J. S. Bell's problem

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The problem of J. S. Bell is studied in inertial and accelerated reference frames in the special theory of relativity. © *1998 American Institute of Physics.* [S1063-7796(98)00205-8]

In J. S. Bell's article[1] "How to teach the special theory of relativity," the following problem is considered.

"Three small spaceships, A, B, and C, drift freely in a region of space remote from other matter, without rotation and without relative motion, with B and C equidistant from A (Fig. 1).

On reception of a signal from A the motors of B and C are ignited and they accelerate gently (Fig. 2).

Let ships B and C be identical, and have identical acceleration programmes. Then (as reckoned by an observer in A) they will have at every moment the same velocity, and so remain displaced one from the other by a fixed distance. Suppose that a fragile thread is tied initially between projections from B and C (Fig. 3). *If it is just long enough to span the required distance initially, then as the rockets speed up, it will become too short, because of its need to Fitzgerald contract, and must finally break. It must break when, at a sufficiently high velocity, the artificial prevention of the natural contraction imposes intolerable stress.*

Is it really so? This old problem came up for discussion once in the CERN canteen. A distinguished experimental physicist refused to accept that the thread would break, and regarded my assertion, that indeed it would, as a personal misinterpretation of special relativity. We decided to appeal to the CERN Theory Division for arbitration, and made a (not very systematic) canvas of opinion in it. There emerged a clear consensus that the thread would not *break!*

Of course many people who give this wrong answer at first get the right answer on further reflection. Usually they feel obliged to work out how things look to observers B or C. They find that B, for example, sees C drifting further and further behind, so that a given piece of thread can no longer span the distance. It is only after working this out, and perhaps only with a residual feeling of unease, that such people finally accept a conclusion which is perfectly trivial in terms of A's account of things, including the Fitzgerald contraction. It is my impression that those with a more classical education, knowing something of the reasoning of Larmor, Lorentz, and Poincaré, as well as that of Einstein, have stronger and sounder instincts."

Let us analyze this problem. Let the length of the thread joining spaceships B and C when they are at rest be equal to the distance between them, l_0. Owing to the Lorentz contraction, when the thread moves, in a stationary reference frame (where its ends are fixed at the same instant of time according to a clock in the stationary frame) its length must become

$$l=l_0\sqrt{1-v^2}.$$

However, since the ends of the thread are attached to the spaceships, the distance between which (l_0) remains constant (because they are moving according to the same law), the thread must break. This is what an observer located in a stationary, inertial reference frame would see.

From the viewpoint of an observer located on one of the spaceships, the thread breaks because the spaceships become farther apart as time evolves. This is easily seen from the invariance of the interval in Minkowski space. Actually, the interval between the two events corresponding to fixing the locations of spaceships B and C at the same instant of time according to a clock in the stationary reference frame is spacelike and equal to $S_{12}^2=-l_0^2$. In a reference frame comoving with the spaceships this interval must have a time part (because events simultaneous in a stationary frame are not simultaneous in the comoving frame). Therefore, the spatial part of the interval (determining the distance between the spaceships) must be larger than l_0 and must grow as the speed of the spaceships increases.

We see from this example that the dynamical nature of the Lorentz transformation is manifested in the case of accelerated motion. The Lorentz contraction, like the actual geometry of flat Minkowski spacetime, has a dynamical nature because it reflects the general dynamical properties of matter—the energy–momentum and angular-momentum conservation laws. It is these general laws, which are universal for any interaction, that lead to a single pseudo-Euclidean geometry common to all forms of matter motion.

Let us discuss this in more detail. First we analyze the motion of a spaceship in the "stationary" inertial reference frame in which spaceships B and C start out. The relativistic covariant equation of motion of each of them involves a constant force f, which produces (in the rest frame of the spaceship) a constant acceleration a and has the form (if the force f is directed along the x axis)

$$m\,\frac{dU_x}{d\tau}=F_x,\quad m\,\frac{dU_0}{d\tau}=F_0. \tag{1}$$

Here U_x and U_0 are the components of the four-velocity:

$$U^\alpha=\left(\frac{1}{\sqrt{1-v^2}},\frac{v}{\sqrt{1-v^2}}\right); \tag{2}$$

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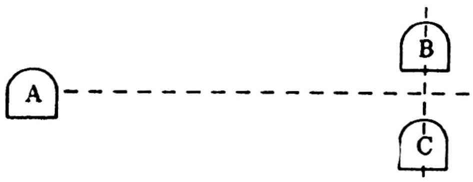

FIG. 1.

$$F^{\alpha}=\left(\frac{fv}{\sqrt{1-v^2}},\frac{f}{\sqrt{1-v^2}}\right); \tag{3}$$

and τ is the proper time. The speed of light is $c=1$.

