Some remarkable charge–current configurations

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We investigate how different magnetization distributions interact with an external electromagnetic field. Strong selectivity to the time dependence of the external electromagnetic field arises for particular magnetizations and suggests that it can be used for practical applications. We review the properties of the known charge–current radiationless configurations. The radiation field of toroidal-like time-dependent current configurations is investigated. Infinitesimal time-dependent configurations are found outside which the electromagnetic field strengths disappear but the potentials survive. For a number of time dependences, their finite radiationless counterparts can be found. In these cases topologically nontrivial (unremovable by a gauge transformation) electromagnetic potentials exist outside the sources. The well-defined rule obtained for constructing time-dependent infinitesimal sources suggests the existence of finite nontrivial radiationless sources with a rather arbitrary time dependence. The latter can be used to carry out time-dependent Aharonov–Bohm-like experiments. Examples are given of nonstatic current configurations generating the static electric field and adequately described by the electric vector potential rather than by the scalar one. © *1998 American Institute of Physics.* [S1063-7796(98)00304-0]

1. INTRODUCTION

It should probably first be explained what is meant by the words "remarkable charge–current configurations" in the title of this paper: charge–current distributions with unusual (paradoxical) properties. For example, it is well known that a point charge radiates electromagnetic energy when it moves with acceleration. However, there are known specific finite-extension configurations of charges which do not radiate when they exhibit acceleration.[1–9]

Further, everybody knows that time-dependent currents emit electromagnetic energy into the surrounding space. However, there are known time-dependent current configurations which do not radiate electromagnetic energy.[6,9–12] Up to now, only those nonradiating time-dependent configurations of charges and currents were known for which the electromagnetic field (EMF) strengths **E**, **H** as well as the electromagnetic potentials **A**, Φ have disappeared outside the finite space region S. It turns out that finite time-dependent configurations of charges and currents exist outside which the electromagnetic field strengths **E**, **H** vanish but nontrivial electromagnetic potentials **A**, Φ differ from zero.[13] By the term "nontrivial" we mean that the physical situation is described adequately by electromagnetic potentials rather than by electromagnetic field strengths.

Further, it is known that electric and magnetic dipoles interact with the electric and magnetic field, respectively. However, there are known finite configurations of magnetic (electric) dipoles whose interaction with an external EMF is proportional to a time derivative (of definite order) of the electric (magnetic) field.[14–19]

It is the goal of the present considerations to study the properties of these remarkable charge–current configurations.

The plan of our exposition is as follows.

In Sec. 2 we study how different configurations of electric and magnetic dipoles interact with the external EMF. It turns out that the selectivity of the interaction to the time dependence of an external EMF can be used for storage and coding of information.

In the same section the classification of current sources according to their interaction with the external field is given.

Consider a metallic ring enclosing a cylindrical solenoid. When the metallic ring becomes superconducting, a supercurrent arises on its surface. This in turn leads to the appearance of a magnetic field in the surrounding space. These quantities are evaluated in Sec. 3.

A review of known radiationless time-dependent sources is given in Sec. 4. The exposition is illustrated by concrete examples of accelerated nonradiating charge distributions.

In Sec. 5 we construct a toroidal charge–current configuration having the property that the time-dependent magnetic field differs from zero only inside the impenetrable torus, while the time-dependent magnetic vector potential (VP) and time-independent electric scalar potential differ from zero everywhere. In the accessible region (i.e., outside the impenetrable torus) the static electric field differs from zero inside the torus hole. Although charged particles may scatter on this electric field, the latter contributes only to the static background. It is just the time variation of the magnetic flux confined to the excluded region that leads to the time dependence of the interference picture. This may be viewed as a new channel for information transfer and can be used to obtain the time-dependent Aharonov–Bohm effect. A classification of more general radiationless sources is given in the same section. Examples are given of current configurations generating a static electric field adequately described by the electric vector potential rather than by the scalar one.

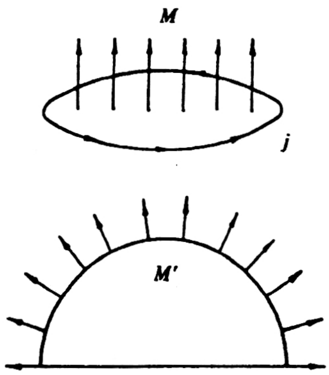

FIG. 1. Two magnetizations **M** and **M**′ corresponding to the same current density **j**.

2. INTERACTION OF MAGNETIZATIONS WITH THE EXTERNAL ELECTROMAGNETIC FIELD

The plan of our exposition is as follows. In Sec. 2.1, we study how the choice of magnetization inside the sample affects its interaction with the external EMF. A generalization of the Ampère hypothesis is discussed in the same section. The physical meaning of the scalar functions entering into the Debye parametrization of the current density is clarified in Sec. 2.2. It turns out that the selectivity of the interaction to the time dependence of an external EMF arises for a specific choice of these functions. This can probably be used for storage and coding of information. In the same section, we give a classification of pointlike and extended current sources according to their interactions with an external EMF.

2.1. Magnetization, toroidization, and generalization of the Ampère hypothesis

Consider a circular current in the $Z=0$ plane (the upper part of Fig. 1):

$$\mathbf{j}=\mathbf{n}_\phi I\,\delta(\rho-d)\,\delta(z)=\frac{1}{d}\,\mathbf{n}_\phi I\,\delta(\rho-d)\,\delta\!\left(\theta-\frac{\pi}{2}\right). \tag{2.1}$$

Since div $\mathbf{j}=0$, the equivalent magnetization can be used instead of **j** (see, e.g., Ref. 20):

$$\mathbf{j}=\text{curl}\,\mathbf{M}, \tag{2.2}$$

$$\mathbf{M}=I\mathbf{n}_z\delta(z)\Theta(d-\rho)=-I\mathbf{n}_\theta\frac{1}{d}\,\Theta(d-r)\,\delta\!\left(\theta-\frac{\pi}{2}\right),$$

$$\text{div}\,\mathbf{M}\neq 0 \tag{2.3}$$

[$\Theta(x)$ is the step function]. This relation is a mathematical expression of the Ampère hypothesis, according to which a closed circular current is equivalent to a magnetized sheet. The magnetic field can be evaluated from either (2.1) or (2.3). For example, the magnetic vector potential is given by

$$\mathbf{A}=\frac{I}{c}\int\frac{1}{|\mathbf{r}-\mathbf{r}|}\,\mathbf{n}'_\phi\delta(\rho'-d)\,\delta(z')\,dV'$$

$$=-\frac{1}{c}\int\frac{\mathbf{r}-\mathbf{r}'}{|\mathbf{r}-\mathbf{r}'|^3}\times\mathbf{M}(\mathbf{r}')\,dV'. \tag{2.4}$$

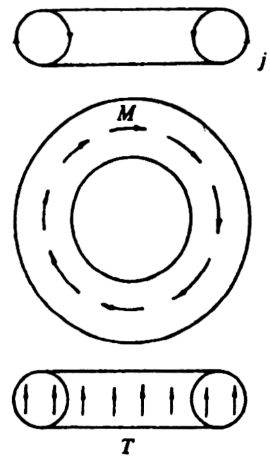

FIG. 2. The poloidal current **j** flowing on the torus surface is equivalent to the magnetization **M**, which in turn is equivalent to the toroidization **T**.

For infinitely small d, the current **j** in Eq. (2.1) is not well defined (the vector $\mathbf{n}_\phi$ loses its meaning at the origin). On the other hand, the magnetization **M** in Eq. (2.3) is still well defined. In the limit $d\to 0$, Eqs. (2.1)–(2.3) mean that the circular current of an infinitely small radius is equivalent to a magnetic dipole.

A more complicated case is a poloidal current flowing on the surface of a torus $(\rho-d)^2+z^2=R^2$ (see Fig. 2). As far as we know, the term "poloidal" is due to Elsasser.[21] To parametrize **j**, it is convenient to introduce the coordinates $\tilde{R},\Psi$ (see Fig. 3):

$$x=(d+\tilde{R}\cos\Psi)\cos\phi,\quad y=(d+\tilde{R}\cos\Psi)\sin\phi,$$

$$z=\tilde{R}\sin\Psi.$$

In these coordinates,

$$\mathbf{j}=\mathbf{n}_\psi\frac{\delta(R-\tilde{R})}{d+\tilde{R}\cos\Psi}\,\frac{j_0}{R^2d}. \tag{2.5}$$

Here $\mathbf{n}_\psi$ is the unit vector tangent to the surface of the torus:

$$\mathbf{n}_\psi=\mathbf{n}_z\cos\Psi-\mathbf{n}_\rho\sin\Psi.$$

It lies in the plane $\phi=$const and defines the direction of **j**. The factor R^2d in the denominator of **j** is introduced for convenience and may be absorbed into j_0. The constant j_0 may be expressed in terms of either the magnetic flux Φ penetrating the solenoid or the number of coils N and the current I in each of them:

$$j_0=\frac{R^2dc\Phi}{8\pi^2(d-\sqrt{d^2-R^2})}=\frac{NIR^2d}{2\pi}.$$

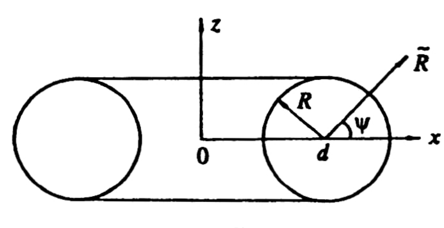

FIG. 3. Geometrical depiction of the $\tilde{R}$, Ψ coordinates used in the text.

Since div $\mathbf{j}=0$, the current $\mathbf{j}$ may be expressed in terms of the magnetization: $\mathbf{j}=\text{curl}\,\mathbf{M}$. It turns out that $\mathbf{M}$ is enclosed inside the torus T and has only a ϕ component:

$$\mathbf{M}=-\mathbf{n}_\phi \frac{\Theta(R-\tilde{R})}{d+\tilde{R}\cos\Psi}\frac{j_0}{R^2 d}. \tag{2.6}$$

Since div $\mathbf{M}=0$, it can be represented as $\mathbf{M}=\text{curl}\,\mathbf{T}$, div $\mathbf{T}\neq 0$, where $\mathbf{T}$ is given by

$$\mathbf{T}=\mathbf{n}_z j_0 \dot{T}/Rd. \tag{2.7}$$

Here

$$T=\ln\frac{d-\sqrt{R^2-z^2}}{d+\sqrt{R^2-z^2}} \tag{2.8}$$

inside the hole of the torus $(0\leqslant\rho\leqslant d-\sqrt{R^2-z^2}, -R\leqslant z \leqslant R)$,

$$T=\ln\frac{\rho}{d+\sqrt{R^2-z^2}} \tag{2.9}$$

inside the torus itself $(d-\sqrt{R^2-z^2}\leqslant\rho\leqslant d+\sqrt{R^2-z^2}, -R\leqslant z\leqslant R)$, and $T=0$ in other space regions. By analogy with the magnetization $\mathbf{M}$, the distribution $\mathbf{T}$ may be called the toroidization (as far as we know, this term has been introduced by Miller.[22]). It follows from Eqs. (2.5)–(2.9) that

$$\mathbf{j}=(\text{curl})^2\mathbf{T}, \quad \text{div}\,\mathbf{T}\neq 0, \tag{2.10}$$

while the vector potential is given by

$$\mathbf{A}=\frac{4\pi}{c}\mathbf{T}(\mathbf{r})+\frac{1}{c}\nabla\int\frac{1}{|\mathbf{r}-\mathbf{r}'|}\text{div}\,\mathbf{T}(\mathbf{r}')dV'. \tag{2.11}$$

The magnetic field strength differs from zero only inside the torus:

$$H_\phi=-\frac{4\pi}{c}\frac{j_0}{dR^2}\frac{1}{\rho}.$$

Physically, Eqs. (2.5)–(2.11) mean that the poloidal current $\mathbf{J}$ given by Eq. (2.5) is equivalent (i.e., produces the same magnetic field) to the toroidal tube with the magnetization $\mathbf{M}$ defined by (2.6) and to the toroidization $\mathbf{T}$ given by (2.7). This is illustrated in Fig. 2.

We consider now the case when the torus dimensions d, R tend to zero. Since R is always less than d, we let R tend to zero first and d later. In the limit $R\to 0$ the current $\mathbf{j}$ (see Fig. 2) becomes ill-defined. On the other hand, $\mathbf{M}$ and $\mathbf{T}$ remain well-defined:

$$\mathbf{M}\to-\mathbf{n}_\phi\frac{\pi}{d^2}j_0\delta(\rho-d)\delta(z) \quad (\text{div}\,\mathbf{M}=0),$$

$$\mathbf{T}\to-\mathbf{n}_z\frac{\pi}{d^2}j_0\Theta(d-\rho)\delta(z) \quad (\text{div}\,\mathbf{T}\neq 0) \quad \text{for } R\to 0. \tag{2.12}$$

After taking the limit $R\to 0$ we let d go to zero. Now it is the turn of the magnetization $\mathbf{M}$ to be ill-defined, but the vector $\mathbf{T}$ is still well-defined:

$$\mathbf{T}\to-\mathbf{n}_z j_0\pi^2\delta^3(\mathbf{r}) \quad (\text{div}\,\mathbf{T}\neq 0) \quad \text{for } d\to 0 \tag{2.13}$$

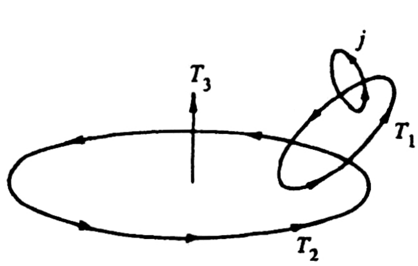

FIG. 4. The family of toroidal solenoids, each turn of which is again a toroidal solenoid (only particular turns are shown).

$[\delta^3(\mathbf{r})=\delta(\rho)\delta(z)/2\pi\rho]$. The VP corresponding to this toroidization is given by[15,23]

$$A_x=-\frac{3\pi^2 j_0}{c}\frac{xz}{r^5}, \quad A_y=-\frac{3\pi^2 j_0}{c}\frac{yz}{r^5},$$

$$A_z=\frac{\pi^2 j_0}{c}\frac{r^2-3z^2}{r^5}-\frac{8\pi^3}{3c}j_0\delta^3(\mathbf{r}). \tag{2.14}$$

We consider now a sequence of toroidal solenoids, each turn of which is again a toroidal solenoid. The simplest configuration is obtained if we take the usual toroidal solenoid (upper part of Fig. 2) and install a new toroidal solenoid into each of its turns. As a result, we arrive at the current configuration shown in Fig. 4. For this case,

$$\mathbf{j}\sim(\text{curl})^3\mathbf{T}(\mathbf{r}), \quad \text{div}\,\mathbf{T}\neq 0, \quad \mathbf{A}\sim\frac{4\pi}{c}\text{curl}\,\mathbf{T}(\mathbf{r}). \tag{2.15}$$

We see that for this current both the vector potential and the magnetic field differ from zero only in those space regions where $\mathbf{T}\neq 0$. When the space region in which $\mathbf{T}\neq 0$ shrinks to a point, the vector potential and the magnetic field differ from zero only at that point.[18,19]

2.1.1. Currents, magnetic dipoles, and monopoles

We rewrite Eq. (2.14) in a condensed form:

$$\mathbf{A}=\frac{1}{r^3}\left[\frac{3}{r^2}\mathbf{r}(\mathbf{mr})-\mathbf{m}\right]+\frac{8\pi}{3}\mathbf{m}\delta^3(\mathbf{r}),$$

$$m_i=-\delta_{iz}\frac{\pi^2 j_0}{c}. \tag{2.16}$$

This equation is an analog of a well-known expression[24] for the magnetic field created by a magnetic dipole $\mathbf{m}$:

$$\mathbf{B}=\frac{1}{r^3}\left[\frac{3}{r^2}\mathbf{r}(\mathbf{mr})-\mathbf{m}\right]+\frac{8\pi}{3}\mathbf{m}\delta^3(\mathbf{r}). \tag{2.17}$$

Sometimes, in the physics literature another representation of $\mathbf{B}$ is used:[25]

$$\mathbf{B}=\frac{1}{r^3}\left[\frac{3}{r^2}\mathbf{r}(\mathbf{mr})-\mathbf{m}\right]-\frac{4\pi}{3}\mathbf{m}\delta^3(\mathbf{r}). \tag{2.18}$$

This difference occurs for the following reason.[25] If we identify the magnetic dipole with an electric current flowing in an infinitely small circular current loop, the VP is given by

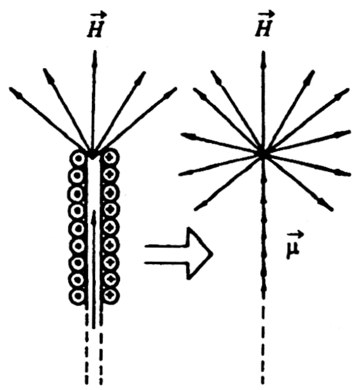

FIG. 5. The magnetic fields of the semi-infinite solenoid and the magnetized filament coincide with the field of a magnetic monopole everywhere except at the position of the solenoid or filament.

$$\mathbf{A}=\frac{1}{cr^3}(\mathbf{m}_e\times\mathbf{r}),\quad \mathbf{m}_e=\frac{1}{2}\int(\mathbf{r}\times\mathbf{j})dV. \tag{2.19}$$

Applying to **A** the curl operator and using the identity (see, e.g., Ref. 26)

$$\frac{\partial}{\partial x_j}\left(\frac{x_i}{r^3}\right)=\frac{1}{r^3}\left(\delta_{ij}-3\frac{x_ix_j}{r^2}\right)+\frac{4\pi}{3}\delta_{ij}\delta^3(\mathbf{r}), \tag{2.20}$$

we get (2.17). On the other hand, if we suggest that magnetic dipoles consist of magnetic monopoles,

$$\mathbf{m}_m=\int\rho_m\mathbf{r}dV,\quad \int\rho_m dV=0, \tag{2.21}$$

then the magnetic induction is obtained from the scalar magnetic potential:

$$\mathbf{B}=-\nabla\Phi_m,\quad \Phi_m=\frac{\mathbf{m}_m\mathbf{r}}{r^3}. \tag{2.22}$$

Again, using the differentiation rule (2.20), we arrive at (2.18). This means that the different coefficients of the $\delta^3(\mathbf{r})$ terms in (2.17) and (2.18) are due to different definitions of magnetic dipoles.

The expression (2.17) leads to the so-called hyperfine contact interaction derived by Fermi. It was observed experimentally by measuring the splitting of hydrogen atomic s levels. Above, we have used the fact that $\mathbf{B}=\mathbf{H}$ in the absence of a medium.

