Path-integral approach for superintegrable potentials on the three-dimensional hyperboloid

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In the present paper on superintegrable potentials on spaces of constant curvature we discuss the case of the three-dimensional hyperboloid. Whereas in many coordinate systems an explicit path-integral solution for the corresponding potential is not possible, we list in the soluble cases the path-integral solutions explicitly in terms of the propagators and the spectral expansions in the wave functions. We find the analogs of the maximally and minimally superintegrable potentials of $\mathbb{R}^3$ on the hyperboloid and many minimally superintegrable potentials which emerge from the subgroup chains corresponding to SO(3,1). Some special care is taken for the proper generalization of the harmonic oscillator and the Kepler problem. [S1063-7796(97)00505-6]

1. INTRODUCTION

Motivation and symmetry methods in physics

The present paper is the fourth in a series concerning superintegrable potentials in spaces of constant curvature. It continues our studies which started from the investigation in two- and three-dimensional Euclidean space, i.e., in $\mathbb{R}^2$ and $\mathbb{R}^3$, on the two- and three-dimensional spheres $S^{(2)}$ and $S^{(3)}$, and on the two-dimensional hyperboloid $\Lambda^{(2)}$. Our goal is to study physical systems in spaces of constant curvature which have *accidental degeneracies*, i.e., systems which have, because of their peculiar features, a so-called *hidden symmetry* or *dynamical group structure*, thus giving rise to degeneracies in the energy spectrum, and additional integrals of motion (respectively, observables).

The best known potential systems of this kind in three-dimensional flat space are the harmonic oscillator with quantum energy spectrum

$$E_N=\hbar\omega\left(N+\tfrac{3}{2}\right),\quad N\in\mathbb{N}_0, \tag{1.1}$$

and the Kepler–Coulomb problem with the quantum energy spectrum

$$E_N=-\frac{Me^4}{2\hbar^2(N+1)^2},\quad N\in\mathbb{N}_0. \tag{1.2}$$

Here, N denotes the principal quantum number, and for fixed N each level E_N for the oscillator is $(N+1)(N+2)/2$-fold degenerate, and in the Coulomb problem $(N+1)^2$-fold degenerate.

The particular symmetry features have the consequence that there are additional constants of the motion in classical mechanics (respectively, observables in quantum mechanics). In comparison, the orbits of a simple integrable system, e.g., a three-dimensional anharmonic oscillator, are generally only periodic with respect to each coordinate, but not globally.[1)] For a physical system in D dimensions just to be integrable, D constants of the motion are required, one of them being the energy E. In classical mechanics these constants of the motion have vanishing Poisson brackets with the Hamiltonian and with each other; in quantum mechanics they are operators which commute with the quantum Hamiltonian and with each other. For instance, for a spherical symmetric system, the constants of the motion are the energy E, the square of the total angular momentum $\mathbf{L}^2$, and the square of the (usually chosen) z component of the angular momentum, L_z^2, in classical mechanics as well as in quantum mechanics.

In systems like the isotropic harmonic oscillator or the Kepler–Coulomb problem in three dimensions, there are two more functionally independent constants of the motion. In the case of the harmonic oscillator the additional constants of the motion correspond to conservation of the quadrupole moment, the so-called Demkov tensor $T_{ik}=p_ip_k+\omega^2x_ix_k$,[12] and, in the case of the Kepler–Coulomb problem, conservation of the square of another component of the angular momentum and the third component of the Pauli–Lenz–Runge vector $A=(1/2M)(L\times P-P\times L)-e^2x/|x|$, and both systems have five constants of the motion (respectively, observables).

A more careful investigation shows that the highly spherical symmetric systems of the isotropic harmonic oscillator and the Kepler–Coulomb problem can be perturbed in various ways by the incorporation of additional potential terms: First, this does not spoil the degeneracy of the energy levels at all, i.e., there are still five observables,[2)] second, one of the observables is removed, i.e., there are four left; and third, only the minimum number of three observables for integrability remains. The first possibility is described by the notion of a maximally superintegrable system, the second is described by the notion of a *minimally superintegrable* system, and the third just describes an *integrable* system.

In this respect, the physical significance of the consideration of separation of variables in several coordinate systems is as follows. The free motion in some space is, of course, the most symmetric one, and the search for the number of coordinate systems which allow the separation of the Hamiltonian is equivalent to the investigation of how many inequivalent sets of observables can be found. In particular, the free motion in various coordinate systems on the hyperboloid has been studied in Refs. 24, 28, 31, and 38. The incorporation of potentials usually removes at least some of the symmetry properties of the space. Well-known examples are spherically symmetric systems, and they are most conveniently studied in spherical coordinates.

All the superintegrable systems have the particular property that all the energy levels of the system are organized in representations of the noninvariance group which contain representations of the dynamical subgroup realized in terms of the wave functions of these energy levels.[20] The additional integrals of the motion also have the consequence that in the case of the superintegrable systems in two dimensions and maximally superintegrable systems in three dimensions all finite trajectories are found to be periodic; in the case of minimally superintegrable systems in three dimensions all finite trajectories are found to be quasiperiodic (Ref. 56).[3)] Of course, in the case of the pure Kepler or the isotropic harmonic oscillator all finite trajectories are periodic.

In general, a physical system in D dimensions is called minimally superintegrable if it has $2D-2$ integrals of the motion, and maximally superintegrable if it has $2D-1$ integrals of the motion (respectively, observables). Therefore, we are led to the search for more (potential) systems which have features of degeneracy and the number of observables similar to those of the radial harmonic oscillator and the Coulomb problem.

A systematic study to classify separable potentials was undertaken by Smorodinsky, Winternitz, and co-workers,[20,71,91] i.e., they looked for potentials which are separable in more than one coordinate system. The separation of a quantum-mechanical potential problem in more than one coordinate system has the consequence that there are additional integrals of the motion and that the spectrum is degenerate. The choice of a coordinate system then emphasizes which observables are considered to be the most appropriate for a particular investigation.

Superintegrable systems

The harmonic oscillator in spaces of constant curvature has been discussed e.g., by Bonatsos *et al.*,[8] Higgs,[43] Lemon,[64] Granovsky *et al.*,[22] and in Refs. 34 and 35. The Coulomb–Kepler problem in spaces of constant curvature was discussed by Higgs[43] and Lemon,[64] and in the general context of the SO(4,2) dynamical algebra by Barut *et al.*,[5] Granovsky *et al.*,[23] Katayama,[55] Pogosyan *et al.*,[79] Otchik and Red'kov,[77] Schrödinger,[80] Stevenson,[85] and Vinitsky *et al.*[88]

The notion of "superintegrability" [16,53,92] can now be introduced in spaces of constant curvature.[34,35] Whereas the general form of potentials which are "superintegrable" in some way has not been clear until now, it is known that the corresponding Higgs oscillator (cf. Bonatsos *et al.*,[8] Granovsky *et al.*,[22] Higgs,[43] Ikeda and Katayama,[45] Katayama,[55] Leemon,[64] Nishino,[74] and Pogosyan *et al.*[79]) and Kepler problems (cf. Granovsky *et al.*,[23] Infeld,[46] Infeld and Schild,[47] Kalnins *et al.*,[54] Kibler *et al.*,[57] Kurochkin and Otchik,[62] Nishino,[74] Otchik and Red'kov,[77] Vinitsky *et al.*,[88,89] and Zhedanov[93]) in spaces of constant curvature do have additional constants of motion: the analogs of the flat space. For the Higgs oscillator this is the Demkov tensor,[12,21,74] and in the Kepler problem a Pauli–Runge–Lenz vector on spaces of constant curvature can be defined (cf. Refs. 23, 43, 62, 64, and 74). Corresponding path-integral considerations are due to Barut *et al.*,[3,4] Otchik and Red'kov,[77] and Refs. 25 (D-dimensional case) and 34 (superintegrable aspects).

Disturbing the spherical symmetry usually spoils it. The first step consists of deforming the ring-shaped feature of the (maximally superintegrable) modified oscillator and Coulomb potential. One gets in the former case a ring-shaped oscillator, and in the latter case the Hartmann potential, which are two minimally superintegrable systems. The number of coordinate systems which allow separation of variables drops from eight to four (namely, spherical, circular polar, oblate spheroidal, and prolate spheroidal coordinates (Kibler *et al.*[57,58]), and from four to three, namely, spherical, parabolic, and prolate spheroidal H coordinates.

Disturbing the system further, one is left with, say, one coordinate system which still allows separation of variables. A constant electric field (Stark effect) allows only the separation in parabolic coordinates.[29] Here it is interesting to note that in the momentum representation of the hydrogen atom the bound-state spectrum is described by free motion on the sphere $S^{(3)}$. To be more precise, the dynamical group O(4) describes the discrete spectrum, and the Lorentz group O(3,1) describes the continuous spectrum.[2] Now there are six coordinate systems on $S^{(3)}$ which separate the corresponding Laplacian. The solution in spherical and cylindrical coordinates corresponds to the spherical and parabolic solution in the coordinate-space representation. The elliptic cylindrical system is of special interest because it enables one to set up a complete classification for the energy levels of the quadratic Zeeman effect (cf. Solov'ev,[83] Brown and Solov'ev,[9] Herrick,[42] and Lakshmann and Hasegawa[63]).

Separation in parabolic coordinates is also possible in the case of a perturbation of the pure Coulomb field with a potential force $\propto z/r$, which allows an exact solution.[27,30] The two-center Coulomb problem turns out to be separable only in spheroidal coordinates (Coulson and Josephson,[10] Coulson and Robinson,[11] and Morse[72]), as was studied first in connection with the hydrogen-molecule ion by Teller.[86]

Another possible way to disturb the spherical symmetry is to remove the invariance with respect to rotations about an axis, e.g., about a uniform magnetic field. Usually, this invariance is used to illustrate the azimuthal quantum number m of the L_z operator. The physical meaning of this quantum number then is that there exists a preferred axis in space. This symmetry can be broken if one considers a Hamiltonian of a nucleus with an electric quadrupole moment Q and spin J in a spatially varying electric field.[66,84] Here sphero-conical coordinates are most convenient, and the projection of the terminus of the angular-momentum vector traces out a cone of elliptic cross section about the z axis.[84] The problem of the asymmetric top (Kramers and Ittmann,[61] Lukač,[65] and Smorodinsky *et al.*[67,82,90]), the symmetric oblate top,[65] or tensor-like potentials (Lukač and Smorodinsky[68]) can also be treated best in sphero-conical coordinates. Therefore, sphero-conical coordinates are most suitable for problems which have spherical symmetry but not sphero-axial symmetry.

A condition for a potential problem to be separated in ellipsoidal coordinates is that the shape of the potential resembles the shape of an ellipsoid. Of course, the anisotropic harmonic oscillator belongs to this class. Introduction of quartic and sextic[87] interaction terms then eventually allows only separation of variables in ellipsoidal coordinates. Another example is the Neumann model,[73] which describes a particle moving on a sphere subject to anisotropic harmonic forces (Babelon and Talon[1] and MacFarlane[69]).

Our first paper[33] dealt with superintegrable potentials in two- and three-dimensional flat space, where we distinguished minimally and maximally superintegrable systems. In two-dimensional Euclidean space there are four (maximally) superintegrable systems,[16] i.e., the (generalized) harmonic oscillator $V_1(\mathbf{x})$, the Holt potential $V_2(\mathbf{x})$, the (generalized) Coulomb potential $V_3(\mathbf{x})$, and a modified Coulomb potential $V_4(\mathbf{x})$.[4)]

In three-dimensional Euclidean space we found five maximally and nine minimally superintegrable systems. Among the maximally superintegrable systems were the (generalized) harmonic oscillator $V_1(\mathbf{x})$, the Holt potential in $\mathbb{R}^B$, $V_2(\mathbf{x})$, and the (generalized) Coulomb potential $V_3(\mathbf{x})$; among the minimally superintegrable systems were a double-ring-shaped oscillator $V_6(\mathbf{x})$, the Hartmann potential $V_7(\mathbf{x})$, a three-dimensional analog of the Holt potential $V_6(\mathbf{x})$, four potentials $V_2(\mathbf{x})$, $V_3(\mathbf{x})$, $V_4(\mathbf{x})$, $V_8(\mathbf{x})$ which emerged from the group chain $E(3)\supset E(2)$ (i.e., they are superintegrable in $\mathbb{R}^2$,) and two potentials $V_1(\mathbf{x})$, $V_9(\mathbf{x})$ which emerged from the group chain $E(3)\supset SO(3)$ (i.e., they are superintegrable on the two-dimensional sphere $S^{(2)}$).

In our second paper[34] we continued our study on the two- and three-dimensional sphere. On $S^{(2)}$ we found only two potentials with the required properties, i.e., the (generalized) Higgs oscillator $V_1(\mathbf{s})$ and the (generalized) Coulomb potential $V_2(\mathbf{s})$. We have not been able to find superintegrable analogs of the Holt potential and the modified Coulomb potential. On the three-dimensional sphere $S^{(3)}$ we have found three maximally superintegrable and four minimally superintegrable potentials. Among the maximally superintegrable potentials were the (generalized) Higgs oscillator $V_1(\mathbf{s})$, the Coulomb potential $V_2(\mathbf{s})$, and, as a third potential, $V_3(\mathbf{s})$, a pure scattering potential which corresponds to $V_4(\mathbf{x})$ in $\mathbb{R}^3$. Among the minimally superintegrable systems there were analogs of the double-ring-shaped oscillator $V_4(\mathbf{s})$ and the Hartmann potential $V_5(\mathbf{s})$ on $S^{(3)}$, and the two remaining potentials $V_6(\mathbf{s})$, $V_7(\mathbf{s})$ emerged from the group chain $SO(4)\supset SO(3)$.

In Ref. 35 we considered superintegrable potentials on the two-dimensional hyperboloid $\Lambda^{(2)}$. We found analogs of the (generalized) harmonic oscillator $V_1(\mathbf{u})$, i.e., the Higgs oscillator in a space of constant negative curvature, the (generalized) Coulomb potential $V_2(\mathbf{u})$, and the Holt potential $V_3(\mathbf{u})$ on $\Lambda^{(2)}$. We also found two more systems $V_3(\mathbf{u})$, $V_4(\mathbf{u})$, which are due to the peculiarity of the hyperboloid that in spaces of constant negative curvature there are generally more orthogonal coordinate systems which separate the Schrödinger (respectively, Boltzmann) equation than in spaces of flat or constant positive curvature. However, we have not been able to find a superintegrable version of the modified Coulomb potential, (cf. $V_4(\mathbf{x})$ in $\mathbb{R}^2$).

