The relativistic theory of gravity and Mach's principle

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This review of the relativistic theory of gravity describes the developments in this theory during the last ten years, for example, the necessity of introducing a graviton mass and the improved statement of the basic propositions of the theory, including the philosophical aspects of choosing a particular spacetime geometry to describe physical phenomena dictated by the universal properties of the motion of matter and the fundamental conservation laws. It is shown that this theory leads to a uniquely defined Lagrangian density and to equations for the gravitational field. Some physical consequences of the theory are discussed. © *1998 American Institute of Physics.* [S1063-7796(98)00101-6]

INTRODUCTION

Since the relativistic theory of gravity (RTG) is constructed on the basis of the special theory of relativity (STR), we shall discuss the latter in some detail, taking the approaches of both Poincaré and Einstein. This analysis will allow a deeper understanding of the differences between these approaches and allow us to state the main points of the theory of relativity.

In analyzing Lorentz transformations, Poincaré showed that these transformations together with all the spatial rotations form a *group* under which the equations of electrodynamics are invariant. Feynman noted that it was Poincaré who proposed the study of what can be done with equations whose form remains invariant under such transformations, and who had the idea of studying the symmetry properties of physical laws. Poincaré did not limit himself to electrodynamics; he discovered the equations of relativistic mechanics and extended Lorentz transformations to all the forces of Nature. The discovery of the group which Poincaré referred to as the Lorentz group allowed Poincaré to introduce four-dimensional spacetime with the *invariant*, later called the interval:

$$d\sigma^2=(dX^0)^2-(dX^1)^2-(dX^2)^2-(dX^3)^2. \qquad (\alpha)$$

From this is it completely obvious that time and spatial length are *relative*.

Further advances in this area were made by Minkowski, who introduced the concepts of timelike and spacelike intervals. Following Poincaré and Minkowski, the basic statement of the theory of relativity is that *all physical phenomena occur in spacetime whose geometry is pseudo-Euclidean and determined by the interval (α)*. Here it is important to stress the fact that *the geometry of spacetime reflects the general dynamical features of matter, which are what make it universal.* In four-dimensional (Minkowski) space, we can take an arbitrary coordinate system

$$X^\nu=f^\nu(x^\mu),$$

effecting a one-to-one correspondence with nonzero Jacobian. Finding the differentials

$$dX^\nu=\frac{\partial f^\nu}{\partial x^\mu}\,dx^\mu$$

and substituting these expressions into (α), we obtain

$$d\sigma^2=\gamma_{\mu\nu}(x)dx^\mu dx^\nu, \qquad (\beta)$$

where

$$\gamma_{\mu\nu}(x)=\epsilon_\sigma\frac{\partial f^\sigma}{\partial x^\mu}\cdot\frac{\partial f^\sigma}{\partial x^\nu},\quad \epsilon_\sigma=(1,-1,-1,-1).$$

It is completely obvious that the transformation to an arbitrary coordinate system did not take us outside the pseudo-Euclidean geometry. However, from this it follows that noninertial coordinate systems can also be used in the STR. The inertial forces arising in transforming to an accelerated coordinate system are expressed by the Christoffel symbols of Minkowski space. The STR representation going back to the studies of Poincaré and Minkowski was more general and turned out to be extremely important for constructing the RTG, since it allowed the introduction of the metric tensor $\gamma_{\mu\nu}(x)$ of Minkowski space in arbitrary coordinates, thereby making it possible to introduce the gravitational field in a covariant fashion by separating the inertial forces and the gravitational forces. Einstein arrived at the theory of relativity by analyzing simultaneity and the concept of synchronization of clocks located at different points in space on the basis of the principle that the speed of light is a constant. "In a coordinate frame 'at rest,' every light ray moves with a certain velocity V, independently of whether this light ray is emitted by a body which is moving or at rest." However, this statement cannot be viewed as a principle, because it presupposes a definite choice of coordinates, whereas a physical principle must be independent of the choice of coordinate frame. In Einstein's approach it is impossible to arrive at noninertial coordinate systems, since in such systems clock synchronization cannot be used and the speed of light cannot be considered constant.

According to the expressions

$$d\sigma^2=d\tau^2-s_{ik}dx^i dx^k,\quad d\tau=\frac{\gamma_{0\alpha}dx^\alpha}{\sqrt{\gamma_{00}}},$$

$$s_{ik}=-\gamma_{ik}+\frac{\gamma_{0i}\gamma_{0k}}{\gamma_{00}},$$

in an accelerated reference frame the proper time $d\tau$ is not a total differential, and so the synchronization of clocks located at different points in space depends on the synchronization path. This implies that this concept is inapplicable to accelerated coordinate systems. It should be emphasized that the coordinates in Eq. (β) themselves do not have a metrical meaning. Physically measurable quantities must be constructed using coordinates and the metric components $\gamma_{\mu\nu}$. However, for a long time all this was not understood in the STR, because the Einstein approach rather than that of Poincaré and Minkowski was usually followed. Therefore, the original statements of Einstein were especially limited, although they did perhaps give the illusion of simplicity. This is why in 1913 Einstein wrote: "Only linear orthogonal transformations are allowed in the conventional theory of relativity." And a little later in the same year he wrote: "In the original theory of relativity the independence of the physical equations from a special choice of reference frame is based on the postulate of the fundamental invariant $ds^2=\Sigma dx_i^2$, whereas now we are speaking about the construction of a theory [here he meant the general theory of relativity—A.L.] in which the role of the fundamental invariant is played by a linear element of the general form

$$ds^2=\sum_{i,k} g_{ik}dx^i dx^k."$$

Again, in 1930 Einstein wrote: "In the special theory of relativity only those coordinate changes (transformations) are allowed for which in the new coordinates the quantity ds^2 (the fundamental invariant) has the form of the sum of the squares of the differentials of the new coordinates. Such transformations are called Lorentz transformations."

We thus see that Einstein's approach did not lead him to the idea of the pseudo-Euclidean geometry of spacetime. When the approaches of Poincaré and Einstein to the construction of the STR are compared, it becomes obvious that the Poincaré approach is deeper and more general, since it is it which determined the pseudo-Euclidean structure of spacetime. The Einstein approach significantly restricted the scope of the STR, but since the expositions of the STR in the literature usually followed Einstein, for a very long time it was assumed that the STR is valid only in inertial reference frames. Here Minkowski space was viewed as a useful geometrical interpretation or as a mathematical statement of the principles of the STR in the Einstein approach. Now let us turn to gravity. In 1905 Poincaré wrote that "forces of any origin, in particular, gravitational forces, behave in translational motion (or under Lorentz transformations) exactly like electromagnetic forces." This is the path we shall follow.

Einstein, in noting that inertial and gravitational mass are equal, became convinced that the forces of inertia and gravity are related, because their action is independent of the mass of the body. In 1913 he concluded that, if in Eq. (α) "we introduce the new coordinates x_1,x_2,x_3,x_4, using an arbitrary substitution, then in the new coordinate system the motion of a point will obey the equation

$$\delta\left\{\int ds\right\}=0,$$

where

$$ds^2=\sum_{\mu,\nu} g_{\mu\nu}dx^\mu dx^\nu";$$

further on, he wrote: "In the new coordinate system the motion of a matter point is determined by the quantities $g_{\mu\nu}$, which according to the preceding paragraphs should be understood as the components of the gravitational field, as soon as we want to view this new system as being 'at rest.' " This identification of the metric field obtained from (α) by coordinate transformations with the gravitational field has no physical foundation, because coordinate transformations do not take us outside the pseudo-Euclidean geometry. From our point of view, it is incorrect to assume that the metric field is the gravitational field, because this would contradict the essential concept of the field as a physical reality. It is therefore impossible to agree with the following arguments of Einstein: "The gravitational field 'exists' with respect to the system K' in the same sense as every other physical quantity which can be defined in some coordinate system, despite the fact that it does not exist in the system K. There is nothing strange about this, and it is easily proved by the following example borrowed from classical mechanics. No one doubts the 'reality' of kinetic energy, because otherwise it would be necessary to reject energy altogether. However, it is clear that the kinetic energy of a body depends on the state of motion of the coordinate system: by a suitable choice of the latter, it is obviously possible to arrange to have the kinetic energy of the translational motion of one body at some instant of time take an *a priori* specified positive or zero value. In the special case where all the masses have velocities in the same direction and of equal magnitude, by suitable choice of the coordinate system it is possible to make the total kinetic energy equal to zero. The analogy is, in my opinion, complete."

We shall see that Einstein rejected the concept of a classical field of the Faraday–Maxwell type possessing energy–momentum density as applying to the case of the gravitational field. This led him to the construction of the general theory of relativity (GTR), to the nonlocalizability of gravitational energy, and to the introduction of the gravitational-field pseudotensor. If the gravitational field is treated as a physical field, it, like all physical fields, is characterized by an energy–momentum tensor $t^{\mu\nu}$. If the gravitational field exists in some coordinate system, for example, K', this implies that some (or all) components of the tensor $t^{\mu\nu}$ are nonzero. The tensor $t^{\mu\nu}$ cannot be brought to zero by a coordinate transformation, i.e., if the gravitational field exists, it is a physical reality and cannot be made to disappear by choice of coordinate system. It is incorrect to compare this gravitational field with kinetic energy, because the latter is not characterized by a covariant quantity. It should be noted that this comparison is inadmissible also in the GTR, because the gravitational field in that theory is characterized by the Riemann curvature tensor. If the latter is nonzero, the gravitational field exists, and it cannot be made to vanish by choice of coordinate system.

Although they have no relation to the essentials of the GTR, accelerated reference frames played an important heuristic role in the work of Einstein. In identifying accelerated reference frames, owing to the equality of inertial and gravitational mass, with the gravitational field, Einstein arrived at the metric tensor of spacetime as a fundamental characteristic of the gravitational field. However, the metric tensor reflects both the intrinsic features of the geometry and the choice of coordinate system. In this way it becomes possible to explain the gravitational force kinematically, by reducing it to an inertial force. However, then the gravitational field cannot be considered a physical field. In 1918 Einstein wrote: "The gravitational field can be specified without introducing energy stresses and density." But this is a huge loss to which we cannot be reconciled. However, as we shall see below, this loss can be avoided in constructing the RTG.

Surprisingly, even in 1933 Einstein wrote: "In the special theory of relativity, as shown by Minkowski, this metric was quasi-Euclidean, i.e., the square of the 'length' ds of a linear element is a definite quadratic function of the coordinate differentials. If other coordinates are introduced using a nonlinear transformation, then ds^2 remains a homogeneous function of the coordinate differentials, but the coefficients of this function ($g_{\mu\nu}$) will no longer be constants, but some functions of the coordinates. Mathematically, this implies that physical (four-dimensional) space possesses a Riemannian metric." This, of course, is incorrect, because it is not possible to transform a pseudo-Euclidean metric into a Riemannian one by coordinate transformations. However, this is a side issue; here the main point is that in this way, thanks to his deep intuition, Einstein arrived at the necessity of introducing Riemannian space after associating it with gravity.

The identity of the Riemannian metric with gravity is the main principle of the general theory of relativity. Fock wrote the following about this principle: "It is the essence of the Einstein theory of gravity." The introduction of Riemannian space allowed the scalar curvature R to be used as the Lagrangian and the Hilbert–Einstein equations to be obtained by the principle of least action. The construction of Einstein's general theory of relativity was then complete. Here, as particularly stressed by Synge, "In Einstein's theory, depending on whether the Riemann tensor is nonzero or zero, the gravitational field is present or absent. This property is absolute, and is not at all related to the world line of any observer."

However, in the GTR difficulties arose with the energy–momentum and angular-momentum conservation laws. Hilbert wrote the following about this: "...I maintain that for the general theory of relativity, i.e., in the case of general invariance of the Hamiltonian, the energy equations, which ... correspond to the energy equations in orthogonal-invariant theories, do not exist at all. I even could note this fact as a characteristic feature of the general theory of relativity." All this is explained by the fact that in Riemannian space there is no ten-parameter group of motions of spacetime, and so in principle it is impossible to introduce energy–momentum and angular-momentum conservation laws like those which exist in any other physical theory.

Another feature of the GTR compared to other theories is the presence of second derivatives in the Lagrangian R. About five or ten years ago, Rosen showed that if the metric $\gamma_{\mu\nu}$ of Minkowski space is introduced along with the Riemann metric $g_{\mu\nu}$, it is possible to construct a scalar density of the gravitational-field Lagrangian which contains derivatives of order no higher than the first. In particular, he constructed such a Lagrangian density which leads to the Hilbert–Einstein equations. This gave rise to the two-metric formalism.

However, this approach complicated the problem of constructing a theory of gravity, because it is possible to write down a large number of scalar densities using the tensors $g_{\mu\nu}$ and $\gamma_{\mu\nu}$, and it is not at all clear which scalar density should be chosen as the Lagrangian density for constructing the theory of gravity. Although the mathematical formalism of the GTR allows the introduction of covariant derivatives of Minkowski space instead of ordinary derivatives, since the metric $\gamma_{\mu\nu}$ does not enter into the Hilbert–Einstein equations, its use in the GTR has no physical meaning, because the solutions for the metric $g_{\mu\nu}$ do not depend on the choice of $\gamma_{\mu\nu}$.

It should be noted that the replacement of the ordinary derivatives by covariant ones in Minkowski space leaves the Hilbert–Einstein equations unchanged. This happens because replacement of the ordinary derivatives in the Riemann curvature tensor by covariant ones in Minkowski space does not change the Riemann tensor. This replacement is nothing but an identity transformation. That is why this freedom of notation for the Riemann tensor cannot be used in the GTR, because the metric tensor of Minkowski space does not enter into the Hilbert–Einstein equations. This freedom of notation for the Riemann tensor turns out to be essential in constructing the RTG. However, here the metric of Minkowski space enters into the gravitational field equations, and the field itself is treated as a physical field in Minkowski space.

In the GTR we deal only with the metric of Riemannian space as a fundamental characteristic of gravity which reflects both the intrinsic properties of the geometry and the choice of coordinate system. When the gravitational interaction is switched off, i.e., when the Riemann curvature tensor is zero, we arrive at Minkowski space. This is why in the GTR the problem arises of the feasibility of the correspondence principle, since it is impossible to determine which coordinate system we end up in when the gravitational field is switched off.

The relativistic theory of gravity,[1] which is described in the present review with some additions and refinements, is constructed as a field theory of the gravitational field within the framework of the special theory of relativity. The starting point is the hypothesis that a universal characteristic of matter, the energy–momentum tensor, is the source of gravity. The gravitational field is viewed as a physical field with spins 2 and 0, the action of which gives rise to an effective Riemannian space. This allows the determination of the gauge group and the unique construction of the Lagrangian density of the gravitational field. The system of equations of this theory is invariant under the Lorentz group. The ideas of Poincaré, Minkowski, Einstein, Hilbert, Rosen, and Fock on the theory of relativity and gravity are developed further.

1. THE GEOMETRY OF SPACETIME

At the beginning of the century, Poincaré wrote as follows in the book *Science and Hypothesis*: "...experiment plays an indispensable role in the origin of geometry, but it would be erroneous to conclude that geometry, even partially, is an experimental science. If it were an experimental science, it would have only a temporary, approximate—and a very crudely approximate—value." Further on, "The subject of geometry includes the study of only a particular 'group' of translations, but the general concept of a group exists beforehand in our mind, at least as a possibility... . Experiment guides us in this choice, but does not make it mandatory for us: it shows us not which geometry is most correct, but which is most convenient... . The experiments which have led us to accept the fundamental conventions of geometry as the most convenient pertain to objects which have nothing in common with the objects studied in geometry; they pertain to the properties of solids, to the rectilinear propagation of light. These are mechanics and optics experiments; they can in no way be viewed as geometry experiments." Later, Poincaré emphasizes the following: "Principles are conventions and hidden definitions. Nevertheless, they are extracted from experimental laws. The latter have been, so to speak, raised to the level of principles to which our mind attributes an absolute meaning."

Somewhat later, in Chap. II entitled "Space and Time" of the book *Recent Thoughts*, Poincaré wrote: "The principle of physical relativity can serve as a definition of space. It gives us, so to speak, a new measuring tool. Let me explain. How can a solid body be used to measure or, more precisely, to construct space? The situation is the following: in moving a solid from one place to another, we notice that it can be matched first with one shape and then with another, and we agree to consider these shapes as the same. Geometry was born from this convention. Geometry is nothing but the study of the relations between these transformations or, expressed in mathematical language, the study of the structure of the group formed by these transformations, i.e., the group of motions of solids.

"Let us now take a different group, the group of transformations which leave our differential equations invariant. We obtain a new method of defining the equality of two shapes. We are no longer saying that two shapes are equal when one solid can be matched both to itself and to the other. We are saying that two shapes are equal when the same mechanical system, far enough away from neighboring systems that it can be viewed as being isolated, when arranged first such that its matter points reproduce the first shape and later such that they reproduce the second shape, behaves in the second case as in the first. Do these two views differ significantly from each other? No.

"A solid body is a mechanical system like any other. The entire difference between our former and our new definition of space is that the latter is broader, as it allows the solid to be replaced by any other mechanical system. Moreover, our new arbitrary convention defines not only space, but also time. It explains to us what are two simultaneous moments, what are two equal time intervals, or what is a time interval twice as large as another interval." In this way, with his discovery of the group of transformations which do not change the Maxwell–Lorentz equations, Poincaré introduced the concept of four-dimensional spacetime with pseudo-Euclidean geometry. This concept of geometry was later developed by Minkowski.

According to the viewpoint of Poincaré, it is natural to base any physical theory on the pseudo-Euclidean geometry of spacetime, as it is the simplest. We shall see below that this conclusion is correct, but we shall strengthen it by additional arguments.

After many years, in 1921, Einstein also turned to the problem of the relation between geometry and physics, and in the article "Geometry and Experiment" wrote: "It is clear that from the system of concepts of axiomatic geometry it is impossible to obtain any ideas about such actually existing objects which we refer to in practice as solids. For such ideas to be possible, we must rid geometry of its formal nature by associating the empty scheme of concepts of axiomatic geometry with the real objects of our experience. For this it is sufficient to add only the assertion that solids behave, in the sense of different possible relative arrangements, as bodies of Euclidean geometry of three dimensions; therefore, the theorems of Euclidean geometry contain statements which determine the behavior in practice of solids. Geometry supplemented by such an assertion obviously becomes a natural science; we can in fact view it as the oldest branch of physics. Its statements rest to a large degree on deductions from experiment, and not only on logical conclusions. Henceforth, we shall refer to geometry supplemented in this manner as 'practical geometry,' in contrast to 'purely axiomatic geometry.' The question of whether practical geometry is Euclidean or not becomes completely clear; the answer can only be given by experiment.

"Every measurement of a length in physics, exactly like geodesic or astronomical measurements, in this sense is a part of the subject of practical geometry, if here we start from the experimental law that light propagates in a straight line, that is, in a straight line in the sense of practical geometry... . If we dispense with the connection between the body of axiomatic Euclidean geometry and a real solid body, we easily arrive at the point of view that was taken by a thinker as original and deep as Henri Poincaré: Euclidean geometry is distinguished from all possible conceivable axiomatic geometries by its simplicity. And since axiomatic geometry by itself contains no statements about actual reality and can contain such statements only together with physical laws, it would appear possible and reasonable to adhere to Euclidean geometry, no matter what properties reality has. If we do find a contradiction between theory and experiment, it will be easier to consent to a change in the physical laws than to a change in axiomatic Euclidean geometry." Further on, he writes: "We feel ourselves forced to adopt the following, more general idea characteristic of the viewpoint of Poincaré. Geometry (Γ) says nothing about the behavior of real objects; this behavior is described only by geometry together with a set of physical laws (Φ). Expressed symbolically, we can say that only the sum $(\Gamma)+(\Phi)$ is the object of experimental verification. Therefore, it is possible to arbitrarily choose both (Γ) and individual parts of (Φ); all these laws are conventions. To avoid contradictions it is necessary to choose the other parts of (Φ) such that (Γ) and the complete (Φ) are together verified experimentally. In this picture, axiomatic geometry, from the viewpoint of the theory of knowledge, is equal in value to the part of the laws of Nature elevated to a convention. In my opinion, this picture of Poincaré is completely correct from the fundamental point of view.''

Although Einstein agreed in principle with Poincaré, he stated his point of view as follows: ''However, I am convinced that, given the current state of theoretical physics, these concepts [by which he meant clocks and a solid body—A.L.] must be used as independent ones, because we are still far from an understanding of the theoretical foundations of atomic science which would allow us to construct theoretically the concepts of solid bodies and clocks from more elementary ones.'' Further on, Einstein stressed that, ''The question of whether this continuum has a Euclidean, a Riemannian, or some other structure is a physical question which must be answered by experiment, rather than a question about convention regarding a choice based on simple expediency.... .

''I attach special importance to this understanding of geometry, because without it I could not have established the theory of relativity. That is, without it, it would be impossible to state that in a reference frame which is rotating relative to some inertial frame, the laws governing the arrangement of solids do not correspond to the laws of Euclidean geometry, owing to Lorentz contraction; therefore, admitting the equally valid existence of noninertial systems, we must reject Euclidean geometry. Any decisive step towards generally covariant equations would also be impossible without this interpretation.'' Einstein thus arrived at the Riemannian geometry of spacetime, which he used as the basis of the general theory of relativity. However, the concepts of homogeneity and isotropy of space are absent in Riemannian space, and so a theory which is constructed on the basis of Riemannian geometry also does not contain the usual energy–momentum and angular-momentum conservation laws.

From the preceding discussion we see that, following Poincaré, it is possible to arrive at a pseudo-Euclidean geometry of spacetime, whereas Einstein later, in the construction of the GTR, arrived at a Riemannian structure of spacetime, thereby putting aside the usual conservation laws. Both Poincaré and Einstein took experimental facts as the basis for choosing a geometry, but with the important difference that Poincaré, understanding the importance of experimental facts, still allowed the possibility of choice, whereas Einstein assumed that the question of geometry is a physical question which must be answered by experiment. This is true in principle. But the question immediately arises: which experiment? There can be many experimental facts. For example, by studying the motion of light and test bodies, it is possible in principle to determine uniquely the geometry of spacetime. Is it necessary to base a physical theory on it? At first sight, this question can be answered in the affirmative. And it would seem that the subject is closed. However, the situation is much more complicated. All forms of matter obey the energy–momentum and angular-momentum conservation laws. It is these laws, arising in the generalization of numerous experimental data, which characterize the general dynamical properties of all forms of matter, introducing the universal characteristics which allow the quantitative description of the transformation of one form of matter into another. All these are also experimental facts established by fundamental physical principles. What about them? The preservation of these principles uniquely determines the structure of geometry, which, it turns out, can be only pseudo-Euclidean. It is in precisely this case that the theory will contain both the law of energy–momentum conservation and the law of angular-momentum conservation for matter and the gravitational field together with the other laws. *Minkowski space reflects the dynamical properties common to all forms of matter, and so it is universal.* However, this implies that everything which is usually ascribed to the kinematics of the theory of relativity is in fact nothing but a reflection of the general dynamical properties of matter.

Therefore, not every experiment can be taken as an initial physical proposition when determining the structure of space. If the structure of space were constructed using experimental facts about the motion of test bodies and light, we would arrive, for example, at Riemannian space, but this would imply that we had lost fundamental principles of the theory: the conservation laws. The choice of pseudo-Euclidean structure of the geometry preserves the fundamental physical principles of the theory, the conservation laws, but to explain the motion of test bodies and light it will be necessary to introduce new physical propositions. We shall discuss this in detail in the next section. According to the ideas of Poincaré, the choice of pseudo-Euclidean geometry can be viewed as a convention which is not at all arbitrary, as it accurately reflects experimental laws elevated to principles.

On the basis of the relativistic theory of gravity,[1] we have postulated the pseudo-Euclidean geometry of spacetime as the fundamental Minkowski space for all the physical fields, including the gravitational field. This proposition is necessary and sufficient to have integrals of the motion: the energy–momentum and angular-momentum conservation laws for matter and the gravitational field in addition to the other laws. Minkowski space must not be assumed to exist *a priori*, because it reflects a property of matter and so is inseparable from it. However, formally, precisely owing to the independence of the structure of space from the form of matter, it is sometimes treated abstractly as separate from matter. In the Galilean coordinates of an inertial frame of Minkowski space, the interval characterizing the structure of the geometry and invariant by construction has the form

$$d\sigma^2=(dx^0)^2-(dx^1)^2-(dx^2)^2-(dx^3)^2.$$

Here dx^ν are the coordinate differentials.

Although the independence of the interval $d\sigma$, as a geometrical characteristic of spacetime, of the choice of coordinate system is specified by construction, up to now even in modern courses on theoretical physics (see, for example, Ref. 2) one can find a ''proof'' that the interval is identical in all inertial coordinate systems, although it is an invariant and is independent of the choice of coordinate system. Even a physicist as great as Mandel'shtam has written,[3] "The questions of how accelerated clocks move and why their motion changes cannot be answered by the special theory of relativity, because it is not at all concerned with the question of accelerated reference frames."

