\equiv E_k(\bar{p})|_{\bar{p}^2=x}$. The stationarity equation for the function (105) has the form

$$\sigma\{\omega(\sigma)-\varphi(\sigma)\}=0, \quad (107)$$

where

$$\omega(\sigma)=\frac{1}{g}+\frac{2\sigma}{\pi}+\frac{1}{\pi}\sqrt{\sigma^2+eH}+\frac{2T}{\pi}\{2\ln(1+\exp[-\beta\sigma])$$
$$+\ln(1+\exp[-\beta\sqrt{\sigma^2+eH}])\}, \qquad (108)$$

$$\varphi(\sigma)=\frac{\sqrt{eH}}{2\pi}\int_{\sigma}^{\sqrt{\sigma^2+eH}}\frac{dE}{\sqrt{E^2-\sigma^2}}\tanh\left(\frac{E}{2T}\right). \qquad (109)$$

It is obvious that $\sigma=0$ is a solution of (107). Some properties of the functions $\omega(\sigma)$ and $\varphi(\sigma)$ are given in Appendix C, where it is shown that $\omega'(\sigma)>\varphi'(\sigma)$ on the interval $(0,\infty)$. There it is also shown that for $\sigma\to\infty$, $\omega(\sigma)\sim 4\sigma/\pi$ and $\varphi(\sigma)\to 0$. It then follows that if $\omega(0)<\varphi(0)$, the expression in curly brackets in (108) vanishes at a single point $\sigma_0\neq 0$, where the function $V_{HT}(\sigma)$ has a global minimum. Therefore, in this case the chiral symmetry of the model is spontaneously broken, and the fermions dynamically acquire a mass σ_0. If $\omega(0)>\varphi(0)$, the fermions are massless.

We thus see that in the plane of the parameters (T,H) the equation

$$\omega(0)=\varphi(0) \qquad (110)$$

determines the critical curve l_c separating the set of parameters (T,H) corresponding to the symmetric ground state from the region where the symmetry (4) is spontaneously broken. Obviously, when the critical curve is crossed a second-order phase transition occurs (a situation analogous to the case of an external magnetic field; see Sec. 1.3). The critical curve can be written as

$$l_c=\{(T,H):T=T_c(H);T,H\geqslant 0\},$$

where the function $T_c(H)$ is called the critical temperature.

To study this function, it is convenient to use the variables

$$t=\frac{\sqrt{eH}}{2T},\quad h=\frac{2\pi}{g\sqrt{eH}}, \qquad (111)$$

in terms of which Eq. (110) has the form

$$h\equiv h(t)=\int_0^t d\tau\,\frac{\tanh\tau}{\tau}$$
$$-2-\frac{2}{t}[2\ln 2+\ln(1+\exp(-2t))]. \qquad (112)$$

The advantage of the parameters (t,h) is that (112) explicitly gives the functional dependence between them. We see from (112) that $h(t)$ is a function which grows monotonically from $-\infty$ to $+\infty$ as t varies from 0 to $+\infty$. Therefore, there exists a $t_0>0$ where $h(t)$ vanishes. The transformation (111) obviously takes the critical curve l_c into the set of points

$$L_+=\{(t,h):h=h(t),\ t>t_0\}$$

if $g>0$, and into the set of points

$$L_-=\{(t,h):h=h(t),\ 0<t<t_0\}$$

if $g<0$. Here values $t\sim t_0$ on the curves $L_\pm$ correspond to large values of the chromomagnetic field. The limits $t\to\infty$ on the curve $L_+(g>0)$ and $t\to 0$ on the curve $L_-(g<0)$ correspond to small values of the external field H. Knowing the properties of the function $h(t)$, it is not difficult to find the behavior of the critical temperature $T_c(H)$ at large and small H.

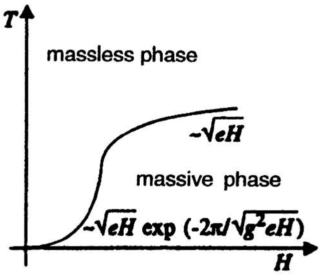

FIG. 4. Phase portrait of the model (73) for $g>0$ in an external non-Abelian chromomagnetic field.

In fact, the limit $t\to t_0$, when $h(t)\to 0$, gives [see (111)] $H\to\infty$ and

$$T_c(H)\sim\sqrt{eH} \qquad (113)$$

for both positive and negative values of the coupling constant g.

Let $g>0$ and $t\to\infty$. This part of the curve L_+ corresponds to small H. It is easy to show that for $t\to\infty$, $h(t)\sim\ln t$. Expressing h and t in this relation in terms of T and H [see (111)], for $H\to 0$ we find

$$T_c(H)\sim\sqrt{eH}\exp(-2\pi/(g\sqrt{eH})). \qquad (114)$$

The phase portrait of the model for $g>0$ is given in Fig. 4.

Now let $g<0$ and $t\to 0$. In this case we also obtain the part of the curve L_- corresponding to small H. Expanding the function (112) in a series in the small parameter t and using (111), for $H\to 0$ we easily find

$$T_c(H)=T_c+(eH)^2/[T_c^3 1728\ln 2]+o((eH)^2), \qquad (115)$$

where T_c is the critical temperature at $H=0$:

$$T_c=-\pi/[6g\ln 2].$$

For $g<0$, $T_c(H)$ has the same qualitative form as the curve shown in Fig. 3.

2.5. The case $H,\mu\neq 0$

Here we study the phase structure of the three-dimensional Gross–Neveu model, taking into account the external chromomagnetic field and the chemical potential μ, but at $T=0$. In this case from (104) we easily find the expression for the effective potential of the model $[V(\sigma)\equiv V_{H\mu}(\sigma)]$:

$$V_{H\mu}(\sigma)=V_H(\sigma)-\frac{N\Theta(\mu-\sigma)}{12\pi}B_1(\sigma)$$
$$-\frac{N\Theta(\mu^2-eH-\sigma^2)}{12\pi}B_2(\sigma). \qquad (116)$$

Here $\Theta(x)$ is the Heaviside step function, and

$$B_1(\sigma)=4\mu^3+8\sigma^3-12\sigma^2\mu+3\mu\sqrt{eH(\mu^2-\sigma^2)}$$
$$-3\sigma^2\sqrt{eH}\ln[(\mu+\sqrt{\mu^2-\sigma^2})/\sigma],$$

$$B_2(\sigma)=2\mu^3-6\mu\sigma^2+4(\sigma^2+eH)^{3/2}-3\mu\sqrt{eH(\mu^2-\sigma^2)}-3eH\sqrt{\sigma^2+eH}+3\sigma^2\sqrt{eH}\ln\left[\frac{\mu+\sqrt{\mu^2-\sigma^2}}{\sqrt{eH}+\sqrt{\sigma^2+eH}}\right]. \quad (117)$$

The functions $B_i(\sigma)$ possess the property (for $\mu^2\geqslant eH$)

$$B_1(\mu)=B_2(\sqrt{\mu^2-eH})=0. \quad (118)$$

Now from (116) we easily obtain the stationarity equation:

$$\frac{\partial V_{H\mu}(\sigma)}{\partial\sigma}=0=\frac{\partial V_H(\sigma)}{\partial\sigma}-\frac{N\Theta(\mu-\sigma)}{12\pi}b_1(\sigma)-\frac{N\Theta(\mu^2-eH-\sigma^2)}{12\pi}b_2(\sigma), \quad (119)$$

where $b_i(\sigma)=\partial B_i(\sigma)/\partial\sigma$. We note that in deriving (119) we have neglected terms proportional to $\delta(\mu-\sigma)$ and $\delta(\mu^2-\sigma^2-eH)$. They obviously vanish, owing to (118).

Let $H\neq 0$. In Sec. 2.2 it was shown that for $\mu=0$ the effective potential $V_H(\sigma)$ must have a global minimum at a point $\sigma_0(H)\neq 0$, i.e., the chiral invariance of the model is spontaneously broken by the external chromomagnetic fields. Let $0<\mu\ll\sigma_0(H)$. Using (116), it is easy to see that here $V_{H\mu}=V_H+\Delta V$, where $\Delta V=O(\mu^2)$. Therefore, for sufficiently small μ the global minimum of the effective potential will still be located at the point $\sigma_0(H)$. We shall show that for sufficiently large μ the chiral invariance of the model is restored. Let $\mu\gg\sqrt{eH}$, $|g|$. We divide the semiaxis $\sigma\geqslant 0$, where Eq. (119) must be solved, into three segments.

(1) Let $\sigma\geqslant\mu$. In this case (119) has the form (89), i.e., $\partial V_H/\partial\sigma=0$. However, it is easily shown that for sufficiently large μ the points $\sigma=0$ and $\sigma_0(H)$, which formally are solutions of (89), do not satisfy the condition $\sigma\geqslant\mu$, and so they are not roots of (119).

(2) Let $\sqrt{\mu^2-eH}\leqslant\sigma\leqslant\mu$. Then the stationarity equation for the potential $V_{H\mu}$ has the form

$$\sigma\left\{\frac{2\pi}{g}+4\mu+\sqrt{eH}\ln\left[\frac{\mu+\sqrt{\mu^2-\sigma^2}}{\sqrt{eH}+\sqrt{\sigma^2+eH}}\right]\right\}=0.$$

It is easily shown that for sufficiently large μ this equation has no solutions satisfying the condition $\sqrt{\mu^2-eH}\leqslant\sigma\leqslant\mu$.

(3) Let $0\leqslant\sigma\leqslant\sqrt{\mu^2-eH}$. In this case the stationarity equation (119) becomes

$$\sigma\{3\mu+\pi/g\}=0.$$

For sufficiently large μ it will have the single solution $\sigma=0$ satisfying the condition $0\leqslant\sigma\leqslant\sqrt{\mu^2-eH}$.

We have therefore shown that for sufficiently large μ, out of the entire set $\sigma\geqslant 0$ only the point $\sigma=0$ is a solution of (119), and at this point the potential (116) will have an absolute minimum. This implies that at fixed H the chiral symmetry (4) of the model, which is spontaneously broken at small μ, is restored at sufficiently large values of the chemical potential.