Consider a semi-infinite cylindrical solenoid of radius R formed either by circular currents or magnetic current dipoles (Fig. 5). The magnetic VP of a particular current j lying in the $z=z_0$ plane is given

$$\mathbf{A}=\frac{1}{c}j\int\frac{d\phi'}{|\mathbf{r}-\mathbf{r}'|}\mathbf{n}_{\phi'},$$

where $\mathbf{n}_\phi=\mathbf{n}_y\cos\phi-\mathbf{n}_x\sin\phi$ is the vector defining the current direction. The only nonvanishing component of the VP is

$$A_\phi(\rho,z)=\frac{2j}{c\sqrt{\rho R}}Q_{1/2}\left(\frac{\rho^2+R^2+(z-z_0)^2}{2\rho R}\right).$$

Here $Q_\nu(x)$ is the Legendre function of the second kind. Using its asymptotic behavior

$$Q_\nu(x)\to\sqrt{\pi}\Gamma(\nu+1)/2^{\nu+1}\Gamma(\nu+3/2)x^{\nu+1},\quad x\to\infty,$$

one obtains, for infinitely small radius R (or large distances),

$$\mathbf{A}=A_\phi\mathbf{n}_\phi,\quad A_\phi\approx\frac{\pi Rj\rho}{c\tilde{r}^3},\quad \tilde{r}=[x^2+y^2+(z-z_0)^2]^{1/2},\quad \operatorname{div}\mathbf{A}=0. \tag{2.23}$$

The nonvanishing components of the magnetic field strength are

$$H_x=\frac{\pi Rj}{c}\frac{\partial^2}{\partial x\partial z}\frac{1}{\tilde{r}},\quad H_y=\frac{\pi Rj}{c}\frac{\partial^2}{\partial y\partial z}\frac{1}{\tilde{r}},$$

$$H_z=\frac{\pi Rj}{c}\left[\frac{\partial^2}{\partial z^2}\frac{1}{\tilde{r}}+4\pi\delta(x)\delta(y)\delta(z-z_0)\right],\quad \operatorname{div}\mathbf{H}=0. \tag{2.24}$$

The magnetic field of the semi-infinite solenoid is obtained by integrating **H** from $z_0=-\infty$ to $z_0=0$. This results in

$$\mathbf{H}=\frac{\pi Rj}{c}\left[\frac{\mathbf{r}}{r^3}+4\pi\mathbf{n}_z\delta(x)\delta(y)\Theta(-z)\right],\quad \operatorname{div}\mathbf{H}=0. \tag{2.25}$$

Thus, an infinitely thin semi-infinite magnetized filament generates the field of a magnetic monopole everywhere except at the position of the filament itself. The equalities

$$\operatorname{div}\mathbf{B}=0,\quad \iint B_n dS=0$$

guarantee the absence of free magnetic charges. Owing to the presence of the δ-function term in (2.25), the monopoles are not true ones.

Earlier, these results were obtained in a qualitative manner in Ref. 22.

2.1.2. Interaction with the external electromagnetic field

Now we explain how the current distributions just obtained interact with an external electromagnetic field $(\mathbf{E}_{\text{ext}},\mathbf{H}_{\text{ext}})$. In the absence of a medium $\mathbf{E}_{\text{ext}}$ and $\mathbf{H}_{\text{ext}}$ satisfy the Maxwell equations

$$\operatorname{div}\mathbf{B}_{\text{ext}}=0,\quad \operatorname{div}\mathbf{D}_{\text{ext}}=4\pi\rho_{\text{ext}},$$

$$\operatorname{curl}\mathbf{E}_{\text{ext}}=-\frac{1}{c}\frac{\partial\mathbf{B}_{\text{ext}}}{\partial t},$$

$$\operatorname{curl}\mathbf{H}_{\text{ext}}=\frac{1}{c}\frac{\partial\mathbf{D}_{\text{ext}}}{\partial t}+\frac{4\pi}{c}\mathbf{j}_{\text{ext}},\quad \mathbf{D}=\mathbf{E},\quad \mathbf{B}=\mathbf{H}.$$

Let **j** be of the form

$$\mathbf{j}\sim(\operatorname{curl})^n\mathbf{T}(\mathbf{r}),\quad \operatorname{div}\mathbf{T}\neq 0, \tag{2.26}$$

where **T** is either confined to a finite region of space or decreases sufficiently fast at infinity. Then the interaction energy of this configuration with an external electromagnetic field is given by

$$U=-\frac{1}{c}\int \mathbf{j}\mathbf{A}_{\text{ext}}\sim\int \mathbf{T}(\text{curl})^{n-1}\mathbf{H}_{\text{ext}}dV. \tag{2.27}$$

The final answer is different for even and odd n. If $n=2k+1$, then

$$U\sim(-1)^k\int \mathbf{T}\left(\frac{1}{c}\frac{\partial}{\partial t}\right)^{2k}\mathbf{H}_{\text{ext}}dV. \tag{2.28}$$

For $n=2k+2$ we have

$$U\sim(-1)^k\int \mathbf{T}\left(\frac{1}{c}\frac{\partial}{\partial t}\right)^{2k+1}\mathbf{E}_{\text{ext}}dV. \tag{2.29}$$

For distances large compared with the dimensions of the particular current configuration, the interaction energy has the form

$$U\sim\mathbf{t}\mathbf{E}_{\text{ext}}^{(2k+1)} \quad \text{and} \quad U\sim\mathbf{t}\mathbf{H}_{\text{ext}}^{(2k)}, \tag{2.30}$$

where the superscripts denote the corresponding time derivatives and $\mathbf{t}=\int\mathbf{T}(\mathbf{r})dV$ is a vector depending on the geometrical dimensions of the treated current, magnetized, or toroidized configuration. The explicit form of $\mathbf{t}$ for a particular toroidal current configuration may be found in Ref. 27. In particular, for the toroidization given by (2.12), $\mathbf{t}=-2\pi^2 j_0\mathbf{n}_z$.

The current configuration corresponding to $k=0$ in (2.29) (poloidal current on the torus surface) was considered by Zeldovich,[14] who referred to it as an anapole. In the modern physics literature, anapoles are associated with radiationless charge–current sources (see, e.g., Refs. 15, 28, and 29), while the charge–current configurations corresponding to Eqs. (2.15) and (2.16) are referred to as toroidal moments (Refs. 4, 30, and 31). The next terms [$k=1$ in Eqs. (2.28) and (2.29)] in the development of the interaction energy were written out in Ref. 19. The general form of the interactions (2.28) and (2.29) was given in Ref. 18.

Thus, we obtain a sequence of current configurations (or magnetizations corresponding to them) which interact with the time-dependent magnetic or electric field. For example, the usual current loop interacts with an external magnetic field in the same way as the magnetic dipole orthogonal to it. The poloidal current shown in the upper part of Fig. 2, the magnetized ring in its middle part, and the toroidal distribution in its lower part all interact with the first derivative of the electric field.

We turn now to Fig. 4. The current distribution $\mathbf{J}$ shown in it is obtained if, instead of each turn of the TS shown in the upper part of Fig. 2, we insert a new TS. The current configuration $\mathbf{j}$, div $\mathbf{j}=0$, of Fig. 4, the magnetization $\mathbf{T}_1$, div $\mathbf{T}_1=0$, distributed over the surface of the torus (in the same way as the current $\mathbf{j}$ in Fig. 2), the toroidization $\mathbf{T}_2$, div $\mathbf{T}_2=0$, confined to the interior of the torus (like the magnetization $\mathbf{M}$ shown in Fig. 2), and the toroidization $\mathbf{T}_3$, div $\mathbf{T}_3\neq 0$, all interact with the second derivative of the magnetic field. The words "interact with the time derivative..." mean that the interaction energy has the form (2.30), i.e., it is proportional to the time derivative (of definite order) of the electric or magnetic field.

Obviously, the equivalence between the current distributions and magnetizations (toroidizations) established in this section is a straightforward generalization of the original Ampère hypothesis.

One may ask why Eqs. (2.28)–(2.30) do not contain the even time derivatives of the electric field and the odd derivatives of the magnetic one. It turns out[15,18,19,32] that the missing terms describe the interaction of closed configurations composed of electric dipoles. To see this, consider electric dipoles distributed inside the space region S with the vector density $\mathbf{d}(\mathbf{r})$. Their interaction with an external EMF is given by

$$U\sim\int \mathbf{d}(\mathbf{r})\mathbf{E}_{\text{ext}}(\mathbf{r})dV. \tag{2.31}$$

Let $\mathbf{d}(\mathbf{r})$ be distributed over the surface of the torus in the same way as the magnetization $\mathbf{M}$ shown in the middle part of Fig. 2. As in the studied case, div $\mathbf{d}=0$, the vector function can be represented in the form $\mathbf{d}=\text{curl}\,\mathbf{T}$, div $\mathbf{T}\neq 0$, where $\mathbf{T}\sim\mathbf{n}_z T$, and T is defined by Eqs. (2.8) and (2.9) and is shown at the bottom of Fig. 2. Substituting $\mathbf{d}$ into (2.31) and integrating by parts, we obtain, for distances large compared with the dimensions of the torus,

$$U\sim\frac{\partial\mathbf{H}_{\text{ext}}(\mathbf{r}_0)}{\partial t}\mathbf{t}, \tag{2.32}$$

where $\mathbf{t}=\int\mathbf{T}dV$ and $\mathbf{r}_0$ is some point inside the torus.

Further, let the electric dipoles be distributed over the torus surface like the current $\mathbf{J}$ in the upper part of Fig. 2. Then

$$\mathbf{d}=(\text{curl})^2\mathbf{T}(\mathbf{r}), \quad \text{div}\ \mathbf{T}\neq 0, \tag{2.33}$$

where T is the same as before [Eqs. (2.8) and (2.9)]. Substituting this $\mathbf{d}$ into (2.18), one easily obtains

$$U\sim\frac{\partial^2\mathbf{E}_{\text{ext}}(\mathbf{r}_0)}{\partial t^2}\mathbf{t}. \tag{2.34}$$

The continuation of this procedure shows that the interaction of electric dipoles with the external EMF is indeed the missing link in Eqs. (2.28)–(2.30). In particular, the term $\mathbf{t}(\partial\mathbf{H}/\partial t)$ describes the interaction of the closed electric dipole ring (see the middle part of Fig. 2, where the distribution M of magnetic dipoles should be replaced by a distribution of electric ones) with the time derivative of an external magnetic field. Corresponding experiments were performed by Tolstoy and Spartakov,[16] and their interpretation was given in Ref. 17.

2.2. Magnetizations and the Debye potential representation

According to the Helmholtz–Neumann theorem (see, e.g., Ref. 33) an arbitrary vector function and, in particular, the current density can be represented as the sum of longitudinal and transverse parts:

$$\mathbf{j}=\mathbf{j}_l+\mathbf{j}_t, \quad \text{curl}\ \mathbf{j}_l=0, \quad \text{div}\ \mathbf{j}_t=0.$$

The terms $\mathbf{j}_l$ and $\mathbf{j}_t$ can be represented in the form

$$\mathbf{j}_l=\nabla\Psi_1, \quad \mathbf{j}_t=\text{curl}(\mathbf{r}\Psi_2)+(\text{curl})^2(\mathbf{r}\Psi_3).$$

As a result, we obtain

$$\mathbf{j}=\nabla\Psi_1+\mathrm{curl}(\mathbf{r}\Psi_2)+(\mathrm{curl})^2(\mathbf{r}\Psi_3). \quad (2.35)$$

The functions Ψ_1, Ψ_2, and Ψ_3 are known as the Debye potentials. They were introduced by Debye[34] in evaluating the light pressure on a sphere of arbitrary material. Various other authors (Thomson, Mie Whittaker, Bromwich, and Sommerfeld) applied these potentials to electromagnetic problems. Earlier, Lamb used the representation (2.35) in studying fluid-mechanics and electromagnetic problems.[35–37]

Comparing (2.35) with (2.1) and (2.2), we obtain

$$\Psi_1=\Psi_3=0, \quad \Psi_2=\frac{1}{2}\,\delta(\rho-d)\Theta\left(\frac{\pi}{2}-\theta\right). \quad (2.36)$$

The corresponding magnetization is given by

$$\mathbf{M}'=\mathbf{n}_r\delta(r-d)\Theta\left(\frac{\pi}{2}-\theta\right). \quad (2.37)$$

This magnetization covers the upper hemisphere of radius d and is directed along its radius (see Fig. 1). It is certainly different from the magnetization (2.3). The magnetizations $\mathbf{M}$ and $\mathbf{M}'$ are connected by the gradient transformation

$$\mathbf{M}'=\mathbf{M}+\nabla\chi, \quad \chi=-\Theta(d-z)\Theta\left(\frac{\pi}{2}-\theta\right),$$

i.e., the function χ differs from zero inside the upper hemisphere. This equation means that the magnetizations $\mathbf{M}$ and $\mathbf{M}'$, despite their different functional forms, lead to the same observable effects. The reason for the appearance of different magnetizations is that the equation curl $\mathbf{M}=\mathbf{J}$ does not fix $\mathbf{M}$ uniquely. We note that the magnetic field strength $\mathbf{H}$ satisfies almost the same equation curl $\mathbf{H}=\mathbf{J}$ but with the auxiliary condition div $\mathbf{H}=0$. These two equations are sufficient for fixing $\mathbf{H}$. In general, the condition div $\mathbf{M}=0$ is not imposed on $\mathbf{M}$. It turns out that the requirement for $\mathbf{M}$ to disappear in the nearest vicinity of $\mathbf{J}$ does not fix $\mathbf{M}$ unambiguously. On the other hand, if both curl $\mathbf{M}=\mathbf{j}$ and div $\mathbf{M}$ are known, then (see, e.g., Ref. 33)

$$4\pi\mathbf{M}=\mathrm{curl}\int \mathrm{curl}\,\mathbf{M}(\mathbf{r}')\,\frac{dV'}{|\mathbf{r}-\mathbf{r}'|}-\nabla\int\frac{\mathrm{div}\,\mathbf{M}(\mathbf{r}')}{|\mathbf{r}-\mathbf{r}'|}. \quad (2.38)$$

Obviously, curl $\mathbf{M}$ and div $\mathbf{M}$ define $\mathbf{M}$ up to a constant vector, which is chosen to be zero in Eq. (2.38).

2.2.1. On the inversion of the Debye parametrization

An interesting question is the inversion of the Debye parametrization (2.35), i.e., the expression of Ψ_1, Ψ_2, and Ψ_3 in terms of the current density $\mathbf{j}$. Rearranging the terms in (2.35), we have

$$\mathbf{j}=\nabla\Psi_1'+(\mathbf{r}\times\nabla)\Psi_2'+\mathbf{r}\Psi_3'. \quad (2.39)$$

This parametrization is used on the same footing as (2.35) (see, e.g., Refs. 38 and 39). To find Ψ_i' one applies to $\mathbf{j}$ the div and curl operators:

$$(\mathbf{r}\cdot\mathbf{j})=r\,\frac{d\Psi_1'}{dr}+r^2\Psi_3',$$

$$r^2\,\mathrm{div}\,\mathbf{j}=(\mathbf{r}\times\nabla)^2\Psi_1'+\frac{d}{dr}[r(\mathbf{r}\cdot\mathbf{j})],$$

$$\mathbf{r}\cdot\mathrm{curl}\,\mathbf{j}=(\mathbf{r}\times\nabla)^2\Psi_2', \quad \mathbf{r}\cdot\mathrm{curl\ curl}\,\mathbf{j}=-(\mathbf{r}\times\nabla)^2\Psi_3'.$$

As a result, the following equations are obtained for ψ_i':

$$(\mathbf{r}\times\nabla)^2\Psi_1'=r^2\,\mathrm{div}\,\mathbf{j}-\frac{d}{dr}[r(\mathbf{r}\cdot\mathbf{j})],$$

$$(\mathbf{r}\times\nabla)^2\Psi_2'=\mathbf{r}\cdot\mathrm{curl}\,\mathbf{j}, \quad (\mathbf{r}\times\nabla)^2\Psi_3'=-\mathbf{r}\cdot\mathrm{curl\ curl}\,\mathbf{j}. \quad (2.40)$$

Consider the equation

$$(\mathbf{r}\times\nabla)^2\Psi=f. \quad (2.41)$$

Its Green function

$$G(\mathbf{n},\mathbf{n}')=-\sum_{l,m}\frac{1}{l(l+1)}\,Y_l^m(\mathbf{n})Y_l^{m*}(\mathbf{n}'), \quad \mathbf{n}=(\theta,\phi),$$

$$l\geqslant 1,$$

satisfies the equation

$$(\mathbf{r}\times\nabla)^2G=\frac{1}{\sin\theta}\,\delta(\theta-\theta')\delta(\phi-\phi')-\frac{1}{4\pi}.$$

Thus,

$$\Psi=\int G(\mathbf{n},\mathbf{n}')f(\mathbf{r}')d\Omega'. \quad (2.42)$$

The functions Ψ_i' are obtained if one substitutes the right-hand sides of (2.40) instead of f. We still need the relations between Ψ_i and Ψ_i'. They are

$$\Delta\Psi_3=\Psi_3', \quad \Psi_2=-\Psi_2', \quad \Psi_1=\Psi_1'+(1+\mathbf{r}\nabla)\Psi_3'. \quad (2.43)$$

These equations are easily solved:

$$\Psi_3=-\frac{1}{4\pi}\int\frac{1}{|\mathbf{r}-\mathbf{r}'|}\,\Psi_3'(\mathbf{r}')dV',$$

$$\Psi_1=\Psi_1'-\frac{1}{4\pi}(1+\mathbf{r}\nabla)\int\frac{1}{|\mathbf{r}-\mathbf{r}'|}\,\Psi_3'(\mathbf{r}')dV'. \quad (2.44)$$

Equations similar to (2.42)–(2.44) were proved with different degrees of rigor many times (see, e.g., Refs. 40–45).

2.2.2. Physical meaning of the Ψ functions

Now we are able to clarify the physical meaning of the functions Ψ_i defining the current density $\mathbf{j}$. For this purpose, we consider the interaction of the pure current density $\mathbf{j}$ (which corresponds to $\Psi_1=0$) with an external electromagnetic field defined by the vector potential $\mathbf{A}_{\mathrm{ext}}$:

$$U=-\frac{1}{c}\int\mathbf{j}\mathbf{A}_{\mathrm{ext}}dV. \quad (2.45)$$

Substituting here $\mathbf{j}$, integrating by parts, and assuming that $\mathbf{j}$ does not overlap with the space region S where $\mathbf{J}_{\mathrm{ext}}\neq 0$, we get

$$U=U_d+U_t, \quad U_d=-\frac{1}{c}\int\mathbf{r}\mathbf{H}\Psi_2dV,$$

$$U_t=-\frac{1}{c^2}\int \mathbf{r}\dot{\mathbf{E}}\Psi_3 dV. \qquad (2.46)$$

Here **H** and **E** are the electromagnetic field strengths of the external field. A dot above a letter denotes the time derivative. Let the dimensions of S be small compared with the distance from the sources of the external field. Then external fields varying rather slowly over S can be approximated by their values taken at some point $\mathbf{r}_0$ inside S:

$$U_d^{(1)}=-\frac{1}{c}\mathbf{H}(0)\int \mathbf{r}\Psi_2 dV,$$

$$U_t^{(1)}=-\frac{1}{c^2}\dot{\mathbf{E}}(0)\int \mathbf{r}\Psi_3 dV. \qquad (2.47)$$

Here $\mathbf{H}(0)=\mathbf{H}(r_0)$ and $\mathbf{E}(0)=\mathbf{E}(r_0)$. It then follows that $\boldsymbol{\mu}_d=\int \mathbf{r}\Psi_2 dV$ and $\boldsymbol{\mu}_t=\int \mathbf{r}\Psi_3 dV$ are the magnetic dipole and toroidal moments (as they interact with the external magnetic field and with the time derivative of the external electric field, respectively) The next terms in the development of U_d are

$$U_d^{(2)}=-\frac{1}{c}\frac{\partial H_i(0)}{\partial x_k}\mu_{ik}, \quad \mu_{ik}=\int q_{ik}^{(2)}\Psi_2 dV,$$

$$q_{ik}^{(2)}=\left(x_i x_k-\frac{1}{3}\delta_{ik}r^2\right),$$

$$U_d^{(3)}=\frac{1}{2c}\frac{\partial^2 H_i(0)}{\partial x_k \partial x_j}\mu_{ijk}-\frac{1}{10c^3}\frac{\partial^2 \mathbf{H}(0)}{\partial t^2}\boldsymbol{\mu}_d^{(2)},$$

$$\boldsymbol{\mu}_d^{(2)}=\int \mathbf{r}r^2\Psi_2 dV,$$

$$\mu_{ijk}=\int q_{ijk}^{(3)}\Psi_2 dV,$$

$$q_{ijk}^{(3)}=\left[x_i x_j x_k-\frac{1}{5}(\delta_{ij}x_k+\delta_{ik}x_j+\delta_{jk}x_i)r^2\right]. \qquad (2.48)$$

Obviously, μ_{ij} and μ_{ijk} are the quadrupole and octupole magnetic moments, respectively. Thus, the function Ψ_2 describes a set of magnetic moments of different multipolarities. Similarly, one obtains the next terms in the expansion of U_t:

$$U_t^{(2)}=-\frac{1}{c^2}\frac{\partial \dot{E}_i(0)}{\partial x_k}t_{ik}, \quad t_{ik}=\int q_{ik}^{(2)}\Psi_3 dV,$$

$$U_t^{(3)}=-\frac{1}{2c^2}\frac{\partial^2 E_i(0)}{\partial x_k \partial x_j}t_{ijk}-\frac{1}{10c}\left(\frac{\partial}{c\partial t}\right)^3 \mathbf{E}(0)\boldsymbol{\mu}_t^{(2)},$$

$$\boldsymbol{\mu}_t^{(2)}=\int \mathbf{r}r^2\Psi_3 dV,$$

$$t_{ijk}=\int q_{ijk}^{(3)}\Psi_3 dV. \qquad (2.49)$$

This means that the function Ψ_3 describes the toroidal moments of higher multipolarities.[15] Their physical realization via toroidal solenoids embedded into each other has been given in Ref. 46.