All this confusion[4] can be attributed to the fact that Minkowski space is viewed by many not as the discovery of the geometry of spacetime, but as a supposedly formal geometrical interpretation of the STR in Einstein's approach. Restricted concepts like the constancy of the speed of light, clock synchronization, and the independence of the speed of light from the motion of the source have been pushed into the spotlight. This has greatly restricted the framework of the STR and hindered understanding of its essential features. *After all, its essential feature is only that the geometry of spacetime in which all physical processes occur is pseudo-Euclidean geometry.*

In an arbitrary coordinate system the interval takes the form

$$d\sigma^2=\gamma_{\mu\nu}(x)dx^\mu dx^\nu,$$

where $\gamma_{\mu\nu}(x)$ is the metric tensor of Minkowski space. We note that in a noninertial coordinate system it is impossible in principle to speak of clock synchronization and constancy of the speed of light.[5] The free motion of a test body in an arbitrary coordinate system occurs along a geodesic of Minkowski space:

$$\frac{DU^\nu}{d\sigma}=\frac{dU^\nu}{d\sigma}+\gamma^\nu_{\alpha\beta}U^\alpha U^\beta=0,$$

where $U^\nu=dx^\nu/d\sigma$, and $\gamma^\nu_{\alpha\beta}(x)$ are the Christoffel symbols, given by

$$\gamma^\nu_{\alpha\beta}(x)=\frac{1}{2}\,\gamma^{\nu\sigma}(\partial_\alpha\gamma_{\beta\sigma}+\partial_\beta\gamma_{\alpha\sigma}-\partial_\sigma\gamma_{\alpha\beta}).$$

We have taken pseudo-Euclidean geometry as the basis of the theory of gravity, but this does not imply that the effective space in the presence of a gravitational field will also be pseudo-Euclidean. Under the action of a gravitational field the effective space will be something different. We shall discuss this point in detail in the following section. The metric of Minkowski space allows the introduction of the concept of a reference length and time interval in the absence of a gravitational field.

2. THE MATTER ENERGY–MOMENTUM TENSOR AS THE SOURCE OF THE GRAVITATIONAL FIELD

Owing to the presence in Minkowski space of the ten-parameter Poincaré group of motions, for any closed physical system there are ten integrals of the motion, i.e., there are energy–momentum and angular-momentum conservation laws. Any physical field in Minkowski space is characterized completely by the energy–momentum tensor $t^{\mu\nu}$, which is a universal characteristic common to all forms of matter. It satisfies both a local and an integral conservation law. In an arbitrary coordinate system the local conservation law is written as

$$D_\mu t^{\mu\nu}=\partial_\mu t^{\mu\nu}+\gamma^\nu_{\alpha\beta}t^{\alpha\beta}=0.$$

Here $t^{\mu\nu}$ is the total conserved density of the energy–momentum tensor of all the matter fields, and D_μ is the covariant derivative in Minkowski space.

Here and below, we shall always deal with densities of scalar and tensor quantities defined by the rule

$$\tilde{\phi}=\sqrt{-\gamma}\phi,\quad \tilde{\phi}^{\mu\nu}=\sqrt{-\gamma}\phi^{\mu\nu},\quad \gamma=\det(\gamma_{\mu\nu}).$$

Densities have been introduced because in arbitrary coordinates the invariant volume element in Minkowski space is given by

$$\sqrt{-\gamma}d^4x,$$

and the invariant volume element in Riemannian space is given by

$$\sqrt{-g}d^4x,\quad g=\det(g_{\mu\nu}).$$

Therefore, the principle of least action is given by

$$\delta S=\delta\int Ld^4x=0,$$

where L is the scalar Lagrangian density of the matter.

In obtaining the Euler equations using the principle of least action we will automatically be dealing with variations of the Lagrangian density. According to Hilbert, the density of the energy–momentum tensor $t^{\mu\nu}$ is expressed in terms of the scalar Lagrangian density L as

$$t^{\mu\nu}=-2\frac{\delta L}{\delta\gamma_{\mu\nu}}, \tag{1}$$

where

$$\frac{\delta L}{\delta\gamma_{\mu\nu}}=\frac{\partial L}{\partial\gamma_{\mu\nu}}-\partial_\sigma\left(\frac{\partial L}{\partial\gamma_{\mu\nu,\sigma}}\right),\quad \gamma_{\mu\nu,\sigma}=\frac{\partial\gamma_{\mu\nu}}{\partial x^\sigma}.$$

Owing to the universality of gravitation, it is natural to assume that the conserved density of the energy–momentum tensor of all the matter fields $t^{\mu\nu}$ is the source of the gravitational field. We shall use the analogy with electrodynamics, in which the source of the electromagnetic field is the conserved density of the charged vector current j^ν, and the field itself has a vector nature and is described by the density of the vector potential $\tilde{A}^\nu$:

$$\tilde{A}^\nu=(\tilde{\phi},\tilde{A}_i).$$

In the absence of gravity, the Maxwell equations of electrodynamics in arbitrary coordinates take the form

$$\gamma^{\alpha\beta}D_\alpha D_\beta\tilde{A}^\nu+\mu^2\tilde{A}^\nu=4\pi j^\nu,$$

$$D_\nu\tilde{A}^\nu=0.$$

Here for generality we have introduced the photon mass μ.

Since we have stated that the source of the gravitational field is the conserved density of the energy–momentum tensor $t^{\mu\nu}$, it is natural to assume that the gravitational field is a tensor field and to describe it by the density of a symmetric tensor $\tilde{\phi}^{\mu\nu}$:

$$\tilde{\phi}^{\mu\nu}=\sqrt{-\gamma}\phi^{\mu\nu},$$

and in complete analogy with Maxwell electrodynamics the gravitational field equations can be written as

$$\gamma^{\alpha\beta}D_\alpha D_\beta\tilde{\phi}^{\mu\nu}+m^2\tilde{\phi}^{\mu\nu}=\lambda t^{\mu\nu}, \tag{2}$$

$$D_\mu\tilde{\phi}^{\mu\nu}=0. \tag{3}$$

Here λ is a constant which, because of the principle of correspondence with the Newtonian law of gravity, must be equal to 16π. Equation (3) excludes spins 1 and $0'$, leaving polarization properties of the field corresponding only to spins 2 and 0.

The density of the matter energy–momentum tensor $t^{\mu\nu}$ consists of the density of the energy–momentum tensor of the gravitational field $t_g^{\mu\nu}$ and the density of the matter energy–momentum tensor $t_M^{\mu\nu}$. By matter we mean all the matter fields except the gravitational field:

$$t^{\mu\nu}=t_g^{\mu\nu}+t_M^{\mu\nu}.$$

The interaction between the gravitational field and matter is included in the density of the matter energy–momentum tensor $t_M^{\mu\nu}$.

In 1913, Einstein wrote:[6] "...the gravitational field tensor $\vartheta_{\mu\nu}$ is the source of field just as much as the tensor of the matter systems $\Theta_{\mu\nu}$. A special role of the gravitational field energy compared to all the other forms of energy would lead to unacceptable consequences." It is this idea of Einstein that we have used as the foundation for constructing the relativistic theory of gravity. When constructing the general theory of relativity, Einstein did not realize this idea, because in the GTR a gravitational-field pseudotensor arose instead of the gravitational-field energy–momentum tensor. This occurred because Einstein did not treat the gravitational field as a physical field (of the Faraday–Maxwell type) in Minkowski space. This is why the equations of the GTR do not contain the metric of Minkowski space. It follows from (2) that these equations will be nonlinear also for the intrinsic gravitational field, because the tensor density $t_g^{\mu\nu}$ is the source of the gravitational field. Equations (2) and (3), which we formally state to be the gravitational equations, by analogy with electrodynamics, will have to be obtained from the principle of least action, because only in this case will we have an explicit expression for the density of the energy–momentum tensor of the gravitational field and the matter fields. However, for this it is necessary to construct the Lagrangian density of the matter and the gravitational field. It is extremely important to perform this construction using general axioms. Only then can we speak of a theory of gravity. The initial scalar Lagrangian density of the matter can be written as

$$L=L_g(\gamma_{\mu\nu},\tilde{\phi}^{\mu\nu})+L_M(\gamma_{\mu\nu},\tilde{\phi}^{\mu\nu},\phi_A),$$

where L_g is the Lagrangian density of the gravitational field, L_M is the Lagrangian density of the matter fields, and ϕ_A are the matter fields.

According to the principle of least action, the equations for the gravitational field and the matter fields take the form

$$\frac{\delta L}{\delta\tilde{\phi}^{\mu\nu}}=0, \tag{4}$$

$$\frac{\delta L_M}{\delta\phi_A}=0. \tag{5}$$

Equation (4) differs from (2) primarily in that the variational derivative of the Lagrangian density is taken with respect to the field $\tilde{\phi}^{\mu\nu}$, whereas (2), according to the definition (1), involves the variational derivative of the Lagrangian density with respect to the metric $\gamma_{\mu\nu}$. In order to have Eq. (4) reduce to (2) for any form of matter, it must be assumed that the tensor density $\tilde{\phi}^{\mu\nu}$ always enters into the Lagrangian density together with the tensor density $\tilde{\gamma}^{\mu\nu}$ through some common density $\tilde{g}^{\mu\nu}$ in the form

$$\tilde{g}^{\mu\nu}=\tilde{\gamma}^{\mu\nu}+\tilde{\phi}^{\mu\nu},\quad \tilde{g}^{\mu\nu}=\sqrt{-g}g^{\mu\nu}. \tag{6}$$

Taking this condition into account, the Lagrangian density L becomes

$$L=L_g(\gamma_{\mu\nu},\tilde{g}^{\mu\nu})+L_M(\gamma_{\mu\nu},\tilde{g}^{\mu\nu},\phi_A).$$

It should be stressed that the condition (6) allows the variational derivative with respect to $\tilde{\phi}^{\mu\nu}$ to be replaced by the variational derivative with respect to $\tilde{g}^{\mu\nu}$, and the variational derivative with respect to $\gamma_{\mu\nu}$ to be expressed in terms of that with respect to $\tilde{g}_{\mu\nu}$ and that with respect to $\gamma_{\mu\nu}$ appearing explicitly in the Lagrangian density L. In fact,

$$\frac{\delta L}{\delta\tilde{\phi}^{\mu\nu}}=\frac{\delta L}{\delta\tilde{g}^{\mu\nu}}=0, \tag{7}$$

$$\frac{\delta L}{\delta\gamma_{\mu\nu}}=\frac{\delta^* L}{\delta\gamma_{\mu\nu}}+\frac{\delta L}{\delta\tilde{g}^{\alpha\beta}}\cdot\frac{\partial\tilde{g}^{\alpha\beta}}{\partial\gamma_{\mu\nu}}. \tag{8}$$

The derivation of the last expression is given in detail in Appendix A. The asterisk in Eq. (8) denotes the variational derivative of the Lagrangian density with respect to the metric $\gamma_{\mu\nu}$ entering explicitly into L. According to (1), Eq. (8) can be written as

$$t^{\mu\nu}=-2\frac{\delta L}{\delta\tilde{g}^{\alpha\beta}}\cdot\frac{\delta\tilde{g}^{\alpha\beta}}{\partial\gamma_{\mu\nu}}-2\frac{\delta^* L}{\delta\gamma_{\mu\nu}}.$$

Using Eq. (7) in this expression, we find

$$t^{\mu\nu}=-2\frac{\delta^* L}{\delta\gamma_{\mu\nu}}. \tag{9}$$

Comparing Eqs. (9) and (2), we obtain the condition

$$-2\frac{\delta^* L}{\delta\gamma_{\mu\nu}}=\frac{1}{16\pi}[\gamma^{\alpha\beta}D_\alpha D_\beta\tilde{\phi}^{\mu\nu}+m^2\tilde{\phi}^{\mu\nu}], \tag{10}$$

which when satisfied ensures that the gravitational field equations (2) and (3) can be obtained directly from the principle of least action. Since the right-hand side of (10) does not contain matter fields, the variation of the Lagrangian density of the matter L_M with respect to the explicitly appearing metric $\gamma_{\mu\nu}$ must be equal to zero. In order that no additional restrictions arise on the motion of the matter described by Eq. (5), it follows directly from this that the tensor $\gamma_{\mu\nu}$ does not enter explicitly into the expression for the matter Lagrangian density L_M. The condition (10) then takes the form

$$-2\frac{\delta^*L_g}{\delta\gamma_{\mu\nu}}=\frac{1}{16\pi}[\gamma^{\alpha\beta}D_\alpha D_\beta\tilde{\phi}^{\mu\nu}+m^2\tilde{\phi}^{\mu\nu}]. \quad (11)$$

Therefore, everything reduces to finding the Lagrangian density of the intrinsic gravitational field L_g which satisfies (11).

Meanwhile, from the above arguments we arrive at the important conclusion that the matter Lagrangian density L has the form

$$L=L_g(\gamma_{\mu\nu},\tilde{g}^{\mu\nu})+L_M(\tilde{g}^{\mu\nu},\phi_A). \quad (12)$$

Thus, from the requirement that the density of the matter energy–momentum tensor be the source of the gravitational field, it naturally follows that the motion of the matter must occur in an effective Riemannian space. It is this fact which will allow us in Sec. 3 below to obtain the gauge group and then construct the Lagrangian density (55) satisfying, according to (B20), the condition (11).

An interesting picture appears: the motion of matter in Minkowski space with the metric $\gamma_{\mu\nu}$ under the action of the gravitational field $\phi^{\mu\nu}$ is identical to the motion of matter in the effective Riemannian space with metric $g_{\mu\nu}$ given by Eq. (6). We refer to this interaction between the gravitational field and matter as the *geometrization principle.* The geometrization principle is a consequence of the original assumption that the source of the gravitational field is a universal characteristic of matter—the density of the energy–momentum tensor. This structure of the matter Lagrangian density indicates that a unique possibility is realized: the gravitational field is coupled directly to the density of the tensor $\tilde{\gamma}^{\mu\nu}$ in the matter Lagrangian density. The effective Riemannian space literally has a field origin owing to the presence of the gravitational field. Let us explain this fundamental property of gravitational forces by comparing them with electromagnetic forces. As is well known, the motion of a charged particle in Minkowski space when there is a uniform magnetic field is, owing to the Lorentz force, a circle in the plane perpendicular to the direction of the magnetic field. However, this motion is far from identical even for charged particles if their charge-to-mass ratios differ. Moreover, there are neutral particles, and their trajectories in a magnetic field are just straight lines. Therefore, since the electromagnetic forces are nonuniversal, their action cannot lead to the geometry of spacetime.

Gravity is different. It is universal; any test body moves along a trajectory, the same trajectory for identical initial conditions. Owing to the hypothesis that the density of the matter energy–momentum tensor is the source of the gravitational field, these trajectories can be described by geodesics in an effective Riemanian spacetime arising from the presence of the gravitational field in Minkowski space. In regions of space where the gravitational field is arbitrarily weak, we have metrical properties of space which to high accuracy approach the properties of pseudo-Euclidean space observed directly. When the gravitational fields are strong, the metrical properties of the effective space become Riemmanian. However, also in this case the pseudo-Euclidean geometry does not completely vanish: it is observable and is manifested in the fact that the motion of bodies in the effective Riemannian space is not free from inertia, but occurs with acceleration with respect to the pseudo-Euclidean space in Galilean coordinates. This is why acceleration in the RTG, as opposed to the GTR, has an absolute meaning. Therefore, the "Einstein elevator" cannot be an inertial coordinate system. This is revealed in the fact that a charge which is at rest in the Einstein elevator will emit electromagnetic waves. This physical phenomenon must also indicate the presence of Minkowski space.

The matter equation of motion does not involve the metric tensor $\gamma_{\mu\nu}$ of Minkowski space. Minkowski space will affect the matter motion only through the metric tensor $g_{\mu\nu}$ of Riemannian space determined, as we shall see below, from the gravitational equations, which involve the metric tensor $\gamma_{\mu\nu}$ of Minkowski space. Since the effective Riemannian metric arises on the basis of a physical field which is specified in Minkowski space, it follows that the effective Riemannian space has a simple topology and is specified on a single sheet. If, for example, the matter is concentrated in an island-like region, then in the Galilean coordinates of the inertial frame the gravitational field $\tilde{\phi}^{\mu\nu}$ cannot fall off more slowly than $1/r$. However, this fact imposes a strong constraint on the asymptotic behavior of the metric $g_{\mu\nu}$ of the effective Riemannian geometry:

$$g_{\mu\nu}=\eta_{\mu\nu}+0\left(\frac{1}{r}\right), \quad \text{with } \eta_{\mu\nu}=(1,-1,-1,-1). \quad (13)$$

If we start simply from the Riemannian metric without assuming that it appeared owing to the action of a physical field, such constraints do not arise, because the asymptote of the metric $g_{\mu\nu}$ depends even on the choice of three-dimensional spatial coordinates. Then, in principle, physical quantities cannot depend on the choice of three-dimensional spatial coordinates. In the RTG no restrictions arise on the choice of coordinate system. The coordinate system can be anything, as long as it establishes a one-to-one correspondence for all the points of the inertial coordinate system of Minkowski space and ensures satisfaction of the inequalities

$$\gamma_{00}>0, \; dl^2=s_{ik}dx^idx^k>0; \; i,k=1,2,3,$$

where

$$s_{ik}=-\gamma_{ik}+\frac{\gamma_{oi}\gamma_{ok}}{\gamma_{oo}},$$

needed for the introduction of the concepts of time and spatial length. In our theory of gravity the geometrical characteristics of Riemannian space arise as field quantities in Minkowski space, and so their transformation properties become tensor ones, even if they were not so earlier. For example, the Christoffel symbols, specified as field values in the Galilean coordinates of Minkowski space, are already rank-three tensors. Similarly, the ordinary derivatives of tensor quantities in the Cartesian coordinates of Minkowski space are also tensors.

The following question may arise. Why not introduce separation of the metric in the form (6) also in the GTR, thus introducing the concept of gravitational field in Minkowski space? The Hilbert–Einstein equations involve only $g_{\mu\nu}$, and so it is impossible to say uniquely which metric $\gamma_{\mu\nu}$ of Minkowski space should be used to find the gravitational field from (6). However, this is not the only difficulty; there is also the problem that the solutions of the Hilbert–Einstein equations are in general specified not on a single sheet, but on a set of sheets. Such solutions for $g_{\mu\nu}$ describe a Riemannian space with complicated topology, whereas the Riemmanian spaces obtained using the representation of the gravitational field in Minkowski space are specified on a single sheet and have a simple topology. It is for these reasons that field representations are not compatible with the GTR, since they are very rigorous. However, this implies that in principle there can be no field formulation of the GTR in Minkowski space, no matter what. The formalism of Riemannian geometry is predisposed to allow the introduction of covariant derivatives in Minkowski space, which we have used in the construction of the RTG. However, to accomplish this it was necessary to introduce the metric of Minkowski space in the gravitational equations, and it thereby became possible to establish a functional relation between the metric of the Riemannian space $g_{\mu\nu}$ and the metric of Minkowski space $\gamma_{\mu\nu}$. We shall discuss this in more detail in the following sections.

3. THE GAUGE GROUP OF TRANSFORMATIONS

Since the matter Lagrangian density has the form

$$L_m(\tilde{g}^{\mu\nu},\phi_A), \tag{14}$$

it is easy to find the gauge group of the transformations under which the matter Lagrangian density changes only by a divergence. For this we use the invariance of the action

$$S_M=\int L_M(\tilde{g}^{\mu\nu},\phi_A)d^4x \tag{15}$$

under an arbitrary infinitesimal change of coordinates

$$x'^{\alpha}=x^{\alpha}+\xi^{\alpha}(x), \tag{16}$$

where ξ^{α} is an infinitesimal translation four-vector.

Under these coordinate transformations the field functions $\tilde{g}^{\mu\nu}$ and ϕ_A change as

$$\tilde{g}'^{\mu\nu}(x')=\tilde{g}^{\mu\nu}(x)+\delta_{\xi}\tilde{g}^{\mu\nu}(x)+\xi^{\alpha}(x)D_{\alpha}\tilde{g}^{\mu\nu}(x),$$

$$\phi'_A(x')=\phi_A(x)+\delta_{\xi}\phi_A(x)+\xi^{\alpha}(x)D_{\alpha}\phi_A(x), \tag{17}$$

where the expressions

$$\delta_{\xi}\tilde{g}^{\mu\nu}(x)=\tilde{g}^{\mu\alpha}D_{\alpha}\xi^{\nu}(x)+\tilde{g}^{\nu\alpha}D_{\alpha}\xi^{\mu}(x)-D_{\alpha}(\xi^{\alpha}\tilde{g}^{\mu\nu}),$$

$$\delta_{\xi}\phi_A(x)=-\xi^{\alpha}(x)D_{\alpha}\phi_A(\tau)+F^{B;\alpha}_{A;\beta}\phi_B(x)D_{\alpha}\xi^{\beta}(x) \tag{18}$$

are the Lie variations.

The operators δ_{ξ} satisfy the conditions of a Lie algebra, i.e., the commutation relation

$$[\delta_{\xi_1},\delta_{\xi_2}](\cdot)=\delta_{\xi_3}(\cdot) \tag{19}$$

and the Jacobi identity

$$[\delta_{\xi_1},[\delta_{\xi_2},\delta_{\xi_3}]]+[\delta_{\xi_3},[\delta_{\xi_1},\delta_{\xi_2}]]+[\delta_{\xi_2},[\delta_{\xi_3},\delta_{\xi_1}]]=0,$$

where

$$\xi_3^{\nu}=\xi_1^{\mu}D_{\mu}\xi_2^{\nu}-\xi_2^{\mu}D_{\mu}\xi_1^{\nu}=\xi_1^{\mu}\partial_{\mu}\xi_2^{\nu}-\xi_2^{\mu}\partial_{\mu}\xi_1^{\nu}. \tag{20}$$

For (19) to hold, the following conditions must be satisfied:

$$F^{B;\mu}_{A;\nu}F^{C;\alpha}_{B;\beta}-F^{B;\alpha}_{A;\beta}F^{C;\mu}_{B;\nu}=f^{\mu\alpha;\tau}_{\nu\beta;\sigma}F^{C;\sigma}_{A;\tau}, \tag{21}$$

where the structure constants f are

$$f^{\mu\alpha;\tau}_{\nu\beta;\sigma}=\delta^{\mu}_{\beta}\delta^{\alpha}_{\sigma}\delta^{\tau}_{\nu}-\delta^{\alpha}_{\nu}\delta^{\mu}_{\sigma}\delta^{\tau}_{\beta}. \tag{22}$$

It is easily checked that they satisfy the Jacobi identity

$$f^{\alpha\nu;\sigma}_{\beta\mu;\tau}f^{\tau\rho;\omega}_{\sigma\epsilon;\delta}+f^{\nu\rho;\sigma}_{\mu\epsilon;\tau}f^{\tau\alpha;\omega}_{\sigma\beta;\delta}+f^{\rho\alpha;\sigma}_{\epsilon\beta;\tau}f^{\tau\nu;\omega}_{\sigma\mu;\delta}=0 \tag{23}$$

and possess the antisymmetry property

$$f^{\alpha\nu;\rho}_{\beta\mu;\sigma}=-f^{\nu\alpha;\rho}_{\mu\beta;\sigma}.$$

Under a coordinate transformation (16) the variation of the action is equal to zero:

$$\delta_c S_M=\int_{\Omega'}L'_M(x')d^4x'-\int_{\Omega}L_M(x)d^4x=0. \tag{24}$$

The first integral in (24) can be written as

$$\int_{\Omega'}L'_M(x')d^4x'=\int_{\Omega}JL'_M(x')d^4x,$$

where

$$J=\det\left(\frac{\partial x'^{\alpha}}{\partial x^{\beta}}\right).$$

In first order in ξ^{α} the determinant J is

$$J=1+\partial_{\alpha}\xi^{\alpha}(x). \tag{25}$$

Using the expansion

$$L'_M(x')=L'_M(x)+\xi^{\alpha}(x)\frac{\partial L_M}{\partial x^{\alpha}},$$

and also (25), the expression for the variation can be written as

$$\delta_c S_M=\int_{\Omega}[\delta L_M(x)+\partial_{\alpha}(\xi^{\alpha}L_M(x))]d^4x=0.$$

Owing to the arbitrariness of the integration volume Ω, we have the identity

$$\delta L_M(x)=-\partial_{\alpha}(\xi^{\alpha}(x)L_M(x)), \tag{26}$$

where the Lie variation δL_M is

$$\delta L_M(x)=\frac{\partial L_M}{\partial\tilde{g}^{\mu\nu}}\delta\tilde{g}^{\mu\nu}+\frac{\partial L_M}{\partial(\partial_{\alpha}\tilde{g}^{\mu\nu})}\delta(\partial_{\alpha}\tilde{g}^{\mu\nu})+\frac{\partial L_M}{\partial\phi_A}\delta\phi_A+\frac{\partial L_M}{\partial(\partial_{\alpha}\phi_A)}\delta(\partial_{\alpha}\phi_A). \tag{27}$$

From this, in particular, it follows that if the scalar density depends only on $\tilde{g}^{\mu\nu}$ and derivatives of it, then under the transformation (18) it will also change by only a divergence:

$$\delta L(\tilde{g}^{\mu\nu}(x))=-\partial_{\alpha}(\xi^{\alpha}(x)L(\tilde{g}^{\mu\nu}(x))), \tag{26a}$$

where the Lie variation δL is

$$\delta L(\tilde{g}^{\mu\nu}(x))=\frac{\partial L}{\partial\tilde{g}^{\mu\nu}}\delta\tilde{g}^{\mu\nu}+\frac{\partial L}{\partial(\partial_{\alpha}\tilde{g}^{\mu\nu})}\delta(\partial_{\alpha}\tilde{g}^{\mu\nu})+\frac{\partial L}{\partial(\partial_{\alpha}\partial_{\beta}\tilde{g}^{\mu\nu})}\delta(\partial_{\alpha}\partial_{\beta}\tilde{g}^{\mu\nu}). \tag{27a}$$

The Lie variations (18) were established within the context of coordinate transformations (16). However, we can adopt a different viewpoint, according to which the transformations (18) can be treated as gauge transformations. In this case an arbitrary infinitesimal four-vector $\xi^\alpha(x)$ will be a gauge vector and not a coordinate translation vector. In what follows, to emphasize the difference between a gauge group and a group of coordinate transformations, we shall use the notation $\epsilon^\alpha(x)$ for the group parameter, and a transformation of the field functions

$$\tilde{g}^{\mu\nu}(x)\to\tilde{g}^{\mu\nu}(x)+\delta\tilde{g}^{\mu\nu}(x),$$
$$\phi_A(x)\to\phi_A(x)+\delta\phi_A(x) \quad (28)$$

with the additions

$$\delta_\epsilon\tilde{g}^{\mu\nu}(x)=\tilde{g}^{\mu\alpha}D_\alpha\epsilon^\nu(x)+\tilde{g}^{\nu\alpha}D_\alpha\epsilon^\mu(x)-D_\alpha(\epsilon^\alpha\tilde{g}^{\mu\nu}),$$
$$\delta_\epsilon\phi_A(x)=-\epsilon^\alpha(x)D_\alpha\phi_A(x)+F_{A;\beta}^{B;\alpha}\phi_B(x)D_\alpha\epsilon^\beta(x) \quad (29)$$

will be referred to as *gauge transformations.*

In complete correspondence with Eqs. (19) and (20), the operators satisfy the same Lie algebra, i.e., the commutation relation

$$[\delta_{\epsilon_1},\delta_{\epsilon_2}](\cdot)=\delta_{\epsilon_3}(\cdot) \quad (30)$$

and the Jacobi identity

$$[\delta_{\epsilon_1},[\delta_{\epsilon_2},\delta_{\epsilon_3}]]+[\delta_{\epsilon_3},[\delta_{\epsilon_1},\delta_{\epsilon_2}]]+[\delta_{\epsilon_2},[\delta_{\epsilon_3},\delta_{\epsilon_1}]]=0. \quad (31)$$

Here, as before, we have

$$\epsilon_3^\nu=\epsilon_1^\mu D_\mu\epsilon_2^\nu-\epsilon_2^\mu D_\mu\epsilon_1^\nu=\epsilon_1^\mu\partial_\mu\epsilon_2^\nu-\epsilon_2^\mu\partial_\mu\epsilon_1^\nu.$$

The gauge group arose from the geometrized structure of the scalar density of the matter Lagrangian $L_M(\tilde{g}^{\mu\nu},\phi_A)$, which, owing to the identity (26), is changed by only a divergence under the gauge transformations (29). Therefore, the geometrization principle, which determined the universal nature of the interaction between matter and the gravitational field, has allowed us to formulate the noncommutative, infinite-dimensional gauge group (29).

