Low-energy expansions in nuclear physics

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The results of recent experimental and theoretical studies of low-energy nucleon–nucleon and nucleon–deuteron collisions are studied and compared with the older data. Special attention is devoted to two problems: that of the extrapolation of the phase shifts, amplitudes, and cross sections of such collisions to experimentally inaccessible energies, and that of separating the contributions of nuclear and electromagnetic interactions in the parameters of low-energy elastic scattering. Various classical and recent approaches to the solution of these problems are analyzed. A method is developed for constructing low-energy expansions which permits information on the nuclear interaction to be extracted from the measured phase shifts in systems of two and three nucleons or nuclei. The role of electromagnetic corrections to the Coulomb interaction in such scattering reactions can also be studied using this method.
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1. INTRODUCTION

Knowledge of the explicit low-energy dependence of the auxiliary functions and physical observables characterizing the collision of quantum-mechanical objects aids in solving many applied and theoretical problems. An example is the problem of extrapolating such observables measured at the accessible low energies of these collisions to even lower energies which are experimentally inaccessible for technical reasons. Another example is one of the most important theoretical problems, namely, that of choosing the functional form and parameters of the interaction in the low-energy limit on the basis of the available experimental data.

The main goal of the present study is to show that the construction of the explicit low-energy asymptotes of the phase shifts in systems of two and three particles and the determination of the coefficients of these asymptotes remains one of the interesting but so far incompletely solved problems of modern scattering theory. One of these coefficients is the scattering length. The concept of the scattering length in the two-body problem, the methods of determining it, and the various approaches to the construction of the low-energy asymptotes in this problem are the subject of Sec. 2 below. In Sec. 3 we discuss the results of experimental and theoretical studies of the scattering lengths, the effective ranges, and the low-energy behavior of the nucleon–nucleon (NN) phase shifts. In Sec. 4 we analyze some attempts to construct the low-energy asymptotes in the three-particle scattering problem ($3\to3$) and ($2\to2$). In Sec. 5 we compare the results of various theoretical studies of nucleon–deuteron (Nd) scattering for the Nd collision energy near zero.

As a rule, in this review we shall use the terminology and notation of the well known mathematical handbooks,[1–4] the books on the theory of potential scattering,[5–10] and the monographs devoted to the methods of phase functions[11,12] and hyperspherical harmonics,[13,14] the theory of the NN interaction,[15–17] and neutron physics.[18,19] All terms denoting relative concepts and all the compact notation used in this review will be defined.

The theory of low-energy expansions has been under construction for a long time. To best describe its current status, in this review we cite both studies which have become classical, and studies which have been published relatively recently. In order to show how the fundamental concepts of the method of low-energy asymptotes have been developed and how the fundamental problems of this method have been solved, the studies devoted to each such concept or problem are discussed in the chronological order in which they appeared.

The present review makes no pretense to completeness, and is essentially a natural extension of our earlier review.[20] There we focused on the analysis of the various methods of estimating the limits of the energy ranges in which short-range (V^s) or long-range (V^l) potentials can or cannot be neglected in the correct description of low-energy collisions within the effectively two-particle problem of scattering by the superposition $V^{cls}=V^c+V^l+V^s$ containing the Coulomb potential V^c. However, these estimates still are not sufficient for extrapolating the phase shifts and cross sections of nucleon–nucleus scattering to experimentally inaccessible low energies. To completely solve this extrapolation problem it is necessary also to know the low-energy expansions: the asymptotic expressions describing the analytic dependence of the extrapolating quantities as the momentum k or the collision energy $E=O(k^2)$ tends to zero.

The various methods of constructing such expansions are analyzed in the present review. The main goal of the analysis is to answer two questions: how to separate the contributions from the nuclear short-range and electromagnetic long-range interactions to the phase shifts, cross sections, and parameters of low-energy NN or Nd scattering, and how then to determine the parameters characterizing only the nuclear interaction.

The solutions of these problems have been sought for a long time. The results which have been obtained are distributed among numerous original studies and are described in the important books[5–10] using different approaches and even different conceptual points of view. We therefore think it is important and interesting to summarize, compare, and evaluate the probable correctness of the results obtained at various

times by various techniques from a single point of view and with a single approach based on the use of low-energy expansions constructed using the method of phase functions. In the present review we attempt to perform the difficult task of systematizing and analyzing the many results.

A recent and detailed review[21] has been devoted to the generalized potential description of the interaction of the lightest nuclei in dd, p^3He, d^3He, and d^4He scattering at relatively low energies. This allowed us to restrict ourselves in the present review to analysis of low-energy potential NN and Nd scattering, but in return forced us to pay special attention to the rigorous mathematical methods of solving the two- and three-body scattering problems with two-body interactions in the form of superpositions of short- and long-range potentials. These methods are applicable for the reliable derivation of the low-energy expansions of the characteristics of elastic scattering in systems of two and three nuclei whose interactions can be described by suitable effective two-body potentials. As a rule, such potentials contain long-range components which fundamentally change the nature of the collision at sufficiently low energies. The concept of "low energy" is rather arbitrary. Here, when we speak of low energies of NN or Nd collisions, we will mean the energy range $0 \leqslant E_{\text{lab}} < 10$ MeV, in which there are few experimental data on such collisions and these occasionally are inconsistent with each other. It is this important fact that motivated the author to restrict the discussion to the known results of studies of NN and Nd collisions only at the low energies defined above, while carefully checking these results for reliability.

2. LOW-ENERGY EXPANSIONS IN THE TWO-BODY PROBLEM

To discuss these expansions, we should first explain what we mean by long- and short-range potentials.

In modern scattering theory,[5,8] a central potential $V^s(r)$ which is finite [$V^s(r) \equiv 0$ for $r > b$, where $0 < b < \infty$] or falls off with increasing distance r at least as fast as the Yukawa potential V^Y,

$$V^s(r) \sim V^Y(r) \text{ or } V^s(r) = o(V^Y(r)), \quad r \to \infty; \tag{1}$$

$$V^Y(r) \equiv V_0^Y \exp(-r/r^Y)(r^Y/r), \quad V_0^Y = \text{const},$$

$$r^Y = \text{const} > 0, \tag{2}$$

is considered to be a short-range interaction, while a potential which falls off more slowly than, for example, an inverse integer power of the distance,

$$V^l(r) \sim V_0^d r^{-d}, \quad V_0^d = \text{const}, \quad d = 1,2,3,..., \quad r \to \infty, \tag{3}$$

is considered long-range. However, the concept of long- or short-range potential is a relative one:[22] a particular potential can be short-range in some cases but long-range in others. Therefore, the range of the potential must be decided in each particular case. In all the situations discussed below, this refinement is not necessary, and so potentials with the asymptotes (1) or (3) will be referred to as short- and long-range, respectively.

For compactness of notation, all functions and quantities characterizing scattering by a potential V^a or by a superposition of potentials $V^{ca} = V^c + V^a$ will be labeled by the superscript a or ca. In the next section we shall systematically discuss several cases: scattering by the potentials $V^s(a=s)$, $V^l(a=l)$, the superposition of these potentials $V^{ls}(a=ls)$, and the superpositions V^{ca} with $a = s,l,ls$. The meanings of all the other subscripts remain the same and coincide with their original definitions in the text. The symbols γ, $\Gamma(x)$, $B(x,y)$, and $\hat{\delta}_{nm}$ everywhere denote the Euler constant, the gamma and beta functions, and the Kronecker delta, while everywhere R is the Bohr radius, $\rho \equiv kr$, and $\eta \equiv 1/2kR$.

2.1. The concept of scattering length and methods of calculating it

The concept of scattering length was first introduced[5,6] for scattering by a potential V^s with rapidly decreasing asymptote (1). The phase shift $\delta_l^s(k)$ for scattering by such a potential in a two-particle quantum state $|k,l\rangle$ with conserved relative momentum k and angular momentum l has a finite limit

$$a_l^s \equiv -\lim_{k \to 0} \tan \delta_l^s(k)/k^{2l+1} \tag{4}$$

(Ref. 5), and the corresponding effective range

$$K_l^s(E) \equiv k^{2l+1} \cot \delta_l^s(k) = k^{2l+1}/\mathscr{K}_l(E) \tag{5}$$

is an entire[7] function of k^2 and has the low-energy ($E \equiv \hbar^2 k^2/2\mu \to 0$) asymptote[6]

$$K_l^s(E) \sim -1/a_l^s + k^2 r_{0l}^s/2 - k^4 r_{0l}^s P_l^s, \quad k \to 0. \tag{6}$$

The function (5) is simply related to the element $\mathscr{K}_l$ of the Heitler matrix[7] (or the $\mathscr{K}$ matrix[5]) in the basis of partial waves $|k,l\rangle$. The finite quantities a_l^s, r_{0l}^s, and P_l^s appearing in (6) are called the scattering parameters[15–17] in the state $|k,l\rangle$, respectively, the scattering length, the effective range, and the shape parameter. Of the known methods[5,6] of calculating the scattering parameters of a short-range potential, the simplest and most convenient algorithm is the method of phase functions. Let us recall it.

In one of the nonlinear versions of the method of phase functions, the quantity $\tan \delta_l^s(k)$ is defined as the $r \to \infty$ limit of the phase function $t_l^s(r;k)$, equal to zero for $r = 0$ and satisfying a nonlinear equation for $r > 0$ (Ref. 12):

$$\partial_r t_l^a(r;k) = -k^{-1} V^a(r)[j_l(\rho) - t_l^a(r;k) n_l(\rho)]^2, \tag{7}$$

where $a = s$ and $\rho \equiv kr$. If the Riccati–Bessel and Riccati–Neumann functions in (7) are replaced by their corresponding expansions[1]

$$j_\lambda(\rho) = k^{\lambda+1} \sum_{n=0}^{\infty} k^{2n} j_{\lambda n}(r),$$

$$j_{\lambda n}(r) \equiv (r/2)^{2n+\lambda+1} \frac{\sqrt{\pi}(-1)^n}{\Gamma(n+1)\Gamma(n+\lambda+3/2)};$$

$$n_l(\rho) = k^{-l} \sum_{n=0}^{\infty} k^{2n} n_{ln}(r),$$

$$n_{ln}(r)\equiv(-1)^{\lambda+1}(2/r)j_{-l,n}(r) \tag{8}$$

with $\lambda=l$ and the ansatz

$$t_l^a(r;k)=k^{2l+1}\sum_{n=0}^{\infty}k^{2n}t_{ln}^a(r) \tag{9}$$

with $a=s$ is used, it becomes possible to separate the parameter k from the argument r and obtain the recursion equations ($n=0,1,\ldots$)

$$\partial_r t_{ln}^a(r)=-V^a(r)\sum_{m'+m=n}Q_{lm'}^a(r)Q_{lm}^a(r),\quad r>0,$$

$$Q_{ln}^a\equiv j_{ln}(r)-\sum_{m'+m=n}t_{lm'}^a(r)n_{lm}(r) \tag{10}$$

for the new unknown functions $t_{ln}^s(r)$, which are zero at $r=0$. The energy dependence and recursive nature of Eqs. (10) greatly simplify the numerical solution of these equations and allow the scattering parameters to be found very accurately from the expressions

$$a_l^a=-t_{l0}^a(\infty),\quad r_{l0}^a=-2t_{l1}^a(\infty)/(a_l^a)^2,$$

$$P_l^a=t_{l2}^a(\infty)/[(a_l^a)^2r_{l0}^a]+a_l^a r_{l0}^a/4, \tag{11}$$

which follow from (5), (6), and (9) in this case ($a=s$).

To avoid possible misunderstandings of terminology, it will help to explain the following phrases frequently encountered in the literature on low-energy potential scattering: "the scattering length exists (is defined)" and "the scattering length does not exist (is not defined)." For example, if for $k=0$ a *particular* effective-range function is not (is) equal to zero, it is said that the scattering length exists (does not exist).

In 1949 Blatt and Jackson[23] proved that for any $l=0,1,\ldots$ the function (5) is an infinite series in even powers of k for a class of potentials larger than (1), namely, for potentials which fall off more rapidly than r^{-d} for any $d>1$. The same authors, studying scattering by long-range potentials (3) not belonging to this class, showed that the function (5) is finite for $k=0$ if $2l+3<d$, and the next term falling off for $k\to0$ is quadratic in k if $2l+5<d$. In other words, for scattering by the potential (3) in the state $|k,l\rangle$, the scattering length is defined for $2l+3<d$, and the effective range exists if $2l+5<d$. For a given l the scattering length can exist while the effective range might not exist. An example of this situation is S-wave scattering ($l=0$) by a polarized potential having the asymptote (3) with the exponent $d=4$:

$$V^p(r)\sim V_0^p r^{-4},\quad V_0^p=-\alpha_e/2|R|=\text{const},\quad r\to\infty. \tag{12}$$

In this case the low-energy asymptote of the effective range given by (5) differs functionally from the asymptote (6), because it has the form[24,25]

$$K_0^p(E)\equiv k\cot\delta_0^p(k)=-\frac{1}{a_0^p}-\frac{\pi V_0^p}{3(a_0^p)^2}|k|$$

$$-\frac{V_0^p}{3a_0^p}k^2\ln\left(\frac{-V_0^pk^2}{16}\right)+O(k^2), \tag{13}$$

where $\delta_0^p(k)$ is the phase shift and a_0^p is the scattering length for the potential (12). The presence of terms linear and logarithmic in k in the sum (13) implies that the effective range r_{00}^s is infinite, i.e., it does not exist. For $d=4$ the condition of Blatt and Jackson $2l+3<d$ is satisfied only for $l=0$, and for all $l>0$ the scattering length and, accordingly, the effective range are not defined. The same conclusion follows from the asymptotic relation uniform in $l>0$ (Ref. 24):

$$\tan\delta_l^p(k)=-\pi V_0^pk^2/[(4l^2-1)(2l+3)]$$

$$+O(k^{2l+1}),\quad k\to0,$$

according to which all phase shifts $\delta_l^p(k)$ with $l>0$ are linear in energy.

Continuing our comparison of the expansions (6) and (13), we see that the slow falloff of the polarization potential (12) compared with any short-range potential (1) generates terms in the low-energy asymptote of the phase shift $\delta_0^p(k)$ which are nonanalytic in k. In 1963 Levy and Keller[26] noticed that if $a=l$ is used in Eqs. (7) and (10), i.e., if V^s is replaced by V^l, the functions $t_{ln}^l(r)$ with subscript n obeying the condition $2n<d-2l-3$ will be finite for $r=\infty$. Therefore, the representation of the function $t_l^l(r;k)$ by an infinite series (9) with $a=l$ becomes meaningless for $r=\infty$, and the expansion correct for all r has the form

$$t_l^l(r;k)=k^{2l+1}\sum_{n=0}^{m}k^{2n}t_{ln}^l(r)+\beta_l(r;k), \tag{14}$$

where m is the maximum possible value of n for which the inequality $2n<d-2l-3$ holds, and the nontrivial term β_l falls off for $k\to0$ more rapidly than $k^{2(l+m)+1}$ but more slowly than $k^{2(l+m+1)+1}$. Substituting (14) into (7), Levy and Keller derived Eq. (10) for functions $t_{ln}^l(r)$ with $n\leqslant m$ bounded for all r and the equation for the function $\beta_l(r;k)$. Studying the latter equation, they proved that the following asymptotic expression is valid for $k\to0$ and any power $d\geqslant3$ of falloff of the potential (3):

$$\beta_l(r;k)\sim-k^{-1}\int_0^r drV^l(r)\left\{[j_l(\rho)]^2\right.$$

$$\left.-\rho^{2l+2}\sum_{n=0}^{m}\tau_{ln}\rho^{2n}\right\},$$

$$\tau_{ln}\equiv\sum_{m'+m=n}d_{lm'}d_{lm},$$

$$d_{ln}\equiv2^l(-1)^n\frac{\Gamma(l+n+1)}{\Gamma(n+1)\Gamma(2l+2n+2)}. \tag{15}$$

Replacing $\beta_l(\infty;k)$ in (14) by the asymptote found for the integral (15), Levy and Keller obtained the first explicit low-energy ($k\to0$) expansions of the tangents of the phase shifts $\delta_l^d(k)$ for the potentials (3) through the first terms nonanalytic in k:

$$\tan\delta_l^d(k)\sim-2^{-d}\pi V_0^d k|k|^{d-3}\frac{\Gamma(d-1)\Gamma(l+3/2-d/2)}{\Gamma^2(d/2)\Gamma(l+1/2+d/2)},$$

$$3<d<2l+3;$$

$$\tan\delta_l^d(k)\sim -V_0^d k|k|^{2l}\ln|k|/[(2l+1)!!]^2,\quad d=2l+3;$$

$$\tan\delta_l^d(k)\sim k^{2l+1}\sum_{n=0}^{m}k^{2n}t_{ln}^l(\infty)-V_0^d k|k|^{d-3}$$
$$\times\int_0^\infty dr r^{-d}\left\{[j_l(r)]^2-r^{2l+2}\sum_{n=0}^{m}\tau_{ln}r^{2n}\right\},$$

$$2l+3<d\neq 5,7,\ldots;$$

$$\tan\delta_l^d(k)\sim k^{2l+1}\sum_{n=0}^{(d-5)/2}k^{2n}t_{ln}^l(\infty)$$
$$+V_0^d\tau_{l,(d-3)/2-l}k|k|^{d-3}\ln|k|,$$

$$2l+3<d=5,7,\ldots. \tag{16}$$

According to (16), all such terms are products of the function $\ln|k|$ and integer powers of k. The next, more rapidly decreasing, terms of the low-energy asymptotes of the functions $\tan\delta_l^d(k)$ are unknown for arbitrary constants V_0^d of the asymptotes (3). If we make the additional assumption that the V_0^d are small, these terms can easily be found in the approximation linear in the small parameter V_0^d by the method of Hinckelman and Sprush.[27] They showed that the phase shift $\delta_0^{ps}(k)$ of the superposition $V^{ps}\equiv V^p+V^s$ of non-overlapping potentials (1) and (12)

$$V^p(r)\equiv 0,\quad r<b;\quad V^s(r)\equiv 0,\quad r\geqslant b;\quad 0<b<\infty, \tag{17}$$

has the following asymptote for $k, V_0^p\to 0$:

$$\tan\delta_0^{ps}(k)=-a_0^{ps}k+(V_0^pk^2/3)[\pi+4a_0^sk\ln|2kb|$$
$$-(3/2)\tau_0k]+(V_0^p/3)(a_0^s)^2k^4[2r_{00}^sk$$
$$\times\ln|2kb|-\pi]+O((V_0^p)^2,k^5). \tag{18}$$

It turned out that the scattering length a_0^{ps} of the superposition (17) and the coefficient τ_0 are related to the scattering length a_0^s and effective range r_{00}^s for the potential V^s cut off at the point $r=b$ as

$$a_0^{ps}=a_0^s+V_0^p(1-\tau+\tau^2/3)/b,\quad \tau\equiv a_0^s/b,$$
$$\tau_0=(a_0^s)^2r_{00}^s/V_0^p+(2b/3)\cdot[1+2(11/3-2\gamma)\tau-(3$$
$$+r_{00}^s(1-\tau)/b)\tau^2]. \tag{19}$$

Equations (18) and (19) allow the scattering parameters for the potential V^s to be found from the measured constant V_0^p and the phase shift $\delta_0^{ps}(k)$ for scattering by the superposition (17). Why are these and other low-energy relations between the characteristics of scattering by the superposition $V^{ls}=V^l+V^s$ and the quantities characterizing the short-range component V^s of particular interest? How can such relations be derived in the general case, and for what functions of the collision momentum can they be derived? Let us begin our discussion of these problems with a remark.

According to (4) and (5), the phase shift $\delta_l^s(k)$ falls off for $k\to 0$ as a power of the momentum k: $-a_l^sk^{2l+1}$. Therefore, the scattering length a_l^s of a single short-range potential V^s has a simple geometrical interpretation: for $k=0$ it is the coefficient of the slope of the graph of the function $\tan\delta^{c,s}(k)$ relative to the horizontal axis, on which $-k^{2l+1}$ is plotted. Apparently, a desire to preserve the analogous meaning in the case of scattering by superpositions V^{ls} motivated the introduction of *modified* scattering parameters in the 1940s. Let us explain this technique for the example of one of the most important superpositions of this type in nuclear physics.

In the low-energy limit, the total phase shift $\delta_l^{cs}(k)$ for the superposition $V^{cs}\equiv V^c+V^s$, where $V^c=1/rR$ is the repulsive Coulomb potential with Bohr radius $R>0$, grows without bound,[1]

$$\delta_l^{cs}(k)\sim\delta_l^c(k)\sim\eta(\ln\eta-1)+\pi/4,\quad n\equiv 1/2kR,$$
$$k\to 0,$$

like the Coulomb phase shift $\delta_l^c(k)\equiv\arg\Gamma(1+l+i\eta)$, and so the analog

$$a_l^{cs}\equiv-\lim_{k\to 0}\tan\delta_l^{cs}(k)/k^{2l+1}=-\lim_{k\to 0}(2Rk^{2l+2})^{-1}$$

of the limit (4) is $-\infty$. This is the first reason why the concept of scattering length requires modification in this case ($a=cs$). The second reason has a clear physical interpretation and amounts to the following. The total phase shift $\delta_l^{cs}(k)$ describes the combined influence of the potentials V^c and V^s on the scattering. In an experimentally studied collision, the interaction V^s in the superposition V^{cs} is, as a rule, unknown. In order to extract information about it, it is necessary to subtract the Coulomb phase shift from the measured phase shift $\delta_l^{cs}(k)$ and to define the effective-range function such that it is a series in integer powers of k^2.