2.6. Discussion of the results

As shown in the preceding section, in the Abelian theory the catalysis effect arises from the interaction between the fermion spins and the magnetic field. In the non-Abelian theory the situation is more complicated because the particles possess an isospin degree of freedom. As a result, for an Abelian-like chromomagnetic field configuration we essentially arrive at the situation discussed earlier in Sec. 1, because in classical language the motions in isospin space and in ordinary space are separate. These motions are not distinct for non-Abelian field configurations, and the particle spectrum is affected not only by the particle spin and isospin, but also by their relative orientation. It is this situation which we have discussed in the present section, based on Refs. 50–52.

Using this result, we have studied the effect of the vacuum gluon condensate on the phase structure of the three-dimensional Gross–Neveu theory (a four-dimensional model of this type essentially describes the quark dynamics at low energies). In four-fermion models the gluon condensate is not a dynamical quantity, but rather is a sort of external parameter which is usually identified with an external color field of, for example, the form (78)–(80), acting on the system. The effect of Abelian-like color fields like (78) on the four-fermion model (73) leads to the same results as for an external magnetic field on the ordinary GN model, in which fermions do not have color degrees of freedom. We have therefore focused on the study of the critical properties of the theory (73) in the background of external non-Abelian fields like (79) and (80). In the true QCD vacuum the gluon condensate at low temperatures must be essentially non-Abelian,[47] and the field (79) [or (80)] is a good candidate for this role.[49] The results of our analysis are the following.

Let an external chromomagnetic field (79) act on the original theory (73). In this case the model is exactly solvable for $N\to\infty$. This implies that in the leading order of the $1/N$ expansion the effective potential is a superposition of elementary functions. This is a unique example of a field-theoretic model in which the effect of an external field can be taken into account exactly (perturbation theory in the external field is usually used, and so there are restrictions on the region where the calculated results are valid), and the response of the system to a chromomagnetic field can be studied in a wide range of field strengths.

We have shown that a non-Abelian (and also an Abelian-like) chromomagnetic field for $g>0$ induces the spontaneous breakdown of chiral invariance (4), while for $g<0$ it stabilizes the unsymmetric vacuum even more. Therefore, a purely chromomagnetic gluon condensate of the form (78) and (79) can serve as a catalyst for the spontaneous breakdown of chiral symmetry in quantum chromodynamics. In Sec. 2.2 we found the behavior of the fermion mass for $H\to\infty$ [see (91)], and also for small H [see (92) for $g>0$ and (93) for $g<0$].

When an external chromoelectric field E of the form (80) acts on the system, the chiral invariance remains unbroken for $g>0$. Let us take the coupling $g<0$. Then the chiral symmetry spontaneously broken at low E is restored at sufficiently high E. We have found the critical value of the chromoelectric field E_c (100) at which a second-order phase transition from the ordered to the disordered phase occurs. In addition, we have found the behavior of the quark mass for $E \to 0$, and also near the critical value of the external field (see Sec. 2.3). It is important to stress the following feature of the non-Abelian chromoelectric condensate. In this case the ground state of the Gross–Neveu theory is stable (in an external Abelian-like chromoelectric field the effective potential of the model has an imaginary part,[36] which indicates that the vacuum is unstable). In our opinion, this can be viewed as evidence for the possible existence of nonzero chromoelectric components of the gluon condensate in QCD.

Now, in addition to the external non-Abelian chromomagnetic field, let the temperature be nonzero. As shown in Sec. 2.4, there is a critical temperature $T_c(H)$ at which a second-order phase transition from the chirally noninvariant to the chirally invariant state of thermodynamical equilibrium occurs. For $H \to \infty$ the critical temperature is proportional to $\sqrt{eH}$, and for $H \to 0$ the behavior of $T_c(H)$ is given by (114) for $g>0$ and by (115) for $g<0$.

Finally, we have considered the case where H and the chemical potential μ are nonzero. We have succeeded in showing that the chiral invariance (4) of the model, spontaneously broken at small μ, is restored at sufficiently large μ.

The effect of external gauge fields on four-fermion models has also been studied in real four-dimensional spacetime.[53,54] The authors of Ref. 53 restricted themselves to Abelian-like vector potentials of the type (78), and, of course, obtained considerably fewer results than in the three-dimensional case. In Ref. 54 it was shown that various types of external chromomagnetic fields in the Nambu–Jona-Lasinio model also lead to spontaneous breakdown of the chiral symmetry.

3. DYNAMICAL GENERATION OF A CHERN–SIMONS TERM IN GENERALIZED FOUR-FERMION MODELS

In the two preceding sections we have remarked upon the importance of studying three-dimensional models with four-fermion interaction in connection with the possibility of using them directly to describe high-temperature superconductivity (HTSC). However, there is another approach to constructing a theory of HTSC based on the use of particles with fractional spin and statistics, as even an ideal gas of such particles possesses superconducting properties.[55] In turn, it is known that fractional statistics occurs only for three-dimensional fields whose dynamics is determined by Lagrangians containing the so-called Chern–Simons (CS) term:

$$L_{CS} = G\varepsilon^{\mu\nu\lambda}\partial_\mu A_\nu A_\lambda, \tag{120}$$

where A_μ is a vector field (see, for example, Ref. 56). It is therefore important to understand the mechanisms by which CS terms arise in a theory. Of course, one can simply add L_{CS} to the original Lagrangian of a theory, but physicists are always interested in obtaining a quantity dynamically, i.e., through radiative corrections, because then physical phenomena can be described with fewer parameters.

Since the Lagrangian (120) is not invariant under P-parity transformations $(t,x,y) \to (t,-x,y)$, one way of obtaining the CS term is by dynamical breaking of P invariance. As noted in Ref. 57, in massless three-dimensional quantum electrodynamics it is not so easy to break P parity. Therefore, a great deal of work has been done on Abelian gauge theories with scalar fields,[58] where it has been shown that the CS term can arise spontaneously. In addition, several authors[16,59,60] have concluded that in very simple three-dimensional models with four-fermion interaction it is also possible to have dynamical P-parity violation which ultimately leads to generation of a CS term.

In this section we continue our study of three-dimensional field theories, using the nonperturbative $1/N$ expansion. Now we consider two models with generalized four-fermion interaction, paying special attention to the conditions sufficient for dynamical generation of a CS term.

3.1. Generalization of the Gross–Neveu model

Let us consider the Lagrangian

$$\begin{aligned} L = {} & \bar\psi_1 i\hat\partial\psi_1 + \bar\psi_2 i\hat\partial\psi_2 + \frac{\tilde G_1}{2N}[(\bar\psi_1\psi_1)^2 + (\bar\psi_2\psi_2)^2] \\ & + \frac{\tilde G_2}{N}\bar\psi_1\psi_1\bar\psi_2\psi_2 + \frac{\tilde H_1}{2N}[(\bar\psi_1\psi_2)^2 + (\bar\psi_2\psi_1)^2] \\ & + \frac{\tilde H_2}{N}\bar\psi_1\psi_2\bar\psi_2\psi_1. \end{aligned} \tag{121}$$

Here $\psi_{1,2}$ are two fundamental multiplets of the $U(N)$ group, each component of which is a two-component Dirac spinor. For simplicity, in (121) we have omitted the sum over $U(N)$ group indices, and so expressions like $\bar\psi_i\psi_j$ are to be understood as

$$\bar\psi_i\psi_j = \sum_{k=1}^{N}\bar\psi_{ik}\psi_{jk}.$$

The 2×2 γ matrices of Appendix A are used in (121). This Lagrangian is invariant under the continuous $U(1)$ gauge group:

$$U(1): \psi_1 \to e^{i\alpha}\psi_1;\quad \psi_2 \to e^{i\alpha}\psi_2, \tag{122}$$

and also under P-parity transformations and the two discrete transformations Γ^3 and Γ^5:

$$\begin{aligned} & P: \psi_{1k}(t,x,y) \leftrightarrow \gamma^1\psi_{2k}(t,-x,y), \\ & \Gamma^3: \psi_{1k}(t,x,y) \leftrightarrow \psi_{2k}(t,x,y), \\ & \Gamma^5: \psi_{1k}(t,x,y) \leftrightarrow i\psi_{2k}(t,x,y). \end{aligned} \tag{123}$$

Other notation is more convenient when studying the field theory (121). Let us construct from ψ_1 and ψ_2 another fundamental multiplet ψ of the $U(N)$ group, in which each component ψ_k will now be a four-component Dirac spinor (see Sec. 1): $\psi_k=\binom{\psi_{1k}}{\psi_{2k}}$, $k=1,\ldots,N$. In terms of ψ, the Lagrangian (121) can be written as

$$L=\bar{\psi}i\hat{\partial}\psi+\frac{G_1}{2N}(\bar{\psi}\psi)^2+\frac{G_2}{2N}(\bar{\psi}\tau\psi)^2+\frac{H_1}{2N}(i\bar{\psi}\Gamma^5\psi)^2$$
$$+\frac{H_2}{2N}(i\bar{\psi}\Gamma^3\psi)^2. \qquad (124)$$

Here we have dropped the summation over $U(N)$ group indices. However, here and below this summation is always understood. In contrast to (121), the Lagrangian (124) contains the 4×4 matrices Γ^μ, Γ^3, and Γ^5, which are given in Appendix A, and the 4×4 matrix τ has the form $\tau=\begin{pmatrix}1&0\\0&-1\end{pmatrix}$. In addition,

$$\tilde{G}_1=G_1+G_2;\quad \tilde{G}_2=G_2-G_1;$$
$$\tilde{H}_1=H_1-H_2;\quad \tilde{H}_2=H_1+H_2.$$

The discrete transformations (123) are easily rewritten using four-component spinors:

$$P:\psi(t,x,y)\to i\Gamma^1\Gamma^5\psi(t,-x,y);$$
$$\Gamma^5:\psi\to\Gamma^5\psi;\quad \Gamma^3:\psi\to\Gamma^3\psi.$$