A significant difference between gauge and coordinate transformations is manifested in a crucial place in the theory when constructing the scalar Lagrangian density of the intrinsic gravitational field. The difference arises because under a gauge transformation the metric tensor $\gamma_{\mu\nu}$ does not change, and so, owing to (6), we have

$$\delta_\epsilon\tilde{g}^{\mu\nu}(x)=\delta_\epsilon\tilde{\phi}^{\mu\nu}(x).$$

On the basis of (19) we obtain the field transformation

$$\delta_\epsilon\tilde{\phi}^{\mu\nu}(x)=\tilde{g}^{\mu\alpha}D_\alpha\epsilon^\nu(x)+\tilde{g}^{\nu\alpha}D_\alpha\epsilon^\mu(x)-D_\alpha(\epsilon^\alpha\tilde{g}^{\mu\nu}),$$

but it differs significantly from the transformation under a coordinate shift:

$$\delta_\xi\tilde{\phi}^{\mu\nu}(x)=\tilde{\phi}^{\mu\alpha}D_\alpha\xi^\nu(x)+\tilde{\phi}^{\nu\alpha}D_\alpha\xi^\mu(x)-D_\alpha(\xi^\alpha\tilde{\phi}^{\mu\nu}).$$

The equations of motion for the matter do not change under the gauge transformations (29), because under any such transformations the Lagrangian density of the matter changes by only a divergence.

4. THE LAGRANGIAN DENSITY AND EQUATIONS OF MOTION FOR THE INTRINSIC GRAVITATIONAL FIELD

As is well known, using only the tensor $g_{\mu\nu}$ it is impossible to construct a Lagrangian density of the intrinsic gravitational field which is a scalar under arbitrary coordinate transformations in the form of a quadratic form of derivatives of order no higher than the first. This Lagrangian density will therefore necessarily involve the metric $\gamma_{\mu\nu}$ along with the metric $g_{\mu\nu}$. However, since the metric $\gamma_{\mu\nu}$ is not changed under a gauge transformation (29), for the Lagrangian density of the intrinsic gravitational field to change by only a divergence under such a transformation, strong constraints on its structure must arise. This is where the fundamental difference between gauge and coordinate transformations appears.

Whereas coordinate transformations impose almost no restrictions on the structure of the scalar Lagrangian density of the intrinsic gravitational field, gauge transformations allow us to find this Lagrangian density. A direct, general method of constructing the Lagrangian is given in Ref. 1.

Here we shall use a simpler method to construct the Lagrangian. On the basis of (26a) we conclude that the simplest scalar densities $\sqrt{-g}$ and $\tilde{R}=\sqrt{-g}R$, where R is the scalar curvature of the effective Riemannian space, change as follows under the gauge transformation (29):

$$\sqrt{-g}\to\sqrt{-g}-D_\nu(\epsilon^\nu\sqrt{-g}), \quad (32)$$
$$\tilde{R}\to\tilde{R}-D_\nu(\epsilon^\nu\tilde{R}). \quad (33)$$

The scalar density $\tilde{R}$ is expressed in terms of the Christoffel symbols

$$\Gamma^\lambda_{\mu\nu}=\frac{1}{2}g^{\lambda\sigma}(\partial_\mu g_{\sigma\nu}+\partial_\nu g_{\sigma\mu}-\partial_\sigma g_{\mu\nu}) \quad (34)$$

as

$$\tilde{R}=-\tilde{g}^{\mu\nu}(\Gamma^\lambda_{\mu\nu}\Gamma^\sigma_{\lambda\sigma}-\Gamma^\lambda_{\mu\sigma}\Gamma^\sigma_{\nu\lambda})-\partial_\nu(\tilde{g}^{\mu\nu}\Gamma^\sigma_{\mu\sigma}-\tilde{g}^{\mu\sigma}\Gamma^\nu_{\mu\sigma}). \quad (35)$$

Since the Christoffel symbols are not tensor quantities, each term in (35) is not a scalar density. However, if we introduce the tensor quantities

$$G^\lambda_{\mu\nu}=\frac{1}{2}g^{\lambda\sigma}(D_\mu g_{\sigma\nu}+D_\nu g_{\sigma\mu}-D_\sigma g_{\mu\nu}), \quad (36)$$

the scalar density can be written identically as

$$\tilde{R}=-\tilde{g}^{\mu\nu}(G^\lambda_{\mu\nu}G^\sigma_{\lambda\sigma}-G^\lambda_{\mu\sigma}G^\sigma_{\nu\lambda})-D_\nu(\tilde{g}^{\mu\nu}G^\sigma_{\mu\sigma}-\tilde{g}^{\mu\sigma}G^\nu_{\mu\sigma}). \quad (37)$$

In (37) each group of terms behaves separately as a scalar density under arbitrary coordinate transformations. We shall see that the formalism of Riemannian geometry is predisposed to the introduction of covariant derivatives in Minkowski space instead of ordinary derivatives, but the metric tensor $\gamma_{\mu\nu}$, used to determine the covariant derivatives, is not fixed at all here.

Taking into account (32) and (33), the expression

$$\lambda_1(\tilde{R}+D_\nu Q^\nu)+\lambda_2\sqrt{-g} \tag{38}$$

changes by only a divergence under arbitrary gauge transformations. Choosing the vector density to be

$$Q^\nu=\tilde{g}^{\mu\nu}G^\sigma_{\mu\sigma}-\tilde{g}^{\mu\sigma}G^\nu_{\mu\sigma},$$

we eliminate from the preceding expression terms with derivatives of order higher than the first and obtain the following Lagrangian density:

$$-\lambda_1\tilde{g}^{\mu\nu}(G^\lambda_{\mu\nu}G^\sigma_{\lambda\sigma}-G^\lambda_{\mu\sigma}G^\sigma_{\nu\lambda})+\lambda_2\sqrt{-g}. \tag{39}$$

We therefore see that the requirement that the Lagrangian density of the intrinsic gravitational field change by only a divergence under a gauge transformation (29) uniquely determines the structure of the Lagrangian density (39). However, it we restrict ourselves to only this density, the gravitational field equations will be gauge-invariant, and the metric of Minkowski space $\gamma_{\mu\nu}$ will not enter into the system of equations determined by the Lagrangian density (39). Since the metric of Minkowski space vanishes in this approach, there is no possibility of representating the gravitational field as a physical field of the Faraday–Maxwell type in Minkowski space.

For the Lagrangian density (39) the introduction of the metric $\gamma_{\mu\nu}$ using Eq. (3) does not save the situation, because physical quantities—the interval and curvature tensor of Riemannian space, and also the gravitational field tensor $t_g^{\mu\nu}$—will depend on the choice of gauge, which is physically inadmissable. For example,

$$\delta_\epsilon R_{\mu\nu}=-R_{\mu\sigma}D_\nu\epsilon^\sigma-R_{\nu\sigma}D_\mu\epsilon^\sigma-\epsilon^\sigma D_\sigma R_{\mu\nu},$$

$$\delta_\epsilon R_{\mu\nu\alpha\beta}=-R_{\sigma\nu\alpha\beta}D_\mu\epsilon^\sigma-R_{\mu\sigma\alpha\beta}D_\nu\epsilon^\sigma-R_{\mu\nu\sigma\beta}D_\alpha\epsilon^\sigma-R_{\mu\nu\alpha\sigma}D_\beta\epsilon^\sigma-\epsilon^\sigma D_\sigma R_{\mu\nu\alpha\beta}.$$

In order to preserve the idea of a field in Minkowski space and eliminate this ambiguity, it is necessary to add to the Lagrangian density of the gravitational field a term violating the gauge group. At first glance it might appear that here there is a great arbitrariness in the choice of Lagrangian density of the gravitational field, because the gauge group can be broken in very different ways. However, it turns out that this is not so, because our physical requirement on the polarization properties of the gravitational field as a field with spins 2 and 0, imposed by Eq. (3), means that the term violating the group (29) must be chosen such that Eqs. (3) follow from the system of equations of the gravitational field and the matter fields, as only then do we not obtain an overdetermined system of differential equations. For this, we introduce into the scalar Lagrangian density of the gravitational field a term of the form

$$\gamma_{\mu\nu}\tilde{g}^{\mu\nu}, \tag{40}$$

which with the conditions (3) changes by a divergence under the transformations (29), but only on the class of vectors satisfying the condition

$$g^{\mu\nu}D_\mu D_\nu\epsilon^\sigma(x)=0. \tag{41}$$

An almost analogous situation occurs in electrodynamics with nonzero photon rest mass. Taking into account (38)–(40), the general scalar Lagrangian density has the form

$$L_g=-\lambda_1\tilde{g}^{\mu\nu}(G^\lambda_{\mu\nu}G^\sigma_{\lambda\sigma}-G^\lambda_{\mu\sigma}G^\sigma_{\nu\lambda})+\lambda_2\sqrt{-g}+\lambda_3\gamma_{\mu\nu}\tilde{g}^{\mu\nu}+\lambda_4\sqrt{-\gamma}. \tag{42}$$

We have introduced the last, constant term in (42) to make the Lagrangian density vanish in the absence of a gravitational field. The restriction of the class of gauge vectors by introduction of the term (40) automatically makes Eqs. (3) be consequences of the gravitational field equations. We shall verify this directly below.

According to the principle of least action, the equations for the intrinsic gravitational field have the form

$$\frac{\delta L_g}{\delta\tilde{g}^{\mu\nu}}=\lambda_1R_{\mu\nu}+\frac{1}{2}\lambda_2 g_{\mu\nu}+\lambda_3\gamma_{\mu\nu}=0, \tag{43}$$

where

$$\frac{\delta L_g}{\delta\tilde{g}^{\mu\nu}}=\frac{\partial L_g}{\partial\tilde{g}^{\mu\nu}}-\partial_\sigma\left(\frac{\partial L}{\partial(\partial_\sigma\tilde{g}^{\mu\nu})}\right),$$

with the Ricci tensor written as

$$R_{\mu\nu}=D_\lambda G^\lambda_{\mu\nu}-D_\mu G^\lambda_{\nu\lambda}+G^\sigma_{\mu\nu}G^\lambda_{\sigma\lambda}-G^\sigma_{\mu\lambda}G^\lambda_{\nu\sigma}. \tag{44}$$

Since in the absence of a gravitational field Eqs. (43) must be satisfied exactly, we find

$$\lambda_2=-2\lambda_3. \tag{45}$$

Now let us find the density of the energy–momentum tensor of the gravitational field in Minkowski space:

$$t_g^{\mu\nu}=-2\frac{\delta L_g}{\delta\gamma_{\mu\nu}}=2\sqrt{-\gamma}\left(\gamma^{\mu\alpha}\gamma^{\nu\beta}-\frac{1}{2}\gamma^{\mu\nu}\gamma^{\alpha\beta}\right)\frac{\delta L_g}{\delta\tilde{g}^{\alpha\beta}}+\lambda_1J^{\mu\nu}-2\lambda_3\tilde{g}^{\mu\nu}-\lambda_4\tilde{\gamma}^{\mu\nu}, \tag{46}$$

where

$$J^{\mu\nu}=D_\alpha D_\beta(\gamma^{\alpha\mu}\tilde{g}^{\beta\nu}+\gamma^{\alpha\nu}\tilde{g}^{\beta\mu}-\gamma^{\alpha\beta}\tilde{g}^{\mu\nu}-\gamma^{\mu\nu}\tilde{g}^{\alpha\beta}) \tag{47}$$

(see Appendix B). If in (46) we take into account the dynamical equations (43), we obtain the equations for the intrinsic gravitational field in the form

$$\lambda_1J^{\mu\nu}-2\lambda_3\tilde{g}^{\mu\nu}-\lambda_4\tilde{\gamma}^{\mu\nu}=t_g^{\mu\nu}. \tag{48}$$

In order that this equation be satisfied identically in the absence of a gravitational field, we must take

$$\lambda_4=-2\lambda_3. \tag{49}$$

Since for the intrinsic gravitational field we always have

$$D_\mu t_g^{\mu\nu}=0, \tag{50}$$

it follows from (48) that

$$D_\mu\tilde{g}^{\mu\nu}=0. \tag{51}$$

Thus, Eqs. (3) determining the polarization states of the field follow directly from (48). Taking into account (51), the field equations (48) can be written as

$$\gamma^{\alpha\beta}D_\alpha D_\beta\tilde{\phi}^{\mu\nu}-\frac{\lambda_4}{\lambda_1}\tilde{\phi}^{\mu\nu}=-\frac{1}{\lambda_1}t_g^{\mu\nu}. \tag{52}$$

In Galilean coordinates this equation has the simple form

$$\Box\tilde{\phi}^{\mu\nu}-\frac{\lambda_4}{\lambda_1}\tilde{\phi}^{\mu\nu}=-\frac{1}{\lambda_1}t_g^{\mu\nu}. \tag{53}$$

It is natural to interpret the numerical factor $-\lambda_4/\lambda_1=m^2$ as the squared graviton mass, and the value of $-1/\lambda_1$ must be equal to 16π according to the correspondence principle. Thus, all the unknown constants entering into the Lagrangian density are determined:

$$\lambda_1=-\frac{1}{16\pi},\quad \lambda_2=\lambda_4=-2\lambda_3=\frac{m^2}{16\pi}. \tag{54}$$

The constructed scalar Lagrangian density of the intrinsic gravitational field will have the form

$$L_g=\frac{1}{16\pi}\tilde{g}^{\mu\nu}(G^\lambda_{\mu\nu}G^\sigma_{\lambda\sigma}-G^\lambda_{\mu\sigma}G^\sigma_{\nu\lambda})-\frac{m^2}{16\pi}\left(\frac{1}{2}\gamma_{\mu\nu}\tilde{g}^{\mu\nu}-\sqrt{-g}-\sqrt{-\gamma}\right). \tag{55}$$

The corresponding dynamical equations for the intrinsic gravitational field can be written as

$$J^{\mu\nu}-m^2\tilde{\phi}^{\mu\nu}=-16\pi t_g^{\mu\nu}, \tag{56}$$

or

$$R^{\mu\nu}-\frac{m^2}{2}(g^{\mu\nu}-g^{\mu\alpha}g^{\nu\beta}\gamma_{\alpha\beta})=0. \tag{57}$$

These equations significantly restrict the class of gauge transformations, leaving only the trivial ones satisfying the Killing conditions in Minkowski space. Such transformations are a consequence of Lorentz invariance and occur in any theory.

The Lagrangian density constructed above leads to Eqs. (57), from which it follows that (51) are their consequences, and so outside the matter we will have ten equations for ten unknown field functions. The equations (51) can be used to easily express the unknown field functions $\phi^{o\alpha}$ in terms of the field functions ϕ^{ik}, where the superscripts i and k take the values 1, 2 and 3. Therefore, in the Lagrangian density of the intrinsic gravitational field the structure of the mass term violating the gauge group is uniquely determined by the polarization properties of the gravitational field.

5. THE EQUATIONS OF MOTION FOR THE GRAVITATIONAL FIELD AND MATTER

The total Lagrangian density of the matter and the gravitational field is

$$L=L_g+L_M(\tilde{g}^{\mu\nu},\phi_A), \tag{58}$$

where L_g is given by (55).

Using (58) with the principle of least action, we obtain the complete system of equations for the gravitational and matter fields:

$$\frac{\delta L}{\delta\tilde{g}^{\mu\nu}}=0, \tag{59}$$

$$\frac{\delta L_M}{\delta\phi_A}=0. \tag{60}$$

Since for an arbitrary infinitesimal change of the coordinates the variation of the action $\delta_c S_M$ is zero,

$$\delta_c S_M=\delta_c\int L_M(\tilde{g}^{\mu\nu},\phi_A)d^4x=0,$$

from this we find the identity [see Appendix C, Eq. (C16)]

$$g_{\mu\nu}\nabla_\lambda T^{\lambda\nu}=-D_\nu\left(\frac{\delta L_M}{\delta\phi_A}F^{B;\nu}_{A;\mu}\phi_B(x)\right)-\frac{\delta L_M}{\delta\phi_A}D_\mu\phi_A(x). \tag{61}$$

Here $T^{\lambda\nu}=-2\delta L_M/\delta g_{\lambda\nu}$ is the density of the matter tensor in Riemannian space, and ∇_λ is the covariant derivative in this space with metric $g_{\lambda\nu}$. It follows from (61) that if the matter equations of motion (60) are satisfied, we have

$$\nabla_\lambda T^{\lambda\nu}=0. \tag{62}$$

When there are four matter equations (60), instead of them we can use the equivalent equations (62). Since henceforth this will always be the case, we shall always use the matter equations in the form (62). The complete set of equations for the gravitational and matter fields will then have the form

$$\frac{\delta L}{\delta\tilde{g}^{\mu\nu}}=0, \tag{63}$$

$$\nabla_\lambda T^{\lambda\nu}=0. \tag{64}$$

The matter will be described by a velocity $\vec{v}$, matter density ρ, and pressure p. The gravitational field is determined by the ten components of the tensor $\phi^{\mu\nu}$.

We therefore have 15 unknowns. To determine them we must add the matter equation of state to the 14 equations (63)–(64). Taking into account the relations

$$\frac{\delta L_g}{\delta\tilde{g}^{\mu\nu}}=-\frac{1}{16\pi}R_{\mu\nu}+\frac{m^2}{32\pi}(g_{\mu\nu}-\gamma_{\mu\nu}), \tag{65}$$

$$\frac{\delta L_M}{\delta\tilde{g}^{\mu\nu}}=\frac{1}{2\sqrt{-g}}\left(T_{\mu\nu}-\frac{1}{2}g_{\mu\nu}T\right), \tag{66}$$

the system of equations (63) and (64) can be written as

$$\left(R^{\mu\nu}-\frac{1}{2}g^{\mu\nu}R\right)+\frac{m^2}{2}\left[g^{\mu\nu}+\left(g^{\mu\alpha}g^{\nu\beta}-\frac{1}{2}g^{\mu\nu}g^{\alpha\beta}\right)\gamma_{\alpha\beta}\right]=\frac{8\pi}{\sqrt{-g}}T^{\mu\nu}, \tag{67}$$

$$\nabla_\lambda T^{\lambda\nu}=0. \tag{68}$$

Owing to the Bianchi identity

$$\nabla_\mu\left(R^{\mu\nu}-\frac{1}{2}g^{\mu\nu}R\right)=0,$$

from (67) we have

$$m^2\sqrt{-g}\left(g^{\mu\alpha}g^{\nu\beta}-\frac{1}{2}g^{\mu\nu}g^{\alpha\beta}\right)\nabla_\mu\gamma_{\alpha\beta}=16\pi\nabla_\mu T^{\mu\nu}. \tag{69}$$

Using the expression

$$\nabla_\mu\gamma_{\alpha\beta}=-G^\sigma_{\mu\alpha}\gamma_{\sigma\beta}-G^\sigma_{\mu\beta}\gamma_{\sigma\alpha}, \tag{70}$$

where $G^\sigma_{\mu\alpha}$ is given by (36), we find

$$\left(g^{\mu\alpha}g^{\nu\beta}-\frac{1}{2}g^{\mu\nu}g^{\alpha\beta}\right)\nabla_\mu\gamma_{\alpha\beta}=\gamma_{\mu\lambda}g^{\mu\nu}(D_\sigma g^{\sigma\lambda}+G^\beta_{\alpha\beta}g^{\alpha\lambda}), \tag{71}$$

but since

$$\sqrt{-g}(D_\sigma g^{\sigma\lambda}+G^\beta_{\alpha\beta}g^{\alpha\lambda})=D_\sigma\tilde{g}^{\lambda\sigma}, \tag{72}$$

Eq. (71) takes the form

$$\sqrt{-g}\left(g^{\mu\alpha}g^{\nu\beta}-\frac{1}{2}g^{\mu\nu}g^{\alpha\beta}\right)\nabla_\mu\gamma_{\alpha\beta}=\gamma_{\mu\lambda}g^{\mu\nu}D_\sigma\tilde{g}^{\lambda\sigma}. \tag{73}$$

Using (73), Eq. (69) can be written as

$$m^2\gamma_{\mu\lambda}g^{\mu\nu}D_\sigma\tilde{g}^{\lambda\sigma}=16\pi\nabla_\mu T^{\mu\nu}.$$

This expression can be rewritten as

$$m^2D_\sigma\tilde{g}^{\lambda\sigma}=16\pi\gamma^{\lambda\nu}\nabla_\mu T^\mu_\nu. \tag{74}$$

Using this, Eq. (68) can be replaced by the equation

$$D_\sigma\tilde{g}^{\nu\sigma}=0. \tag{75}$$

Therefore, the system of equations (67) and (68) reduces to a system of gravitational equations of the form

$$\left(R^{\mu\nu}-\frac{1}{2}g^{\mu\nu}R\right)+\frac{m^2}{2}\left[g^{\mu\nu}+\left(g^{\mu\alpha}g^{\nu\beta}-\frac{1}{2}g^{\mu\nu}g^{\alpha\beta}\right)\gamma_{\alpha\beta}\right]=\frac{8\pi}{\sqrt{-g}}T^{\mu\nu}, \tag{76}$$

$$D_\mu\tilde{g}^{\mu\nu}=0. \tag{77}$$

These equations are form-invariant under Lorentz transformations, i.e., phenomena are described by identical equations in any inertial (Galilean) coordinate system.