It is this method of constructing the low-energy scattering theory for nuclear particles of identical charge ($R>0$) that was chosen by Breit, Condon, and Present[28] for constructing the first theory of proton–proton (pp) scattering, and it has been used up to the present time. These authors introduced the concept of the Coulomb–nuclear phase shift

$$\delta_l^{c,s}(k)=\delta_l^{cs}(k)-\delta_l^c(k), \tag{20}$$

as the contribution from the nuclear pp potential V^s to the total phase shift $\delta_l^{cs}(k)$ for the superposition V^{cs}, and showed that a suitably defined Coulomb–nuclear effective-range function

$$K_l^{c,s}(E)\equiv[k^lC_l(\eta)]^2[k\cot\delta_l^{c,s}(k)+h^c(\eta)] \tag{21}$$

must contain the Coulomb factors[1]

$$C_l(\eta)\equiv 2^l\exp(-\pi\eta/2)|\Gamma(l+1+i\eta)|/\Gamma(2l+2);$$
$$h^c(\eta)\equiv h(\eta)/RC_0^2(\eta),\quad h(\eta)\equiv\mathrm{Re}\,\psi(i\eta)-\ln\eta, \tag{22}$$

in order to obtain the desired type of asymptote:

$$K_l^{c,s}(E)\sim -1/a_l^{c,s}+k^2r_{0l}^{c,s}/2-k^4r_{0l}^{c,s}P_l^{c,s},\quad k\to 0. \tag{23}$$

Since the expansions (23) and (6) have the same functional dependence on k^2, the coefficients

$$a_l^{c,s}(k)\equiv-\lim_{k\to 0}\tan\delta_l^{c,s}(k)/k[k^lC_l(\eta)]^2, \tag{24}$$

$r_{0l}^{c,s}$, and $P_l^{c,s}$ have come to be called the Coulomb–nuclear (or modified) scattering length, effective range, and, accordingly, shape parameter. Along with the term "Coulomb–

nuclear'' scattering parameter,[16] the term ''nuclear–Coulomb'' scattering parameter is also used.[9] The physical meaning of the parameters is most accurately reflected by the former term, meaning a ''parameter of nuclear scattering in a Coulomb field.''

Interest in the problem of determining the Coulomb–nuclear scattering parameters revived in 1949, owing to the study by Landau and Smorodinskiĭ.[29] They found a simple approximate expression

$$1/a_0^{c,s} \approx 1/a_0^s + R^{-1}[\ln(r_s/R) + 2\gamma - 0.824], \tag{25}$$

relating the Coulomb–nuclear scattering length $a_0^{c,s}$ for the superposition V^{cs} to the scattering length a_0^s for the short-range component V^s of this superposition, the range of this component r_s, and the Bohr radius R.

It would be impossible to mention even briefly all the many studies following that of Landau and Smorodinskiĭ and devoted to the problem of determining the Coulomb–nuclear scattering parameters, but it is worth classifying the known methods and comparing their simplicity and reliability. For this we need to recall several formulas.

The total amplitude of elastic ($k=k'$) scattering $f(\theta;k)$ by a central potential $V(r)$ from an initial state $|\vec{k}\rangle$ to a final state $|\vec{k}'\rangle$ depends on one scattering angle $\theta \equiv \arccos[(\vec{k}\cdot\vec{k}')/kk']$ and is an infinite series[5]

$$f(\theta;k) = \sum_{l=0}^{\infty} f_l(k)[(2l+1)P_l(\cos\theta)] \tag{26}$$

in Legendre polynomials $P_l(\cos\theta)$ and partial-wave amplitudes

$$f_l(k) = (2ik)^{-1}[\exp(2i\delta_l(k)) - 1]. \tag{27}$$

If $V = V^{ca}$, the splitting

$$\delta_l^{ca}(k) = \delta_l^c(k) + \delta_l^{c,a}(k) \tag{28}$$

of each ($l=0,1,\ldots$) phase shift $\delta_l(k) \equiv \delta_l^{ca}(k)$ into the Coulomb shift $\delta_l^c(k)$ and the non-Coulomb shift $\delta_l^{c,a}(k)$ generates the splitting

$$f^{ca}(\theta;k) = f^c(\theta;k) + f^{c,a}(\theta;k),$$

$$f_l^{ca}(k) = f_l^c(k) + f_l^{c,a}(k) \tag{29}$$

of the total (26) and partial-wave (27) scattering amplitudes into the total,[9]

$$f^c(\theta;k) = -R^{-1}[2k\sin(\theta/2)]^{-2}\exp(2i(\delta_0^c(k) - \eta\ln\sin(\theta/2))), \tag{30}$$

and partial-wave $[f_l^c(k)]$ amplitudes of scattering by the Coulomb potential, and the corresponding amplitudes

$$f^{c,a}(\theta;k) = \sum_{l=0}^{\infty} f_l^{c,a}(k)[(2l+1)P_l(\cos\theta)], \tag{31}$$

$$f_l^{c,a}(k) = (2ik)^{-1}\exp(2i\delta_l^c(k))[\exp(2i\delta_l^{c,a}(k)) - 1], \tag{32}$$

generated by the potential V^a in scattering in a Coulomb field. The contributions $\sigma_l^{c,a}$, $d\sigma^{c,a}/d\theta$, and $\sigma^{c,a}$ of this potential to the partial-wave, differential, and total cross sections for scattering by the superposition V^{ca} are defined as

$$\sigma_l^{c,a}(E) \equiv 4\pi|f_l^{c,a}(k)|^2, \tag{33}$$

$$d\sigma^{c,a}(\theta;E)/d\theta \equiv |f^{c,a}(\theta;k)|^2 \sin\theta, \tag{34}$$

$$\sigma^{c,a}(E) \equiv \sum_{l=0}^{\infty} \sigma_l^{c,a}(E) = 2\pi\int_0^{\pi} d\theta(d\sigma^{c,a}(\theta;E)/d\theta). \tag{35}$$

In the traditional approach,[30–34] the Coulomb–nuclear ($a=s$) amplitudes (31) and (32) are given by the matrix elements

$$f^{c,s}(\theta;k) = -\frac{\mu}{2\pi}\langle\vec{k}|T^{c,s}(z)|\vec{k}'\rangle,$$

$$f_l^{c,s}(k) = -\frac{\mu}{2\pi}T_l^{c,s}(k,k;z) \tag{36}$$

of the total ($T^{c,s}$) and partial-wave ($T_l^{c,s}$) Coulomb–nuclear operators effecting the passage to the energy shell ($k=k'$, $z=E+i0=\hbar^2k^2/2\mu+i0$), and the determination of the Coulomb–nuclear parameters reduces to the calculation of the limits for $z\equiv E+i0\to 0+$ of the function $T_l^{c,s}(k,k;z)$ and its derivatives with respect to the energy. These limits are taken from positive energies, where the matrix $\langle\vec{k}|T^{c,s}(z)|\vec{k}'\rangle$ and all its partial-wave components $T_l^{c,s}(k,k';z)$ are complex functions. Therefore, to determine the real values (of the Coulomb–nuclear scattering parameters) it is necessary first to find the complex solution $T_l^{c,s}(k,k';z)$ of the partial-wave Lippmann–Schwinger equation. This is the first reason why the traditional method is not so easy to realize computationally, especially in the case of Coulomb attraction ($V^c<0$), when all the Coulomb functions $T_l^c(k,k';z)$ have an infinite number of poles corresponding to Coulomb bound states and accumulating at the point $z=0$. The second reason why the traditional method is not very effective is that it is necessary to calculate the limits mentioned above. Numerically, this reduces to the problem of extrapolating to the point $E=0$ the partial-wave amplitude (32), (36), and its derivatives with respect to the energy found for the decreasing sequence $E_1>E_2>\ldots>E_N>0$ of positive energies. As is well known,[3] any extrapolation method leads to an additional loss of accuracy.

In Ref. 35 Navrotskiĭ showed that the limits determining the Coulomb–nuclear parameters can be taken from negative energies. To start with, he used the representation[36]

$$T^{c,s}(z) = [1+T^c(z)G_0(z)]\tilde{T}^{c,s}(z)[1+T^c(z)G_0(z)], \tag{37}$$

containing the free Green function G_0, the auxiliary Coulomb–nuclear operator $\tilde{T}^{c,s}$, and the explicitly known Coulomb matrix[37]

$$\langle\vec{k}|T^c(z)|\vec{k}'\rangle = \frac{2\pi\hbar^2}{\mu R|\vec{k}-\vec{k}'|^2}[1+\omega^{-1}F_{i\eta}(\tau) - \omega^{-1}F_{i\eta}(\tau^{-1})],$$

$$\omega\equiv-\left[1+\frac{(k^2-p^2)(k'^2-p^2)}{p^2|\vec{k}-\vec{k}'|^2}\right]^{1/2},\quad \tau\equiv\frac{\omega+1}{\omega-1};$$

$$p^2\equiv 2\mu z/\hbar^2, \tag{38}$$

where $F_{i\eta}(x)\equiv {}_2F_1(1,i\eta;1+i\eta;x)$ is the hypergeometric function.[1] The operator $\tilde{T}^{c,s}$, in contrast to the operator $T^{c,s}$, satisfies the equation

$$\tilde{T}^{c,s}(z)=V^s+V^sG^c(z)\tilde{T}^{c,s}(z) \tag{39}$$

with compact kernel containing V^c only via the Coulomb Green function:

$$G^c(z)=G_0(z)[1+T^c(z)G_0(z)]. \tag{40}$$

The above reformulation (37)–(40) of the original problem (the Lippmann–Schwinger equation for the operator $T^{c,s}$ with $V^c<0$) was followed by analysis of the kernel of Eq. (39) in Ref. 35. This analysis was begun by proving the representation of the Coulomb matrix (38) as the sum

$$\langle\vec{k}|T^c(z)|\vec{k}'\rangle=\langle\vec{k}|w(z)|\vec{k}'\rangle+i\coth(\pi\eta)\langle\vec{k}|u(z)|\vec{k}'\rangle, \tag{41}$$

where the functions $\langle\vec{k}|w(z)|\vec{k}'\rangle$ and $\langle\vec{k}|u(z)|\vec{k}'\rangle$ are continuous in energy at the point $z=0$, and the factor $\coth(\pi\eta)$ explicitly describes all the singularities which are poles in the energy and correspond to Coulomb bound states. The representation (41) allowed separation of the singular (G^u) and smooth (G^w) parts from the function (40):

$$G^c=G^u+i\coth(\pi\eta)G^w,\quad G^u\equiv G_0uG_0,$$

$$G^w\equiv G_0(1+wG_0), \tag{42}$$

and then reduction of Eq. (39) to two equations, the equation

$$\tilde{T}^{c,s}(z)=T^w(z)+i\coth(\pi\eta)T^w(z)G^u(z)\tilde{T}^{c,s}(z) \tag{43}$$

with explicitly isolated Coulomb singularities, and the equation

$$T^w(z)=V^s+V^sG^w(z)T^w(z) \tag{44}$$

for the auxiliary operator T^w, whose matrix has no singularities of Coulomb origin and is continuous at the point $z=0$. Further constructions were made only for zero l. It turned out that the Coulomb–nuclear S-wave scattering length is equal to the zero-energy limit,

$$a_0^{c,s}=\frac{\mu}{2\pi}T_{00}(0), \tag{45}$$

of the matrix element of the nonsingular part

$$T_{kk'}(z)\equiv\langle\phi_k|T^{\rm reg}(z)|\phi_{k'}\rangle,$$

$$T^{\rm reg}(z)=\begin{cases}\tilde{T}^{c,s}(z), & V^c>0,\\ T^w(z), & V^c<0\end{cases} \tag{46}$$

of the operator $\tilde{T}^{c,s}$ taken between the renormalized Coulomb functions $\phi_k(r)\equiv F_0(\rho,\eta)/\rho C_0(\eta)$. In the case $V^c>0$ the function $T_{kk'}(z)$ thus obtained obeys the equation

$$T_{kk'}(z)=V^s_{kk'}+\int_0^\infty\frac{dpp^2C^2(\eta_p)}{2\pi^2(z-E_p)}V^s_{kp}T_{pk'}(z),$$

$$V^s_{kk'}\equiv\langle\phi_k|V^s|\phi_{k'}\rangle,\quad \eta_p\equiv 1/2pR,\quad E_p\equiv\hbar^2p^2/2\mu, \tag{47}$$

and the effective range is given by

$$r_{00}^{c,s}=2R/3+[1/2\pi(a^{c,s})^2][\lim_{z\to0}\partial_zT_{00}(z)$$

$$+4\mu\lim_{k\to0}\partial_{k^2}T_{k0}(0)]. \tag{48}$$

Thus, in contrast to the traditional approach, in the Navrotskiĭ method *all* the singularities of Coulomb origin are *sequentially* isolated, and the limits (45) and (48) determining the Coulomb–nuclear parameters can be taken from negative energies, when the function (46) is real. These two advantages simplify the calculation of the Coulomb–nuclear parameters. However, the scheme of calculations using Eqs. (37)–(48) requires finding many auxiliary operators and is therefore rather complicated. This is mainly due to the choice of working in momentum space, in which the Coulomb matrix (38) is not only singular, but is also a very complicated function. Starting from the Schrödinger equation in coordinate space and using simple objects, the functions F_l and G_l, instead of this matrix, it is possible to construct more reliable algorithms for determining the Coulomb–nuclear parameters. Let us recall two such approaches.

After applying the method of evolution in the coupling constant[38] to the problem of Schrödinger scattering with $l=0$ and interaction V^{cs}, Kirzhnits and Pen'kov[39] found the following integral representation for the difference of the functions (5) and (21):

$$\Delta(E)\equiv K_0^s(E)-K_0^{c,s}(E)=R^{-1}\operatorname{cosec}^2\delta_0^s(k)$$

$$\times\operatorname{Im}\int_0^\infty dr\,\ln(r/|R|)\partial_r[u_0^{+s}(r;k)u_0^{-s}(r;k)$$

$$+i[u_0^{+s}(r;k)]^2-\sin(\rho+\delta_0^s(k))\exp(i(\rho$$

$$+\delta_0^s(k)))], \tag{49}$$

which allows the S-wave Coulomb–nuclear parameters to be determined after calculation of the phase shift $\delta_0^s(k)$ and the regular (u_0^{+s}) and irregular (u_0^{-s}) functions describing scattering by the potential V^s. For $k=0$ the representation (49) takes the form

$$1/a_0^{c,s}-1/a_0^s=\Delta(0) \tag{50}$$

and allows the Coulomb–nuclear scattering length $a_0^{c,s}$ to be found from the known scattering length a_0^s and the integral $\Delta(0)$. Equation (50) is consistent with (25), but, in contrast to the latter, does not depend on the range r_s of the potential V^s. The parameters $r_{00}^{c,s}$ and $P_0^{c,s}$ of the expansion (23) are expressed in terms of the first and second derivatives of the function $\Delta(E)$ with respect to the energy E at the point $E=0$.

The method proposed by Babikov[40] is simpler, because there it is not necessary to calculate any derivatives or limits with respect to the energy. First, Babikov reformulated the original Schrödinger problem with the potential V^{cs} in such a way that the tangent of the phase shift $\delta_l^{c,s}(k)$ was equal to

the $r\to\infty$ limit of the auxiliary function $t_l^{c,s}(r;k)$, equal to zero at $r=0$ and for $r>0$ satisfying the nonlinear equation

$$\partial_r t_l^{c,s}(r;k)=-k^{-1}V^s(r)[F_l(\rho,\eta)+t^{c,s}(r;k)G_l(\rho,\eta)]^2. \tag{51}$$

Then, using two successive substitutions

$$t_l^{c,s}(r;k)=(2l+1)k[k^lC_l(\eta)]^2\sum_{n=0}^{\infty}k^{2n}A_{ln}^{c,s}(r;h(\eta)),$$

$$k\to 0; \tag{52}$$

$$A_{ln}^{c,s}(r;h(\eta))=t_{ln}^{c,s}(r)/\tau_l(r;h(\eta)),\quad n=0,1,$$

$$\tau_l(r;h(\eta))\equiv 1+t_{l0}^{c,s}(r)h(\eta)/R,$$

$$A_{l2}^{c,s}=b_l^{c,s}(r;h(\eta))/\tau_l(r;h(\eta)),\quad t_{l2}^{c,s}(r)=b_l^{c,s}(r;0) \tag{53}$$

and the Bessel–Clifford representation[1] for F_l and G_l, Babikov reduced Eq. (51) to a recursion chain of rather complicated but energy-independent equations for the functions $t_{ln}^{c,s}(r)$, $n=0,1,2$, in terms of which the Coulomb–nuclear parameters with $l=0$ were expressed by Eq. (11) with $a=c,s$.

We note that the functions (53) of the low-energy expansion (52) depend parametrically on the nonanalytic function $h(\eta)$ [see (22)]. Babikov was not able to isolate this dependence explicitly for a simple reason: the original equation (51) that he used is nonlinear.

One can attempt to develop an even more reliable and simple method of finding the Coulomb–nuclear parameters, using the following scheme. First, the original scattering problem for the superposition V^{cs} is reduced to a system of *linear* equations for certain auxiliary functions in terms of which the Coulomb–nuclear phase shift is expressed. Then all the nonanalytic factors and terms of Coulomb origin are explicitly isolated from the desired auxiliary functions, and the remaining parts of these functions are represented as series in known functions of the momentum parameter k and new unknown functions of the distance argument r. Finally, this representation is used to reduce the equations to energy-independent equations for the new unknown functions, in terms of which all the Coulomb–nuclear parameters are then expressed.

The method developed along these lines by the present author is briefly described in the next section. It allows the unique determination of both the Coulomb–nuclear scattering parameters for the superposition V^{cs}, and the nuclear scattering parameters for the superposition V^{cls} with $V^c>0$.

The problem of determining the nuclear scattering length for the superposition V^{cls} turned out to be rather complicated and took a long time to solve. The total phase shift $\delta_l^{cls}(k)$ of this superposition can be split in two different ways:

$$\delta_l^{cls}(k)=\delta_l^{(c)}(k)+\delta_l^{c,ls}(k); \tag{54}$$

$$\delta_l^{cls}(k)=\delta_l^{cl}(k)+\delta_l^{cl,s}(k),\quad \delta_l^{cl}(k)=\delta_l^{c}(k)+\delta_l^{c,l}(k), \tag{55}$$

thereby isolating the components with different physical meaning. The phase shift $\delta_l^{c,ls}(k)$ characterizes the combined effect of two interactions (V^l and V^s) on scattering in the Coulomb field V^c, and the phase shift $\delta_l^{cl,s}(k)$ describes the contribution of a single nuclear potential V^s to the total phase shift $\delta_l^{cls}(k)$. It is the low-energy asymptote of this contribution which can be used to judge the structure of the nuclear interaction in the low-energy limit. Only the total phase shift and not its components can be determined experimentally. Then, two problems arise: how to isolate the component $\delta_l^{cl,s}(k)$ from the known total phase shift, and how to construct the low-energy expansion of this component. So far, these incompletely solved problems of potential scattering remain rather general and very important from both the theoretical and the applied point of view. Such problems unavoidably arise in the theoretical study of the role of a given potential V^s describing the nuclear interaction in molecular, atomic, and nuclear collisions occurring at energies which are superlow on the nuclear scale. They also arise in the solution of the inverse problem, i.e., the determination of the shape and parameters of the nuclear forces V^s from the experimental observables characterizing such collisions.

The main features of the low-energy scattering of a charged particle by a target with non-pointlike distribution of electric and (or) magnetic charges (an atom, ion, molecule, or nucleus) can be predicted using an effective two-particle approximation in which the particle and the target are treated as pointlike, but interacting via a suitable effective potential containing terms of various origin which effectively take into account the non-pointlike nature of the particle and the target. In a fairly general case the effective two-particle interaction is represented as the superposition V^{cls} of the Coulomb potential V^c, the long-range electromagnetic correction V^l to this potential, and a short-range potential V^s describing the nuclear interaction in particular. The electromagnetic corrections have the asymptotes (3) with exponent $d=2,3,4,\ldots$.

For example, the interaction of the nucleon magnetic moment $\vec{\mu}=2\vec{s}\mu_N$ with the electric charge Ze of the target has the asymptotic form[19]

$$V_{ls}^m\sim V_{ls,0}^m(\vec{l}\cdot\vec{s})r^{-3},\quad V_{ls,0}^m\equiv\mu_N Ze^2/2m_Nc^2,\quad r\to\infty, \tag{56}$$

where $\vec{s}$ is the nucleon spin operator, and m_N and μ_N are the nucleon mass and magnetic moment.

For $r\to\infty$ the noncontact part of the interaction of the magnetic moments $\vec{\mu}_1$ and $\vec{\mu}_2$ of two nucleons is given by[19]

$$V^m(r)\sim-\frac{m_N}{\hbar^2}[3(\vec{\mu}_1\cdot\vec{r})(\vec{\mu}_2\cdot\vec{r})r^{-2}-(\vec{\mu}_1\cdot\vec{\mu}_2)]r^{-3}. \tag{57}$$

The polarization interaction of a charged particle with the deuteron electric moment[15] has the asymptote (12), where $\alpha_e=0.007$ F is the deuteron electric polarizability constant[41] and R is the Bohr radius of the particle+deuteron system.