To study the phase structure of the model (121)–(124), we introduce the auxiliary Lagrangian

$$\tilde{L}=\bar{\psi}_i\hat{\partial}\psi+\sigma_1(\bar{\psi}\psi)+\sigma_2(\bar{\psi}\tau\psi)+\phi_1(i\bar{\psi}\Gamma^5\psi)$$
$$+\phi_2(i\bar{\psi}\Gamma^3\psi)-\frac{N}{2}\sum_{k=1}^{2}\left(\frac{\sigma_k^2}{G_k}+\frac{\phi_k^2}{H_k}\right), \qquad (125)$$

where σ_i and ϕ_k are auxiliary real scalar and pseudoscalar fields, respectively. The field theories (124) and (125) are equivalent because the equations of motion can be used to eliminate the fields σ_i and ϕ_k from (125), which gives the Lagrangian (124). It is easily shown that the auxiliary fields transform as follows under the discrete symmetries (123):

$$P:\sigma_1\to\sigma_1;\quad \sigma_2\to-\sigma_2;\quad \phi_1\to-\phi_1;\quad \phi_2\to\phi_2;$$
$$\Gamma^5:\sigma_1\to-\sigma_1;\quad \sigma_2\to\sigma_2;\quad \phi_1\to-\phi_1;\quad \phi_2\to\phi_2;$$
$$\Gamma^3:\sigma_1\to-\sigma_1;\quad \sigma_2\to\sigma_2;\quad \phi_1\to\phi_1;\quad \phi_2\to-\phi_2. \qquad (126)$$

Using the Lagrangian (125), we can find the effective action of the theory, which in the one-loop approximation [equivalent to the leading order of the $1/N$ expansion in the model (124)] has the form

$$S_{\text{eff}}(\sigma,\phi)=-N\sum_{k=1}^{2}\left(\frac{\sigma_k^2}{2G_k}+\frac{\phi_k^2}{2H_k}\right)-i\,\mathrm{Tr}\ln\hat{\Delta}, \qquad (127)$$

where

$$\hat{\Delta}=i\hat{\partial}+\sigma_1+\sigma_2\tau+i\phi_1\Gamma^5+i\phi_2\Gamma^3.$$

Here the fields σ_i and ϕ_k depend on spacetime points. To obtain the effective potential of the model, we must use the definition

$$V(\sigma,\phi)=-S_{\text{eff}}(\sigma,\phi)|_{\sigma,\phi=\text{const}}, \qquad (128)$$

where it is assumed that the boson fields no longer depend on the coordinates. Equations (127) and (128) can be used to find the following expression for the effective potential of the model (124) in the leading order of the $1/N$ expansion:

$$V(\sigma,\phi)=N\sum_{k=1}^{2}\left(\frac{\sigma_k^2}{2G_k}+\frac{\phi_k^2}{2H_k}-\int\frac{d^3p}{(2\pi)^3}\ln(p^2+M_k^2)\right), \qquad (129)$$

where

$$M_{1,2}=|\sigma_2\pm\sqrt{\sigma_1^2+\phi_1^2+\phi_2^2}|. \qquad (130)$$

Equation (129) is derived like the effective potential in Ref. 61. Integrating over the region $0\leqslant p^2\leqslant\Lambda^2$ in (129), we find

$$V(\sigma,\phi)=N\sum_{k=1}^{2}\left[\frac{\sigma_k^2}{2}\left(\frac{1}{G_k}-\frac{2\Lambda}{\pi^2}\right)+\frac{\phi_k^2}{2}\left(\frac{1}{H_k}-\frac{2\Lambda}{\pi^2}\right)+\frac{M_k^3}{6\pi}\right]. \qquad (131)$$

Now, to eliminate the cutoff parameter Λ from (131), we introduce renormalized coupling constants, using the normalization conditions ($i=1,2$)

$$\frac{1}{N}\frac{\partial^2V}{(\partial\sigma_i)^2}\bigg|_{\sigma_i=m}=\frac{1}{G_i}-\frac{2\Lambda}{\pi^2}+\frac{2m}{\pi}\equiv\frac{1}{g_i(m)},$$
$$\frac{1}{N}\frac{\partial^2V}{(\partial\phi_i)^2}\bigg|_{\phi_i=m}=\frac{1}{H_i}-\frac{2\Lambda}{\pi^2}+\frac{2m}{\pi}\equiv\frac{1}{h_i(m)}. \qquad (132)$$

Here it should be noted that in the first expression in (132) we use the value of the function $\partial^2V/(\partial\sigma_i)^2$ at the point $\phi_{1,2}=0$, $\sigma_i=m$, $\sigma_j=0$ $(i\neq j)$. In the second expression only the component $\phi_i=m$ is nonzero at the normalization point of the function $\partial^2V/(\partial\phi_i)^2$. The equations (132) allow us to rewrite the effective potential in terms of ultraviolet-finite quantities:

$$V(\sigma,\phi)=N\sum_{k=1}^{2}\left[\frac{g_k}{2}\sigma_k^2+\frac{h_k}{2}\phi_k^2+\frac{M_k^3}{6\pi}\right], \qquad (133)$$

where

$$g_i=\frac{1}{g_i(m)}-\frac{2m}{\pi},\quad h_j=\frac{1}{h_j(m)}-\frac{2m}{\pi}. \qquad (134)$$

We recall that the bare coupling constants G_i and H_j are independent of the normalization mass m. It therefore follows from (132)–(134) that the constants g_i and h_j also are independent of m, i.e., the effective potential (133) is a renormalization-invariant quantity.

We have thus actually shown that the model (121) is renormalizable in the leading order of the $1/N$ expansion. The complete proof that the field theory (121) is renormalizable is not the subject of this discussion. However, we think that this is true, on the basis of Refs. 7 and 62, where it was shown that very simple three-dimensional models with four-fermion interaction are renormalizable within the nonperturbative $1/N$ expansion.

3.2. The phase structure of the model

Here we shall study the dependence of the phase structure of the model on the renormalization-invariant constants g_i and h_j (134). For this it is necessary to find the vacuum expectation values of the auxiliary fields, which, as is well known, are determined from the global minimum of the effective potential. For this we write down the stationarity equations for the function (133):

$$\frac{\partial V}{\partial \sigma_1}=N\sigma_1\left[g_1+\frac{M_1+M_2}{2\pi}+\frac{2\sigma_2^2}{\pi(M_1+M_2)}\right]=0,$$

$$\frac{\partial V}{\partial \sigma_2}=N\sigma_2\left[g_2+\frac{M_1+M_2}{2\pi}+\frac{2(\sigma_1^2+\phi_1^2+\phi_2^2)}{\pi(M_1+M_2)}\right]=0,$$

$$\frac{\partial V}{\partial \phi_i}=N\phi_i\left[h_i+\frac{M_1+M_2}{2\pi}+\frac{2\sigma_2^2}{\pi(M_1+M_2)}\right]=0. \qquad (135)$$

In the last equation $i=1,2$. We will need to find all the solutions of the system (135) and use those at which the potential is a minimum. This will give the vacuum expectation values $\langle\sigma_i\rangle$ and $\langle\phi_j\rangle$. In order not to encumber the text with tedious calculations, we shall not present the detailed solution of the system here, as we have studied similar equations in Ref. 61. We therefore immediately write down the point at which the potential is a global minimum, its symmetry properties, and its dependence on the constants g_i and h_j.

We divide the coupling-constant plane (h_1,h_2) into three regions:

$$I_1=\{h_1,h_2>0\},\quad I_2=\{h_1>h_2,h_2<0\},$$
$$I_3=\{h_1<h_2,h_1<0\}, \qquad (136)$$

and henceforth use the notation $\langle\sigma_i\rangle\equiv\sigma_i$ and $\langle\phi_j\rangle\equiv\phi_j$.

Let $(h_1,h_2)\in I_1$. In this case the coupling constants (g_1,g_2) can be located in one of the three regions

$$a_1=\{g_1,g_2>0\},\quad b_1=\{g_2<0,\ g_2<g_1\},$$
$$c_1=\{g_1<0,\ g_2>g_1\}, \qquad (137)$$

where the vacuum expectation values have the form

$$(h_1,h_2)\in I_1,\quad (g_1,g_2)\in a_1:\sigma_1=\sigma_2=\phi_1=\phi_2=0,$$
$$(h_1,h_2)\in I_1,\quad (g_1,g_2)\in b_1:\sigma_1=\phi_{1,2}=0,\sigma_2=-\pi g_2,$$
$$(h_1,h_2)\in I_1,\quad (g_1,g_2)\in c_1:\sigma_2=\phi_{1,2}=0,\sigma_1=-\pi g_1.$$

Let $(h_1,h_2)\in I_2$. The plane of the parameters (g_1,g_2) is divided into three regions:

$$a_2=\{g_1,g_2>h_2\},\quad b_2=\{g_2<h_2,g_2<g_1\},$$
$$c_2=\{g_1<h_2,g_2>g_1\}, \qquad (138)$$

in which the vacuum expectation values of the auxiliary fields have the form

$$(h_1,h_2)\in I_2,\quad (g_1,g_2)\in a_2:\sigma_1=\sigma_2=\phi_1=0,$$
$$\phi_2=-\pi h_2,$$
$$(h_1,h_2)\in I_2,\quad (g_1,g_2)\in b_2:\sigma_1=\phi_1=\phi_2=0,$$
$$\sigma_2=-\pi g_2,$$
$$(h_1,h_2)\in I_2,\quad (g_1,g_2)\in c_2:\sigma_2=\phi_1=\phi_2=0,$$
$$\sigma_1=-\pi g_1. \qquad (139)$$

Let $(h_1,h_2)\in I_3$. In this case the coupling constants (g_1,g_2) must be located in one of the three regions

$$a_3=\{g_1,g_2>h_1\},\quad b_3=\{g_2<h_1,g_2<g_1\},$$
$$c_3=\{g_1<h_1,g_2>g_1\}, \qquad (140)$$

in which the absolute minimum of the effective potential occurs at the points

$$(h_1,h_2)\in I_3,\quad (g_1,g_2)\in a_3:\sigma_1=\sigma_2=\phi_2=0,$$
$$\phi_1=-\pi h_1, \qquad (141)$$
$$(h_1,h_2)\in I_3,\quad (g_1,g_2)\in b_3:\sigma_1=\phi_1=\phi_2=0,$$
$$\sigma_2=-\pi g_2, \qquad (142)$$
$$(h_1,h_2)\in I_3,\quad (g_1,g_2)\in c_3:\sigma_2=\phi_1=\phi_2=0,$$
$$\sigma_1=-\pi g_1. \qquad (143)$$

Therefore, for any fixed values of the coupling constants $g_{1,2}$ and $h_{1,2}$ we can give the point at which the effective potential has a global minimum, and also the symmetry properties of the ground state of the original theory. For this it is necessary to find the transformations from (122) and (123) under which this point is invariant.