A particular inertial Galilean coordinate system is distinguished by the actual statement of the physical problem (the initial and boundary conditions). The description of a given physical problem is, of course, different in different inertial (Galilean) coordinate systems, but this does not contradict the relativity principle. If we introduce the tensor

$$N^{\mu\nu}=R^{\mu\nu}-\frac{m^2}{2}[g^{\mu\nu}-g^{\mu\alpha}g^{\nu\beta}\gamma_{\alpha\beta}],\quad N=N^{\mu\nu}g_{\mu\nu},$$

the system of equations (76) and (77) can be written as

$$N^{\mu\nu}-\frac{1}{2}g^{\mu\nu}N=\frac{8\pi}{\sqrt{-g}}T^{\mu\nu}, \tag{76a}$$

$$D_\mu\tilde{g}^{\mu\nu}=0. \tag{77a}$$

It can also be written as

$$N^{\mu\nu}=\frac{8\pi}{\sqrt{-g}}\left(T^{\mu\nu}-\frac{1}{2}g^{\mu\nu}T\right), \tag{78}$$

$$D_\mu\tilde{g}^{\mu\nu}=0, \tag{79}$$

or

$$N_{\mu\nu}=\frac{8\pi}{\sqrt{-g}}\left(T_{\mu\nu}-\frac{1}{2}g_{\mu\nu}T\right), \tag{78a}$$

$$D_\mu\tilde{g}^{\mu\nu}=0. \tag{79a}$$

It should be stressed in particular that the metric tensor of Minkowski space enters into both the system (78) and the system (79).

Coordinate transformations under which the metric of Minkowski space is form-invariant relate physically equivalent reference frames. The simplest of these will be inertial frames. Therefore, allowed gauge transformations satisfying the Killing conditions

$$D_\mu\epsilon_\nu+D_\nu\epsilon_\mu=0$$

do not take us outside the class of physically equivalent reference frames.

If we imagine the possibility of experimentally measuring the characteristics of Riemannian space and the motion of matter with arbitrarily good accuracy, then we can use Eqs. (78a) and (79a) to determine the metric of Minkowski space and find the Galilean (inertial) coordinate systems. Therefore, Minkowski space is, in principle, observable.

The existence of Minkowski space is reflected in the conservation laws, and so the verification of these laws in physical phenomena is equivalent to verification of the structure of spacetime.

It should be noted in particular that the metric tensor of Minkowski space enters into both the system of equations (76) and the system (77). It is well known that in the GTR the presence of the cosmological term in the equations is not required, and this question is still under discussion. In the RTG the equations must contain the cosmological term. However, this term arises in (76) in combination with a term associated with the metric of Minkowski space $\gamma_{\mu\nu}$, with the same constant equal to half the squared graviton mass. The presence of the term in (76) associated with the metric $\gamma_{\mu\nu}$ significantly changes the nature of the collapse and evolution of the Universe. According to (76), in the absence of matter and a gravitational field the metric of space becomes the Minkowski metric, and it exactly coincides with the metric chosen earlier in formulating the physical problem. If the Minkowski metric were not present in the gravitational field equations, it would be completely unclear which coordinate system of Minkowski space we ended up in, the absence of matter and a gravitational field.

The introduction of the graviton mass is of fundamental importance for this theory, because only with it is it possible to construct a theory of gravity in Minkowski space. The graviton mass violates the gauge group or, in other words, it lifts the degeneracy. It is therefore impossible to exclude the possibility that the graviton mass tends to zero in the final results when gravitational effects are studied. However, the theory with a massive graviton and the theory with broken gauge group[7] (with the graviton mass later taken to zero) are in principle different theories. For example, whereas in the former the Universe is homogeneous and isotropic, in the latter it is impossible to have this type of Universe.

Let us now discuss the correspondence principle. Any physical theory must satisfy the correspondence principle. Gravitational interactions change the matter equations of motion. The correspondence principle reduces to the requirement that when the gravitational interaction is switched off, i.e., when the Riemann curvature tensor vanishes, these equations of motion become the usual equations of motion of the STR in the selected coordinate system.

In formulating the physical problem in the RTG we choose a coordinate system with the metric tensor of Minkowski space $\gamma_{\mu\nu}(x)$. In the RTG the equation of motion of the matter in the effective Riemannian space with metric tensor $g_{\mu\nu}(x)$ determined from the gravitational field equations (76) and (77) has the form

$$\nabla_\mu T^{\mu\nu}(x)=0. \qquad (\sigma)$$

As an example, let us take dust-like matter with energy–momentum tensor $T^{\mu\nu}$ equal to

$$T^{\mu\nu}(x)=\rho U^\mu U^\nu,\quad U^\nu=\frac{dx^\nu}{ds},$$

where ds is the interval in Riemannian space.

On the basis of (σ) and using the expression for $T^{\mu\nu}$, we find the equation for a geodesic in Riemannian space:

$$\frac{dU^\nu}{ds}+\Gamma^\nu_{\alpha\beta}(x)U^\alpha U^\beta=0.$$

When the gravitational interaction is switched off, i.e., when the Riemann curvature tensor vanishes, it follows from the gravitational field equations (76) and (77) that the Riemann metric $g_{\mu\nu}(x)$ becomes the Minkowski metric $\gamma_{\mu\nu}(x)$ chosen earlier. The matter equation of motion (σ) then takes the form

$$D_\mu t^{\mu\nu}(x)=0. \qquad (\lambda)$$

Here the energy–momentum tensor $t^{\mu\nu}(x)$ is

$$t^{\mu\nu}(x)=\rho u^\mu u^\nu,\quad u^\nu=\frac{dx^\nu}{d\sigma},$$

where $d\sigma$ is the interval in Minkowski space.

On the basis of (λ), using the expression for $t^{\mu\nu}$ we find the geodesic equations in Minkowski space:

$$\frac{du^\nu}{d\sigma}+\gamma^\nu_{\alpha\beta}u^\alpha u^\beta=0,$$

i.e., we obtain the usual equations for the free motion of particles in the STR with the coordinate system chosen earlier with metric tensor $\gamma_{\mu\nu}(x)$. Therefore, when the gravitational interaction is switched off, i.e., when the Riemann curvature tensor vanishes, the equation of motion of matter in a gravitational field in the selected coordinate system automatically becomes the equation of motion of matter in Minkowski space in the same coordinate system with metric tensor $\gamma_{\mu\nu}(x)$, i.e., the correspondence principle is satisfied. This is a general statement in the RTG, because when the Riemann tensor vanishes the Lagrangian density of matter in a gravitational field $L_M(\tilde{g}^{\mu\nu},\Phi_A)$ becomes the usual Lagrangian density $L_M(\tilde{\gamma}^{\mu\nu},\Phi_A)$ of the STR in the chosen coordinate system.

In the GTR the matter equation of motion also has the form (σ). However, since the Hilbert–Einstein equations do not involve the metric tensor of Minkowski space, when the gravitational interaction is switched off, i.e., when the Riemann curvature tensor vanishes, it is impossible to say which coordinate system (inertial or accelerated) of Minkowski space we will end up in, and so it is impossible to determine which matter equation of motion in Minkowski space we obtain when the gravitational interaction is switched off. This is why it is impossible to satisfy the correspondence principle in the GTR while remaining within this theory.

We conclude by noting that in the RTG all the concepts of classical Newtonian mechanics and the special theory of relativity (inertial reference frames, the law of inertia, acceleration relative to space, the energy–momentum and angular-momentum conservation laws) remain valid, whereas Einstein was forced to reject them in constructing the general theory of relativity. Inertial and gravitational forces cannot even be locally identical, because they have completely different origins. Whereas the former can be eliminated by a choice of coordinate system, the latter cannot be eliminated by any choice of coordinate system.

6. THE CAUSALITY PRINCIPLE IN THE RTG

The RTG was constructed within the framework of the STR just like the theories of other physical fields. According to the STR, any motion of any point test body always occurs inside the light cone of causality in Minkowski space. Therefore, noninertial reference frames realized by test bodies also must be located inside the causality cone of pseudo-Euclidean spacetime. The entire class of allowed noninertial reference frames is thereby determined. The three-dimensional inertial force and gravity acting on a matter point will be equal locally if the light cone of the effective Riemannian space does not leave the confines of the light cone of causality in Minkowski space. Only in this case can the three-dimensional force of the gravitational field acting on the test body be locally canceled, leading to an allowed noninertial reference frame associated with this body.

If the light cone of the effective Riemannian space left the confines of the causality light cone in Minkowski space, this would imply that for such a "gravitational field" there is no allowed noninertial reference frame in which this "force field" acting on a matter point could be canceled. In other words, local cancellation of the gravitational 3-force by an inertial force is possible only when the gravitational field as a physical field acting on particles does not take the particle world lines outside the confines of the causality cone of pseudo-Euclidean spacetime. This condition should be viewed as the causality principle allowing the selection of solutions of the system of equations (76) and (77) which have a physical meaning and correspond to gravitational fields.

The causality principle is not satisfied automatically. This is because the gravitational interaction enters into the coefficients of the second derivatives in the field equations, i.e., it changes the original geometry of spacetime. Only the gravitational field possesses this property. The interactions of all the other known physical fields usually do not affect the second derivatives of the field equations, and so do not change the original pseudo-Euclidean geometry of spacetime.

Let us now give an analytic formulation of the causality principle in the RTG. Since in the RTG the motion of matter under the influence of a gravitational field in pseudo-Euclidean spacetime is equivalent to the motion of matter in the corresponding effective Riemannian spacetime, for causally connected events (world lines of particles and light) we must have, on the one hand, the condition

$$ds^2=g_{\mu\nu}dx^\mu dx^\nu\geqslant 0, \tag{80}$$

and, on the other, satisfaction of the inequality

$$d\sigma^2=\gamma_{\mu\nu}dx^\mu dx^\nu\geqslant 0. \tag{81}$$

The selected reference frame realized by physical bodies satisfies the condition

$$\gamma_{oo}>0. \tag{82}$$

In Eq. (81) we separate the timelike and spacelike parts:

$$d\sigma^2=\left(\sqrt{\gamma_{oo}}dt+\frac{\gamma_{oi}dx^i}{\sqrt{\gamma_{oo}}}\right)^2-s_{ik}dx^i dx^k, \tag{83}$$

where the Latin indices i and k take the values 1, 2, 3, and

$$s_{ik}=-\gamma_{ik}+\frac{\gamma_{oi}\gamma_{ok}}{\gamma_{oo}} \tag{84}$$

is the metric tensor of three-dimensional space in four-dimensional pseudo-Euclidean spacetime. The squared spatial distance is given by

$$dl^2=s_{ik}dx^i dx^k. \tag{85}$$

We write the velocity $v^i=dx^i/dt$ as $v^i=ve^i$, where v is the magnitude of the velocity and e^i is an arbitrary unit vector in three-dimensional space:

$$s_{ik}e^i e^k=1. \tag{86}$$

In the absence of a gravitational field, the speed of light in the selected reference frame is easily found by setting (83) equal to zero:

$$\left(\sqrt{\gamma_{oo}}dt+\frac{\gamma_{oi}dx^i}{\sqrt{\gamma_{oo}}}\right)^2=s_{ik}dx^i dx^k.$$

From this we find

$$v=\sqrt{\gamma_{oo}}\bigg/\left(1-\frac{\gamma_{oi}e^i}{\sqrt{\gamma_{oo}}}\right). \tag{87}$$

Therefore, an arbitrary four-dimensional isotropic vector u^ν in Minkowski space is

$$u^\nu=(1,ve^i). \tag{88}$$

To satisfy simultaneously the conditions (80) and (81) it is necessary and sufficient that for any isotropic vector

$$\gamma_{\mu\nu}u^\mu u^\nu=0 \tag{89}$$

the causality condition is satisfied,

$$g_{\mu\nu}u^\mu u^\nu\leqslant 0, \tag{90}$$

which also implies that the light cone of effective Riemannian space does not go outside the causality light cone of pseudo-Euclidean spacetime. The causality conditions can be written as

$$g_{\mu\nu}v^\mu v^\nu=0, \tag{89a}$$

$$\gamma_{\mu\nu}v^\mu v^\nu\geqslant 0. \tag{90a}$$

In the GTR, the solutions of the Hilbert–Einstein equations which have physical meaning are those satisfying the inequality

$$g<0$$

at each point in spacetime, and also a requirement referred to as the energy-dominance condition and stated as follows. Any timelike vector K_ν must satisfy the inequality

$$T^{\mu\nu}K_\mu K_\nu\geqslant 0,$$

and the quantity $T^{\mu\nu}K_\nu$ for the vector K_ν must be a non-spacelike vector.

In our theory only those solutions of Eqs. (78a) and (79a) which, along with these requirements, also satisfy the causality condition (89a) and (90a) have a physical meaning. Using (78a), the latter condition can be written as

$$R_{\mu\nu}K^\mu K^\nu\leqslant\frac{8\pi}{\sqrt{-g}}\left(T_{\mu\nu}-\frac{1}{2}g_{\mu\nu}T\right)K^\mu K^\nu+\frac{m^2}{2}g_{\mu\nu}K^\mu K^\nu. \tag{91}$$

If the density of the matter energy–momentum tensor is written as

$$T_{\mu\nu}=\sqrt{-g}[(\rho+p)U_\mu U_\nu-pg_{\mu\nu}],$$

then on the basis of (78a) it is possible to establish the following relation between the interval of Minkowski space $d\sigma$ and the interval of effective Riemannian space ds:

$$\frac{m^2}{2}d\sigma^2=ds^2\left[4\pi(\rho+3p)+\frac{m^2}{2}-R_{\mu\nu}U^\mu U^\nu\right],$$

where $U^\mu=dx^\mu/ds$.

Owing to the causality principle, we have the inequality

$$R_{\mu\nu}U^\mu U^\nu<4\pi(\rho+3p)+\frac{m^2}{2},$$

which is a special case of the inequality (91) or

$$\sqrt{-g}R_{\mu\nu}v^\mu v^\nu\leqslant 8\pi T_{\mu\nu}v^\mu v^\nu. \tag{91a}$$

In 1918 Einstein stated the equivalence principle as follows: "Inertia and gravity are identical: from this statement and the results of the special theory of relativity it unavoidably follows that the symmetric ⟨⟨fundamental tensor⟩⟩ $g_{\mu\nu}$

determines the metrical properties of space, the motion of bodies owing to inertia in this space, and also the effect of gravity.'' The GTR identification of the gravitational field with the metric tensor $g_{\mu\nu}$ of Riemannian space allows the coordinate system to be chosen such that all the components of the Christoffel symbol are equal to zero at all the points of an arbitrary line. However, also in the GTR the gravitational field cannot be eliminated by a choice of coordinate system, since the motion of two closely spaced matter points will not be free, owing to the presence of the curvature tensor, which because of the tensor properties cannot be made to vanish by a choice of coordinate system.

In the RTG the gravitational field is a physical field in the Faraday–Maxwell sense, and so the gravitational force is described by a four-vector. Therefore, inertial forces can be made to balance the three-dimensional part of the gravitational force by a choice of coordinate system only when the conditions (89) and (90) are satisfied. The motion of a matter point in a gravitational field, independently of the coordinate system, can never be free. This is especially obvious when we write the geodesic equation in the form[8]

$$\frac{DU^\nu}{d\sigma} = -G^\rho_{\alpha\beta}U^\alpha U^\beta(\delta^\nu_\rho - U^\nu U_\rho).$$

Here

$$d\sigma^2 = \gamma_{\mu\nu}dx^\mu dx^\nu, \quad U^\nu = \frac{dx^\nu}{d\sigma}.$$

Free motion in Minkowski space is described by the equation

$$\frac{DU^\nu}{d\sigma} = \frac{dU^\nu}{d\sigma} + \gamma^\nu_{\mu\lambda}U^\mu U^\lambda = 0,$$

where $\gamma^\nu_{\mu\lambda}$ are the Christoffel symbols of Minkowski space.

We see that motion along a geodesic of Riemannian space is motion of a test body under the action of the force

$$F^\nu = -G^\rho_{\alpha\beta}U^\alpha U^\beta(\delta^\nu_\rho - U^\nu U_\rho),$$

which is a four-vector. If the test body were charged, it would emit electromagnetic waves because it is accelerated. This situation is exactly the same as that for the other known physical forces.

In the STR there is a fundamental difference between inertial forces and physical forces (electromagnetic, nuclear, and so on). Inertial forces can always be made to vanish by a simple choice of reference frame, whereas physical forces in principle cannot be made to vanish by any choice of reference frame, because they have a vector nature in Minkowski space. Since in the RTG all forces, including gravitational ones, have a vector nature, this implies that they cannot be made to vanish by a choice of coordinate system. It is only possible to cancel a three-dimensional force of any origin, including gravitational, by an inertial force by a choice of the coordinate system. In the GTR, as noted by Synge,[9] ''the concept of gravitational force is absent, because gravitational properties are inherent in the structure of spacetime and are manifested in the curvature of spacetime, i.e., in the fact that the Riemann tensor R_{ijkm} is nonzero.'' In the same book, Synge wrote, ''According to a well known legend, Newton was inspired to create his theory of gravity one day in observing the fall of an apple from the branch of a tree, and students of Newtonian physics even now claim that the acceleration (980 cm/sec^2) of a falling apple is due to the gravitational field. According to the theory of relativity [here he means the general theory of relativity—A.L.], this viewpoint is completely incorrect. We shall undertake a careful study of this problem and show that, in fact, the gravitational field (i.e., the Riemann tensor) plays an extremely small role in the phenomenon of free fall, and the acceleration of 980 cm/sec^2 is actually due to the curvature of the world line of the tree branch.''

According to the RTG, the gravitational field is a physical field, and so, in contrast to the GTR, it fully preserves the concept of gravitational force. Owing to the gravitational force, the free fall of a body occurs, and in fact everything occurs, just as in Newtonian physics. Moreover, all the gravitational effects in the solar system (the perihelion shift of Mercury, bending of light by the Sun, time delay of a radio signal, gyroscope precession) are just due to the action of the gravitational force, and not to the curvature tensor of spacetime, which is quite small in the solar system.

Einstein saw the local identification of inertia with gravity as the main reason for the equality of inertial and gravitational mass. However, in our opinion, the reason for this equality, as seen from Eq. (2), is that the source of the gravitational field is the conserved total density of the matter and gravitational field tensor. It is for this reason that the equality of inertial and gravitational mass does not require local identification of the gravitational force with inertial forces.

7. MACH'S PRINCIPLE

In formulating the laws of mechanics, Newton introduced the concept of absolute space, which always remains unchanged and stationary. It is with respect to this space that he defined the acceleration of a body. This acceleration was absolute in nature. The introduction of the abstraction of absolute space proved to be extremely fruitful. In particular, it gave rise to the concepts of inertial reference frames throughout space and the relativity principle for mechanical processes, and it led to the idea of physically distinguished states of motion. In 1923 Einstein wrote as follows on this topic: ''The coordinate systems found in such states of motion are distinguished by the fact that the laws of nature formulated in these coordinates take the simplest form.'' Further on, ''...according to classical mechanics, there exists a relativity of velocity, but not a relativity of acceleration.'' The theory therefore used a representation of inertial reference frames in which matter points not subjected to forces do not undergo acceleration, but remain in their state of rest or uniform rectilinear motion. However, the absolute space of Newton or inertial reference frames were actually introduced *a priori*, separately from the nature of the matter distribution in the Universe.

Mach was rather bold in seriously criticizing the main tenets of Newtonian mechanics. As he himself later wrote, he was able to publish his ideas only with great difficulty. Although Mach also did not construct a physical theory free from the defects he had pointed out, he had an enormous influence on the development of physical theory. He drew the attention of scientists to the analysis of fundamental physical concepts.

Let us quote some of Mach's statements,[10] which have become known as Mach's principle in the literature. "No one can say anything about absolute space and absolute motion; this is something only imaginary, and not experimentally observable." Further on: "Instead of referring a moving body to space (to some coordinate system), we shall directly study its relation to the bodies of the universe, in terms of which the only determination of reference frame is possible. ...Even in the simplest case, when we would study the interaction of only two masses, it is impossible to isolate our system from the rest of the world... . If a body is rotating relative to the fixed stars, there are centrifugal forces, and if it is rotating relative to another body and not relative to the fixed stars, there are no centrifugal forces. I have nothing against calling the first type of rotation absolute, as long as it is remembered that this means nothing but rotation relative to the fixed stars."

Then Mach wrote: "...there is no need to relate the law of inertia to any special absolute space. The most natural approach of a modern natural scientist is the following: first to study the law of inertia as an approximation, to relate it spatially to the fixed stars, ... and then to expect corrections to or evolution of our knowledge on the basis of a future experiment. Lange has recently published a critical article in which he explains how it would be possible, following his principles, to introduce a new system of coordinates if the usual crude association with the fixed stars turned out to be no longer applicable, owing to more accurate astronomical observations. Lange and I are of the same opinion regarding the theoretical, formal value of Lange's conclusions, namely, that at present the fixed stars are the only reference frame applicable in practice, and also regarding the method of defining a new reference frame by making successive corrections." Later Mach quotes S. Neumann: "Since all motions must be referred to an alpha frame (an inertial frame), this frame obviously represents an indirect link between all the processes occurring in the Universe and, therefore, it can be said to contain as many mysteries as a complex universal law." Regarding this, Mach notes: "I think that everyone agrees with this."

It is obvious from these statements of Mach that, since he is speaking of the law of inertia, according to which, as Newton says, "Every body will continue in its state of rest or of uniform motion in a straight line except insofar as it is compelled to change that state by impressed force," the question naturally arises of inertial reference frames and their relation to the distribution of matter. Mach and his contemporaries clearly understood that such a relation must exist in nature. In what follows we shall include this idea in the concept of Mach's principle.

Mach wrote: "Although I also think that astronomical observations will at first make only very insignificant corrections necessary, I still suppose that the law of inertia in the simple form stated by Newton has for us humans only a limited and transient value." As we shall see below, here Mach was wrong. Mach did not give his ideas a mathematical form, and so very often different authors have superimposed their own ideas on Mach's principle. Here we shall try to preserve the meaning that Mach himself had in mind.

Poincaré and later Einstein generalized the principle of relativity to all physical phenomena. In his formulation,[11] Poincaré refers to "...the principle of relativity, according to which the laws of physical phenomena must be identical for a stationary observer and for an observer in uniform translational motion, so that we do not have and cannot have any means of determining whether we are in such a state of motion or not." The application of this principle to electromagnetic phenomena led Poincaré, and later Minkowski, to the discovery of the pseudo-Euclidean geometry of spacetime and thereby to an even greater degree strengthened the hypothesis of the existence of inertial reference frames in all space. Such reference frames are physically distinguishable, and so acceleration relative to them has an absolute meaning.

In the general theory of relativity there are no inertial reference frames in all space. Einstein wrote the following about this in 1929: "The starting point of the theory is the statement that a physically distinguished state of motion does not exist, i.e., not only velocity but also acceleration has no absolute meaning."

The Mach principle as stated in the GTR turned out to be uncalled for. However, it should be noted that ideas about inertial reference frames in all space have a solid experimental foundation, since, for example, in going from a reference frame attached to the Earth to one attached to the Sun, and then to one attached to the metagalaxy we more and more accurately approach an inertial reference frame. There is therefore no serious justification for rejecting such an important concept as that of an inertial reference frame. On the other hand, the presence of the fundamental energy–momentum and angular-momentum conservation laws also necessarily leads to the existence of inertial reference frames in all space. The pseudo-Euclidean geometry of spacetime reflects the general dynamical properties of matter and at the same time introduces inertial reference frames. Although the pseudo-Euclidean geometry of spacetime arose in studying matter and therefore is inseparable from it, it is still possible to speak formally of Minkowski space in the absence of matter. However, just as earlier in Newtonian mechanics, in the special theory of relativity there is no answer to the question of how inertial reference frames are related to the distribution of matter in the Universe.

The discovery of the pseudo-Euclidean geometry of space and time made it possible to study not only inertial but also accelerated reference frames from a unified viewpoint. A large difference appeared between inertial forces and forces generated by physical fields. This was that inertial forces can always be made to vanish by choice of an appropriate reference frame, whereas forces generated by physical fields in principle cannot be made to vanish by choice of reference frame, because they have a vector nature in four-dimensional spacetime. Since in the RTG the gravitational field is a physical field in the spirit of the Faraday–Maxwell field, the forces generated by this field cannot be made to vanish by choice of reference frame.

A different situation occurs in the general theory of relativity. There, gravitational forces are not vectors in four-dimensional spacetime, and so they can be made to vanish locally by choice of reference frame. Owing to the fact that the gravitational field has a rest mass, the main equations of the RTG, (76) and (77), contain, along with the Riemann metric, the metric tensor of Minkowski space. However, this implies that in principle the metric of this space can be expressed in terms of the geometrical characteristics of the effective Riemannian space, and also in terms of quantities characterizing the matter distribution in the Universe. This is easily accomplished by changing from contravariant to covariant quantities in (76). In this manner we obtain

$$\frac{m^2}{2}\gamma_{\mu\nu}(x)=\frac{8\pi}{\sqrt{-g}}\left(T_{\mu\nu}-\frac{1}{2}g_{\mu\nu}T\right)-R_{\mu\nu}+\frac{m^2}{2}g_{\mu\nu}. \tag{92}$$

From this we see that the right-hand sides of the equations contain only the geometrical characteristics of the effective Riemannian space and quantities determining the matter distribution in this space.

By experimentally studying the motion of particles and light in Riemannian space, it is possible in principle to find the metric tensor of Minkowski space and, therefore, to construct an inertial reference frame. The RTG constructed on the basis of the special theory of relativity thus permits a mathematical statement of Mach's principle. We shall see that a special principle of relativity has a universal value independently of the form of matter.

The requirements on the gravitational field are expressed as the condition that Eqs. (76) and (77) be form-invariant under the Lorentz group. The Lorentz-invariance of physical equations is the most important physical principle in the construction of the theory, because it is this principle which makes it possible to introduce universal characteristics for all forms of matter.