A fundamental theoretical study of the low-energy scattering by the superposition V^{cls} with V^l of the type (3) was performed in Refs. 42 and 43. Berger and Spruch[42] showed that for $k\to 0$, $V^c>0$, $d\geq 3$, and any $l=0,1,\ldots$ the higher-

order terms of the asymptotes of the components $\delta_l^{c,ls}(k)$ and $\delta_l^{c,l}(k)$ of the splittings (54) and (55) are given by the Born integral

$$\delta_l^{c,ls}(k)\sim\delta_l^{c,l}(k)\sim-k^{-1}\int_b^\infty drV^l(r)F_l^2(\rho,\eta),$$

$$0<b<\infty, \tag{58}$$

from which we find the asymptotes *uniform* in l,

$$\delta_l^{c,d}(k)\sim\frac{-V_0^d k}{2l(l+1)(2l+1)}[2l+1-2\eta\chi_l(\eta)],$$

$$d=3; \tag{59}$$

$$\delta_l^{c,d}(k)\sim\frac{-V_0^d}{2R^2}\frac{2[3\eta^2+l(l+1)]\chi_l(\eta)-(6l+3)\eta}{l(l+1)(4l^2-1)(2l+3)},$$

$$d=4; \tag{60}$$

$$\chi_l(\eta)\equiv\pi/2-\operatorname{Im}\psi(l+1+i\eta), \tag{61}$$

and the asymptotic expression valid for all $l\ll\eta$ and $d\geq3$:

$$\delta_l^{c,ls}(k)\sim\delta_l^{c,l}(k)\sim(-V_0^d/2R^{1-d})k^{2d-3}B(d-1,1/2),$$

$$k\to0. \tag{62}$$

As noted by Berger and Spruch, owing to the slow, i.e., power-law, falloff of the phase shift $\delta_l^{c,ls}(k)$ (62) with k, the analog

$$a_l^{c,ls}\equiv-\lim_{k\to0}\tan\delta_l^{c,ls}(k)/k[k^lC_l(\eta)]^2$$

$$=\text{const}\cdot\lim_{k\to0}k^{2(d-l-1)}\exp(\pi/kR) \tag{63}$$

of the Coulomb–nuclear scattering length (24) is infinite. The problem of determining the *nuclear* scattering length of the superposition V^{cls} thus arose. To solve it, Berger, Snodgrass, and Spruch[43] used the splitting (55). Introducing a suitable analog

$$K_l^{cl,s}(E)\equiv[k^lC_l^{cl}(\eta)]^2[k\cot\delta_l^{cl,s}(k)+h^{cl}(\eta)] \tag{64}$$

of the Coulomb–nuclear effective-range function (21) and proving that

$$K_l^{cl,s}(E)\sim-1/a_l^{cl,s}+k^2r_{0l}^{cl,s}/2-k^4r_{0l}^{cl,s}P_l^{cl,s},\quad k\to0, \tag{65}$$

they defined the modified scattering length

$$a_l^{cl,s}\equiv-\lim_{k\to0}\tan\delta_l^{cl,s}(k)/k[k^lC_l^{cl}(\eta)]^2, \tag{66}$$

the effective range $r_{0l}^{cl,s}$, and the shape parameter $P_l^{cl,s}$ as the quantities appearing in the coefficients of the asymptote (65). These parameters characterize the effect of the single potential V^s on scattering by the superposition V^{cls} and are therefore referred to as the nuclear parameters of scattering in the field V^{cl}.

Although the problem of determining the nuclear scattering length was solved from the conceptual point of view, the integral representations obtained in Refs. 42 and 43 for the coefficients of the expansion (65) and the nonanalytic functions $C_l^{cl}(\eta)$ and $h_l^{cl}(\eta)$ of the parameter η turned out to be too awkward.

The discussion of the scattering length of the superposition V^{cls} resumed only in 1984. It began with the report[44] of Kvitsinsky and Merkuriev, who showed in a later study[45] that owing to the pd polarization interaction (12), the doublet ($^2a_{pd}$) and quartet ($^4a_{pd}$) pd scattering lengths determined by substitution of the corresponding pd phase shift into (63) instead of the phase shift $\delta_0^{c,ls}(k)$ turn out to be infinite. After this was noticed, polarization effects in low-energy nuclear collisions were studied intensively by various methods. The main results of these studies are discussed in Ref. 20.

The most complete theoretical investigation of scattering by the superposition V^{cls} in the case $V^c>0$, $V^l=V^p$, and $l=0$ was performed by Bencze *et al.*,[46] using the original linear version[47] of the method of phase functions. The basic conclusion of this study is that, at least in the calculation of the nuclear scattering length in Eq. (66), it is possible to replace $C_l^{cl}(\eta)$ by $C_l(\eta)$ and to use the resulting expression

$$a_l^{cl,s}\approx-\lim_{k\to0}\tan\delta_l^{cl,s}(k)/k[k^lC_l(\eta)]^2 \tag{67}$$

as an approximation for this scattering length.

The theory constructed by Bencze *et al.* is far from complete, because the authors did not succeed in answering the two important questions of how to construct the functions $C_l^{cl}(\eta)$ and $h^{cl}(\eta)$ from (64) and how to calculate the quantities $r_{0l}^{cl,s}$ and $P_l^{cl,s}$ from (65).

The present author recently found the answers to these questions. The approach will be described in detail in a separate article. A schematic description of it for the case $V^c>0$ is given in the following section.

2.2. Low-energy expansions for scattering by the superposition V^{cls}

Following Ref. 48, we introduce the dimensionless argument $x\equiv r/R$ and parameter $q\equiv kR$, and we write the Schrödinger scattering problem in the form of differential equations

$$[\partial_x^2-l(l+1)x^{-2}-V^c(x)-V(x)+q^2]u_l^\pm(x;q)=0,$$

$$x\in\mathscr{R}^+, \tag{68}$$

for the desired regular (u_l^+) and irregular (u_l^-) wave functions subject to the boundary conditions

$$u_l^\pm(x;q)=O(x^{\pm(l+1/2)+1/2}),\quad x\to0, \tag{69}$$

$$u_l^\pm(x;q)\sim\sin[\rho-\eta\ln2\rho-(2l+1\mp1)\pi/4+\delta_l^c(q)$$

$$+\delta_l(q)],\quad x\to\infty. \tag{70}$$

Here $\mathscr{R}^+\equiv\{x:0\leq x\leq\infty\}$, $\rho=kr=qx$, $\eta=1/2q$, and $\delta_l(q)$ is the phase shift generated by the potential V in the Coulomb field $V^c=1/x$. The only restriction

$$I_l(b,x)\equiv(2\pi/(l+1))^{1/2}\int_b^x dt\,t|V(t)|<\infty,$$

$$0\leq b\leq x\leq\infty, \tag{71}$$

imposed on the potential V is quite general.[8] It is satisfied by most of the short-range potentials used in nuclear physics for describing the strong interaction,[16,17] all the long-range corrections (3) with $d>2$, and superpositions V^{ls} of such nuclear and long-range potentials. Below, when necessary, we shall specify precisely the class of potentials V (V^s, V^l, or V^{ls}) for which a particular low-energy ($q\to 0$) expansion is valid. Most of the expansions are asymptotic[4] infinite series

$$S(x;q)=N(q)\sum_{n=0}^{\infty} q^{2n}S_n(x), \quad x\in\mathscr{R}^+, \quad q\to 0, \tag{72}$$

with normalization constant $N(q)$ and argument x separated from the parameter q tending to zero. We shall split the series (72) as

$$S=S^{(M)}+{}^{(M)}S, \quad S^{(M)}(x;q)\equiv N(q)\sum_{n=0}^{M} q^{2n}S_n(x), \tag{73}$$

where $S^{(M)}$ is a finite subsum and ${}^{(M)}S$ is the remainder.

A fairly simple method of studying the *various* solutions of the Schrödinger equation (68) is the linear version of the method of phase functions,[11] which is essentially equivalent to the method of varying "constant" coefficients.[2] Calogero[47] formulated this version in terms of amplitude functions (the "constant" coefficients) for calculating the regular solution u_l^+ of the problem (68)–(70) and the phase shift $\delta_l(q)$. Recently, this original version was supplemented by a method of constructing the irregular solution[48] developed for studying artificial and physical resonance states[49] and combined with the method of complex rotation of the coordinate[50] for calculating the Jost function.[51]

Let us extend the linear version for constructing the low-energy expansions of functions related to the problem (68)–(71) in the form (72) and (73). First we recall how this problem was restated in Ref. 48.

The desired wave functions are written as

$$u_l^+(x;q)=N_l^+(q)U_l^+(x;q),$$

$$u_l^-(x;q)=\widetilde{u}_l^-(x;q)+\alpha_l(q)u_l^+(x;q),$$

$$\widetilde{u}_l^-(x;q)=N_l^-(q)U_l^-(x;q); \tag{74}$$

$$U_l^{\pm}(x;q)\equiv c_l^{\pm}(x;q)F_l(\rho,\eta)+s_l^{\pm}(x;q)G_l(\rho,\eta). \tag{75}$$

The cotangent of the phase shift $\delta_l(q)$ and the normalization factors $N_l^{\pm}(q)$ and $\alpha_l(q)$ ensuring the asymptote (70) are defined as the limits for $x\to\infty$ of the corresponding functions:

$$\cot\delta_l(x;q)\equiv c_l^+(x;q)/s_l^+(x;q), \tag{76}$$

$$N_l^{\pm}(x;q)\equiv[(c_l^+(x;q))^2+(s_l^+(x;q))^2]^{\mp 1/2}, \tag{77}$$

$$\alpha_l(x;q)\equiv -c_l^+(x;q)c_l^-(x;q)-s_l^+(x;q)s_l^-(x;q). \tag{78}$$

Two uncoupled systems of equations are derived for the amplitude functions (the first system is for c_l^+ and s_l^+, and the second is for c_l^- and s_l^-):

$$\partial_x\begin{Bmatrix} c_l^{\pm}(x;q)\\ s_l^{\pm}(x;q)\end{Bmatrix}=q^{-1}V(x)U_l^{\pm}(x;q)\begin{Bmatrix} +G_l(\rho,\eta)\\ -F_l(\rho,\eta)\end{Bmatrix} \tag{79}$$

with the corresponding initial ($x\to 0$) conditions:

$$\begin{Bmatrix} c_l^+(x;q)\\ s_l^+(x;q)\end{Bmatrix}\sim\begin{Bmatrix} 1\\ 0\end{Bmatrix}+q^{-1}\int_0^z dtV(t)F_l(\rho',\eta)\times\begin{Bmatrix} +G_l(\rho',\eta)\\ -F_l(\rho',\eta)\end{Bmatrix}; \tag{80}$$

$$c_l^-(x;q)\sim c_l^-(x_0;q)+q^{-1}\int_{x_0}^x dtV(t)G_l^2(\rho',\eta),$$

$$s_l^-(x;q)\sim 1-q^{-1}\int_0^x dtV(t)F_l(\rho',\eta)G_l(\rho',\eta), \tag{81}$$

ensuring the asymptotic behavior (69) of the functions (74). In (80) and (81), $\rho'\equiv tq$; if the first integral (81) exists for $x_0=0$, it is assumed that $x_0=0$ and $c_l^-(0;q)=0$. Otherwise, the parameter x_0 is chosen so that two inequalities are satisfied: $x_0>x$ and $x_0q\ll 1$.

Therefore, for constructing the solutions $u_l^{\pm}$ of the original problem (68)–(71) using Eqs. (74) and (75), it is necessary to solve the problem (79)–(81) and then calculate the limits of each function (76)–(78) for $x\to\infty$.

Let us turn to the construction of low-energy expansions of the type (72) for *all* the functions associated with the problem (68)–(71).

First we consider the case $V\equiv 0$, when $u_l^+=F_l$ and $u_l^-=G_l$. We rewrite the Lambert expression [Eq. (3.25) in Ref. 52] as

$$G_l(\rho,\eta)=\widetilde{G}_l(\rho,\eta)+h^c(q)F_l(\rho,\eta),$$

$$\widetilde{G}_l(\rho,\eta)\equiv\Theta_l(x,q)/C_l^2(q), \tag{82}$$

$$C_l(q)\equiv q^lC_l(\eta), \quad h^c(q)\equiv h(\eta)/qC_0(q), \tag{83}$$

where $C_l(\eta)$ and $h(\eta)$ are given by (22), and Θ_l is an entire function of q^2.

The well known Bessel–Clifford series [see Eqs. (14.4.1)–(14.4.4) in Ref. 1] contain polynomials $b_n(\eta)$ in the parameter k^2 and modified Bessel functions $I_n(z)$ and $K_n(z)$ in the variable $z\equiv 2x^{1/2}$. Combining the terms in these series with identical powers of the parameter k^2, we obtain series of the type (72):

$$F_l(\rho,\eta)=qC_l(q)\sum_{n=0}^{\infty} q^{2n}f_{ln}(x),$$

$$\widetilde{G}_l(\rho,\eta)=C_l^{-1}(q)\sum_{n=0}^{\infty} q^{2n}g_{ln}(x), \tag{84}$$

$$\begin{Bmatrix} 2f_{ln}(x)\\ (2l+1)g_{ln}(x)\end{Bmatrix}\equiv 2^{-2n}\sum_{m=2n}^{3n} a_{nm}z^{m+1}\times\begin{Bmatrix} I_{2l+m+1}(z)\\ (-1)^{-m}K_{2l+m+1}(z)\end{Bmatrix}. \tag{85}$$

Here a_{nm} are energy-independent coefficients obeying recursion chains ($m=2n,\ldots,3n$ for each $n=1,2,\ldots$) of equations

$$2ma_{nm}+2(2l+m)a_{n-1,m-2}+a_{n-1,m-3}=0, \tag{86}$$

with $a_{00}\equiv 1$ and $a_{nm}\equiv 0$ if $n>0$ and $m<2n$ or $m>3n$.

As is well known,[4] the estimates[1]

$$^{(M)}F_l(\rho,\eta)=O(q^{2M+3}C_l(q)),$$

$$^{(M)}\tilde{G}_l(\rho,\eta)=O(q^{2M+2}/C_l(q)) \tag{87}$$

of the remainder terms in the splittings (73) of the series (84) are uniform in ρ if

$$\rho\ll\rho_l^c\equiv\eta[1+(1+l(l+1)/\eta)^{1/2}]. \tag{88}$$

Let us construct series of the type (72) for the amplitude functions. For this we substitute these functions in the form of the desired series

$$c_l^{\pm}(x;q)=\tilde{c}_l^{\pm}(x;q)-h^c(q)s_l^{\pm}(x;q), \tag{89}$$

$$\begin{Bmatrix}\tilde{c}_l^{\pm}(x;q)\\ s_l^{\pm}(x;q)\end{Bmatrix}=\begin{Bmatrix}(qC_l^2(q))^{(-1\pm1)/2}\\ (qC_l^2(q))^{(+1\pm1)/2}\end{Bmatrix}\sum_{n=0}^{\infty}q^{2n}\begin{Bmatrix}c_{ln}^{\pm}(x)\\ s_{ln}^{\pm}(x)\end{Bmatrix} \tag{90}$$

into (75) and (79)–(81), and we represent the functions F_l and G_l in the form (82)–(86). Separating q from x in the resulting expressions, we find the representations

$$U_l^{\pm}(x;q)=q^{(1\pm1)/2}C_l^{\pm1}(q)\sum_{n=0}^{\infty}q^{2n}U_{ln}^{\pm}(x),$$

$$U_{ln}^{\pm}(x)\equiv\sum_{m'+m=n}[c_{lm'}^{\pm}(x)f_{lm}(x)+s_{lm'}^{\pm}(x)g_{lm}(x)], \tag{91}$$

the infinite ($n=0,1,\ldots$) chain of energy-independent equations

$$\partial_x\begin{Bmatrix}c_{ln}^{\pm}(x)\\ s_{ln}^{\pm}(x)\end{Bmatrix}=V(x)\sum_{m'+m=n}U_{lm'}^{\pm}(x)\begin{Bmatrix}+g_{lm}(x)\\ -f_{lm}(x)\end{Bmatrix} \tag{92}$$

for the desired functions $c_{ln}^{\pm}$ and $s_{ln}^{\pm}$, and their asymptotes for $x\to0$:

$$\begin{Bmatrix}c_{ln}^{+}(x)\\ s_{ln}^{+}(x)\end{Bmatrix}\sim\begin{Bmatrix}\hat{\delta}_{n0}\\ 0\end{Bmatrix}+\sum_{m'+m=n}\int_0^x dtV(t)f_{lm'}(t)\times\begin{Bmatrix}+g_{lm}(t)\\ -f_{lm}(t)\end{Bmatrix}; \tag{93}$$

$$c_{ln}^{-}(x)\sim c_{ln}^{-}(x_0)+\sum_{m'+m=n}\int_{x_0}^x dtV(t)g_{lm'}(t)g_{lm}(t),$$

$$s_{ln}^{-}(x)\sim\hat{\delta}_{n0}-\sum_{m'+m=n}\int_0^x dtV(t)f_{lm'}(t)g_{lm}(t). \tag{94}$$

If the first integral in (94) exists for $x_0=0$, it is assumed that $x_0=0$ and $c_{ln}^{-}(0)=0$. Otherwise, it is assumed that $x<x_0\ll1$.

Substituting $U_l^{\pm}$ in the form (91) into (74), we obtain the desired series:

$$u_l^{+}(x;q)=qC_l(q)N_l^{+}(q)\sum_{n=0}^{\infty}q^{2n}U_{ln}^{+}(x),$$

$$\tilde{u}_l^{-}(x;q)=(C_l(q)N_l^{+}(q))^{-1}\sum_{n=0}^{\infty}q^{2n}U_{ln}^{-}(x). \tag{95}$$

Using Eq. (87), it can be shown that the following estimates are valid with the condition (88) for the remainder terms of the splittings (73) of the series (90) and (95):

$$\begin{Bmatrix}^{(M)}\tilde{c}_l^{\pm}(x;q)\\ ^{(M)}s_l^{\pm}(x;q)\end{Bmatrix}=O\left(q^{2M+2}\begin{Bmatrix}(qC_l^2(q))^{(-1\pm1)/2}\\ (qC_l^2(q))^{(+1\pm1)/2}\end{Bmatrix}\right), \tag{96}$$

$$^{(M)}u_l^{\pm}(x;q)=O(q^{2M+3}C_l(q)),$$

$$^{(M)}\tilde{u}_l^{-}(x;q)=O(q^{2M+2}/C_l(q)). \tag{97}$$

Let us now turn to the construction of the effective-range functions. We shall introduce two interrelated superscripts a and a' in order that the expressions take a more compact form. We use $A^{ca}(x,q)$ with the superscript ca, where $a=s,l,ls$, to denote the studied function $A(x,q)$ characterizing scattering by the corresponding superposition V^{ca} with $a=s,l,ls$. Symbols of the type $A^{a'}(q)$ with primed superscript $a'=c,s$; $a=c,l$; $a'=cl,s$, containing a comma, will be used to denote the contributions $\delta_l^{c,s}(k)$, $\delta_l^{c,l}(k)$, and $\delta_l^{cl,s}(k)$ to the total phase shift $\delta_l^{ca}(k)$, the effective-range functions $K_l^{c,s}$ or $K_l^{cl,s}$, and the coefficients of their expansions for $q\to0$.

First let us consider scattering by the superposition V^{cs}. In this case $V=V^s$, $a=s$, and $a'=c,s$. Using (83), we write the Coulomb–nuclear effective-range function (21) in dimensionless form:

$$K_l^{a'}(q)\equiv R^{2l+1}K_l^{a'}(E). \tag{98}$$

We shall prove the asymptotic representation

$$K_l^{a'}(q)=\sum_{n=0}^{M}q^{2n}K_{ln}^{a'}+O(q^{2M+2}),\quad M<\infty,\quad q\to0 \tag{99}$$

and, as a by-product, derive expressions for its first three coefficients

$$K_{l0}^{a'}=c_{l0}^{+ca}(\infty)/s_{l0}^{+ca}(\infty),$$

$$K_{l1}^{a'}=[c_{l1}^{+ca}(\infty)-K_{l0}^{a'}s_{l1}^{+ca}(\infty)]/s_{l0}^{+ca}(\infty),$$

$$K_{l2}^{a'}=[c_{l2}^{+ca}(\infty)-K_{l0}^{a'}s_{l2}^{+ca}(\infty)-K_{l1}^{a'}s_{l1}^{+ca}(\infty)]/s_{l0}^{+ca}(\infty) \tag{100}$$

and the ratios

$$a_l^{a'}=-R^{2l+1}/K_{l0}^{a'},\quad r_{0l}^{a'}=2R^{1-2l}K_{l1}^{a'},$$

$$P_l^{a'}=-R^{3-2l}K_{l2}^{a'}/r_{0l}^{a'}, \tag{101}$$

determining the Coulomb–nuclear scattering parameters ($a'=c,s$), as simple algebraic combinations (100) of the limiting ($x\to\infty$) values $c_{ln}^{+ca}(\infty)$ and $s_{ln}^{+ca}(\infty)$ of the solutions c_{ln}^{+} and s_{ln}^{+} to problems (92) and (93) with $V=V^s$ and $n=0,1,2$.

We substitute c_l^{+} in the form of the difference (89) into (76). We write the series (90) for the functions $\tilde{c}_l^{+}$ and s_l^{+} in the form (72) and (73) and take $x\to\infty$. As shown above, in this limit the functions c_{ln}^{+} and s_{ln}^{+} are finite and correct estimates (96). Therefore, the limit of the function (76) for $x\to\infty$ gives the following low-energy ($q\to0$) representation:

$$\cot\delta_l^{c,s}(q)=q^{-1}C_l^{-2}(q)\left[\frac{\sum_{n=0}^{M}q^{2n}c_{ln}^{+}(\infty)}{\sum_{n=0}^{M}q^{2n}s_{ln}^{+}(\infty)}+O(q^{2M+2})\right]-h^c(q).$$

Equations (99) and (101) follow from this representation. Comparing them with the standard expression (23), we obtain (101). According to them, the calculation of the Coulomb–nuclear scattering parameters $a_l^{c,s}$, $r_{0l}^{c,s}$, and $P_l^{c,s}$ reduces to the successive ($n=0,1,2$) integration of three energy-independent problems (92) and (93) with $V=V^s$. In the case $V=V^{ls}$ the calculation of the modified scattering parameters is more complicated.

Now let $V=V^l+V^s$, where V^s satisfies (1) and V^l has the asymptote (3) with $d\geqslant 3$. We shall prove that the function

$$K_l^{cl,s}(q)=[C_l(q)N_l^{+cl}(q)]^2[q\cot\delta_l^{cl,s}(q)+q\alpha_l(q)] \tag{102}$$

is related to the modified effective-range function (64) by (98) and has the asymptote (99) with coefficients (100), where now $a=ls$ and $a'=cl,s$, and $c_{ln}^{+cls}(\infty)$ and $s_{ln}^{+cls}(\infty)$ are certain finite quantities which we shall determine in the course of the proof.