Let Ψ_2 be of the form

$$\Psi_2=\Delta\Psi_2^{(1)}. \qquad (2.50)$$

Then

$$U_d=-\frac{1}{c^3}\frac{\partial^2 \mathbf{H}(0)}{\partial t^2}\int \mathbf{r}\Psi_2^{(1)}dV. \qquad (2.51)$$

It follows from this that such a current configuration interacts neither with a stationary nor with a linearly growing (with time) external magnetic field. It interacts with a magnetic field with polynomial growth not slower than t^2. Further, if Ψ_2 is represented in the form

$$\Psi_2=(\Delta)^n\Psi_2^{(n)}, \quad n\geq 1, \qquad (2.52)$$

then

$$U_d=-\frac{1}{c}\left(\frac{\partial}{c\partial t}\right)^{2n}\mathbf{H}(0)\int \mathbf{r}\Psi_2^{(n)}dV. \qquad (2.53)$$

Such a current distribution interacts with a magnetic field with polynomial growth not slower than t^{2n}. If the external magnetic field grows as t^α (where α is not integer), then the interaction energy decreases as a function of time for $\alpha<2n$ and increases for $\alpha>2n$.

Now we turn to the toroidal moments. Taking into account the Maxwell equations and the fact that at large distances $\mathbf{j}_{\text{ext}}$ does not overlap with S, we rewrite U_t as

$$U_t=-\frac{1}{c^2}\int \mathbf{r}\dot{\mathbf{E}}\Psi_3 dV=-\frac{1}{c^2}\dot{\mathbf{E}}(0)\int \mathbf{r}\Psi_3 dV. \qquad (2.54)$$

Now let Ψ_3 be of the form

$$\Psi_3=(\Delta)^n\Psi_3^{(n)}, \quad n\geq 1. \qquad (2.55)$$

Then

$$U_t=-\frac{1}{c}\left(\frac{\partial}{c\partial t}\right)^{2n+1}\mathbf{E}(0)\int \mathbf{r}\Psi_3^{(n)}dV. \qquad (2.56)$$

This means that this current configuration interacts with a polynomial electric field which grows not slower than t^{2n+1}.

It then follows that a magnetized sample consisting of magnetic dipoles, all of which are combined into ring-like structures (thus realizing toroidal magnetic moments), does not interact with a spatially uniform magnetic field $\mathbf{H}_0$ (although each of the magnetic dipoles does interact with $\mathbf{H}_0$). This sample interacts with curl $\mathbf{H}_0$ (or, equivalently, with the time derivative of the electric field). A magnetized sample, all the magnetic moments of which are organized into toroidal moments of higher multipolarities, interacts with higher derivatives of the electric and magnetic fields. Thus, we obtain a one-to-one correspondence between the hierarchy of magnetic structures and the electromagnetic fields interacting with them. This selectivity of the interaction can probably be used for storage and coding of information. There have been some first practical attempts in this direction (see, e.g., Ref. 47).

When representing Ψ_2 or Ψ_3 in the form (2.52) or (2.55), we have implicitly assumed that $\Psi_2^{(n)}$ or $\Psi_3^{(n)}$ are confined to a finite space region or that they decrease sufficiently fast at large distances. This is required for the disappearance of the surface integrals arising when the transition from (2.50) to (2.51), or from (2.55) to (2.56), is performed. In fact, every function Ψ can be represented in the form

$$\Psi=\Delta f, \quad \text{where } f=-\frac{1}{4\pi}\int|\mathbf{r}-\mathbf{r}'|^{-1}\Psi(\mathbf{r}')dV',$$

but there is no guarantee that f decreases sufficiently fast (which is needed for its physical meaning). As a result, Eqs. (2.51), (2.53), (2.54), and (2.56) are valid for very specific current configurations.

We elucidate now what magnetic field corresponds to the choice of functions in the form (2.50) and (2.55). A convenient parametrization of the VP corresponding to the stationary current density has been found in Ref. 45 [see Eqs. (2.10) and (2.14) therein]. Substituting the current parametrization (2.35) into it, we get, outside the space region S to which the current density is confined,

$$\mathbf{A}=\frac{4\pi}{c}\sum\frac{1}{2l+1}r^{-l-1}(\mathbf{r}\times\nabla)Y_l^m\int r^lY_l^{m*}\Psi_2dV$$
$$+\frac{4\pi}{c}\nabla\sum\frac{l}{2l+1}r^{-l-1}Y_l^m\int r^lY_l^{m*}\Psi_3dV. \tag{2.57}$$

The magnetic field $\mathbf{H}$ disappears if

$$\int r^lY_l^{m*}\Psi_2dV=0. \tag{2.58}$$

This relation is automatically satisfied if Ψ_2 has the form (2.52). The condition for the vector potential to vanish is (2.58) and

$$\int r^lY_l^{m*}\Psi_3dV=0. \tag{2.59}$$

Obviously, it is satisfied if Ψ_3 has the form (2.55). Thus, the simultaneous fulfillment of Eqs. (2.52) and (2.55) leads to the disappearance of both the VP and the magnetic field outside the space region S to which the current configuration $\mathbf{J}$ is confined.

The representation (2.57) of the VP, valid only outside S, disappears for specific current distributions defined by Eqs. (2.52) and (2.55). This does not mean that the VP vanishes everywhere. Inside S one should use either the general formula

$$\mathbf{A}=\frac{1}{c}\int\frac{1}{|\mathbf{r}-\mathbf{r}'|}\mathbf{j}(\mathbf{r}')dV'$$

(as was done in Sec. 3.1) or its expansion in vector spherical harmonics. The latter certainly differs from (2.16) inside S. It follows from this that no experiments performed outside S (including Aharonov–Bohm-like experiments) can give information on the studied current distribution inside S.

Obviously, Eqs. (2.53) and (2.56) generalize Eqs. (2.28) and (2.29) obtained earlier. In fact, Eqs. (2.53) and (2.56) contain two arbitrary functions Ψ_2 and Ψ_3, while only one function T enters into (2.28) and (2.29).

Inspection of Eqs. (2.46)–(2.56) shows that there are two degrees of freedom. The first of them is due to the appearance of definite multipoles in the expansions of $\mathbf{E}$ and $\mathbf{H}$ [see Eqs. (2.47)–(2.49)]. Let Ψ_2 (or Ψ_3) transform according to a particular representation of the rotation group with a fixed value l_2 (l_3) of the angular momentum. Then only terms with these angular momenta survive in the expansion of U_d (or U_t). In particular, for $l_2=l_3=1$ we have

$$U_d(l_2=1)=-\frac{1}{c}\mathbf{H}(0)\int\mathbf{r}\Psi_2dV$$
$$-\frac{1}{10c^3}\ddot{\mathbf{H}}(0)\int\mathbf{r}r^2\Psi_2dV-\frac{1}{280c^5}\mathbf{H}^{(4)}(0)$$
$$\times\int\mathbf{r}r^4\Psi_2dV-\ldots,$$
$$U_t(l_3=1)=-\frac{1}{c^2}\dot{\mathbf{E}}(0)\int\mathbf{r}\Psi_3dV-\frac{1}{10c^4}\mathbf{E}^{(3)}(0)$$
$$\times\int\mathbf{r}r^2\Psi_3dV-\frac{1}{280c^6}\mathbf{E}^{(5)}(0)$$
$$\times\int\mathbf{r}r^4\Psi_3dV-\ldots.$$

Let $l_2=l_3=2$. Then

$$U_d(l=2)=-\frac{1}{c}\frac{\partial H_i(0)}{\partial x_k}\mu_{ik}$$
$$-\frac{1}{42c^3}\frac{\partial\ddot{H}_i(0)}{\partial x_k}\int q_{ik}^{(2)}r^2\Psi_2dV,$$
$$U_t(l=2)=-\frac{1}{c^2}\frac{\partial\dot{E}_i(0)}{\partial x_k}t_{ik}$$
$$-\frac{1}{42c^4}\frac{\partial E_i^{(3)}(0)}{\partial x_k}\int q_{ik}^{(2)}r^2\Psi_3dV.$$

The second degree of freedom is due to the fact that for a given multipole it is possible to change the interaction with an external electromagnetic field by choosing Ψ_2 and Ψ_3 in the form (2.52) and (2.55), respectively.

Thus, we have a wonderful electromagnetic object with a number of interesting properties. It does not act on a test charge or on a magnetic needle. On the other hand, it interacts with a time-dependent external electromagnetic field. The difficult question of the equality of action and reaction is beyond the scope of the present considerations. The question arises of practical realizations of this object. One of them is the family of toroidal solenoids considered in Sec. 2.1 (when each turn of a solenoid is replaced by a toroidal solenoid). The ambiguity in the magnetization implies that this realization is not unique.

2.2.3. Transition to point-like sources

For a point-like current source carrying a magnetic moment of multipolarity l_2 and a toroidal moment of multipolarity l_3 we have[13,18]

$$\Psi_2^{(k_2,l_2)}=f_2(t)\Delta^{k_2}(Q^{(l_2)}\nabla)\delta^3(\mathbf{r}),$$
$$\Psi_3^{(k_3l_3)}=f_3(t)\Delta^{k_3}(Q^{(l_3)}\nabla)\delta^3(\mathbf{r}). \tag{2.60}$$

Here $f_2(t)$ and $f_3(t)$ are functions of the time only,

$$(Q^{(l)}\nabla)=Q^{(l)}_{i_1i_2\ldots i_l}\nabla_{i_1}\nabla_{i_2}\ldots\nabla_{i_l}$$

(a summation over repeated indices is understood), and $\nabla_i=\partial/\partial x_i$. Further, $Q^{(l)}_{i_1i_2\ldots i_l}(n_k)$ is a traceless symmetric form of order l in the unit vectors n_i, $i=1,\ldots,3$, defining the orientation of the current configuration [e.g., $Q^1_i=n_i$, $Q^{(2)}_{i,j}=n_in_j-\delta_{ij}/3$ $(i=1,\ldots,3)$, etc.]. Then the point-like analogs of the interaction energies defined by (2.46) are given by

$$U_d=(-1)^{l_2+1}f_2(t)c^{-2k_2-1}l_2(\mathbf{v}_{l_2}\mathbf{H}^{(2k_2)}),$$

$$U_t=(-1)^{l_3+1}f_3(t)c^{-2k_3-1}l_3(\mathbf{v}_{l_3}\mathbf{E}^{(2k_3+1)}).$$

Here $\mathbf{v}$ is a vector with Cartesian components $(\mathbf{v}_l)_i=Q^{(l)}_{ii_2\ldots i_l}\nabla_{i_2}\ldots\nabla_{i_l}$. Superscripts on $\mathbf{E}$ and $\mathbf{H}$ denote time derivatives. We shall write out a few particular terms.

The choices $l_2=1$, $k_2=0$ and $l_3=1$, $k_3=0$ correspond to dipole magnetic and toroidal moments, respectively. Their interaction with the external EMF is given by

$$U^{(1)}_d=f_2(\mathbf{nH})/c \quad \text{and} \quad U^{(1)}_t=f_3(\mathbf{n}\dot{\mathbf{E}})/c^2.$$

The quadrupole magnetic and toroidal moments correspond to $l_2=2$, $k_2=0$ and $l_3=2$, $k_3=0$, respectively. The interaction energies are

$$U^{(2,0)}_d=-\frac{2}{c}f_2(\mathbf{n}\nabla)(\mathbf{nH}), \quad U^{(2,0)}_t=-\frac{2}{c^2}f_3(\mathbf{n}\nabla)(\mathbf{n}\dot{\mathbf{E}}).$$

Further, for $l_2=1$, $k_2=1$ and $l_3=1$, $k_3=1$ one gets

$$U^{(1,1)}_d=\frac{1}{c^3}f_2(\mathbf{n}\ddot{\mathbf{H}}), \quad U^{(1,1)}_t=\frac{1}{c^3}f_3(\mathbf{n}\mathbf{E}^{(3)}).$$

Again, we see that the indices l and k describe different degrees of freedom. The index l defines the particular multipole, while k shows how much the magnetic distribution is "toroidized."

2.2.4. Interaction of charge densities with an external field

The reader may wonder why we have confined ourselves to the consideration of pure current configurations imbedded in the external electromagnetic field. The obvious generalization including charge density is (see, e.g., Refs. 24 and 28)

$$U=\int\rho\phi_{\text{ext}}dV-\frac{1}{c}\int\mathbf{j}\mathbf{A}_{\text{ext}}dV. \tag{2.61}$$

We rewrite this equation as

$$U=U_q+U_d+U_t. \tag{2.62}$$

Here U_t and U_d were defined earlier [see (2.46)], and

$$U_q=\int\rho\phi_{\text{ext}}dV-\frac{1}{c}\int\mathbf{j}_l\mathbf{A}_{\text{ext}}dV. \tag{2.63}$$

Here $\mathbf{j}_l$ is the longitudinal part of $\mathbf{j}$ ($\mathbf{j}_l=\nabla\Psi_1$, $\operatorname{div}\mathbf{j}=-\dot{\rho}$). Expanding U_q, we get[18,31]

$$U_q=e\phi_{\text{ext}}(0)+(\mathbf{d}\nabla)\phi_{\text{ext}}(0)+\frac{1}{2}q_{ik}\frac{\partial^2\phi_{\text{ext}}(0)}{\partial x_i\partial x_k}$$

$$-\frac{1}{c}\dot{\mathbf{d}}\mathbf{A}_{\text{ext}}(0)-\frac{1}{2c}\dot{q}_{ik}\frac{\partial(A_{\text{ext}})_i(0)}{\partial x_k}$$

$$-\frac{1}{2}\boldsymbol{\mu}_l\mathbf{H}_{\text{ext}}(0)+\ldots, \tag{2.64}$$

where $\mathbf{d}=\int\rho\mathbf{r}dV$, $q_{ik}=\int x_ix_k\rho dV$, and $\boldsymbol{\mu}_l=(1/2c)\int(\mathbf{r}\times\mathbf{j}_l)dV$ are the electric dipole, electric quadrupole, and longitudinal magnetic dipole moments, respectively.

Suppose that the function Ψ_1 entering into the Debye parametrization (2.35) of $\mathbf{j}_l$ ($\mathbf{j}_l=\nabla\Psi_1$) decreases sufficiently fast outside the region S to which the current $\mathbf{j}$ is confined. Then $\boldsymbol{\mu}_l$ disappears. If, in addition, the external field is a pure induction (i.e., it is generated by a pure current density), then

$$\phi_{\text{ext}}=0, \quad \mathbf{A}_{\text{ext}}=\frac{1}{c}\int R^{-1}\mathbf{j}_{\text{ext}}(\mathbf{r}',t-R/c)dV',$$

$$\mathbf{E}_{\text{ext}}=-\dot{\mathbf{A}}_{\text{ext}}/c,$$

$$\mathbf{H}_{\text{ext}}=\text{curl }\mathbf{A}_{\text{ext}}, \quad R=|\mathbf{r}-\mathbf{r}'|.$$

It follows from this that

$$U_q=-\frac{1}{c}\dot{\mathbf{d}}\mathbf{A}_{\text{ext}}-\frac{1}{2c}\frac{\partial A_i}{\partial x_k}\dot{q}_{ik}. \tag{2.65}$$

On the other hand, for a charge configuration carrying an electric dipole moment $\mathbf{d}$ and a quadrupole moment q_{ik} the appearance of the interaction term

$$-\mathbf{d}\mathbf{E}_{\text{ext}}-\frac{1}{2}\frac{\partial(E_{\text{ext}})_i}{\partial x_k}q_{ik} \tag{2.66}$$

is intuitively expected. But these terms are absent in (2.65). According to Kobe and Yang (see, e.g., Ref. 49 and references therein), this is due to the gauge noninvariance of the interaction energy:

$$\phi_{\text{ext}}\to\phi'_{\text{ext}}=\phi_{\text{ext}}-\dot{\chi}/c, \quad \mathbf{A}_{\text{ext}}\to\mathbf{A}'_{\text{ext}}=\mathbf{A}_{\text{ext}}+\nabla\chi,$$

$$U\to U'=U-\frac{1}{c}\frac{d}{dt}\int\rho\chi dV.$$

It follows from this that the interaction energy is a gauge-invariant quantity for a pure current density ($\rho=0$). There is no ambiguity like the one mentioned above.

We note also that the transformed interaction energy differs from the original one by a total time derivative. This means that both of them should lead to the same equations of motion. In particular, after the insertion of (2.65) into the Lagrangian and subsequent variation relative to $\dot{d}_i$ and $\dot{q}_{ik}$ we obtain the usual expression for the Lorentz force acting on the dipole and quadrupole moments.

3. ON THE SUPERCURRENT ARISING IN A SUPERCONDUCTING RING

Consider a closed circular metallic ring C encircling an infinite cylindrical solenoid with a constant flux Φ_0 in it (Fig. 6). Suppose that initially there is no current in C. Let the ring C be cooled. At some temperature T_c its transition to

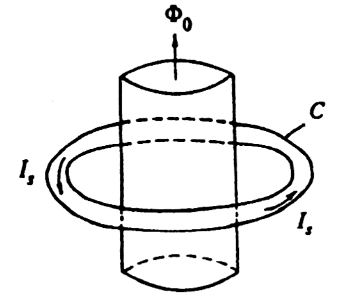

FIG. 6. The cylindrical solenoid with magnetic flux Φ_0 is encircled by the metallic ring C. When C becomes superconducting, the supercurrent I_s arises on its surface (although C is in the region where the electromagnetic field strengths are zero).

the superconducting state occurs. The following two properties were observed experimentally[50–52] and explained theoretically:[53–55]

1) The magnetic field $\mathbf{H}$ vanishes inside C (it is therefore assumed that the penetration depth is zero).

2) The total magnetic flux trapped by C is an integer (in units of $hc/2e$).

The appearance of the supercurrent flowing on the surface of C (despite its location in a field-free region where $\mathbf{E}=\mathbf{H}=0$) for $T<T_c$ was predicted in Refs. 56 and 57. Indeed, as the flux inside the cylindrical solenoid is not in general an integer, the supercurrent in C arises, making the total flux an integer.

This supercurrent was, in fact, observed in Tonomura's experiments (see Refs. 59 and 62, where this fact was clearly stated). It is our aim to evaluate explicitly the distribution of supercurrent on the surface of C and the resulting magnetic field.