In 1950 Einstein wrote: "...is it not necessary in the end to test whether the concept of inertial frame is preserved, after abandoning all attempts to explain the fundamental feature of gravitational phenomena which is manifested in the Newtonian system as the equivalence of inertial and gravitational mass?" In the RTG the concept of inertial frame is preserved, and at the same time it is shown that the equivalence of inertial and gravitational mass is a direct consequence of the hypothesis that the conserved density of the matter energy–momentum tensor is the source of the gravitational field. Therefore, the equality of inertial and gravitational mass does not at all contradict the existence of an inertial reference frame. Moreover, these ideas intrinsically supplement each other and form the basis of the RTG.

In contrast to our view, Einstein answered the question he posed as follows. "One who believes in the comprehensibility of nature must answer no." The existence of inertial reference frames eliminates the Mach paradox, because in this case is it possible to speak of acceleration relative to space. Fock wrote: "As far as the Mach paradox is concerned, it is based, as is well known, on the study of a rotating fluid body having the shape of an ellipsoid and a nonrotating body having the shape of a sphere. A paradox arises here only if the concept of 'rotation relative to space' is considered devoid of meaning; then both bodies (rotating and nonrotating) actually are equally justified, and it becomes unclear why one is like a sphere while the other is not. However, the paradox disappears as soon as we adopt the idea of 'acceleration relative to space.' "

Mach's ideas deeply influenced Einstein's views on gravity in the construction of the general theory of relativity. In one of his papers, Einstein writes, "Mach's principle: the G field is completely determined by the masses of bodies." However, it turns out that even this is not true in the GTR, because there are solutions even when matter is absent. An attempt to eliminate this problem by the introduction of a λ term did not give the desired result. It turned out that the equations with the λ term also have nonzero solutions in the absence of matter. We shall see that Einstein understood something completely different by "Mach's principle." However, even his understanding of Mach's principle did not fit into the GTR.

Does the Einstein formulation of Mach's principle have any place in the RTG? In contrast to the GTR, in the RTG there are spacelike surfaces in all space (global Cauchy surfaces), owing to the principle of causality. And if matter is absent on one of these surfaces, it will be absent everywhere, owing to the requirement of energy dominance imposed on the matter tensor.[12] Since matter exists in nature, it follows that a system of gravitational equations homogeneous throughout space has no solutions realized in nature. In other words, all the solutions of this system are devoid of physical meaning in the present stage of development of the Universe. This discarding of solutions of the system of homogeneous gravitational equations became possible not only because of the equations themselves, but primarily because of the nature of the real Universe.

In principle, the equations of the theory do not reject universes constructed from the gravitational field without matter. They are rejected by the actual evolution of the matter. The fact that our universe turns out to contain matter does not yet have any theoretical explanation. Only solutions of the system of inhomogeneous gravitational equations for which there is matter in a part or all of space have a physical meaning. This implies that the gravitational field and the effective Riemannian space in the real Universe could not have arisen without the matter generating them. We shall see that the Einstein formulation of Mach's principle is realized in the relativistic theory of gravity.

However, there is a significant difference in the concept of the G field in our theory and in the GTR. Einstein understood the G field to be the Riemannian metric, while in our approach the gravitational field is a physical field. This field enters into the Riemannian metric along with the flat-space metric, and so the metric does not vanish in the absence of matter and the gravitational field, but remains the metric of Minkowski space.

In the literature there are also other statements of Mach's principle which differ in meaning from the ideas of both Mach and Einstein. However, since in our opinion they have not been stated very precisely, we have not considered them. Because the gravitational forces in the RTG arise from a physical field of the Faraday–Maxwell type, there cannot in principle be anything in common between inertial and gravitational forces.

Sometimes Mach's principle is viewed as if it says that inertial forces are determined by interaction with the matter of the Universe. From the field point of view this cannot occur in nature. The point is that although inertial reference frames, as we saw above, are associated with the distribution of matter in the Universe, inertial forces are not a result of interaction with the matter of the Universe, because any effect of the matter is possible only via physical fields. However, this implies that the forces due to these fields cannot be made to vanish by choice of reference frame, owing to their vectorial nature. Therefore, inertial forces are directly determined not by physical fields, but by the strictly defined structure of the geometry and the choice of reference frame.

The pseudo-Euclidean geometry of spacetime reflecting the dynamical properties common to all forms of matter on the one hand confirmed the hypothesis of the existence of inertial reference frames and, on the other, showed that the inertial forces arising for suitable choice of reference frame are expressed in terms of the Christoffel symbols of Minkowski space. They are therefore independent of the nature of the body. This all became clear when it was shown that the special theory of relativity is applicable not only in inertial reference frames, but also in noninertial (accelerated) ones.

This allowed us to give a more general statement of the relativity principle in Ref. 5: "Whatever physical reference frame (inertial or noninertial) we choose, it can always be shown that there is an infinite set of other reference frames in which all physical phenomena occur in exactly the same way as in the original reference frame, and so we do not have and cannot have any experimental way of distinguishing in precisely which reference frame of this infinite set we find ourselves." This is expressed mathematically as follows. Let the interval in some coordinate system of Minkowski space be

$$d\sigma^2=\gamma_{\mu\nu}(x)dx^\mu dx^\nu.$$

Then there exists another coordinate system x',

$$x'^\nu=f^\nu(x),$$

in which the interval takes the form

$$d\sigma^2=\gamma_{\mu\nu}(x')dx'^\mu dx'^\nu,$$

where the metric components $\gamma_{\mu\nu}$ have the same functional form as in the original coordinate system.

In this case it is said that *the metric is form-invariant* under such transformations, and that all *the physical equations are also form-invariant*, i.e., they have the same form in both the primed and unprimed coordinate systems. The coordinate transformations leaving the metric form-invariant form a group. In the case of Galilean coordinates these are the usual Lorentz transformations in an inertial frame.

In the RTG, inertial and gravitational forces differ significantly in that the gravitational field becomes weaker with increasing distance from a body, while inertial forces can be arbitrarily large, depending on the choice of reference frame. They are equal to zero only in an inertial frame. It is therefore incorrect to assume that inertial forces cannot be distinguished from gravitational forces. In everyday life the difference between them is almost obvious.

The construction of the RTG allowed a connection to be established between an inertial reference frame and the distribution of matter in the Universe, and thereby led to a deeper understanding of the nature of inertial forces and how they differ from matter forces. In our theory, inertial forces are assigned the same role as in any other field theory.

8. THE POST-NEWTONIAN APPROXIMATION

The post-Newtonian approximation is completely adequate for studying gravitational effects in the solar system. In this section we shall construct this approximation. Many of the technical details of this construction are borrowed from Fock.[13]

We write the basic equations of the theory as (see Appendix D)

$$\tilde{\gamma}^{\alpha\beta}D_\alpha D_\beta\tilde{\Phi}^{\epsilon\lambda}+m^2\sqrt{-\gamma}\tilde{\Phi}^{\epsilon\lambda}=-16\pi g(T_M^{\epsilon\lambda}+\tau_g^{\epsilon\lambda}), \tag{93}$$

$$D_\lambda\tilde{\Phi}^{\epsilon\lambda}=0, \tag{94}$$

where $T_M^{\epsilon\lambda}$ is the matter energy–momentum tensor and $\tau_g^{\epsilon\lambda}$ is the energy–momentum tensor of the gravitational field.

The expression for the energy–momentum tensor of the gravitational field can be written as

$$\begin{aligned}-16\pi g\tau_g^{\epsilon\lambda}&=\frac{1}{2}\left(\tilde{g}^{\epsilon\alpha}\tilde{g}^{\lambda\beta}-\frac{1}{2}\tilde{g}^{\epsilon\lambda}\tilde{g}^{\alpha\beta}\right)\left(\tilde{g}_{\nu\sigma}\tilde{g}_{\tau\mu}-\frac{1}{2}\tilde{g}_{\tau\sigma}\tilde{g}_{\nu\mu}\right)\\&\times D_\alpha\tilde{\Phi}^{\tau\sigma}D_\beta\tilde{\Phi}^{\mu\theta}+\tilde{g}^{\alpha\beta}\tilde{g}_{\tau\sigma}D_\alpha\tilde{\Phi}^{\epsilon\tau}D_\beta\tilde{\Phi}^{\lambda\sigma}\\&-\tilde{g}^{\epsilon\beta}\tilde{g}_{\tau\sigma}D_\alpha\tilde{\Phi}^{\lambda\sigma}D_\beta\tilde{\Phi}^{\alpha\tau}-\tilde{g}^{\lambda\alpha}\tilde{g}_{\tau\sigma}D_\alpha\tilde{\phi}^{\beta\sigma}\\&\times D_\beta\tilde{\Phi}^{\epsilon\tau}+\frac{1}{2}\tilde{g}^{\epsilon\lambda}\tilde{g}_{\tau\sigma}D_\alpha\tilde{\Phi}^{\sigma\beta}D_\beta\tilde{\Phi}^{\alpha\tau}+D_\alpha\tilde{\Phi}^{\epsilon\beta}\\&\times D_\beta\tilde{\Phi}^{\lambda\alpha}-\tilde{\Phi}^{\alpha\beta}D_\alpha D_\beta\tilde{\Phi}^{\epsilon\lambda}-m^2\Big(\sqrt{-g}\tilde{g}^{\epsilon\lambda}\\&-\sqrt{-\gamma}\tilde{\Phi}^{\epsilon\lambda}+\tilde{g}^{\epsilon\tau}\tilde{g}^{\lambda\beta}\gamma_{\alpha\beta}-\frac{1}{2}\tilde{g}^{\epsilon\lambda}\tilde{g}^{\alpha\beta}\gamma_{\alpha\beta}\Big).\end{aligned} \tag{95}$$

This is written in an arbitrary coordinate system in Minkowski space. All the calculations that follow will be performed in the Galilean coordinates of the inertial frame

$$\gamma_{\mu\nu}=(1,-1,-1,-1). \tag{96}$$

In constructing the perturbation series, it is natural to use a small parameter ϵ such that

$$v\sim\epsilon,\ U\sim\epsilon^2,\ \Pi\sim\epsilon^2,\ p\sim\epsilon^2. \tag{97}$$

Here U is the Newtonian gravitational-field potential, Π is the specific internal energy of the body, and p is the specific pressure.

For the solar system the parameter ϵ^2 is of order

$$\epsilon^2 \sim 10^{-6}. \tag{98}$$

We shall start from the expansions of the components of the tensor density:

$$\tilde{g}^{00} = 1 + \overset{(2)}{\tilde{\Phi}}{}^{00} + \overset{(4)}{\tilde{\Phi}}{}^{00} + \ldots, \tag{99}$$

$$\tilde{g}^{0i} = \overset{(3)}{\tilde{\Phi}}{}^{0i} + \overset{(5)}{\tilde{\Phi}}{}^{0i} + \ldots, \tag{100}$$

$$\tilde{g}^{ik} = \tilde{\gamma}^{ik} + \overset{(2)}{\tilde{\Phi}}{}^{ik} + \overset{(4)}{\tilde{\Phi}}{}^{ik} + \ldots. \tag{101}$$

As the matter model we take an ideal fluid with energy–momentum tensor

$$T^{\epsilon\lambda} = [p + \rho(1+\Pi)]u^\epsilon u^\lambda - p g^{\epsilon\lambda}, \tag{102}$$

where ρ is the invariant density of the ideal fluid, p is the specific isotropic pressure, and u^λ is the velocity four-vector.

Now we write down the expansion in the small parameter ϵ for the matter energy–momentum tensor:

$$T_M^{00} = \overset{(0)}{T}{}^{00} + \overset{(2)}{T}{}^{00} + \ldots, \tag{103}$$

$$T_M^{0i} = \overset{(1)}{T}{}^{0i} + \overset{(3)}{T}{}^{0i} + \ldots, \tag{104}$$

$$T_M^{ik} = \overset{(2)}{T}{}^{ik} + \overset{(4)}{T}{}^{ik} + \ldots. \tag{105}$$

In the Newtonian approximation, i.e., when we neglect gravitational forces, for the velocity four-vector we have

$$u^0 = 1 + 0(\epsilon^2), \quad u^i = v^i(1 + 0(\epsilon^2)). \tag{106}$$

Using these expressions in (102), we find

$$\overset{(0)}{T}{}^{00} = \rho, \quad \overset{(1)}{T}{}^{0i} = \rho v^i, \quad \overset{(0)}{T}{}^{ik} = 0. \tag{107}$$

In this approximation we have

$$\partial_0 \rho + \partial_i(\rho v^i) = 0. \tag{108}$$

From this we see that in the Newtonian approximation the total inertial mass of a body is conserved:

$$M = \int \rho d^3x. \tag{109}$$

In the Newtonian approximation, from Eqs. (93) we have

$$\Delta \overset{(2)}{\tilde{\Phi}}{}^{00} = -16\pi\rho, \tag{110}$$

$$\Delta \overset{(3)}{\tilde{\Phi}}{}^{0i} = -16\pi\rho v^i, \tag{111}$$

$$\Delta \overset{(2)}{\tilde{\Phi}}{}^{ik} = 0. \tag{112}$$

Since it is small, the graviton mass does not play any role in the inertial coordinate system for effects in the solar system, and so we have neglected it in deriving Eqs. (110)–(112). In the general case of a noninertial reference frame or for strong gravitational fields, the term involving the graviton mass m cannot be neglected. For example, even for a static body in the region close to the Schwarzschild sphere, the effect of the graviton mass is very large, and so it cannot be ignored.

The solution of Eqs. (110)–(112) has the form

$$\overset{(2)}{\tilde{\Phi}}{}^{00} = 4U, \quad U = \int \frac{\rho}{|x - x'|} d^3x', \tag{113}$$

$$\overset{(3)}{\tilde{\Phi}}{}^{0i} = -4V^i, \quad V^i = -\int \frac{\rho v^i}{|x - x'|} d^3x', \tag{114}$$

$$\overset{(2)}{\tilde{\Phi}}{}^{ik} = 0. \tag{115}$$

On the basis of (94) we have

$$\partial_0 \overset{(2)}{\tilde{\Phi}}{}^{00} + \partial_i \overset{(3)}{\tilde{\Phi}}{}^{0i} = 0. \tag{116}$$

Substituting (113) and (114) into this equation, we find

$$\partial_0 U - \partial_i V^i = 0. \tag{117}$$

From this it is obvious that the order of smallness in ϵ is increased when the potential U is differentiated with respect to time. We shall use this fact in what follows for calculating the energy–momentum tensor of the gravitational field $\tau_g^{\epsilon\lambda}$. We note that Eq. (117) is satisfied identically, owing to (108).

It follows from (114) and (115) that of all the components of the tensor density $\tilde{\Phi}^{\epsilon\lambda}$, only the one component $\overset{(2)}{\tilde{\Phi}}{}^{00}$ given by (113) remains in the second approximation. It is this fact which significantly simplifies the method of finding the post-Newtonian approximation, where at each stage of the construction we use the densities of tensor quantities.

From (113)–(115) through second order we obtain

$$\sqrt{-g}g^{00} = 1 + 4U, \quad \sqrt{-g}g^{11} = \sqrt{-g}g^{22} = \sqrt{-g}g^{33} = -1. \tag{118}$$

From this we find

$$-g = 1 + 4U. \tag{118a}$$

and therefore

$$g_{00} = 1 - 2U, \quad g_{11} = g_{22} = g_{33} = -(1 + 2U). \tag{119}$$

We see from (118) that in the Newtonian approximation, where we restrict ourselves to only one component of the matter tensor density $\tilde{T}^{00}$, the gravitational field is described by only the component $\tilde{\Phi}^{00}$, while, according to (119), the metric tensor $g_{\mu\nu}$ has several components, as expected. The use of the field components $\tilde{\Phi}^{\mu\nu}$ rather than the metric tensor $g_{\mu\nu}$ significantly simplifies the entire process of constructing the post-Newtonian approximation. This is why the introduction of the tensor density of the gravitational field $\tilde{\Phi}^{\mu\nu}$ has not only a general theoretical but also a practical value. The metric tensor of the effective Riemannian space is then

$$g_{00} = 1 - 2U, \quad g_{0i} = 4\gamma_{ik}V^k, \quad g_{ik} = \gamma_{ik}(1 + 2U). \tag{120}$$

It follows from Eq. (113) for U that the inertial mass (109) is equal to the active gravitational mass. As we have seen, in the RTG this equation arose because the source of the gravitational field is the energy–momentum tensor.

Let us now turn to the construction of the next approximation for the component of the metric tensor g_{00}. For this we shall find the contribution from the energy–momentum tensor of the gravitational field. Since in (95) only $\overset{(2)}{\tilde{\Phi}}{}^{00}$ is involved in the derivative, the first term of (95) gives a contribution

$$2(\text{grad } U)^2, \tag{121}$$

and the second gives

$$-16(\text{grad } U)^2. \tag{122}$$

The contribution from all the other terms in this approximation will be zero. We have also discarded terms involving time derivatives of the potential U, since, owing to (117), they are also of higher order in the small parameter ϵ. From (121) and (122) we find

$$-16\pi g \tau_g^{00} = -14(\text{grad } U)^2. \tag{123}$$

Taking into account (123), Eq. (93) in this approximation for the component $\tilde{\Phi}^{00}$ takes the form

$$\Delta\tilde{\Phi}^{00} = 16\pi g T^{00} + 14(\text{grad } U)^2 + 4\partial_0^2 U. \tag{124}$$

Since from (120) in second order in ϵ the interval is

$$ds = dt\left(1 - U + \frac{1}{2} v_i v^i\right), \tag{125}$$

we find

$$u^0 = \frac{dt}{ds} = 1 + U - \frac{1}{2} v_i v^i. \tag{126}$$

Substituting this expression into (102), we find

$$\overset{(2)}{T}{}^{00} = \rho[2U + \Pi - v_i v^i]. \tag{127}$$

On the basis of (118a) and (127), from Eq. (124) we obtain

$$\Delta \overset{(4)}{\tilde{\Phi}}{}^{00} = -96\pi\rho U + 16\pi\rho v_i v^i + 14(\text{grad } U)^2 - 16\pi\rho\Pi + 4\partial_0^2 U. \tag{128}$$

We use the obvious identity

$$(\text{grad } U)^2 = \frac{1}{2}\Delta U^2 - U\Delta U. \tag{129}$$

However, since

$$\Delta U = -4\pi\rho, \tag{130}$$

Eq. (128), after using (129) and (130), takes the form

$$\Delta(\overset{(4)}{\tilde{\Phi}}{}^{00} - 7U^2) = 16\pi\rho v_i v^i - 40\pi\rho U - 16\pi\rho\Pi + 4\partial_0^2 U. \tag{131}$$

From this we obtain

$$\overset{(4)}{\tilde{\Phi}}{}^{00} = 7U^2 + 4\Phi_1 + 10\Phi_2 + 4\Phi_3 - \frac{1}{\pi}\partial_0^2 \int \frac{U}{|x-x'|} d^3x'. \tag{132}$$

where

$$\Phi_1 = -\int \frac{\rho v_i v^i}{|x-x'|} d^3x',$$

$$\Phi_2 = \int \frac{\rho U}{|x-x'|} d^3x', \quad \Phi_3 = \int \frac{\rho\Pi}{|x-x'|} d^3x'. \tag{133}$$

Therefore, in the post-Newtonian approximation we have

$$\tilde{g}^{00} = 1 + 4U + 7U^2 + 4\Phi_1 + 10\Phi_2 + 4\Phi_3 - \frac{1}{\pi}\partial_0^2 \int \frac{U}{|x-x'|} d^3x'. \tag{134}$$

Now we need to find the value of the determinant g in the post-Newtonian approximation. For this we write $\tilde{g}^{ik}$ as

$$\tilde{g}^{ik} = \tilde{\gamma}^{ik} + \overset{(4)}{\tilde{\Phi}}{}^{ik}. \tag{135}$$

It should be especially stressed that the calculation of the determinant g is simplest if we use the tensor density $\tilde{g}^{\mu\nu}$ and take into account the fact that

$$g = \det(\tilde{g}^{\mu\nu}) = \det(g_{\mu\nu}). \tag{136}$$

From (134) and (135) we find

$$\sqrt{-g} = 1 + 2U + \frac{3}{2}U^2 + 2\Phi_1 + 5\Phi_2 + 2\Phi_3 - \frac{1}{2}\Phi - \frac{1}{2\pi}\partial_0^2 \int \frac{U}{|x-x'|} d^3x'. \tag{137}$$

Here

$$\Phi = \overset{(4)}{\tilde{\Phi}}{}^{11} + \overset{(4)}{\tilde{\Phi}}{}^{22} + \overset{(4)}{\tilde{\Phi}}{}^{33}. \tag{138}$$

Since in this approximation $g_{00}g^{00} = 1$, from Eqs. (134) and (137) we find

$$g_{00} = 1 - 2U + \frac{5}{2}U^2 - 2\Phi_1 - 5\Phi_2 - 2\Phi_3 - \frac{1}{2}\Phi + \frac{1}{2\pi}\partial_0^2 \int \frac{U}{|x-x'|} d^3x'. \tag{139}$$

To determine g_{00} we need to calculate Φ. Since Φ is obtained by summation, we can use Eq. (93) and then obtain the equations for Φ directly by summation.

From (95) by summation we obtain the following expression from the first term:

$$-16\pi g \tau_g^{ii} = -2(\text{grad } U)^2. \tag{140}$$

All the other terms entering into (95) do not contribute in this approximation. Using Eq. (102) for the matter energy–momentum tensor, we find

$$-16\pi g \overset{(2)}{\tilde{T}}{}^{ii} = -16\pi\rho v_i v^i + 48\pi p. \tag{141}$$

Taking into account (140) and (141), we write the equation for Φ as

$$\Delta\Phi = 16\pi\rho v_i v^i - 48\pi p + 2(\text{grad } U)^2. \tag{142}$$

Using the identity (129) and Eq. (130), we find

$$\Delta(\Phi - U^2) = 16\pi\rho v_i v^i + 8\pi\rho U - 48\pi p. \tag{143}$$

From this we obtain

$$\Phi=U^2+4\Phi_1-2\Phi_2+12\Phi_4, \quad \Phi_4=\int \frac{p}{|x-x'|}\,d^3x'. \tag{144}$$

Substituting (144) into (139), we have

$$g_{00}=1-2U+2U^2-4\Phi_1-4\Phi_2-2\Phi_3-6\Phi_4 + \frac{1}{2\pi}\partial_0^2\int \frac{U}{|x-x'|}\,d^3x'. \tag{145}$$

Using the identity

$$\frac{1}{2\pi}\int \frac{U}{|x-x'|}\,d^3x'=-\int \rho|x-x'|d^3x',$$

we write (145) as

$$g_{00}=1-2U+2U^2-4\Phi_1-4\Phi_2-2\Phi_3-6\Phi_4 - \partial_0^2\int \rho|x-x'|d^3x'. \tag{146}$$

If we make the transformation

$$x'^0=x^0+\eta^0(x), \quad x'^i=x^i, \tag{147}$$

the metric coefficients change as

$$g'_{00}=g_{00}-2\partial_0\eta^0, \quad g'_{0i}=g_{0i}-\partial_i\eta^0, \quad g'_{ik}=g_{ik}. \tag{148}$$

The transformation (147) does not take us outside the inertial reference frame, because this transformation is nothing but a different choice of clocks. All physically measurable quantities are independent of this choice.

Taking the function η^0 to be

$$\eta^0=-\frac{1}{2}\partial_0\int \rho|x-x'|d^3x' \tag{149}$$

and taking into account the identity

$$\partial_i\eta^0=\frac{1}{2}(\gamma_{ik}V^k-N_i),$$

$$N_i=\int \frac{\rho v^k(x_k-x'_k)(x_i-x'_i)}{|x-x'|^3}\,d^3x', \tag{150}$$

after substitution of (120) for g_{0i} and g_{ik} and also (146) for g_{00} into (148), using (149) and (150) we find the metric components of the effective Riemannian space in the so-called canonical form:

$$g_{00}=1-2U+2U^2-4\Phi_1-4\Phi_2-2\Phi_3-6\Phi_4,$$

$$g_{0i}=\frac{7}{2}\gamma_{ik}V^k+\frac{1}{2}N_i,$$

$$g_{ik}=\gamma_{ik}(1+2U). \tag{151}$$

On the basis of (151), the post-Newtonian Nordtvedt–Will parameters in the RTG are

$$\gamma=1, \quad \beta=1, \quad \alpha_1=\alpha_2=\alpha_3=\xi_1=\xi_2=\xi_3=\xi_4=\xi_W=0.$$

We have calculated the metric components (151) in an inertial reference frame in the RTG. Let us now give the expressions for the components of the matter energy–momentum tensor, in comparison with (107), in the next approximation. Using Eq. (126) for u^0, and also

$$u^i=\frac{dx^i}{ds}=v^i\left(1+U-\frac{1}{2}v_kv^k\right), \tag{152}$$

from Eq. (102) we find

$$\overset{(3)}{T}{}^{0i}=\rho v^i(2U+\Pi-v_kv^k)+pv^i, \tag{153}$$

$$\overset{(2)}{T}{}^{ik}=\rho v^iv^k-p\gamma^{ik}. \tag{154}$$

The component $\overset{(2)}{T}{}^{00}$ is given by Eq. (127). From Eq. (151), using the geodesic equations we can calculate all the effects in the solar system.