Interbasis expansions

An important aspect of group path integration (see below) in quantum mechanics is the so-called interbasis expansion technique for problems which allow the representation of the wave functions in various coordinate-space representations. The basic formula is quite simple:

$$|\mathbf{k}\rangle=\int dE_{\mathbf{l}}C_{\mathbf{p},\mathbf{k}}|\mathbf{p}\rangle+\sum_{\mathbf{n}} C_{\mathbf{n},\mathbf{k}}|\mathbf{n}\rangle, \tag{1.3}$$

where $|\mathbf{k}\rangle$ stands for a basis of eigenfunctions of the Hamiltonian in the coordinate-space representation $\mathbf{k}$, and $\int dE_{\mathbf{l}}$ is the spectral expansion with respect to the coordinate-space representation $\mathbf{l}$ with coefficients $C_{\mathbf{p},\mathbf{k}}$ and $C_{\mathbf{n},\mathbf{k}}$, which can be discrete, continuous, or both. The main difficulty is, when one has two coordinate-space representations in the quantum numbers $\mathbf{k}$ and $\mathbf{p}$, $\mathbf{n}$, respectively, to find the expansion coefficients $C_{\mathbf{p},\mathbf{k}}$ and $C_{\mathbf{n},\mathbf{k}}$. There are well-known expansions which involve Cartesian coordinates and polar coordinates. In the simple case of free quantum motion in Euclidean space, this means that exponentials representing plane waves are expanded in terms of Bessel functions and spherical waves, in a discrete interbasis expansion, i.e., $e^{z\cos\psi}=\sum_{\nu\in Z}e^{i\nu\psi}I_\nu(z)$.

This general method of changing a coordinate basis in quantum mechanics can now be used in the path integral. We assume that we can expand the short-time kernel (respectively, the exponential $e^{z\mathbf{x}_{j-1}\cdot\mathbf{x}_j}$) in terms of matrix elements of a group[31] by choosing a specific coordinate basis. We can then change the coordinate basis by means of (1.3). Because of the unitarity of the expansion coefficients $C_{l,k}$, the short-time kernel is expanded in the new coordinate basis, and the orthonormality of the basis allows us to perform explicitly the path integral, in exactly the same way as in the original coordinate basis. However, to find the dynamical group and its corresponding coordinate-space representation in a superintegrable system—one of the principal problems—is not always very easy. From the two (or more) different equivalent coordinate-space representations, formulas and path-integral identities can be derived, and, at the same time, interbasis coefficients. These identities actually correspond to integral and summation identities between special functions. The case of the expansion from Cartesian coordinates to polar coordinates has been studied by Peak and Inomata,[78] who obtained the solution of the isotropic harmonic oscillator as well. The path-integral solution of the isotropic harmonic oscillator in turn enables one to calculate numerous path-integral problems related to the radial harmonic oscillator—actually, problems which are of the so-called Besselian type, including the Coulomb problem. Furthermore, a (path-integral) solution in a particular coordinate-space representation can serve as a starting point for a perturbative analysis in cases where a system separates, say, in only one coordinate system, but is not exactly solvable. Then knowledge of the wave functions and interbasis coefficients of the corresponding exactly solvable model is of paramount importance (see, e.g., Refs. 32 and 70).

Path-integral approach

In our investigations the path integral turns out to be a very convenient tool to formulate and solve the superintegrable potentials on spaces of constant curvature, in particular, on the hyperboloid. The subsequent separation of variables in each problem can be performed in a straightforward and easy way. We start by considering the classical Lagrangian corresponding to the line element $ds^2=g_{ab}dq^a dq^b$ of the classical motion in a D-dimensional Riemannian space (e.g., Refs. 13, 17, 39, 59, and 81 and the bibliography cited there):

$$\mathscr{L}_{\rm Cl}(\mathbf{q},\dot{\mathbf{q}})=\frac{M}{2}\left(\frac{ds}{dt}\right)^2-V(\mathbf{q})=\frac{M}{2}\,g_{ab}(\mathbf{q})\dot{q}^a\dot{q}^b-V(\mathbf{q}). \tag{1.4}$$

The quantum Hamiltonian is *constructed* by means of the Laplace–Beltrami operator

$$H=-\frac{\hbar^2}{2M}\,\Delta_{\rm LB}+V(\mathbf{q})$$
$$=-\frac{\hbar^2}{2M}\frac{1}{\sqrt{g}}\frac{\partial}{\partial q^a}\,g^{ab}\sqrt{g}\,\frac{\partial}{\partial q^b}+V(\mathbf{q}) \tag{1.5}$$

as a *definition* of the quantum theory on a curved space. Here $g=\det(g_{ab})$, $(g^{ab})=(g_{ab})^{-1}$, and $\Delta_{\rm LB}=g^{-1/2}\partial_a g^{ab}g^{1/2}\partial_b$. The scalar product for wave functions on the manifold reads $(f,g)=\int dq\sqrt{g}f^*(q)g(q)$, and the momentum operators, which are Hermitian with respect to this scalar product, are given by

$$p_a=\frac{\hbar}{i}\left(\frac{\partial}{\partial q^a}+\frac{\Gamma_a}{2}\right),\quad \Gamma_a=\frac{\partial\ln\sqrt{g}}{\partial q^a}. \tag{1.6}$$

In terms of the momentum operators (1.6), we can rewrite H by using an ordering prescription called product ordering, where we assume that $g_{ab}=h_{ac}h_{cb}$ [we do not discuss other lattice formulations like the important midpoint prescription (MP), corresponding to Weyl ordering in the Hamiltonian]. We then obtain the following expression, for the Hamiltonian (1.5):

$$H=-\frac{\hbar^2}{2M}\,\Delta_{\rm LB}+V(\mathbf{q})$$
$$=\frac{1}{2M}\,h^{ac}p_a p_b h^{cb}+\Delta V(\mathbf{q})+V(\mathbf{q}), \tag{1.7}$$

and for the path integral,

$$K(\mathbf{q}'',\mathbf{q}';T)=\int_{\mathbf{q}(t')=\mathbf{q}'}^{\mathbf{q}(t'')=\mathbf{q}''}\mathscr{D}\mathbf{q}(t)\sqrt{g(\mathbf{q})}\times\exp\left\{\frac{i}{\hbar}\int_{t'}^{t''}\left[\frac{M}{2}\,h_{ac}(\mathbf{q})h_{cb}(\mathbf{q})\dot{q}^a\dot{q}^b-V(\mathbf{q})-\Delta V(\mathbf{q})\right]dt\right\}\equiv\lim_{N\to\infty}\left(\frac{M}{2\pi i\epsilon\hbar}\right)^{ND/2}\prod_{k=1}^{N-1}\times\int d\mathbf{q}_k\sqrt{g(\mathbf{q}_k)}\exp\left\{\frac{i}{\hbar}\sum_{j=1}^{N}\left[\frac{m}{2\epsilon}\,h_{bc}(\mathbf{q}_j)\times h_{ac}(\mathbf{q}_{j-1})\Delta q_j^a\Delta q_j^b-\epsilon V(\mathbf{q}_j)-\epsilon\Delta V(\mathbf{q}_j)\right]\right\}. \tag{1.8}$$

Here $\Delta V_{\rm PF}$ denotes the well-defined quantum potential

$$\Delta V_{\rm PF}(\mathbf{q})=\frac{\hbar^2}{8M}\,[g^{ab}\Gamma_a\Gamma_b+2(g^{ab}\Gamma_b)_{,b}+g^{ab}{}_{ab}+2h^{ac}h^{bc}{}_{,ab}-h^{ac}{}_{,a}h^{bc}{}_{,b}-h^{ac}{}_{,b}h^{bc}{}_{,a}]. \tag{1.9}$$

Here we have used the abbreviations $\epsilon=(t''-t')/N\equiv T/N$, $\Delta\mathbf{q}_j=\mathbf{q}_j-\mathbf{q}_{j-1}$, $\bar{q}_j=\frac{1}{2}(\mathbf{q}_j+\mathbf{q}_{j-1})$ for $\mathbf{q}_j=\mathbf{q}(t'+j\epsilon)$ $(t_j=t'+\epsilon j,\ j=0,\ldots,N)$, and we interpret the limit $N\to\infty$ as equivalent to $\epsilon\to0$, with T fixed. The lattice representation can be obtained by exploiting the composition law for the time-evolution operator $U=\exp(-iHT/\hbar)$ (respectively, its semigroup property), and the discretized path integral emerges in a natural way. The classical Langrangian is modified into an effective Lagrangian via $\mathscr{L}_{\rm eff}=\mathscr{L}_{\rm Cl}-\Delta V$. We use this path-integral formulation throughout the paper. For the technique of space-time transformations we refer to Refs. 15, 18, 39, and 59 and the bibliography cited there.

Presentation of results

The contents of this paper is as follows. In the next section we give an introduction to the formulation and construction of coordinate systems on the three-dimensional hyperboloid. This includes an enumeration of the coordinate systems according to Refs. 31, 50, 52, and 75. The enumeration includes the explicit statement of the quantity $\mathbf{u}=(u_0,u_1,u_2,u_3)$ in terms of the coordinate variables $\boldsymbol{\rho}=(\rho_1,\rho_2,\rho_3)$, the line element $ds^2=ds^2(\boldsymbol{\rho})$, the momentum operators P_{ρ_i}, the Hamiltonian H_0, the form that a potential $V(\mathbf{u})$ must have for the Schrödinger equation $H\Psi=(H_0+V)\Psi=E\Psi$ to be separable, and the corresponding integrals of motion (respectively, observables).

In Sec. 3 we present the three maximally superintegrable potentials on the three-dimensional hyperboloid, including an analog of a Stark-effect potential which is, however, in contrast to the case of $\mathbb{R}^3$, only minimally superintegrable. The maximally superintegrable systems have five integrals of the motion. For instance, in the pure Coulomb problem in $\mathbb{R}^3$ they are the energy E, the square of the absolute value of the angular momentum $\mathbf{L}^2$, L_z^2, an observable corresponding to the semihyperbolic system, and the third component of the

Pauli–Lenz–Runge vector $\mathbf{A}$ (the whole set E, $\mathbf{L}^2$, L_z^2, $\mathbf{A}$ is not functionally independent). Actually, the first three of these constants of the motion are typical for each radial problem, and the minimum number of three observables is required for a three-dimensional system to be separable at all (in Ref. 16 a systematic listing of these constants of the motion was presented). We treat the first two potentials, i.e., the Higgs oscillator and the Coulomb potential on $\Lambda^{(3)}$, in some detail. The relevant observables are listed in the tables.

In Sec. 4 we discuss the minimally superintegrable potentials on $\Lambda^{(3)}$. We find four potentials which have counterparts in three-dimensional Euclidean space. The remaining potentials emerge from the subgroup structure of SO(3,1), i.e., we find four potentials corresponding to the chain $\mathrm{SO}(3,1)\supset E(2)$, two potentials corresponding to the chain $\mathrm{SO}(3,1)\supset\mathrm{SO}(3)$, one of which is, however, equivalent to a previous one, and five potentials corresponding to the chain $\mathrm{SO}(3,1)\supset\mathrm{SO}(2,1)$. This yields 15 minimally superintegrable potentials on $\Lambda^{(3)}$. We do not explicitly list each solution again, because this would expand our paper too much; instead, we refer to our previous work concerning the superintegrable potentials in flat space,[33] on the sphere,[34] and on the two-dimensional hyperboloid.[35] In Secs. 3 and 4 we make frequent use of the path-integral formulations of the Pöschl–Teller, the modified Pöschl–Teller, and the Rosen–Morse potential, whose solutions can be found in Refs. 33–35 and in the references cited there (e.g., Böhm and Junker,[7] (Refs. 31, 39, and 40, Fischer *et al.*,[18] Inomata *et al.*,[48] and Kleinert and Mustapic.[60]

In Sec. 5 we summarize and discuss our results. Here we also establish a correspondence of maximally and minimally superintegrable potentials in two and three dimensions in the three spaces of constant curvature, i.e., Euclidean space, the sphere, and the hyperboloid. In addition, we suggest analogs of the Holt potential on the two- and three-dimensional spheres and on the two- and three-dimensional hyperboloids. However, these potentials turn out to be only integrable. On the sphere the corresponding separating coordinate systems are the $k=k'=1/\sqrt{2}$ particular case of the rotated elliptic (respectively, rotated prolate spheroidal) systems. On the hyperboloid the separating coordinate systems are the semihyperbolic systems. The flat-space limits of these systems are parabolic coordinates in two and three dimensions.

2. COORDINATE SYSTEMS ON HYPERBOLOIDS

In this section we construct the coordinate systems on the three-dimensional hyperboloid. However, first we cite some useful information concerning the construction of coordinate systems on the most important spaces of constant curvature. These are Euclidean spaces, spheres, and hyperboloids.

For the classification of coordinate systems in a homogeneous space, and hence for sets of inequivalent observables, we need second-order differential operators I_i ($i\in J$; J is an index set) which are at most quadratic in the derivatives. If they are to characterize a coordinate system which separates the Hamiltonian, we must require that they commute with the Hamiltonian and with each other, i.e., $[H,I_i]=[I_i,I_j]=0$. This property characterizes them as observables (in classical mechanics, as constants of the motion). In two-dimensional spaces we have one characteristic operator I which corresponds to the additional observable, and in three-dimensional spaces there are two characteristic operators I_1, and I_2, which correspond to the two separation constants that appear for each coordinate system. The problem of finding all inequivalent sets of $\{I\}$ (respectively, $\{I_1,I_2\}$) is equivalent to that of finding all inequivalent sets of observables for the Hamiltonian of the free motion. Because the operators $I_{1,2}$ commute with the Hamiltonian and with each other, one can find simultaneously eigenfunctions of H, I (respectively, H, I_1, I_2).

Before discussing the coordinate systems on the three-dimensional hyperboloid in some detail, let us start with some remarks concerning harmonic analysis on $\Lambda^{(3)}$ and cite some results from Refs. 52 and 90.