Now let us introduce new notation. Let A_i be the set $(i=1,2,3)$ of values of the coupling constants g_k and h_l such that $(g_1,g_2)\in a_i$ and $(h_1,h_2)\in I_i$. Here the I_i are given in (136), and the factors a_i are easily found from (137), (138), and (140), i.e.,

$$A_i=\{g_1,g_2,h_1,h_2:(g_1,g_2)\in a_i,\ (h_1,h_2)\in I_i\}. \qquad (144)$$

Similarly, we define the following parameter sets $(i=1,2,3)$:

$$B_i=\{g_1,g_2,h_1,h_2:(g_1,g_2)\in b_i,\ (h_1,h_2)\in I_i\}, \qquad (145)$$

$$C_i=\{g_1,g_2,h_1,h_2:(g_1,g_2)\in c_i,\ (h_1,h_2)\in I_i\}. \qquad (146)$$

Finally, we introduce the two sets

$$B=\bigcup_{i=1}^{3}B_i,\quad C=\bigcup_{i=1}^{3}C_i. \qquad (147)$$

It follows from this discussion that the generalized four-fermion field theory (121) describes a system which can be in five different phase states, where the gauge symmetry (122) is unbroken in all of them.

(1) If the coupling constants are located in the region A_1, the original symmetry is unbroken, and the vacuum expectation values of the auxiliary fields are zero.

(2) In the coupling-constant region B the vacuum expectation values have the form (142). Using (126), it is easily shown that here we have a phase with only P-parity violation.

(3) Let us assume that we are in region C (147). Here the vacuum is P-even, but the Γ^5 and Γ^3 symmetries are spontaneously broken. The vacuum expectation values of the fields have the form (143).

(4) We assume that the coupling constants belong to the set A_2. In this case the auxiliary fields have vacuum expectation values of the form (139), and the ground state of the theory is invariant under P and Γ^5. However, the Γ^3 symmetry is spontaneously broken.

(5) Finally, let us consider the case where the coupling constants are located in region A_3. Here we have the phase in which both P parity and the chiral symmetry Γ^5 are broken, and the vacuum expectation values are given in (141). This is one of the most interesting results of this section, because this phase is absent in all the simplest four-fermion models.[7,16,60]

3.3. The mass spectrum of the model

Since in the model some symmetries are spontaneously broken, and some of the auxiliary fields acquire nonzero vacuum expectation values, the fermions become massive. It is easiest to see this if in (125) we make the shift $\sigma_i \to \sigma_i + \langle\sigma_i\rangle$, $\phi_j \to \phi_j + \langle\phi_j\rangle$, because this gives rise to a fermion mass term in the Lagrangian. Depending on which phase the theory is in, the fermion mass term has the form

$$\begin{aligned} &A_2: -\pi h_2(i\bar\psi\Gamma^3\psi) = -i\pi h_2(\bar\psi_1\psi_2 - \bar\psi_2\psi_1),\\ &A_3: -\pi h_1(i\bar\psi\Gamma^5\psi) = \pi h_1(\bar\psi_1\psi_2 + \bar\psi_2\psi_1),\\ &B: \ -\pi g_2(\bar\psi\tau\psi) = -\pi g_2(\bar\psi_1\psi_1 + \bar\psi_2\psi_2),\\ &C: \ -\pi g_1(\bar\psi\psi) = -\pi g_1(\bar\psi_1\psi_1 - \bar\psi_2\psi_2), \end{aligned} \tag{148}$$

while in phase A_1 the fermions are obviously massless. The value of the mass of the fermionic fields in each of these phases is easily found from (148).

To find the boson mass spectrum in the leading order of the $1/N$ expansion, let us consider the effective action (127). It is known that this is the generating functional of one-particle- irreducible (1PI) Green functions of the bosonic fields. Therefore, to find the 1PI Green function of, for example, two σ_1 fields, it is necessary to differentiate $S_{\rm eff}$ twice with respect to the field σ_1, and then equate all the fields in the resulting expression to their vacuum expectation values.

Let us consider phase C, in which $\langle\sigma_1\rangle \neq 0$, while the vacuum expectation values of the other fields are zero. (To avoid introducing additional mathematical symbols, we shall refer to each phase by the symbol for the region of coupling constants corresponding to it.) In this case

$$\begin{aligned} \Gamma_{\sigma_1\sigma_1}(x,y) &\equiv \frac{\delta^2 S_{\rm eff}(\sigma,\phi)}{N\delta\sigma_1(x)\delta\sigma_1(y)}\\ &= -i\,{\rm Tr}[\hat\Delta_{xy}^{-1}\hat\Delta_{yx}^{-1}] + \frac{\delta(x-y)}{G_1}, \end{aligned} \tag{149}$$

where

$$(\hat\Delta^{-1})_{xy}^{\alpha\beta} = \int \frac{d^3p}{(2\pi)^3}\left[\frac{\hat p + \langle\sigma_1\rangle}{\langle\sigma_1\rangle^2 - p^2}\right]^{\alpha\beta} \exp(ip(x-y)). \tag{150}$$

The symbol Tr denotes the trace over spinor indices. It can be shown that the Fourier transform of the function (149) has the form

$$\Gamma_{\sigma_1\sigma_1}(p) = \frac{4\langle\sigma_1\rangle^2 - p^2}{2\pi\sqrt{-p^2}}\Gamma(p),\quad \Gamma(p) = \tan^{-1}\left[\frac{\sqrt{-p^2}}{2\langle\sigma_1\rangle}\right]. \tag{151}$$

Here, as in (150), momentum space has the Minkowski metric. Similarly, it can be shown that in this phase

$$\Gamma_{\sigma_2\sigma_2}(x,y) = \frac{\delta(x-y)}{G_2} - i\,{\rm Tr}[\tau\hat\Delta_{xy}^{-1}\tau\hat\Delta_{yx}^{-1}],$$

$$\Gamma_{\phi_2\phi_2}(x,y) = \frac{\delta(x-y)}{H_2} + i\,{\rm Tr}[\Gamma^3\hat\Delta_{xy}^{-1}\Gamma^3\hat\Delta_{yx}^{-1}],$$

$$\Gamma_{\phi_1\phi_1}(x,y) = \frac{\delta(x-y)}{H_1} + i\,{\rm Tr}[\Gamma^5\hat\Delta_{xy}^{-1}\Gamma^5\hat\Delta_{yx}^{-1}],$$

which corresponds to the following momentum representation of these functions ($k = 1,2$):

$$\Gamma_{\sigma_2\sigma_2}(p) = g_2 - g_1 + \frac{4\langle\sigma_1\rangle^2 - p^2}{2\pi\sqrt{-p^2}}\Gamma(p), \tag{152}$$

$$\Gamma_{\phi_k\phi_k}(p) = h_k - g_1 + \frac{\sqrt{-p^2}}{2\pi}\Gamma(p). \tag{153}$$

We note that here, and also in the other phases of the original model, mixed 1PI Green functions of two fields ($\Gamma_{\sigma_1\sigma_2}$, $\Gamma_{\sigma_1\phi_1}$, and so on) are zero. The functions inverse to (151)–(153) are the propagators of the corresponding bosonic fields. It is well known that the singularities of the propagators in the variable p^2 determine the boson mass spectrum. The analytic properties of functions like (151)–(153) have been studied in Ref. 7, where it was shown that for coupling constants from the region

$$\tilde C = \{h_1 \geq g_1,\ h_2 \geq g_1,\ g_2 \geq g_1\}$$

these functions do not vanish for $p^2 < 0$. Since $\hat C \supset C$ [see (147)], it is obvious that in phase C the propagators do not have a tachyon singularity. It can also be shown that the scalar field σ_1 corresponds to a stable particle with twice the fermion mass, which is $\langle\sigma_1\rangle$. The particle σ_2 is a pseudoscalar resonance, corresponding to a pole of the propagator on the second sheet of its region of analyticity. The fields $\phi_{1,2}$ correspond to two stable fermion bound states with nonzero binding energy. One of these particles is a scalar (ϕ_1), and the other is a pseudoscalar (ϕ_2).

To obtain the 1PI two-point Green functions in phase B, where only $\langle\sigma_2\rangle \neq 0$, it is necessary to make the replacement $\sigma_1 \leftrightarrow \sigma_2$, $g_1 \leftrightarrow g_2$ in Eqs. (151)–(153).

If we are in one of the phases $A_{2,3}$, where $\langle\phi_i\rangle \neq 0$, and the vacuum expectation values of the fields ϕ_j, $\sigma_{1,2}$ are zero ($i \neq j$), the change of notation in (151)–(153) $\sigma_1 \leftrightarrow \phi_i$, $\sigma_2 \leftrightarrow \phi_j$, $g_1 \leftrightarrow h_i$, $g_2 \leftrightarrow h_j$ gives us the two-point 1PI Green functions in the corresponding phase of the theory. The boson mass spectrum in the phases B, A_2, and A_3 can be described as for phase C. In addition, they do not contain tachyons.

Finally, we note that in phase A_1, where the fermions are massless, the boson-field propagators have a singularity of the type $\sqrt{-p^2}$, which indicates the presence of massless bosons in the mass spectrum of the theory.

3.4. Some special cases

In this section we study the conditions under which the original model (121)–(124) is invariant under continuous symmetries. First, we note that without any restrictions the theory is symmetric under the gauge group

$$U(1): \psi \to e^{i\alpha}\psi. \tag{154}$$

This symmetry remains unbroken for any values of the coupling constants. If restrictions are placed on the model parameters, the theory acquires additional continuous symmetries. Let us consider a few cases.