In the first stage of the proof we find the *auxiliary* functions $u_l^{\pm cl}$, defined as the solutions $u_l^{\pm}$ of the problems (68)–(71) with the nuclear interaction switched off, i.e., when $V=V^l$, $a=l$, and $a'=c,l$. We construct the functions $u_l^{\pm cl}$ and their low-energy expansions by the method described above. In the end we obtain

$$\begin{aligned}
&u_l^{+cl}(x;q)=N_l^{+cl}(q)U_l^{+cl}(x;q),\\
&u_l^{-cl}(x;q)=\tilde{u}_l^{-}(x;q)+\alpha_l^{cl}(q)u_l^{+cl}(x;q),\\
&\tilde{u}_l^{-}(x;q)=N_l^{-cl}(q)U_l^{-cl}(x;q),\\
&U_l^{\pm cl}(x;q)\equiv c_l^{\pm cl}(x;q)F_l(\rho,\eta)+s_l^{\pm cl}(x;q)G_l(\rho,\eta);
\end{aligned} \tag{103}$$

$$u_l^{+cl}(x;q)=qC_l(q)N_l^{+cl}(q)\sum_{n=0}^{\infty}q^{2n}U_{ln}^{+cl}(x),$$

$$\tilde{u}_l^{-cl}(x;q)=[C_l(q)N_l^{+cl}(q)]^{-1}\sum_{n=0}^{\infty}q^{2n}U_{ln}^{-cl}(x),$$

$$U_{ln}^{\pm cl}(x)\equiv\sum_{m'+m=n}[c_{lm'}^{\pm cl}(x)f_{lm}(x)+s_{lm'}^{\pm cl}(x)g_{lm}(x)]. \tag{104}$$

In (103), $c_l^{\pm cl}$ and $s_l^{\pm cl}$ denote the solutions $c_l^{\pm}$ and $s_l^{\pm}$ of the problems (79)–(81) with $V=V^l$, and the factors $N_l^{\pm cl}(q)$ and $\alpha_l^{cl}(q)$ are equal to the limits of the corresponding functions (77) and (78) for $x\to\infty$. The functions $c_{ln}^{\pm cl}$ and $s_{ln}^{\pm cl}$ appearing in (104) denote the solutions $c_{ln}^{\pm}$ and $s_{ln}^{\pm}$ of the problems (92)–(94) with $V=V^l$.

Let us now turn to the next stage ($a=ls$, $a'=cl,s$) and construct the regular solution u_l^{+cls} of the original problem (68)–(70) with $V=V^{ls}$, using the generalized linear version of the method of phase functions[11] as follows. First we write the asymptote (70) of the function u_l^{+ca} with $a=ls$ in the form corresponding to the splitting (55) of the total phase shift:

$$\begin{aligned}
u_l^{+cls}(x;q)\sim&\cos\delta_l^{cl,s}(q)u_l^{+cl}(x;q)\\
&+\sin\delta_l^{cl,s}(q)u_l^{-cl}(x;q)\sim\sin(\rho-\eta\ln 2\rho\\
&+\delta_l^{cl}(q)+\delta_l^{cl,s}(q)),\quad qx\to\infty.
\end{aligned} \tag{105}$$

The functions (103), which, owing to (70), satisfy the Wronskian

$$u_l^{-cl}(x;q)\partial_x u_l^{+cl}(x;q)+u_l^{+cl}(x;q)\partial_x u_l^{-cl}(x;q)=q,\quad x\in\mathscr{R}^+, \tag{106}$$

will now be used as the reference functions instead of F_l and G_l, and we shall seek u^{+cls} in the form

$$\begin{aligned}
&u^{+cls}(x;q)=N_l^{+cls}(q)U_l^{+cls}(x;q),\\
&U_l^{+cls}(x;q)\equiv c_l^{+cls}(x;q)u_l^{+cl}(x;q)\\
&\qquad+s_l^{+cls}(x;q)u_l^{-cl}(x;q),
\end{aligned} \tag{107}$$

where the unknown generalized amplitude functions c_l^{+cls} and s_l^{+cls} by definition satisfy the Lagrange identity

$$\partial_x c_l^{+cls}(x;q)u_l^{+cl}(x;q)+\partial_x s_l^{+cls}(x;q)u_l^{-cl}(x;q)\equiv 0,\quad x\in\mathscr{R}^+. \tag{108}$$

Substituting (107) into (68)–(70) and taking into account (106) and (108), we derive the equations

$$\partial_x\begin{Bmatrix}c_l^{+cls}(x;q)\\ s_l^{+cls}(x;q)\end{Bmatrix}=q^{-1}V^s(x)U_l^{+cls}(x;q)\begin{Bmatrix}+u_l^{-cl}(x;q)\\ -u_l^{+cl}(x;q)\end{Bmatrix} \tag{109}$$

with simple initial conditions: $c_l^{+cls}(0;q)=1$ and $s_l^{+cls}(0;q)=0$. The asymptotes of the functions (107) for $x\to\infty$ will have the required form (105) if we define

$$\cot\delta_l^{cl,s}(q)\equiv\lim_{x\to\infty}c_l^{+cls}(x;q)/s_l^{+cls}(x;q). \tag{110}$$

Using the expansions (104) and the ansatz

$$c_l^{+cls}(x,q)=\tilde{c}_l^{+cls}(x;q)-\alpha_l^{cl}(q)s_l^{+cls}(x;q), \tag{111}$$

$$\tilde{c}_l^{+cls}(x;q)=\sum_{n=0}^{\infty}q^{2n}c_{ln}^{+cls}(x),$$

$$s_l^{+cls}(x;q)=q[C_l(q)N_l^{+cl}(q)]^2\sum_{n=0}^{\infty}q^{2n}s_{ln}^{+cls}(x), \tag{112}$$

we reduce the problem (109) to a recursion ($n=0,1,\ldots$) chain of coupled pairs of energy-independent equations

$$\partial_x\begin{Bmatrix}c_{ln}^{+cls}(x)\\ s_{ln}^{+cls}(x)\end{Bmatrix}=V^s(x)\sum_{m'+m=n}U_{lm'}^{+cls}(x)\begin{Bmatrix}+U_{lm}^{-cl}(x)\\ -U_{lm}^{+cl}(x)\end{Bmatrix},$$

$$\begin{aligned}
U_{ln}^{+cls}(x)\equiv&\sum_{m'+m=n}[U_{lm'}^{+cl}(x)c_{lm}^{+cls}(x)\\
&+U_{lm'}^{-cl}(x)s_{lm}^{+cls}(x)]
\end{aligned} \tag{113}$$

for the functions c_{ln}^{+cls} and s_{ln}^{+cls} such that $c_{ln}^{+cls}(0)=\hat{\delta}_{n0}$ and $s_{ln}^{+cls}(0)=0$. Using the iteration method,[2] it can be shown that all these functions are bounded on $\mathscr{R}^+$ if V^s satisfies (1). Substituting c_l^{+cls} in the form (111) into (110) and taking $x\to\infty$, we arrive at an expression rewritten in the form (102), determining the dimensionless function $K_l^{cl,s}$ in terms of the functions $\tilde{c}_l^{cls}$ and s_l^{+cls} taken at $x=\infty$. Splitting the series (112) into terms of the type (73) and using the estimates (97) for the finite sums of the series (104), we prove Eqs. (99) and (100), where $a=ls$, $a'=cl,s$, and $c_{ln}^{+cls}(\infty)$ and $s_{ln}^{+cls}(\infty)$ denote the limiting values of the solutions of (113). Comparing (64) and (102), we obtain the relations

$$C_l^{cl}(\eta)=C_l(q)N_l^{+cl}(q), \quad h^{cl}(\eta)=k\alpha_l^{cl}(q) \tag{114}$$

and Eq. (101) determining the modified scattering parameters in this case, $a=ls$, $a'=cl,s$. According to (99), (101), and (102), the equation for the nuclear scattering length has the form

$$a_l^{cl,s}\equiv-\lim_{k\to0}\tan\delta_l^{cl,s}(k)/k[k^lC_l(\eta)N_l^{+cl}(k)]^2$$

and differs from the approximate expression (67) obtained in Ref. 46 by the presence of the factor N_l^{+cl}. The latter is close to unity if V^l is sufficiently small [see the estimates (37) in Ref. 48]. It is only in this case that Eq. (67) gives a plausible but still *approximate* value of the scattering length $a_l^{cl,s}$.

As shown above, to calculate C_l^{cl} and h_l^{cl} from (64) using Eq. (114), first the problems (79) and (80) with $V=V^l$ are solved. The subsequent calculation of the modified scattering parameters reduces to the solution of three ($n=0,1,2$) energy-independent and recursion problems (92)–(94) with $V=V^l$ and three ($n=1,2,3$) problems (113). All these problems are systems of linear differential equations with simple boundary conditions, and their practical and accurate solution does not present any difficulty. In the method we are proposing the calculation of the parameters for nuclear scattering by a central potential V^s in a field V^{cl} does not require any assumption of small V^l, but it allows the study of only *model* scattering in systems of two nucleons or nuclei. The point is that, according to current ideas, the nuclear NN interaction in particular is not a central potential, although the coordinate dependence of all its components is described by short-range potentials with the asymptotes (1).

It has been found, first empirically[15] and later using the meson theory,[16] that a realistic NN interaction should contain spin–orbit and tensor terms in addition to a spherically symmetric component. At present, the potentials which are viewed as being the most well justified theoretically and which allow the description of a large set of data on the NN and $3N$ systems are the Paris,[53] the Bonn,[54] the Nijmegen,[55] and the Urbana–Argonne[56] potentials. There is also a series of hybrid potentials obtained from quark models and reflecting meson exchange.[57] A recent study[58] was devoted to the investigation of the contributions of $\pi\pi$ and $\pi\rho$ exchange to the NN interaction. Despite the large number of variants of the NN potential, the construction of a theory of nuclear forces and the refinement of the ideas about the nature of these forces continues. An important aspect of this construction is the derivation of low-energy expansions of the characteristics of NN scattering by the superpositions V^{ls} and V^{cls} with a realistic nuclear potential and electromagnetic correction V^l. To understand why, we shall recall the results of the old studies of the effects of charge-symmetry breaking in the NN forces and analyze how the parameters of low-energy NN scattering were improved.

3. LOW-ENERGY *NN* SCATTERING

In 1932 Heisenberg[59] introduced the concept of isospin, largely as an elegant computational device. According to this concept, the neutron and the proton are two manifestations of the same particle, the nucleon, which by definition has isospin 1/2. The neutron has isospin projection $-1/2$, and the proton has isospin projection $+1/2$. However, the two states are physically distinguishable, owing to the proton charge and the neutron–proton mass difference. It is therefore not obvious that the concept of isospin as a nongeometrical (isospin) symmetry has any genuine meaning. From the experimental point of view a consequence of this symmetry is the equivalence (charge independence) of the nuclear pp, np, and nn interactions. Equality of the pp and nn interactions (charge symmetry) is a less restricted concept. The question of the sources of violation of charge symmetry and the effects arising from this violation remains one of the fundamental problems of the modern theory of the nucleus and elementary particles. Charge symmetry is violated by the electromagnetic and weak interactions, and the latter implies the presence of quarks of different colors ($u,d,s,c,\ldots$). Another source of violation of this symmetry is the mass difference of the u and d quarks, which is approximately 3 MeV.

One of the indications of the nn and pp asymmetry of the nuclear interaction known[60–63] since the early 1960s is the difference between the masses of the mirror nuclei ^{3}H and ^{3}He, called the Okamoto–Nolen–Schiffer anomaly. In 1971 Negele[64] explained it by assuming that the nuclear NN interaction is charge-asymmetric, and the nuclear nn interaction V_{nn}^s is slightly more attractive ($|V_{nn}^s|>|V_{pp}^s|$) than the nuclear pp interaction V_{pp}^s. After this assumption, several preliminary versions[65–68] of charge-asymmetric nuclear $^1S_0$ potentials taking into account ρw and $\pi^0\eta$ mixing were constructed. The systematic use of such potentials allowed the explanation of the main effects due to violation of the NN charge symmetry in the ^{3}H and ^{3}He nuclei: the mass difference[69] and the difference between the energy dependences of the elastic form factors.[70]

The experimental and theoretical study of charge-symmetry violation effects in low-energy NN scattering is quite complicated. Protons are charged, so that they are easier to accelerate and detect than neutrons. Measurement of the characteristics of pp collisions is still the best source of quantitative information about the nuclear forces at energies which are not too low ($E_{\text{lab}}>10$ MeV). It becomes more and more difficult to perform pp experiments as the energy decreases, owing to the replusive Coulomb barrier, which strongly suppresses the probability for the protons to be at a separation within the nuclear-force range. The Cou-

lomb interaction also complicates the theoretical analysis[71–74] of the experimental pp data. The analysis becomes even more difficult with decreasing pp collision energy, owing to the increasingly important role[75,76] of various electromagnetic corrections. In contrast to the pp and nn systems, two types of state are possible in the np system: states which are singlets and triplets in the total isospin. The need to include both of these states is the main reason why analysis of the experimental np data and the theoretical determination of the low-energy np scattering parameters using these data are problems which are enormously more complicated than in the case of pp or nn collisions. Direct observation of nn collisions has still not been achieved, although a realistic scheme for an nn experiment in a neutron beam has been discussed in Ref. 77. At present, the nn scattering parameters can only be determined experimentally by analyzing the kinematics of systems of three or more particles in the final state, for example, $D(\pi^-,\gamma)2n$, $D(n,nn)p$, $D(t,^3\mathrm{He})2n$, and $D(\mu^-,\nu_\mu)2n$. However, any theoretical description of a system of three or more particles interacting via the strong interaction requires *modeling* of the problem and, as a rule, an *approximation* algorithm for solving it. Both of these approximations, logical and numerical, unavoidably introduce a number of theoretical uncertainties which are difficult to control. Neglecting them can lead to incorrect conclusions. It is useful to give an example which demonstrates this.

In the literature of the 1970s (see the review of Ref. 78), one often encounters the value of the $^1S_0$ nn scattering length $a^s_{nn}=(-16.6\pm0.6)$ F, obtained by simple averaging of a large number of data on the $D(n,nn)p$ reaction. The theoretical uncertainties in the analysis of this reaction with three hadrons in the final state were not taken into account in the averaging, and the average value turned out to be smaller in absolute value than the $^1S_0$ pp scattering length $a^{c,s}_{pp}=(-17.1\pm0.2)$ F extracted from the pp data.[79] As explained later (see Ref. 80), because of the neglect of these uncertainties the average value $a^s_{nn}=(-16.6\pm0.6)$ F, like the later result $a^s_{nn}=(-16.9\pm0.6)$ F from study[81] of the $D(n,nn)p$ reaction, is unreliable. Both of these values contradict Negele's assumption that $|V^s_{nn}|>|V^s_{pp}|$ and are inconsistent with the value $a^s_{nn}=(-18.5\pm0.4)$ F obtained in Refs. 82 and 83 by a complete kinematical analysis of the photon spectrum of the reaction $D(\pi^-,\gamma)2n$ measured with large statistics. The study of this reaction is of particular interest, because all three particles can be detected in the final state, and only two of them interact via nuclear forces. The latter fact is especially important: it presents the rare possibility of studying all the kinematical regions, which allows determination of not only the scattering length a^s_{nn} but also the effective range r^s_{nn} of singlet nn scattering. The $D(\pi^-,\gamma)2n$ reaction is also attractive theoretically. The initial state of this reaction can be described by three-particle differential Faddeev equations, and the study of the final state reduces to the solution of a simple two-body (neutrons) problem. For these reasons all the theoretical uncertainties in the analysis of the $D(\pi^-,\gamma)2n$ data are minimized, which allows the determination of the nn scattering parameters with higher accuracy and reliability than when data on reactions with three hadrons in the final state are used. In 1987 de Téramond and Gabioud[80] carefully studied the neutron interaction in the final state of the π^-D capture reaction and confirmed the values measured earlier:[83] $a^s_{nn}=(-18.5\pm0.4)$ F and $r^s_{nn}=(2.80\pm0.11)$ F. By 1992 the values considered most reliable[84] were $a^s_{nn}=(-18.5\pm0.4)$ F and $r^s_{nn}=(2.75\pm0.11)$ F, obtained by averaging all the data obtained up to that time,[85–87] and the pp scattering length was taken to be[87] $a^{c,s}_{pp}=(-17.3\pm0.3)$ F. These values of the scattering lengths are consistent with the inequality $|V^s_{nn}|>|V^s_{pp}|$ and provide an explanation of the Okamoto–Nolen–Schiffer anomaly: if the difference $a^{c,s}_{pp}-a^s_{nn}$ is assumed to be 1.3 F or 1.5 F, then the corresponding mass difference of the ^{3}H and ^{3}He nuclei is found to be[88] 59 keV or 63 keV, which are comparable to the experimental[89] value of (62 ± 3) keV.

A careful analysis[90] of the experimental data[91] on the $D(n,nn)p$ reaction performed in 1993 using the W-matrix AGS method[92] with a modified Paris NN potential[53] produced the value $a_{nn}=(-17.0\pm1.0)$ F.

From the chronological review presented above, we see that during the last three decades the values of the $^1S_0$ parameters of low-energy nn scattering have constantly been improving. Now it is understood how to obtain even greater accuracy in the experimental[80] and theoretical[90] determination of these S-wave parameters. At energies $E_{\mathrm{lab}}<20$ MeV the contribution of the S-wave phase shifts to the NN scattering cross section dominates, and this significantly complicates the experimental determination of the energy dependence of the ^3P_j phase shifts ($j=0,1,2$) in this energy range. Another reason why the ^3P_j phase shifts have been studied experimentally only for $E_{\mathrm{lab}}>10$ MeV is that their unique determination requires measurement of five independent NN amplitudes.[71,73,75] The measurement of only two such amplitudes even at fairly high energies, for example,[93] the np spin observables K_{osko} and K_{osso} at energies (230–530) MeV, is a difficult experimental problem. As pointed out in Ref. 84, at present there are no data on all five NN observables, and there will not be any in the near future. Given this experimental situation, the theoretical determination of the P-wave ($l=1$) NN scattering parameters and the development of a reliable method of extrapolating the ^3P_j phase shifts to low energies appear to be acute problems. Their solution would allow improvement of the structure and magnitude of the P-wave terms in the current NN potentials. Such improvement is needed both for analyzing the experimental data on the $D(\pi^-,\gamma)2n$ and $D(n,nn)p$ reactions, and for the reliable theoretical description of the Nd spin observables, in particular, the pd analyzing power $A_{y,pd}$, which, in contrast to the NN analyzing power $A_{y,NN}$, is extremely sensitive to the value of the P-wave terms in the NN interaction. For example,[84] for a plausible description of the angular dependence $A_{y,pd}(\theta;E)$ measured[94] at the energies 5.0 MeV and 8.5 MeV within the three-body problem, it is necessary to modify the Bonn potential[54] by multiplying its $^3P_0$ terms by a factor differing significantly from unity, that is, 1.04, 0.86, and 0.75 for the np, pp, and nn interactions, respectively.[95] We shall present several arguments which suggest that an alternative to this *empirical* modification might be the cor-

rect inclusion of the electromagnetic corrections to the nuclear NN potential.

For a long time, the role of such corrections in NN collisions has been studied by phase-shift analysis of the experimental NN data. In 1950 Schwinger[96] drew attention to the fact that the contact, short-range parts of the interaction of the neutron and proton magnetic moments and the proton and proton magnetic moments in $^1S_0$ states differ from each other in both absolute value and sign:

$$V_{np}^{\delta}(\vec{r})=4\pi\mu_n(2\mu_p-\mu_0)\delta(\vec{r})<0,$$
$$V_{pp}^{\delta}(\vec{r})=2\pi[(2\mu_p^2-\mu_0)^2+\mu_0^2]\delta(\vec{r})>0, \quad (115)$$

where $\mu_n=-1.913\mu_0$ and $\mu_p=2.793\mu_0$ are the neutron and proton magnetic moments, $\mu_0\equiv e\hbar/2m_Nc$ is the nuclear magneton, and $\delta(\vec{r})$ is the delta function. Schwinger assumed that the short-range parts of the np and pp interactions are superpositions $V_{np}^{s\delta}\equiv V^Y+V_{np}^{\delta}$ and $V_{pp}^{s\delta}\equiv V^Y+V_{pp}^{\delta}$ of identical particular nuclear Yukawa potentials (2) and corrections (115). Estimating the contributions $a_{np}^{s\delta}-a_{np}^{s}$ and $a_{pp}^{c,\delta s}-a_{pp}^{c,s}$ from these corrections to the scattering lengths $a_{np}^{s\delta}$ and $a_{pp}^{c,\delta s}$ of singlet np and pp scattering by the proposed superpositions, Schwinger showed that this assumption completely explains the difference known at that time between the short-range parts of the np and pp interactions.

For just as long a time, apparently since 1955, it has been known[97] that the inclusion of the vacuum-polarization potential[98]

$$V^v(r)=\frac{2e^2}{3\pi\hbar c}V^c(r)\int_1^{\infty}dt\,\exp(-tr/\lambda_e)t^{-4}(2t^2+1)\times(t^2-1)^{1/2} \quad (116)$$

allows a significant improvement of the fit to the energy dependence of the $^1S_0$ phase shift of pp scattering at low ($E_{\rm lab}<10$ MeV) energies. The potential (116) is short-range,

$$V^v(r)\sim(3\sqrt{2\pi}/4)t^{-3/2}\exp(-t),\quad \lambda_e\equiv/2m_ec,$$
$$t\equiv r/\lambda_e\to\infty,$$

but owing to the large Compton wavelength of the electron ($\lambda_e\approx 193$ F) it falls off more slowly than the nuclear pp interaction, the range of which is $r_s\approx 4$ F. Therefore, even for low energies the phase shifts $\delta_l^{c,v}(k)$ with $l>0$ cannot be neglected when analyzing pp scattering. Durand[98] has shown that in order to extract the correct value of the nuclear $^1S_0$ phase shift from the experimental pp data, it is necessary to perform the following procedure based on Eqs. (26)–(32) and (55).