On the basis of (2) we have

$$(U_0)^2-(U_x)^2=1. \tag{4}$$

From (2) and (3) we find

$$F_0=fU_x, \quad F_x=fU_0. \tag{5}$$

Using (5), we write (1) as

$$\frac{dU_x}{d\tau}=aU_0, \quad \frac{dU_0}{d\tau}=aU_x, \quad a=\frac{f}{m}. \tag{6}$$

Owing to (4), these equations have the simple solution

$$U_x=\frac{dx}{d\tau}=\sinh a\tau, \quad U_0=\frac{dt}{d\tau}=\cosh a\tau. \tag{7}$$

On the basis of (2) and (7) we obtain

$$v=\tanh a\tau, \quad \cosh a\tau=\frac{1}{\sqrt{1-v^2}}. \tag{8}$$

If at the initial instant of time ($\tau=0$) one of the spaceships (B) is located on the x axis at the point x_B^0, and the other is at the point x_C^0, the trajectory of the motion of the first spaceship will be determined parametrically from (7) (in terms of the proper time τ):

$$x_B=\frac{1}{a}(\cosh a\tau-1)+x_B^0, \tag{9}$$

$$t=\frac{1}{a}\sinh a\tau. \tag{10}$$

The trajectory of the second spaceship C will be given by a similar expression. Uniformly accelerated motion is usually called hyperbolic motion in relativistic mechanics.

From (8) and (10) we find

$$\cosh a\tau=\frac{1}{\sqrt{1-v^2}}=\sqrt{1+a^2t^2}. \tag{11}$$

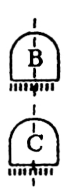

FIG. 2.

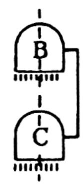

FIG. 3.

Using (11), we obtain the explicit dependence of the spaceship coordinates and velocity on the time t:

$$x_B=\frac{1}{a}(\sqrt{1+a^2t^2}-1)+x_B^0, \tag{12}$$

$$x_C=\frac{1}{a}(\sqrt{1+a^2t^2}-1)+x_C^0. \tag{13}$$

Using (8), (10), and (11) we find

$$v_C(t)=v_B(t)=\frac{at}{\sqrt{1+a^2t^2}}. \tag{14}$$

Therefore, in a "stationary" inertial reference frame, the distance between the two spaceships remains constant and equal to the original separation:

$$l_0=x_B(t)-x_C(t)=x_B^0-x_C^0. \tag{15}$$

This is apparently the reason why most of the physicists polled by Bell maintained that the thread connecting the spaceships would not break. Space–time relations in Minkowski space are determined by the interval between events. In the problem of the motion of spaceships B and C, a "stationary" observer in an inertial reference frame is, according to the statement of the problem, dealing with a spacelike interval equal to

$$S_{BC}^2=-(x_B(t)-x_C(t))^2=-(x_B^0-x_C^0)^2=-l_0^2.$$

This observer cannot determine from an experiment whether or not the separation of the spaceships increases in their reference frame when they undergo uniformly accelerated motion. However, if an experiment could be performed using a special device which sends a light signal when the thread breaks, then, upon observing the light signal, the stationary observer would note the breaking of the thread. However, to explain this phenomenon he would have to discover the pseudo-Euclidean geometry of spacetime. To answer the question posed above, the motion of spaceships B and C must be observed in a comoving reference frame. By comparing the spacelike part of the interval in the comoving reference frame with l_0, it is possible to establish the fact that the spaceships B and C become farther apart as time increases.