(a) We assume that the bare coupling constants G_2 and H_2 are arbitrary, and

$$G_1 = H_1 \tag{155}$$

(and so $g_1 = h_1 \equiv g$). Then, in addition to the $U(1)$ symmetry (154), the theory will be invariant under chiral transformations:

$$U_5(1): \psi \to \exp(i\alpha\Gamma^5)\psi \equiv \begin{pmatrix} \cos\alpha\psi_1 + \sin\alpha\psi_2 \\ \cos\alpha\psi_2 - \sin\alpha\psi_1 \end{pmatrix}. \tag{156}$$

It follows from Sec. 3.2 that in this case only the phases A_1, A_2, B, and C will be realized in the model [phase A_3 is chirally equivalent to phase C, i.e., vacuum expectation values of the form (141) can be brought to the form (143) by using transformations from the group (156)]. We denote the regions of coupling constants (g, g_2, h_2) corresponding to these phases by A_{15}, A_{25}, B_5, and C_5, where

$$A_{15} = \{g, h_2, g_2 > 0\}, \quad A_{25} = \{g, g_2 > h_2; h_2 < 0\},$$

$$C_5 = \{h_2, g_2 > g; g < 0\}, \tag{157}$$

while the region B_5 is formed from the values of the coupling constants not belonging to the sets (157). It is easy to see that only C_5 is located on the boundary of region C (147), while the sets A_{15}, A_{25}, and B_5 lie inside the corresponding regions A_1, A_2, and B (144)–(147).

It is now obvious that in the original model with the constraint (155) the continuous chiral invariance (156) is spontaneously broken only in phase C, and, as follows from the preceding section, the propagator of the field ϕ_1 is singular at the point $p^2 = 0$ [see (153) for $g_1 = h_1$], i.e., a Goldstone particle of zero mass appears in the theory.

(b) If $G_1 = H_2$ and G_2 and H_1 are arbitrary, then in addition to the $U(1)$ symmetry (154) the model has another continuous symmetry:

$$U_3(1): \psi \to \exp(i\alpha\Gamma^3)\psi. \tag{158}$$

This case can be described like (a), and so we shall not dwell on it.

(c) If we want the model to be invariant under transformations from the group $U(1) \times U_\tau(1)$, where

$$U_\tau(1): \psi \to \exp(i\alpha\tau)\psi, \tag{159}$$

it is necessary to impose the constraint $H_1 = H_2 \equiv H$. Obviously, now the two-component spinors and auxiliary fields transform as

$$U_\tau(1): \psi_1 \to e^{i\alpha}\psi_1; \quad \psi_2 \to e^{-i\alpha}\psi_2;$$

$$(\phi_1 - i\phi_2) \to \exp(-2i\alpha)(\phi_1 - i\phi_2) \tag{160}$$

[the fields σ_i are invariant under the transformations (159)]. Clearly, in this case the model is also exactly solvable in the leading order of the $1/N$ expansion, and, using the results of the preceding sections, it is easy to obtain its phase portrait, which consists of the four phases A_1, A_2, B, and C. [The vacuum expectation value $\langle \phi_1 \rangle \neq 0$ of phase A_3 can be transformed, using (160), to $\langle \phi_2 \rangle \neq 0$, i.e., into the vacuum expectation value of the bosonic fields of phase A_2. Therefore, these phases are unitarily equivalent to each other under the transformations (159), and so we do not include A_3 in the phase portrait of the model for the above constraint on the coupling constant.] Using (160), it is easily seen that in phase A_2 the $U_\tau(1)$ symmetry is spontaneously broken. For $H = 0$ we obtain the theory already studied in connection with the phenomenon of high-temperature superconductivity,[16,60] but in it the $U_\tau(1)$ symmetry is not broken.

(d) There is another class of models of the form

$$L_{VP} = \bar{\psi} i\hat{\partial}\psi + V(\bar{\psi}\Gamma^\mu\psi)^2 + P(\bar{\psi}\Gamma^\nu\tau\psi)^2, \tag{161}$$

invariant under $U_\tau(1)$. This Lagrangian is also a special case of the generalized model (121)–(124) for $N = 1$, i.e., when the spinors ψ, ψ_1, and ψ_2 do not carry $U(N)$ group indices. In fact, using Fierz transformations (A6) for two-component Dirac spinors (see Appendix A), it is easy to show that

$$L_{VP} = \bar{\psi}_1 i\hat{\partial}\psi_1 + \bar{\psi}_2 i\hat{\partial}\psi_2 - 3(V+P)[(\bar{\psi}_1\psi_1)^2 + (\bar{\psi}_2\psi_2)^2]$$

$$+ (P - V)[4\bar{\psi}_1\psi_2\bar{\psi}_2\psi_1 + 2\bar{\psi}_1\psi_1\bar{\psi}_2\psi_2]. \tag{162}$$

Now, introducing the bare constants G_i and H_j [see (124)], for the Lagrangian (162) we have $H_2 = H_1$ and $G_2 = H_1 + G_1$. Therefore, for describing the phase structure of the model it is sufficient to know the values of the parameters h_1 and g_1. Then for $h_1 > 0$ and $g_1 > 0$ we will have phase A_1, for $h_1 > 0$ and $g_1 < 0$ we will have phase C, for $h_1 < 0$ and $g_1 < 0$ we will have phase B, and for $h_1 < 0$ and $g_1 > 0$ we will have phase A_2 with spontaneous breakdown of the $U_\tau(1)$ invariance.

(e) Finally, we can have the situation where the theory is invariant under $U(2)$ transformations:

$$U(2): \psi \equiv \begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix} \to U \begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix}. \tag{163}$$

[Here U are 2×2 unitary matrices.] It is easy to see that in this case $G_1 = H_1 = H_2 \equiv H$, i.e., $g_1 = h_1 = h_2 \equiv h$, as follows from (132)–(134). We therefore have only two independent coupling constants, and so the phase portrait of the model is especially simple. For example, the massless phase is realized for $g_2, h > 0$. The model is in phase B when the coupling constants satisfy the relations $g_2 < 0$, $h > g_2$. The $U(2)$ symmetry remains unbroken. If $h < 0$ and $g_2 > h$, we have phase C, in which the $U(2)$ symmetry is spontaneously broken to $U(1) \times U_\tau(1)$, and the mass spectrum of the theory contains

Goldstone bosons. The other features of the model located in phase B or C remain unchanged and are given in the preceding sections.

If we take $P=0$ in the Lagrangian (161), we obtain a theory with vector–vector interaction of the fermions. For $N=1$ it is a special case of the generalized Gross–Neveu model (121)–(124) in which $G_1=H_1=H_2\equiv H$ and $G_2=2H$, i.e., there is $U(2)$ invariance. Obviously, here in addition to A_1 it is possible to have only phase B without breakdown of the $U(2)$ invariance.

3.5. Dynamical generation of the Chern–Simons term

Let us continue our study of the generalized model (121) and consider the conditions under which a Chern–Simons term (120) can be spontaneously generated in it. For this we make the global continuous gauge symmetry (122) local, i.e., we introduce into the theory a vector gauge field A_μ, the Lagrangian for whose interaction with fermions has the form

$$L_{\text{int}}\sim \bar{\psi}\Gamma^\mu\psi A_\mu\equiv(\bar{\psi}_1\gamma^\mu\psi_1+\bar{\psi}_2\gamma^\mu\psi_2)A_\mu .$$

We shall assume that the full Lagrangian of the system does not contain the Maxwell kinetic term for the gauge field, i.e.,

$$L_{\text{tot}}=L+L_{\text{int}}, \qquad (164)$$

where L is given in (121). We also assume that the interaction constants are located in region B or C (147), where only the fields σ_i can have nonzero vacuum expectation values. Making the shift $\sigma_i\to\sigma_i+\langle\sigma_i\rangle$ in (164), we obtain a diagonal fermion mass term in L_{tot}:

$$L_{\text{tot}}=\sum_{k=1}^{2}\bar{\psi}_k(i\hat{\partial}+m_k)\psi_k+\dots . \qquad (165)$$

Here m_k denotes the mass of the kth spinor multiplet. Using this Lagrangian, which will correspond to the Feynman rules with massive fermionic propagators, we can find the radiative corrections to the effective action, which have a term proportional to A_μ^2:

$$S_{\text{eff}}=\int d^3p\tilde{A}_\mu(-p)\pi^{\mu\nu}\tilde{A}_\nu(p)+\dots, \qquad (166)$$

where $\hat{A}_\mu(p)$ is the Fourier transform of the field $A_\mu(x)$, and $\pi^{\mu\nu}$ is the polarization operator of the gauge field, which in the one-loop approximation at small momenta has the form

$$\pi^{\mu\nu}\sim\varepsilon^{\mu\nu\alpha}p_\alpha\sum_{k=1}^{2}\ \text{sign}\ m_k . \qquad (167)$$

Substituting (167) into (166) and going to coordinate space, we obtain a CS term which is generated dynamically, i.e., by radiative corrections:

$$S_{\text{CS}}\sim(\text{sign}\ m_1+\text{sign}\ m_2)\int d^3x\varepsilon^{\mu\nu\alpha}A_\mu\partial_\nu A_\alpha . \qquad (168)$$

We see that in phase B of the model, where $m_1=m_2$ [see (148)] and the P parity is spontaneously broken, the CS term is nonzero; however, in phase C, where the fermion masses have opposite signs, $S_{\text{CS}}=0$.

Let us now consider the case where the coupling constants lie in the regions $A_{2,3}$ (144). Here a nondiagonal fermion mass term arises spontaneously [see (148)], and so Eq. (168) is not applicable. In this case we can make a unitary transformation of the fermion fields:

$$\begin{pmatrix}\psi_1\\ \psi_2\end{pmatrix}=U\begin{pmatrix}f_1\\ f_2\end{pmatrix},$$

which, for example, for the region A_3 has the form $U=(1/\sqrt{2})\begin{pmatrix}1&1\\-1&1\end{pmatrix}$. In terms of the spinors f_i, the mass terms of the model in the phases $A_{2,3}$ will now be diagonal, but the masses will have opposite signs. Therefore, according to (168) the CS term does not appear dynamically in either the A_2 or the A_3 phase.

We thus have an example of a phenomenon where the P parity is spontaneously broken (in phase A_3), but the CS term is not generated.