The homogeneous Lorentz group SO(3,1) consists of those proper, real, linear transformations which leave the following hyperboloid ($u_0>0$) invariant:

$$\mathbf{u}\cdot\mathbf{u}=\mathbf{u}^2=u_0^2-(u_1^2+u_2^2+u_3^2)=u_0^2-\mathbf{u}^2=R^2. \tag{2.1}$$

The Lie algebra is six-dimensional, and is generated by the spatial-rotation generators

$$L_1=\frac{\hbar}{i}\left(u_2\frac{\partial}{\partial u_3}-u_3\frac{\partial}{\partial u_2}\right),\quad L_2=\frac{\hbar}{i}\left(u_1\frac{\partial}{\partial u_3}-u_3\frac{\partial}{\partial u_1}\right),\quad L_3=\frac{\hbar}{i}\left(u_1\frac{\partial}{\partial u_2}-u_2\frac{\partial}{\partial u_1}\right) \tag{2.2}$$

(note the sign convention, in contrast to the case of the sphere) and the Lorentz-transformation generators

$$K_1=\frac{\hbar}{i}\left(u_0\frac{\partial}{\partial u_1}+u_1\frac{\partial}{\partial u_0}\right),\quad K_2=\frac{\hbar}{i}\left(u_0\frac{\partial}{\partial u_2}+u_2\frac{\partial}{\partial u_0}\right),\quad K_3=\frac{\hbar}{i}\left(u_0\frac{\partial}{\partial u_3}+u_3\frac{\partial}{\partial u_0}\right). \tag{2.3}$$

The commutation relations are

$$[L_i,L_j]=-i\hbar\varepsilon_{ijk}L_k,$$

$$[L_i,K_j]=-i\hbar\varepsilon_{ijk}K_k,\quad [K_i,K_j]=i\hbar\varepsilon_{ijk}K_k. \tag{2.4}$$

The Hamiltonian on $\Lambda^{(3)}$ can then be written as [$V(\mathbf{u})$ is a potential on $\Lambda^{(3)}$]

$$H=H_0+V(\mathbf{u}),$$

$$H_0=-\frac{\hbar^2}{2MR^2}\Delta_{\mathrm{LB}}=\frac{1}{2MR^2}(\mathbf{K}^2-\mathbf{L}^2). \tag{2.5}$$

The irreducible representations of the identity component of SO(3,1) are labeled by two numbers (j_0,σ), where j_0 is an integer or half-integer, and σ is complex. The eigenvalues of the Schrödinger operator H_0 are found to have the form

$$E_{\sigma,j_0}=-\frac{\hbar^2}{2MR^2}[j_0^2+\sigma(\sigma+2)],\quad \begin{cases}\text{continuous spectrum:} & j_0=0,\ \sigma=-1+ip,\\ \text{discrete spectrum:} & j_0=2n\ (n\in\mathbb{N}),\ \sigma=-1\end{cases}. \tag{2.6}$$

Actually, the discrete spectrum is not present in our case; for instance, it must be taken into account for the quantum motion on the single-sheeted hyperboloid,[31] and on the SU(1,1) (Ref. 7) and O(2,2) group manifolds.[31,51] We have the following expression for the energy spectrum of the free quantum motion on $\Lambda^{(3)}$:

$$E_p=\frac{\hbar^2}{2MR^2}(p^2+1),\quad p>0. \tag{2.7}$$

In the following enumeration we give in each case the definition of the coordinate system, the metric, the momentum operators, the Hamiltonian, and the observables I_1, I_2.

In the sequel we consider only orthogonal coordinate systems on the three-dimensional hyperboloid; $\mathbf{u}\in\Lambda^{(3)}$ is expressed as $\mathbf{u}=u(\varrho)$, where $\varrho=(\varrho_1,\varrho_2,\varrho_3)$ are three-dimensional coordinates on $\Lambda^{(3)}$. Following Olevskiĭ,[75] the line element is found to have the form

$$\begin{aligned}ds^2=\epsilon_a g_{aa}d\varrho_a^2=\frac{1}{4}\Bigg[&\frac{(\varrho_1-\varrho_2)(\varrho_1-\varrho_3)}{P(\varrho_1)}d\varrho_1^2\\ &+\frac{(\varrho_2-\varrho_3)(\varrho_2-\varrho_1)}{P(\varrho_2)}d\varrho_2^2\\ &+\frac{(\varrho_3-\varrho_1)(\varrho_2-\varrho_1)}{P(\varrho_3)}d\varrho_3^2\Bigg],\end{aligned} \tag{2.8}$$

which must be a positive-definite quantity; hence $\varepsilon_a=-1$, with $\alpha=1,2,3$, and $P(\varrho)$ is the characteristic polynomial corresponding to the coordinate system. In algebraic form, a coordinate system on $\Lambda^{(3)}$ is described in the following way:

$$\left.\begin{aligned}u_0^2&=R^2\frac{(\varrho_1-a_1)(\varrho_2-a_1)(\varrho_3-a_1)}{(a_2-a_1)(a_3-a_1)(a_4-a_1)},\\ u_1^2&=-R^2\frac{(\varrho_1-a_2)(\varrho_2-a_2)(\varrho_3-a_2)}{(a_1-a_2)(a_3-a_2)(a_4-a_2)},\\ u_2^2&=-R^2\frac{(\varrho_1-a_3)(\varrho_2-a_3)(\varrho_3-a_3)}{(a_1-a_3)(a_2-a_3)(a_4-a_3)},\\ u_3^2&=-R^2\frac{(\varrho_1-a_4)(\varrho_2-a_4)(\varrho_3-a_4)}{(a_1-a_4)(a_2-a_4)(a_3-a_4)},\end{aligned}\right\} \tag{2.9}$$

and we have, for the characteristic polynomial,

$$P(\varrho)=(\varrho-a_1)(\varrho-a_2)(\varrho-a_3)(\varrho-a_4). \tag{2.10}$$

Specification of the numbers α_i, with $i=1,2,3,4$, and of the range of ϱ specifies a coordinate system. For the metric tensor we have

$$g_{ab}=G_{ik}\frac{\partial u_i}{\partial\varrho_a}\frac{\partial u_k}{\partial\varrho_b}, \tag{2.11}$$

where G_{ik} is the metric tensor of the ambient space, which in the present case has the form $G_{ik}=\mathrm{diag}(1,-1,-1,-1)$, and for the line element $ds^2=\Sigma_{ab}\varepsilon_{ab}g_{ab}dq^a dq^b$ to be positive definite, appropriate $\varepsilon_{aa}=\pm1$ must be taken into account. Actually, $\varepsilon_{ab}=\varepsilon_{aa}=-1,\ \forall_{a,b}$. In the following, we state for convenience only the explicit form of ds^2. In Table I we summarize the results on the coordinate systems on $\Lambda^{(3)}$ according to Refs. 49, 52, and 75. The coordinate systems on $\Lambda^{(3)}$ are summarized in Table I. The potentials $V_1,\ldots,V_{20}$ refer to Secs. 3–5.

3. PATH-INTEGRAL FORMULATION OF THE MAXIMALLY SUPERINTEGRABLE POTENTIALS ON $\Lambda^{(3)}$

In Table II we list the superintegrable potentials on the three-dimensional hyperboloid together with the separating coordinate systems. The cases in which an explicit path integration is possible are underlined.

3.1. The oscillator

We consider the generalized Higgs oscillator on the hyperboloid ($k_{1,2,3}>0$),

$$\begin{aligned}V_1(\mathbf{u})=&\frac{M}{2}\omega^2R^2\frac{u_1^2+u_2^2+u_3^2}{u_0^2}\\ &+\frac{\hbar^2}{2M}\left(\frac{k_1^2-\frac14}{u_1^2}+\frac{k_2^2-\frac14}{u_2^2}+\frac{k_3^2-\frac14}{u_3^2}\right),\end{aligned} \tag{3.1}$$

which in the 14 separating coordinate systems has the form

Cylindrical ($\tau_{1,2}>0$, $\varphi\in(0,\pi/2)$):

$$\begin{aligned}V_1(\mathbf{u})=&\frac{M}{2}\omega^2R^2\left(1-\frac{1}{\cosh^2\tau_1\cosh^2\tau_2}\right)\\ &+\frac{\hbar^2}{2MR^2}\left(\frac{1}{\sinh^2\tau_1}\left(\frac{k_1^2-\frac14}{\cos^2\varphi}+\frac{k_2^2-\frac14}{\sin^2\varphi}\right)\right.\\ &\left.+\frac{k_3^2-\frac14}{\cosh^2\tau_1\sinh^2\tau_2}\right).\end{aligned} \tag{3.2}$$

Sphero-elliptic ($\tau>0$, $\tilde\alpha\in(0,K)$, $\tilde\beta\in(0,K')$):

$$\begin{aligned}=&\frac{M}{2}\omega^2R^2\tanh^2\tau+\frac{\hbar^2}{2MR^2\sinh^2\tau}\left(\frac{k_1^2-\frac14}{\mathrm{sn}^2\tilde\alpha\,\mathrm{dn}^2\tilde\beta}\right.\\ &\left.+\frac{k_2^2-\frac14}{\mathrm{cn}^2\tilde\alpha\,\mathrm{cn}^2\tilde\beta}+\frac{k_3^2-\frac14}{\mathrm{dn}^2\tilde\alpha\,\mathrm{sn}^2\tilde\beta}\right).\end{aligned} \tag{3.3}$$

Equidistant-elliptic ($\tau>0$, $\alpha\in(iK',iK'+K)$, $\beta\in(0,K')$):

$$\begin{aligned}=&\frac{M}{2}\omega^2R^2\left(1-\frac{1}{\cosh^2\tau}\frac{1}{\mathrm{sn}^2\alpha\,\mathrm{dn}^2\beta}\right)-\frac{\hbar^2}{2MR^2}\\ &\times\left[\frac{1}{\cosh^2\tau}\left(\frac{k_1^2-\frac14}{\mathrm{cn}^2\alpha\,\mathrm{cn}^2\beta}+\frac{k_2^2-\frac14}{\mathrm{dn}^2\alpha\,\mathrm{sn}^2\beta}\right)-\frac{k_3^2-\frac14}{\sinh^2\tau}\right].\end{aligned} \tag{3.4}$$

TABLE I. Coordinate systems on the three-dimensional hyperboloid.

Coordinate system Observables I_1, I_2	Coordinates	Separates the potential
I. Cylindrical $\tau_{1,2}\in\mathbb{R},\ \varphi\in[0,2\pi)$ $I_1=K_3^2$ $I_2=L_3^2$	$u_0=R\cosh\tau_1\cosh\tau_2$ $u_1=R\sinh\tau_1\cos\varphi$ $u_2=R\sinh\tau_1\sin\varphi$ $u_3=R\cosh\tau_1\sinh\tau_2$	V_1
II. Horicyclic $x_{1,2}\in\mathbb{R},\ y>0$ $I_1=(K_1+L_2)^2$ $I_2=(K_2-L_1)^2$	$u_0=R[y+(x_1^2+x_2^2)/y+1/y]/2$ $u_1=Rx_1/2y$ $u_2=Rx_2/2y$ $u_3=R[y+(x_1^2+x_2^2)/y-1/y]/2$	V_8, V_9, V_{10}
III. Sphero-elliptic $\tau>0,\ \tilde{\alpha}\in[-K,K]$ $\tilde{\beta}\in[-2K',2K']$ $I_1=\mathbf{L}^2,\ I_2=L_1^2+k'^2L_2^2$	$u_0=R\cosh\tau$ $u_1=R\sinh\tau\,\mathrm{sn}\,\tilde{\alpha}\,\mathrm{dn}\tilde{\beta}$ $u_2=R\sinh\tau\,\mathrm{cn}\,\tilde{\alpha}\,\mathrm{cn}\tilde{\beta}$ $u_3=R\sinh\tau\,\mathrm{dn}\,\tilde{\alpha}\,\mathrm{sn}\tilde{\beta}$	V_1, V_2, V_7 $V_7{}^{\dagger *}$ $V_{13}{}^{*}$
IV. Equidistant-elliptic $\tau\in\mathbb{R},\ \alpha\in(iK',iK'+2K)$ $\beta\in[0,4K')$ $I_1=K_1^2+K_2^2-L_3^2$ $I_2=L_3^2+\sinh^2 fK_1^2$	$u_0=R\cosh\tau\,\mathrm{sn}\,\alpha\,\mathrm{dn}\beta$ $u_1=iR\cosh\tau\,\mathrm{cn}\,\alpha\,\mathrm{cn}\beta$ $u_2=iR\cosh\tau\,\mathrm{dn}\,\alpha\,\mathrm{sn}\beta$ $u_3=R\sinh\tau$	V_1, V_4^+, V_{14} V_{15}^+
V. Equidistant-hyperbolic $\tau\in\mathbb{R},\ \mu\in(iK',iK'+2K)$ $\eta\in[0,4K')$ $I_1=K_1^2+K_2^2-L_3^2$ $I_2=K_1^2-\sin^2\alpha L_3^2$	$u_0=-R\cosh\tau\,\mathrm{cn}\mu\,\mathrm{cn}\eta$ $u_1=iR\cosh\tau\,\mathrm{sn}\mu\,\mathrm{dn}\eta$ $u_2=iR\cosh\tau\,\mathrm{dn}\mu\,\mathrm{sn}\eta$ $u_3=R\sinh\tau$	V_1, V_{14}
VI. Equidistant-semihyperbolic $\tau\in\mathbb{R},\ \mu_{1,2}>0$ $I_1=K_1^2+K_2^2-L_3^2$ $I_2=\{L_3K_2\}$	$u_0=\dfrac{R}{\sqrt{2}}\cosh\tau(\sqrt{(1+\mu_1^2)(1+\mu_2^2)}+\mu_1\mu_2+1)^{1/2}$ $u_1=\dfrac{R}{\sqrt{2}}\cosh\tau(\sqrt{(1+\mu_1^2)(1+\mu_2^2)}-\mu_1\mu_2-1)^{1/2}$ $u_2=R\cosh\tau\sqrt{\mu_1\mu_2}$ $u_3=R\sinh\tau$	V_4, V_{15}
VII. Equidistant-elliptic parabolic $\tau,a\in\mathbb{R},\ \vartheta\in(-\pi/2,\pi/2)$ $I_1=K_1^2+K_2^2-L_3^2$ $I_2=(K_2-L_3)^2+K_1^2$	$u_0=R\cosh\tau\dfrac{\cosh^2 a+\cos^2\vartheta}{2\cosh a\cos\vartheta}$ $u_1=R\cosh\tau\dfrac{\sinh^2 a-\sin^2\vartheta}{2\cosh a\cos\vartheta}$ $u_2=R\cosh\tau\tan\vartheta\tanh a$ $u_3=R\sinh\tau$	V_4, V_{15}, V_{17}
VIII. Equidistant-hyperbolic-parabolic $\tau\in\mathbb{R},\ b>0,\ \vartheta\in(0,\pi)$ $I_1=K_1^2+K_2^2-L_3^2$ $I_2=(K_2-L_3)^2-K_1^2$	$u_0=R\cosh\tau\dfrac{\cosh^2 b+\cos^2\vartheta}{2\sinh b\sin\vartheta}$ $u_2=R\cosh\tau\dfrac{\sinh^2 b-\sin^2\vartheta}{2\sinh b\sin\vartheta}$ $u_1=R\cosh\tau\cot\vartheta\coth b$ $u_3=R\sinh\tau$	V_{17}
IX. Equidistant-semicircular-parabolic ($\tau\in\mathbb{R},\ \xi,\eta>0$) $I_1=K_1^2+K_2^2-L_3^2$ $I_2=\{K_1,K_2\}-\{K_1,L_3\}$	$u_0=R\cosh\tau\dfrac{(\xi^2+\eta^2)^2+4}{8\xi\eta}$ $u_1=R\cosh\tau\dfrac{(\xi^2+\eta^2)^2-4}{8\xi\eta}$ $u_2=R\cosh\tau\dfrac{\eta^2-\xi^2}{2\xi\eta}$ $u_3=R\sinh\tau$	V_{16}, V_{17}, V_{18}

*After a rotation with $I_2'=\cos 2fL_3^2-1/2\sin 2f\{L_1,L_3\}$.
†After a rotation with $I_2'=\cosh 2fL_3^2-1/2\sinh 2f\{K_2,L_3\}$.