The density of the current J_s, flowing on the surface of C and providing $\mathbf{H}=0$ inside C was obtained in Ref. 58. Let the surface of C be given by

$$(\rho-d)^2+z^2=R^2.$$

It is convenient to introduce toroidal coordinates

$$\rho=a\,\frac{\sinh\mu}{\cosh\mu-\cos\theta},\quad z=a\,\frac{\sin\theta}{\cosh\mu-\cos\theta},\quad \phi=\phi. \tag{3.1}$$

For a given value of μ the points ρ, z, ϕ [where ρ, z, ϕ are defined in (3.1)] fill the surface of the torus with the parameters $d=a\coth\mu$, $R=a/\sinh\mu$ ($a=\sqrt{d^2-R^2}$) (see Fig. 7). Let $\mu=\mu_0$ correspond to the surface of C. Then the surface current providing the vanishing of $\mathbf{H}$ inside C is given by[58]

$$\mathbf{J}_s=\delta(\mu-\mu_0)j(\theta)\mathbf{n}_\phi,$$

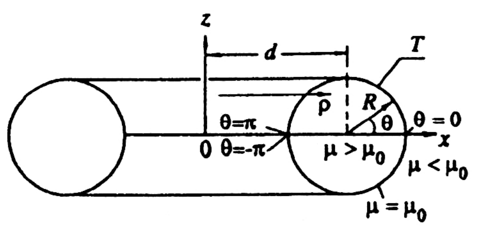

FIG. 7. Geometrical depiction of toroidal coordinates.

$$j(\theta)=-\frac{C_0}{2\sqrt{2}\pi^2a^2}\,\frac{(\cosh\mu_0-\cos\theta)^{5/2}}{\sinh\mu_0}\times\sum\frac{\cos n\theta}{1+\delta_{n0}}[P^1_{n-1/2}(\cosh\mu_0)]^{-1}.$$

This current gives the VP

$$A_\phi=C_0\,\frac{\cosh\mu-\cos\theta}{\sinh\mu}$$

inside C ($\mu>\mu_0$) and

$$A_\phi=C_0\,\frac{\sqrt{2}}{\pi}(\cosh\mu-\cos\theta)^{1/2}\times\sum\frac{\cos n\theta}{1+\delta_{n0}}\,\frac{1}{n^2-1/4}\,\frac{Q^1_{n-1/2}(\cosh\mu_0)}{P^1_{n-1/2}(\cosh\mu_0)}\times P^1_{n-1/2}(\cosh\mu)$$

outside C ($\mu<\mu_0$). In particular, on the circle $z=0$, $\rho=d-R$ (that is, for $\mu=\mu_0$, $\theta=\pi$) we have

$$A_\phi=C_0\,\frac{1+\cosh\mu_0}{\sinh\mu_0}.$$

The integral

$$\oint A_\phi dl=2\pi C_0a$$

taken along the same circle is equal to the flux Φ_s of the magnetic field produced by the supercurrent J_s. The total magnetic flux trapped by the superconducting ring is the sum of the flux Φ_0 in the cylindrical solenoid and the supercurrent flux Φ_s:

$$2\pi C_0a+\Phi_0=\frac{hcn}{2e},$$

where n is the integer nearest to $2e\Phi_0/hc$. From this we find C_0:

$$C_0=-\left(\Phi_0-\frac{hcn}{2e}\right)/2\pi a.$$

The corresponding magnetic field is given by

$$H_\mu=\frac{(\cosh\mu-\cos\theta)^2}{a\sinh\mu}\,\frac{\partial}{\partial\theta}\left(\frac{\sinh\mu A_\phi}{\cosh\mu-\cos\theta}\right),$$

$$H_\theta=-\frac{(\cosh\mu-\cos\theta)^2}{a\sinh\mu}\,\frac{\partial}{\partial\mu}\left(\frac{\sinh\mu A_\phi}{\cosh\mu-\cos\theta}\right).$$

At large distances the VP and the field strengths fall off like r^{-2} and r^{-3}, respectively:

$$A_\phi\sim\frac{2a^2}{\pi r^2}\sin\theta_s\cdot\text{const},$$

$$\text{const}=C_0\sum\frac{1}{1+\delta_{n0}}\,\frac{Q_{n-1/2}(\cosh\mu_0)}{P_{n-1/2}(\cosh\mu_0)},$$

$$H_r\sim\frac{4a^2}{\pi r^3}\cos\theta_s\cdot\text{const},\quad H_\theta\sim\frac{2a^2}{\pi r^3}\sin\theta_s\cdot\text{const}.$$

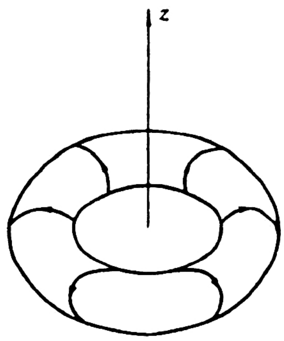

FIG. 8. The lines with arrows denote the poloidal current flowing on the torus surface.

Here r and θ_s are usual spherical coordinates.

It turns out that cooling of the ring C below the critical temperature T_c inevitably leads to the appearance of a magnetic field in the space surrounding C. Obviously, the appearance of a supercurrent in C is a pure quantum effect, as the ring C is located in a region where $\mathbf{E}=\mathbf{H}=0$. But for the creation of a supercurrent in C, energy is needed. Where does it come from? Theory says[59] that for $T>T_c$ the electrons in C are in chaotic motion and the average current is zero. For $T<T_c$ the external vector potential correlates the phases of the electron wave functions. As a result, a macroscopic flow of electrons arises in C.

It would be interesting to observe this supercurrent experimentally. This is not an easy task, as the quantity $\Phi_0 - hcn/2e$ entering into the definition of the vector potential and field strengths is rather small (it is of order $hc/2e$).

Theoretically, in Tonomura experiments the reason for the quantization of the total magnetic flux penetrated by the toroidal solenoid is the appearance (for $T<T_c$) of the poloidal supercurrent on the torus surface. But the poloidal supercurrent (Fig. 8) produces no magnetic field outside the toroidal solenoid. Thus, the magnetic flux quantization observed in Tonomura experiments is only indirect evidence for the supercurrent. On the other hand, the supercurrent arising in a circular turn embracing either cylindrical or toroidal solenoids may be observed by the detection of the magnetic field created by this supercurrent. There have been many experiments in which the dependence of the physical parameters (e.g., resistivity) of a multi-connected sample embracing the magnetic flux (but lying outside the region where $\mathbf{H}=0$) was studied as a function of the magnetic flux (see, e.g., Ref. 60). As in Tonomura experiments, the supercurrent is not measured directly, but its existence is needed for the explanation of the experimental data.

4. RADIATIONLESS TIME-DEPENDENT CHARGE-CURRENT SOURCES

It is usually believed that a charged body radiates when it accelerates. We shall demonstrate now that this intuition is not always correct. We follow closely Ref. 5.

First, we clarify under what conditions the accelerated configuration of charge $\rho(\mathbf{r},t)$ and current $\mathbf{j}(\mathbf{r},t)$ densities does not radiate. The corresponding VP is given by

$$\mathbf{A}=\frac{1}{c}\int \frac{1}{R}\mathbf{j}(\mathbf{r}',t')\delta\left(t'-t+\frac{R}{c}\right)dV'dt'.$$

Here $R=|\mathbf{r}-\mathbf{r}'|$. Obviously, only terms of order not higher than r^{-1} contribute to the radiation field. Expanding the VP in powers of r'/r and neglecting terms of order r'^2/r^2 and higher, we get

$$\mathbf{A}=\frac{1}{cr}\int \mathbf{j}(\mathbf{r}',t')\delta\left(t'-t+\frac{r}{c}-\frac{1}{c}\mathbf{n}_r\mathbf{r}'\right)dV'dt'. \quad (4.1)$$

Here $\mathbf{n}_r=\mathbf{r}/r$. Now, making a Fourier transformation of $\mathbf{j}$,

$$\mathbf{j}(\mathbf{r},t)=\int \mathbf{j}(\mathbf{k},\omega)e^{i(\mathbf{kr}-\omega t)}d^3k\,d\omega,$$

and inserting this into (4.1), we obtain

$$\mathbf{A}=\frac{(2\pi)^3}{cr}\int \mathbf{j}\left(\mathbf{n}_r\frac{\omega}{c},\omega\right)e^{-i\omega(t-r/c)}d\omega.$$

Obviously, $\mathbf{A}$ vanishes if

$$\mathbf{j}\left(\mathbf{n}_r\frac{\omega}{c},\omega\right)=0. \quad (4.2)$$

Now, let $\mathbf{j}(\mathbf{r},t)$ be a periodic function of time with period T. This can be achieved most easily if we choose

$$\mathbf{j}(\mathbf{k},\omega)=\sum \mathbf{j}_n(\mathbf{k})\delta(\omega-\omega_n),$$

$$\rho(\mathbf{k},\omega)=\sum \frac{1}{\omega_n}(\mathbf{k}\mathbf{j}_n(\mathbf{k}))\delta(\omega-\omega_n),$$

$$\omega_n=\omega_1 n,\quad \omega_1=2\pi/T.$$

Then

$$\mathbf{j}(\mathbf{r},t)=\sum_n \int d^3k\,e^{i(\mathbf{kr}-\omega_n t)}\mathbf{j}_n(\mathbf{k}).$$

From this we find $\mathbf{j}_n(\mathbf{k})$:

$$\mathbf{j}_n(\mathbf{k})=\frac{1}{(2\pi)^3}\frac{1}{T}\int \mathbf{j}(\mathbf{r},t)e^{-i(\mathbf{kr}-\omega_n t)}d^3x\,dt.$$

It turns out that the condition (4.2) reduces to

$$\mathbf{j}_n\left(\mathbf{n}_r\frac{\omega}{c},\omega\right)=0. \quad (4.3)$$

Let $\rho(\mathbf{r},t)$ be centered around the time-dependent position $\mathbf{a}(t)$, which is a periodic function of time. That is, we suppose that $\rho(\mathbf{r},t)$ and $\mathbf{j}(\mathbf{r},t)$ are of the form

$$\rho(\mathbf{r},t)=ef(\mathbf{r}-\mathbf{a}(t)),\quad \mathbf{j}(\mathbf{r},t)=e\dot{\mathbf{a}}f(\mathbf{r}-\mathbf{a}(t)). \quad (4.4)$$

The Fourier components are given by

$$\mathbf{j}(\mathbf{k},t)=\frac{1}{(2\pi)^3}\int \mathbf{j}(\mathbf{r},t)e^{-i\mathbf{kr}}d^3x$$

$$=\frac{1}{(2\pi)^3}e\dot{\mathbf{a}}e^{i\mathbf{ka}(t)}\int f(\mathbf{z})e^{-i\mathbf{kz}}d^3z.$$

Here $\mathbf{z}=\mathbf{r}-\mathbf{a}(t)$. Let $f(\mathbf{z})$ be spherically symmetric: $f(\mathbf{z})=f(|\mathbf{z}|)=f(z)$. Then

$$I(k)=\int f(\mathbf{z})e^{-i\mathbf{kz}}d^3z=\int f(z)z^2dz\,\frac{\sin kz}{kz}.$$

Obviously, this expression should vanish for $k=\omega_n/c$.

Consider particular choices of $f(z)$. Let

$$f(z)=\frac{1}{4\pi r^2}\,\delta(z-R). \tag{4.5}$$

Then

$$I_n=I(\omega_n)=\frac{c}{\omega_n R}\sin\frac{\omega_n R}{c}.$$

It can be seen that I_n vanishes if

$$\omega_1 R=l\pi c \quad (l \text{ is an integer}). \tag{4.6}$$

This means that the charge–current distributions (4.4) and (4.5), where $\mathbf{a}_n(t)$ is an arbitrary vector periodic function of time, do not radiate if the condition (4.6) is fulfilled.

The charge–current configuration (4.5) corresponds to a surface distribution. Nonradiating volume distributions are also easily found. Let

$$\rho(y)=A\Theta(b-y)z^{-1}\cos\omega_m y. \tag{4.7}$$

Here A is a constant, $\Theta(x)$ is the step function, and m is an integer. Then

$$I_{n,m}=4\pi A\int_0^b dy\sin\omega_n y\cos\omega_m y$$

$$=2\pi Ac\left[\frac{1-\cos(\omega_n-\omega_m)^{b/c}}{\omega_n-\omega_m}+\frac{1-\cos(\omega_n+\omega_m)^{b/c}}{\omega_n+\omega_m}\right].$$

Clearly, $I_{n,m}$ vanishes and, correspondingly, the accelerated volume distribution (4.7) does not radiate if the condition $\omega_1 b=2\pi c$ is satisfied.

In the same way, the spherically symmetric configuration

$$\rho(y)=\sum_m A_m\Theta(b-y)y^{-1}\cos\omega_m y$$

does not radiate if $\omega_1 b=2\pi c$.

Examples of nonradiating spherically nonsymmetric distributions can also be presented. Let

$$\rho(\mathbf{z})=R_{l,m}(z)Y_{lm}(\theta_z,\phi_z), \tag{4.8}$$

where $R_{l,m}(z)=A_{lm}\Theta(b-z)$, $A_{lm}=\text{const}$. Then the condition for the absence of radiation is

$$I_{lm}(\omega_n)=A_{lm}\int_0^b dzz^2 j_l\left(\frac{\omega_n z}{c}\right)=0$$

[$j_l(x)$ is the spherical Bessel function]. For $l=1$ we have

$$I_{1m}(\omega_n)\sim A_{1m}\left[2\left(1-\cos\frac{\omega_n b}{c}\right)-\frac{\omega_n b}{c}\sin\frac{\omega_n b}{c}\right]\frac{c^3}{\omega_n^3}.$$

It is easy to see that $I_{1m}=0$ if $\omega_1 b=2\pi c$.

Another example[9] of a nonradiating charge distribution is the uniformly charged sphere

$$\rho=\sigma\delta(r-R),\quad \sigma=e/4\pi R^2,$$

which oscillates around a fixed axis with angular velocity

$$\mathbf{\Omega}=U(\omega)\cos\omega t\mathbf{n}_\omega.$$

Here $\mathbf{n}_\omega$ is a constant unit vector. The current is given by

$$\mathbf{j}=\sigma\delta(r-R)(\mathbf{\Omega}(t)\times\mathbf{r})\cos\omega t.$$

The corresponding VP is

$$\mathbf{A}_\omega=\frac{ekR}{cr}(\mathbf{\Omega}(t)\times\mathbf{r})j_1(kr_<)h_1(kr_>)\cos\omega t,$$

where $r_<=\min(r,R)$, $r_>=\max(r,R)$; $j_1(x)$ and $h_1(x)$ are the spherical Bessel and Hankel functions of the first order. Thus, outside the charged sphere we have

$$\mathbf{A}_\omega=\frac{ekR}{cr}(\mathbf{\Omega}(t)\times\mathbf{r})j_1(kR)h_1(kr)\cos\omega t.$$

We observe that the considered oscillating charge distribution does not radiate when $x=kR$ coincides with the zero of $j_1(x)$, i.e., when x satisfies the equation $\tan x=x$.

We summarize: There are charge distributions of finite extension which do not radiate when they exhibit arbitrary periodic accelerated motion described by a time-dependent vector $\mathbf{a}(t)$.

As we have seen, the condition for nonradiation of the studied charge–current configuration is

$$\mathbf{j}(\mathbf{k},\omega)|_{k=\omega/c}=0.$$

Now we apply this condition to a uniformly moving charge. In this case $\mathbf{j}(\mathbf{r},t)=\mathbf{v}\rho(\mathbf{r}-\mathbf{v}t)$ and

$$\mathbf{j}(\mathbf{k},\omega)=2\pi\mathbf{v}\delta(\omega-\mathbf{kv})\rho(\mathbf{k}).$$

Consider first motion in a vacuum. For $\omega=ck$ we have

$$\delta(\omega-\mathbf{kv})=\omega^{-1}\delta(1-\beta\cos\theta),\quad \beta=v/c.$$

Since in a vacuum $\beta<1$, the argument of the δ function is always greater than 1 and the nonradiation condition is satisfied.

Let the charged particle move uniformly in a medium. Then the conditions for the absence of radiation are $\omega=kc_n$, $c_n=c/n$ (c_n is the light velocity in the medium, and n is the refraction index), and

$$\mathbf{j}(\mathbf{k},kc_n)=2\pi\mathbf{v}\rho(\mathbf{k})\delta(1-\beta_n\cos\theta)=0.$$

It can be seen that the nonradiation condition is satisfied everywhere except at the angle $\cos\theta_n=1/\beta_n$. For arbitrary density the quantity

$$\rho(\mathbf{k})|_{\omega=kc_n,\cos\theta=1/\beta_n}$$

differs from zero, and this is just the reason for the appearance of the Cherenkov radiation. This takes place, e.g., for a point-like charge and for an arbitrary spherically symmetric charge distribution confined to a finite region of space. Now we prove the existence of nonradiating finite charge distributions moving with superluminal velocity in a medium. We choose ρ in the form

$$\rho(\mathbf{r})=\rho(r)P_l(\cos\theta_{rv}),$$

where θ_{rv} is the angle between the charge velocity $\mathbf{v}$ and the radius vector $\mathbf{r}$, and P_l is the Legendre polynomial. The Fourier transform of this density is

$$\rho(\mathbf{k})=\frac{1}{2\pi^2}(-i)^l P_l(\cos\theta_{kv})\int j_l(kr)\rho(r)r^2dr.$$

Since the Cherenkov radiation differs from zero only at the definite angle $\cos\theta_{kv}=1/\beta_n$, the nonradiation condition is

$$P_l(1/\beta_n)=0.$$

Let $l=2$. The function $P_2(x)$ has a zero at $x=1/\sqrt{3}$, which corresponds to $\beta_n=\sqrt{3}$. This means that the charge distribution $\rho_2(\mathbf{r})=\rho(r)P_2(\cos\theta_{rv})$ does not radiate if it moves with velocity $\beta_n=\sqrt{3}$. Similarly, the charge distribution $\rho(r)P_3(\cos_{rv})$ does not radiate when its velocity in a medium is equal to $\beta_n=\sqrt{5/3}$. Further, there are two velocities for which the charge distribution $\rho(r)P_4(\cos\theta_{rv})$ does not radiate. These interesting results were obtained in Ref. 7.