3.6. The vacuum structure of the $(\bar{\psi}\lambda^a\psi)^2$ theory

Here we shall use the leading order of the $1/N$ expansion to study the phase structure of a three-dimensional four-fermion field theory of the form

$$L_\psi=i\bar{\psi}\hat{\partial}\psi+m\bar{\psi}\psi+g(\bar{\psi}\lambda^a\psi)^2/(2N), \qquad (169)$$

where

$$\bar{\psi}\psi=\sum_{i,\alpha}\bar{\psi}_{i\alpha}\psi_{i\alpha},\quad \bar{\psi}\lambda^a\psi=\sum_{i,\alpha}\bar{\psi}_{i\alpha}\lambda^a_{\alpha\beta}\psi_{i\beta}. \qquad (170)$$

The summation over i runs from 1 to N, and that over α and β runs from 1 to K. For all fixed values of i and α, $\psi_{i\alpha}$ is a two-component Dirac spinor and λ^a are the generators of the $SU(K)$ group $(a=1,\dots,K^2-1)$. Therefore, ψ transforms under the fundamental representations of the $U(N)$ and $U(K)$ groups, and the theory (169), which is not a special case of the generalized model (121), is invariant under $U(N)\times U(K)$. We shall perform all the calculations in the leading order of the $1/N$ expansion. Here it is understood that the rank of the $U(N)$ group is very large (i.e., $1/N$ is a small parameter). The structure of the Lagrangian (169) is such that the $U(N)$ invariance of the theory is not broken at any values of m and g. Therefore, when we speak of the residual symmetry of the vacuum, we shall always mean its invariance under $U(N)$.

To obtain the effective potential of the original model, it is convenient to use an equivalent theory with auxiliary scalar fields:

$$L_\sigma=i\bar{\psi}\hat{\partial}\psi+m\bar{\psi}\psi+\sigma^a(\bar{\psi}\lambda^a\psi)-N(\sigma^a)^2/(2g)$$
$$=i\bar{\psi}\hat{\partial}\psi+m\bar{\psi}\psi+\bar{\psi}\varphi\psi-N\,\text{Tr}(\varphi^2)/(4g), \qquad (171)$$

where $\varphi=\sigma^a\lambda^a$ and $\text{Tr}\ \varphi=0$. [Using the equations of motion, it is easy to eliminate σ^a from (171) and obtain L_ψ (169) as a result.] It can be shown (see Ref. 63) that in the theory (171) the effective potential in the leading order of the $1/N$ expansion has the form

$$\frac{2\pi}{N}V(\varphi)=\sum_{i=1}^{K}\left\{\frac{a}{2}\varphi_i^2+\frac{1}{3}|\varphi_i+m|^3\right\}, \qquad (172)$$

where $a=\pi/g$, and the variables φ_i satisfy the condition

$$\varphi_1+\ldots+\varphi_K=0. \qquad (173)$$

In deriving (172) we have used dimensional regularization. The advantage of this in three-dimensional spacetime is that the regularized expression for the effective potential in the leading order in $1/N$ does not contain ultraviolet divergences. Therefore, the bare coupling constants g and the mass m are finite quantities.

Using the results of Ref. 63 and also of numerical calculations that we performed for $N\leqslant 1000$, it can be stated that in the massless case $(m=0)$ the global minimum of the potential (172) possesses $U(n)\times U(n)$ symmetry for $K=2n$ or $U(n+1)\times U(n)$ symmetry for $K=2n+1$, if $g<0$. If $g>0$, the original $U(K)$ invariance of the model is not broken.

Now let us discuss the properties of the potential (172) in the massive case. For this we introduce dimensionless quantities, dividing both sides of (172) by m^3:

$$\frac{2\pi}{m^3N}V(w)=\sum_{i=1}^{K}\left\{\frac{b}{2}(w_i-1)^2+\frac{1}{3}|w_i|^3\right\}, \qquad (174)$$

where $b=a/m$ and $w_i=1+\varphi_i/m$. The new variables obviously satisfy the relation

$$w_1+\ldots+w_K=K. \qquad (175)$$

From (174) we find the stationarity equations for $V(w)$ $(i=1,\ldots,K-1)$:

$$bw_i-bw_K+|w_i|w_i-|w_K|w_K=0. \qquad (176)$$

In deriving (176), we have used the first $(K-1)$ components of w_i as the independent variables, and the w_K are defined by (175).

Let $b>0$ (i.e., $g>0$). Solving Eqs. (175) and (176), it is easy to show that in this case the function $V(w)$ has only a single stationary point $\Omega_K=(1,\ldots,1)$, at which its global minimum occurs. Obviously, Ω_K corresponds to the $U(K)$ symmetry of the ground state of the theory (169).

Now let us consider the case $b<0$ (i.e., $g<0$). First, we show that an arbitrary point $\Omega=(w_1,\ldots,w_K)$, where all the w_i satisfy (175) and (176), has no more than three different components.

This fact follows directly from (176), from whose solution we find that for arbitrary fixed $i<K$ the corresponding component w_i of the stationary point Ω can take one of three values (without loss of generality, here we assume that $w_K>0$):

$$\tilde{w}_1=w_K,\quad \tilde{w}_2=-b-w_K,\quad \tilde{w}_3=\frac{b}{2}-\sqrt{\frac{b^2}{4}-bw_K-w_K^2} \qquad (177)$$

for $0<w_K<-b$, or

$$\tilde{w}_1,\tilde{w}_3,\tilde{w}_4=\frac{b}{2}+\sqrt{\frac{b^2}{4}-bw_K-w_K^2} \qquad (178)$$

for $-b<w_K<-b(1+\sqrt{2})/2$. For other positive values of w_K, all the components w_i are equal to w_K. To determine the component w_K of the corresponding stationary point it is necessary to use the condition (175).

We shall use the following procedure to find the point where the potential has a global minimum. First, we use (177) and (178) to construct all possible stationary points with two and also three different components, determining w_K from the condition (175) each time. Next we find the value of the function $V(w)$ at each of these points and select the one at which V is a minimum. Naturally, as K increases the number of stationary points of the potential $V(w)$ grows significantly. We therefore restricted ourselves to a few values of K. Omitting the details of the calculations, we present only the final results.

(A) Let $K=2$. It follows from the system (175), (176) that the global minimum of the function (174) is located at the point

$$\Omega_{11}=\left(1+\frac{b}{2}-\frac{1}{2}\sqrt{b^2-4},\ 1-\frac{b}{2}+\frac{1}{2}\sqrt{b^2-4}\right),$$

if $b<-2$. In this case the original $U(2)$ symmetry is spontaneously broken to $U(1)\times U(1)$. If $b>-2$, the symmetry remains unbroken.

(B) Let $K=3$. Numerical calculations (here and in the following cases) show that $b<b_3^*=-1.5388\ldots$, and the global minimum occurs at $\Omega_{21}=(x_1,x_1,y_1)$, where $y_1=3-2x_1$ and

$$x_1=-[3(b-4)-\sqrt{9b^2-12b-36}]/10.$$

Obviously, Ω_{21} and, accordingly, the vacuum of the model, possess $U(2)\times U(1)$ symmetry. For $b>b_3^*$ the theory is $U(3)$-invariant.

(C) Let $K=4$. In this case for $b<b_4^*=-4.0694\ldots$ the global minimum of the potential occurs at the point $\Omega_{22}=(x_2,x_2,y_2,y_2)$, where $y_2=2-x_2$ and

$$x_2=1-\frac{b}{2}+\frac{1}{2}\sqrt{b^2-4}.$$

Clearly, in this region of the parameter b the symmetry of the model is spontaneously broken to $U(2)\times U(2)$.

For $b_4^*<b<b_4^{**}=-1.3572\ldots$ the absolute minimum of the function (174) occurs at the point $\Omega_{31}=(x_3,x_3,x_3,y_3)$, where $y_3=4-3x_3$ and

$$x_3=\frac{6-b}{5}+\frac{1}{5}\sqrt{b^2-2b-4}.$$

Obviously, Ω_{31} is invariant under the group $U(3)\times U(1)$. For $b>b_4^{**}$ the original $U(4)$ symmetry is unbroken.

(D) Finally, for $K=5$ the model is located in the $U(3)\times U(2)$-symmetric phase if $b<b_5^*=-2.5300\ldots$. For values $b_5^*<b<b_5^{**}=-1.2654\ldots$ the vacuum state is invariant under the group $U(4)\times U(1)$. At the same time, the original $U(5)$ symmetry is not broken for $b>b_5^{**}$. The points corresponding to these phases where the effective potential has a global minimum are of the form

$$U(3)\times U(2):\Omega_{32}=(x_4,x_4,x_4,y_4,y_4);$$

$$y_4=(5-3x_4)/2,$$

$$x_4=[15-5b+\sqrt{25b^2-20b-100}]/13,$$

$$U(4)\times U(1):\Omega_{41}=(x_5,x_5,x_5,x_5,y_5);\ y_5=5-4x_5,$$

$$x_5=[40-5b+\sqrt{25b^2-60b-100}]/34,$$

$$U(5):\Omega_5=(1,1,1,1,1).$$

In summary, it can be stated that for $K>3$ the phase structure in the massive model (169) is much richer than in the massless case.

3.7. Dynamical generation of a CS term

Here we shall discuss the possibility of using the model (169) for dynamically generating a topological mass of the vector field, which corresponds to a CS-like term (120) in the Lagrangian. For this we introduce the minimal interaction of a statistical vector field A_μ with spinor fields. The modified Lagrangian (171) takes the form

$$L_\sigma\to\tilde{L}\sigma=\bar{\psi}i(\hat{\partial}+ie\hat{A})\psi+m\bar{\psi}\psi+\bar{\psi}\varphi\psi$$
$$-N\,\mathrm{Tr}(\varphi^2)/(4g). \quad (179)$$

In the leading order of the $1/N$ expansion the model (179) possesses the same effective potential as the model (169)–(171). Therefore, the results of the preceding section are applicable to the theory (179), and it can be stated that the vacuum expectation values of the scalar fields $\langle\varphi_{\alpha\beta}\rangle$ from $\tilde{L}_\sigma$ are in general nonzero, i.e., they coincide with the coordinates of the point where the effective potential has a global minimum. In (179) we make the change of variable $\varphi_{\alpha\beta}\to\varphi_{\alpha\beta}+\langle\varphi_{\alpha\beta}\rangle$. Using the fact that $\langle\varphi_{\alpha\beta}\rangle=\mathrm{diag}(\varphi_1,\ldots,\varphi_K)$, the fermion mass term takes the form

$$\tilde{L}_\sigma=(\ldots)+\sum_{i=1}^{K}(m+\varphi_i)\bar{\psi}_i\psi_i, \quad (180)$$

where we have explicitly indicated only the summation over the $U(K)$ group indices, and summation over the $U(N)$ indices is understood. From this we see that in the leading order of the $1/N$ expansion the ith multiplet of spinor fields has mass $M_i=m+\varphi_i$, which, in general, is different from the bare mass m of the original Lagrangian. The values M_i obviously depend significantly on the vacuum properties of the theory, i.e., on which phase the theory is in.