Equidistant-hyperbolic ($\tau>0$, $\mu\in(iK',iK'+K)$, $\eta\in(0,K')$):

$$=\frac{M}{2}\omega^2R^2\left(1-\frac{1}{\cosh^2\tau}\frac{1}{\mathrm{cn}^2\mu\,\mathrm{cn}^2\eta}\right)-\frac{\hbar^2}{2MR^2}$$

$$\times\left[\frac{1}{\cosh^2\tau}\left(\frac{k_1^2-\frac14}{\mathrm{sn}^2\mu\,\mathrm{dn}^2\eta}+\frac{k_2^2-\frac14}{\mathrm{dn}^2\mu\,\mathrm{sn}^2\eta}\right)-\frac{k_3^2-\frac14}{\sinh^2\tau}\right].\quad(3.5)$$

Spherical ($\tau>0$, $\vartheta\in(0,\pi/2)$, $\varphi\in(0,\pi/2)$):

TABLE I. (*Continued.*)

Coordinate system	Coordinates	Separates the potential	Limiting systems
X. Spherical	$u_0=R\cosh\tau$	V_1, V_2, V_3	Spherical
$\tau>0,\ \vartheta\in(0,\pi),\ \varphi\in[0,2\pi)$	$u_1=R\sinh\tau\sin\vartheta\cos\varphi$	V_5, V_6, V_7	
$S_1=\mathbf{L}^2$	$u_2=R\sinh\tau\sin\vartheta\sin\varphi$	V_{12}, V_{13}	
$S_2=L_3^2$	$u_3=R\sinh\tau\cos\vartheta$		
XI. Equidistant-cylindrical	$u_0=R\cosh\tau_1\cosh\tau_2$	V_1, V_3, V_4	Circular-polar
$\tau_{1,2}\in\mathbb{R},\ \varphi\in[0,2\pi)$	$u_1=R\cosh\tau_1\sinh\tau_2\cos\varphi$	V_5, V_{14}, V_{15}	
$S_1=\Delta_{\mathrm{LB}}^{\Lambda(2)}$	$u_2=R\cosh\tau_1\sinh\tau_2\sin\varphi$		
$S_2=L_3^2$	$u_3=R\sinh\tau_1$		
XII. Equidistant	$u_0=R\cosh\tau_1\cosh\tau_2\cosh\tau_3$	V_1, V_{14}, V_{17}	Cartesian
$r_{1,2,3}\in\mathbb{R}$	$u_1=R\cosh\tau_1\cosh\tau_2\sinh\tau_3$	V_{18}	
$S_1=\Delta_{\mathrm{LB}}^{\Lambda(2)}$	$u_2=R\cosh\tau_1\sinh\tau_2$		
$S_2=K_1^2$	$u_3=R\sinh\tau_1$		
XIII. Equidistant-horicyclic	$u_0=R\cosh\tau(y+x^2/y+1/y)/2$	V_{15}, V_{17}	Cartesian
$r,\ x\in\mathbb{R},\ y>0$	$u_1=R\sinh\tau$		
$S_1=\Delta_{\mathrm{LB}}^{\Lambda(2)}$	$u_2=Rx\cosh\tau/y$		
$S_2=(K_2-L_3)^2$	$u_3=R\cosh\tau(y+x^2/y-1/y)/2$		
XIV. Horicyclic-cylindrical	$u_0=R(y+e^2/y+1/y)/2$	V_8, V_{10}	Circular-polar
$y,\varrho>0,\ \varphi\in[0,2\pi)$	$u_1=R\varrho\cos\varphi/y$		
$S_1=(K_1+M_2)^2+(K_2-M_1)^2$	$u_2=R\varrho\sin\varphi/y$		
$S_2=L_3^2$	$u_3=R(y+\varrho^2/y-1/y)/2$		
XV. Horicyclic-elliptic	$u_0=R[y+(\cosh^2\mu-\sin^2\nu)/y+1/y]/2$	V_8, V_{11}*	Circular-elliptic
$y,\mu>0,\ \nu\in(-\pi,\pi)$	$u_1=R\cosh\mu\cos\nu/y$		
$S_1=(K_1+M_2)^2+(K_2-M_1)^2$	$u_2=R\sinh\mu\sin\nu/y$		
$S_2=L_3^2+(K_1+L_2)^2$	$u_3=R[y+(\cosh^2\mu-\sin^2\nu)/y-1/y]/2$		
XVI. Horicyclic-parabolic	$u_0=R[y(\xi^2+\eta^2)^2/y+1/y]/2$	V_9, V_{10}, V_{11}	Circular-parabolic
$y,\eta>0,\ \xi\in\mathbb{R}$	$u_1=R(\eta^2-\xi^2)/2y$		
$S_1=(K_1+M_2)^2+(K_2-M_1)^2$	$u_2=R\xi\eta/y$		
$S_2=\{L_3K_1+L_2\}$	$u_3=R[y+(\xi^2+\eta^2)^2/y-1/y]/2$		
XVII. Elliptic-cylindrical 1	$u_0=R\,\mathrm{sn}\alpha\,\mathrm{dn}\beta$	$V_1, V_2,$* V_3	Prolate-spheroidal
$\alpha\in(iK',iK'+2K)$	$u_1=iR\,\mathrm{dn}\,\alpha\,\mathrm{sn}\beta\cos\varphi$	V_5, V_6*	
$\beta\in[0,4K'),\ \varphi\in[0,2\pi)$	$u_2=iR\,\mathrm{dn}\,\alpha\,\mathrm{sn}\beta\sin\varphi$		Prolate-spheroidal II*
$S_1=L_3^2$	$u_3=iR\,\mathrm{cn}\alpha\,\mathrm{cn}\beta$		
$S_2=K_1^2+K_2^2+k^2K_3^2-k'^2L_3^2$			
XVIII. Elliptic-cylindrical 2	$u_0=R\,\mathrm{sn}\alpha\,\mathrm{dn}\beta$	V_1, V_3, V_5	Oblate-spheroidal
$\alpha\in(iK',iK'+2K)$	$u_1=iR\,\mathrm{cn}\alpha\,\mathrm{cn}\beta\cos\varphi$		
$\beta\in[0,4K'),\ \varphi\in[0,2\pi)$	$u_2=iR\,\mathrm{cn}\alpha\,\mathrm{cn}\beta\sin\varphi$		
$S_1=L_3^2$	$u_3=R\,\mathrm{dn}\alpha\,\mathrm{sn}\beta$		
$S_2=K_3^2+k^2(K_1^2+K_2^2)+k'^2L_3^2$			
XIX. Elliptic-cylindrical 3	$u_0=R\,\mathrm{sn}\alpha\,\mathrm{dn}\beta\cosh\tau$	V_1	Circular-elliptic
$\tau\in\mathbb{R},\ \alpha\in(iK',iK'+2K),\ \beta\in[0,4K')$	$u_1=R\,\mathrm{sn}\alpha\,\mathrm{dn}\beta\sinh\tau$		
$S_1=K_1^2$	$u_3=iR\,\mathrm{dn}\alpha\,\mathrm{sn}\beta$		
$S_2=K_2^2-L_3^2+k^2(K_3^2-L_2^2)-(1+k^2)K_1^2$	$u_2=iR\,\mathrm{cn}\alpha\,\mathrm{cn}\beta$		
XX. Hyperbolic-cylindrical 1	$u_0=-R\,\mathrm{cn}\mu\,\mathrm{cn}\eta\cosh\tau$	V_1	Cartesian
$\tau\in\mathbb{R},\ \mu\in(iK',iK'+2K)$	$u_1=-R\,\mathrm{cn}\mu\,\mathrm{cn}\eta\sinh\tau$		
$\eta\in[0,4K')$	$u_2=iR\,\mathrm{dn}\mu\,\mathrm{sn}\eta$		
$S_1=K_1^2,\ S_2=K_3^2-M_2^2+k^2(K_1^2-M_1^2)$	$u_3=iR\,\mathrm{sn}\mu\,\mathrm{dn}\eta$		

*After an appropriate rotation.

$$=\frac{M}{2}\omega^2R^2\tanh^2\tau+\frac{\hbar^2}{2MR^2\sinh^2\tau}\times\left(\frac{1}{\sin^2\vartheta}\left(\frac{k_1^2-\frac14}{\cos^2\varphi}+\frac{k_2^2-\frac14}{\sin^2\varphi}\right)+\frac{k_3^2-\frac14}{\cos^2\vartheta}\right).\tag{3.6}$$

Equidistant-cylindrical ($\tau_{1,2}>0,\ \varphi\in(0,\pi/2)$):

$$=\frac{M}{2}\omega^2R^2\left(1-\frac{1}{\cosh^2\tau_1\cosh^2\tau_2}\right)+\frac{\hbar^2}{2MR^2}\times\left(\frac{1}{\cosh^2\tau_1\sinh^2\tau_2}\left(\frac{k_1^2-\frac14}{\cos^2\varphi}+\frac{k_2^2-\frac14}{\sin^2\varphi}\right)+\frac{k_3^2-\frac14}{\sinh^2\tau_1}\right).\tag{3.7}$$

Equidistant ($\tau_{1,2,3}>0$):

$$=\frac{M}{2}\omega^2R^2\left(1-\frac{1}{\cosh^2\tau_1\cosh^2\tau_2\cosh^2\tau_3}\right)+\frac{\hbar^2}{2MR^2}$$

TABLE I. (*Continued.*)

Coordinate system	Coordinates	Separates the potential	Limiting systems
XXI. Hyperbolic-cylindrical 2 $\mu\in(iK',iK'+2K)$ $\eta\in[0,4K')$, $\varphi\in[0,2\pi)$ $S_1=L_3^2$, $S_2=K_3^2+L_3^2-k^2(L_1^2+L_2^2)$	$u_0=-R\,\mathrm{cn}\mu\,\mathrm{cn}\eta$ $u_1=\mathrm{i}R\,\mathrm{sn}\mu\,\mathrm{dn}\eta\cos\varphi$ $u_2=\mathrm{i}R\,\mathrm{dn}\mu\,\mathrm{sn}\eta\sin\varphi$ $u_3=\mathrm{i}R\,\mathrm{sn}\mu\,\mathrm{dn}\eta$	V_1, V_3, V_5	Circular-polar
XXII. Semi-hyperbolic $\mu_{1,2}>0$, $\varphi\in[0,2\pi)$ $S_1=L_3^2$ $S_2=\{K_1,L_2\}+\{K_2,L_1\}$	$u_0=\frac{R}{\sqrt{2}}(\sqrt{(1+\mu_1^2)(1+\mu_2^2)}+\mu_1\mu_2+1)^{1/2}$ $u_1=R\sqrt{\mu_1\mu_2}\cos\varphi$ $u_2=R\sqrt{\mu_1\mu_2}\sin\varphi$ $u_3=\frac{R}{\sqrt{2}}(\sqrt{(1+\mu_1^2)(1+\mu_2^2)}-\mu_1\mu_2-1)^{1/2}$	V_2	Parabolic Circular-polar
XXIII. Elliptic-parabolic 1 $a,\varrho\in\mathbb{R}$, $\vartheta\in(-\pi/2,\pi/2)$ $S_1=(K_1+L_2)^2$ $S_2=2K_1^2+K_2^2+K_3^2+L_1^2-\{K_1,L_2\}-\{K_2,L_1\}$	$u_0=R\frac{\cosh^2 a+\cos^2\vartheta+\varrho^2}{2\cosh a\cos\vartheta}$ $u_1=R\varrho/\cosh a\cos\vartheta$ $u_2=R\tanh a\tan\vartheta$ $u_3=R\frac{\cosh^2 a+\cos^2\vartheta-\varrho^2-2}{2\cosh a\cos\vartheta}$	$V_9^{(\omega=0)}$	Circular-parabolic
XXIV. Hyperbolic-parabolic 1 $b>0$, $\varrho\in\mathbb{R}$, $\vartheta\in(0,\pi)$ $S_1=(K_1+L_2)^2$ $S_2=2L_2^2+L_1^2+K_2^2-K_3^2-\{K_2,L_1\}-\{K_1,L_2\}$	$u_0=R\frac{\sinh^2 b-\sin^2\vartheta+\varrho^2+2}{2\sinh b\sin\vartheta}$ $u_1=R\varrho/\sinh b\sin\vartheta$ $u_2=R\coth b\cot\vartheta$ $u_3=R\frac{\sinh^2 b-\sin^2\vartheta-\varrho^2}{2\sinh b\sin\vartheta}$	$V_9^{(\omega=0)}$	Cartesian
XXV. Elliptic-parabolic 2 $a\in\mathbb{R}$, $\vartheta\in(-\pi/2,\pi/2)$, $\varphi\in[0,2\pi)$ $S_1=L_3^2$ $S_2=L_1^2+L_2^2+K_1^2+K_2^2+K_3^2-\{K_1L_2\}-\{K_2L_1\}$	$u_0=R\frac{\cos^2\vartheta+\cosh^2 a}{2\cosh a\cos\vartheta}$ $u_1=R\tanh a\tan\vartheta\cos\varphi$ $u_2=R\tanh a\tan\vartheta\sin\varphi$ $u_3=R\frac{\sinh^2 a-\sin^2\vartheta}{2\cosh a\cos\vartheta}$	V_2 $V_9^{(\omega=0)}$	Parabolic
XXVI. Hyperbolic-parabolic 2 $b>0$, $\vartheta\in(0,\pi)$, $\varphi\in[0,2\pi)$ $S_1=L_3^2$ $S_2=L_1^2+L_2^2+K_1^2+K_2^2-K_3^2+\{K_1,L_2\}+\{K_2L_1\}$	$u_0=R\frac{\cosh^2 b+\cos^2\vartheta}{2\sinh b\sin\vartheta}$ $u_1=R\coth b\cot\vartheta\cos\varphi$ $u_2=R\coth b\cot\vartheta\sin\varphi$ $u_3=R\frac{\sin^2\vartheta-\sinh^2 b}{2\sinh b\sin\vartheta}$	$V_9^{(\omega=0)}$	Circular-polar
XXVII. Semicircular-parabolic $\varrho\in\mathbb{R}$, $\xi,\eta>0$ $S_1=(K_1+L_2)^2$ $S_2=\{L_3,K_1+L_2\}+\{K_3,K_2-L_1\}$	$u_0=R\frac{(\eta^2-\xi^2)^2+4\varrho^2+4}{8\xi\eta}$ $u_1=R\frac{\eta^2-\xi^2}{2\xi\eta}$ $u_2=R\frac{\varrho}{\xi\eta}$ $u_3=R\frac{(\eta^2-\xi^2)^2+4\varrho^2-4}{8\xi\eta}$	V_{19}, V_{20}	Cartesian
XXVIII. Ellipsoidal $0<1<\varrho_3<b<\varrho_2<a<\varrho_1$ $S_1=abK_1^2+aK_2^2+bK_3^2$ $S_2=(a+b)K_1^2+(a+1)K_2^2+(b+1)K_3^2$ $-aL_3^2-bL_2^2-L_1^2$	$u_0^2=R^2\frac{\varrho_1\varrho_2\varrho_3}{ab}$ $u_1^2=R^2\frac{(\varrho_1-1)(\varrho_2-1)(\varrho_3-1)}{(a-1)(b-1)}$ $u_2^2=-R^2\frac{(\varrho_1-b)(\varrho_2-b)(\varrho_3-b)}{(a-b)(b-1)b}$ $u_3^2=R^2\frac{(\varrho_1-a)(\varrho_2-a)(\varrho_3-a)}{(a-b)(a-1)a}$	V_1	Ellipsoidal