First, the symmetrized "electric" amplitude

$$f^{cv}(\theta;k)\equiv f^c(\theta;k)+f^c(\pi-\theta;k)+f^{c,v}(\theta;k)+f^{c,v}(\pi-\theta;k) \quad (117)$$

is subtracted from the experimentally measured total amplitude $f^{cvs}(\theta;k)$ of singlet pp scattering by the superposition $V^{cvs}\equiv V^c+V^v+V^s$. The Coulomb amplitude in this expression is calculated from Eq. (30), and the amplitude $f^{c,v}$ for scattering by the potential (116) in a Coulomb field, given by Eqs. (31) and (32) with $a=v$, is approximated as

$$f^{c,v}(\theta;k)\sim k^{-1}\sum_{n=0}^{6}(2l+1)\delta_l^{c,v}(k)\exp(2i\delta_l^c(k))P_l(\cos\theta),$$
$$k\to 0. \quad (118)$$

The resulting symmetrized difference amplitude $f^{cv,s}$ for scattering by the potential V^s in the "electric" field $V^{cv}\equiv V^c+V^v$ is replaced by twice its S-wave component $f_0^{cv,s}$:

$$f^{cv,s}(\theta;k)=f^{cvs}(\theta;k)-f^{cv}(\theta;k)\approx 2f_0^{cv,s}(k)\equiv(ik)^{-1}\exp(2i\delta_0^{c,v}(k))[(\exp(2i\delta_0^{cv,s}(k))-1]\approx 2k^{-1}\exp(2i\delta_0^{c,v}(k))\delta_0^{cv,s}(k). \quad (119)$$

The experimental value of the $^1S_0$ phase shift $\delta_0^{cv,s}(k)$ generated only by the nuclear potential V^s in scattering in the field V^{cv} is thereby determined.

After carrying out this procedure, Heller[99] showed that if the potential (116) is neglected in the analysis of the $^1S_0$-wave pp effective-range function, the shape parameter $P_0^{c,s}$ turns out to be 0.02 times smaller than the modified shape parameter $P_0^{cv,s}$ obtained by taking into account the potential (116) within the Born approximation according to Eqs. (117)–(119).

In 1949 Schwinger[100] showed that the interaction (56) of the neutron magnetic moment $\vec{\mu}$ with the Coulomb field of the nucleus causes the neutron–nucleus differential cross section to diverge in forward scattering. This effect was later studied many times, both theoretically and experimentally.[18] In the series of studies of Refs. 101–103 carried out in the 1950s, it was shown that the inclusion of the interaction (56) of the magnetic moment of a single proton with the Coulomb field of another in the plane-wave Born approximation markedly improves the fit to the experimental values of the pp analyzing power $A_{y,pp}(\theta;E)$ at high energies ($E_{\rm lab}>150$ MeV) and small angles ($\theta<15°$). To obtain a satisfactory fit to the values of the function $A_{y,pp}(\theta;E)$ measured[104] by 1975 at a lower energy ($E_{\rm lab}=10$ MeV), the authors of Ref. 105 were forced to include Coulomb distortion effects by replacing the plane-wave approximation in the Born correction (56) by regular Coulomb functions. The resulting approximation has become traditional, and is still used to take into account the interactions (56), (57), and (116) in phase-shift analyses of pp data. In 1990 Stoks and de Swart[75] used this approximation to show that the inclusion of the pp corrections (56) gives agreement between the angular dependence of the function $A_{y,pp}(\theta;E)$ that they calculated and the value[106,107] measured experimentally at two values of the energy: $E_{\rm lab}=9.75$ MeV and $E_{\rm lab}=5.05$ MeV. It turned out that in the first case the inclusion of the correction (56) improves the agreement in the region $\theta<30°$, while in the second it improves it for all $\theta<90°$. Therefore, the upper limit on the range of scattering angles where the inclusion of the correction (56) significantly changes the function $A_{y,pp}(\theta;E)$ increases with decreasing energy. Comparison of the functions $A_{y,pp}(\theta;E)$ and $A_{y,np}(\theta;E)$, shown in Figs. 1 and 2 of Ref. 75, shows that the effect of the

interaction (56) between the neutron magnetic moment and the Coulomb field of the proton on the angular dependence of the function $A_{y,np}(\theta;E)$, measured in Ref. 108 at the energy $E_{\rm lab}=10.03$ MeV close to the energy $E_{\rm lab}=9.75$ MeV, is manifested in a wider range of angles, namely, for $\theta<60°$. Measurement of the functions $A_{y,NN}(\theta;E)$ and of the ^3P_j phase shifts of NN scattering at lower energies is particularly interesting because the corrections (56) and (57) radically change the behavior of these observables in the low-energy limit. As a result of this change, the commonly used[17] definitions of the ^3P_j NN scattering lengths, also referred to as the NN scattering volumes, become meaningless. Let us begin our proof of this important statement with triplet pp scattering.

Let $\delta_{l,j}^{c,ms}(k)$ be the difference between the total phase shift $\delta_{l,j}^{cms}(k)$ of scattering by the superposition $V_{pp}^{cms}\equiv V^c+V_{ls}^m+V^s$ and the Coulomb phase shift $\delta_l^c(k)$, $l=j$. Owing to (59) and (62),

$$\delta_{1,j}^{c,ms}(k)\sim -3^{-1}[j(j+1)-4]R^2V_{ls,0}^m k^3, \quad k\to 0.$$

Therefore, the ^3P_j phase shifts $\delta_{1,j}^{c,ms}(k)$ fall off for $k\to 0$ much more slowly than the Coulomb–nuclear ^3P_j phase shifts $\delta_{1,j}^{c,s}(k)$. The ^3P_j pp scattering volume $a_{1,j}^{c,ms}$ defined by analogy with the Coulomb–nuclear volume (24), i.e., as

$$a_{1,j}^{c,ms}=\lim_{k\to 0}\tan\delta_{j1}^{c,ms}(k)/k^3C_1^2(\eta),$$

is then infinite for any $j=0,1,2,$.

As shown in Refs. 109 and 110, the correction (57) to the nuclear nn interaction leads to falloff of the ^3P_j phase shifts which is linear in k,

$$\delta_{1,j}^{ms}(k)\sim\frac{(-1)^{j+1}\mu_n^2m_nk}{\max\{1,j\}(2j+1-j\hat{\delta}_{1j})}, \quad k\to 0,$$

for nn scattering by the superposition $V^{ms}=V^m+V^s$. This means that the definition

$$a_{1,j}^{ms}=\lim_{k\to 0}k^{-3}\tan\delta_{1,j}^{ms}(k) \tag{120}$$

of the ^3P_j nn scattering volume, like the standard definition (4), leads to $|a_{1,j}^{ms}|=\infty$.

Using the method of Ref. 110 to analyze triplet np scattering by the superposition $V^{ms}=V_{ls}^m+V^s$, it can be shown that all the ^3P_j phase shifts $\delta_{1,j}^{ms}(k)$ for this scattering have asymptotes linear in k:

$$\delta_{1,j}^{ms}(k)\sim -2^{-3}[j(j+1)-4]V_{ls,0}^m k, \quad k\to 0. \tag{121}$$

Accordingly, the np limits (120) with $j=0,1,2$ are infinite, and the parametrization

$$k^3\cot\Delta_{ls}^{ms}(k)=-1/a_{ls}^{ms}$$

used in Ref. 75 for fitting the energy dependence of the combination

$$12\Delta_{ls}^{ms}(k)\equiv -2\delta_{1,0}^{ms}(k)-3\delta_{1,1}^{rms}(k)+5\delta_{1,2}^{ms}(k) \tag{122}$$

of ^3P_j phase shifts of np scattering is not applicable for $k\to 0$, when the parameter a_{ls}^{ms} becomes infinite. Owing to Eqs. (121) and (122), in this limit the well known equation[111]

$$(d\sigma_{np}^{ms}(\theta;E)/d\theta)A_{y,np}^{ms}(\theta;E)$$
$$=[\sin\delta_0^s(k)/2k]^2 12\Delta_{ls}^{ms}(k)\sin\theta$$

is transformed into the asymptotic expression

$$(d\sigma_{np}^{ms}(\theta;E)/d\theta)A_{y,np}^{ms}(\theta;E)\sim(-3/4)(a_0^s)^2V_{ls,0}^m k\sin\theta,$$
$$k\to 0,$$

relating the differential cross section and the analyzing power of np scattering to the $^1S_0$ scattering length a_0^s and the constant $V_{ls,0}^m$ of the interaction (56).

Therefore, for $k\to 0$ the behavior of all the ^3P_j phase shifts of pp, np, or nn scattering are independent of the nuclear interaction and on the whole are determined by the corresponding corrections (56) or (57). This must be taken into account both when extrapolating the ^3P_j phase shifts to zero energy, and for the *correct* description of the characteristics of slow collisions in three-nucleon systems. Such a mathematically rigorous description is necessary also for other reasons which are just as important. Let us explain some of them.

It is well known[14] that the continuum wave functions describing the final state in the total-breakup reactions

$$\pi^+ + {}^3\mathrm{He}\to 3p, \quad \pi^- + {}^3\mathrm{H}\to 3n, \quad \mu^- + {}^3\mathrm{H}\to 3n+\nu_\mu$$

are greatly changed when the interaction between all the reaction products is taken into account, and the corresponding cross sections turn out to be very sensitive to the choice of the type of two-body interaction. Therefore, the study of such breakups and the combined analysis of $(2\to 2,3)$ and $(3\to 3)$ collisions in a system of three nucleons or nuclei is an effective means of selecting the most realistic variant of the NN force. The construction of a theory of low-energy nuclear processes $(2\to 2,3)$ and $(3\to 3)$ is especially interesting and important also from the applied point of view, as a way to solve many problems in nuclear astrophysics and thermonuclear fusion. Two approaches to the construction of such a theory are discussed in the next section.

4. LOW-ENERGY EXPANSIONS IN THE THREE-BODY PROBLEM

The studies devoted to the three-body problem traditionally use not only different notation for the coordinates, quantum numbers, wave vectors, and multiple indices, but also different ansatzes for the auxiliary functions. Therefore, it is not possible to discuss the low-energy expansions obtained by various authors without introducing more compact and unified notation. Let us introduce such notation for a system of three spinless particles with potentials V_i acting in each of the three $(i=1,2,3)$ pairs of particles and possessing spherical symmetry: $V_i(\vec{x}_i)=V_i(x_i)$. The quantum states of this system are characterized by sets $\varepsilon=\{l,m,E,\sigma\}$ of four conserved quantum numbers. They are the total angular momentum $\vec{\ell}=\vec{\ell}_{x_i}+\vec{\ell}_{y_i}$; its third projection m; E, the squared six-dimensional momentum $\vec{k}\equiv(\vec{p}_i,\vec{q}_i)$; and the parity $\sigma=\pm 1$ under inversion $\vec{r}_i\to-\vec{r}_i$ of any $(i=1,2,3)$ six-dimensional vector $\vec{r}_i\equiv(\vec{x}_i,\vec{y}_i)$ whose components are three-dimensional reduced Jacobi vectors. By definition, the vector $\vec{x}_i$ is conju-

gate to the angular momentum $\vec{l}_{x_i}$ and the momentum $\vec{p}_i$ of the pair of particles labeled i, and the vector $\vec{y}_i$ describing the location of the third particle in the three-particle coordinate space $\mathscr{R}^6 \equiv \mathscr{R}^3_{x_i} \oplus \mathscr{R}^3_{y_i}$ corresponds to the angular momentum $\vec{l}_{y_i}$ and the momentum $\vec{q}_i$. The set of Jacobi coordinates $\vec{r}_i = (\vec{x}_i, \vec{y}_i)$ corresponds to the set of hyperspherical coordinates $\vec{r}_i = (r, \Omega_i)$: the hyperradius $r = \sqrt{x_i^2 + y_i^2}$ and the set $\Omega_i \equiv \{\hat{\vec{x}}_i, \hat{\vec{y}}_i, \varphi_i\}$ of two pairs $\hat{\vec{\xi}} \equiv \{\theta_\xi, \varphi_\xi\}$ of spherical angles of the Jacobi vectors $\vec{\xi} = \vec{x}_i, \vec{y}_i$ and the hyperangle $\varphi_i \equiv \arctan(y_i/x_i)$. There is an analogous correspondence for the total momentum $\vec{k} = (k, \Omega_{k_i})$. In the Schrödinger theory[8] it is assumed that the wave function $\Psi^\varepsilon(\vec{r}, \vec{k})$ of the three-particle state ε under study belongs to the linear space $\mathscr{A}^\varepsilon$ of functions possessing the set of quantum numbers ε which are twice continuously differentiable everywhere in $\mathscr{R}^6$ and with respect to all their six arguments. In the space $\mathscr{A}^\varepsilon$ the bispherical harmonics

$$\mathscr{Y}^\varepsilon_\alpha(\hat{x}_i, \hat{y}_i) \equiv [Y^{m_{x_i}}_{l_{x_i}}(\hat{x}_i) \otimes Y^{m_{y_i}}_{l_{y_i}}(\hat{y}_i)]^{l,m} = (-1)^{l_{x_i}+l_{y_i}} \mathscr{Y}^\varepsilon_\alpha (-\hat{x}_i, -\hat{y}_i) \tag{123}$$

possessing quantum numbers l_{x_i}, l_{y_i}, l, m, and σ form an orthonormal basis on the torus $\mathscr{T}^4 \equiv \mathscr{S}^2(\hat{x}_i) \oplus \mathscr{S}^2(\hat{y}_i)$, where $\mathscr{S}^2(\hat{\xi})$ is the unit sphere of the vector $\vec{\xi}$ in $\mathscr{R}^3$. In (123) the components of the double index $\alpha \equiv \{l_{x_i}, l_{y_i}\}$ obey triangle inequalities, in particular, $l_{x_i} - l_{y_i} \leq l \leq |l_{x_i} + l_{y_i}$, and the additional condition that $l_{x_i} + l_{y_i}$ is an even (odd) number if $\sigma = +1(-1)$. As the orthonormal basis of the space $\mathscr{A}^\varepsilon$ on the unit sphere $\mathscr{S}^5$ in $\mathscr{R}^6$ it is convenient to take the set of polyspherical hyperharmonics with quantum numbers l_{x_i}, l_{y_i}, l, m, σ, and L. Introducing the triple index $\nu \equiv \{\alpha, L\}$, we write these harmonics as products

$$Y^\varepsilon_\nu(\Omega_i) \equiv 2 \operatorname{cosec} 2\varphi_i W_\nu(\varphi_i) \mathscr{Y}^\varepsilon_\alpha(\hat{x}_i, \hat{y}_i) \tag{124}$$

of the bispherical harmonics (123) and the functions

$$W_\nu(\varphi_i) \equiv N_\nu (\sin \varphi_i)^{l_{y_i}+1} (\cos \varphi_i)^{l_{x_i}+1} P_n^{(l_{y_i}+1/2, l_{x_i}+1/2)} \times (\cos 2\varphi_i). \tag{125}$$

These functions contain the Jacobi polynomials $P_n^{(a,b)}$ and have unit $\mathscr{L}^2_{[0,\pi/2]}$ norm for suitable choice of the factor N_ν, which is always assumed to be the case below. The hyperangular momentum $L = l_{x_i} + l_{y_i} + 2n$, where $n = 0, 1, \ldots$, is an even (odd) number if $\sigma = +1(-1)$.

Now let us recall how the boundary-value problem for the wave function Ψ^ε is stated in the method of hyperharmonics.[112] First, from the three Jacobi sets of hyperspherical coordinates we choose the one set of coordinates (r, Ω_i) which is most convenient for describing the process of interest. In what follows the subscript i will usually be dropped. The function Ψ^ε is expanded in the basis (124) and represented by the matrix product

$$\Psi^\varepsilon(\vec{r}; \vec{k}) = r^{-5/2} Y^\varepsilon(\Omega) u(r; \vec{k}) \equiv r^{-5/2} \sum_{\nu=\nu_-}^{\nu_+} Y^\varepsilon_\nu(\Omega) u_\nu(r; \vec{k}) \tag{126}$$

of row Y of hyperharmonics Y^ε_ν and column u of unknown partial components u_ν. In the ansatz (126), $\nu_\pm \equiv \{\alpha_\pm, L_\pm\}$, $\alpha_\pm \equiv \{l_{x\pm}, l_{y\pm}\}$, and $L_\pm \equiv l_{x\pm} + l_{y\pm} + 2n$. By definition, all the components of the multiple index ν_- have the minimal values of all those possible for the state ε. In general, the replacement of the infinite upper limits of the summation over the components of the multiple index ν by finite limits ($\nu_+ < \infty$, i.e., l_{x+}, l_{y+}, $L_+ < \infty$) is the *fundamental* approximation of the method of hyperharmonics. It allows calculations to be done and will always be used in what follows. The boundary-value problem for Ψ^ε, i.e., the Schrödinger equation in $\mathscr{R}^6$,

$$(H_0 + E)\Psi^\varepsilon = \left(-\sum_{i=1}^{3} V_i \right) \Psi^\varepsilon, \tag{127}$$

supplemented by regular asymptotic conditions at the triple collision point ($r = 0$) and on the axes $x = 0$, $y = 0$ and by the boundary condition $\Psi^\varepsilon(\vec{r}; \vec{k}) \sim \Psi^{\text{as}}(\vec{r}; \vec{k})$ for $r \to \infty$ is projected onto the finite-dimensional linear space of hyperharmonics (124) with $\nu = \nu_-, \ldots, \nu_+$. Thus, we obtain the one-dimensional matrix Schrödinger problem

$$[(\partial_r^2 + k^2)I + D_0 r^{-2} - V(r)] u(r; \vec{k}) = 0, \quad r \in \mathscr{R}^+, \tag{128}$$

$$u(0; \vec{k}) = 0; \quad u(r; \vec{k}) \sim u^{\text{as}}(r; \vec{k}), \quad r \to \infty. \tag{129}$$

Here and below, I is the unit matrix and $D_0 \equiv \operatorname{diag}\{\lambda(\lambda+1)\}$ is a diagonal matrix; the half-integer index λ takes all values from $L_- + 3/2$ to $L_+ + 3/2$ in unit steps. The elements $V_{\nu\nu'}$ of the interaction matrix V and the elements u^{as}_ν of the column u^{as} are the corresponding integrals over the hypersphere $\mathscr{S}^5$:

$$V_{\nu\nu'}(r) \equiv \int_{\mathscr{S}^5} d\Omega (Y^\varepsilon_\nu(\Omega))^* \sum_{i=1}^{3} V_i(x_i) Y^\varepsilon_{\nu'}(\Omega) \equiv \left\langle Y^\varepsilon_\nu \middle| \sum_{i=1}^{3} V_i(x_i) \middle| Y^\varepsilon_{\nu'} \right\rangle, \tag{130}$$

$$u^{\text{as}}_\nu(r; \vec{k}) \equiv r^{5/2} \langle Y^\varepsilon_\nu | \Psi^{\text{as}}(\vec{r}; \vec{k}) \rangle. \tag{131}$$

The asymptotic three-particle functions $\Psi^{\text{as}}(\vec{r}; \vec{k})$ with real, nonzero momentum $\vec{k}$ have been studied fairly thoroughly: they have been constructed using the Faddeev equations[10,113,114] for all types of collisions possible in systems of three neutral or charged particles. Therefore, no problem arises with the formulation of the problem within the approximation standard for the method of hyperharmonics ($\nu_+ < \infty$). The entire formulation in fact reduces to the calculation of the integrals (130) and (131). The system of equations (128) for the boundary-value problem (128) and (129) looks exactly like the system of two-particle Schrödinger equations with a potential which does not conserve the two-particle angular momentum.[12] Therefore, the matrix generalizations of various versions of the method of phase functions known in the two-particle problem can be fairly simply modified for reformulating the three-particle problem (128) and (129). How promising is this approach, based on

combining the theory of hyperharmonics and the nonlinear[115–119] or linear[49,120–122] versions of the method of phase functions, for constructing the low-energy asymptotes in the three-body problem? What are the basic difficulties, and how can they be overcome? The next section is devoted to the discussion of these questions.

4.1. The method of phase functions in the three-body problem

The problem (128) and (129) has one special feature: independently of whether the two-body interactions $V_i=V_i^a$ are short- ($a=s$) or long-range ($a=l$) potentials (1) or (3), or superpositions ($a=ls$) of such potentials, the elements (130) fall off slowly:[123,124]

$$V^a(r)\sim\sum_{n=n_-}^{\infty} r^{-n}V_n^a,\quad V_n^a\equiv\|V_{\nu\nu';n}\|,\quad r\to\infty, \tag{132}$$

where $n_-\geqslant 3$ for $a=s$ and $n_-=d$ for $a=l,ls$. For example,[124] for the state $\varepsilon=\{0,0,1,E\}$ of three identical particles interacting via Gaussian potentials

$$V_i(x_i)=V_0\exp(-(\tau x_i)^2),\quad V_0=\text{const},\quad \tau=\text{const}>0$$

the elements (130) with $\nu=\nu'=\nu_-=\{0,0,0\}$ fall off for $r\to\infty$ as

$$V_{\nu_-\nu_-}(r)=\frac{6V_0}{\pi\sqrt{\pi}}\sum_{n=1}^{\infty}(\tau r)^{-2n-1}\Gamma^{-1}(n+1)\times\left[\frac{\Gamma(1/2+n)}{\Gamma(1/2-n)}-\frac{\Gamma(5/2+n)}{\Gamma(5/2-n)}\right].$$

The slow falloff of the elements of the potential matrix should be given very close attention when studying the low-energy asymptotes of all functions associated with the problem (128) and (129). Otherwise, incorrect conclusions are unavoidable. To understand how such incorrect conclusions can arise, let us see how reliable are the basic constructions of Ref. 115, devoted to the "pure" three-particle process ($3\to 3$) in the state $\varepsilon=\{0,0,1,E\}$ of a system of three pairwise unbound identical particles with short-range two-body potentials. In this case $\nu_-=\{0,0,0\}$, and in (132) $a=s$ and $n_-=3$.