Let us conclude by giving a somewhat more detailed comparison of the RTG and the GTR in analyzing effects in a weak gravitational field. The system of equations (93) and (94) together with the equation of state determines all the physical quantities of a particular gravitational problem. All the above calculations of the post-Newtonian approximation were carried out in an inertial reference frame. In the GTR there is no inertial frame in principle. Einstein wrote: "The starting point of the theory is the statement that a physically distinguished state of motion does not exist, i.e., not only velocity but also acceleration has no absolute meaning." However, if there is no inertial reference frame, to what reference frame should the calculations performed in the GTR be referred?

Fock used harmonic conditions in Cartesian coordinates in calculating gravitational effects. He referred to these as coordinate conditions. In 1939 he wrote:[13] "In solving the Einstein equations we have used a coordinate system which we term harmonic, but which deserves to be referred to as inertial." Later in the same article he noted: "It seems to us that the possibility of introducing a particular inertial coordinate system into the general theory of relativity in a unique way is worthy of note." Finally, in Ref. 14 he wrote: "The principle of relativity expressed by Lorentz transformations is also possible in an inhomogeneous space; but a general principle of relativity is impossible."

Fock made all these statements in an attempt to inject clarity into the physical meaning of the GTR, after freeing it from the physically meaningless concept of general relativity. However, here Fock actually went beyond the framework of the GTR. It is for this reason that he arrived at the surprising statement that the principle of relativity is valid in inhomogeneous space. However, in order to achieve this it is necessary to introduce ideas about the gravitational field in Minkowski space. Where did Fock go beyond the GTR? When using the harmonic conditions he introduced the Cartesian coordinates

$$\frac{\partial \tilde{g}^{\mu\nu}}{\partial x^\mu}=0, \tag{155}$$

where x^μ are the Cartesian coordinates.

In these coordinates $\gamma(x)=\det\gamma_{\mu\nu}=-1$.

Therefore, according to the tensor transformation law we have

$$\tilde{g}^{\mu\nu}(x)=\frac{\partial x^{\mu}}{\partial y^{\alpha}}\cdot\frac{\partial x^{\nu}}{\partial y^{\beta}}\cdot\frac{\tilde{g}^{\alpha\beta}(y)}{\sqrt{-\gamma(y)}}. \tag{156}$$

We write Eq. (155) as

$$\partial_{\mu}\tilde{g}^{\mu\nu}=\frac{\partial u^{\tau}}{\partial x^{\mu}}\cdot\frac{\partial\tilde{g}^{\mu\nu}}{\partial y^{\tau}}. \tag{157}$$

For the rest of the calculations we use the expressions

$$\frac{\partial}{\partial y^{\tau}}\left(\frac{1}{\sqrt{-\gamma(y)}}\right)=-\frac{1}{\sqrt{-\gamma}}\gamma^{\lambda}_{\tau\lambda},$$

$$\gamma^{\nu}_{\alpha\beta}=\frac{\partial^{2}x^{\sigma}}{\partial y^{\alpha}\partial y^{\beta}}\cdot\frac{\partial y^{\nu}}{\partial x^{\sigma}}. \tag{158}$$

After substituting (156) into (157) and using (158), we find

$$\partial_{\mu}\tilde{g}^{\mu\nu}(x)=\frac{1}{\sqrt{-\gamma}}\frac{\partial x^{\nu}}{\partial y^{\sigma}}\cdot\frac{\partial\tilde{g}^{\alpha\sigma}}{\partial y^{\alpha}}+\frac{1}{\sqrt{-\gamma}}\tilde{g}^{\alpha\beta}\frac{\partial^{2}x^{\nu}}{\partial y^{\alpha}\partial y^{\beta}}=0. \tag{159}$$

We write the factor in the second term as

$$\frac{\partial^{2}x^{\nu}}{\partial y^{\alpha}\partial y^{\beta}}=\frac{\partial x^{\nu}}{\partial y^{\sigma}}\cdot\frac{\partial y^{\sigma}}{\partial x^{\tau}}\cdot\frac{\partial^{2}x^{\tau}}{\partial y^{\alpha}\partial y^{\beta}}=\frac{\partial x^{\nu}}{\partial y^{\sigma}}\cdot\gamma^{\sigma}_{\alpha\beta}.$$

Substituting this expression into the preceding one, we find

$$\partial_{\mu}\tilde{g}^{\mu\nu}(x)=\frac{1}{\sqrt{-\gamma}}\cdot\frac{\partial x^{\nu}}{\partial y^{\sigma}}\left(\frac{\partial\tilde{g}^{\alpha\sigma}}{\partial y^{\alpha}}+\gamma^{\sigma}_{\alpha\beta}\tilde{g}^{\alpha\beta}\right)=0,$$

i.e., we have

$$\partial_{\mu}\tilde{g}^{\mu\nu}(x)=\frac{1}{\sqrt{-\gamma}}\cdot\frac{\partial x^{\nu}}{\partial y^{\sigma}}D_{\mu}\tilde{g}^{\mu\sigma}=0. \tag{160}$$

We have thus shown that the tensor density $\tilde{g}^{\mu\sigma}(y)$ in arbitrary coordinates automatically satisfies the generally covariant expression

$$D_{\lambda}\tilde{g}^{\lambda\sigma}=0$$

if the original harmonicity condition (155) is written in Cartesian coordinates. However, this implies that the harmonicity condition is not a coordinate condition, but a field equation in Minkowski space. Therefore, the use of the harmonic condition in Cartesian coordinates is not an innocent operation, but involves going beyond the confines of the GTR by introducing Minkowski space.

Did Fock attempt to study the gravitational field in Minkowski space? No, he was far from thinking of this, and wrote:[13] "We mention it here only in connection with the sometimes observed striving (which we do not at all take part in) to build a theory of gravity within the framework of Euclidean space." As we have seen, the use of the harmonic conditions in Cartesian coordinates takes us outside the GTR. However, this implies that the system of gravitational equations studied by Fock differs from the system of GTR equations, i.e., that the Fock theory of gravity based on the harmonic conditions in Cartesian coordinates and the Einstein general theory of relativity are different theories. The Fock approach proved to be closer to the RTG ideas. Everything that Fock tried to introduce into the theory of gravity (inertial frames, acceleration relative to space) is fully contained in the RTG, but in the latter this is achieved by treating the gravitational field like all the other physical fields in Minkowski space. All the geometrical characteristics of Riemannian space are already field quantities in Minkowski space.

In analyzing gravitational effects in the solar system, Fock actually used Minkowski space, since he referred all the calculated gravitational effects to an inertial reference frame. It is this fact which allowed him to obtain the correct expressions for these effects. For example, he wrote:[14] "How should a straight line be defined: as a light ray or as a line in the Euclidean space in which the harmonic coordinates x_1, x_2, x_3 serve as the Cartesian coordinates? It seems to us that only the second definition is correct. We have in fact used it when we stated that a light ray near the Sun has the form of a hyperbola," and later on, "the idea that a straight line is more directly observable as a light ray has no value; the decisive factor in definitions is not direct observability, but correspondence to nature, even if this correspondence is established by indirect conclusions."

When calculating gravitational effects (for example, the deflection of a light ray or the time delay of a radio signal) in the GTR, it is necessary to compare the motion of light and a radio signal along a geodesic in Riemannian space with their motion in the absence of a gravitational field. This is how a gravitational effect must be determined. However, since in the GTR it is impossible to say which coordinate system (inertial or noninertial) we are in when the gravitational field is switched off, such a comparison is in principle ambiguous.[15] Fock overcame this difficulty by using the harmonic conditions in Cartesian coordinates. However, this led him outside the GTR.

Gravitational effects are determined unambiguously in the RTG, because according to Eqs. (93) and (94) written in the Galilean coordinates of an inertial frame, the motion of light or of a test body when the gravitational field is switched off does actually occur along a straight line, a geodesic in Minkowski space. It is quite obvious that in a noninertial coordinate system a geodesic in Minkowski space will no longer be a straight line. However, this implies that in the RTG, in order to find a gravitational effect in a noninertial coordinate system, the motion in the effective Riemannian space must be compared with precisely this motion.

In calculating the effects of gravity in the solar system, where the effect of the graviton mass can be neglected, the system of equations (93) and (94) of the RTG in Galilean coordinates coincides with the system of equations solved by Fock. Therefore, the post-Newtonian approximation (151) in the RTG coincides with the analogous approximation in the Fock theory of gravity. As we noted earlier, the harmonic conditions in Cartesian coordinates used successfully by Fock took him outside the framework of the Einstein GTR. This was noticed by Infeld, who in 1957 wrote, "Thereby, for Fock the choice of harmonic coordinate condition becomes a sort of fundamental law of nature, changing the very nature of the Einstein general theory of relativity and transforming it into a theory of the gravitational field valid only in inertial reference frames.'' However, even when the harmonic conditions in Cartesian coordinates are not used in the GTR, we still obtain similar expressions for the post-Newtonian approximation. What is going on? The reason for this is that again we have introduced Minkowski space in Galilean coordinates and have actually treated the gravitational field as a physical field in this space.

The metric of Minkowski space in Galilean coordinates is taken as the zeroth-order approximation for the Riemann metric. To it are added various potentials with arbitrary post-Newtonian parameters, each of which falls off as $0(1/r)$. It is here that gravity is treated as a physical field in Minkowski space whose behavior is described by the gravitational potentials introduced. This requirement on the nature of the behavior of the metric of Riemannian space does not follow from the GTR, since in general the asymptote of the metric is very arbitrary and even depends on the choice of three-dimensional spatial coordinates.

By substituting the Riemann metric $g_{\mu\nu}$ in this form into the Hilbert–Einstein equations, we can determine the values of the post-Newtonian parameters and again arrive at the same post-Newtonian approximation. It should be noted that the Hilbert–Einstein equations themselves do not determine the post-Newtonian approximation as the unique solution of these equations.

The reason for the nonuniqueness of the solution in the GTR lies in the form-invariance of the Hilbert–Einstein equations under coordinate transformations. Because of it, an unbounded number of solutions exists in a single coordinate system for Riemann metric of the space $g_{\mu\nu}$. In 1914 Einstein wrote as follows about this set of solutions: ''We shall consider a finite part Σ of the space in which matter processes do not occur. Then the physical events in the region Σ are completely determined if the quantities $g_{\mu\nu}$ as functions of the coordinates x_ν are specified with respect to the coordinate system K used for the description. The set of these functions will be denoted symbolically by $G(x)$.

''Let us introduce a new coordinate system K' coinciding with K outside the region Σ, but differing from K inside Σ such that the quantities $g'_{\mu\nu}$ in the system K', like the $g_{\mu\nu}$ (together with their derivatives), are everywhere continuous. We denote the set $g'_{\mu\nu}$ symbolically by $G'(x')$. The quantities $G'(x')$ and $G(x)$ describe the gravitational field itself. Let us express the coordinates x'_ν entering into $g'_{\mu\nu}$ in terms of the coordinates x_ν, i.e., we form $G'(x)$, which will then give a different description of the gravitational field in the system K, which, however, is not the same as the existing (or specified) gravitational field.

''Now we assume that the differential equations of the gravitational field are generally covariant; then their solution will be the set $G'(x')$ (in the system K') if in the system K the solution set is $G(x)$. The functions $G'(x)$ will now also satisfy these equations in the system K. Therefore, relative to the system K there exist solutions $G(x)$ and $G'(x)$ differing from each other, despite the fact that at the boundary of the region both sets of solutions coincide, i.e., *for generally covariant differential equations of the gravitational field the sequence of events can be nonunique*. If we require that the evolution of events in a gravitational field be completely determined by established laws, then it is necessary to restrict the set of coordinate systems in such a way that it is impossible to introduce a new coordinate system K' of the form described above without violating this restriction. The extension of the coordinate system inside a region Σ cannot be arbitrary.''

In order to understand the conclusion reached by Einstein at the end of this quotation, let us turn to electrodynamics. In some inertial frame (Galilean coordinates) let the solution of the equations have the form $F_{\mu\nu}(x)$ for the current $j^\mu(x)$. If we go to an arbitrary coordinate system, owing to general covariance the solution $F'_{\mu\nu}(x')$ of the Maxwell equations in the new coordinates will correspond to the current $j'^\mu(x')$. We note that $F'_{\mu\nu}(x)$ will not satisfy the Maxwell equations in the old coordinates for the current $j'^\mu(x)$, because the Maxwell equations are not form-invariant under arbitrary coordinate transformations. The Maxwell equations are form-invariant under Lorentz transformations, and so in the new Lorentz coordinates x' the solution $F'_{\mu\nu}(x')$ corresponds to the current $j'^\mu(x')$, and therefore in the old coordinates the solution $F'_{\mu\nu}(x')$ corresponds to the current $j'^\mu(x)$. We see that in electrodynamics the field $F'_{\mu\nu}(x)$ is a solution of the Maxwell equations in the old coordinates x, but for the current $j'^\mu(x)$ instead of the current $j^\mu(x)$. In the GTR, owing to the form-invariance of the Hilbert–Einstein equations under arbitrary coordinate transformations, $G(x)$ and $G'(x)$ are solutions for the same matter tensor $T_{\mu\nu}(x)$. In arriving at his conclusion, Einstein apparently intuitively represented gravity as a field in Minkowski space. However, since he retained only Riemannian space, this idea was not developed in his studies.

In the GTR the gravitational field is the Riemann tensor. The geometry of Riemannian space is described by the solution of the Hilbert–Einstein equations in arbitrary coordinates. When the gravitational field is switched off, the Riemann tensor vanishes, and the geometry of spacetime becomes pseudo-Euclidean, but it is impossible to say what coordinate system we are in (inertial or accelerated). How then can the correspondence principle be satisfied? In the RTG the gravitational equations (76) and (77) are generally covariant but not form-invariant under arbitrary transformations. They are form-invariant under Lorentz transformations. However, this implies that in Lorentz coordinates, if there is a solution $G(x)$ for the matter tensor $T_{\mu\nu}(x)$, then in the new Lorentz coordinates x' there is a solution $G'(x')$ for the matter tensor $T'_{\mu\nu}(x')$, and therefore in the coordinates x the solution $G'(x)$ is possible only for the matter tensor $T'_{\mu\nu}(x)$.

Everything that Einstein wrote about is thus realized in the RTG, and it is not necessary to abandon the general covariance of the theory. However, this was all obtained starting from the representation of the gravitational field as a physical field of spins 2 and 0 in Minkowski space. Riemannian space arises as an effective space due to the presence of the gravitational field. In the RTG there is a one-to-one correspondence between the Riemann metric and the Minkowski metric, which in calculations of gravitational effects allows the comparison of motion in the presence and absence of a gravitational field. In the RTG, when the gravitational field is switched off the Riemann tensor vanishes, and at the same time there is a transition from the Riemann metric to the Minkowski metric chosen earlier for stating the physical problem. This ensures that the correspondence principle is satisfied in the RTG.

To calculate the effect of gravity it is necessary to compare motion in Riemannian space with motion in the absence of a gravitational field. This is how the effect of gravity is determined. If the set of solutions for $g_{\mu\nu}$ is related to a particular inertial frame, it is completely obvious that we obtain a set of different values for the gravitational effect. How do we choose between them? Since the Minkowski metric is absent in the Hilbert–Einstein equations, it is impossible to observe the correspondence principle, because it is impossible to determine whether we are in an inertial or noninertial reference frame when the gravitational field is switched off. Therefore, in the GTR it is impossible to determine uniquely the post-Newtonian approximation, and, accordingly, impossible to describe the known gravitational effects in the solar system. This is achieved in the GTR only by additional assumptions which take us outside this theory.

As an illustration of the nonuniqueness of the GTR predictions, let us consider the problem of determining the delay time of a radio signal under the action of the Sun as the signal travels from the Earth to Mercury and back. For a spherically symmetric static body in a particular coordinate system, the Hilbert–Einstein equations have a set of solutions which includes the Schwarzschild solution

$$ds_1^2=\left(1-\frac{2M}{r}\right)dt^2-\left(1-\frac{2M}{r}\right)^{-1}dr^2-r^2d\Omega^2 \qquad (161)$$

and the harmonic solution

$$ds_2^2=\frac{r-M}{r+M}\,dt^2-\frac{r+M}{r-M}\,dr^2-(r+M)^2d\Omega^2. \qquad (162)$$

Let r_e and r_p be the distances from the emission and reflection points of the radio signal to the center of the source of gravitational field (the Sun) and R be the distance between the points e and p. To determine the effect of gravity it is necessary to compare the motion in Riemannian space with that along a straight line in an inertial frame. We thereby obtain the delay time of a radio signal for the Schwarzschild solution

$$\Delta t_1=2GM\ \ln\frac{r_e+r_p+R}{r_e+r_p-R}-2GM \qquad (163)$$

and for the harmonic solution

$$\Delta t_2=2GM\ \ln\frac{r_e+r_p+R}{r_e+r_p-R}. \qquad (164)$$

They differ by an amount $2GM$, equal to tens of microseconds in the case of the Sun. Which expression, (163) or (164), should be used to find the delay time of a radio signal traveling from the Earth to Mercury and back? In principle, the GTR does not answer this question, because the metric of Minkowski space does not enter into the Hilbert–Einstein equations. In the GTR the gravitational field is characterized not by a single metric tensor, but by a class of equivalent diffeomorphic metrics $G(x), G'(x),\ldots$, obtained using coordinate transformations. From this point of view, in Minkowski space the metrics obtained by coordinate transformation from Eq. (α) in the Introduction also form an equivalence class, but physically they are different, because some correspond to inertial reference frames and others to accelerated ones. In the GTR it is impossible to determine which metric of Minkowski space the Riemann metric from the equivalence class should be referred to in order to determine the gravitational effect. It is therefore impossible to observe an obvious principle: the correspondence principle.

According to the RTG, the delay time of a radio signal is determined uniquely as Eq. (164) if Galilean coordinates are used in the inertial frame. In the RTG it would also be possible to take the Schwarzschild solution, but in this case, according to Eqs. (76) and (77) of the RTG, the metric of Minkowski space would be slightly different. Because of this change of the Minkowski metric, the Schwarzschild solution (161) also leads to the delay of a radio signal due to the Sun given by (164), i.e., it gives the same result.

We conclude this section by noting that the post-Newtonian approximation (151) satisfies the causality principle (90).

9. SOME PHYSICAL CONCLUSIONS OF THE RTG

It follows from the results obtained in the preceding section that the RTG explains all the known gravitational experiments in the solar system. All the effects in the solar system are calculated in the inertial reference frame to which all the observational data pertain. Since the source of the gravitational field is the conserved total density of the matter and gravitational-field tensor, it follows that the inertial mass of a static body is exactly equal to its active gravitational mass. This equality does not presuppose even local identity of inertial and gravitational forces.

An important physical conclusion can also be drawn regarding the evolution of a homogeneous and isotropic universe. Using equations (76) and (77) for such a universe, we obtain the expressions

$$\left(\frac{1}{R}\cdot\frac{dR}{d\tau}\right)^2=\frac{8\pi}{3}\,\rho(\tau)-\frac{1}{6}\,m^2\left(1-\frac{3}{2a^4R^2}+\frac{1}{2R^6}\right), \qquad (165)$$

$$\frac{1}{R}\cdot\frac{d^2R}{d\tau^2}=-\frac{4\pi}{3}\,\rho(\tau)-4\pi p(\tau)-\frac{1}{6}\,m^2\left(1-\frac{1}{R^6}\right). \qquad (166)$$

The interval of the effective Riemannian space has the form

$$ds^2=d\tau^2-a^4R^2(\tau)(dr^2+r^2d\Omega^2), \qquad (167)$$

where a is the constant from integrating (77). Owing to the causality condition (90), we have

$$R(\tau)\leqslant a. \qquad (168)$$

The constant is $a>1$. It is easy to establish the following inequality:

$$\left(1-\frac{3}{2a^4R^2}+\frac{1}{2R^6}\right)>\frac{1}{R^6}(R^2-1)^2\left(R^2+\frac{1}{2}\right). \qquad (169)$$

Taking into account (169), it can be shown on the basis of (165) that a homogeneous and isotropic universe evolves cyclically from some minimum nonzero value $R_{\min}$ to some maximum value $R_{\max}$, and so on. To satisfy the causality condition (168) it is necessary to take

$$a=R_{\max}. \qquad (170)$$

We introduce the function

$$H(\tau)=\frac{1}{R}\cdot\frac{dR}{d\tau}. \qquad (171)$$

Then for the present time τ_p, by reconstructing explicitly the dependence on the Newtonian gravitational constant G and the speed of light c, on the basis of (165) we obtain

$$\rho(\tau_p)=\rho_c+\rho_g, \qquad (172)$$

where the critical density ρ_c is determined by the Hubble constant $H(\tau_p)$ and has the value

$$\rho_c=\frac{3H^2}{8\pi G}, \qquad (173)$$

and the constant ρ_g determined by the graviton mass is

$$\rho_g=\frac{1}{16\pi G}\left(\frac{mc^2}{\hbar}\right)^2. \qquad (174)$$

The homogeneous and isotropic universe is infinite, its three-dimensional geometry is Euclidean, and it evolves cyclically from some maximum density $\rho_{\max}$ to the minimum $\rho_{\min}$, equal to

$$\rho_{\min}=\rho_g, \qquad (175)$$

and so on.

The cosmological constant Λ is expressed in terms of the graviton mass:

$$\Lambda=\frac{m^2}{2}. \qquad (176)$$

It is obvious from Eqs. (172) and (174) that if, for example, the graviton mass is less than or equal to 10^{-66} g, which corresponds to the cosmological constant $\Lambda\leqslant 4.5\times 10^{-58}$ cm^{-2}, the matter density in the Universe at the present time must be close to the critical density ρ_c. If the graviton mass is equal to 10^{-65} g, which corresponds to the cosmological constant $\Lambda=4.5\times 10^{-56}$ cm^{-2}, the density ρ_g gives a significant contribution to the matter density of the universe ρ, equal to

$$\rho_g\simeq 2.8\cdot 10^{-29}\ \text{g/cm}^3. \qquad (177)$$

In any case, we have the inequality

$$\rho\geqslant\rho_c. \qquad (178)$$

Since the observed matter density in the Universe is considerably smaller than the critical density ρ_c, the RTG predicts the existence of a large "hidden" mass in the Universe. According to the RTG, the deceleration parameter is

$$q(\tau_p)=\frac{1}{2}\left[1+3\frac{\rho_g}{\rho_c}\right].$$

From this we see that ρ_g and, consequently, the graviton mass m can be expressed in terms of measurable quantities: the deceleration parameter q and the Hubble constant H. If the deceleration parameter turns out to be larger than 1/2, the graviton mass will be nonzero, according to the theory. Then the Universe will not be closed, as occurs in the GTR, but will be flat.

It should be particularly noted that, in contrast to the GTR, the Friedmann model of the Universe discussed above does not contain any of the well known problems such as singularities, causality, Euclidicity, flatness, and entropy. The form of the Friedmann universe is independent of the ratio of the current matter density and the critical density determined by the Hubble constant.

For isotropic or timelike vectors U^ν we have

$$g_{\mu\nu}U^\mu U^\nu\geqslant 0,$$

and for the metric of a homogeneous and isotropic universe from (92) we find

$$R_{\mu\nu}U^\mu U^\nu<0 \quad \text{for } R=R_{\min}$$

and

$$R_{\mu\nu}U^\mu U^\nu>0 \quad \text{for } R=R_{\max}.$$

From this we see that the conditions of the Penrose–Hawking theorems regarding the singularity problem are not satisfied in our theory.

Therefore, in contrast to the GTR, in the RTG a homogeneous and isotropic universe can only be flat and cannot have singularities. Another important consequence of the RTG is a significant change in the nature of the collapse. It turns out that in the collapse of a spherically symmetric body of arbitrarily large mass, the process of contraction into a region near the size of the Schwarzschild sphere stops and is replaced by subsequent expansion. This stopping of the contraction occurs owing to the presence of the mass term with the Minkowski metric tensor in Eq. (76). It is this term which also stops the contraction process in a homogeneous, isotropic universe. Therefore, according to the RTG the existence of black holes (objects without matter boundaries which are cut off from the external world) is completely excluded. It was shown in Ref. 16 that for a spherically symmetric, static body the metric components of Riemannian space have the following form in the vicinity of the Schwarzschild sphere:

$$ds^2=U(Z)dt^2-V(Z)dZ^2-Z^2d\Omega^2,$$

$$V(Z)=\frac{Z}{Z-Z_g},\quad U(Z)=(1+2mM)\frac{Z-Z_g}{Z}+qm^2M^2,\quad q>0. \qquad (179)$$

A singularity in the function V arises at the point Z_g equal to

$$Z_g=2M+\nu m^2M^3\ln\frac{1}{mM},\quad \nu>0. \qquad (180)$$

The sphere of radius Z_g is singular, and this singularity cannot be eliminated by a choice of coordinate system. We see from Eqs. (179) and (180) that in the region close to the Schwarzschild sphere the effect of the graviton mass m is large and fundamentally changes the nature of the solution in this region. It therefore cannot be neglected. If we transform to a synchronous system of freely falling test particles having zero velocity at infinity, using the transformations

$$\tau=t+\int dZ\left[\frac{V(1-U)}{U}\right]^{1/2},$$

$$R=t+\int dz\left[\frac{V}{U(1-U)}\right]^{1/2},$$

we obtain the following expression for the interval:

$$ds^2=d\tau^2-(1-U)dR^2-Z^2(d\Theta^2+\sin^2\Theta d\Phi^2).$$

The radial velocity of a particle falling along a radial line is

$$\frac{dZ}{d\tau}=-\sqrt{\frac{1-U}{UV}}.$$

From (179), in the vicinity of the Schwarzschild sphere we have

$$\frac{dZ}{d\tau}=-\frac{1}{\sqrt{qmM}}\sqrt{\frac{Z-Z_g}{Z}}.$$

From this it is obvious that the point $Z=Z_g$ is a turning point for the radial motion of the particles. Therefore, the presence of a graviton mass m, independently of its value, leads to the repulsion of matter particles[17] from a sphere close to the Schwartzschild sphere. Since the solution inside the body cannot be matched to the exterior solution, a sphere with radius $Z=Z_g$ cannot be located outside matter.