TABLE I. (*Continued.*)

Coordinate system	Coordinates	Separates the potential	Limiting systems
XXIX. Hyperboloidal	$u_0^2=-R^2\frac{(\varrho_1-1)(\varrho_2-1)(\varrho_3-1)}{(a-1)(b-1)}$	V_1	Cartesian
$\varrho_3<0<1<b<\varrho_2<a<\varrho_1$	$u_1^2=-R^2\frac{\varrho_1\varrho_2\varrho_3}{ab}$		
$S_1=abK_1^2-aL_3^3-bL_2^2$	$u_2^2=-R^2\frac{(\varrho_1-b)(\varrho_2-b)(\varrho_1-b)}{(a-b)(b-1)b}$		
$S_2=(a+b)K_1^2-(a+1)L_3^2-(b+1)L_2^2+aK_2^2+bK_3^2-L_1^2$	$u_3^2=R^2\frac{(\varrho_1-a)(\varrho_2-a)(\varrho_3-a)}{(a-b)(a-1)a}$		
XXX. Paraboloidal	$(u_1+iu_0)^2=2R^2\frac{(\varrho_1-a)(\varrho_2-a)(\varrho_3-a)}{(a-b)(b-1)b}$		Paraboloidal
$\varrho_3<0<\varrho_2<1<\varrho_1$	$u_2^2=R^2\frac{(\varrho_1-1)(\varrho_2-1)(\varrho_3-1)}{(a-1)(b-1)}$		
$a=b^*=\alpha+i\beta,\ \alpha,\beta\in\mathbb{R}$	$u_3^2=-R^2\frac{\varrho_1\varrho_2\varrho_3}{ab}$		
$S_1=-\lvert a\rvert^2L_1^2+\alpha(K_3^2-L_2^2)-\beta\{K_3L_2\}$ $S_2=-2\alpha L_1^2+(\alpha+1)(K_3^2-L_2^2)+\alpha(K_2^2-L_3^2)$ $+\beta(\{K_2,L_3\}-\{K_3,L_2\})$			
XXXI.	$(u_0+u_1)^2=R^2\frac{\varrho_1\varrho_2\varrho_3}{a}$		
$0<\varrho_3<1<\varrho_2<a<\varrho_1$	$(u_0^2-u_1^2)=R^2\frac{a(\varrho_1\varrho_2+\varrho_1\varrho_3+\varrho_2\varrho_3)-(a+1)\varrho_1\varrho_2\varrho_3}{a^2}$		
$S_1=(K_3+L_2)^2-a(K_2+L_3)^2+aK_1^2$	$u_2^2=-R^2\frac{(\varrho_1-1)(\varrho_2-1)(\varrho_3-1)}{(a-1)}$		
$S_2=(a+1)K_1^2+K_3^2-L_2^2+a(L_3^2-K_2^2)+(K_2+L_3)^2$ $+(K_3+L_2)^2$	$u_3^2=R^2\frac{(\varrho_1-a)(\varrho_2-a)(\varrho_3-a)}{a^2(a-1)}$		
XXXII.	$(u_0+u_1)^2=-R^2\frac{\varrho_1\varrho_2\varrho_3}{a}$		
$-\varrho_3<0<1<\varrho_2<a<\varrho_1$	$(u_0^2-u_1^2)=R^2\frac{a(\varrho_1\varrho_2+\varrho_2\varrho_3+\varrho_1\varrho_3)-(a+1)\varrho_1\varrho_2\varrho_3}{a^2}$		
$S_1=-(K_3+L_2)^2+a(K_2+L_3)^2+aK_1^2$	$u_2^2=-R^2\frac{(\varrho_1-1)(\varrho_2-1)(\varrho_3-1)}{(a-1)}$		
$S_2=(a+1)K_1^2-K_3^2+L_2^2+a(K_2^2-L_3^2)-(K_2+L_3)^2$ $-(K_3+L_2)^2$	$u_3^2=R^2\frac{(\varrho_1-a)(\varrho_2-a)(\varrho_3-a)}{a^2(a-1)}$		
XXXIII.	$(u_0+u_1)^2=-R^2\frac{\varrho_1\varrho_2\varrho_3}{a}$		
$\varrho_3<-1<0<\varrho_2<a<\varrho_1$	$(u_0^2-u_1^2)=R^2\frac{a(\varrho_1\varrho_3+\varrho_1\varrho_2+\varrho_2\varrho_3)-(a-1)\varrho_1\varrho_2\varrho_3}{a^2}$		
$S_1=aK_1^2-(K_2+L_3)^2+a(K_2+L_3)^2$	$u_2^2=R^2\frac{(\varrho_1-a)(\varrho_2-a)(\varrho_3-a)}{a^2(a+1)}$		
$S_2=(a-1)K_1^2-K_2^2+L_3^2+a(L_2^2-K_3^2)$ $-(K_2+L_3)^2+(K_3+L_2)^2$	$u_3^2=-R^2\frac{(\varrho_1+1)(\varrho_2+1)(\varrho_3+1)}{(a+1)}$		
XXXIV.	$(u_0-u_1)^2=-R^2\varrho_1\varrho_2\varrho_3$		
$\varrho_3<0<\varrho_2<1<\varrho_1$	$2u_2(u_1-u_0)=R^2(\varrho_1\varrho_2+\varrho_2\varrho_3+\varrho_1\varrho_3-\varrho_1\varrho_2\varrho_3)$		
$S_1=(L_2-K_3)^2-K_1(K_2-L_3)-(K_2-L_3)K_1$	$u_1^2+u_2^2-u_0^2=R^2(-\varrho_1\varrho_2\varrho_3+\varrho_1\varrho_2+\varrho_2\varrho_3+\varrho_1\varrho_3-\varrho_1-\varrho_2-\varrho_3)$		
$S_2=L_2^2-K_3^2-L_1^2-(L_2-K_3)^2-\{L_1,L_2-K_3\}$	$u_3^2=R^2(\varrho_1-1)(\varrho_2-1)(\varrho_3-1)$		

$$\times\left[\frac{1}{\cosh^2\tau_1}\left(\frac{k_1^2-\frac14}{\cosh^2\tau_2\,\sinh^2\tau_3}+\frac{k_2^2-\frac14}{\sinh^2\tau_2}\right)+\frac{k_3^2-\frac14}{\sinh^2\tau_1}\right]. \tag{3.8}$$

Prolate elliptic $(\alpha\in(iK',iK'+K),\quad \beta\in(0,K'),\ \varphi\in(0,\pi/2))$:

$$=\frac{M}{2}\,\omega^2R^2\left(1-\frac{1}{\mathrm{sn}^2\alpha\,\mathrm{dn}^2\beta}\right)-\frac{\hbar^2}{2MR^2}\left[\frac{1}{\mathrm{dn}^2\alpha\,\mathrm{sn}^2\beta}\times\left(\frac{k_1^2-\frac14}{\cos^2\varphi}+\frac{k_2^2-\frac14}{\sin^2\varphi}\right)+\frac{k_3^2-\frac14}{\mathrm{cn}^2\alpha\mathrm{cn}^2\beta}\right]. \tag{3.9}$$

TABLE II. Maximally superintegrable potentials on $\Lambda^{(3)}$.

Potential $V(\mathbf{u})$	Coordinate systems	Observables
$V_1(\mathbf{u})=\frac{M}{2}\omega^2R^2\frac{u_1^2+u_2^2+u_3^2}{u_0^2}+\frac{\hbar^2}{2M}\left(\frac{k_1^2-\frac{1}{4}}{u_1^2}+\frac{k_2^2-\frac{1}{4}}{u_2^2}+\frac{k_3^2-\frac{1}{4}}{u_3^2}\right)$	Cylindrical Sphero-elliptic	$I_1=\frac{1}{2MR^2}(\mathbf{K}^2-\mathbf{L}^2)+V_1(\mathbf{u})$
	Equidistant-elliptic Equidistant-hyperbolic	$I_2=\frac{1}{2M}L_3^2+\frac{\hbar^2}{2M}\left(\frac{k_1^2-\frac{1}{4}}{\cos^2\varphi}+\frac{k_2^2-\frac{1}{4}}{\sin^2\varphi}\right)$
	Spherical Equidistant-cylindrical	$I_3=\frac{1}{2M}\mathbf{L}^2+\frac{\hbar^2}{2M}\left(\frac{1}{\sin^2\vartheta}\left(\frac{k_1^2-\frac{1}{4}}{\cos^2\varphi}+\frac{k_2^2-\frac{1}{4}}{\sin^2\varphi}\right)+\frac{k_3^2-\frac{1}{4}}{\cos^2\vartheta}\right)$
	Equidistant Prolate Elliptic	$I_4=\frac{1}{2M}K_3^2-\frac{M}{2}\frac{\omega^2R^4}{\cosh^2\tau_2}+\frac{\hbar^2}{2M}\frac{k_3^2-\frac{1}{4}}{\sinh^2\tau_2}$
	Oblate Elliptic Elliptic-cylindrical	$I_5=\frac{1}{2M}(L_1^2+k'^2L_2^2)+\frac{\hbar^2}{2MR^2\sinh^2\tau}\left(\frac{k_1^2-\frac{1}{4}}{\mathrm{sn}^2\tilde\alpha\,\mathrm{dn}^2\tilde\beta}+\frac{k_2^2-\frac{1}{4}}{\mathrm{cn}^2\tilde\alpha\,\mathrm{cn}^2\tilde\beta}+\frac{k_3^2-\frac{1}{4}}{\mathrm{dn}^2\tilde\alpha\,\mathrm{sn}^2\tilde\beta}\right)$
	Hyperbolic-cylindrical 1 Hyperbolic-cylindrical 2 Ellipsoidal Hyperboloidal	
$V_2(\mathbf{u})=-\frac{\alpha}{R}\left(\frac{u_0}{\sqrt{u_1^2+u_2^2+u_3^2}}-1\right)+\frac{\hbar^2}{2M}\left(\frac{k_1^2-\frac{1}{4}}{u_1^2}+\frac{k_2^2-\frac{1}{4}}{u_2^2}\right)$	Sphero-elliptic	$I_1=\frac{1}{2MR^2}(\mathbf{K}^2-\mathbf{L}^2)+V_2(u)$
$\mathbf{A}=\frac{1}{2R}(\mathbf{L}\times\mathbf{K}-\mathbf{K}\times\mathbf{L})-\frac{\alpha\mathbf{u}}{\lvert\mathbf{u}\rvert},\quad \mathbf{u}=(u_1,u_2,u_3)$	Spherical	$I_2=\frac{1}{2M}L_3^2+\frac{\hbar^2}{2M}\left(\frac{k_1^2-\frac{1}{4}}{\cos^2\varphi}+\frac{k_2^2-\frac{1}{4}}{\sin^2\varphi}\right)$
	Prolate elliptic II	$I_3=\frac{1}{2M}\mathbf{L}^2+\frac{\hbar^2}{2M\sin^2\vartheta}\left(\frac{k_1^2-\frac{1}{4}}{\cos^2\varphi}+\frac{k_2^2-\frac{1}{4}}{\sin^2\varphi}\right)$
	Semi-hyperbolic	$I_4=\frac{1}{2M}(L_1^2+k'^2L_2^2)+\frac{\hbar^2}{2MR^2\sinh^2\tau}\left(\frac{k_1^2-\frac{1}{4}}{\mathrm{sn}^2\tilde\alpha\,\mathrm{dn}^2\tilde\beta}+\frac{k_1^2-\frac{1}{4}}{\mathrm{cn}^2\tilde\alpha\,\mathrm{cn}^2\tilde\beta}\right)$
	Elliptic parabolic II	$I_5=\frac{1}{2M}A_3+\frac{\hbar^2}{2M\sinh^2\tau\sin^2\vartheta}\left(\frac{k_1^2-\frac{1}{4}}{\cos^2\varphi}+\frac{k_2^2-\frac{1}{4}}{\sin^2\varphi}\right)$