In Ref. 115 Jibuti and Sigua assumed that the asymptotic configuration of the ($3\to 3$) collision is the sum of plane and spherical waves:

$$\Psi^{\text{as}}(\vec{r};\vec{k})\equiv(2\pi)^{-3}\exp(i\vec{p}\vec{x}+i\vec{q}\vec{y})+\mathscr{F}(\Omega;\vec{k})r^{-5/2}\exp(ikr). \tag{133}$$

The asymptote (133) *postulated* by those authors does not contain a term corresponding to double rescattering processes.[10] It is not known whether or not this term dominates over the contribution from the outgoing spherical wave in the limit $k\to 0$. Therefore, the neglect of double rescattering is the first unjustified simplification made by those authors. The next simplification that they made for the *construction* of the "pure" three-particle scattering amplitude $\mathscr{F}$ concerned the problem (128), (129) and (133) already stated and amounted to the unjustified assumption that the matrix V is diagonal:

$$V_{\nu\nu'}(r)=V_{\nu\nu}(r)\hat{\delta}_{\nu\nu'}. \tag{134}$$

This "diagonal approximation" causes the system (128) to break up into a set of uncoupled one-dimensional equations

$$[\partial_r^2-\lambda(\lambda+1)r^{-2}+k^2-V_{\nu\nu'}(r)]u_\nu(r;\vec{k})=0, \tag{135}$$

which allowed the authors to use one of the nonlinear versions of the method of phase functions[12] and reduce each ($\nu=\nu_-,\ldots,\nu_+$) equation (135) by an appropriate substitution

$$u_\nu(r;\vec{k})=r^{1/2}A_\nu(r;\vec{k})U_\nu(r;\vec{k}),\quad \rho\equiv kr,$$
$$U_\nu(r;\vec{k})\equiv\cos\delta_\nu(r;k)j_\lambda(\rho)-\sin\delta_\nu(r;k)n_\lambda(\rho) \tag{136}$$

to two equations: a nonlinear equation

$$\partial_r\delta_\nu(r;k)=-k^{-1}V_{\nu\nu}(r)U_\nu(r;k) \tag{137}$$

for the phase function $\delta_\nu(r;k)$, and a linear equation determining the amplitude function $A_\nu(r;k)$.

It is known[2] that for

$$\int_0^\infty dx_i x_i|V_i(x_i)|<\infty,\quad i=1,2,3, \tag{138}$$

the asymptote of the solution of Eq. (137) in a small ($0\leqslant r\leqslant b$, $bk\ll 1$) semineighborhood of the point $r=0$ is given by the first iteration of the equivalent integral equation and has the form

$$\delta_\nu(r;k)\sim-k^{-1}\int_0^\tau dtV_{\nu\nu}(t)j_\lambda^2(\rho'),\quad \rho'\equiv kt,\quad r\to 0. \tag{139}$$

For the condition (132) with $n_-=3$ the solution of this equation (137) has[2] a *finite* limit for $r\to\infty$. Therefore, Jibuti and Sigua referred to the finite value $\delta_\nu(k)\equiv\delta_\nu(\infty;k)$ as the partial "true" three-particle phase shift and, assuming that each amplitude function is bounded at $r=\infty$, they wrote the asymptotes of the functions (136) as

$$u_\nu(r;k)\sim A_\nu(\infty;k)\sin(\rho-\pi\lambda/2+\delta_\nu(k)),\quad r\to\infty,$$
$$\lambda\equiv L+3/2. \tag{140}$$

Then, using (140) and taking without explanation

$$A_\nu(\infty;k)\equiv 1,\quad \nu=\nu_-,\ldots,\nu_+, \tag{141}$$

those authors wrote down the asymptote of the function (126) for $r\to\infty$ in terms of the partial phase shifts $\delta_\nu(k)$ and showed that this asymptote will have the required form (133) if the amplitude $\mathscr{F}$ is defined by the series

$$\mathscr{F}(\Omega;\vec{k})=\sum_{\nu=\nu_-}^{\nu_+}Y_\nu^e(\Omega)(Y_\nu^e(\Omega_k))^*\mathscr{F}_\nu(k) \tag{142}$$

in the partial amplitudes

$$\mathscr{F}_\nu(k)\equiv\sqrt{\frac{2}{\pi}}[-\exp(-i\pi/4)k^{-3/2}]\times(2ik)^{-1}[\exp(2i\delta_\nu(k))-1]. \tag{143}$$

The main error in the study by Jibuti and Sigua[115] is the statement that the leading term of the low-energy asymptote of the phase shift $\delta_\nu(k)$ coincides with the leading term of its coordinate asymptote (139) taken at the point $r=b$, $bk\ll 1$, and therefore has a power-law dependence on k:

$$\delta_\nu(k)\sim -\frac{\pi k^{2L+4}}{[2^{L+5/2}\Gamma(L+3)]^2}\int_0^b dr V_{\nu\nu}(r)r^{2L+5},$$

$$k\to 0. \tag{144}$$

Equations (142)–(144) led the authors to the incorrect relations

$$\mathscr{F}_\nu(k)=O(k^{2L+3/2}),\quad \mathscr{F}(\Omega;\vec{k})=O(k^{2L_-+3/2}),\quad k\to 0 \tag{145}$$

and, most disappointingly, to the incorrect conclusion that the amplitude $\mathscr{F}$ in the case $l=0$ and $\nu_-=\{0,0,0\}$ has an $E^{3/4}$ threshold behavior. This dependence and its qualitative explanation were later mentioned in Ref. 13. Therefore, before discussing how to overcome the restrictions (133), (134), and (141) imposed in Ref. 115, it is necessary to obtain the correct asymptotic relations with these restrictions. For this, we use the substitution $t_\nu(r;k)\equiv\tan\delta_\nu(r;k)$ to reduce Eq. (137) to the equation

$$\partial_r t_\nu(r;k)=-k^{-1}V_{\nu\nu}(r)[j_\lambda(\rho)-t_\nu(r;k)n_\lambda(\rho)]^2 \tag{146}$$

with the initial condition $t_\nu(0;k)=0$. The resulting equation differs functionally from the two-particle equation (7) only in that it contains the functions j_λ and n_λ with half-integer index $\lambda=L+3/2$ instead of the functions j_l and n_l with integer l. This difference does not hinder the application of the method of Levy and Keller,[26] described in Sec. 2.1 above, to Eq. (146), or the proof that for any ν the Born integral [the analog of the function (15) with $m=0$]

$$\beta_\nu(\infty;k)\equiv -k^{-1}\int_0^\infty dr V_{\nu\nu}(r)j_\lambda^2(\rho) \tag{147}$$

generated by the first iteration of this equation determines the leading term of the desired asymptote:

$$\tan\delta_\nu(k)\sim\beta_\nu(\infty,k)\sim -\frac{2V_{\nu\nu;3}}{(2L+3)(2L+5)}k,\quad k\to 0. \tag{148}$$

The same dependence, linear in k, of all the partial phase shifts in the limit $k\to 0$ is obtained (see Ref. 125) in the WKB approximation[4] and causes all the partial amplitudes (143) to diverge:

$$\mathscr{F}_\nu(k)\sim\frac{2V_{\nu\nu;3}\exp(-i\pi/4)}{(2L+3)(2L+5)}k^{-3/2},\quad k\to 0.$$

The complete amplitude (142) also has an analogous $k^{-3/2}$ singularity for any *finite* values of all three components of the multiple index ν_+. Therefore, with this condition any characteristic of a $(3\to 3)$ collision proportional to $|\mathscr{F}|^2$, for example, the differential cross section, must diverge as $O(k^{-3})$ for $k\to 0$. We arrive at the same conclusion if we replace the phase shift $\delta_{\nu_-}(k)$ by its asymptote (148) in the expression[116]

$$\sigma_{\nu_-}=264k^{-5}[\pi\sin\delta_{\nu_-}(k)]^2$$

describing the cross section for the $3\to 3$ collision of three neutrons in the $^3P_{1/2}$ state ($\varepsilon=\{1,m,-1,E\}$) within the minimal $\nu=\nu_-=\{0,1,1\}$ approximation.

Let us turn to the qualitative explanations. If in (147) we take

$$V_{\nu\nu}(r)\equiv 0,\quad r\geqslant b,\quad b<\infty, \tag{149}$$

then for any finite b it will be possible to choose k to be so small that the relation $kr\ll 1$ will hold, which allows the function $j_\lambda(\rho)$ to be approximated by the leading term of its asymptote for $\rho\to 0$. This replacement gives Eq. (144), which generates (145). Therefore, the asymptotes (144) and (145) obtained in Ref. 115 are valid in the case of an *artificial* cutoff (149) of all the elements $V_{\nu\nu}$ of the matrix V for *any* finite value b of the hyperradius r. However, owing to the slow falloff (132) of the elements $V_{\nu\nu}$, it is precisely this region of large values of the hyperradius ($kr\gg 1$) which determines the leading term (148) for $k\to 0$ of the asymptote of the integral (147).

The method proposed in Ref. 120 makes it possible in principle to go beyond the assumptions (133), (134), and (141). It can be used to construct the low-energy expansion of the function of interest Ψ^ε with arbitrary *a priori* specified asymptote Ψ^{as} and for any finite values of r and ν_+ and two-body potentials subject to two rather general conditions: the integrability condition (138) and the condition of absolute convergence of the series

$$V_i(x_i)=\sum_{n=-1}^{\infty}x_i^n V_i^{(n)},\quad x_i\to 0,\quad i=1,2,3,$$

ensuring the absolute convergence of the expansions

$$V_{\nu\nu'}(r)=\sum_{n=-1}^{\infty}r^n V_{\nu\nu'}^{(n)},\quad r\to 0 \tag{150}$$

of all the matrix elements (130) near the triple collision point.

The advantages and disadvantages of this method become obvious if we apply it to the most general case, where the coefficients $V_{\nu\nu'}^{(-1)}$ of all $(\nu,\nu'=\nu_-,\ldots,\nu_+)$ the series (150) are nonzero.

The first step is the definition and construction of the fundamental[2] matrix $U=\|U_{\nu\nu'}\|$ of regular solutions of Eq. (128). Each of its columns $U^{\nu'}$ $(\nu'=\nu_-,\ldots,\nu_+)$ is defined as a solution of this equation, i.e., the equation

$$[(\partial_r^2+k^2)I+D_0r^{-2}-V(r)]U^{\nu'}(r;k)=0 \tag{151}$$

with the *special* initial $(r\to 0)$ condition

$$U_\nu^{\nu'}(r;k)\equiv U_{\nu\nu'}(r;k)=r^{\lambda'+1}[\hat{\delta}_{\nu\nu'}+O(r)(1-\hat{\delta}_{\nu\nu'})],\quad \nu=\nu_-,\ldots,\nu_+. \tag{152}$$

As shown by Palumbo,[126] the conditions (152) guarantee the linear independence of the columns $U^{\nu'}$, which implies that it is possible[2] to represent the general regular solution u of Eq. (128) as a linear combination

$$u(r;\vec{k})=U(r;k)N(\vec{k}) \tag{153}$$

of columns $U_\nu^{\nu'}$ and coefficients $N_\nu(\vec{k})$ arranged as columns $N(\vec{k})$. The particular solution u with given physical asymptote (131) is obtained by choosing the column $N(\vec{k})$ to be the solution of the linear matrix equation

$$U(r;k)N(\vec{k})\sim u^{\mathrm{as}}(r;\vec{k}),\quad r\to\infty. \tag{154}$$

Therefore, the main advantage of the method is its universality in the following sense: if the fundamental matrix is constructed at a given energy, then Eqs. (124)–(126), (153), and (154) can be used to calculate the wave function describing any process possible at this energy in the three-particle system in question. However, this advantage is difficult to realize computationally, owing to the rapid growth of the number of equations of the system (151) when even one component of the multiple index ν_+ is increased.

It is convenient to use the matrix linear version of the method of phase functions[120] to study the $k\to 0$ asymptote of the solution U of the problem (151) and (152). Here for U we use the ansatz

$$U(r;k)=j(\rho)c(r;k)-n(\rho)s(r;k), \tag{155}$$

which contains the diagonal matrices $j\equiv\mathrm{diag}\{j_\lambda\}$ and $n\equiv\mathrm{diag}\{n_\lambda\}$ and the unknown amplitude matrices $c\equiv\|c_{\nu\nu'}\|$ and $s\equiv\|s_{\nu\nu'}\|$. The substitution (155) reduces the problem (151) and (152) to the matrix equations

$$\partial_r\begin{Bmatrix}c(r;k)\\ s(r;k)\end{Bmatrix}=-k^{-1}\begin{Bmatrix}n(\rho)\\ j(\rho)\end{Bmatrix}V(r)U(r;k) \tag{156}$$

with the initial ($r\to 0$) conditions

$$k^{\lambda+1}c_{\nu\nu'}(r;k)\sim\hat{\delta}_{\nu\nu'}/\sigma_\lambda-\tau_\lambda V_{\nu\nu'}^{(-1)}\times\begin{Bmatrix}r^\tau/\tau, & \tau\equiv\lambda'-\lambda+1\neq 0\\ \ln r, & \tau=0\end{Bmatrix},$$

$$k^{-\lambda}s_{\nu\nu'}(r;k)\sim-\sigma_\lambda V_{\nu\nu'}^{(-1)}(\lambda+\lambda'+2)^{-1}r^{\lambda+\lambda'+2}, \tag{157}$$

ensuring the required behavior (152) of the functions (155).

For deriving Eq. (157) it is necessary to take into account the condition (152), to replace the elements of the matrix V in (156) by the series (150), to approximate the functions j_λ and n_λ by their asymptotes $j_\lambda(\rho)\sim\sigma_\lambda\rho^{\lambda+1}$ and $n_\lambda(\rho)\sim\tau_\lambda\rho^{-\lambda}$ for $\rho\to 0$, and, finally, to find the first iteration of the resulting equations.

The second disadvantage of this method is the singular initial conditions (157) for the functions $c_{\nu\nu'}(r;k)$ in the case $\lambda'+1\leqslant\lambda$. This is not very important, but it does complicate the numerical solution of the problem (156) and (157).

The next difficulty generated by the construction of the function $n_\lambda(\rho)$ with half-integer index is easily overcome. The function $j_\lambda(\rho)$ with half-integer index $\lambda=L+3/2$ is represented in the form (8), and the expansion[1] of the function n_λ contains the energy-dependent factor $(2/\pi)\ln(\rho/2)$. The separation of r and k in the problem (156) and (157) begins by writing this factor in an identical form involving the arbitrary parameter τ (Ref. 121):

$$\frac{2}{\pi}\ln\left(\frac{\rho}{2}\right)\equiv h^s(k\tau)+\frac{2}{\pi}\ln\left(\frac{r}{2\tau}\right),\quad h^s(k\tau)\equiv\frac{2}{\pi}\ln(k\tau).$$

This identity allows the factor $h^s(k\tau)$ nonanalytic in k to be separated from the function $n_\lambda(\rho)$, which can then be written as

$$n_\lambda(\rho)=\tilde{n}_\lambda(\rho)+h^s(k\tau)j_\lambda(\rho),$$

$$\tilde{n}_\lambda(\rho)=k^{-\lambda}\sum_{n=0}^{\infty}k^{2n}n_{\lambda n}(r),$$

$$n_{\lambda n}(r)\equiv-(r/2)^{2n-\lambda}\frac{\Gamma(\lambda+1/2-n)}{\sqrt{\pi}\Gamma(n+1)},\quad n=0,\ldots,\lambda-1/2;$$

$$n_{\lambda n}(r)\equiv(1/\pi)\left[\ln\left(\frac{r}{2\tau}\right)-\psi(n+1)-\psi(n-\lambda+1/2)\right]\times j_{\lambda,n-\lambda-1/2}(r),\quad n=\lambda+1/2,\ldots. \tag{158}$$

Then in (155)–(157) the functions j_λ and n_λ are replaced by their known sums (8) and (158), while the matrices c and s are replaced by the desired matrix series

$$c(r;k)=\tilde{c}(r;k)+h^s(k\tau)s(r;k),$$

$$\tilde{c}_{\nu\nu'}(r;k)=k^{-\lambda-1}\sum_{n=0}^{\infty}k^{2n}c_{\nu\nu';n}(r),$$

$$s_{\nu\nu'}(r;k)=k^{\lambda}\sum_{n=0}^{\infty}k^{2n}s_{\nu\nu';n}(r). \tag{159}$$

In the end we obtain the formal asymptotic expansion

$$U(r;k)=\sum_{n=0}^{\infty}k^{2n}U_n(r),$$

$$U_n(r)\equiv\sum_{m'+m=n}[j_{m'}(r)c_m(r)-n_{m'}(r)s_m(r)] \tag{160}$$

containing the matrices $c_n\equiv\|c_{\nu\nu';n}\|$ and $s_n\equiv\|s_{\nu\nu';n}\|$ obeying a recursion ($n=0,1,\ldots$) chain of energy-independent equations

$$\partial_r\begin{Bmatrix}c_n(r)\\ s_n(r)\end{Bmatrix}=-\sum_{m'+m=n}\begin{Bmatrix}n_{m'}(r)\\ j_{m'}(r)\end{Bmatrix}V(r)U_m(r) \tag{161}$$

with the initial ($r\to 0$) conditions

$$c_{\nu\nu';n}(r)\sim\hat{\delta}_{n0}\hat{\delta}_{\nu\nu'}/\sigma_\lambda-\hat{\delta}_{n0}\tau_\lambda V_{\nu\nu'}^{(-1)}\times\begin{Bmatrix}r^\tau/\tau, & \tau\equiv\lambda'-\lambda+1\neq 0,\\ \ln r, & \tau=0,\end{Bmatrix},$$

$$s_{\nu\nu';n}(r)\sim-\hat{\delta}_{n0}\sigma_\lambda V_{\nu\nu'}^{(-1)}(\lambda+\lambda'+2)^{-1}r^{\lambda+\lambda'+2}. \tag{162}$$

Owing to the slow falloff (132) of the elements $V_{\nu\nu'}(r)$, all the functions $s_{\nu\nu';n}(r)$ satisfying Eq. (161) and the boundary conditions (162) are unbounded for $r=\infty$. Therefore, the approximation of the infinite series (159) by their finite subsums is valid only in the interior region $0\leqslant r\leqslant b_-$, $kb_-\ll 1$ and allows the following asymptotic representation of the series (160) to be obtained only in this region:

$$U(r;k)=\sum_{n=0}^{M<\infty} k^{2n}U_n(r)+O(k^{2M+2}),\quad k\to 0.$$

As shown in Ref. 49, it is not difficult to construct the asymptote of the matrix U for $\rho\to\infty$ if the ansatz (155) and the key problem (156) and (157) are rewritten in terms of the diagonal matrices $h^{\pm}\equiv j\pm in=\mathrm{diag}\{h_\lambda^{\pm}\}$ composed of Riccati–Hankel functions $h_\lambda^{\pm}=j_\lambda\pm in_\lambda$ and the amplitude matrices $f^{\pm}\equiv c\pm is$. Then the ansatz (155) and the system (156) take the form

$$U(r;k)=2^{-1}[h^{+}(\rho)f^{+}(r;k)+h^{-}(\rho)f^{-}(r;k)],\quad (163)$$

$$\partial_r f^{\pm}(r;k)=(\mp i/2k)h^{\mp}(\rho)V(r)U(r;k).\quad (164)$$

If in (164) we substitute $V=V^a$ in the form (132), replace the matrices $h^{\pm}$ by the series[1]

$$h^{\pm}(\rho)=\exp(\pm i\rho)\sum_{n=0}^{\infty}\rho^{-n}h_n^{\pm},\quad \rho\to\infty,$$

$$h_n^{\pm}\equiv(\mp 2i)^{-n}\,\mathrm{diag}\{(\lambda+1/2,n)\exp(\mp i\pi(\lambda+1/2)/2)\},\quad (165)$$

and seek the matrices $f^{\pm}$ in the form

$$f^{\pm}(\rho)=\sum_{n=0}^{\infty}\rho^{-n}[C_{1n}^{\pm}(k)+\exp(\mp 2i\rho)\rho^{-3}C_{2n}^{\pm}(k)],\quad (166)$$

then for the matrices $C_{jn}^{\pm}$ we obtain the recursion ($n=1,2,..$; $j=1,2$) chain of equations

$$2(n-1)iC_{j,n-1}^{\pm}(k)-\sum_{m'+m=n}k^{m'-2}V_{m'}D_{jm}^{\pm}(k)=0,$$

$$8iC_{j,n}^{\mp}+4(1-n)C_{j,n-1}^{\mp}+\sum_{m'+m=n}k^{m'-2}V_{m'}D_{jm}^{\mp}(k)=0;$$

$$D_{jn}^{\pm}\equiv\sum_{m'+m=n}[\tau_{1m'}C_{jm}^{\pm}+(-1)^{m'}\tau_{2m'}C_{jm}^{\mp}],$$

$$\tau_{jn}\equiv(-2i)^{-n}\sum_{m'+m=n}(-1)^{(j-1)m'}h_{m'}^{+}h_{m}^{-},\quad (167)$$

determining all such matrices with $n>0$ in terms of the matrices $C_{j0}^{\pm}$. The latter can be found by matching, at a sufficiently distant ($kb_+\gg 1$) point $r=b_+$,

$$f^{\pm}(b_+-0;k)=f^{\pm(M)}(b_++0;k),$$

$$\partial_r f^{\pm}(b_+-0;k)=\partial_r f^{\pm(M)}(b_++0;k),$$

the functions $f^{\pm}$, calculated by integrating (156) or (164) over a finite range $0\leqslant r\leqslant b_+$, to the functions $f^{\pm(M)}$ generated by approximating the series (166) by their finite ($M<\infty$) subsums. If now in (163) the series (165) and (166), (167) are approximated by their finite subsums, we obtain the desired asymptotic representation

$$U(r;k)=(1/2)[U^{+(M)}(r;k)+U^{-(M)}(r;k)]+O(\rho^{-M-1}),\quad \rho\to\infty,$$

$$U^{\pm(M)}(r;k)\equiv\exp(\pm i\rho)\sum_{n=0}^{M}\rho^{-n}\sum_{m'+m=n}[h_{m'}^{\pm}C_{1m}^{\pm}(k)+\rho^{-3}h_{m'}^{\mp}C_{2m}^{\mp}(k)],$$

allowing the assumption (149) to be dropped, i.e., making it possible to include all two-body interactions for any $r\geqslant b_+$.