At large distances r from the body the metric components have the form

$$U(r)=1-\frac{2M}{r}e^{-mr},\quad V(r)=1+\frac{2M}{r}e^{-mr},$$

$$Z(r)=r\left(1+\frac{M}{r}e^{-mr}\right).$$

Let us discuss the problem of studying weak gravitational waves when there is a graviton mass. It has long been known that in linear tensor theory the introduction of a graviton mass is always accompanied by "ghosts." However, it was shown in Ref. 18 that the intensity of the gravitational emission of massive gravitons in the nonlinear theory is a positive-definite quantity, equal to

$$\frac{dI}{d\Omega}=\frac{2}{\pi}\int_{\omega_{\min}}^{\infty}d\omega\omega^2q\left\{|T_2^1|^2+\frac{1}{4}|T_1^1-T_2^2|^2+\frac{m^2}{\omega^2}(|T_3^1|^2+|T_3^2|^2)+\frac{3m^4}{4\omega^4}|T_3^3|^2\right.,\qquad(181)$$

where $q=(1-m^2/\omega^2)^{1/2}$.

In the RTG, as in the GTR, outside matter the density of the energy–momentum tensor of the gravitational field in Riemannian space is equal to zero:

$$T_g^{\mu\nu}=-2\frac{\delta L_g}{\delta g_{\mu\nu}}=0.\qquad(182)$$

However, this implies that the energy flux of the gravitational field is not determined by the components of the tensor density T_g^{0i} calculated for solutions of (182), because they are equal to zero. The problem of determining the energy flux in the theory of gravity, in contrast to other theories, requires a different approach. The author of Ref. 18 sought a solution in the form

$$\tilde{\Phi}^{\mu\nu}=\chi^{\mu\nu}+\psi^{\mu\nu},\qquad(183)$$

where the quantities $\chi^{\mu\nu}$ and $\psi^{\mu\nu}$ are of the same order in the small parameter, and $\psi^{\mu\nu}$ describes outgoing waves, while $\chi^{\mu\nu}$ characterizes the background. Energy is transferred only by the outgoing waves. The author of Ref. 18 showed that in fact the flux of gravitational energy is determined by the quantity $T_g^{0i}(\psi)$ calculated not using the solutions of (182), but using only the part of the solutions describing outgoing waves $\psi^{\mu\nu}$. He takes into account the fact that gravitons propagate not in Minkowski space, as always occurs in the linear theory, but in an effective Riemannian space. Therefore, the following equation holds in the linear approximation:

$$\gamma_{\mu\nu}\frac{dx^\mu}{ds}\cdot\frac{dx^\nu}{ds}-1=\frac{d\sigma^2-ds^2}{ds^2}\simeq-\frac{1}{2}\gamma_{\mu\nu}\Phi^{\mu\nu}+\Phi^{\mu\nu}\frac{dx^\alpha}{d\sigma}\cdot\frac{dx^\beta}{d\sigma}\gamma_{\mu\alpha}\gamma_{\nu\beta}.$$

It is the systematic inclusion of this fact in the process of finding the intensity that leads the author of Ref. 18 to the positive-definite energy flux given by Eq. (181). The result of Ref. 18 is of fundamental importance because it changes the accepted ideas. It must therefore be analyzed further.

The system of gravitational equations (76) and (77) is hyperbolic, and the principle of causality ensures that there exists in all space a spacelike surface which every nonspacelike curve in Riemannian space intersects only once. In other words, there exists a global Cauchy surface on which the initial physical conditions are specified for a particular problem. Penrose and Hawking[12] proved theorems about the existence of singularities in the GTR under certain general conditions. On the basis of (78a), outside matter the isotropic vectors of Riemannian space satisfy the following inequality, owing to the causality conditions (91a):

$$R_{\mu\nu}v^\mu v^\nu\leqslant 0.\qquad(184)$$

Because of (184), the conditions for those theorems are not satisfied in the RTG, and so they cannot be used.

In the RTG, spacelike events in the absence of a gravitational field can never become timelike under the action of a gravitational field. On the basis of the causality principle, the effective Riemannian space in the RTG will possess isotropic and timelike geodesic completeness. According to the RTG, an inertial reference frame is determined from the distribution of matter and gravitational field in the Universe (Mach's principle).

In the GTR, inertial and gravitational fields are indistinguishable. Einstein wrote: "...there is no real difference between inertia and gravity, because the answer to the question of whether a body at a particular moment is acted on exclusively by inertia or by a combination of inertia and gravity depends on the coordinate system, i.e., on the method of observation." Inertial fields satisfy the Hilbert–Einstein equations. In the RTG, the gravitational field and inertial fields determined by the metric tensor of Minkowski space are distinguishable; they have nothing in common. They have different origins. Inertial fields are not solutions of the RTG equations (76) and (77). In the RTG, inertial fields are specified by the metric tensor $\gamma_{\mu\nu}$, and the gravitational field $\Phi^{\mu\nu}$ is determined from the gravitational equations (76) and (77).

On the basis of the RTG we can draw the following general conclusion:

The universal integral laws of energy–momentum conservation and universal properties of matter such as gravitational interactions are reflected in the metrical properties of spacetime. Whereas the former are embodied in the pseudo-Euclidean geometry of spacetime, the latter are reflected in the effective Riemannian geometry of spacetime arising owing to the presence of a gravitational field in Minkowski space. The structure of the effective geometry can give rise to everything which is common to all matter. However, Minkowski space must necessarily be present here, which leads to integral energy–momentum and angular-momentum conservation laws, and also ensures that the correspondence principle holds when the gravitational field is switched off.

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APPENDIX A

Let us prove the relation

$$\frac{\delta L}{\delta\gamma_{\mu\nu}}=\frac{\delta L}{\delta g_{\alpha\beta}}\cdot\frac{\partial g_{\alpha\beta}}{\partial\gamma_{\mu\nu}}+\frac{\delta^* L}{\delta\gamma_{\mu\nu}}, \tag{A1}$$

where

$$\frac{\delta L}{\delta\gamma_{\mu\nu}}=\frac{\partial L}{\partial\gamma_{\mu\nu}}-\partial_\sigma\left(\frac{\partial L}{\partial\gamma_{\mu\nu,\sigma}}\right), \tag{A2}$$

$$\frac{\delta L}{\delta g_{\mu\nu}}=\frac{\partial L}{\partial g_{\alpha\beta}}-\partial_\sigma\left(\frac{\partial L}{\partial g_{\alpha\beta,\sigma}}\right), \tag{A3}$$

and the asterisk in the first expression denotes the variational derivative of the Lagrangian density with respect to the metric $\gamma_{\mu\nu}$ entering explicitly into L. After differentiation we obtain

$$\frac{\partial L}{\partial\gamma_{\mu\nu}}=\frac{\partial^* L}{\partial\gamma_{\mu\nu}}+\frac{\partial L}{\partial g_{\alpha\beta,\sigma}}\cdot\frac{\partial g_{\alpha\beta,\sigma}}{\partial\gamma_{\mu\nu}}+\frac{\partial L}{\partial g_{\alpha\beta}}\cdot\frac{\partial g_{\alpha\beta}}{\partial\gamma_{\mu\nu}}, \tag{A4}$$

$$\frac{\partial L}{\partial\gamma_{\mu\nu,\sigma}}=\frac{\partial^* L}{\partial\gamma_{\mu\nu,\sigma}}+\frac{\partial L}{\partial g_{\alpha\beta,\tau}}\cdot\frac{\partial g_{\alpha\beta,\tau}}{\partial\gamma_{\mu\nu,\sigma}}. \tag{A5}$$

Let us substitute these expressions into Eq. (A2):

$$\begin{aligned}\frac{\partial L}{\partial\gamma_{\mu\nu}}-\partial_\sigma\left(\frac{\partial L}{\partial\gamma_{\mu\nu,\sigma}}\right)&=\frac{\delta^* L}{\delta\gamma_{\mu\nu}}+\frac{\partial L}{\partial g_{\alpha\beta,\sigma}}\times\frac{\partial g_{\alpha\beta,\sigma}}{\partial\gamma_{\mu\nu}}\\&+\frac{\partial L}{\partial g_{\alpha\beta}}\times\frac{\partial g_{\alpha\beta}}{\partial\gamma_{\mu\nu}}-\partial_\sigma\left(\frac{\partial L}{\partial g_{\alpha\beta,\tau}}\right.\\&\left.\times\frac{\partial g_{\alpha\beta,\tau}}{\partial\gamma_{\mu\nu,\sigma}}\right)=\frac{\delta^* L}{\delta\gamma_{\mu\nu}}+\frac{\partial L}{\partial g_{\alpha\beta}}\\&\times\frac{\partial g_{\alpha\beta}}{\partial\gamma_{\mu\nu}}-\partial_\sigma\left(\frac{\partial L}{\partial g_{\alpha\beta,\tau}}\right)\times\frac{\partial g_{\alpha\beta,\tau}}{\partial\gamma_{\mu\nu,\sigma}}\\&+\frac{\partial L}{\partial g_{\alpha\beta,\sigma}}\left[\frac{\partial g_{\alpha\beta,\sigma}}{\partial\gamma_{\mu\nu}}\right.\\&\left.-\partial_\rho\left(\frac{\partial g_{\alpha\beta,\sigma}}{\partial\gamma_{\mu\nu,\rho}}\right)\right].\end{aligned} \tag{A6}$$

We consider the expression

$$\frac{\partial g_{\alpha\beta,\sigma}}{\partial\gamma_{\mu\nu}}-\partial_\rho\left(\frac{\partial g_{\alpha\beta,\sigma}}{\partial\gamma_{\mu\nu,\rho}}\right). \tag{A7}$$

We write the derivative $g_{\alpha\beta,\sigma}$ as

$$g_{\alpha\beta,\sigma}=\frac{\partial g_{\alpha\beta}}{\partial\gamma_{\lambda\omega}}\partial_\sigma\gamma_{\lambda\omega}+\frac{\partial g_{\alpha\beta}}{\partial\Phi_{\lambda\omega}}\partial\sigma\Phi_{\lambda\omega}, \tag{A8}$$

from which we easily find

$$\frac{\partial g_{\alpha\beta,\sigma}}{\partial\gamma_{\mu\nu,\rho}}=\frac{\partial g_{\alpha\beta}}{\partial\gamma_{\mu\nu}}\cdot\delta^\rho_\sigma. \tag{A9}$$

Differentiating this expression, we find

$$\partial_\rho\left(\frac{\partial g_{\alpha\beta,\sigma}}{\partial\gamma_{\mu\nu,\rho}}\right)=\frac{\partial^2 g_{\alpha\beta}}{\partial\gamma_{\mu\nu}\partial\gamma_{\lambda\omega}}\partial_\sigma\gamma_{\lambda\omega}+\frac{\partial^2 g_{\alpha\beta}}{\partial\gamma_{\mu\nu}\partial\Phi_{\lambda\omega}}\partial_\sigma\Phi_{\lambda\omega}. \tag{A10}$$

On the other hand, differentiating (A8) with respect to $\gamma_{\mu\nu}$, we find

$$\frac{\partial g_{\alpha\beta,\sigma}}{\partial\gamma_{\mu\nu}}=\frac{\partial^2 g_{\alpha\beta}}{\partial\gamma_{\mu\nu}\partial\gamma_{\lambda\omega}}\partial_\sigma\gamma_{\lambda\omega}+\frac{\partial^2 g_{\alpha\beta}}{\partial\gamma_{\mu\nu}\partial\Phi_{\lambda\omega}}\partial_\sigma\Phi_{\lambda\omega}. \tag{A11}$$

Comparing (A10) and (A11), we obtain

$$\frac{\partial g_{\alpha\beta,\sigma}}{\partial\gamma_{\mu\nu}}-\partial_\rho\left(\frac{\partial g_{\alpha\beta,\sigma}}{\partial\gamma_{\mu\nu,\rho}}\right)=0. \tag{A12}$$

Using this expression in (A6), we find

$$\frac{\delta L}{\delta\gamma_{\mu\nu}}=\frac{\delta^* L}{\delta\gamma_{\mu\nu}}+\frac{\partial L}{\partial g_{\alpha\beta}}\cdot\frac{\partial g_{\alpha\beta}}{\partial\gamma_{\mu\nu}}-\partial_\sigma\left(\frac{\partial L}{\partial g_{\alpha\beta,\tau}}\right)\cdot\frac{\partial g_{\alpha\beta,\tau}}{\partial\gamma_{\mu\nu,\sigma}}. \tag{A13}$$

Substituting (A9) into (A13), we obtain

$$\frac{\delta L}{\delta\gamma_{\mu\nu}}=\frac{\delta^* L}{\delta\gamma_{\mu\nu}}+\left[\frac{\partial L}{\partial g_{\alpha\beta}}-\partial_\sigma\left(\frac{\partial L}{\partial g_{\alpha\beta,\sigma}}\right)\right]\frac{\partial g_{\alpha\beta}}{\partial\gamma_{\mu\nu}}, \tag{A14}$$

that is,

$$\frac{\delta L}{\delta\gamma_{\mu\nu}}=\frac{\delta^* L}{\delta\gamma_{\mu\nu}}+\frac{\delta L}{\delta g_{\alpha\beta}}\cdot\frac{\partial g_{\alpha\beta}}{\partial\gamma_{\mu\nu}}. \quad (A15)$$

In a similar way we can calculate

$$\frac{\delta L}{\delta g_{\alpha\beta}}=\frac{\delta L}{\delta\tilde{g}^{\lambda\rho}}\cdot\frac{\partial\tilde{g}^{\lambda\rho}}{\partial g_{\alpha\beta}}. \quad (A16)$$

Using (A16), Eq. (A15) can be written as

$$\frac{\delta L}{\delta\gamma_{\mu\nu}}=\frac{\delta^* L}{\delta\gamma_{\mu\nu}}+\frac{\delta L}{\delta\tilde{g}^{\lambda\rho}}\cdot\frac{\partial\tilde{g}^{\lambda\rho}}{\partial\gamma_{\mu\nu}}. \quad (A17)$$

APPENDIX B

The Lagrangian density of the intrinsic gravitational field has the form

$$L_g=L_{g0}+L_{gm}, \quad (B1)$$

$$L_{g0}=\frac{1}{16\pi}\tilde{g}^{\alpha\beta}(G^\tau_{\lambda\alpha}G^\lambda_{\tau\beta}-G^\tau_{\alpha\beta}G^\lambda_{\tau\lambda}), \quad (B2)$$

$$L_{gm}=-\frac{m^2}{16\pi}\left(\frac{1}{2}\gamma_{\alpha\beta}\tilde{g}^{\alpha\beta}-\sqrt{-g}-\sqrt{-\gamma}\right). \quad (B3)$$

The rank-three tensor $G^\tau_{\alpha\beta}$ is

$$G^\tau_{\alpha\beta}=\frac{1}{2}g^{\tau\lambda}(D_\alpha g_{\beta\lambda}+D_\beta g_{\alpha\lambda}-D_\lambda g_{\alpha\beta}). \quad (B4)$$

It is expressed in terms of the Christoffel symbols of Riemannian space and Minkowski space:

$$G^\tau_{\alpha\beta}=\Gamma^\tau_{\alpha\beta}-\gamma^\tau_{\alpha\beta}. \quad (B5)$$

Let us calculate the variational derivative of L_g with respect to the Minkowski metric $\gamma_{\mu\nu}$ appearing explicitly:

$$\frac{\delta^* L_{g0}}{\delta\gamma_{\mu\nu}}=\frac{\partial L_{g0}}{\partial\gamma_{\mu\nu}}-\partial_\sigma\left(\frac{\partial L_{g0}}{\partial\gamma_{\mu\nu,\sigma}}\right). \quad (B6)$$

For this we perform some preliminary calculations:

$$\frac{\partial G^\lambda_{\alpha\beta}}{\partial\gamma_{\mu\nu}}=-\frac{\partial\gamma^\lambda_{\alpha\beta}}{\partial\gamma_{\mu\nu}}=\frac{1}{2}(\gamma^{\lambda\mu}\gamma^\nu_{\alpha\beta}+\gamma^{\lambda\nu}\gamma^\mu_{\alpha\beta}), \quad (B7)$$

$$\frac{\partial G^\lambda_{\alpha\lambda}}{\partial\gamma_{\mu\nu}}=-\frac{\partial\gamma^\lambda_{\alpha\lambda}}{\partial\gamma_{\mu\nu}}=\frac{1}{2}(\gamma^{\lambda\mu}\gamma^\nu_{\lambda\alpha}+\gamma^{\lambda\nu}\gamma^\mu_{\alpha\lambda}),$$

$$\frac{\partial G^\lambda_{\alpha\beta}}{\partial\gamma_{\mu\nu,\sigma}}=-\frac{\partial\gamma^\lambda_{\alpha\beta}}{\partial\gamma_{\mu\nu,\sigma}}=-\frac{1}{4}[\gamma^{\lambda\mu}(\delta^\nu_\alpha\delta^\sigma_\beta+\delta^\sigma_\alpha\delta^\nu_\beta)+\gamma^{\lambda\nu}(\delta^\mu_\alpha\delta^\sigma_\beta+\delta^\sigma_\alpha\delta^\mu_\beta)-\gamma^{\lambda\sigma}(\delta^\mu_\alpha\delta^\nu_\beta+\delta^\nu_\alpha\delta^\mu_\beta)],$$

$$\frac{\partial G^\lambda_{\alpha\lambda}}{\partial\gamma_{\mu\nu,\sigma}}=-\frac{\partial\gamma^\lambda_{\alpha\lambda}}{\partial\gamma_{\mu\nu,\sigma}}=-\frac{1}{2}\gamma^{\mu\nu}\delta^\sigma_\alpha. \quad (B8)$$

Differentiating, we obtain

$$\frac{\partial L_{g0}}{\partial\gamma_{\mu\nu}}=-\frac{1}{16\pi}\tilde{g}^{\alpha\beta}\left[\frac{\partial G^\tau_{\alpha\lambda}}{\partial\gamma_{\mu\nu}}G^\lambda_{\tau\beta}+G^\tau_{\lambda\alpha}\frac{\partial G^\lambda_{\tau\beta}}{\partial\gamma_{\mu\nu}}-\frac{\partial G^\tau_{\alpha\beta}}{\partial\gamma_{\mu\nu}}G^\lambda_{\tau\lambda}-G^\tau_{\alpha\beta}\frac{\partial G^\lambda_{\tau\lambda}}{\partial\gamma_{\mu\nu}}\right].$$

Using Eq. (B7) in this expression, we find

$$\frac{\partial L_{g0}}{\partial\gamma_{\mu\nu}}=-\frac{1}{16\pi}\tilde{g}^{\alpha\beta}\left\{G^\tau_{\lambda\alpha}\gamma^{\lambda\mu}\gamma^\nu_{\tau\beta}+G^\tau_{\lambda\alpha}\gamma^{\lambda\nu}\gamma^\mu_{\tau\beta}-\frac{1}{2}G^\lambda_{\tau\lambda}\gamma^{\tau\mu}\gamma^\nu_{\alpha\beta}-\frac{1}{2}G^\lambda_{\tau\lambda}\gamma^{\tau\nu}\gamma^\mu_{\alpha\beta}-\frac{1}{2}G^\tau_{\alpha\beta}\gamma^{\lambda\mu}\gamma^\nu_{\tau\lambda}-\frac{1}{2}G^\tau_{\alpha\beta}\gamma^{\lambda\nu}\gamma^\mu_{\tau\lambda}\right\}=\frac{1}{32\pi}B^{\mu\nu}. \quad (B9)$$

Using the derivatives (B8), we obtain

$$\frac{\partial L_{g0}}{\partial\gamma_{\mu\nu,\sigma}}=\frac{1}{32\pi}A^{\sigma\mu\nu},\quad A^{\sigma\mu\nu}=\gamma^{\tau\mu}(G^\sigma_{\tau\beta}\tilde{g}^{\nu\beta}+G^\nu_{\tau\beta}\tilde{g}^{\sigma\beta}-G^\lambda_{\tau\lambda}\tilde{g}^{\sigma\nu})+\gamma^{\tau\nu}(G^\sigma_{\tau\beta}\tilde{g}^{\mu\beta}+G^\mu_{\tau\beta}-\tilde{g}^{\sigma\beta}-G^\lambda_{\tau\lambda}\tilde{g}^{\sigma\mu})+\gamma^{\tau\sigma}(G^\lambda_{\tau\lambda}\tilde{g}^{\mu\nu}-G^\mu_{\tau\beta}\tilde{g}^{\nu\beta}-G^\nu_{\tau\beta}\tilde{g}^{\mu\beta})-\gamma^{\mu\nu}G^\sigma_{\alpha\beta}\tilde{g}^{\alpha\beta}, \quad (B10)$$

where the tensor density $A^{\sigma\mu\nu}$ is symmetric in the indices μ and ν. The ordinary derivative of the tensor density can be written as

$$\partial_\sigma A^{\sigma\mu\nu}=D_\sigma A^{\sigma\mu\nu}-\gamma^\mu_{\sigma\rho}A^{\sigma\rho\nu}-\gamma^\nu_{\sigma\rho}A^{\sigma\mu\rho}.$$

Substituting (B9) and (B10) into (B6), we find

$$\frac{\delta^* L_{g0}}{\delta\gamma_{\mu\nu}}=\frac{1}{32\pi}B^{\mu\nu}-\frac{1}{32\pi}D_\sigma A^{\sigma\mu\nu}+\frac{1}{32\pi}\gamma^\mu_{\sigma\rho}A^{\sigma\rho\nu}+\frac{1}{32\pi}\gamma^\nu_{\sigma\rho}A^{\sigma\mu\rho}. \quad (B11)$$

We write the tensor density $A^{\sigma\rho\nu}$ as

$$A^{\sigma\rho\nu}=(G^\sigma_{\tau\beta}\gamma^{\tau\rho}\tilde{g}^{\nu\beta}-G^\rho_{\tau\beta}\gamma^{\tau\sigma}\tilde{g}^{\nu\beta})+(G^\nu_{\tau\beta}\gamma^{\tau\rho}\tilde{g}^{\sigma\beta}-G^\nu_{\tau\beta}\gamma^{\tau\sigma}\tilde{g}^{\rho\beta})-(G^\lambda_{\tau\lambda}\gamma^{\tau\rho}\tilde{g}^{\sigma\nu}-G^\lambda_{\tau\lambda}\gamma^{\tau\sigma}\tilde{g}^{\rho\nu})+G^\sigma_{\tau\beta}\gamma^{\tau\nu}\tilde{g}^{\rho\beta}+G^\rho_{\tau\beta}\gamma^{\tau\nu}\tilde{g}^{\sigma\beta}-G^\lambda_{\tau\lambda}\gamma^{\tau\nu}\tilde{g}^{\sigma\rho}-G^\sigma_{\alpha\beta}\gamma^{\rho\nu}\tilde{g}^{\alpha\beta},$$

where the expressions in the parentheses are antisymmetric in the indices σ and ρ. This notation makes it easier to find the expression for $\gamma^\mu_{\sigma\rho}A^{\sigma\rho\nu}$, because the terms antisymmetric in σ and ρ automatically disappear:

$$\gamma^\mu_{\sigma\rho}A^{\sigma\rho\nu}=2G^\sigma_{\tau\beta}\gamma^\mu_{\sigma\rho}\gamma^{\tau\nu}\tilde{g}^{\rho\beta}-G^\lambda_{\tau\lambda}\gamma^\mu_{\sigma\rho}\gamma^{\tau\nu}\tilde{g}^{\sigma\rho}-G^\sigma_{\alpha\beta}\gamma^\mu_{\sigma\rho}\gamma^{\nu\rho}\tilde{g}^{\alpha\beta}. \quad (B12)$$