Using (10) and (11), it is easy to express τ in terms of t:

$$\tau=\frac{1}{a}\ln(at+\sqrt{1+a^2t^2}).$$

From this we see that the proper time increases much more slowly than the time t in a stationary inertial reference frame. The original inertial reference frame, in which the force field

F^α produces an acceleration a of test bodies which is constant in their reference frame, can be transformed by a coordinate transformation into a uniformly accelerated reference frame. In a stationary inertial reference frame the interval ds characterizing the metrical properties of spacetime has the form

$$ds^2=dt^2-dx^2-dy^2-dz^2. \tag{16}$$

It is in this coordinate system that the force field (3) with components (F_0,F_x) acts. To see more clearly why the thread breaks, let us go to a uniformly accelerated reference frame.[2] We introduce the variable ρ, using the transformation

$$\rho=x-\frac{1}{a}[\cosh a\tau-1], \tag{17}$$

leaving the other coordinates τ, y, and z unchanged. Under this transformation, test bodies in a comoving noninertial reference frame are at rest. According to (10), we have

$$dt=d\tau\cosh a\tau. \tag{18}$$

Taking into account (17) and (18), we transform (16) to

$$ds^2=d\tau^2-d\rho^2-2d\rho d\tau\sinh a\tau-dy^2-dz^2. \tag{19}$$

We see that we have transformed the inertial reference frame (16) with the force field F_x into a uniformly accelerated reference frame (19) with metric coefficients

$$g_{00}=1,\quad g_{\rho\rho}=-1,\quad g_{0\rho}=-\sinh a\tau,$$
$$g_{yy}=-1,\quad g_{xx}=-1.$$

The effect of the force field will now be manifested in the metrical properties of spacetime. To determine the physical time and distance in a uniformly accelerated reference frame, we separate the timelike and spacelike parts in ds^2:

$$ds^2=[d\tau-d\rho\sinh a\tau]^2-d\rho^2\cosh^2 a\tau-dy^2-dz^2. \tag{20}$$

In this expression the quantity

$$d\sigma=d\tau-d\rho\sinh a\tau \tag{21}$$

determines the interval of physical time in a uniformly accelerated reference frame.[2] It should be noted that the time $d\sigma$ depends on the action of the force field producing the constant acceleration a. In a noninertial reference frame the quantity $d\sigma$, as also seen in the present example, is not a total differential. This means that in a noninertial reference frame it is not possible to have a single synchronization of clocks located at different spatial points which is preserved as time passes, because the synchronization depends on the synchronization path. Such a synchronization can be achieved only in an inertial reference frame. It is in this case that $d\sigma$ is a total differential. The metrical properties of the three-dimensional space orthogonal to the time $d\sigma$ are determined by the quantity

$$dl^2=d\rho^2\cosh^2 a\tau+dy^2+dz^2. \tag{22}$$

Here dl depends on the effect of the force field a. We have thereby transformed the action of the field in an inertial frame into a uniformly accelerated reference frame of Minkowski space, and thus into the metrical properties of spacetime. A uniformly accelerated reference frame is not "rigid," because in it the distance between fixed points changes with time. Taking into account (21) and (22), the interval (20) becomes

$$ds^2=d\sigma^2-dl^2. \tag{23}$$

Since for light the interval ds is equal to zero, measurement of the length dl reduces to measurement of the time $d\sigma$ for a light signal to travel the distance dl. It follows from this that distances through which light travels in the same time are equal distances.

From (22) we find that the element of length along the ρ axis is

$$dl_\rho=d\rho\cosh a\tau,$$

or, using (11),

$$dl_\rho=\frac{d\rho}{\sqrt{1-v^2}}=d\rho\sqrt{1+a^2t^2}. \tag{24}$$

This is the expression which arises owing to the Lorentz contraction of the length. Therefore, the distance between the spaceships B and C in a uniformly accelerated reference frame is given by

$$l_\rho=(x_B-x_C)\sqrt{1+a^2t^2}=l_0\sqrt{1+a^2t^2}. \tag{25}$$

However, this implies that in an inertial frame K there is (contrary to the conclusion of Ref. 4) a Lorentz contraction of the length of a rod at rest in the system K_N: $x_B-x_C=l_\rho\sqrt{1-v^2}$. The distance l_0 between the spaceships, which is fixed at the same instants of time ($t_B=t_C$) by a stationary observer in an inertial reference frame, cannot be the distance in a comoving reference frame: the instants of time $t_B=t_C$ correspond to different times in the comoving reference frame, because the physical time $d\sigma$ is not zero. Equation (25) shows that, owing to the presence of the force field producing the acceleration a, the distance between the spaceships B and C increases with time, causing the thread joining the ships to break.