Consider the current $\mathbf{j}$ flowing on the cylinder surface:

$$\mathbf{j}=\mathbf{n}_\phi j\delta(\rho-R).$$

Let j be a periodic function of time: $j=j_0\cos\omega t$. Then, outside the cylinder the VP and the field strengths vanish for a discrete set of frequencies satisfying the equation[6,61]

$$J_1(kR)=0$$

(J_1 is the Bessel function). The same is true for the sphere. Suppose that on its surface (of radius R) there flows the current

$$\mathbf{j}=\mathbf{n}_\phi jP_l^1(\cos\theta)\delta(\rho-R),$$

which is a periodic function of time ($j=j_0\cos\omega t$). Then the VP and the field strengths vanish outside the sphere for the infinite set of frequencies satisfying the equation

$$j_1(kR)=0,\quad j_l(x)=(\pi/2x)^{1/2}J_{l+1/2}(x).$$

In Ref. 22 a point-like electric solenoid was considered by using the following nonstatic point charge and current densities:

$$\rho=D\exp(-i\omega t)\Delta\delta^3(\mathbf{r}),$$

$$\mathbf{j}=i\omega D\exp(-i\omega t)\nabla\delta^3(\mathbf{r}). \tag{4.9}$$

Here D is a constant. By the electric solenoid we mean a charge–current configuration generating a magnetic field equal to zero everywhere, with an electric field confined to a finite region of space. The corresponding electromagnetic potentials are

$$\Phi=-\exp(-i\omega t)D\left[4\pi\delta^3(\mathbf{r})+\frac{k^2}{r}\exp(ikr)\right],$$

$$\mathbf{A}=ikD\exp(-i\omega t)\nabla\frac{\exp(ikr)}{r}. \tag{4.10}$$

Only the electric field is nonzero:

$$\mathbf{E}=-\nabla\Phi-\frac{1}{c}\frac{\partial\mathbf{A}}{\partial t}=4\pi D\exp(-i\omega t)\nabla\delta^3(\mathbf{r}). \tag{4.11}$$

These relations are easily generalized to the case of charge and current distributions of finite size.[12,32] We choose ρ and $\mathbf{j}$ in the form

$$\rho=\exp(-i\omega t)\Delta f,\quad \mathbf{j}=i\omega\exp(-i\omega t)\nabla f. \tag{4.12}$$

The following potentials and field strengths correspond to these sources:

$$\Phi=-\exp(-i\omega t)\left[4\pi f+k^2\int G(\mathbf{r},\mathbf{r}')fdV'\right],$$

$$\mathbf{A}=ik\exp(-i\omega t)\nabla\int GfdV',$$

$$\mathbf{E}=4\pi\exp(-i\omega t)\nabla f,\quad \mathbf{H}=0,\quad G=\frac{\exp(ik|\mathbf{r}-\mathbf{r}'|}{|\mathbf{r}-\mathbf{r}'|}. \tag{4.13}$$

The factor $\exp(-i\omega t)$ will be omitted below when it is obvious.

Equations (4.9)–(4.11) are obtained for the choice $f=D\delta^3(\mathbf{r})$. It follows from (4.13) that if the function f is nonzero inside some region of space, $\mathbf{H}=0$ everywhere, while $\mathbf{E}\neq0$ only in the region where $f\neq0$. On the other hand, the electromagnetic potentials differ from zero everywhere. Thus, Eqs. (4.12) and (4.13) realize a nonstatic electric solenoid. In particular, f can be chosen to be nonzero inside the torus $(\rho-d)^2+z^2=R^2$. For this it is sufficient to take $f=D\Theta(R-\sqrt{(\rho-d)^2+z^2})$, where D is a constant. As an example, consider a spherical capacitor, which is obtained for a special choice of the function f. We have

$$\rho=\frac{e}{4\pi r^2}[\delta(r-r_1)-\delta(r-r_2)],$$

$$\mathbf{j}=\frac{i\omega e}{4\pi r^3}\mathbf{r}\Theta(r-r_1)\Theta(r_2-r),\quad r_1<r_2. \tag{4.14}$$

This spherical capacitor consists of two oppositely charged spheres and a radial current between them. Using the general expressions

$$\Phi=\int G\rho(\mathbf{r}')dV',\quad \mathbf{A}=\frac{1}{c}\int G\mathbf{j}(\mathbf{r}')dV',$$

we easily find the scalar and vector potentials (only the radial component of the vector potential is nonzero):

$$\Phi=ikeh_0^{(1)}(kr)[j_0(1)-j_0(2)],$$

$$A_r=-keh_1^{(1)}(kr)[j_0(1)-j_0(2)]\quad \text{for } r>r_2,$$

$$\Phi=ikej_0(kr)[h_0^{(1)}(1)-h_0^{(1)}(2)],$$

$$A_r=-kej_1(kr)[h_0^{(1)}(1)-h_0^{(1)}(2)]\quad \text{for } r<r_1$$

and

$$\Phi=ike[h_0^{(1)}(kr)j_0(1)-j_0(kr)h_0^{(1)}(2)],$$

$$A_r=ek[j_1(kr)h_0^{(1)}(2)-h_1^{(1)}(kr)j_0(1)]-\frac{ie}{kr^2}\quad \text{for } r_1<r<r_2.$$

Here we put

$$j_l(x)=\sqrt{\frac{\pi}{2x}}J_{l+1/2}(x),\quad h_l^{(1)}(x)=\sqrt{\frac{\pi}{2x}}H_{l+1/2}^{(1)}(x),$$

$$j_l(1)=j_l(kr_1),\quad \text{etc.}$$

The magnetic field is zero everywhere, while the electric field $\mathbf{E}=e\mathbf{r}/r^3$ differs from zero only inside the spherical capacitor (i.e., for $r_1<r<r_2$).

It can be seen that waves of electromagnetic potentials appear outside a nonstatic electric solenoid. The question arises of the physical meaning of such waves and the possibility of detecting them experimentally. Let the region S in which $\mathbf{E}$ and $\mathbf{H}$ are nonzero be inaccessible to observation. Can an observer located outside S verify the existence of electromagnetic potential waves?

Since $\mathbf{E}=\mathbf{H}=0$ in these waves, they do not carry energy. Therefore, they can be detected only at the quantum level. This is the case because the Schrödinger equation

$$i\hbar\frac{\partial\Psi}{\partial t}=H\Psi,\quad H=-\frac{\hbar^2}{2m}\left(\nabla-\frac{ie}{\hbar c}\mathbf{A}\right)^2+e\Phi$$

describing the scattering of charged particles on the waves of electromagnetic potentials involves the potentials Φ and $\mathbf{A}$ rather than the fields $\mathbf{E}$ and $\mathbf{H}$. The gauge transformation

$$\Psi\to\Psi'=\Psi\exp(-ie\chi/\hbar c),$$

$$\chi=ik\exp(-i\omega t)\int GfdV' \tag{4.15}$$

eliminates the electromagnetic potentials outside S. If χ is a single-valued function outside S, Eq. (4.15) is a unitary transformation between the single-valued wave functions in the presence and absence of electromagnetic potentials outside S. In this case the presence of electromagnetic potential waves outside S does not lead to observable consequences. On the other hand, if χ is discontinuous outside S (which, in turn, depends on the choice of the source function f), the possibility in principle arises of observing electromagnetic potential waves, e.g., by observing a phase difference acquired by the wave function of a charged particle as the particle travels around a closed contour. A necessary condition is that the region of space accessible to charged test particles be multiply connected (as nontrivial electromagnetic potentials corresponding to $\mathbf{E}=\mathbf{H}=0$ are allowed only in non-simply connected spaces).

Up to now we have considered only nonradiating charge–current sources outside which the electromagnetic field strengths $\mathbf{E}$, $\mathbf{H}$ disappeared. No attention was paid to the existence of electromagnetic potentials in the surrounding space. In the next section we will be interested in studying charge–current distributions outside which $\mathbf{E}=\mathbf{H}=0$ but $\mathbf{A},\Phi\neq0$. To be observable the nonvanishing electromagnetic potentials must be nontrivial, i.e., unremovable by a gauge transformation. The static analog of such distributions is a TS with a constant current in its winding. Outside such a TS, $\mathbf{E}=\mathbf{H}=\Phi=0$, but $\mathbf{A}\neq0$. This static VP was observed in Tonomura experiments.[62] The existence of nontrivial, nonstatic electromagnetic potentials with the properties mentioned above makes the observation of the time-dependent Aharonov–Bohm effect possible.

5. ELEMENTARY TIME-DEPENDENT TOROIDAL SOURCES

Interest in time-dependent currents flowing in toroidal coils is due to the following remark made by James Clerk Maxwell in his memoir "On physical lines of force":[63]

"Let B, Fig. 3, be a circular ring of uniform section, lapped uniformly with covered wire. It may be shewn that if an electric current is passed through this wire, a magnet placed within the coil of wire will be strongly affected, but no magnetic effect will be produced on any external point. The effect will be that of a magnet bent round till its two poles are in contact."

"If the coil is properly made, no effect on a magnet placed outside it can be discovered, whether the current is kept constant or made to vary in strength; but if a conducting wire C be made to embrace the ring any number of times, an electromotive force will act on this wire whenever the current in the coil is made to vary; and if the circuit be closed, there will be an actual current in the wire C."

Figure 3 mentioned in this passage shows the torus with a poloidal winding on its surface (see our Fig. 7). At the present time, it is known that in general this Maxwell assertion is not correct. It turns out that for a time-dependent current in the toroidal coil the electromagnetic field strengths appear outside it. Qualitatively, this was shown by Mitkevich[64] and Page.[65] The corresponding experiments were performed by Mitkevich,[64] Ryazanov,[66] Bartlett and Ward,[67] and many others. Quantitative results were obtained in Ref. 58, where the electromagnetic fields were evaluated for a number of time dependences of the current flowing in the toroidal coil. After all, experimentalists widely use toroidal transformers for their own purposes without philosophizing on this subject. The sole exception for which Maxwell's claim holds is when a current rising linearly in time flows in the toroidal coil. In this case $\mathbf{H}=0$ and $\mathbf{E}$ is independent of time outside the torus (see, e.g., Miller[22,68]). The question of the energy transfer into the wire C embracing the torus was considered by Heald[69] (the difficulty is that the Poynting vector is zero for a linearly growing current).

In Sec. 3, we studied the electromagnetic field of static toroidal-like configurations, their interactions with an external electromagnetic field, and possible physical applications. It is our next goal to study nonstatic current configurations. Perhaps it would be appropriate to explain the meaning of the words "elementary toroidal sources" in the title of this section. The words "toroidal source" mean the poloidal current flowing in the winding of the toroidal solenoid (TS), which in turn may be an element of a more complex configuration. When the dimensions of this configuration tend to zero, we obtain an "elementary toroidal source." The reason for the treatment of an elementary toroidal source is the considerable simplification of the theoretical considerations. A TS with finite dimensions has a number of nontrivial topological properties (see, e.g., the reviews of Refs. 12, 61, and

70). Suppose that these properties survive when the TS dimensions tend to zero. Thus, if we find some interesting property for an elementary toroidal source, there is a chance that it will survive for a finite toroidal configuration. This is confirmed for the simplest toroidal configurations for which analytical solutions can be found. As an example, we mention the configuration consisting of a TS with a linearly growing current flowing in its winding and a double charged layer at the hole of the TS (see Sec. 5.3). Outside this configuration, there is a time-dependent vector potential. The electromagnetic field strengths vanish everywhere, except for a static electric field filling the torus hole. Thus, it becomes possible to perform a time-dependent Aharonov–Bohm-like experiment. However, the linear time dependence of the current is unrealistic. It is the aim of this study to find elementary charge–current configurations possessing the radiationless properties mentioned above but with a rather arbitrary time dependence.

The plan of our exposition is as follows. The radiation of elementary time-dependent toroidal-like configurations, in the winding of which a time-dependent current flows, is studied in Sec. 5.1. It turns out that two different types of these configurations generate essentially different electromagnetic fields. On the other hand, the current sources of the same type generate the same electromagnetic field if their time dependences are properly adjusted. In Secs. 5.2 and 5.3 we give examples of an elementary radiationless charge–current source having the property that the electromagnetic field strengths vanish outside it, but the time-dependent potentials survive there. In Sec. 5.4 examples are given of current configurations generating a static electric field adequately described by an electric vector potential rather than by a scalar one. In Sec. 5.5 these results are used to study the time-dependent Aharonov–Bohm effect. Extended toroidal-like current sources are considered in Sec. 5.6. By using the Neumann–Helmholtz parametrization for the current density, convenient formulas for the time-dependent electromagnetic fields are obtained. On the basis of them, more general elementary radiationless charge–current sources of different multipolarities are constructed in Sec. 5.7. These elementary configurations have their finite counterparts. Those which can be treated analytically are radiationless and have nontrivial electromagnetic potentials outside them. Although the electromagnetic field of more complicated finite configurations cannot be obtained in a closed form, the electromagnetic field of their infinitesimal analogs can. The well prescribed rule for the construction of elementary radiationless configurations found in Sec. 5.7 suggests that their finite radiationless counterparts will also possess nontrivial electromagnetic potentials. A brief discussion of the results and a summary are given in Sec. 5.8.

5.1. The radiation of elementary toroidal sources

5.1.1. A pedagogical example: time-dependent circular current

According to the Ampère hypothesis, a distribution of magnetic dipoles $\mathbf{M}(\mathbf{r})$ is equivalent to a current distribution $\mathbf{J}(\mathbf{r}) = \text{curl}\,\mathbf{M}(\mathbf{r})$. For example, a circular current flowing in the $Z=0$ plane (the upper part of Fig. 1),

$$\mathbf{J} = I\mathbf{n}_\phi \delta(\rho - d)\delta(z), \tag{5.1}$$

is equivalent to the magnetization

$$\mathbf{M} = I\mathbf{n}\Theta(d-\rho)\delta(z), \tag{5.2}$$

which is different from zero in the same plane and directed along its normal $\mathbf{n}$ [$\Theta(x)$ is the step function]. When the radius d of the circumference along which the current flows tends to zero, the current $\mathbf{J}$ becomes ill-defined (it is not clear what the vector $\mathbf{n}_\phi$ means at the origin). On the other hand, the vector $\mathbf{M}$ is still well-defined. In this limit the elementary current (5.1) turns out to be equivalent to a magnetic dipole oriented normally to the plane of this current. It is convenient to introduce $I/\pi d^2$ instead of I in Eqs. (5.1) and (5.2). Then in the limit $d \to 0$ we have

$$\mathbf{M} = I\mathbf{n}\delta^3(\mathbf{r}) \quad [\delta^3(\mathbf{r}) = \delta(\rho)\delta(z)/2\pi\rho] \tag{5.3}$$

and

$$\mathbf{J} = I\ \text{curl}(\mathbf{n}\delta^3(\mathbf{r})). \tag{5.4}$$

Equations (5.3) and (5.4) define the magnetization and current density corresponding to the elementary magnetic dipole. These questions were considered in detail in Sec. 3. Now suppose that the intensity of the elementary current changes with time:

$$\mathbf{J}_0 = f_0(t)\text{curl}(\mathbf{n}\delta^3(\mathbf{r})) \tag{5.5}$$

(the factor I is absorbed into f_0). The VP corresponding to this current is easily obtained:

$$\mathbf{A}_0 = -\frac{D_0}{c^2 r^2}(\mathbf{r}\times\mathbf{n}), \quad D_0 = D(f_0) = \dot{f}_0 + \frac{c}{r} f_0. \tag{5.6}$$

From now on, a time derivative will be denoted either by a dot above the letter or (especially for higher derivatives) by a superscript. For example, $f^{(2)} = \ddot{f} = d^2 f/dt^2$. The argument of the f functions, if not indicated, will be $t - r/c$ throughout this section. The electromagnetic field strengths are

$$\mathbf{E}_0 = \frac{1}{c^3 r^2}(\mathbf{r}\times\mathbf{n})\dot{D}_0, \quad \mathbf{H}_0 = \frac{\mathbf{rn}}{c^3 r^3}\mathbf{r}F_0 - \frac{1}{c^3 r}\mathbf{n}G_0, \tag{5.7}$$

where for brevity we put

$$F_0 = F(f_0) = \ddot{f}_0 + 3\frac{c}{r}\dot{f}_0 + 3\frac{c^2}{r^2} f_0,$$

$$G_0 = G(f_0) = f_0^{(2)} + \frac{c}{r}\dot{f}_0 + \frac{c^2}{r^2} f_0.$$

The flux of electromagnetic energy through a sphere of radius r is

$$S = \int P_r r^2 d\Omega = \frac{2}{3c^5}\dot{D}_0 G_0, \quad \mathbf{P} = \frac{c}{4\pi}(\mathbf{E}_0 \times \mathbf{H}_0). \tag{5.8}$$

This flux is positive at large distances and is determined by the second derivative of f_0: $S_0 = (2/3c^5)|\ddot{f}_k|^2$. These results are well known and can be found in many textbooks (see, e.g., Stratton's book[71]).

In the subsequent discussion the following notation will be used:

$$D_k=D(f_k)=\dot{f}_k+\frac{c}{r}f_k,\quad F_k=F(f_k)=\ddot{f}_k+3\frac{c}{r}\dot{f}_k+3\frac{c^2}{r^2}f_k,$$

$$G_k=G(f_k)=f_k^{(2)}+\frac{c}{r}\dot{f}_k+\frac{c^2}{r^2}f_k.$$

From classical electrodynamics it is known[24,48] that there are two types of multipole radiation. For multipole radiation of magnetic type $\mathbf{rE}=0$, $\mathbf{rH}\neq 0$, while for radiation of electric type $\mathbf{rH}=0$, $\mathbf{rE}\neq 0$ (it is therefore assumed that the origin lies within the region where $\rho,\mathbf{j}\neq 0$). It follows from (5.7) that $\mathbf{rE}_0=0$, $\mathbf{rH}_0\neq 0$. Thus, the radiation field of a time-dependent current flowing in a circular loop is of magnetic type.

5.1.2. The elementary radiating toroidal solenoid

The case next in complexity is the radiation of a current flowing in the winding of an elementary (i.e., infinitely small) toroidal solenoid. As stated in Sec. 3 (see the upper part of Fig. 2), this elementary current is given by

$$\mathbf{j}_1=f_1(t)\mathrm{curl}^{(2)}(\mathbf{n}\delta^3(\mathbf{r})),\tag{5.9}$$

where $\mathrm{curl}^{(2)}=\mathrm{curl\,curl}$ and $\mathbf{n}$ denotes the normal to the equatorial plane of the TS. The electromagnetic potentials and field strengths are

$$\phi_1=0,\quad \mathbf{A}_1=-\mathbf{n}\frac{1}{c^3r}G_1+\frac{1}{c^3r^3}\mathbf{r}(\mathbf{rn})F_1,$$

$$\mathbf{E}_1=\mathbf{n}\frac{1}{c^4r}\dot{G}_1-\frac{1}{c^4r^3}\mathbf{r}(\mathbf{rn})\dot{F}_1,\quad \mathbf{H}_1=\frac{1}{c^4r^2}\ddot{D}_1(\mathbf{r}\times\mathbf{n}).\tag{5.10}$$

In this and the following equations of this section we omit the δ-function terms giving the field values at the origin (to which the current is confined). Thus, Eqs. (5.10) are valid everywhere except at the origin. Since $\mathbf{rH}_1=0$, $\mathbf{rE}_1\neq 0$, the electromagnetic field radiated by a time-dependent current flowing in the winding of a TS is of electric type.

5.1.3. More complicated elementary toroidal sources

We consider now a hierarchy of TSs, each turn of which is again a TS. The simplest example is the usual TS [which is obtained by installing an infinitely thin TS in a single turn with the current (5.5) in it]. We denote this TS by T_1 [the initial current source (5.5) will be denoted by T_0]. The case next in complexity is obtained when each turn of T_1 is replaced by an infinitely thin TS with alternating current in its winding. This current configuration is denoted by T_2 (Fig. 4). When its dimensions tend to zero, we get (see Sec. 2)

$$\mathbf{j}_2=f_2(t)\mathrm{curl}^{(3)}\mathbf{n}\delta^3(\mathbf{r}).\tag{5.11}$$

The corresponding VP and field strengths are given by

$$\mathbf{A}_2=\frac{1}{c^4r^2}D_2^{(2)}(\mathbf{r}\times\mathbf{n}),\quad \mathbf{E}_2=-\frac{1}{c^5r^2}D_2^{(3)}(\mathbf{r}\times\mathbf{n}),$$

$$\mathbf{H}_2=\mathbf{n}\frac{1}{c^5r}G_2^{(2)}-\frac{1}{c^5r^3}\mathbf{r}(\mathbf{rn})F_2^{(2)}.\tag{5.12}$$

By comparing Eqs. (5.6) and (5.7) with (5.12) we conclude that the electromagnetic fields coincide for the current configurations T_0 and T_2 everywhere except at the origin if the following relation between the time-dependent intensities is fulfilled: $f_2^{(2)}=-f_0/c^2$. This means, in particular, that the electromagnetic field of a static magnetic dipole ($f_0=\mathrm{const}$) coincides with that of the current configuration T_2 if the current in it varies quadratically with time ($f_2=-f_0c^2t^2/2$). It follows from this that the magnetic field of the usual magnetic dipole can be compensated everywhere (except at the origin) by a time-dependent current flowing in T_2.