Similarly, for $A^{\sigma\mu\rho}$ we have

$$A^{\sigma\mu\rho}=(G^\sigma_{\tau\beta}\gamma^{\tau\rho}\tilde{g}^{\mu\beta}-G^\rho_{\tau\beta}\gamma^{\tau\sigma}\tilde{g}^{\mu\beta})+(G^\mu_{\tau\beta}\gamma^{\tau\rho}\tilde{g}^{\sigma\beta}-G^\mu_{\tau\beta}\gamma^{\tau\sigma}\tilde{g}^{\rho\beta})+(G^\lambda_{\tau\lambda}\gamma^{\tau\sigma}\tilde{g}^{\mu\rho}-G^\lambda_{\tau\lambda}\gamma^{\tau\rho}\tilde{g}^{\sigma\mu})+G^\sigma_{\tau\beta}\gamma^{\tau\mu}\tilde{g}^{\rho\beta}+G^\rho_{\tau\beta}\gamma^{\tau\mu}\tilde{g}^{\sigma\beta}-G^\lambda_{\tau\lambda}\gamma^{\tau\mu}\tilde{g}^{\sigma\rho}-G^\sigma_{\alpha\beta}\gamma^{\mu\rho}\tilde{g}^{\alpha\beta},$$

where the parentheses again contain terms antisymmetric in the indices σ and ρ. From this we obtain

$$\gamma^\nu_{\sigma\rho}A^{\sigma\mu\rho}=2G^\sigma_{\tau\beta}\gamma^\nu_{\sigma\rho}\gamma^{\tau\mu}\tilde{g}^{\rho\beta}-G^\lambda_{\tau\lambda}\gamma^\nu_{\sigma\rho}\gamma^{\tau\mu}\tilde{g}^{\sigma\rho}$$

$$-G^{\sigma}_{\alpha\beta}\gamma^{\nu}_{\sigma\rho}\gamma^{\mu\rho}\tilde{g}^{\alpha\beta}. \quad \text{(B13)}$$

Adding (B12) and (B13), we easily prove the equation

$$\gamma^{\mu}_{\sigma\rho}A^{\sigma\rho\nu}+\gamma^{\nu}_{\sigma\rho}A^{\sigma\mu\rho}=-B^{\mu\nu}. \quad \text{(B14)}$$

Taking this into account, Eq. (B11) can be written as

$$\frac{\delta L_{g0}}{\delta\gamma_{\mu\nu}}=-\frac{1}{32\pi}D_{\sigma}A^{\sigma\mu\nu}. \quad \text{(B15)}$$

Using the equations

$$G^{\lambda}_{\tau\lambda}=\frac{1}{2}g^{\lambda\rho}D_{\tau}g_{\lambda\rho},\ D_{\tau}\sqrt{-g}=\sqrt{-g}G^{\lambda}_{\tau\lambda},$$

we find

$$G^{\sigma}_{\tau\beta}\tilde{g}^{\nu\beta}+G^{\nu}_{\tau\beta}\tilde{g}^{\sigma\beta}-G^{\lambda}_{\tau\lambda}\tilde{g}^{\sigma\nu}=-D_{\tau}\tilde{g}^{\nu\sigma},$$

$$G^{\sigma}_{\tau\beta}\tilde{g}^{\mu\beta}+G^{\mu}_{\tau\beta}\tilde{g}^{\sigma\beta}-G^{\lambda}_{\tau\lambda}\tilde{g}^{\sigma\mu}=-D_{\tau}\tilde{g}^{\mu\sigma},$$

$$G^{\nu}_{\tau\beta}\tilde{g}^{\mu\beta}+G^{\mu}_{\tau\beta}\tilde{g}^{\nu\beta}-G^{\lambda}_{\tau\lambda}\tilde{g}^{\mu\nu}=-D_{\tau}\tilde{g}^{\mu\nu}. \quad \text{(B16)}$$

Substituting these expressions into (B10), we obtain

$$A^{\sigma\mu\nu}=\gamma^{\tau\sigma}D_{\tau}\tilde{g}^{\mu\nu}+\gamma^{\mu\nu}D_{\tau}\tilde{g}^{\tau\sigma}-\gamma^{\tau\mu}D_{\tau}\tilde{g}^{\nu\sigma}-\gamma^{\tau\nu}D_{\tau}\tilde{g}^{\mu\sigma}.$$

Using this expression in (B15), we find

$$\frac{\delta^{*}L_{g0}}{\delta\gamma_{\mu\nu}}=\frac{1}{32\pi}J^{\mu\nu}, \quad \text{(B17)}$$

where

$$J^{\mu\nu}=-D_{\sigma}D_{\tau}(\gamma^{\tau\sigma}\tilde{g}^{\mu\nu}+\gamma^{\mu\nu}\tilde{g}^{\tau\sigma}-\gamma^{\tau\mu}\tilde{g}^{\nu\sigma}-\gamma^{\tau\nu}\tilde{g}^{\mu\sigma}).$$

From (B3) we have

$$\frac{\delta^{*}L_{gm}}{\delta\gamma_{\mu\nu}}=-\frac{m^2}{32\pi}(\tilde{g}^{\mu\nu}-\tilde{\gamma}^{\mu\nu})=-\frac{m^2}{32\pi}\tilde{\Phi}^{\mu\nu}. \quad \text{(B18)}$$

Therefore, taking into account (B1) and using (B17) and (B18), we find

$$\frac{\delta^{*}L_{g}}{\delta\gamma_{\mu\nu}}=\frac{1}{32\pi}(J^{\mu\nu}-m^2\tilde{\Phi}^{\mu\nu}), \quad \text{(B19)}$$

and, accordingly,

$$-2\frac{\delta^{*}L_{g}}{\delta\gamma_{\mu\nu}}=\frac{1}{16\pi}(-J^{\mu\nu}+m^2\tilde{\Phi}^{\mu\nu}). \quad \text{(B20)}$$

APPENDIX C

For any given Lagrangian density L, the variation of the action

$$S=\int L d^4x$$

under an infinitesimal change of coordinates will vanish. We shall calculate the variation of the action from the matter Lagrangian density L_M,

$$S_M=\int L_M(\tilde{g}^{\mu\nu},\Phi_A)d^4x,$$

and establish a strong identity. Under the coordinate transformation

$$x'^{\mu}=x^{\mu}+\xi^{\mu}(x), \quad \text{(C1)}$$

where $\xi^{\mu}(x)$ is an infinitesimal four-vector shift, the variation of the action under a coordinate transformation will be

$$\delta_c S_M=\int d^4x\left(\frac{\delta L_M}{\delta\tilde{g}^{\mu\nu}}\delta_L\tilde{g}^{\mu\nu}+\frac{\delta L_M}{\delta\Phi_A}\delta_L\Phi_A+\text{div}\right)=0, \quad \text{(C2)}$$

where div denotes the divergence terms, which are not important for us.

The Euler variation is defined in the usual manner:

$$\frac{\delta L}{\delta\Phi}\equiv\frac{\partial L}{\partial\Phi}-\partial_{\mu}\frac{\partial L}{\partial(\partial_{\mu}\Phi)}+\partial_{\mu}\partial_{\nu}\frac{\partial L}{\partial(\partial_{\mu}\partial_{\nu}\Phi)}.$$

The Lie variations $\delta_L\tilde{g}^{\mu\nu}$ and $\delta_L\Phi_A$ resulting from a change of coordinates are easily calculated by using the transformation law for $g^{\mu\nu}$ and Φ_A:

$$\delta_L\tilde{g}^{\mu\nu}=\tilde{g}^{\lambda\mu}D_{\lambda}\xi^{\nu}+\tilde{g}^{\lambda\nu}D_{\lambda}\xi^{\mu}-D_{\lambda}(\xi^{\lambda}\tilde{g}^{\mu\nu}),$$

$$\delta_L\Phi_A=-\xi^{\lambda}D_{\lambda}\Phi_A+F^{B;\lambda}_{A;\sigma}\Phi_B D_{\lambda}\xi^{\sigma}, \quad \text{(C3)}$$

where D_{λ} are the covariant derivatives in Minkowski space. Substituting these expressions into (C2) and integrating by parts, we obtain

$$\delta S_M=\int d^4x\left\{-\xi^{\lambda}\left[D_{\alpha}\left(2\frac{\delta L_M}{\delta\tilde{g}^{\lambda\nu}}\tilde{g}^{\alpha\nu}\right)-D_{\lambda}\left(\frac{\delta L_M}{\delta\tilde{g}^{\alpha\beta}}\right)\tilde{g}^{\alpha\beta}+D_{\sigma}\left(\frac{\delta L_M}{\delta\Phi_A}F^{B;\sigma}_{A;\lambda}\Phi_B\right)+\frac{\delta L_M}{\delta\Phi_A}D_{\lambda}\Phi_A\right]+\text{div}\right\}=0. \quad \text{(C4)}$$

Because the vector ξ^{λ} is arbitrary, from this equation we find a strong identity valid independently of whether or not the equations of motion for the fields are satisfied. It has the form

$$D_{\alpha}\left(2\frac{\delta L_M}{\delta\tilde{g}^{\lambda\nu}}\tilde{g}^{\alpha\nu}\right)-D_{\lambda}\left(\frac{\delta L_M}{\delta\tilde{g}^{\alpha\beta}}\right)\tilde{g}^{\alpha\beta}=-D_{\sigma}\left(\frac{\delta L_M}{\delta\Phi_A}F^{B;\sigma}_{A;\lambda}\Phi_B\right)-\frac{\delta L_M}{\delta\Phi_A}D_{\lambda}\Phi_A. \quad \text{(C5)}$$

We introduce the notation

$$T_{\mu\nu}=2\frac{\delta L_M}{\delta g^{\mu\nu}},\ T^{\mu\nu}=-2\frac{\delta L_M}{\delta g_{\mu\nu}}=g^{\mu\alpha}g^{\nu\beta}T_{\alpha\beta},$$

$$T=T^{\mu\nu}g_{\mu\nu},$$

$$\tilde{T}_{\mu\nu}=2\frac{\delta L_M}{\delta\tilde{g}^{\mu\nu}},\ \tilde{T}^{\mu\nu}=-2\frac{\delta L_M}{\delta\tilde{g}_{\mu\nu}}=\tilde{g}^{\mu\alpha}\tilde{g}^{\nu\beta}\tilde{T}_{\alpha\beta},$$

$$\tilde{T}=\tilde{T}^{\alpha\beta}\tilde{g}_{\alpha\beta}. \quad \text{(C6)}$$

With this notation, the left-hand side of the identity (C5) can be written as

$$D_{\alpha}(\tilde{T}_{\lambda\nu}\tilde{g}^{\alpha\nu})-\frac{1}{2}\tilde{g}^{\alpha\beta}D_{\lambda}\tilde{T}_{\alpha\beta}=\partial_{\alpha}(\tilde{T}_{\lambda\nu}\tilde{g}^{\alpha\nu})-\frac{1}{2}\tilde{g}^{\alpha\beta}\partial_{\lambda}\tilde{T}_{\alpha\beta}.$$

The right-hand side of this equation is easily brought to the form

$$\partial_\alpha(\tilde{T}_{\lambda\nu}\tilde{g}^{\alpha\nu})-\frac{1}{2}\tilde{g}^{\alpha\beta}\partial_\lambda\tilde{T}_{\alpha\beta}=\tilde{g}_{\lambda\nu}\nabla_\alpha\left(\tilde{T}^{\alpha\nu}-\frac{1}{2}\tilde{g}^{\alpha\nu}\tilde{T}\right). \tag{C7}$$

Here ∇_α is the covariant derivative in Riemannian space. Now we write the expression inside the derivative in terms of the tensor density $T^{\alpha\nu}$, using Eq. (A16):

$$\frac{\delta L_M}{\delta g_{\mu\nu}}=\frac{\delta L_M}{\delta\tilde{g}^{\alpha\beta}}\cdot\frac{\partial\tilde{g}^{\alpha\beta}}{\partial g_{\mu\nu}}, \tag{C8}$$

where

$$\frac{\partial\tilde{g}^{\alpha\beta}}{\partial g_{\mu\nu}}=\sqrt{-g}\,\frac{\partial g^{\alpha\beta}}{\partial g_{\mu\nu}}-\frac{1}{2\sqrt{-g}}\cdot\frac{\partial g}{\partial g_{\mu\nu}}\,g^{\alpha\beta}. \tag{C9}$$

Using the relations

$$g^{\alpha\beta}g_{\beta\sigma}=\delta^\alpha_\sigma,$$

we find

$$\frac{\partial g^{\alpha\beta}}{\partial g_{\mu\nu}}=-\frac{1}{2}(g^{\alpha\mu}g^{\nu\beta}+g^{\alpha\nu}g^{\mu\beta}). \tag{C10}$$

From the rule for differentiating a determinant we find

$$dg=gg^{\mu\nu}dg_{\mu\nu}, \tag{C11}$$

from which we have

$$\frac{\partial g}{\partial g_{\mu\nu}}=gg^{\mu\nu}. \tag{C12}$$

Substituting Eqs. (C10) and (C12) into (C9), we obtain

$$\frac{\partial\tilde{g}^{\alpha\beta}}{\partial g_{\mu\nu}}=-\frac{1}{2}\sqrt{-g}[g^{\alpha\mu}g^{\beta\nu}+g^{\alpha\nu}g^{\beta\mu}-g^{\mu\nu}g^{\alpha\beta}]. \tag{C13}$$

Using this expression in (C8), we find

$$\frac{\delta L_M}{\delta g_{\mu\nu}}=\sqrt{-g}\left(\frac{\delta L_M}{\delta\tilde{g}^{\alpha\beta}}g^{\alpha\mu}g^{\beta\nu}-\frac{1}{2}\frac{\delta L_M}{\delta\tilde{g}^{\alpha\beta}}g^{\alpha\beta}g^{\mu\nu}\right). \tag{C14}$$

With the notation (C6) this expression can be written as

$$\sqrt{-g}T^{\mu\nu}=\tilde{T}^{\mu\nu}-\frac{1}{2}\tilde{g}^{\mu\nu}\tilde{T}. \tag{C15}$$

On the basis of this equation, the strong identity (C5) taking into account (C7) takes the form

$$g_{\lambda\nu}\nabla_\alpha T^{\alpha\nu}=-D_\sigma\left(\frac{\delta L_M}{\delta\Phi_A}F^{B;\sigma}_{A;\lambda}\Phi_B\right)-\frac{\delta L_M}{\delta\Phi_A}D_A\Phi_A,$$

or

$$\nabla_\alpha T^\alpha_\lambda=-D_\sigma\left(\frac{\delta L_M}{\delta\Phi_A}F^{B;\sigma}_{A;\lambda}\Phi_B\right)-\frac{\delta L_M}{\delta\Phi_A}D_\lambda\Phi_A, \tag{C16}$$

APPENDIX D

The rank-two curvature tensor $R_{\mu\nu}$ can be written as

$$\begin{aligned}R_{\mu\nu}=\frac{1}{2}\Big[&\tilde{g}^{\alpha\beta}\left(\tilde{g}_{\mu\kappa}\tilde{g}_{\nu\rho}-\frac{1}{2}\tilde{g}_{\mu\nu}\tilde{g}_{\kappa\rho}\right)D_\alpha D_\beta\tilde{g}^{\kappa\rho}\\&-\tilde{g}_{\nu\rho}D_\kappa D_\mu\tilde{g}^{\kappa\rho}-\tilde{g}_{\mu\kappa}D_\nu D_\rho\tilde{g}^{\kappa\rho}\Big]\\&+\frac{1}{2}\tilde{g}_{\nu\omega}\tilde{g}_{\rho\tau}D_\mu\tilde{g}^{\kappa\rho}D_\kappa\tilde{g}^{\omega\tau}\\&+\frac{1}{2}\tilde{g}_{\mu\omega}\tilde{g}_{\rho\tau}D_\nu\tilde{g}^{\kappa\rho}D_\kappa\tilde{g}^{\omega\tau}\\&-\frac{1}{2}\tilde{g}_{\mu\omega}\tilde{g}_{\nu\rho}D_\tau\tilde{g}^{\omega\kappa}D_\kappa\tilde{g}^{\rho\tau}-\frac{1}{4}\Big(\tilde{g}_{\omega\rho}\tilde{g}_{\kappa\tau}\\&-\frac{1}{2}\tilde{g}_{\omega\tau}\tilde{g}_{\kappa\rho}\Big)D_\mu\tilde{g}^{\kappa\rho}D_\nu\tilde{g}^{\omega\tau}-\frac{1}{2}\tilde{g}^{\alpha\beta}\tilde{g}_{\rho\tau}\Big(\tilde{g}_{\mu\kappa}\tilde{g}_{\nu\omega}\\&-\frac{1}{2}\tilde{g}_{\mu\nu}\tilde{g}_{\kappa\omega}\Big)D_\alpha\tilde{g}^{\kappa\rho}D_\beta\tilde{g}^{\omega\tau}.\end{aligned} \tag{D1}$$

Raising the indices by multiplying by $g^{\epsilon\mu}g^{\lambda\nu}$ and using the equation

$$D_\mu\tilde{g}^{\mu\nu}=0, \tag{D2}$$

we obtain

$$\begin{aligned}-gR^{\epsilon\lambda}=&\frac{1}{2}\tilde{g}^{\alpha\beta}D_\alpha D_\beta\tilde{g}^{\epsilon\lambda}-\frac{1}{4}\tilde{g}^{\epsilon\lambda}\tilde{g}_{\kappa\rho}\tilde{g}^{\alpha\beta}D_\alpha D_\beta\tilde{g}^{\kappa\rho}\\&+\frac{1}{2}\tilde{g}_{\rho\tau}\tilde{g}^{\epsilon\mu}D_\mu\tilde{g}^{\kappa\beta}D_\kappa\tilde{g}^{\lambda\tau}\\&+\frac{1}{2}\tilde{g}_{\rho\tau}\tilde{g}^{\lambda\nu}D_\nu\tilde{g}^{\kappa\rho}D_\kappa\tilde{g}^{\epsilon\tau}-\frac{1}{2}D_\tau\tilde{g}^{\epsilon\kappa}D_\kappa\tilde{g}^{\lambda\tau}\\&-\frac{1}{4}\Big(\tilde{g}_{\omega\rho}\tilde{g}_{\kappa\tau}\\&-\frac{1}{2}\tilde{g}_{\omega\tau}\tilde{g}_{\kappa\rho}\Big)\tilde{g}^{\epsilon\mu}\tilde{g}^{\lambda\nu}D_\mu\tilde{g}^{\kappa\rho}D_\nu\tilde{g}^{\omega\tau}\\&-\frac{1}{2}\tilde{g}_{\mu\tau}\tilde{g}^{\alpha\beta}D_\alpha\tilde{g}^{\epsilon\rho}D_\beta\tilde{g}^{\lambda\tau}\\&+\frac{1}{4}\tilde{g}_{\rho\tau}\tilde{g}^{\epsilon\lambda}\tilde{g}_{\kappa\omega}\tilde{g}^{\alpha\beta}D_\alpha\tilde{g}^{\kappa\rho}D_\beta\tilde{g}^{\omega\tau}.\end{aligned} \tag{D3}$$

From this we find

$$\begin{aligned}-gR=&\frac{1}{2}g_{\epsilon\lambda}\tilde{g}^{\alpha\beta}D_\alpha D_\beta\tilde{g}^{\epsilon\lambda}-g_{\kappa\rho}\tilde{g}^{\alpha\beta}D_\alpha D_\beta\tilde{g}^{\kappa\rho}\\&+\frac{1}{2}g_{\rho\tau}D_\mu\tilde{g}^{\kappa\rho}D_\kappa\tilde{g}^{\mu\tau}+\frac{1}{2}g_{\rho\tau}D_\epsilon\tilde{g}^{\kappa\rho}D_\kappa\tilde{g}^{\epsilon\tau}\\&-\frac{1}{2}g_{\epsilon\lambda}D_\tau\tilde{g}^{\epsilon\kappa}D_\kappa\tilde{g}^{\lambda\tau}-\frac{1}{4}\Big(\tilde{g}_{\omega\rho}\tilde{g}_{\kappa\tau}\\&-\frac{1}{2}\tilde{g}_{\omega\tau}\tilde{g}_{\kappa\rho}\Big)\sqrt{-g}\tilde{g}^{\mu\nu}D_\mu\tilde{g}^{\kappa\rho}D_\nu\tilde{g}^{\omega\tau}\\&-\frac{1}{2}\tilde{g}_{\rho\tau}\tilde{g}^{\alpha\beta}g_{\epsilon\lambda}D_\alpha\tilde{g}^{\epsilon\beta}D_\beta\tilde{g}^{\lambda\tau}\\&+\tilde{g}_{\rho\tau}g_{\kappa\omega}\tilde{g}^{\alpha\beta}D_\alpha\tilde{g}^{\kappa\rho}D_\beta\tilde{g}^{\omega\tau}.\end{aligned} \tag{D4}$$

Using Eqs. (D3) and (D4), we obtain

$$-g\left(R^{\epsilon\lambda}-\frac{1}{2}g^{\epsilon\lambda}R\right)$$

$$=-\frac{1}{2}\left\{\frac{1}{2}\left(\tilde{g}_{\nu\sigma}\tilde{g}_{\tau\kappa}-\frac{1}{2}\tilde{g}_{\nu\kappa}\tilde{g}_{\tau\sigma}\right)\tilde{g}^{\epsilon\alpha}\tilde{g}^{\lambda\beta}D_{\alpha}\tilde{g}^{\sigma\tau}D_{\beta}\tilde{g}^{\nu\kappa}\right.$$

$$-\frac{1}{4}\tilde{g}^{\epsilon\lambda}\tilde{g}^{\alpha\beta}\left(\tilde{g}_{\nu\sigma}\tilde{g}_{\tau\kappa}-\frac{1}{2}\tilde{g}_{\nu\kappa}\tilde{g}_{\tau\sigma}\right)D_{\alpha}\tilde{g}^{\tau\sigma}D_{\beta}\tilde{g}^{\nu\kappa}$$

$$+\tilde{g}^{\alpha\beta}\tilde{g}_{\sigma\tau}D_{\alpha}\tilde{g}^{\epsilon\tau}D_{\beta}\tilde{g}^{\lambda\sigma}-\tilde{g}^{\epsilon\beta}\tilde{g}_{\tau\sigma}D_{\alpha}\tilde{g}_{\lambda\alpha}D_{\beta}\tilde{g}^{\alpha\tau}$$

$$-\tilde{g}^{\lambda\alpha}\tilde{g}_{\tau\sigma}D_{\alpha}\tilde{g}^{\beta\sigma}D_{\beta}\tilde{g}^{\epsilon\tau}+\frac{1}{2}\tilde{g}^{\epsilon\lambda}\tilde{g}_{\tau\sigma}D_{\alpha}\tilde{g}^{\beta\sigma}D_{\beta}\tilde{g}^{\alpha\tau}$$

$$\left.+D_{\alpha}\tilde{g}^{\epsilon\beta}D_{\beta}\tilde{g}^{\lambda\alpha}-\tilde{g}^{\alpha\beta}D_{\alpha}D_{\beta}\tilde{g}^{\epsilon\lambda}\right\}. \tag{D5}$$

It should be specially stressed that we have used Eq. (D2) in finding Eq. (D5). Substituting (D5) into Eq. (76) and writing the resulting equation in the form (93), we find the expression for $-16\pi g\tau_g^{\epsilon\lambda}$:

$$-16\pi g\tau_g^{\epsilon\lambda}=\frac{1}{2}\left(\tilde{g}^{\epsilon\alpha}\tilde{g}^{\lambda\beta}-\frac{1}{2}\tilde{g}^{\epsilon\lambda}\tilde{g}^{\alpha\beta}\right)\left(\tilde{g}_{\nu\sigma}\tilde{g}_{\tau\mu}\right.$$

$$\left.-\frac{1}{2}\tilde{g}_{\tau\sigma}\tilde{g}_{\nu\mu}\right)D_{\alpha}\tilde{\Phi}^{\tau\sigma}D_{\beta}\tilde{\Phi}^{\mu\nu}$$

$$+\tilde{g}^{\alpha\beta}\tilde{g}_{\tau\sigma}D_{\alpha}\tilde{\Phi}^{\epsilon\tau}D_{\beta}\tilde{\Phi}^{\lambda\sigma}$$

$$-\tilde{g}^{\epsilon\beta}\tilde{g}_{\tau\sigma}D_{\alpha}\tilde{\Phi}^{\lambda\sigma}D_{\beta}\tilde{\Phi}^{\alpha\tau}$$

$$-\tilde{g}^{\lambda\alpha}\tilde{g}_{\tau\sigma}D_{\alpha}\tilde{\Phi}^{\beta\sigma}D_{\beta}\tilde{\Phi}^{\epsilon\tau}$$

$$+\frac{1}{2}\tilde{g}^{\epsilon\lambda}\tilde{g}_{\tau\sigma}D_{\alpha}\tilde{\Phi}^{\sigma\beta}D_{\beta}\tilde{\Phi}^{\alpha\tau}+D_{\alpha}\tilde{\Phi}^{\epsilon\beta}D_{\beta}\tilde{\Phi}^{\lambda\alpha}$$

$$-\tilde{\Phi}^{\alpha\beta}D_{\alpha}D_{\beta}\tilde{\Phi}^{\epsilon\lambda}-m^2\left(\sqrt{-g}\tilde{g}^{\epsilon\lambda}-\sqrt{-g}\tilde{\Phi}^{\epsilon\lambda}\right.$$

$$\left.+\tilde{g}^{\epsilon\alpha}-\tilde{g}^{\lambda\beta}\gamma_{\alpha\beta}-\frac{1}{2}\tilde{g}^{\epsilon\lambda}\tilde{g}^{\alpha\beta}\gamma_{\alpha\beta}\right). \tag{D6}$$

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