Oblate elliptic ($\alpha\in(iK',iK'+K)$, $\beta\in(0,K')$, $\varphi\in(0,\pi/2)$):

$$=\frac{M}{2}\omega^2R^2\left(1-\frac{1}{\mathrm{sn}^2\alpha\,\mathrm{dn}^2\beta}\right)-\frac{\hbar^2}{2MR^2}\times\left[\frac{1}{\mathrm{cn}^2\alpha\,\mathrm{cn}^2\beta}\left(\frac{k_1^2-\frac{1}{4}}{\cos^2\varphi}+\frac{k_2^2-\frac{1}{4}}{\sin^2\varphi}\right)+\frac{k_3^2-\frac{1}{4}}{\mathrm{dn}^2\alpha\,\mathrm{sn}^2\beta}\right]. \tag{3.10}$$

Elliptic-cylindrical ($\alpha\in(iK',iK'+K)$, $\beta\in(0,K')$, $\tau>0$):

$$=\frac{M}{2}\omega^2R^2\left(1-\frac{1}{\mathrm{sn}^2\alpha\,\mathrm{dn}^2\beta\cosh^2\tau}\right)+\frac{\hbar^2}{2MR^2}\left(\frac{k_1^2-\frac{1}{4}}{\mathrm{sn}^2\alpha\,\mathrm{dn}^2\beta\sinh^2\tau}-\frac{k_2^2-\frac{1}{4}}{\mathrm{dn}^2\alpha\,\mathrm{sn}^2\beta}-\frac{k_3^2-\frac{1}{4}}{\mathrm{cn}^2\alpha\,\mathrm{cn}^2\beta}\right). \tag{3.11}$$

Hyperbolic-cylindrical 1 ($\mu\in(iK',iK'+K)$, $\eta\in(0,K')$, $\tau>0$):

$$=\frac{M}{2}\omega^2R^1\left(1-\frac{1}{\mathrm{cn}^2\mu\,\mathrm{cn}^2\eta\cosh^2\tau}\right)+\frac{\hbar^2}{2MR^2}\left(\frac{k_1^2-\frac{1}{4}}{\mathrm{cn}^2\mu\,\mathrm{cn}^2\eta\sinh^2\tau}-\frac{k_2^2-\frac{1}{4}}{\mathrm{sn}^2\mu\,\mathrm{dn}^2\eta}-\frac{k_3^2-\frac{1}{4}}{\mathrm{dn}^2\mu\,\mathrm{sn}^2\eta}\right). \tag{3.12}$$

Hyperbolic-cylindrical 2 ($\mu\in(iK',iK'+K)$, $\eta\in(0,K')$, $\varphi\in(0,\pi/2)$):

$$=\frac{M}{2}\omega^2R^2\left(1-\frac{1}{\mathrm{cn}^2\mu\,\mathrm{cn}^2\eta}\right)-\frac{\hbar^2}{2MR^2}\times\left[\frac{1}{\mathrm{sn}^2\mu\,\mathrm{dn}^2\eta}\left(\frac{k_1^2-\frac{1}{4}}{\cos^2\varphi}+\frac{k_2^2-\frac{1}{4}}{\sin^2\varphi}\right)+\frac{k_3^2-\frac{1}{4}}{\mathrm{dn}^2\mu\,\mathrm{sn}^2\eta}\right]. \tag{3.13}$$

Ellipsoidal ($a_{ij}=a_i-a_j$, $a_1=0$, $a_2=1$, $a_3=b$, $a_4=a$):

$$=\frac{M}{2}\omega^2R^2\left[a_{14}a_{24}a_{34}\left(\frac{1}{(\varrho_1-\varrho_3)(\varrho_2-\varrho_3)}\frac{1}{\varrho_3-a_4}\right.\right.$$

TABLE II. (*Continued.*)

Potential $V(\mathbf{u})$	Coordinate systems	Observables
$V_3(\mathbf{u})=\frac{\hbar^2}{2M}\left[-\frac{k_0^2-\frac14}{u_0^2}+\frac{k_3^2-\frac14}{u_3^2}+\frac{1}{\sqrt{u_1^2+u_2^2}}\times\left(\frac{k_1^2-\frac14}{\sqrt{u_1^2+u_2^2}+u_1}+\frac{k_2^2-\frac14}{\sqrt{u_1^2+u_2^2}-u_1}\right)\right]$	Spherical	$I_1=\frac{1}{2MR^2}(\mathbf{K}^2-\mathbf{L}^2)+V_3(\mathbf{u})$
	Equidistant-cylindrical	$I_2=\frac{1}{2M}L_3^2+\frac{\hbar^2}{8M}\left(\frac{k_1^2-\frac14}{\cos^2\frac{\varphi}{2}}+\frac{k_2^2-\frac14}{\sin^2\frac{\varphi}{2}}\right)$
	Prolate elliptic	$I_3=\frac{1}{2M}\mathbf{L}^2+\frac{\hbar^2}{2M}\left(\frac{1}{4\sin^2\vartheta}\left(\frac{k_1^2-\frac14}{\cos^2\frac{\varphi}{2}}+\frac{k_2^2-\frac14}{\sin^2\frac{\varphi}{2}}\right)+\frac{k_3^2-\frac14}{\cos^2\vartheta}\right)$
	Oblate elliptic	$I_4=\frac{1}{2M}(K_1^2+K_2^2-L_3^2)+\frac{\hbar^2}{8M\sinh^2\tau_2}\left(\frac{k_1^2-\frac14}{\cos^2\frac{\varphi}{2}}+\frac{k_2^2-\frac14}{\sin^2\frac{\varphi}{2}}\right)$
	Hyperbolic-cylindrical 2	$I_5=\frac{1}{2M_2}[K_3^2+L_3^2-k^2(L_1^2+L_2^2)]-\frac{\hbar^2}{2M}\left[\frac{1}{4\mathrm{sn}^2\mu\,\mathrm{dn}^2\nu}\left(\frac{k_1^2-\frac14}{\cos^2\frac{\varphi}{2}}+\frac{k_2^2-\frac14}{\sin^2\frac{\varphi}{2}}\right)+\frac{k_3^2-\frac14}{\mathrm{dn}^2\mu\,\mathrm{sn}^2\nu}\right]$
Minimally superintegrable potential on $\Lambda^{(3)}$ (with maximally superintegrable analog in $\mathbb{R}^3$)		
$V_4(\mathbf{u})=\frac{\hbar^2}{4M\sqrt{u_1^2+u_2^2}}\left(\frac{k_1^2-\frac14}{\sqrt{u_1^2+u_2^2}+u_1}+\frac{k_2^2-\frac14}{\sqrt{u_1^2+u_2^2}-u_1}\right)+k_3u_3$	Equidistant-elliptic*	$I_1=\frac{1}{2MR^2}(\mathbf{K}^2-\mathbf{L}^2)+V_4(\mathbf{u})$
	Equidistant-semihyperbolic	$I_2=\frac{1}{2M}(K_1^2+K_2^2-L_3^2)+\frac{\hbar^2}{8M\sinh^2\tau_2}\left(\frac{k_1^2-\frac14}{\sin^2\frac{\varphi}{2}}+\frac{k_2^2-\frac14}{\cos^2\frac{\varphi}{2}}\right)$
	Equidistant-elliptic-parabolic	$I_3=\frac{1}{2M}L_3^2+\frac{\hbar^2}{8M}\left(\frac{k_1^2-\frac14}{\sin^2\frac{\varphi}{2}}+\frac{k_2^2-\frac14}{\cos^2\frac{\varphi}{2}}\right)$
	Equidistant-cylindrical	$I_4=\frac{1}{2M}[(K_2-L_3)^2+K_1^2]+\frac{\hbar^2}{2M}\frac{\cosh^2a\cos^2\vartheta}{\cosh^2a-\cos^2\vartheta}\left(\frac{k_1^2-\frac14}{\sin^2\vartheta}+\frac{k_2^2-\frac14}{\sinh^2a}\right)$

*After an appropriate rotation, $\sin^2 f=k^2$.

$$+\frac{1}{(\varrho_1-\varrho_2)(\varrho_3-\varrho_2)}\frac{1}{\varrho_2-a_4}$$
$$\left.+\frac{1}{(\varrho_2-\varrho_1)(\varrho_3-\varrho_1)}\frac{1}{\varrho_1-a_4}\right)-1\Bigg]$$
$$+\frac{\hbar^2}{2MR^2}\left\{\frac{1}{(\varrho_1-\varrho_3)(\varrho_2-\varrho_3)}\left[a_{31}a_{21}a_{41}\frac{k_1^2-\frac14}{\varrho_3-a_1}\right.\right.$$
$$\left.+a_{12}a_{32}a_{42}\frac{k_2^2-\frac14}{\varrho_3-a_2}+a_{13}a_{23}a_{43}\frac{k_3^2-\frac14}{\varrho_3-a_3}\right]$$
$$+\frac{1}{(\varrho_1-\varrho_2)(\varrho_3-\varrho_2)}\left[a_{31}a_{21}a_{41}\frac{k_1^2-\frac14}{\varrho_2-a_1}\right.$$
$$\left.+a_{12}a_{32}a_{42}\frac{k_2^2-\frac14}{\varrho_2-a_2}+a_{13}a_{23}a_{43}\frac{k_3^2-\frac14}{\varrho_2-a_3}\right]$$
$$+\frac{1}{(\varrho_2-\varrho_1)(\varrho_3-\varrho_1)}\left[a_{31}a_{21}a_{41}\frac{k_1^2-\frac14}{\varrho_1-a_1}\right.$$
$$\left.\left.+a_{12}a_{32}a_{42}\frac{k_2^2-\frac14}{\varrho_1-a_2}+a_{13}a_{23}a_{43}\frac{k_3^2-\frac14}{\varrho_1-a_3}\right]\right\}. \qquad (3.14)$$

Hyperboloidal ($a_{ij}=a_i-a_j$, $a_1=0$, $a_2=1$, $a_3=b$, $a_4=a$):

$$=\frac{M}{2}\omega^2R^2\left[a_{14}a_{24}a_{34}\left(\frac{1}{(\varrho_1-\varrho_3)(\varrho_2-\varrho_3)}\frac{1}{\varrho_3-a_4}\right.\right.$$
$$+\frac{1}{(\varrho_1-\varrho_2)(\varrho_3-\varrho_2)}\frac{1}{\varrho_2-a_4}$$
$$\left.\left.+\frac{1}{(\varrho_2-\varrho_1)(\varrho_3-\varrho_1)}\frac{1}{\varrho_1-a_4}\right)-1\right]$$

$$-\frac{\hbar^2}{2MR^2}\left\{\frac{1}{(\varrho_1-\varrho_3)(\varrho_2-\varrho_3)}\left[a_{31}a_{21}a_{41}\frac{k_2^2-\frac14}{\varrho_3-a_1}+a_{12}a_{32}a_{42}\frac{k_1^2-\frac14}{\varrho_3-a_2}-a_{13}a_{23}a_{43}\frac{k_3^2-\frac14}{\varrho_3-a_3}\right]\right.$$
$$-\frac{1}{(\varrho_1-\varrho_2)(\varrho_3-\varrho_2)}\left[a_{31}a_{21}a_{41}\frac{k_2^2-\frac14}{\varrho_2-a_1}+a_{12}a_{32}a_{42}\frac{k_1^2-\frac14}{\varrho_2-a_2}-a_{13}a_{23}a_{43}\frac{k_3^2-\frac14}{\varrho_2-a_3}\right]$$
$$\left.-\frac{1}{(\varrho_2-\varrho_1)(\varrho_3-\varrho_1)}\left[a_{31}a_{21}a_{41}\frac{k_2^2-\frac14}{\varrho_1-a_1}+a_{12}a_{32}a_{42}\frac{k_1^2-\frac14}{\varrho_1-a_2}-a_{13}a_{23}a_{43}\frac{k_3^2-\frac14}{\varrho_1-a_3}\right]\right\}. \quad (3.15)$$

In the following we do not display all path-integral representations for all potentials in all separable coordinate systems. To save space, we display explicitly only those path-integral representations for which an analytic solution is available.

Cylindrical Coordinates. For the oscillator on $\Lambda^{(3)}$ we obtain the following path-integral representation ($\lambda_1=2m\mp k_1\mp k_2+1$, $\lambda_2=2l\mp k_3-\nu+1$, $\nu^2=M^2\omega^2R^4/\hbar^2+1/4$):

$$K^{(V_1)}(\mathbf{u}'',\mathbf{u}';T)=R^{-3}\exp\left[-\frac{i}{\hbar}T\left(\frac{\hbar^2}{2MR^2}+\frac{M}{2}R^2\omega^2\right)\right]\int_{\tau_1(t')=\tau_1'}^{\tau_1(t'')=\tau_1''}\mathcal{D}\tau_1(t)\sinh\tau_1\cosh\tau_1\int_{\tau_2(t')=\tau_2'}^{\tau_2(t'')=\tau_2''}\mathcal{D}\tau_2(t)$$
$$\times\int_{\varphi(t')=\varphi'}^{\varphi(t'')=\varphi''}\mathcal{D}\varphi(t)\exp\left\{\frac{i}{\hbar}\int_{t'}^{t''}\left[\frac{M}{2}R^2\left(\dot\tau_1^2+\cosh^2\tau_1\dot\tau_2^2+\sinh^2\tau_1\dot\varphi^2+\frac{\omega^2}{\cosh^2\tau_1\cosh^2\tau_2}\right)-\frac{\hbar^2}{2MR^2}\left(\frac{1}{\sinh^2\tau_1}\left(\frac{k_1^2-\frac14}{\cos^2\varphi}+\frac{k_2^2-\frac14}{\sin^2\varphi}-\frac14\right)\right)\right.\right.$$
$$\left.\left.+\frac{1}{\cosh^2\tau_1}\left(\frac{k_3^2-\frac14}{\sinh^2\tau_2}+\frac14\right)\right]dt\right\} \quad (3.16)$$
$$=\sum_{m=0}^{N_m}\left\{\sum_{l=0}^{N_l}\left[\sum_{n=0}^{N_n}e^{-iE_NT/\hbar}\Psi_{nlm}^{(V_1)}(\tau_1'',\tau_2'',\varphi'';R)\Psi_{nlm}^{(V_1)}(\tau_1',\tau_2',\varphi';R)+\int_0^\infty dp\,e^{-iE_pT/\hbar}\Psi_{plm}^{(V_1)}(\tau_1'',\tau_2'',\varphi'';R)\Psi_{plm}^{(V_1)*}(\tau_1',\tau_2',\varphi';R)\right.\right.+\int_0^\infty dk$$
$$\times\int_0^\infty dp\,e^{-iE_pT/\hbar}\Psi_{pkm}^{(V_1)}(\tau_1'',\tau_2'',\varphi'';R)\Psi_{pkm}^{(V_1)*}(\tau_1',\tau_2',\varphi';R). \quad (3.17)$$