In the general case of accelerated motion, the squared interval has the form

$$ds^2=g_{\mu\nu}dx^\mu dx^\nu.$$

As before, we split the squared interval into timelike and spacelike parts:

$$ds^2=d\sigma^2-dl^2,\quad\text{where}\quad d\sigma=\frac{g_{0\lambda}dx^\lambda}{\sqrt{g_{00}}},$$

$$dl^2=x_{ik}dx^i dx^k,\quad x_{ik}=-g_{ik}+\frac{g_{0i}g_{0k}}{g_{00}}.$$

The quantity $d\sigma$ characterizes the physical time, which is independent of the choice of time variable. In fact, let us, for example, introduce a new variable x'^0 according to the law

$$x'^0=x'^0(x^0,x^i),\quad x'^i=x^i.$$

Then on the basis of the tensor transformation

$$g'_{\mu\nu}=g_{\alpha\beta}\frac{\partial x^\alpha}{\partial x'^\mu}\cdot\frac{\partial x^\beta}{\partial x'^\nu}$$

we have

$$g'_{00}=g_{00}\left(\frac{\partial x^0}{\partial x'^0}\right)^2,\quad g'_{0\lambda}=g_{0\beta}\frac{\partial x^0}{\partial x'^0}\cdot\frac{\partial x^\beta}{\partial x'^\lambda},$$

$$dx'^\lambda=\frac{\partial x'^\lambda}{\partial x^\sigma}dx^\sigma,$$

and using

$$\frac{\partial x^\beta}{\partial x'^\lambda}\cdot\frac{\partial x'^\lambda}{\partial x^\sigma}=\delta^\beta_\sigma,$$

we find

$$d\sigma=\frac{g'_{0\lambda}dx'^\lambda}{\sqrt{g'_{00}}}=\frac{g_{0\sigma}dx^\sigma}{\sqrt{g_{00}}}.$$

We have therefore shown that the physical time $d\sigma$ does not depend on the method of choosing the coordinate time variable. The coordinate time variable does not characterize the time evolution of a physical process, because it depends on an arbitrary choice of clock. The physical time determines the time behavior of a physical process, but the quantity $d\sigma$ is local in nature, because in a noninertial reference frame it is not a total differential, and so the variable σ does not exist. In an inertial Galilean coordinate system $d\sigma$ coincides with the differential of the coordinate variable and is a total differential. The quantity dl^2 is the squared distance between points; it is independent of the choice of coordinate variables and is local in nature.

It should also be noted that the transformation (21) is nonholonomic, and so the variable σ does not exist. In a comoving noninertial reference frame the interval (23) will have the form

$$ds_c^2=d\tau^2-d\rho^2\cosh^2 a\tau-dy^2-dz^2. \tag{23a}$$

Owing to the nonholonomic nature of the transformation (21), this interval will be Riemannian.[3] Using (23a), we easily find the Christoffel symbols:

$$\Gamma^\lambda_{\mu\nu}=\tfrac{1}{2}g^{\lambda\rho}(\partial_\mu g_{\nu\rho}+\partial_\nu g_{\mu\rho}-\partial_\rho g_{\mu\nu}).$$

In our case they are

$$\Gamma^0_{11}=a\sinh a\tau\cosh a\tau,\quad \Gamma^1_{01}=a\frac{\sinh a\tau}{\cosh a\tau}.$$

It follows from this that the components of the Riemann curvature tensor

$$R^\lambda_{\nu\rho\sigma}=\partial_\rho\Gamma^\mu_{\nu\sigma}-\partial_\sigma\Gamma^\mu_{\nu\rho}+\Gamma^\mu_{\tau\rho}\Gamma^\tau_{\nu\sigma}-\Gamma^\mu_{\tau\sigma}\Gamma^\tau_{\nu\rho}$$

for the metric (23a) will be

$$R^1_{010}=-a^2,\quad R^0_{110}=-a^2\cosh^2 a\tau.$$

Using the well known expression for the geodesic deviation

$$\frac{d^2\delta x^\mu}{d\tau^2}+R^\mu_{\sigma\lambda\nu}U^\sigma U^\nu\delta x^\lambda=0,$$

we obtain

$$\frac{d^2\delta x^1}{d\tau^2}-a^2\delta x^1=0,$$

which shows that test bodies in a comoving noninertial reference frame move away from each other with increasing time.