As our next example following the derivation of the correct asymptote (148) of the phase shift $\delta_\nu(k)$ and showing how important it is to take it into account, we present the main conclusion of Ref. 49: the artificial cutoff (149) of the matrix V at any finite value of the hyperradius $r=b$ can generate three-particle resonances which vanish in the limit $b\to\infty$ and are therefore termed artificial.

Now let the two-body interactions be described by the superpositions $V_i=V_i^c+V_i^a$ of Coulomb potentials $V_i^c=1/R_ix_i$ and potentials V_i^a, $a=s,l,ls$, which fall off more rapidly. Then the potential matrix V is the sum $V=V^c+V^a$ of the Coulomb matrix V^c with the elements

$$V_{\nu\nu'}^c(r)=\sum_{i=1}^{3}\langle Y_\nu|(R_ix_i)^{-1}|Y_{\nu'}\rangle=(R_{\nu\nu'}r)^{-1}\quad (168)$$

and the matrix V^a, the elements of which fall off according to the law (132) in any of the three cases ($a=s,l,ls$). The application of the method of phase functions to the matrix problem (128) and (129) with $V=V^{ca}$, with the Coulomb functions chosen as the reference functions, is hindered by the fact that the Coulomb matrix (168) is not diagonal. All the results listed below were obtained with the assumption (134), which causes the system (128) to break up into uncoupled equations (135). Each such equation is a one-dimensional Schrödinger equation with *half-integer* angular momentum λ and the superposition $V_{\nu\nu}=V_{\nu\nu}^{ca}=V_{\nu\nu}^c+V_{\nu\nu}^a$ of Coulomb and long-range potentials. It is therefore easily reformulated in either the nonlinear[117] or linear[122] version of the method of phase functions.

Analytic expressions for the coefficients $R_{\nu\nu'}$ of the Coulomb interactions (168) and the regular and irregular solutions of Eq. (135) in the case $V_{\nu\nu}=V_{\nu\nu}^c$, i.e., the Coulomb functions $F_\lambda(\rho,\eta_\nu)$ and $G_\lambda(\rho,\eta_\nu)$ with half-integer angular momentum λ and parameter $\eta_\nu=1/2kR_{\nu\nu}$, were obtained in Ref. 127 and discussed in Ref. 119.

By analogy with the two-particle expression (20), we can subtract the Coulomb phase shift $\delta_\nu^c(k)$ from the phase shift $\delta_\nu^{ca}(k)$ generated by the superposition $V_{\nu\nu}^{ca}$ to obtain the phase shift $\delta_\nu^{c,a}(k)$:

$$\delta_\nu^{c,a}(k)=\delta_\nu^{ca}(k)-\delta_\nu^c(k),\quad \delta_\nu^c(k)=\arg\Gamma(\lambda+1+i\eta_\nu).\quad (169)$$

In the case of Coulomb repulsion and artificial cutoff (149) of the potential $V_{\nu\nu}^a(r)$ at any finite value of the hyperradius $r=b$, the effective-range function corresponding to the splitting (169) has the form[122]

$$K_\nu^{c,a}(E)\equiv[k^\lambda C_\lambda(\eta_\nu)]^2[k\cot\delta_\nu^{c,a}(k)+h^c(\eta_\nu)],$$

where C_λ and h^c are the functions (22) for $l=\lambda$ and $\eta=\eta_\nu$. If we use the potential $V_{\nu\nu}^a(r)$ without a cutoff, taking $b\to\infty$ in (149), and then consider the limit $a^{c,a}$ of the function $-1/K_\nu^{c,a}(E)$ for $E\to 0$, we obtain $|a_\nu^{c,a}|=\infty$. This attempt to

determine the modified three-particle scattering length $a_\nu^{c,a}$ proved to be unsound for the same reason as in the case of the scattering of two particles by a superposition of Coulomb and long-range potentials. The point is that, owing to the slow falloff of the potential $V_{\nu\nu}^a(r)$, the low-energy asymptote of the phase shift $\delta_\nu^{c,a}(k)$ does not contain the factor $k^{2\lambda+1}C_\nu^2(\eta_\nu)$ and for any λ has the form[125]

$$\delta_\nu^{c,a}(k)\sim -V_{\nu\nu;n_-}(2R_{\nu\nu})^{n_--1}(\pi/8\tau)^{1/2}\cdot\Gamma(n_--1)(\tau^2-1)^{(3-2n_-)/4}P_{-1/2}^{3/2-n_-}(1/\tau),$$

$$\tau^2\equiv 1+4(kR_{\nu\nu})^2\lambda(\lambda+1),$$

where n_- is the exponent of the leading term of the asymptote (132) of the function $V_{\nu\nu}^a(r)$, and P_ν^μ is the Legendre function of the first kind.[1]

Therefore, a combination of the method of phase functions and the theory of hyperharmonics leads to progress in the study of low-energy expansions in the three-particle problem (126)–(131) with a truncated ($\nu_+<\infty$) basis of hyperharmonics (124). It is not known how to go to the complete basis, i.e., to the limit $\nu_+\to\infty$ in this combination of methods. An alternative is to replace the discrete variable, the hyperangular momentum L, by its continuous analog, the hyperangle φ_i. This replacement is possible and leads to integro-differential Faddeev equations.

4.2. The method of integro-differential equations in the three-particle problem

As shown in Refs. 128–130, if we project onto the basis of hyperharmonics (124) not the Schrödinger equation (127), but the system of three differential Faddeev equations corresponding to it,

$$(H_0-E)\Psi_i^\varepsilon=-V_i\Psi^\varepsilon=-V_i\sum_{i=1}^3\Psi_i^\varepsilon,\quad i=1,2,3,\quad (170)$$

for the Faddeev components

$$\Psi_i^\varepsilon(r,\Omega_i;E)=r^{-2}\sum_{\nu=\nu_-}^\infty u_{i\nu}(r;E)Y_\nu^\varepsilon(\Omega_i)$$

of the wave function Ψ^ε, and then sum the resulting equations for the unknown functions $u_{i\nu}$ over the index L, we obtain the integro-differential Faddeev equations for the components $\Psi_{i\alpha}^\varepsilon$ of the expansions

$$\Psi_i^\varepsilon(r,\Omega_i;E)=2r^{-2}\operatorname{cosec}2\varphi_i\sum_{\alpha=\alpha_-}^\infty\Psi_{i\alpha}(r,\varphi_i;E)\mathscr{Y}_\alpha^\varepsilon(\vec{x}_i,\vec{y}_i)\quad (171)$$

in the basis (123). The analytic properties of the integro-differential equations have been studied fairly thoroughly. The asymptotes of the components $\Psi_{i\alpha}^\varepsilon$ for $r\to\infty$ are known[113] for all possible processes in a system of three neutral or charged particles with *nonzero* collision energy. The asymptotes of the components $\Psi_{i\alpha}^\varepsilon$ for $r\to 0$ can be constructed in various ways.[131–133] The spectral representation[129,130,134] of the most complex objects of the integro-differential equations—the integrals of the operators $h_{\alpha\alpha'}^l$ over the variable φ_i—and the compact expressions for their kernels[130] made it possible to carry out a series of analytic studies.[135–140] A simple method was suggested in Ref. 135 for regularizing the integro-differential equations at the end points $\varphi_i=0,\ \pi/2$ of their angular arguments φ_i, $i=1,2,3$. Differential and integral representations of the Raynal–Revai coefficients were obtained in Ref. 136. Some exact solutions of the problem of three identical particles with S-wave pair potentials $V_i=\text{const}\,x_i^{-2}$ were found in Ref. 137. Following Ref. 141, spurious solutions of the integro-differential equations for the same three-particle system but with oscillator interactions $V_i=\omega x_i^2$ were studied in Ref. 138. The criterion for the existence of spurious solutions of the equations (170) in the case of central potentials V_i was proved in Refs. 139 and 140, and a method for constructing such solutions explicitly was suggested and explained by examples in Ref. 140.

The spectrum of the spurious solutions coincides with that of the free three-particle Hamiltonian.[142] The spurious solutions do not correspond to any physical configuration of the three-particle system, because by definition such solutions make the right- and left-hand sides of the system (170) vanish simultaneously and are independent of the interactions V_i. Therefore, it is impossible to extract any information from the explicitly known spurious solutions for the three-nucleon system[143] which is useful in the construction of low-energy expansions of physical solutions of the integro-differential Faddeev equations. Nevertheless, these equations are attractive for constructing such expansions from the computational point of view, because they combine relatively simple boundary conditions with sparseness of the matrix of the discrete problem. The efficient algorithms[144–150] for solving this problem are based on approximating the desired solutions by various splines.[3] The tensor representation[147] of the auxiliary matrices considerably simplifies the calculations.

In order to understand the obstacles to deriving three-particle low-energy expansions within the integro-differential Faddeev problem, let us consider a case where this problem not only looks as simple as possible, but also has been solved several times by completely different methods.

In the case of S-wave nuclear potentials V_i^s, the problem of pd scattering in the quantum state $\varepsilon=\{l,m,S,E\}$ with $l=0$, total spin $S=3/2$, and pd collision energy $E=(3/4m_N)(\hbar k)^2$ less than the deuteron binding energy $B(d)$ [$\tau^2(k)\equiv B(d)-E>0$] is reduced by the separation of the discrete spin–isospin variables, as described in detail in Ref. 151, to an infinite system of equations coupled through the index α (Ref. 10). The coupling is a consequence of the fact that the matrix of the Coulomb pp interaction $V_1^c=1/Rx_1$ is nondiagonal in the basis of bispherical harmonics (123). If in the series (171) for the spatial parts of the Faddeev components we limit ourselves to the first term, i.e., if we take[10]

$$\Psi_i^\varepsilon(r,\Omega_i;E)=2r^{-2}\operatorname{cosec}2\varphi\Psi_{i,\alpha_-}^\varepsilon(r,\varphi;E)\mathscr{Y}_{\alpha_-}^\varepsilon(\vec{x}_i,\vec{y}_i)$$

$$=(2r^2\pi)^{-1}\operatorname{cosec}2\varphi U(r,\varphi;k),$$

$\alpha_-=\{0,0\}$,

there remains a single integro-differential equation[10]

$$[\partial_r^2+r^{-1}\partial_r+r^{-2}\partial_\varphi^2+\tau^2(k)-V^c_{\alpha_-\alpha_-}(r,\varphi)]U(r,\varphi;k)$$

$$=V^s(r\cos\varphi)\left[1+\frac{4}{\sqrt{3}}\int_{t_-(\varphi)}^{t_+(\varphi)}d\varphi' U(r,\varphi';k)\right] \quad (172)$$

with variable integration limits[144]

$$t_\pm(\varphi)\equiv\min\{|\varphi\pm\pi/3|,2\pi/3-\varphi\} \quad (173)$$

and matrix element

$$V^c_{\alpha_-\alpha_-}(r,\varphi)=\frac{1}{rR_0(\varphi)},$$

$$R_0(\varphi)\equiv\frac{R}{2}\begin{cases}\cos(\varphi), & \varphi<\pi/6\\ \sqrt{3}\sin\varphi, & \varphi\geq\pi/6\end{cases} \quad (174)$$

of the Coulomb interaction V_1^c.

It should be noted that the existence and uniqueness of the regular solution

$$U(0,\varphi;k)=0;\quad U(r,\varphi;k)=0,\quad \varphi=0,\pi 2, \quad (175)$$

of this equation have been proved rigorously[10] only in a single case: when k is a *nonzero* fixed parameter, and the asymptote of U for $r\to\infty$ has a completely determined form. Using the splitting (28) of the pd phase shift $\delta_0^{cs}(k)$ and Eqs. (29)–(32) for the total pd amplitude f^{ca} with $a=s$, this asymptote can be written as

$$U(r,\varphi;k)=k^{-1}u_d(x)[\sin\theta^c(t;\eta)$$
$$+\tan\delta_0^{c,s}(k)\cos\theta^c(t;\eta)+O(y^{-1})]$$
$$+[A^{br}(\varphi;k)+O(r^{-1})]r^{-1/2}$$
$$\times\exp(-\tau(k)r),\quad r\to\infty. \quad (176)$$

Here $x\equiv x_1=r\cos\varphi$, $y\equiv y_1=r\sin\varphi$, $u_d(x)$ is the radial $^3S_1$ component of the deuteron wave function, and the trigonometric functions of the argument θ^c,

$$\theta^c(t;\eta)\equiv t-2\eta\ln t+\delta_0^c(k),\quad t\equiv ky,$$
$$\eta\equiv 1/2kR_0(\pi/2),$$

coincide with the *leading* terms of the asymptotes

$$F_0(t,\eta)=\sin\theta^c(t;\eta)(1+O(t^{-1})),$$
$$G_0(t,\eta)=\cos\theta^c(t;\eta)(1+O(t^{-1}))$$

of the Coulomb functions[1] for $t=ky\to\infty$.

This agreement apparently stimulated the authors of Refs. 152 and 153 to replace the asymptote (176) by the, in general, *inequivalent* asymptote

$$U(r,\varphi;k)=k^{-1}u_d(x)[F_0(t,\eta)+\tan\delta_0^{c,s}(k)G_0(t,\eta]$$
$$+[A^{br}(\varphi;k)+O(r^{-1})]r^{-1/2}$$
$$\times\exp(-\tau(k)r),\quad r\to\infty. \quad (177)$$

It is inequivalent because the asymptote (177), in contrast to (176), contains all the terms of the expansions of the Coulomb functions in infinite series[1] in inverse integer powers of the argument ky. The existence and uniqueness of the solution of this problem (172)–(175), (177) so far have not been proved. However, the *postulated* asymptotic condition (177) motivated the authors of Refs. 152 and 153 to go to the zero-energy limit ($k\to 0$) and obtain in this limit the boundary condition

$$U(r,\varphi;0)\sim f_{00}(\xi)-a_0^{c,s}g_{00}(\xi),\quad \xi\equiv 2\sqrt{y/R_0(\pi/2)} \quad (178)$$

with finite coefficient $a_0^{c,s}$, which they identified with the quartet Coulomb–nuclear pd scattering length ($^4a_{pd}$), and the first terms f_{00} and g_{00} of the series (84). Equation (172) with $k=0$ and the boundary conditions (175) and (178) obtained in this manner allows this scattering length to be found directly, i.e., without calculating the phase shift $\delta_0^{c,s}(k)$ by solving the energy-dependent problem (172)–(175), (177) for a sequence of values of k which converge to zero but are nonzero and then finding $a_0^{c,s}$ by extrapolating the effective-range function (21) to the point $k=0$.

We note that this passage from the problem (172)–(175), (177) with $k>0$ to the problem (172)–(175), (178) with $k=0$ was accomplished in two steps, which require yet another assumption. First, Eq. (172) and the boundary conditions (175) and (177) are divided by the Coulomb barrier factor $C_0(\eta)$. Then we take $k\to 0$ and replace the functions $\tan\delta_0^{c,s}(k)$ and F_0, G_0 in (177) by their expansions (21) and (84). If it is assumed that for $k\to 0$ the amplitude of virtual deuteron breakup $A^{br}(\varphi;k)$ falls off more quickly than $C_0(\eta)$, we obtain (178). There are two obstacles to justifying this assumption and constructing the low-energy expansions of the correctly posed problem (172)–(176). First, there is the absence of complete asymptotic expansions of the Faddeev components for fixed $k>0$ and $r\to\infty$. Second, there is the absence of any asymptotic expansions of these components for $kr\to\infty$. The derivation of the second type of asymptote is an important aspect of the three-body problem. The complete solution of this problem would allow energy-independent boundary conditions for $r\to\infty$ to be specified for each equation obtained by the substitution of

$$U(r,\varphi;k)=N(k)\sum_{n=0}^{\infty}k^{2n}U_n(r,\varphi) \quad (179)$$

into (172) followed by division by k. The problem thus formulated for the functions $U_n(r)$ with $n=0,1,\ldots,M<\infty$, which is energy-independent and recursive in the index n, could then be solved numerically. Then the unknown normalization factor $N(k)$ and the phase shift $\delta_0^{c,s}(k)$ could be found by matching the finite ($M<\infty$) subsum of the series (179) at some sufficiently distant point $r=b<\infty$ to the constructed asymptotic expansion of the function U for $kr\to\infty$.

These remarks about the incompleteness of the scattering theory for a three-particle system should be borne in mind when comparing the results of studies devoted to finding the Nd scattering parameters, which are discussed below.

5. LOW-ENERGY *Nd* SCATTERING

The scattering of thermal neutrons by deuterons has been well studied experimentally.[154–158] The results of many the-

oretical studies are in satisfactory agreement with the experimental results and have been discussed many times in reviews[153,159–162] and reports.[163,164] Therefore, in discussing the current ideas about nd scattering in the limit of zero nd collision energy, it is reasonable to limit ourselves to mentioning several recent studies, while devoting more attention to studies of the correlation dependences in the nnp system. This helps not only to explain why the construction of such dependences is a promising method of determining the parameters of low-energy extrapolation formulas for Nd scattering, but also to better understand why the problem of finding the pd scattering lengths is still unsolved.

In spite of the real advances made in the theoretical description of the dynamics of $3N$ systems, there is always the important question of the degree to which the $3N$ data obtained are consistent with each other within the context of the present ideas about NN forces. The traditional method of solving this problem is to calculate $3N$ parameters for different variants of the NN potentials, whose parameters are fixed by the low-energy data on two-nucleon scattering. As a rule, there is a sizable spread of the values calculated for different NN potentials. This method of trial and error requires rather cumbersome calculations, and does not shed light on the general situation, i.e., it does not allow systematization of the results known earlier or reliable prediction of new results. However, the results of early nnp calculations performed by various methods but taken all together led Phillips[159] in 1968 to establish the presence of a linear correlation dependence of the doublet nd scattering length ${}^2a_{nd}$ on the tritium binding energy $B({}^3\mathrm{H})$. That curve is called the Phillips nd line, and study of the correlation dependences of two characteristics of the three-particle system was accepted in Ref. 160 as a method in few-particle theory allowing the prediction of the value of one unknown characteristic from the known value of another.

In the construction of low-energy $3N$ expansions, it is particularly interesting to understand why parameters of these expansions which are difficult to calculate have a correlation dependence on the relatively easily calculated parameters of $2N$ scattering. Knowledge of this dependence would not only permit the systematization of the known calculations, but also the accurate prediction of the results of new calculations of the $3N$ parameters from the known values of the $2N$ parameters. The reason why both the scattering length ${}^4a_{nd}$ and the effective range ${}^4r_{nd}$ of quartet nd scattering depend on the scattering length ${}^3a_{np}$ and the effective range ${}^3r_{np}$ of triplet np scattering was explained in Ref. 165. The authors found the functional dependences

$$
\begin{aligned}
&{}^4a_{nd}(\tau)/{}^3a_{np}\approx 1.179066-0.03595\tau-0.007448\tau^2,\\
&{}^4r_{nd}(\tau)/{}^3a_{np}\approx -0.0383+1.0558\tau, \quad \tau\equiv {}^3r_{np}/{}^3a_{np}
\end{aligned}
\tag{180}
$$

of the nd parameters on the ratio τ and obtained the expressions

$$
\begin{aligned}
&{}^4a_{nd}\approx[6.327+1.57({}^3a_{np}-5.414)-0.32({}^3r_{np}-1.705)]\ \mathrm{F},\\
&{}^4r_{nd}\approx[1.736+0.92({}^3a_{np}-5.414)+0.44({}^3r_{np}-1.705)]\ \mathrm{F},
\end{aligned}
\tag{181}
$$

relating these parameters to the deviations of the np parameters from their average experimental values[166] ${}^3a_{np}^{\mathrm{exp}}=(5.414\pm0.005)$ F and ${}^3r_{np}^{\mathrm{exp}}=(1.750\pm0.005)$ F. Let us give the most interesting qualitative results of Ref. 165. According to (180), ${}^4a_{nd}(\tau)$ is a decreasing function of the ratio τ, reaching its maximum (≈ 6.383 F) in the limit ${}^3r_{np}\to 0$. This value is an upper limit and differs insignificantly from the value 6.316 F obtained by substituting the averaged experimental data ${}^3a_{np}=5.414$ F and ${}^3r_{np}=1.705$ F into (180). The fact that this difference is insignificant implies that the scattering length ${}^4a_{nd}$ is highly model-independent. In contrast, the effective range ${}^4r_{nd}(\tau)$ depends significantly on the NN-force model used: if ${}^3r_{np}=0$, then ${}^4r_{nd}\approx -0.207$ F; for all ${}^3r_{np}\neq 0$ we have ${}^4r_{nd}(\tau)\approx {}^3r_{np}$. The correlation dependences (180) and (181) systematize all the numerous calculations of the scattering length ${}^4a_{nd}$ mentioned in the review of Ref. 160, and allow both the results of three-particle calculations and the results of future experiments to be predicted from the known two-particle data. For example, it follows[165] from (181) that the preferred experimental value of the parameter ${}^4r_{nd}$ must lie in the range from 1.73 F to 1.74 F.