The bound-state wave functions are given by the following expression, with $m=0,\ldots,N_m=[1/2(2l\mp k_2-\nu)]$, $l=0,\ldots,N_l=[(\nu\mp k_3-1)/2]$, $N=0,\ldots,N_{\max}=[1/2(\nu\mp k_1\mp k_2\mp k_3)]$:

$$\Psi_{nlm}^{(V_1)}(\tau_1,\tau_2,\varphi;R)=(\sinh\tau_1\cosh\tau_1)^{-1/2}S_n^{(\lambda_1,\lambda_2)}(\tau_1;R)\psi_l^{(\pm k_3,\nu)}(\tau_2)\times\phi_m^{(\pm k_2,\pm k_1)}(\varphi), \quad (3.18)$$

where

$$S_n^{(\lambda_1,\lambda_2)}(\tau_1;R)=\frac{1}{\Gamma(1+\lambda_1)}\left[\frac{2(\lambda_2-\lambda_1-2n-1)\Gamma(n+1+\lambda_1)\Gamma(\lambda_2-n)}{R^3\Gamma(\lambda_2-\lambda_1-n)n!}\right]^{1/2}\times(\sinh\tau_1)^{1/2+\lambda_1}(\cosh\tau_1)^{2n+1/2-\lambda_2}{}_2F_1(-n,\lambda_2-n;1+\lambda_1;\tanh^2\tau_1), \quad (3.19)$$

$$\psi_l^{(\pm k_3,\nu)}(\tau_2)=\frac{1}{\Gamma(1\pm k_3)}\left[\frac{2(\nu\mp k_3-2l-1)\Gamma(l+1\pm k_3)\Gamma(\nu-l)}{\Gamma(\nu\mp k_3-l)l!}\right]^{1/2}\times(\sinh\tau_2)^{1/2\pm k_3}(\cosh\tau_2)^{2l+1/2-\nu}{}_2F_1(-l,\nu-l;1\pm k_3;\tanh^2\tau_2), \quad (3.20)$$

$$\phi_m^{(\pm k_2,\pm k_1)}(\varphi)=\left[2(1+2m\pm k_1\pm k_2)\frac{m!\Gamma(m\pm k_1\pm k_2+1)}{\Gamma(1+m\pm k_1)\Gamma(1+m\pm k_2)}\right]^{1/2}\times(\sin\varphi)^{1/2\pm k_2}(\cos\varphi)^{1/2\pm k_1}\times P_m^{(\pm k_2,\pm k_1)}(\cos 2\varphi). \quad (3.21)$$

The bound-state energy spectrum is given by ($N=m+l+n+3$ is the principal quantum number)

$$E_N=-\frac{\hbar^2}{2MR^2}[(2N\pm k_1\pm k_2\pm k_3-\nu)^2-1]+\frac{M}{2}\omega^2R^2. \quad (3.22)$$

In the limit $R\to\infty$ we obtain

$$E_N\simeq\hbar\omega(2N\pm k_1\pm k_2\pm k_3),\quad N\in\mathbb{N}_0, \quad (3.23)$$

which gives the correct spectrum for the corresponding superintegrable flat-space oscillator, i.e., the generalized oscillator in $\mathbb{R}^3$ (Ref. 33).

For the first set of continuum states we find

$$\Psi^{(V_1)}_{plm}(\tau_1,\tau_2,\varphi;R)=(\sinh\tau_1\cosh\tau_1)^{-1/2}S_p^{(\lambda_1,\lambda_2)}\times(\tau_1;R)\psi_l^{(\pm k_3,\nu)}(\tau_2)\phi_m^{(\pm k_2,\pm k_1)}(\varphi), \tag{3.24}$$

where

$$S_p^{(\lambda_1,\lambda_2)}(\tau_1;R) =\frac{1}{\Gamma(1+\lambda_1)}\sqrt{\frac{p\sinh\pi p}{2\pi^2R^3}}\,\Gamma\left(\frac{\lambda_2-\lambda_1+1-ip}{2}\right)\times\Gamma\left(\frac{\lambda_1-\lambda_2+1-ip}{2}\right)(\tanh\tau_1)^{1/2+\lambda_1}(\cosh\tau_1)^{ip}\times{}_2F_1\left(\frac{\lambda_2+\lambda_1+1-ip}{2},\frac{1+\lambda_1-\lambda_2-ip}{2};\times 1+\lambda_1;\tanh^2\tau_1\right), \tag{3.25}$$

with the $\psi_l^{(\pm k_3,\nu)}(\tau_2)$ and $\phi_m^{(\pm k_2,\pm k_1)}(\varphi)$ as in (3.20) and (3.21), and the continuous spectrum has the form

$$E_p=\frac{\hbar^2}{2MR^2}(p^2+1)+\frac{M}{2}\omega^2R^2. \tag{3.26}$$

In the limiting case $\omega\to 0$ we obtain

$$E_p=\frac{\hbar^2}{2MR^2}(p^2+1), \tag{3.27}$$

which corresponds to the case where only a radial part is present and has the same feature as the spectrum of the free motion on $\Lambda^{(3)}$; i.e., there is no discrete spectrum in this case.

For the second set of continuum states we find

$$\Psi^{(V_1)}_{pkm}(\tau_1,\tau_2,\varphi;R)=(\sinh\tau_1\cosh\tau_1)^{-1/2}S_p^{(\lambda_1,ik)}\times(\tau_1;R)\psi_k^{(\pm k_3,\nu)}(\tau_2)\times\phi_m^{(\pm k_2,\pm k_1)}(\varphi), \tag{3.28}$$

where

$$S_p^{(\lambda_1,ik)}(\tau_1;R)=\frac{1}{\Gamma(1+\lambda_1)}\sqrt{\frac{p\sinh\pi p}{2\pi^2R^3}}\times\Gamma\left(\frac{ik-\lambda_1+1-ip}{2}\right)\Gamma\left(\frac{\lambda_1-ik+1-ip}{2}\right)\times(\tanh\tau_1)^{1/2+\lambda_1}(\cosh\tau_1)^{ip}{}_2F_1\times\left(\frac{ik+\lambda_1+1-ip}{2},\frac{1+\lambda_1-ik-ip}{2};1+\lambda_1;\tanh^2\tau_1\right), \tag{3.29}$$

$$\psi_k^{(\pm k_3,\nu)}(\tau_2)=\frac{1}{\Gamma(1\pm k_3)}\sqrt{\frac{k\sinh\pi k}{2\pi^2}}\times\Gamma\left(\frac{\nu\mp k_3+1-ik}{2}\right)\Gamma\left(\frac{\pm k_3-\nu+1-ik}{2}\right)\times(\tanh\tau_2)^{1/2\pm k_3}(\cosh\tau_2)^{ik}{}_2F_1\times\left(\frac{\nu\pm k_3+1-ik}{2},\frac{1\pm k_3-\nu-ik}{2};1\pm k_3;\tanh^2\tau_2\right), \tag{3.30}$$

with the $\phi_m^{(\pm k_1,\pm k_2)}(\varphi)$ as in (3.21).

Spherical and sphero-elliptic coordinates. In spherical coordinates we have the path-integral representation ($\lambda_1=2m\mp k_1\mp k_2+1$, $\lambda_2=2l\mp k_3+\lambda_1+1$, $\nu^2=M^2\omega^2R^4/\hbar^2+1/4$)

$$K^{(V_1)}(u'',u';T)=\frac{e^{-i\hbar T/2MR^2}}{R^3}\int_{\tau(t')=\tau'}^{\tau(t'')=\tau''}\mathcal{D}\tau(t)\sinh^2\tau\times\int_{\vartheta(t')=\vartheta'}^{\vartheta(t'')=\vartheta''}\mathcal{D}\vartheta(t)\sin\vartheta\int_{\varphi(t')=\varphi'}^{\varphi(t'')=\varphi''}\mathcal{D}\varphi(t)\times\exp\left\{\frac{i}{\hbar}\int_{t'}^{t''}\left[\frac{M}{2}R^2(\dot\tau^2+\sinh^2\tau(\dot\vartheta^2+\sin^2\vartheta\dot\varphi^2)-\omega^2\tanh^2\tau)-\frac{\hbar^2}{2MR^2}\frac{1}{\sinh^2\tau}\left(\frac{1}{\sin^2\vartheta}\left(\frac{k_1^2-\frac14}{\cos^2\varphi}+\frac{k_2^2-\frac14}{\sin^2\varphi}-\frac14\right)+\frac{k_3^2-\frac14}{\cos^2\vartheta}-\frac14\right)\right]dt\right\} \tag{3.31}$$

$$=\sum_{m=0}^{\infty}\sum_{l=0}^{\infty}\left\{\sum_{n=0}^{N_n}e^{-iE_NT/\hbar}\Psi^{(V_1)}_{nlm}\times(\tau'',\vartheta'',\varphi'';R)\Psi^{(V_1)}_{nlm}(\tau',\vartheta',\varphi';R)+\int_0^\infty dp\,e^{-iE_pT/\hbar}\Psi^{(V_1)}_{plm}(\tau'',\vartheta'',\varphi'';R)\times\Psi^{(V_1)*}_{plm}(\tau',\vartheta',\varphi';R)\right\}. \tag{3.32}$$

The bound-state wave functions are given by the following expression, with $N=0,\dots,N_{\max}=[1/2(\nu\mp k_1\mp k_2\mp k_3)]$:

$$\Psi^{(V_1)}_{nlm}(\tau,\vartheta,\varphi;R)=(\sinh^2\tau\sin\vartheta)^{-1/2}S_n^{(\lambda_2,\nu)}\times(\tau;R)\phi_l^{(\lambda_1,\pm k_3)}(\vartheta)\phi_m^{(\pm k_2,\pm k_1)}(\varphi), \tag{3.33}$$

where

$$\phi_l^{(\lambda_1,\pm k_3)}(\vartheta) = \left[2(1+2l\pm k_3+\lambda_1)\frac{l!\Gamma(l+\lambda_1\pm k_3+1)}{\Gamma(1+l\pm k_3)\Gamma(1+l+\lambda_1)}\right]^{1/2} \times(\sin\vartheta)^{1/2+\lambda_1}(\cos\vartheta)^{1/2\pm k_3}P_l^{(\lambda_1,\pm k_3)}(\cos 2\vartheta), \tag{3.34}$$

in which $[n=0,...,N_n<(\nu-\lambda_2-1)/2]$

$$S_n^{(\lambda_2,\nu)}(\tau;R) = \frac{1}{\Gamma(1+\lambda_2)} \times\left[\frac{2(\nu-\lambda_2-2n-1)\Gamma(n+1+\lambda_2)\Gamma(\nu-n)}{R^3\Gamma(\nu-\lambda_2-n)n!}\right]^{1/2} \times(\sinh\tau)^{\lambda_2+1/2}(\cosh\tau)^{2n+1/2-\nu}{}_2F_1 \times(-n,\nu-n;1+\lambda_2;\tanh^2\tau). \tag{3.35}$$

The scattering states are

$$\Psi_{plm}^{(V_1)}(\tau,\vartheta,\varphi;R)=(\sinh^2\tau\sin\vartheta)^{-1/2}S_p^{(\lambda_2,\nu)}(\tau;R)\phi_l^{(\lambda_1,\pm k_3)}(\vartheta)\phi_m^{(\pm k_2,\pm k_1)}(\varphi), \tag{3.36}$$

where

$$S_p^{(\lambda_2,\nu)}(\tau;R) = \frac{1}{\Gamma(1+\lambda_2)}\sqrt{\frac{p\sinh\pi p}{2\pi^2R^3}}\Gamma\left(\frac{\nu-\lambda_2+1-ip}{2}\right) \times\Gamma\left(\frac{\lambda_2-\nu+1-ip}{2}\right)(\tanh\tau)^{\lambda_2+1/2}(\cosh\tau)^{ip}{}_2F_1 \times\left(\frac{\nu+\lambda_2+1-ip}{2},\frac{\lambda_2-\nu+1-ip}{2}; 1+\lambda_2;\tanh^2\tau\right). \tag{3.37}$$

Here $\phi_m^{(\pm k_2,\pm k_1)}(\varphi)$ denote the same wave functions as in (3.21). The discrete and continuous energy spectra E_N and E_p are, of course, the same as in (3.22) and (3.26), respectively.

The path-integral solution in terms of sphero-elliptic coordinates is very similar to that for spherical coordinates, and the bound-state wave functions for the sphero-elliptic coordinates are given by

$$\Psi_{nlh}^{(V_1)}(\tau,\tilde\alpha,\tilde\beta;R) =(\sinh^2\tau\,\mathrm{sn}\tilde\alpha\,\mathrm{cn}\tilde\alpha\,\mathrm{dn}\tilde\alpha\,\mathrm{sn}\tilde\beta\,\mathrm{cn}\tilde\beta\,\mathrm{dn}\tilde\beta)^{-1/2} \times S_n^{(\lambda_2,\nu)}(\tau;R)\Xi_{lh}^{(\pm k_1,\pm k_2,\pm k_3)}(\tilde\alpha,\tilde\beta), \tag{3.38}$$

with the same energy spectrum as in the previous case. The wave functions $\Xi_{lh}^{(\pm k_1,\pm k_2,\pm k_3)}(\tilde\alpha,\tilde\beta)$ were determined in Ref. 34; they correspond to the free wave function on the six-dimensional sphere in a cylindrical-elliptic coordinate system. They are not yet explicitly known, and therefore the above solution in sphero-elliptic coordinates remains on a somewhat formal level. We nevertheless present it for completeness. The continuous spectrum has the form

$$\Psi_{plh}^{(V_1)}(\tau,\tilde\alpha,\tilde\beta;R) =(\sinh^2\tau\,\mathrm{sn}\tilde\alpha\,\mathrm{cn}\tilde\alpha\,\mathrm{dn}\tilde\alpha\,\mathrm{sn}\tilde\beta\,\mathrm{cn}\tilde\beta\,\mathrm{dn}\tilde\beta)^{-1/2}S_p^{(\lambda_2,\nu)}(\tau;R) \times\Xi_{lh}^{(\pm k_1,\pm k_2,\pm k_3)}(\tilde\alpha,\tilde\beta), \tag{3.39}$$

and E_p is the same as in the previous case.