We now compare the periodic currents flowing in T_0 and T_2: $f_0=f_{00}\cos\omega t$ and $f_2=f_{20}\cos\omega t$. It turns out that the electromagnetic fields of T_0 and T_2 coincide if $f_{20}=f_{00}c^2/\omega^2$. Obviously, the radiation emitted by T_2 is of magnetic type.

Now we are able to write out the electromagnetic field for a point-like toroidal configuration of arbitrary order. Let

$$\mathbf{j}_m=f_m(t)\mathrm{curl}^{(m+1)}(\mathbf{n}\delta^3(\mathbf{r})).\tag{5.13}$$

Then for even m ($m=2k$, $k\geqslant 0$) we have

$$\mathbf{A}_{2k}=(-1)^{k+1}\frac{1}{c^{2k+2}r^2}D_{2k}^{(2k)}(\mathbf{r}\times\mathbf{n}),$$

$$\mathbf{E}_{2k}=(-)^k\frac{1}{c^{2k+3}r^2}D_{2k}^{(2k+1)}(\mathbf{r}\times\mathbf{n}),$$

$$\mathbf{H}_{2k}=(-1)^k\frac{1}{c^{2k+3}}\left[\frac{1}{r^3}\mathbf{r}(\mathbf{rn})F_{2k}^{(2k)}-\mathbf{n}\frac{1}{r}G_{2k}^{(2k)}\right].\tag{5.14}$$

From the facts that (i) $\mathbf{A}$ transforms like a vector under space rotations, (ii) the VP changes sign under space reflections, and (iii) $\mathbf{rE}_{2k}=0$, $\mathbf{rH}_{2k}\neq 0$ it follows[24,48] that a toroidal configuration of even order emits radiation of magnetic type.

The flux of electromagnetic energy through a sphere of radius r is

$$S=\frac{2}{3c^{4k+5}}G_{2k}^{(2k)}D_{2k}^{(2k+1)}.$$

On the other hand, for odd m ($m=2k+1$, $k\geqslant 0$) we have

$$\mathbf{A}_{2k+1}=(-1)^k\frac{1}{c^{2k+3}}\left[\frac{1}{r^3}\mathbf{r}(\mathbf{rn})F_{2k+1}^{(2k)}-\mathbf{n}\frac{1}{r}G_{2k+1}^{(2k)}\right],$$

$$\mathbf{E}_{2k+1}=(-1)^{k+1}\frac{1}{c^{2k+4}}\left[\frac{1}{r^3}\mathbf{r}(\mathbf{rn})F_{2k+1}^{(2k+1)}-\mathbf{n}\frac{1}{r}G_{2k+1}^{(2k+1)}\right],$$

$$\mathbf{H}_{2k+1}=(-1)^k\frac{1}{c^{2k+4}r^2}D_{2k+1}^{(2k+2)}(\mathbf{r}\times\mathbf{n}),$$

$$S=\frac{2}{3c^{4k+7}}G_{2k+1}^{(2k+1)}D_{2k+1}^{(2k+2)}.\tag{5.15}$$

From the facts that (i) the VP $\mathbf{A}$ in (5.15) transforms like a vector under rotations, (ii) the VP does not change sign under space reflections, and (iii) $\mathbf{rH}_{2k}=0$, $\mathbf{rE}_{2k}\neq 0$ it follows that the electromagnetic field (5.15) is of electric type.

We see that there are two types of toroidal point-like currents generating essentially different electromagnetic fields. A representative of the first type is the usual magnetic dipole. The electromagnetic field of the kth member of this family reduces to that of a circular current if the time dependences of these currents are properly adjusted:

$$f_{2k}^{(2k)}=(-1)^k f_0(t)/c^{2k} \quad (k\geqslant 0). \tag{5.16}$$

We recall that the lower index of the f functions selects a particular member of the first type, while the upper one denotes the time derivative.

A representative of the second type is the elementary TS. Again, the electromagnetic fields of this family are the same if the time dependences of the currents are properly adjusted:

$$f_{2k+1}^{(2k)}=(-1)^k f_1(t)/c^{2k} \quad (k\geqslant 0). \tag{5.17}$$

From the equations defining the energy flux it follows that, for high frequencies, toroidal emitters of higher order are more effective (as the time derivatives of higher orders contribute to the energy flux).

Earlier, the electromagnetic fields of the T_0, T_1, and T_2 current configurations were considered by Nevessky.[11] Further, the radiation field originating from the instantaneous change of dipole moments (i.e., the radiation emitted by the current configuration T_1 for the very particular choice of f_1) was given by Dubovik and Shabanov.[72]

5.1.4. Toroidal solenoids of higher multipolarities

So far, we have used the usual TS as a cornerstone for constructing more complicated current configurations. By the term "usual" we mean the torus $(\rho-d)^2+z^2=R^2$ with the poloidal current flowing on its surface. The VP corresponding to this current falls off as r^{-3} at large distances:

$$\mathbf{A}\sim\frac{3\mathbf{r}(\mathbf{rn})-\mathbf{n}r^2}{r^5} \quad \text{for} \quad r\to\infty. \tag{5.18}$$

Here $\mathbf{n}$ is the unit vector normal to the TS equatorial plane. This VP can be represented in a slightly different form:

$$A_i\sim r^{-5}\sum Q_{ik}(x)n_k,$$

where $Q_{ik}(x)=x_ix_k-\delta_{ik}r^2/3$ is the symmetric traceless tensor of the second rank.

It was shown in Ref. 46 that it is possible to distribute the currents inside the torus in such a way (for the same magnetic flux) as to cancel the leading term $(\sim r^{-3})$ in the expansion of the VP. It turns out that the first nonvanishing term in the expansion of the VP has the form

$$A_i\sim r^{-9}\sum n_jn_kn_lQ_{ijkl}^{(4)}(x), \tag{5.19}$$

where $Q_{ijkl}^{(4)}(x)$ is the symmetric traceless tensor of the fourth rank:

$$Q_{ijkl}^{(4)}(x)=x_ix_jx_kx_l-\frac{1}{7}(\delta_{ij}x_kx_l+\delta_{ik}x_jx_l+\delta_{il}x_jx_k+\delta_{jk}x_ix_l+\delta_{jl}x_ix_k+\delta_{kl}x_ix_j)r^2+\frac{1}{35}(\delta_{ij}\delta_{kl}+\delta_{ik}\delta_{jl}+\delta_{il}\delta_{jk})r^4.$$

This VP falls off like r^{-5} for $r\to\infty$ and carries the same magnetic flux as the initial solenoid with asymptotic behavior r^{-3} of the VP. With this TS taken as a cornerstone, and using the procedure described above, we can construct a new hierarchy of TSs. This game may be continued further. More complicated current configurations may be found inside the torus for which the VP falls off like r^{-7} (Ref. 46). This current configuration may in turn be used as a cornerstone for the construction of TSs installed in each other. These cornerstone current configurations correspond to higher-order toroidal multipoles.[15] At large distances these VPs have the following asymptotic behavior:

$$A_i^{(l)}\sim r^{-2l-1}\sum Q_{i_1,i_2,\ldots,i_l}^l(x)n_{i_1}n_{i_2}\ldots n_{i_l}. \tag{5.20}$$

Here $Q_{i_1,i_2,\ldots,i_1}^l$ is the symmetric traceless form of order l. Correspondingly, the VP $\mathbf{A}^{(l)}$ falls off as r^{-l-1} for $r\to\infty$. Only even values of l correspond to the finite configurations of poloidal currents found in Ref. 46. As the asymptotic form (5.20) satisfies the conditions div $\mathbf{A}=0$, curl $\mathbf{A}=0$ for any l, the question arises of the possible existence of finite current toroidal-like configurations (i.e., ones outside of which $\mathbf{E}=\mathbf{H}=0$) corresponding to odd l. So far, we have not identified them.

5.2. On radiationless topologically nontrivial sources of electromagnetic fields

Consider an electric dipole oriented in the $\mathbf{n}$ direction. Its charge density is

$$\rho_d=e[\delta^3(\mathbf{r}+a\mathbf{n})-\delta^3(\mathbf{r}-a\mathbf{n})].$$

For small separation a this reduces to

$$\rho_d=2ea(\mathbf{n}\nabla)\delta^3(\mathbf{r}).$$

Suppose that the strength of this dipole changes with time:

$$\rho_d=f_d(t)(\mathbf{n}\nabla)\delta^3(\mathbf{r})$$

(the factor $2ea$ is absorbed into f_d). The corresponding current density is given by

$$\mathbf{j}_d=-\dot{f}_d(t)\mathbf{n}\delta^3(\mathbf{r}).$$

These densities generate the following potentials and field strengths (see, e.g., Ref. 73):

$$\phi_d=-\frac{1}{cr^2}(\mathbf{nr})D_d, \quad \mathbf{A}_d=-\mathbf{n}\dot{f}_d/rc,$$

$$\mathbf{H}_d=\frac{1}{c^2r^2}(\mathbf{r}\times\mathbf{n})\dot{D}_d, \quad E_d=\frac{1}{c^4r}\mathbf{n}G_d-\frac{1}{c^2r^3}(\mathbf{nr})\mathbf{r}F_d. \tag{5.21}$$

Evidently, the radiation emitted by the oscillating electric dipole is of electric type.

From a comparison of Eqs. (5.10) and (5.21) we conclude that the field strengths of a time-dependent current flowing in the winding of an infinitely small TS can be compensated by that of an electric dipole if their time dependences are properly adjusted: $f_d=-\dot{f}_1/c^2$. Then the total charge–current densities are

$$\rho=-\frac{1}{c^2}\dot{f}_1\cdot(\mathbf{n}\nabla)\delta^3(\mathbf{r}),$$

$$\mathbf{j}=f_1(t)\text{curl}^{(2)}\mathbf{n}\delta^3(\mathbf{r})+\frac{1}{c^2}\ddot{f}_1\mathbf{n}\delta^3(\mathbf{r}). \tag{5.22}$$

In the surrounding space $\mathbf{E}=\mathbf{H}=0$, but the potentials differ from zero:

$$\phi=\frac{1}{c^3r^2}(\mathbf{nr})\dot{D}_1,\quad \mathbf{A}=-\frac{1}{c^2r^2}\mathbf{n}D_1+\frac{1}{c^3r^3}\mathbf{r}(\mathbf{rn})F_1. \tag{5.23}$$

Thus, outside this composite object (electric dipole and TS placed at the same point) there are nonvanishing time-dependent electric and vector potentials, despite the vanishing of the field strengths. The simplest example corresponds to $f_1=\text{const}$. Then

$$\phi=0,\quad \mathbf{A}=f_1[3\mathbf{r}(\mathbf{rn})-\mathbf{n}r^2]/cr^5,$$

which coincides with the VP of an elementary (i.e., infinitely small) static TS. The case next in complexity is the composite object consisting of a static electric dipole ($f_d=f=\text{const}$) and a current which changes linearly with time in the winding of a TS:

$$\rho=f(\mathbf{n}\nabla)\delta^3(\mathbf{r}),\quad \mathbf{j}=-c^2ft\ \text{curl}^{(2)}\mathbf{n}\delta^3(\mathbf{r}),$$

$$\mathbf{E}=\mathbf{H}=0,\quad \phi=-f(\mathbf{nr})/r^3,$$

$$\mathbf{A}=-ctf[3\mathbf{r}(\mathbf{nr})-\mathbf{n}r^2)/r^5. \tag{5.24}$$

A counterpart of (5.24) with finite dimensions is a linearly rising (with time) current flowing in the winding of a TS and a double charged layer filling the hole of the same TS. Outside this configuration the electromagnetic field strengths vanish, but a nontrivial (that is, unremovable by a gauge transformation) VP exists.

Another interesting case is the compensation of the electromagnetic field generated by an oscillating electric dipole by that of a periodic current flowing in the winding of a TS:

$$\rho=\rho_d=f\cos\omega t(\mathbf{n}\nabla)\delta^3(\mathbf{r}),$$

$$\mathbf{j}=\mathbf{j}_d+\mathbf{j}_1=f\omega\sin\omega t\left[\mathbf{n}\delta^3(\mathbf{r})-\frac{c^2}{\omega^2}\text{curl}^{(2)}\mathbf{n}\delta^3(\mathbf{r})\right],$$

$$\mathbf{E}=\mathbf{H}=0,\quad \phi=\frac{f}{cr^2}(\mathbf{nr})\left(\omega\sin\Omega-\frac{c}{r}\cos\Omega\right),$$

$$\Omega=\omega(t-r/c),$$

$$\mathbf{A}=\frac{f}{r^2}\mathbf{n}\left(\cos\Omega+\frac{c}{\omega r}\sin\omega t\right)+\frac{\omega f}{cr^3}(\mathbf{rn})\mathbf{r}\left(\sin\Omega-3\frac{c}{\omega r}\cos\Omega-3\frac{c^2}{\omega^2r^2}\sin\omega t\right).$$

It turns out that the field strengths are compensated if the phase of the charge density of the electric dipole is shifted by $\pi/2$ relative to the phase of the current flowing in the winding of a toroidal solenoid.

In the wave zone the equivalence of the EMF radiated by an oscillating electric dipole to that produced by a periodic current flowing in the winding of a TS was established earlier in Ref. 30. There is no equivalence in the whole space if the finite-dimensional counterparts of the aforementioned charge–current configurations are nontrivial. In this case there is no global gauge transformation between the corresponding potentials, and this could in principle be observable. The following sections illustrate this. There are references (Refs. 6, 10, 12, 22, 32, 45, 74, and 75) in which nonradiating sources were treated. Outside these sources both the electromagnetic field strengths and potentials were zero, and, thus, they are of no interest to us. Hitherto, it was not known whether nontrivial nonradiating time-dependent sources can exist in principle. As far as we know, the first such example was given in Ref. 13. Nontrivial time-dependent electromagnetic potentials can be used as a new channel for information transfer (by modulating the phase of the charged-particle wave function) and for the performance of time-dependent Aharonov–Bohm-like experiments.

5.3. On current configurations generating a static electric field

Consider a poloidal current (Fig. 8) on the surface of a torus $[(\rho-d)^2+z^2=R^2]$ which increases linearly with time: $\mathbf{j}=\mathbf{j}_0t$. To parametrize $\mathbf{j}_0$ it is convenient to introduce the coordinates $\tilde{R}$, Ψ (Fig. 3):

$$x=(d+\tilde{R}\cos\Psi)\cos\phi,\quad y=(d+\tilde{R}\cos\Psi)\sin\phi,$$

$$z=\tilde{R}\sin\Psi.$$

In these coordinates,

$$\mathbf{j}_0=\mathbf{n}_\psi\frac{j_0t}{R^2}\frac{\delta(R-\tilde{R})}{d+R\cos\Psi}.$$

Here $\mathbf{n}_\psi$ is the unit vector tangential to the surface of the torus: $\mathbf{n}_\psi=\mathbf{n}_z\cos\psi-\mathbf{n}_\rho\sin\psi$. It lies in the plane $\phi=\text{const}$ on the surface of the torus ($\tilde{R}=R$) and defines the direction of $\mathbf{j}$. It turns out[58,68] that for this current only the electric field strength $\mathbf{E}$ differs from zero outside the torus. For simplicity we consider an infinitely thin torus ($R\ll d$). The following representation for the VP is valid:[76,77]

$$A_x=\frac{\Phi_0t}{4\pi}\frac{\partial^2\alpha}{\partial x\partial z},\quad A_y=\frac{\Phi_0t}{4\pi}\frac{\partial^2\alpha}{\partial y\partial z},$$

$$A_z=-\frac{\Phi_0t}{4\pi}\left(\frac{\partial^2\alpha}{\partial x^2}+\frac{\partial^2\alpha}{\partial y^2}\right),\quad \text{div}\,\mathbf{A}=0.$$

Here

$$\Phi_0=-\frac{4\pi^2j_0}{cd},\quad \alpha=\int\int\frac{dx'dy'}{|\mathbf{r}-\mathbf{r}'|}. \tag{5.25}$$

The integration in α is performed over the circle $z=0$, $\rho\leqslant d$ coinciding with the hole of the infinitely thin torus

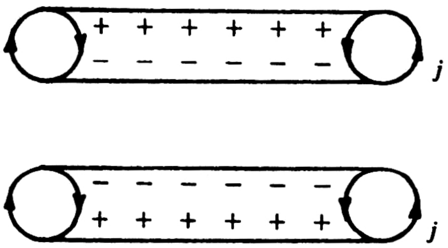

FIG. 9. The poloidal current j growing linearly with time is equivalent to the doubly charged layer (the upper part of the figure). The lower part of this figure illustrates that the electric field of the current may be compensated by that of the doubly charged layer.

$(R \ll d)$. It was shown in Ref. 77 that the VP has no singularities, except for the line $z=0$, $\rho=d$ into which the torus T degenerates. The electromagnetic field strengths are

$$H_\rho=H_z=0,\quad H_\phi=\Phi_0 t\,\delta(z)\,\delta(d-\rho),$$
$$E_x=-\frac{\Phi_0}{4\pi c}\frac{\partial^2\alpha}{\partial x\partial z},\quad E_y=-\frac{\Phi_0}{4\pi c}\frac{\partial^2\alpha}{\partial y\partial z},$$
$$E_z=\frac{\Phi_0}{4\pi c}\left(\frac{\partial^2\alpha}{\partial x^2}+\frac{\partial^2\alpha}{\partial y^2}\right). \tag{5.26}$$

On the other hand, the electric field produced by two oppositely charged layers ($\rho\leq d$, $z=\pm\epsilon$) filling the hole of the torus is given by

$$E_x^d=\frac{2e\epsilon}{\pi d^2}\frac{\partial^2\alpha}{\partial x\partial z},\quad E_y^d=\frac{2e\epsilon}{\pi d^2}\frac{\partial^2\alpha}{\partial y\partial z},$$
$$E_z^d=\frac{2e\epsilon}{\pi d^2}\frac{\partial^2\alpha}{\partial z^2}=\frac{2e\epsilon}{\pi d^2}\left(\frac{\partial^2\alpha}{\partial x^2}+\frac{\partial^2\alpha}{\partial y^2}\right)-\frac{8e\epsilon}{d^2}\,\delta(z)\Theta(d-\rho). \tag{5.27}$$

We see that E_z^d has a singularity on the circle $z=0$, $\rho\leq d$. It follows from Eqs. (5.26) and (5.27) that if

$$\frac{\Phi_0}{4\pi c}=\frac{2e\epsilon}{\pi d^2},$$

then the electric field of a linearly growing poloidal current is compensated by that of a double layer everywhere except at the position of the layer itself (see Fig. 9). The electromagnetic potentials and field strengths of this combined configuration are given by

$$\phi=-\frac{\Phi_0}{4\pi c}\frac{\partial\alpha}{\partial z},\quad A_x=\frac{\Phi_0 t}{4\pi}\frac{\partial^2\alpha}{\partial x\partial z},$$
$$A_y=\frac{\Phi_0 t}{4\pi}\frac{\partial^2\alpha}{\partial y\partial z},\quad A_z=-\frac{\Phi_0 t}{4\pi}\left(\frac{\partial^2\alpha}{\partial x^2}+\frac{\partial^2\alpha}{\partial y^2}\right),$$
$$E_x=E_y=0,\quad E_z=-\frac{1}{c}\Phi_0\delta(z)\Theta(d-\rho),$$
$$H_\rho=H_z=0,\quad H_\phi=\Phi_0 t\,\delta(z)\,\delta(d-\rho). \tag{5.28}$$

We observe that the time-independent electric field $\mathbf{E}$ differs from zero only inside the hole of the torus ($\rho\leq d$, $z=0$), with a magnetic field $\mathbf{H}\neq 0$ only on the filament $\rho=d$, $z=0$ coinciding with an infinitely thin torus.