This prediction is in conflict with the value ${}^4r_{nd}=1.91$ F calculated by Yakovlev and Filikhin[167] using the strong channel-coupling method for the problem (172)–(176) with the MTI-III version of the potential V^s (Ref. 168). Nevertheless, the value ${}^4a_{nd}=6.44$ F obtained by the same authors is very close to the values ${}^4a_{nd}=6.4$ F (Refs. 113 and 169) and ${}^4a_{nd}=6.43$ F (Ref. 170) calculated earlier for the same NN potentials.

The results of recent three-particle calculations[171,172] of the scattering lengths ${}^2a_{nd}$ and ${}^4a_{nd}$ together with $B({}^3\mathrm{H})$ are in good agreement with the experimental values ${}^2a_{nd}^{\mathrm{exp}}=(0.65\pm0.04)$ F, ${}^4a_{nd}^{\mathrm{exp}}=(6.35\pm0.02)$ F (Ref. 154), and $B({}^3\mathrm{H})=(8.481855\pm0.000013)$ MeV (Ref. 89). The inclusion of three-particle forces in the description of the nd dynamics[171,172] does not change the earlier linear correlation dependence[160] between ${}^2a_{nd}$ and $B({}^3\mathrm{H})$ calculated with realistic two-particle NN interactions. At the present time it is considered a well-established and tested fact that the Phillips nd line passes through many points calculated in different ways and, moreover, through the point corresponding to the experimental values of ${}^2a_{nd}$ and $B({}^3\mathrm{H})$.

Since the data on mirror nuclear systems contain valuable information about the charge symmetry of NN forces, it is especially interesting to use the same approach to perform a parallel study of the process mirroring nd scattering, i.e., elastic pd collisions. This mirror process is much less well studied, both experimentally and theoretically. The reason for the former is the repulsive Coulomb interaction, which suppresses the probability for the proton to approach the deuteron the more, the smaller the pd-collision energy E. This makes the direct measurement of the pd scattering length at $E=0$ impossible, and the measurement of the pd scattering cross section at $E_{\mathrm{lab}}<10$ MeV is so complicated that by the present time only three sets of experimental values for the doublet and triplet scattering lengths have been published:

$$^2a_{pd}^{\exp}=(1.3\pm0.2)\ \text{F},$$

$$^4a_{pd}^{\exp}=\left(11.4^{+1.8}_{-1.2}\right)\ \text{F (Ref. 173)};$$

$$^2a_{pd}^{\exp}=(2.73\pm0.10)\ \text{F},$$

$$^4a_{pd}^{\exp}=\left(11.88^{+0.4}_{-0.1}\right)\ \text{F (Ref. 174)},$$

$$^2a_{pd}^{\exp}=(4.0\pm0.66)\ \text{F},$$

$$^4a_{pd}^{\exp}=(11.11\pm0.24)\ \text{F (Ref. 175)}. \qquad (182)$$

The first[173] and second[174] were obtained by extrapolating data measured at proton energies above 1 MeV, and the third[175] was obtained by extrapolation from energies above 0.4 MeV. According to (182), there is a significant discrepancy between the experimental values of the doublet scattering length obtained in Refs. 173–175 and a much smaller discrepancy between the corresponding values of the quartet scattering length. A possible reason for this is the use of the traditional low-energy expansion (23) for extrapolating data measured at high energies to zero energy.

What physical effects should be included when deriving the correct pd extrapolation formulas? We shall begin our answer to this question with a discussion of the objective reasons which complicate the theoretical analysis of pd scattering, and shall compare the known approaches to calculating the pd scattering lengths in order to select their most probable values.

The main factor which significantly complicates the numerical solution of the three-particle pd problem and the theoretical derivation of the extrapolation formulas essential for the correct analysis of the experimental data is the presence of the long-range Coulomb interaction. Owing to the singularities in the matrices (37) and (38) of the two-body pp subsystem generated by this interaction, the kernels of the original Faddeev equations[176] for the pnp system are non-Fredholm, and for this reason the equations themselves have many solutions. In 1970 Veselova[177] explicitly isolated the singular contributions of Coulomb origin from the kernels of the Faddeev integral equations. In 1978 Alt, Sandhas, and Zeigelmann[178] used her renormalization procedure to derive modified equations whose kernels are compact only for pd energy E lower than the threshold for breakup into three particles $[E\leqslant B(d)]$. In 1967 Noble[179] proposed a different approach to modifying the Faddeev equations, based on the inclusion of all Coulomb interactions in the unperturbed three-particle Hamiltonian. In 1975 Kharchenko and Storozhenko[36] used his scheme to derive the Faddeev–Noble equations for the pd scattering problem at $E\leqslant B(d)$.

It should be emphasized that the integral pd equations obtained in Refs. 36 and 178 are so complicated that their numerical solution in the case of *nonseparable* nuclear interactions requires a series of auxiliary approximations and is therefore an unreliable method which does not allow sufficient control over the uncertainties arising from these approximations. The Faddeev differential equations[10] for a system of three charged particles are fairly simple, in contrast to the physical boundary conditions[114] which, when added to these equations, give a uniquely solvable boundary-value problem.

For these reasons there are not many three-particle calculations of the pd scattering lengths. The pd scattering lengths were calculated in Refs. 180–184 for the simplest $3N$ models with separable nuclear potentials. In Refs. 181 and 182 they were found by extrapolating the results of calculations at $E>400$ keV and $E>200$ keV, respectively. To avoid solving an additional extrapolation problem, the authors of Ref. 183 integrated the pd equations[36] at $E=0$ and obtained values of the pd scattering lengths considerably different from those calculated in Ref. 182. As noted in Ref. 184, it is not possible to explain the disagreement because different methods and auxiliary approximations were used to calculate the scattering lengths in Refs. 182 and 183. In contrast to the Phillips nd line, the dependences obtained in Ref. 183 between the mirror lines (the $^2a_{nd}-{}^2a_{pd}$ and $^4a_{nd}-{}^4a_{pd}$ correlations) relate characteristics of the $3N$ system at the same, namely, zero value of the Nd collision energy. Therefore, such correlations allow the reliable prediction of the real values of the pd scattering lengths.

The analysis of a *single* correlation dependence can, in general, lead to an erroneous estimate of the reliablity of the results of three-particle calculations of the characteristics of the $3N$ system. A correct assessment of the reliability can be obtained only by careful analysis of a *set* of different correlation dependences between the characteristics of the nd system and the mirror pd system. As a clear example illustrating this warning, let us compare the behavior of the three curves constructed in Figs. 1, 2, 4a, and 4b of Ref. 183 using the results obtained in that study and in two earlier studies.[152,182] According to Fig. 1, the three Phillips pd lines corresponding to the results of Refs. 152, 182, and 183 and describing the dependence of $^2a_{pd}$ on $B(^3\text{He})$ are shifted relative to each other, but do not intersect. Therefore, on the basis of only this correlation dependence it is difficult to judge which of the three sets of results is the most reliable. Comparison of the three curves describing the correlation dependences of $^2a_{pd}$ on $^2a_{nd}$ and shown in Fig. 2 of Ref. 183 only strengthens this doubt: these three curves do not intersect and, moreover, the curve of Ref. 152 passes quite close to that of Ref. 182. Any lingering doubt of the author of the present review about the reliability of the results of Ref. 182 was removed by Figs. 4a and 4b: the curve of the calculated correlation dependence of $^4a_{pd}$ on $^4a_{nd}$ shown there is perpendicular to the curve constructed from the data of Refs. 152 and 183. A possible reason for this clear qualitative disagreement might be sought in the complexity of the three-particle calculations, were it not for the simple analytic formula[39]

$$1/{}^4a_{pd}=(1-\tau)/{}^4a_{nd}+(\ln\tau+2\gamma-1/2)/R.$$

This is the Nd analog of the Np formula (25) and describes the dependence of the Coulomb–nuclear scattering length $^4a_{pd}$ on the scattering length $^4a_{nd}$ and the combination $\tau\equiv\xi r_d/R$ of the parameter ξ from the matching $(1<\xi<3)$ of the fragments of the model wave function of quartet nd scattering, the deuteron size r_d, and the Bohr pd radius R.

The first calculations of the pd scattering lengths[152,185,186] performed using the integro-differential Faddeev equations with the same Malfliet–Tjon local S-wave NN potential V^s (Ref. 168) but by different methods turned out to be inconsistent with each other:

$$^2a_{pd}=0.15 \text{ F}, \quad ^4a_{pd}=13.8 \text{ F (Refs. 152 and 185).} \tag{183}$$

$$^2a_{pd}=1.03 \text{ F}, \quad ^4a_{pd}=11.96 \text{ F (Ref. 186),} \tag{184}$$

The scattering lengths (183) were calculated directly, i.e., by solving the pd equations for $E=0$ on the basis of approximation of the desired Faddeev components of the pd function by bicubic splines.[3] The scattering lengths (184) were obtained by extrapolation to the point $E=0$ of the Coulomb–nuclear effective-range function (21) with $\delta_0^{c,s}(k)$ replaced by the pd phase shift. The latter was calculated by solving the pd equations, using the finite-difference method[3] at energies $E>200$ keV. The qualitative explanation[44] of the difference between the scattering lengths (183) and (184) is based on the assumption that the pd amplitude contains a pole at energies in the keV range. When there is such a pole at the point $E=iE_0$, the expansion (23) for the function (21) with $l=0$ must be replaced by the expression

$$K^{c,s}(E)\sim[-1/a^{c,s}(E_0)+k^2r_{00}^{c,s}(E_0)/2]/[1+(E/E_0)^2], \quad k\to 0. \tag{185}$$

For $E\gg E_0$ the asymptotes (23) and (185) are close to each other, and for $E\ll E_0$ they have different slopes relative to the energy axis. Therefore, even for the same doublet pd phase shift $\delta_0^{c,s}(k)$ the values of the doublet pd scattering lengths $a_0^{c,s}$ and $a_0^{c,s}(E_0)$ calculated in Ref. 185 using Eq. (185) and in Ref. 186 using Eq. (23) must differ from each other. A subsequent careful study[153] confirmed the existence of the pole and showed that $^2a_{pd}$ should be determined by extrapolation of the asymptote (185) to zero energy. Thus, the values (183) calculated earlier in Ref. 152 and the inequality $^2a_{pd}<{}^2a_{nd}$ were confirmed in Ref. 153. This example showing how important it is to know the correct extrapolation formula becomes even more convincing when we recall the significant difference between the values $r_{00}^{c,s}=11000$ F and $r_{00}^{c,s}(E_0)=-450$ F calculated in Ref. 167 using the corresponding expressions (21) and (185) at $E_0=25$ keV.

The results of the first calculations[162,171] of the pd scattering lengths carried out with various realistic two-particle potentials and including and neglecting the $3N$ forces showed that the $3N$ forces do not change the correlation dependence of $^2a_{pd}$ on $B(^3\text{He})$, and the value $B(^3\text{He})=7.68$ MeV close to the experimental value $B(^3\text{He})=(7.718109\pm 0.000010)$ MeV (Ref. 89) corresponds to a negative value $^2a_{pd}\approx -0.1$ F, which disagrees with the available experimental data (182). This latter conclusion was confirmed by a careful study[184] of the correlation dependence of $^2a_{pd}$ on $^2a_{nd}$ by direct solution of the modified Faddeev integral equations with separable NN potentials and zero Nd collision energy. The correlation curve obtained accurately passes through the points corresponding to the values of the Nd scattering lengths calculated earlier in Refs. 152, 162, and 171, but lies considerably below the points corresponding to the experimental values (182). The calculated dependence of $^2a_{pd}$ on $^2a_{nd}$ turned out to be weakly sensitive to the choice of NN-force model, which provided the authors of Ref. 184 with good grounds for the following conclusion: according to the correlation dependence, the value $^2a_{pd}=-0.1$ F corresponds to the experimental value $^2a_{nd}^{\text{exp}}=0.65$ F. In spite of this likely and quite convincing conclusion, the final answer to the question of the magnitude of the pd scattering lengths requires redetermination of these scattering lengths followed by the solution of many problems. These are due to the polarized pd interaction[187] with the asymptote (12). The formalism of integral pd equations in momentum space developed in Refs. 36 and 178 was modified in Ref. 188 in order to isolate the induced polarization interactions. Up to now no calculations have been carried out using this method. At present there is not a single three-particle calculation of the modified pd scattering lengths ($^2a_{pd}^{cp,s}$ and $^4a_{pd}^{cp,s}$) which systematically takes into account polarization effects. The two main reasons for the absence of such a calculation are practical and theoretical.

Insurmountable computational difficulties arise in the direct solution of the standard Schrödinger or Faddeev equations with long-range potentials and parameter (collision energy) tending to zero.[20,147,148] It is well known[20] that the accuracy of solving the problem of two-particle scattering by the superposition $V^c+V^p+V^s$, $V^c>0$, which allows study of the influence of the polarization interaction on the low-energy dependence of the S-wave phase shifts $\delta_0^{c,ps}(k)$ and $\delta_0^{cp,s}(k)$, approaches zero as $O(\alpha k^5R^2)$ for $k\to 0$. Therefore, even when solving the model problem (170)–(176) the accuracy of calculating the S-wave Faddeev component U describing the pd configuration in pd scattering with collision energy $E\approx 10$ keV must have order of magnitude 10^{-12}. Such high accuracy of the solution of the three-particle equations describing threshold pd scattering cannot be obtained even using modern supercomputers. According to the estimates of Cornelius *et al.*,[148] existing computers allow the solution of Faddeev $3N$ problems with realistic NN forces[53,54] only with an accuracy of about 0.01.

In the solution of the Faddeev problem for a weakly bound state of three particles with Coulomb pair potentials, high accuracy can be obtained by using the well known method[147] of including information about the slowly decreasing asymptotes of the Faddeev components at large values of the hyperradius. This method consists of the substitution

$$r'\equiv 1-\exp(-\tau r), \quad \tau=\text{const}>0, \tag{186}$$

which maps the semiaxis $\mathscr{R}^+$ of values of the hyperradius r onto the unit segment of variation of the new hyperradial coordinate r'. Then, the following ansatz is made for the Faddeev components Ψ_i:

$$\Psi_i(r,\Omega;E)=\exp(-\sqrt{|E|}r')\Psi_i'(r',\Omega;E),$$

where the first factor is the reference function for large r explicitly describing the asymptotes of these components for $r\to\infty$. The extension of this method to scattering in the three-particle system requires, in addition to rescaling of the hyperradius (186), the solution of the very complicated prob-

lem of constructing a pair of reference functions for each of the three Faddeev components Ψ_i. These functions must *simultaneously* describe, in the two-particle asymptotic region ($x_i<\infty$, $y_i\to\infty$), waves which converge and diverge in the Jacobi variable y_i, while in the three-particle asymptotic region ($x\sim y$; $x,y\to\infty$) they must reproduce the asymptote of an outgoing spherical wave in the variable r. This problem in the theory of scattering in a three-particle system has not yet been solved. A possible approach is a combination of the method of phase functions and the Faddeev integro-differential equations. The advantages of such a combination[189,190] have been convincingly demonstrated by a study of the dependence of the nd scattering lengths[189] and the low-energy properties of three bosons in the resonance region[190] on the cutoff radius of a one-term separable NN potential.

As shown in Secs. 4.1 and 4.2 above, the method of constructing three-particle low-energy expansions which reduces the Schrödinger or Faddeev problem to a recursion chain of energy-independent equations convenient for numerical solution is at present in a rudimentary state.

The only method of studying polarization effects in pd scattering available at present is low-energy analysis of the effectively two-particle pd problem (68)–(70) with the superposition $V=V^{cps}$ of effective Coulomb, nuclear (1), and polarization (12) pd interactions. The results of studies performed using this approximation for S-wave pd collisions have been discussed in detail in Ref. 20. A physically interesting question arose in the discussion: can the mutual effect of the nuclear and polarization potentials on each other, which generates minima[191] in the S-wave partial Coulomb-polarized nuclear ($a=ps$) doublet and quartet pd scattering cross sections, produce minima (pd analogs of the Ramsauer effect[192]) in the corresponding total cross sections for this scattering? We shall conclude our review by answering this question.

We begin with the case $a=p$, where the nuclear pd interaction is not taken into account and in (68) $V\equiv V^p$, while in (70) $\delta_l(k)\equiv\delta_l^{c,p}(k)$. According to (60) and (61), the phase shifts $\delta_l^{c,p}(k)$ with large l fall off for $q\equiv kR\to 0$ much more slowly,

$$\tan\delta_l^{c,p}(q)\sim-\pi V_0^p q^2/R^2(4l^2-1)(2l+3),\quad l\gg\eta, \tag{187}$$

than the phase shifts (62) with small l. It therefore follows from (62) and (187) that the falloff of the leading terms of the low-energy asymptotes of the partial amplitudes (32) with $a=p$ varies continuously in q in going from the region $l\ll\eta$ to the region $l\gg\eta$:

$$f_l^{c,p}(q)=O(q^4),\quad l\ll\eta;\quad f_l^{c,p}(q)=O(q),\quad l\gg\eta.$$

Owing to this variation, the representation (31) which is discrete in l is not convenient for deriving the asymptote of the total amplitude $f^{c,p}$.

Using the method of Ref. 193, Kvitsinskiĭ[194] replaced the sum (31) by an integral over a continuous variable l and found the asymptote of this integral for $q\to 0$ by the stationary-phase method.[4] It turned out that in this limit

$$f^{c,p}(\theta;q)\sim-2V_0^pR^{-1}q^3\exp(2i\eta(\ln(\eta\operatorname{cosec}\theta/2)-1))b_4(\theta),$$
$$b_4(\theta)\equiv\sin\theta(1+\cos\theta)^{-3}[(2-\cos\theta)(\pi-\theta)-3\sin\theta], \tag{188}$$

and the main contribution to the sum (31) comes from the partial amplitudes (32) with angular momentum l close to the stationary-phase point $l_0\equiv\eta\cot(\theta/2)-1$.

Substituting (188) into (34), we obtain the asymptote

$$d\sigma^{c,p}(\theta;E)/d\theta\sim 8\pi(V_0^pR^{-1}q^3)^2b_4^2(\theta)\sin\theta,\quad q\to 0, \tag{189}$$

of the contribution of the polarization potential to the differential cross section for pd scattering by the superposition V^{cp}. According to (189), at sufficiently low energies this contribution depends on the angle θ as $O(\theta^3)$ in forward scattering ($\theta\to 0$) and as $O(\pi-\theta)$ in backward scattering ($\theta\to\pi$), and for any fixed q reaches its maximum at the point $\theta\approx\pi/2$.

Replacing the contribution $d\sigma^{c,p}/d\theta$ in (35) by its asymptote (189), we find

$$\sigma^{c,p}(E)\sim 8\pi(V_0^pR^{-1}q^3)^2\int_0^\pi d\theta b_4^2(\theta)\sin\theta,\quad q\to 0, \tag{190}$$

which implies that the contribution of the polarization potential to the total pd collision cross section falls off in the zero-energy limit as $O(E^6)$. Now let $a=ps$ and the nuclear pd interaction V^s be included along with the polarization potential (12), but only in the state with $l=0$. Then

$$\delta_l^{c,s}(q)\sim-a_0^{c,s}qR^{-1}C_0^2(\eta)\hat\delta_{l0},\quad l=0,1,\ldots,\quad q\to 0. \tag{191}$$

Taking into account these equations and using Eqs. (31), (32), and (35), we obtain a representation, convenient for numerical analysis, of the combined contribution of the nuclear and polarization interactions to the total cross section for pd scattering by the superposition V^{cps}:

$$\sigma^{c,ps}(E)=\sigma_0^{c,ps}(E)+[\sigma^{c,p}(E)-\sigma_0^{c,p}(E)]. \tag{192}$$

To calculate this contribution using Eqs. (32), (33), (190), and (191), we used the same pd scattering lengths as in Ref. 191, namely, $a_0^{c,s}=1.03$ F for the doublet and $a^{c,s}=11.96$ F for the quartet pd scattering length. It turned out that the contribution of the difference $[\sigma^{c,p}(E)-\sigma_0^{c,p}(E)]$ to the sum (192) is the main one,

$$[\sigma^{c,p}(E)-\sigma_0^{c,p}(E)]/\sigma_0^{c,ps}(E)>0.9, \tag{193}$$

in the doublet case for $E<{}^2E_1=3.1$ keV and in the quartet case for $E<{}^4E_1=2.0$ keV. The component $\sigma_0^{c,ps}$ of the cross section (192) vanishes[20] at the point ${}^2E_{\min}\approx 2.2$ keV in the doublet case and at the point ${}^4E_{\min}\approx 1.2$ keV in the quartet case. Since ${}^nE_{\min}<{}^nE_1$, $n=2,4$, in both cases the contribution $\sigma^{c,ps}(E)$ cannot have a significant local minimum in the energy range $E<{}^nE_1$. According to (193), at these energies this contribution arises from scattering by the polarization potential in the Coulomb field in pd states with $l>0$.

6. CONCLUSION

Unsolved problems in the theory of low-energy expansions in nuclear physics have been described in each section of this review. Therefore, let us end with two very important conclusions.

It has been shown that the study of low-energy collisions in NN and $3N$ systems continues to be an important problem in nuclear physics in connection with checking the possibility of describing a complicated system using modern ideas about the *total* interaction between the nucleons and the solution of the inverse problem—extraction of information about the nuclear interaction by analyzing all the experimental data and theoretical results on NN and $3N$ systems.

Studies of collisions of neutrons and protons with the lightest nuclei at energies which are superlow on the nuclear scale play an important role in nuclear astrophysics and are essential for applications associated with achieving thermonuclear fusion. This is yet another reason why in this review we have focused on proving the statement that the analysis of such collisions must be based on mathematically justified low-energy expansions obtained from rigorous dynamical equations of motion, which allow the systematic inclusion of the Coulomb interaction and the most important electromagnetic corrections to it.

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