We can obtain another illustration of the nature of the relative motion of the "Bell spaceships" by studying the question of whether or not observers located on these spaceships can maintain radio contact with each other. To analyze this problem, we must use the transformations (9) and (10) to write the squared interval in the variables of the accelerated reference frame:

$$S_{12}^2=(t_2-t_1)^2-(x_2-x_1)^2=\left[\frac{2}{a}\sinh\frac{a(\tau_2-\tau_1)}{2}\right.$$
$$\left.-l_0\sinh\frac{a(\tau_1+\tau_2)}{2}\right]^2-l_0^2\cosh^2\frac{a(\tau_1+\tau_2)}{2},$$

where $l_0=x_2^0-x_1^0$.

The time of arrival of a radio signal sent by one observer to the other is determined by the condition $S_{12}^2=0$. If the signal is emitted from the head of spaceship B at time τ_1, it is easily shown from the expression for the interval that it will be received by an observer on spaceship C at time τ_2 given by

$$e^{a\tau_2}=e^{a\tau_1}+al_0.$$

Therefore, a radio signal emitted from spaceship B will always reach spaceship C. From the viewpoint of a stationary observer at $a\tau_1\gg 1$ the condition takes the simple form

$$t_2\simeq t_1+\frac{l_0}{2}.$$

A completely different result is obtained when studying the propagation of a radio signal emitted from the tail of spaceship C. The time of its arrival at the head of the spaceship τ_1 will be determined from the equation

$$e^{-a\tau_1}=e^{-a\tau_2}(1-al_0e^{a\tau_2}).$$

This equation has a solution only when the following inequality holds:

$$al_0e^{a\tau_2}<1.$$

If this condition is not satisfied, an observer in spaceship B cannot obtain any information from spaceship C. Therefore, for an observer at the head of spaceship B, the back of spaceship C must vanish beyond the event horizon as time evolves. To obtain information from spaceship C at spaceship B it is necessary to turn off the engine of spaceship B. All this will actually be manifested in the acceleration of charged particles in a constant electric field.

For example, let an electron bunch be accelerated in a linear accelerator with average electric field strength E to an energy $\varepsilon\gg m$. Then by the end of the acceleration process, from the condition

$$al_0e^{a\tau_2}=al_0(at_2+\sqrt{1+a^2t_2^2})<1$$

we will have the inequality

$$l_0 \leq \frac{(mc^2)^2}{2eE\varepsilon}.$$

It follows from this that only those charged particles which at the start of the acceleration were located at a distance less than $(mc^2)^2/2eE\varepsilon$ will by the end of the acceleration affect the leading edge of the bunch. For the planned linear colliders with $eE = 100$ MeV/m and energy $\varepsilon = 200$ GeV, this distance is $l_0 \simeq 6\times 10^{-7}$ cm.

Bell's problem suggests a conceptual method of constructing a uniformly accelerated reference frame. Let there be a single synchronization of clocks at each point of space in an inertial reference frame. Then a uniformly accelerated reference frame can be realized by means of an infinite number of "Bell spaceships" starting simultaneously from different points of a stationary inertial reference frame with identical and constant acceleration a. In this way, we can conceptually create a uniformly accelerated reference frame in an arbitrarily large volume of space. It is such a reference frame which should be compared with a uniform gravitational field. However, one sometimes uses a "rigid" uniformly accelerated reference frame in which the distance between fixed points is independent of time. Such a system was introduced by Møller. In it the coordinates (η,ρ) are related to the coordinates of an inertial frame (t,x) as

$$t = \rho \sinh a\eta, \quad x = \rho \cosh a\eta \tag{26}$$

(to make the discussion as simple as possible, we restrict ourselves to a single spatial coordinate x). The interval between events in the Møller frame, which in accordance with Ref. 4 will be denoted by K_a, is