The situation remains essentially the same for a TS with a finite value of R. Suppose that a linearly rising current flows in its winding. The corresponding VP is $\mathbf{A}_{TS}=t\mathbf{A}_0$, where $\mathbf{A}_0$ is independent of time and, apart from an inessential constant, coincides with the VP of a static TS. The corresponding electric field strength is $\mathbf{E}_{TS}=-\mathbf{A}_0/c$. It is known[58,77] that A_0 is an everywhere continuous function of the coordinates. Further, outside the solenoid $\mathbf{A}_0$ can be written as a gradient of some function χ: $\mathbf{A}_0=\mathrm{grad}\,\chi$. This representation is valid everywhere except in the circle $\rho\leq d-R$, $z=0$ filling the TS hole. The function χ suffers a finite jump from the value $\chi=\Phi_0$ on the lower side ($z=0^-$) of this circle to the value $\chi=-\Phi_0$ on its upper side ($z=0^+$). Here $\Phi_0=d\Phi/dt$ is the magnetic flux change per unit time. Obviously, it does not depend on the time. Now we identify $-\chi/c$ with the scalar potential of some electric field. The corresponding electric field strength is

$$\mathbf{E}_d=-\mathrm{grad}(-\chi/c)=\frac{1}{c}\,\mathrm{grad}\,\chi=\frac{1}{c}\,\mathbf{A}_0-\frac{1}{c}\,\Theta(d-\rho)\,\delta(z)\Phi_0\mathbf{n}_z.$$

The associated charge density

$$\rho_d=(1/4\pi)\mathrm{div}\,\mathbf{E}_d=-(1/4\pi c)\Theta(d-\rho)\dot{\delta}(z)\Phi_0$$

describes the electric dipole layer filling the TS hole. The total electric field is

$$\mathbf{E}=\mathbf{E}_{TS}+\mathbf{E}_d=-\frac{1}{c}\,\Theta(d-\rho)\delta(z)\Phi_0\mathbf{n}_z.$$

This means that the EMF of a TS with a linearly rising current can be compensated by the EMF of a static electric dipole layer filling the TS hole everywhere, except for the TS hole itself.

5.3.1. On the current electrostatics

Although a toroidal solenoid with a linearly growing current and a double charged layer produce the same electric field in the space surrounding them, they in fact represent quite different systems. The following example illustrates this. Consider an arbitrary closed curve C, at each point of which we install (perpendicular to this curve) an infinitely thin toroidal solenoid with a current that grows linearly with time. The whole set of these solenoids forms a toroidal-like surface S. The magnetic field strength is everywhere zero except on the surface S. The electric field strength and the time-dependent magnetic VP will be different from zero only inside the tube T surrounded by the surface S. It seems at first that this contradicts the vanishing of the VP outside S (the VP should be everywhere continuous). The reason is the same as the discontinuity of the usual electric scalar potential on the surface of a double charged layer: it turns out that the surface S is an example of a double current layer. This construction (Fig. 10) realizes a pure current capacitor (the static electric field produced by the time-dependent current is confined to the interior of the tube T). If the set of charged layers (instead of TS) were installed on the same curve C, the

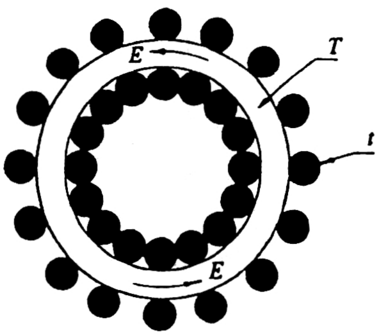

FIG. 10. The torus T is densely covered by infinitely thin toroidal solenoids t (only a few of them are shown), in the windings of which there is a current rising linearly with time. The magnetic field **H** differs from zero only inside t (that is, at the surface of T in the limit of infinitely thin T), while the time-independent electric field **E** differs from zero inside T. The scalar electric potential is zero everywhere. The vector magnetic potential is zero outside T and t. Although the electromagnetic potentials and field strengths are zero outside T and t, there is a nonzero electric vector potential ($\mathbf{E} = \mathrm{curl}\, \boldsymbol{\alpha}$) there. The Stokes theorem (see the text) ensures that $\boldsymbol{\alpha}$ cannot be eliminated by a gauge transformation.

electric field strength would vanish inside the tube T. However, the nontrivial electric induction will be different from zero there.[32]

Consider a semi-infinite cylinder C densely covered by infinitely thin toroidal solenoids (Fig. 11). For simplicity, consider the case when the radius of C tends to zero. In the limit, one obtains a semi-infinite filament composed of toroidal moments μ_t. The VP of a particular toroidal moment lying at $z=z_0$ is

$$A_x=\mu_t \frac{\partial^2}{\partial x \partial z}\frac{1}{\tilde{r}}, \quad A_y=\mu_t \frac{\partial^2}{\partial y \partial z}\frac{1}{\tilde{r}},$$

$$A_z=\mu_t\left[\frac{\partial^2}{\partial z^2}\frac{1}{\tilde{r}}+4\pi\delta(x)\delta(x)\delta(z-z_0)\right],$$

$$\tilde{r}=\sqrt{x^2+y^2+(z-z_0)^2}, \quad \mathrm{div}\,\mathbf{A}=0.$$

To obtain the VP of a semi-infinite filament composed of toroidal moments, we integrate these equations from $z_0=-\infty$ to $z_0=0$:

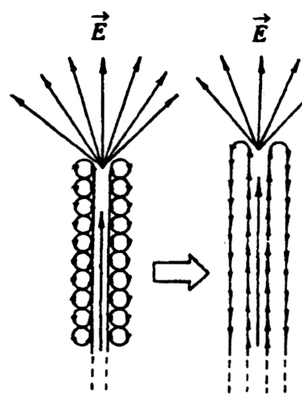

FIG. 11. A semi-infinite set of infinitely thin TSs with linearly rising currents in their windings (left part of the figure) and linearly rising currents flowing along the semi-infinite parallel cylindrical surfaces (right part) generate the field of an electric charge everywhere, except at the position of the cylinder.

$$A_x=\mu_t \frac{x}{r^3}, \quad A_y=\mu_t \frac{y}{r^3},$$

$$A_z=\mu_t\left[\frac{z}{r^3}+4\pi\delta(x)\delta(y)\Theta(-z)\right], \quad \mathrm{div}\,\mathbf{A}=0.$$

Suppose that in the windings of toroidal solenoids covering the surface of C there is a current that rises linearly with time. The VP of a particular infinitely small solenoid located at $z=z_0$ was obtained in Ref. 58. It is given by

$$A_x=t\dot{\mu}_t \frac{\partial^2}{\partial x \partial z}\frac{1}{\tilde{r}}, \quad A_y=t\dot{\mu}_t \frac{\partial^2}{\partial y \partial z}\frac{1}{\tilde{r}},$$

$$A_z=t\dot{\mu}_t\left[\frac{\partial^2}{\partial z^2}\frac{1}{\tilde{r}}+4\pi\delta(x)\delta(x)\delta(z-z_0)\right], \quad \mathrm{div}\,\mathbf{A}=0.$$

Here $\dot{\mu}_t$ is the constant characterizing the rate of change of the current. The total VP of a semi-infinite filament densely covered by infinitely small toroidal solenoids with time-dependent currents in their windings is obtained by integrating these equations from $z_0=-\infty$ to $z_0=0$:

$$A_x=t\dot{\mu}_t \frac{x}{r^3}, \quad A_y=t\dot{\mu}_t \frac{y}{r^3},$$

$$A_z=t\dot{\mu}_t\left[\frac{z}{r^3}+4\pi\delta(x)\delta(y)\Theta(-z)\right], \quad \mathrm{div}\,\mathbf{A}=0.$$

This semi-infinite filament corresponds to the static electric field

$$\mathbf{D}=\mathbf{E}, \quad E_x=-\dot{\mu}_t \frac{x}{cr^3}, \quad E_y=-\dot{\mu}_t \frac{y}{cr^3},$$

$$E_z=-\frac{\dot{\mu}_t}{c}\left[\frac{z}{r^3}+4\pi\delta(x)\delta(y)\Theta(-z)\right], \quad \mathrm{div}\,\mathbf{E}=0,$$

and a singular magnetic field confined to the negative z semi-axis:

$$\mathbf{B}=\mathbf{H}=H_\phi \mathbf{n}_\phi, \quad H_\phi=-4\pi t\dot{\mu}_t \frac{d}{d\rho}\frac{\delta(\rho)}{2\pi\rho}, \quad \mathrm{div}\,\mathbf{H}=0.$$

The resulting EMF coincides with that of a point electric charge $e=-\dot{\mu}_t/c$ everywhere except on the semi-infinite filament (the left part of Fig. 11).

Above, we have used the fact that $\mathbf{D}=\mathbf{E}$, $\mathbf{B}=\mathbf{H}$ in the absence of a medium.

The same electric field may also be realized via two linearly rising currents flowing in opposite directions along cylindrical surfaces parallel to the z axis (the right part of Fig. 11). The equalities

$$\mathrm{div}\,\mathbf{D}=0, \quad \int D_n d\Omega=0$$

guarantee the absence of free charges. Obviously, the resulting electric charges are not true ones (owing to the presence of the δ-function term).

In a qualitative manner these results were obtained earlier by Miller,[68] who pointed out the possibility of simulating charge distributions by time-dependent currents. He referred to it as to "current electrostatics." The present investigation may be viewed as a concrete realization of these ideas. Excellent measurements of the static electric fields produced by time-dependent currents have been reported in Ref. 66.

There have been attempts (see, e.g., Ref. 78 and references therein) to measure the electric field arising from stationary currents. Maxwell's theory negates the existence of this field. On the other hand, we have seen that there exist nonstatic current configurations generating a static electric field.

5.4. On the electric vector potentials

As we have learned from the previous section, it is possible to find current configurations producing a static electric field $\mathbf{E}$ inside a tube T. As $\mathbf{E}$ is due to the currents, so $\operatorname{div}\mathbf{E}=0$, and it can be represented in the form $\mathbf{E}=\operatorname{curl}\mathbf{A}_e$. The possibility of such a representation for a free electromagnetic field was pointed out earlier by Stratton.[71] The integral $\int\mathbf{E}dS$ taken over the tube cross section differs from zero. Then the Stokes theorem $\int\mathbf{E}dS=\oint\mathbf{A}_e d\mathbf{l}$ (the line integral is taken along a contour embracing the tube T but lying outside it) tells us that $\mathbf{A}_e$ differs from zero outside T. In other words, there is a nontrivial electric VP outside T (Fig. 10). The same is valid for a closed chain of electric dipoles.[32] A drawback of the present considerations is that we have not taken into account singular fields in the infinitely thin layer on the surface of T (where the currents flow). It may happen that they exactly compensate the flux of $\mathbf{E}$ inside T. Then the total flux of the electric field strength will be zero, and there will be no need to introduce the electric vector potential. To clarify this point, we turn again to the closed chain of the TS, installed along the closed curve C perpendicular to it. The total VP and electric field strength are given by

$$\mathbf{A}(\mathbf{r})=\int\mathbf{A}_{\mathrm{TS}}(\mathbf{r},\mathbf{r}_0(s))ds,\quad \mathbf{E}=-\dot{\mathbf{A}}/c. \tag{5.29}$$

Here $\mathbf{A}_{\mathrm{TS}}$ is the VP of the particular infinitely thin TS with its center at the point $\mathbf{r}_0(s)$. The integration in (5.29) is performed along the curve C defined as $\mathbf{r}=\mathbf{r}_0(s)$. For the studied case the time-dependent VP is given by[58] $\mathbf{A}_{\mathrm{TS}}(t)=t\mathbf{A}^0_{\mathrm{TS}}$, where $\mathbf{A}^0_{\mathrm{TS}}$ is the VP of a TS with a static current. However, we are unable to evaluate the integral (5.29) along an arbitrary closed curve. Instead, we integrate along the infinite straight line parallel to the TS symmetry axis. In the special gauge the VP of a TS with its axis parallel to the Z axis is[76,77] $\mathbf{A}^0_{\mathrm{TS}}=-g\mathbf{n}_zT$, where T is given by Eqs. (2.8) and (2.9) with $z-z_0$ instead of z (z_0 is the position of the TS center); $g=\Phi_0[2\pi(d-\sqrt{d^2-R^2})]^{-1}$; Φ_0 is the magnetic flux inside the TS, and d and R are its geometrical parameters. Now we integrate this VP along the Z axis:

$$A_z=\int(\mathbf{A}^0_{\mathrm{TS}})_z dz_0.$$

It turns out that

$$A_z=\Phi_0\quad\text{for}\quad \rho<d-R,$$

$$A_z=\Phi_0-2g\xi\ln\rho+2g\int_0^\xi dz\ln(d-\sqrt{R^2-z^2})\quad\text{for}\quad d-R<\rho<d,$$

$$\xi=\sqrt{R^2-(\rho-d)^2},$$

$$A_z=-2g\xi\ln\rho+2g\int_0^\xi dz\ln(d+\sqrt{R^2-z^2})$$

$$\times\quad\text{for}\quad d<\rho<d+R,$$

and $A_z=0$ for $\rho>d+R$. The flux of the VP $\mathbf{A}$ is obtained by integration over the cross section of the cylinder C:

$$\int A_z\rho d\rho d\phi=\pi^2gdR^2.$$

In the limit $R\to0$ this expression becomes $\pi d^2\Phi_0$, which coincides with the integral of the VP taken over the interior of the cylindrical tube without taking into account the singular magnetic field concentrated on the surface of the cylinder. This means that the surface magnetic field contributes nothing in the limit $R\to0$.

Thus, we have proved that for the studied current configuration (a TS continuously distributed over the surface of the cylinder C) the VP is equal to zero outside C, but its flux over the cross section of C differs from zero. As $\operatorname{div}\mathbf{A}=0$, we may put $\mathbf{A}=\operatorname{curl}\boldsymbol{\alpha}$. Using the Stokes theorem, we see that there is a nontrivial vector function $\boldsymbol{\alpha}$ outside C, although $\mathbf{A}=0$ there. The main problem is that $\boldsymbol{\alpha}$ does not enter into the Schrödinger or the Dirac equation. Nevertheless, such a current configuration interacts with an external electromagnetic field (see Sec. 2.2) and, in particular, with that of an incoming charged particle.

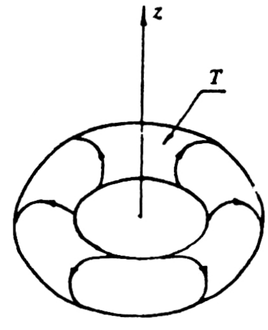

FIG. 12. The torus T is densely covered by magnetized rings (only a few of them are shown). The magnetic field strength $\mathbf{H}$ differs from zero at the surface of T, while the magnetic vector potential $\mathbf{A}$ differs from zero only inside T and at its surface. Outside T there is nontrivial (that is, unremovable by the gauge transformation) vector $\boldsymbol{\alpha}$ whose curl is the vector potential $\mathbf{A}$.

The existence of a nontrivial (that is, unremovable by a gauge transformation) vector $\boldsymbol{\alpha}$ whose curl is the VP may be proved without recourse to such rather complicated nonstatic current configurations. Consider a set of closed magnetized filaments uniformly distributed over the surface of a torus T (see Fig. 12, where the lines on the torus surface indicate the magnetized filaments). This configuration can be assembled from the ferromagnetic rings used in Tonomura experiments[62] testing the existence of the Aharonov–Bohm effect. The VP differs from zero only inside the torus T, although the magnetic field strength $\mathbf{H}$ vanishes there (it differs from zero on the surface of T). Then arguments similar to the previous ones prove the existence of a vector $\boldsymbol{\alpha}$ in the space external to T. It appears that $\boldsymbol{\alpha}$ cannot be eliminated by a gauge transformation.

5.5. Time-dependent Aharonov–Bohm effect

Consider the scattering of charged particles by the charge–current configuration shown in the lower part of Fig. 9. It consists of an impenetrable toroidal solenoid with a layer of electric dipoles filling the hole of a torus. The corresponding Schrödinger equation is

$$i\hbar\frac{\partial\psi}{\partial t}=\left[-\frac{\hbar^2}{2m}\left(\boldsymbol{\nabla}-\frac{ie}{\hbar c}\mathbf{A}\right)^2+e\phi\right]\psi. \quad (5.30)$$

To prevent particle penetration into the interior of the torus, it can be made impenetrable. Outside it, the magnetic field is $\mathbf{H}=0$ everywhere, and the electric field is also everywhere zero, except in the hole of the torus, where it has a δ-type singularity. The static scalar potential and linearly growing (with time) vector potential differ from zero everywhere. The integral $\oint A_l dl$ taken along a closed path passing through the hole of the torus also grows linearly with time. The question arises of the extent to which the electromagnetic potentials can be removed from the Schrödinger equation (5.30).

But first we recall the situation for the usual infinitely thin static magnetic toroidal solenoid without a double charged layer.[76,77] In this case

$$\Phi=0,\quad A_x=\frac{\Phi_0}{4\pi}\frac{\partial^2\alpha}{\partial x\partial z},\quad A_y=\frac{\Phi_0}{4\pi}\frac{\partial^2\alpha}{\partial y\partial z},$$

$$A_z=-\frac{\Phi_0}{4\pi}\left(\frac{\partial^2\alpha}{\partial x^2}+\frac{\partial^2\alpha}{\partial y^2}\right),$$

$$\mathbf{E}=0,\quad \mathbf{H}=\mathbf{n}_\phi\Phi_0\delta(\rho-d)\delta(z)$$

[Φ_0 is the magnetic flux inside the TS, and α is defined in Eq. (5.25)]. The gauge transformation

$$\mathbf{A}\rightarrow\mathbf{A}'=\mathbf{A}-\boldsymbol{\nabla}\chi,\quad \psi\rightarrow\psi'=\psi\exp(ie\chi/\hbar c),$$

$$\chi=\frac{1}{4\pi}\Phi_0\frac{\partial\alpha}{\partial z}$$

leads to a VP filling the hole of the torus:

$$A'_x=A'_y=0,\quad A'_z=\Phi_0\delta(z)\Theta(d-\rho),$$

$$i\hbar\frac{\partial\psi'}{\partial t}=-\frac{\hbar^2}{2m}\left[\nabla_x^2+\nabla_y^2+\left(\nabla_z-\frac{ie}{\hbar c}\Phi_0\delta(z)\times\Theta(d-\rho)\right)^2\right]\Psi'. \quad (5.31)$$

The VP cannot be eliminated from this equation by a gauge transformation, and this leads to a shift of the interference pattern on a screen installed behind the TS. Corresponding experiments have been performed by Tonomura,[62] and their theoretical description is given in Ref. 79.