Using the Lagrangian (180), it can be shown (see Sec. 3.5) that in the leading order in N a term of the CS form (120) is dynamically generated in the field theory (179) and (180), where

$$G=e^2\sum_{i=1}^{K}\mathrm{sign}(m+\varphi_i)\equiv e^2\tilde{G}. \quad (181)$$

The parameter $\tilde{G}$ is directly related both to the topological mass of the vector field and to the values of the fractional spin and statistics of the matter fields.

First we assume that $m=0$. In this case for $K=2n$ a CS term will not be generated at all. This is easy to understand from the form of the point where the potential has a global minimum: it has identical numbers of positive and negative components.[63] If $K=2n+1$, then in the $U(n+1)\times U(n)$-symmetric phase of the theory a CS term necessarily arises dynamically, in spite of the fact that it is absent in the massless phase.

The situation in the massive case ($m\neq 0$) is more interesting. Here, as follows from (181) and the results of the preceding subsection, the statistical parameter $\tilde{G}$ takes the following values, depending on the phase of the theory: $K,K-2,\ldots,K-2n$ ($n=K/2$ for even K and $n=(K-1)/2$ for odd K). We have $\tilde{G}=K$ in the phase with the maximum symmetry $U(K)$, and $\tilde{G}$ is a minimum (i.e., zero or unity) in the phase with minimum symmetry [for even $K=2n$, $U(n)\times U(n)$, or, for odd $K=2n+1$, $U(n+1)\times U(n)$]. It is important to note that the values of $\tilde{G}$ and the possible symmetry groups of the vacuum of the theory are in one-to-one correspondence. If the particle statistics (or spin) is known, the invariance group of the vacuum is known, and vice versa.

We thus arrive at the following conclusions:

(1) Instead of a set of vacuum expectation values of the scalar fields $\langle\varphi_{\alpha\beta}\rangle$ in the theory, it is sufficient to have only a single order parameter which arises dynamically. This is the statistical parameter $\tilde{G}$ (181).

(2) A transition from one phase of the theory to another is accompanied by a jump (transmutation) in the spin and statistics of the matter fields.

All the above discussion pertains to a massive field theory (169)–(171) interacting with a vector field and, strictly speaking, cases with $K\leq 5$. However, since they are so obvious, these results appear to be valid also for $K>5$.

3.8. Discussion of the results

In this section we use the results of Refs. 61, 63, and 66 to study two three-dimensional four-fermion models of a nonstandard type. The first model (121)–(124) is referred to as the generalized GN model, because for $G_1\neq 0, G_2, H_1, H_2=0$ it coincides with the ordinary Gross–Neveu theory of Sec. 1. The Lagrangian (121) is symmetric under discrete P, Γ^3, and Γ^5 transformations (123). We have shown that the generalized theory can exist in five different phases, two of which, A_2 and A_3, have not been observed earlier in very simple models of the Gross–Neveu type. In the phase A_2 only the Γ^3 symmetry is spontaneously broken, and in the phase A_3 the P and Γ^5 symmetries are broken simultaneously. The properties of the ground state of the latter phase are similar to those of the QCD vacuum, where both chiral and CP invariance are broken spontaneously.

In contrast to the simplest GN models, the boson mass spectrum of the theory (121) contains stable particles with nontrivial binding energy, and also resonances with finite lifetime.

We have studied several special cases, including a theory with vector–vector coupling of the fermion fields, and also a model (for $H_1=H_2$) with continuous global $U_\tau(1)$ symmetry (160), which is spontaneously broken in the phase A_2.

We assume that the fields ψ_1 and ψ_2 describe particles with opposite electric charges. Then the generator of the

$U_{\tau}(1)$ group corresponds physically to the electric-charge operator of the theory. It can be stated that the generalized model (121) has both chiral phase transitions (when the Γ^5 invariance is broken) and phase transitions of the superconducting type. In the latter case the gauge group of the electric charge $U_E(1)\equiv U_{\tau}(1)$ is spontaneously broken [phase A_2 of the model (121)–(124) for $H_1=H_2$; see Sec. 3.4], and scalar charged particles $\phi\sim\bar{\psi}_1\psi_2$ (analogs of Cooper pairs in the theory of ordinary superconductivity) appear.

We have also shown that in the model (121) the CS term arises dynamically only in the phase B, when P parity is spontaneously broken. In addition, in Sec. 3.5 we showed that violation of P parity is not sufficient for the generation of a CS term (in the phase A_3 the P parity is broken, but there is no CS term in the theory).

In the second model, studied here by means of the nonperturbative $1/N$ expansion, the spinor fields have internal degrees of freedom corresponding to the $U(K)$ symmetry [we do not include the $U(N)$ group for $N\to\infty$ in the calculation, because it is auxiliary]. The physical justification for studying this model is the same as for the other three-dimensional theories. However, here there is also additional motivation.

It has recently been suggested[64] that the electric charge has a topological origin. This idea has been tested, in particular, for the gauge theory based on the Lagrangian (169). The possibility of spontaneous breakdown of the original $U(K)$ symmetry to $U(K-1)\times U(1)$ was very important in that study. There, this possibility arose in an analysis of the Schwinger–Dyson equations, which the masses of the fermions in the model (169) must satisfy.[65] In our opinion, the method of Schwinger–Dyson equations is not reliable for the phase analysis of a field theory with fairly complicated structure of the interaction Lagrangian. The reason is that the mass equations have, as a rule, several solutions, from which it is very difficult to select the one corresponding to the ground state of the theory. We have studied the vacuum properties of the model (169) by a more suitable method, the effective-potential method, and have shown that the ground state can be $U(K-1)\times U(1)$-invariant only in the massive theory (169).

Finally, for the model (169) we included the minimal interaction of matter fields ψ with the vector statistical field A_μ and studied the possibility of dynamical generation of a CS term, which causes the statistics of the fields ψ to become fractional. We have shown that in the massive case ($m\neq 0$), instead of the multicomponent point where the potential has a global minimum, we can take as the order parameter a single parameter, the topological mass of the vector field (the coefficient of the CS term). The particle statistics undergoes a jump in transitions from one phase to another.

CONCLUSION

In this review we have carried out a detailed study of the critical features of several three-dimensional field theories with four-fermion interaction, taking into account external factors (temperature, magnetic fields, and so on) which significantly affect the vacuum structure.

This study has revealed some dynamical effects which were previously unknown. For example, we have shown that an external magnetic field can spontaneously break the symmetry[22,27,41] (this effect is now referred to as the catalysis of spontaneous symmetry breaking in an external magnetic field). In addition, we have shown that the chromomagnetic gluon condensate in QCD can also catalyze the spontaneous breakdown of chiral invariance.[50–52] The effect of catalysis of spontaneous symmetry breaking by external gauge fields is just as important as the well-studied effects generated by (anti-)self-dual fields.[20] All these phenomena have a common source: modification of the infrared regime due to interaction of the fermion spin with an external gauge field.

By studying the radiative corrections to the effective action, we have obtained the conditions sufficient for the dynamical generation of a Chern–Simons term,[63,66] which plays an important role in the anyon theory of high-temperature superconductivity. Finally, we have demonstrated the possibility in principle that, in addition to chiral transitions, phase transitions of the superconducting type occur in three-dimensional four-fermion models.[61] [In the latter case the $U(1)$ gauge symmetry is broken.]

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APPENDICES

A. Algebra of the γ matrices in three dimensions

Two-component Dirac spinors in three-dimensional spacetime realize an irreducible two-dimensional representation of the Lorentz group. In this case the 2×2 γ matrices have the form

$$\gamma^0=\sigma_3=\begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix},\quad \gamma^1=i\sigma_1=\begin{pmatrix}0 & i\\ i & 0\end{pmatrix},$$

$$\gamma^2=i\sigma_2=\begin{pmatrix}0 & 1\\ -1 & 0\end{pmatrix}. \tag{A1}$$

These matrices possess the properties

$$\mathrm{Tr}(\gamma^\mu\gamma^\nu)=2g^{\mu\nu};\quad [\gamma^\mu,\gamma^\nu]=-2i\varepsilon^{\mu\nu\alpha}\gamma_\alpha;$$

$$\gamma^\mu\gamma^\nu=-i\varepsilon^{\mu\nu\alpha}\gamma_\alpha+g^{\mu\nu}, \tag{A2}$$

where $g^{\mu\nu}=g_{\mu\nu}=\mathrm{diag}(1,-1,-1)$, $\gamma_\alpha=g_{\alpha\beta}\gamma^\beta$, and $\varepsilon^{012}=1$. In addition,

$$\mathrm{Tr}(\gamma^\mu\gamma^\nu\gamma^\alpha)=-2i\varepsilon^{\mu\nu\alpha}. \tag{A3}$$

The matrix elements of the γ matrices satisfy Fierz identities:

$$(\gamma^\mu)_{mn}(\gamma_\mu)_{\dot m\dot n}=\tfrac{3}{2}(1)_{m\dot n}(1)_{\dot m n}-\tfrac{1}{2}(\gamma^\nu)_{m\dot n}(\gamma_\nu)_{\dot m n}, \tag{A4}$$

where $m,n,\dot m,\dot n=1,2$, and summation over the indices μ, ν from 0 to 2 is understood. It follows from (A4) that

$$\bar{\psi}_1\gamma^\mu\psi_2\bar{\psi}_3\gamma_\mu\psi_4=-\tfrac{3}{2}\bar{\psi}_1\psi_4\bar{\psi}_3\psi_2+\tfrac{1}{2}\bar{\psi}_1\gamma^\nu\psi_4\bar{\psi}_3\gamma_\nu\psi_2, \tag{A5}$$

where ψ_k $(k=1,2,3,4)$ are arbitrary anticommuting two-component spinors. If we again apply the Fierz identity (A4) to the right-hand side of (A5), we find

$$\bar{\psi}_1\gamma^\mu\psi_2\bar{\psi}_3\gamma_\mu\psi_4=-2\bar{\psi}_1\psi_4\bar{\psi}_3\psi_2-\bar{\psi}_1\psi_2\bar{\psi}_3\psi_4. \tag{A6}$$

Using this relation, we can easily transform a vector–vector coupling of spinors into a scalar–scalar coupling.