$$ds^2 = dt^2 - dx^2 = a^2\rho^2 d\eta^2 - d\rho^2. \tag{27}$$

From this it follows that the proper-time interval in the frame K_a is $d\tau = a\rho d\eta$, and the distance between adjacent (fixed) points is constant and equal to $d\rho$. In this sense the frame K_a is rigid. To explain how it differs from the coordinate system (9), (10) (or, in the notation of Ref. 4, from the frame K_N), let us consider the question of what set of objects can produce the frame K_a. It follows from (26) that a point $\rho = \rho_0$ fixed in the frame K_a moves in a stationary inertial reference frame according to the law

$$x = \rho_0 \sqrt{1 + \frac{t^2}{\rho_0^2}}, \tag{28}$$

i.e., it starts at time $t = 0$ from the point $x_0 = \rho_0$ on the x axis with constant acceleration $a_0 = 1/\rho_0$. This means that the frame K_a is realized by a set of "Møller spaceships" starting simultaneously in a stationary inertial reference frame with different accelerations $a(x)$ depending on their locations on the x axis: $a(x) = 1/|x|$. If in the stationary inertial reference frame Møller spaceships separated by a distance dx start simultaneously, and the distance between them in the accelerated frame K_a is equal to $d\rho$, then in the stationary reference frame the distance dx will decrease with time according to the law

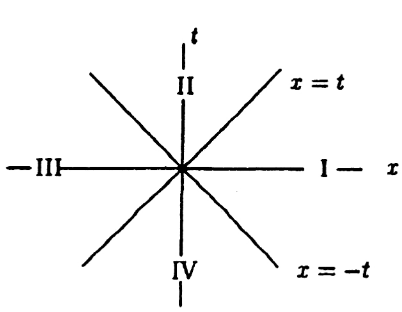

FIG. 4.

$$dx = \frac{d\rho}{\sqrt{1 + \frac{t^2}{\rho^2}}}, \tag{29}$$

which corresponds to a Lorentz contraction. The arguments given above clearly show that the frame K_a (in contrast to K_N) cannot model a uniform force field, all of whose points are assumed to move according to the hyperbolic law (9) with identical constant acceleration a. In principle, it is impossible to construct a rigid reference frame which would mimic a uniform force field. The introduction of a rigid reference frame not only leads to violation of the uniformity of the force field, but it also necessarily leads to incomplete mapping of the spacetime of the inertial reference frame onto the accelerated frame. Therefore, the preference for K_a over K_N shown by the authors of Ref. 4 in their discussion of the equivalence principle appears unjustified.

For the example of analyzing Bell's problem, we have clearly seen that the special theory of relativity is also applicable to noninertial reference frames.[2] This is natural and obvious, because the essence of the special theory of relativity is just that space and time form a continuously connected four-dimensional continuum in which the measure of the distance between events is the square of the interval between them. Statements that the special theory of relativity cannot be used to describe phenomena in noninertial reference frames are just as absurd as the statement that only Cartesian and not curvilinear coordinates can be used on the plane. The confusion has arisen because unjustified importance has been given to the concept of simultaneity at different spatial points and the process of clock synchronization at different spatial points. Since these concepts are limited and make sense only in inertial reference frames, it has apparently seemed that the special theory of relativity cannot be used in noninertial reference frames. The interval (16) by construction is independent of the choice of reference frame in Minkowski space. Transformations from the coordinates of an inertial frame to those of a noninertial frame must be single-valued and possess the property that the inverse transformation cover the entire Minkowski space. Only in this case can physical processes occurring in an inertial reference frame be fully described in a noninertial reference frame.

It should especially be noted that the transformations (26) do not map all the points of Minkowski space onto an accelerated reference frame, because the determinant of the metric tensor

$$g = -a\rho$$

vanishes at the point $\rho = 0$. Since from (26) we have

$$x^2 - t^2 = (x-t)(x+t) = \rho^2,$$

this implies that the transformations (26) map only the part of Minkowski space located between the lines $x = t$ and $x = -t$. This is region I in Fig. 4.

Therefore, physical processes which in an inertial frame (Galilean coordinates) occur in regions lying outside region I cannot be described in the Møller frame.

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Translated by Patricia A. Millard