For the studied time-dependent case the gauge transformation which partially eliminates the electromagnetic potentials is

$$\mathbf{A}\rightarrow\mathbf{A}'=\mathbf{A}-\boldsymbol{\nabla}\chi,\quad \phi\rightarrow\phi'=\phi+\dot{\chi}/c,$$

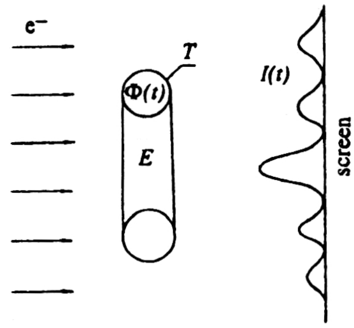

FIG. 13. The magnetic time-dependent AB effect. For the charge–current configuration discussed in the text, the time-dependent magnetic flux differs from zero only inside the impenetrable torus T. Outside T the time-independent electric field strength $\mathbf{E}$ differs from zero only inside the torus hole. It is the time-dependent magnetic flux inside T that leads to the time variation of the intensity of scattered charged particles.

$$\psi\rightarrow\psi'=\psi\exp(ie\chi/\hbar c),\quad \chi=\frac{1}{4\pi}\Phi_0 t\frac{\partial\alpha}{\partial z}.$$

After this transformation,

$$\phi'=A'_x=A'_y=E'_x=E'_y=0,\quad A'_z=\Phi_0 t\delta(z)\Theta(d=\rho),$$

$$E'_z=E_z=-\frac{1}{c}\Phi_0\delta(z)\Theta(d-\rho),$$

$$H'_\phi=H_\phi=\Phi_0 t\delta(z)\delta(d-\rho),$$

$$i\hbar\frac{\partial\psi'}{\partial t}=-\frac{\hbar^2}{2m}\left[\nabla_x^2+\nabla_y^2+\left(\nabla_z-\frac{ie}{\hbar c}\Phi_0 t\delta(z)\times\Theta(d-\rho)\right)^2\right]\Psi'. \quad (5.32)$$

Equations (5.31) and (5.32) have essentially the same form. Likewise, the static VP cannot be removed from Eq. (5.31), and the time-dependent VP cannot be removed from Eq. (5.32). This means that a changing (with time) interference pattern inevitably arises on a screen installed behind the impenetrable toroidal solenoid (Fig. 13). The static electric field $\mathbf{E}$ filling the hole of the torus certainly deflects incoming charged particles (via the Lorentz force). The charged-particle scattering cross section evaluated according to the laws of classical mechanics does not depend on the time. The time dependence of the interference pattern is a pure quantum effect. It is due to the time-dependent magnetic flux enclosed in the impenetrable torus. We observe that effects of excluded fields (a time-dependent magnetic field confined to the impenetrable torus) are observed against a background of accessible ones (i.e., the static electric field filling the hole of the torus). This agrees with the standard definition of the Aharonov–Bohm effect as the observable effects of enclosed (or inaccessible) fields (see, e.g., Ref. 62). For cylindrical geometry, the magnetic time-dependent AB effect was considered recently in Refs. 80 and 81.

5.6. Finite toroidal-like configurations

5.6.1. The Debye parametrization for the electromagnetic potentials and field strengths

Consider now a time-dependent current distribution confined to a finite region of space:

$$\mathbf{j}(\mathbf{r},t)=f(t)\mathbf{j}(\mathbf{r}). \tag{5.33}$$

An arbitrary vector function and, in particular, the current distribution can be represented in the form (Debye parametrization)

$$\mathbf{j}(\mathbf{r})=\nabla\Psi_1+\text{curl}(\mathbf{r}\Psi_2)+\text{curl}^{(2)}(\mathbf{r}\Psi_3). \tag{5.34}$$

It turns out that the VP corresponding to the current density (5.33) in the Lorentz gauge ($\text{div}\,\mathbf{A}+\dot{\Phi}/c=0$) is given by

$$\mathbf{A}=\nabla a_1+\text{curl}(\mathbf{r}a_2)+\text{curl}^2(\mathbf{r}a_3). \tag{5.35}$$

Clearly, Eq. (5.35) is the Debye parametrization of the VP. The functions entering into it are

$$a_k=I_k/c,\quad I_k=\int\frac{1}{R}f(t-R/c)\Psi_k(\mathbf{r}')dV'. \tag{5.36}$$

Here $R=|\mathbf{r}-\mathbf{r}'|$. For completeness, we write out the corresponding scalar electric potential:

$$\phi=-\dot{I}_1/c+4\pi F(t)\Psi_1(\mathbf{r})+\phi_{\text{stat}}. \tag{5.37}$$

Here a dot above I_k denotes the time derivative, $F(t)=\int^t f(t)dt$, and ϕ_{stat} is the scalar potential arising from the time-independent part of the charge density (if it exists): $\phi_{\text{stat}}=\int R^{-1}\rho_{\text{stat}}(\mathbf{r}')dV'$. It is convenient to represent the field strengths in the same form as $\mathbf{j}$ and $\mathbf{A}$:

$$\mathbf{E}=\nabla e_1+\text{curl}(\mathbf{r}e_2)+\text{curl}^{(2)}(\mathbf{r}e_3),$$
$$\mathbf{H}=\nabla h_1+\text{curl}(\mathbf{r}h_2)+\text{curl}^{(2)}(\mathbf{r}h_3). \tag{5.38}$$

It turns out that

$$e_1=-\phi_{\text{stat}}-4\pi F(t)\Psi_1(\mathbf{r}),\quad e_2=-\dot{I}_2/c^2,\quad e_3=-\dot{I}_3/c^2,$$
$$h_1=0,\quad h_2=-\ddot{I}_3/c^3+4\pi f(t)\Psi_3(\mathbf{r})/c,\quad h_3=I_2/c. \tag{5.39}$$

These representations are convenient because the potentials and field strengths are obtained from relatively simple integrals, and their time and space derivatives.

We know from Sec. 3 that the functions Ψ_2 and Ψ_3 carry information about the magnetic and toroidal (electric) moments, respectively. Thus, putting $\Psi_2(\mathbf{r})=\psi_2(r)Y_{lm}(\theta,\phi)$ and $\Psi_3(\mathbf{r})=\psi_3(r)Y_{lm}(\theta,\phi)$, we obtain formulas describing the radiation of particular magnetic and toroidal (electric) multipoles. The functions ψ_2 and ψ_3 define the radial distribution of the current sources. Expanding the function $g=f(t-R/c)/R$ in spherical harmonics,

$$g=4\pi\sum\frac{1}{2l+1}g_l(r,r',t)Y_{lm}(\theta,\phi)Y^*_{lm}(\theta',\phi'), \tag{5.40}$$

we obtain, for the particular lm multipole,

$$I_{lm}=\frac{4\pi}{c}\frac{1}{2l+1}Y_{lm}(\theta,\phi)\int g_l(r,r',t)\psi_k(r')r'^2dr' \tag{5.41}$$

(with no sum over l,m here).

5.6.2. Transition to the point-like limit

Equation (5.41) defines the integrals for a finite spatial current distribution. To obtain the point current limit we follow the method used by Rowe[82] for the evaluation of the integral I_1 entering into the definition of ϕ [see Eq. (5.37)]. One simply puts

$$\Psi_k(\mathbf{r})\sim Y_{lm}(-\nabla)\delta^3(\mathbf{r}). \tag{5.42}$$

We shall clarify the meaning of $Y_{lm}(-\nabla)$ on the right-hand side of this equation. We write

$$Y_{lm}(x)=r^lY_{lm}(\theta,\phi), \tag{5.43}$$

where $Y_{lm}(\theta,\phi)$ is the usual spherical harmonic. Clearly, $Y_{lm}(x)$ is a homogeneous function (of order l) in the Cartesian variables x,y,z. For example,

$$Y_{20}(x)\sim 2z^2-x^2-y^2. \tag{5.44}$$

To obtain $Y_{lm}(-\nabla)$ we replace x_i by $-\partial/\partial x_i$ in Eq. (5.43). In particular,

$$Y_{20}(-\nabla)\sim 2\frac{\partial^2}{\partial z^2}-\frac{\partial^2}{\partial x^2}-\frac{\partial^2}{\partial y^2}. \tag{5.45}$$

Many of the properties of the functions $Y_{lm}(x)$ and their physical applications are collected in Ref. 83. Now we substitute (5.42) into (5.36) and integrate by parts:

$$I_k\sim Y_{lm}(\nabla)f(t-r/c)/r. \tag{5.46}$$

Inserting this expression into Eqs. (5.35) and (5.38), we obtain the electromagnetic potentials and field strengths describing the elementary source.

5.7. More general radiationless sources

Having obtained explicit expressions for extended and point-like sources, we now try to construct radiationless sources of higher multipolarities. Consider the charge and current densities corresponding to an oscillating quadrupole moment:

$$\rho_q=f_q(t)\left[(\mathbf{n}\nabla)^2-\frac{1}{3}\Delta\right]\delta^3(\mathbf{r}),$$
$$\mathbf{j}_q=-\dot{f}_q\left[\mathbf{n}(\mathbf{n}\nabla)-\frac{1}{3}\nabla\right]\delta^3(\mathbf{r}). \tag{5.47}$$

On the other hand, consider the pure current density (5.34) with

$$\Psi_1=\Psi_2=0,\quad \Psi_3=\left[(\mathbf{n}\nabla)^2-\frac{1}{3}\Delta\right]\delta^3(\mathbf{r}),$$
$$\mathbf{j}_c=f_c(t)\text{curl}^2(\mathbf{r}\Psi_3). \tag{5.48}$$

It turns out that the oscillating quadrupole charge–current configuration (5.47) and the pure current configuration (5.48) placed at the same point generate total field strengths equal to zero everywhere, except at the origin, if the following relation is fulfilled: $f_q=2\dot{f}_c/c^2$. The total charge–current densities are

$$\rho=\frac{2}{c^2}\dot{f}_c(t)\left[(\mathbf{n}\nabla)^2-\frac{1}{3}\Delta\right]\delta^3(\mathbf{r}),$$

$$\mathbf{j}=f_c(t)\mathrm{curl}^2(\mathbf{r}\Psi_3)-\frac{2}{c^2}\ddot{f}_c\left[\mathbf{n}(\mathbf{n}\nabla)-\frac{1}{3}\nabla\right]\delta^3(\mathbf{r}). \quad (5.49)$$

Nevertheless, the electromagnetic potentials are not zero:

$$\phi=\phi_q=\frac{2}{c^4r^3}\left[(\mathbf{nr})^2-\frac{1}{3}r^2\right]\dot{F}_c,$$

$$\mathbf{A}=\mathbf{A}_q+\mathbf{A}_c=-\frac{4}{c^3r^3}\left[(\mathbf{nr})\mathbf{n}-\frac{1}{3}\mathbf{r}\right]F_c+\frac{2}{c^4r^4}\mathbf{r}\left[(\mathbf{nr})^2-\frac{1}{3}r^2\right]\left(f_c^{(3)}+6\frac{c}{r}f_c^{(2)}+15\frac{c^2}{r^2}\dot{f}_c+15\frac{c^3}{r^3}f_c\right),$$

$$\left[F_c=F(f_c)=\ddot{f}_c+3\frac{c}{r}\dot{f}_c+3\frac{c^2}{r^2}f_c\right]. \quad (5.50)$$

For $f_c=\text{const}$, $f_q=0$ we get the following static configuration:

$$\mathbf{j}=f_c(t)\mathrm{curl}^2(\mathbf{r}\Psi_3),$$

$$\mathbf{A}=-\frac{12}{cr^5}f_c\left[(\mathbf{nr})\mathbf{n}-\frac{1}{3}\mathbf{r}\right]+\frac{30}{cr^7}f_c\mathbf{r}\left[(\mathbf{nr})^2-\frac{1}{3}r^2\right]. \quad (5.51)$$

This VP, falling off at large distances as r^{-4}, corresponds to $l=3$ in Eq. (5.20). As we mentioned at the end of Sec. 6.1.3, we did not succeed in identifying the finite static current configuration whose infinitesimal limit coincides with (5.51) and corresponds to odd l in (5.20).

The case next in complexity corresponds to octupole oscillations of the charge density:

$$\rho_q=f_q(t)(\mathbf{n}\nabla)\left[(\mathbf{n}\nabla)^2-\frac{3}{5}\Delta\right]\delta^3(\mathbf{r}),$$

$$\mathbf{j}_q=-\dot{f}_q\mathbf{n}\left[(\mathbf{n}\nabla)^2-\frac{3}{5}\Delta\right]\delta^3(\mathbf{r}). \quad (5.52)$$

The elementary toroidal current distribution giving the same field strengths corresponds to

$$\Psi_1=\Psi_2=0,\quad \Psi_3=f_c(t)(\mathbf{n}\nabla)\left[(\mathbf{n}\nabla)^2-\frac{3}{5}\Delta\right]\delta^3(\mathbf{r}),$$

$$f_q=-3\dot{f}_c/c^2. \quad (5.53)$$

The finite poloidal current distribution whose infinitesimal limit coincides with Eq. (5.53) was obtained in Ref. 46. The asymptotic behavior of the corresponding VP is determined by Eq. (5.19).

Now we are able to write out more general radiationless charge–current configurations. The extension of Eqs. (5.47) and (5.52) to an arbitrary multipolarity l is given by

$$\rho_q=f_q(t)(\mathbf{v}\nabla)\delta^3(\mathbf{r}),\quad \mathbf{j}_q=-\dot{f}_q(t)\mathbf{v}\delta^3(\mathbf{r}). \quad (5.54)$$

Here $\nabla_i=\partial/\partial x_i$, while $\mathbf{v}$ is the vector whose Cartesian components are

$$v_i=\sum_{i_2\ldots i_1}Q^{(l)}_{i,i_2,\ldots,i_1}\nabla_{i_2}\ldots\nabla_{i_1},$$

where $Q^{(l)}_{i,i_2,\ldots,i_1}$ is the symmetric traceless tensor (see Sec. 3.2.2) of rank l in the variables n_x, n_y, n_z defining the direction of the fixed 3-vector (this vector can be identified with the direction of the TS axis). The electromagnetic potentials and field strengths corresponding to these densities are

$$\phi_q=(\mathbf{v}\nabla)\frac{f_q}{r},\quad \mathbf{A}_q=-\mathbf{v}\frac{1}{c}\frac{\dot{f}_q}{r},$$

$$E_q=-\nabla(\mathbf{v}\nabla)\frac{f_q}{r}+\frac{1}{c^2}\mathbf{v}\frac{\ddot{f}_q}{r},\quad \mathbf{H}_q=-\frac{1}{c}(\nabla\times\mathbf{v})\frac{\dot{f}_q}{r} \quad (5.55)$$

(we recall that the argument of the f functions, if not indicated, is $t-r/c$).

On the other hand, a pure current configuration generalizing Eqs. (5.48) and (5.53) is given by

$$\rho_c=0,\quad \mathbf{j}_c=f_c(t)\mathrm{curl}^{(2)}(\mathbf{r}\Psi_3),\quad \Psi_3=(\mathbf{v}\nabla)\delta^3(\mathbf{r}). \quad (5.56)$$

The corresponding electromagnetic potentials and field strengths are

$$\phi_c=0,$$

$$\mathbf{A}_c=-\frac{l}{c}\nabla(\mathbf{v}\nabla)\frac{f_c}{r}+\frac{l}{c^3}\mathbf{v}\frac{\ddot{f}_c}{r}+\frac{4\pi}{c}f_c(t)\mathbf{r}(\mathbf{v}\nabla)\delta^3(\mathbf{r}),$$

$$\mathbf{E}_c=\frac{l}{c^2}\nabla(\mathbf{v}\nabla)\frac{\dot{f}_c}{r}-\frac{l}{c^4}\mathbf{v}\frac{f_c^3}{r}-\frac{4\pi}{c^2}\dot{f}_c(t)\mathbf{r}(\mathbf{v}\nabla)\delta^3(\mathbf{r}),$$

$$\mathbf{H}_c=\frac{l}{c^3}(\nabla\times\mathbf{v})\frac{\ddot{f}_c}{r}-\frac{4\pi l}{c}f_c(t)(\mathbf{r}\times\nabla)(\mathbf{v}\nabla)\delta^3(\mathbf{r}). \quad (5.57)$$

Now we place the charge–current densities (5.54) and (5.56) at the same point. It turns out that if $f_q=\dot{f}_c/c^2$, then the total electromagnetic field strengths are everywhere zero, except at the origin:

$$\mathbf{H}=-\frac{4\pi l}{c}f_c(t)(\nabla\times\mathbf{v})\delta^3(\mathbf{r}),\quad \mathbf{E}=\frac{4\pi l}{c^2}\dot{f}_c(t)\mathbf{v}\delta^2(\mathbf{r}). \quad (5.58)$$

Nevertheless, the electromagnetic potentials differ from zero in the whole space:

$$\phi=-\dot{\chi}/c,\quad \mathbf{A}=\nabla\chi-\frac{4\pi l}{c}f_c(t)\mathbf{v}\delta^3(\mathbf{r}),$$

$$\chi=-\frac{l}{c}(\mathbf{v}\nabla)\frac{f_c}{r}. \quad (5.59)$$

Evidently, these equations generalize the particular cases considered earlier.

5.8. Concluding remarks on toroidal radiationless sources

In a previous section we found elementary charge–current configurations with the property that the electromagnetic field strengths, not the potentials, vanish outside them. Turning to Eq. (5.59), we observe that outside the source $\mathbf{A}=\nabla\chi$ and $\phi=-\dot{\chi}/c$, that is, the electromagnetic potential can be represented there as a 4-gradient of a singular function χ. Does this mean that the electromagnetic potentials can be eliminated by a gauge transformation? One cannot comment on the topological nontriviality of the electromagnetic potentials without going beyond the framework of the elementary source. This is due to the fact that it is not clear what is a topologically nontrivial point-like source. As an illustration, consider the vector potential (5.18) of the usual static elementary toroidal solenoid. It turns out that outside the origin (where the TS is placed) the VP can be represented as the gradient of the singular function $\chi=-f_1(\mathbf{nr})/r^3$. On the other hand, outside a finite TS [whose infinitesimal counterpart is the elementary source (5.18)] the VP cannot be eliminated by a gauge transformation (despite the fact that $\mathbf{E}=\mathbf{H}=0$ there). This leads to numerous experimental consequences and, in particular, to the static magnetic Aharonov–Bohm effect. Experiments in which electrons were scattered on an impenetrable magnetized ring were performed by Tonomura *et al.*[62]

Now we turn again to Eqs. (5.54) and (5.56). We know[46] how to find finite counterparts of the elementary sources (5.52). For time dependences for which the VP can be found in a closed form, the rules (6.50) and (5.57) lead to topologically nontrivial electromagnetic potentials outside radiationless sources. The uniformity of these prescriptions suggests that nontrivial potentials should exist for an arbitrary time dependence. To the best of our knowledge, the nontrivial radiationless sources considered in Ref. 13 are their first concrete realizations.

Further, it turns out that the field strengths vanish in the space surrounding radiationless sources. Since the electromagnetic field strengths generated by oscillating charge densities and elementary toroidal sources are the same (if their time dependences are properly adjusted), particular terms of the multipole expansions defining these strengths coincide and have the double names known in the physics literature as electric (see, e.g., Refs. 24 and 48) or toroidal[15,30,31] multipoles. Despite the coincidence of the electromagnetic field strengths, the corresponding potentials may be physically different. In those cases the multipole expansion of the field strengths does not describe the whole physical situation (since the same multipole expansion of the field strengths corresponds to physically different electromagnetic potentials which can be distinguished experimentally).

We briefly summarize the main results obtained in this section:

1. The radiation fields of toroidal-like current configurations have been investigated. There are two different representatives which generate essentially different electromagnetic fields. These representatives are a circular turn and a toroidal solenoid with time-dependent currents flowing in them.

2. There are elementary time-dependent charge–current configurations outside which the electromagnetic field strengths vanish but the potentials survive. In the solvable cases their finite-dimensional counterparts have nontrivial (i.e., unremovable by a gauge transformation) electromagnetic potentials outside them. This can be used for performing time-dependent Aharonov–Bohm-like experiments and for information transfer (modulating the phase of the charged-particle wave function).

3. Using the Debye parametrization of the current density, we represent the electromagnetic field of an arbitrary time-dependent charge–current density in a form convenient for applications. The contributions of different multipoles in it are explicitly separated.

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