The use of a reducible four-dimensional representation of the Lorentz group for spinor fields has recently become popular. The corresponding gamma matrices have the form $\Gamma^\mu=\mathrm{diag}(\gamma^\mu,-\gamma^\mu)$, where the γ^μ are given in (A1). It is easily shown that $(\mu,\nu=0,1,2)$

$$\mathrm{Tr}(\Gamma^\mu\Gamma^\nu)=4g^{\mu\nu};\quad \Gamma^\mu\Gamma^\nu=\sigma^{\mu\nu}+g^{\mu\nu};$$

$$\sigma^{\mu\nu}=\tfrac{1}{2}[\Gamma^\mu,\Gamma^\nu]=\mathrm{diag}(-i\varepsilon^{\mu\nu\alpha}\gamma_\alpha,-i\varepsilon^{\mu\nu\alpha}\gamma_\alpha). \tag{A7}$$

The dimension of the algebra of the matrices acting in four-dimensional spinor space is 16, and the generators of this algebra are the Γ^μ $(\mu=0,1,2)$ and the matrix Γ^3 which anticommutes with them:

$$\Gamma^3=\begin{pmatrix}0 & I\\ I & 0\end{pmatrix}, \tag{A8}$$

where I is the 2×2 unit matrix. There is another matrix which anticommutes with all the Γ^μ and with Γ^3. It has the form

$$\Gamma^5=i\begin{pmatrix}0 & -I\\ I & 0\end{pmatrix}. \tag{A9}$$

B. Estimates for $F(0)$ from Sec. 1.3

Let us make the change of variables $eH\alpha=\tau$, $\sigma^2=eH\tilde{\sigma}^2$, and $T^2=eH\tilde{T}^2$ in (47). Then

$$F(\sigma)=\frac{T}{\pi}\sum_n\int_0^\infty d\tau\,\exp(-\tau\tilde{\sigma}^2-\tau(2n+1)^2\pi^2\tilde{T}^2)\times[\coth\tau-1/\tau]. \tag{B1}$$

Let $\varphi(\tau)=[\coth\tau-1/\tau]$. Obviously, $\varphi'(\tau)>0$, and so $\varphi(\tau)$ is a monotonically increasing function. Now, taking τ to infinity, we have $\varphi(\tau)<1$ on the entire semiaxis $0\leqslant\tau<\infty$. Using this, from (B1) we find

$$F(\sigma)<\frac{T}{\pi}\sum_n\int_0^\infty d\tau\,\exp(-\tau\tilde{\sigma}^2-\tau(2n+1)^2\pi^2\tilde{T}^2)$$
$$=\frac{T}{\pi}\sum_n(\tilde{\sigma}^2+(2n+1)^2\pi^2\tilde{T}^2)^{-1}$$
$$=\frac{eH}{2\pi\sigma}\tanh\left(\frac{\sigma}{2T}\right), \tag{B2}$$

where in deriving the last equality we have used the summation formula[35]

$$\sum_{n\geqslant0}[(2n+1)^2+a^2]^{-1}=\frac{\pi}{4a}\tanh\left(\frac{\pi a}{2}\right).$$

We see from (B2) that for $\sigma\to0$

$$F(0)<\frac{eH}{4\pi T}\equiv\Phi_1(T,H). \tag{B3}$$

From this it follows that for $T\neq0$, $F(0)=\mathrm{const}<\infty$, and for $T\to\infty$, $F(0)\to0$ (H fixed).

Now let us obtain a lower bound on $F(0)$. We see from (B1) that all the terms of this series are positive. Therefore, discarding all terms except the one corresponding to $n=0$, we have

$$F(0)>\frac{T}{\pi}\int_0^\infty d\tau\,\exp(-\tau\pi^2\tilde{T}^2)[\coth\tau-1/\tau]$$
$$>\frac{T}{\pi}\int_1^\infty d\tau\,\exp(-\tau\pi^2\tilde{T}^2)\varphi(\tau)$$
$$>\frac{CT}{\pi}\int_1^\infty d\tau\,\exp(-\tau\pi^2\tilde{T}^2), \tag{B4}$$

where $C=\varphi(1)$. Calculating the last integral in (B4), we find

$$F(0)>\frac{CeH}{T\pi^3}\exp\left(-\frac{\pi^2T^2}{eH}\right)\equiv\Phi_2(T,H). \tag{B5}$$

From this we see that $F(0)\to\infty$ for fixed external magnetic field and $T\to0$.

C. Properties of the functions $\omega(\sigma)$ and $\varphi(\sigma)$ from Sec. 2.4

Let us consider some properties of the functions $\omega(\sigma)$ and $\varphi(\sigma)$, defined in Eqs. (108) and (109), respectively. First we show that $\omega'(\sigma)\geqslant\varphi'(\sigma)$ on the interval $(0,\infty)$. From (108) it is obvious that $(\beta=1/T)$

$$\omega'(\sigma)=\frac{2}{\pi}\tanh(\beta\sigma/2)+\frac{\sigma}{\pi\tilde{\sigma}}\tanh(\beta\tilde{\sigma}/2), \tag{C1}$$

where $\tilde{\sigma}=\sqrt{\sigma^2+eH}$. We write the function (109) as

$$\frac{2\pi}{\sqrt{eH}}\varphi(\sigma)=\int_\sigma^{\tilde{\sigma}}\frac{dE}{\sqrt{E^2-\sigma^2}}[\tanh(\beta E/2)-\tanh(\beta\sigma/2)]+\tanh(\beta\sigma/2)\int_\sigma^{\tilde{\sigma}}\frac{dE}{\sqrt{E^2-\sigma^2}}. \tag{C2}$$

This expression is more convenient than (109) for differentiation with respect to σ:

$$\varphi'(\sigma)=\frac{\sigma\sqrt{eH}}{2\pi}\int_\sigma^{\tilde{\sigma}}\frac{dE}{(E^2-\sigma^2)^{3/2}}[\tanh(\beta E/2)-\tanh(\beta\sigma/2)]+\frac{\sigma}{2\pi\tilde{\sigma}}\tanh(\beta\tilde{\sigma}/2)-\frac{\tilde{\sigma}}{2\pi\sigma}\tanh(\beta\sigma/2). \tag{C3}$$

The first term on the right-hand side of (C3) can be estimated by using the inequality $(x>x_0)$

$$\tanh x-\tanh x_0<[\tanh x_0]'(x-x_0). \tag{C4}$$

As a result, we have

$$\varphi'(\sigma)<\tilde{\varphi}'(\sigma)\equiv\frac{\beta(\tilde{\sigma}-\sigma)}{4\pi\cosh^2(\beta\sigma/2)}+\frac{\sigma}{2\pi\tilde{\sigma}}\tanh(\beta\tilde{\sigma}/2)-\frac{\tilde{\sigma}}{2\pi\sigma}\tanh(\beta\sigma/2). \quad (C5)$$

We introduce two new functions:

$$A(\sigma)\equiv\omega'(\sigma)-\frac{\sigma}{2\pi\tilde{\sigma}}\tanh(\beta\tilde{\sigma}/2)=\frac{\sigma}{2\pi\tilde{\sigma}}\tanh(\beta\tilde{\sigma}/2)+\frac{2}{\pi}\tanh(\beta\sigma/2), \quad (C6)$$

$$\tilde{A}(\sigma)\equiv\tilde{\varphi}'(\sigma)-\frac{\sigma}{2\pi\tilde{\sigma}}\tanh(\beta\tilde{\sigma}/2)=-\frac{\beta\sigma}{4\pi\cosh(\beta\sigma/2)}-\frac{\tilde{\sigma}[\sinh(\beta\sigma)-\beta\sigma]}{8\pi\sigma\cosh^2(\beta\sigma/2)}. \quad (C7)$$

Obviously, $A(\sigma)\geqslant 0$ and $\tilde{A}(\sigma)\leqslant 0$ [because the expression in square brackets in (C7) is positive] for all σ in the range $(0,\infty)$, i.e., $A(\sigma)\geqslant\tilde{A}(\sigma)$. Then from the definitions (C6) and (C7) and the inequality (C5) it follows that

$$\omega'(\sigma)\geqslant\tilde{\varphi}'(\sigma)>\varphi'(\sigma) \quad (C8)$$

for all $\sigma\in(0,\infty)$.

Now let us consider the behavior of the functions $\omega(\sigma)$ and $\varphi(\sigma)$ for $\sigma\to\infty$. From (108) it is easy to show that for $\sigma\to\infty$

$$\omega(\sigma)\sim 4\sigma/\pi. \quad (C9)$$

For the function $\varphi(\sigma)$ it is again convenient to use the representation (C2), from which, taking into account the inequality (C4), we have

$$\varphi(\sigma)<\frac{\beta\sqrt{eH}}{4\pi\cosh^2(\beta\sigma/2)}\int_\sigma^{\tilde{\sigma}}\frac{dE(E-\sigma)}{\sqrt{E^2-\sigma^2}}+\frac{\sqrt{eH}}{2\pi}\tanh\left(\frac{\beta\sigma}{2}\right)\ln\left[\frac{\sqrt{eH}+\tilde{\sigma}}{\sigma}\right]. \quad (C10)$$

Since $\varphi(\sigma)>0$, it follows from (C10) that for $\sigma\to\infty$, $\varphi(\sigma)\to 0$. Equations (C8)–(C10) allow us to make an important statement: if $\omega(0)<\varphi(0)$, then the equation $\omega(\sigma)=\varphi(\sigma)$ has a unique solution $\sigma_0(H)\neq 0$. In fact, let $\alpha(\sigma)\equiv\omega(\sigma)-\varphi(\sigma)$. Then from (C8)–(C10) we see that $\alpha(\sigma)$ is a monotonically increasing function on the interval $(0,\infty)$, with $\alpha(\infty)=\infty$. If $\alpha(0)<0$, there obviously exists a single point on the σ axis at which $\alpha(\sigma)$ vanishes.

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