Equidistant-cylindrical coordinates. In equidistant-cylindrical coordinates we obtain the path-integral solution $(\lambda_1=2m\mp k_1\mp k_2+1,\ \lambda_2=2l+\lambda_1-\nu+1,\ \nu^2=M^2\omega^2R^4/\hbar^2+1/4)$

$$K^{(V_1)}(\mathbf{u}'',\mathbf{u}';T)=R^{-3}\exp\left[-\frac{i}{\hbar}T\left(\frac{\hbar^2}{2MR^2}+\frac{M}{2}R^2\omega^2\right)\right] \times\int_{\tau_1(t')=\tau_1'}^{\tau_1(t'')=\tau_1''}\mathscr{D}\tau_1(t)\cosh^2\tau_1\int_{\tau_2(t')=\tau_2'}^{\tau_2(t'')=\tau_2''} \times\mathscr{D}\tau_2(t)\sinh\tau_2\int_{\varphi(t')=\varphi'}^{\varphi(t'')=\varphi''}\mathscr{D}\varphi(t) \times\exp\left\{\frac{i}{\hbar}\int_{t'}^{t''}\left[\frac{M}{2}R^2\left(\dot\tau_1^2+\cosh^2\tau_1(\dot\tau_2^2 +\sinh^2\tau_2\dot\varphi^2)+\frac{\omega^2}{\cosh^2\tau_1\cosh^2\tau_2}\right) -\frac{\hbar^2}{2MR^2}\left(\frac{1}{\cosh^2\tau_1}\left(\frac{1}{\sinh^2\tau_2}\left(\frac{k_1^2-\frac14}{\cos^2\varphi} +\frac{k_2^2-\frac14}{\sin^2\varphi}-\frac14\right)+\frac14\right)+\frac{k_3^2-\frac14}{\sinh^2\tau_1}\right)\right]dt\right\} \tag{3.40}$$

$$=\sum_{m=0}^{\infty}\left\{\sum_{l=0}^{N_l}\left[\sum_{n=0}^{N_n}e^{-i\hbar E_N T/\hbar}\Psi_{nlm}^{(V_1)} \times(\tau_1'',\tau_2'',\varphi'';R)\Psi_{nlm}^{(V_1)}(\tau_1',\tau_2',\varphi';R) +\int_0^\infty dpe^{-iE_pT/\hbar}\Psi_{plm}^{(V_1)}(\tau_1'',\tau_2'',\varphi'';R) \times\Psi_{plm}^{(V_1)*}(\tau_1',\tau_2',\varphi';R)\right] +\int_0^\infty dk\int_0^\infty dpe^{-iE_pT/\hbar}\Psi_{pkm}^{(V_1)} \times(\tau_1'',\tau_2'',\varphi'';R)\Psi_{pkm}^{(V_1)*}(\tau_1',\tau_2',\varphi';R)\right\}. \tag{3.41}$$

We obtain one set of bound-state wave functions and two sets of scattering wave functions. The bound-state wave functions are $[l+m=0,...,\ N_l=1/2[(\nu\mp k_1\mp k_2-2)],\ N=0,...,\ N_{\max}=1/2(\nu\mp k_1\mp k_2\mp k_3)]$

$$\Psi^{(V_1)}_{nlm}(\tau_1,\tau_2,\varphi;R)=(\cosh^2\tau_1\sinh\tau_2)^{-1/2}S_n^{(\pm k_3,\lambda_2)}\times(\tau_1;R)\psi_l^{(\lambda_1,\nu)}(\tau_2)\phi_m^{(\pm k_2,\pm k_1)}(\varphi), \tag{3.42}$$

and E_N is the same as in (3.22). The two sets of continuum states are

$$\Psi^{(V_1)}_{pkm}(\tau_1,\tau_2,\varphi;R)=(\cosh^2\tau_1\sinh\tau_2)^{-1/2}S_p^{(\pm k_3,\lambda_2)}\times(\tau_1;R)\psi_l^{(\lambda_1,\nu)}(\tau_2)\phi_m^{(\pm k_2,\pm k_1)}\times(\varphi), \tag{3.43}$$

$$\Psi^{(V_1)}_{plm}(\tau_1,\tau_2,\varphi;R)=(\cosh^2\tau_1\sinh\tau^2)^{-1/2}S_p^{(\pm k_3,ik)}\times(\tau_1;R)\psi_k^{(\lambda_1,\nu)}(\tau_2)\phi_m^{(\pm k_2,\pm k_1)}\times(\varphi). \tag{3.44}$$

The continuous spectrum has the form

$$E_p=\frac{\hbar^2}{2MR^2}(p^2+1)+\frac{M}{2}\omega^2R^2. \tag{3.45}$$

Our notation here is as follows:

- The wave functions $\phi_m^{(\pm k_2,\pm k_1)}(\varphi)$ are the same as in (3.21);
- the wave functions $\psi_{l,k}^{(\lambda_1,\nu)}(\tau_2)$ are the same as in (3.20) and (3.30) with $\pm k_3\to\lambda_1$;
- the wave functions $S_{n,p}^{(\pm k_3,\lambda_2)}(\tau_1;R)$ are the same as in (3.19) and (3.25) with $\lambda_1\to\pm k_3$, respectively.

Equidistant coordinates. As the last system for which an explicit solution is possible, we consider the equidistant system and obtain the path-integral solution ($\lambda_1=2m\mp k_1-\nu+1$, $\lambda_2=2l\mp k_2-\lambda_1+1$, $\nu^2=M^2\omega^2R^4/\hbar^2+1/4$)

$$\begin{aligned}K^{(V_1)}(\mathbf{u}'',\mathbf{u}';T)&=R^{-3}\exp\left[-\frac{i}{\hbar}T\left(\frac{\hbar^2}{2MR^2}+\frac{M}{2}R^2\omega^2\right)\right]\\&\times\int_{\tau_1(t')=\tau_1'}^{\tau_1(t'')=\tau_1''}\mathscr{D}\tau_1(t)\cosh^2\tau_1\int_{\tau_2(t')=\tau_2'}^{\tau_2(t'')=\tau_2''}\mathscr{D}\tau_2(t)\\&\times\cosh\tau_2\int_{\tau_3(t')=\tau_3'}^{\tau_3(t'')=\tau_3''}\mathscr{D}\tau_3(t)\exp\left\{\frac{i}{\hbar}\right.\\&\times\int_{t'}^{t''}\left[\frac{M}{2}R^2\left(\dot\tau_1^2+\cosh^2\tau_1(\dot\tau_2^2+\cosh^2\tau_1\dot\tau_3^2)\right.\right.\\&\left.+\frac{\omega^2}{\cosh^2\tau_1\cosh^2\tau_2\cosh^2\dot\tau_3}\right)-\frac{\hbar^2}{2MR^2}\\&\times\left(\frac{k_3^2-\frac14}{\sinh^2\tau_1}+\frac{1}{\cosh^2\tau_1}\left(\frac{1}{\cosh^2\tau_2}\left(\frac{k_1^2-\frac14}{\sinh^2\tau_3}+\frac14\right)\right.\right.\\&\left.\left.\left.\left.+\frac{k_2^2-\frac14}{\sinh^2\tau_2}+\frac14\right)\right)\right]dt\right\}\end{aligned} \tag{3.46}$$

$$\begin{aligned}&=\sum_{m=0}^{N_m}\left\{\sum_{l=0}^{N_l}\left[\sum_{n=0}^{N_n}e^{-iE_NT/\hbar}\Psi^{(V_1)}_{nlm}\times(\tau_1'',\tau_2'',\tau_3'';R)\Psi^{(V_1)}_{nlm}(\tau_1',\tau_2',\tau_3';R)+\int_0^\infty dp\,e^{-iE_pT/\hbar}\right.\right.\\&\left.\times\Psi^{(V_1)}_{plm}(\tau_1'',\tau_2'',\tau_3'';R)\Psi^{(V_1)*}_{plm}(\tau_1',\tau_2',\tau_3';R)\right]\\&+\int_0^\infty dk\int_0^\infty dp\,e^{-iE_pT/\hbar}\Psi^{(V_1)}_{pkm}(\tau_1'',\tau_2'',\tau_3'';R)\\&\left.\times\Psi^{(V_1)*}_{pkm}(\tau_1',\tau_2',\tau_3';R)\right\}\\&+\int_0^\infty d\varrho\int_0^\infty dk\int_0^\infty dp\,e^{-iE_pT/\hbar}\\&\times\Psi^{(V_1)}_{pk\varrho}(\tau_1'',\tau_2'',\tau_3'';R)\Psi^{(V_1)*}_{pk\varrho}(\tau_1',\tau_2',\tau_3';R).\end{aligned} \tag{3.47}$$

We obtain one set of bound-state wave functions and three sets of scattering wave functions. The bound-state wave functions are [$m=0,\dots,N_m<(\nu\mp k_1-1)/2$, $l=0,\dots,N_l<(\lambda_1\mp k_2-1)/2$, $n=0,\dots,N_n<(\lambda_2\mp k_3-1)/2$]

$$\Psi^{(V_1)}_{nlm}(\tau_1,\tau_2,\tau_3;R)=(\cosh^2\tau_1\cosh_2)^{-1/2}S_n^{(\pm k_3,\lambda_2)}\times(\tau_1;R)\psi_l^{(\pm k_2,\lambda_1)}(\tau_2)\times\psi_m^{(\pm k_1,\nu)}(\tau_3), \tag{3.48}$$

and E_N is the same as in (3.22). The three sets of continuum states are

$$\Psi^{(V_1)}_{plm}(\tau_1,\tau_2,\tau_3;R)=(\cosh^2\tau_1\cosh\tau_2)^{-1/2}S_p^{(\pm k_3,\lambda_2)}\times(\tau_1;R)\psi_l^{(\pm k_2,\lambda_1)}(\tau_2)\times\psi_m^{(\pm k_1,\nu)}(\tau_3), \tag{3.49}$$

$$\Psi^{(V_1)}_{mkp}(\tau_1,\tau_2,\tau_3;R)=(\cosh^2\tau_1\cosh\tau_2)^{-1/2}S_p^{(\pm k_3,ik)}\times(\tau_1;R)\psi_k^{(\pm k_2,\lambda_1)}(\tau_2)\times\psi_m^{(\pm k_1,\nu)}(\tau_3), \tag{3.50}$$

$$\Psi^{(V_1)}_{\varrho kp}(\tau_1,\tau_2,\tau_3;R)=(\cosh^2\tau_1\cosh\tau_2)^{-1/2}S_p^{(\pm k_3,ik)}\times(\tau_1;R)\psi_k^{(\pm k_2,i\varrho)}(\tau_2)\times\psi_\varrho^{(\pm k_1,\nu)}(\tau_3), \tag{3.51}$$

and E_p is the same as in (3.45). Here we use the following notation:

- The wave functions $\psi_m^{(\pm k_1,\nu)}(\tau_3)$ are the same as in (3.20) with $\tau_2\to\tau_3$, $l\to m$, and $\pm k_3\to\pm k_1$;
- The wave functions $\psi_\varrho^{(\pm k_1,\nu)}(\tau_3)$ are the same as in (3.30) with $\tau_2\to\tau_3$, $k\to\varrho$, and $\pm k_3\to\pm k_1$;
- The wave functions $\psi_l^{(\pm k_2,\lambda_1)}(\tau_2)$ are the same as in (3.19) with $\tau_1\to\tau_2$, $n\to l$, $(\lambda_1,\lambda_2)\to(\pm k_2,\lambda_1)$, and $R=1$;
- The wave functions $\psi_k^{(\pm k_2,\lambda_1)}(\tau_2)$ are the same as in (3.25) with $\tau_1\to\tau_2$, $p\to k$, $(\lambda_1,\lambda_2)\to(\pm k_2,\lambda_1)$, and $R=1$;

• The wave functions $\psi_k^{(\pm k_2, i\varrho)}(\tau_2)$ are the same as in (3.29) with $\tau_1 \to \tau_2$, $(p,k) \to (k,\varrho)$, $\lambda_1 \to \pm k_2$, and $R=1$;

• The wave functions $S_{n,p}^{(\pm k_3,\lambda_2)}(\tau_1;R)$ and $S_p^{(\pm k_3,ik)} \times(\tau_1;R)$ are the same as in (3.19), (3.25), and (3.29) with $\lambda_1 \to \pm k_3$, respectively.

We note that the wave functions have been normalized in the domain $\varphi \in (0,\pi/2)$, $\vartheta \in (0,\pi/2)$, and $\tau > 0$ in the spherical system and in the domain $\tau_{1,2,3} > 0$ in the equidistant system. The positive sign for the k_i must be used whenever $k_i \geqslant 1/2$ $(i=1,2,3)$; i.e., the potential term is repulsive at the origin, and the motion takes place only in the indicated domains. If $0 < |k_i| < 1/2$, i.e., if the potential term is attractive at the origin, then both the positive and the negative sign must be taken into account in the solution. This is indicated by the notation $\pm k_i$ in the formulas. This also has the consequence that for each k_i the motion can take place in the entire domains of the variables on $\Lambda^{(3)}$. In the present case this means that we must, e.g., in the equidistant system, distinguish eight cases: i) $\tau_{1,2,3} > 0$; ii) $\tau_{1,2} > 0$, $\tau_2 \in \mathbb{R}$; iii) $\tau_1 \in \mathbb{R}$, $\tau_{2,3} > 0$; iv) $\tau_2 \in \mathbb{R}$, $\tau_{1,3} > 0$; v) $\tau_{1,2} \in \mathbb{R}$, $\tau_2 > 0$; vi) $\tau_1 > 0$, $\tau_{2,3} \in \mathbb{R}$; vii) $\tau_2 > 0$, $\tau_{1,3} \in \mathbb{R}$; viii) $\tau_{1,2,3} \in \mathbb{R}$. In polar coordinates the same feature is recovered by the observation that the Pöschl–Teller barriers are absent for $|k_i| < 1/2$.

In elliptic coordinates this feature is taken into account in the following way. Because $\alpha \in (iK', iK'+K)$, we have $\mathrm{sn}(\alpha,k)$, i $\mathrm{cn}(\alpha,k) > k'/k$, i $\mathrm{dn}(\alpha,k) \geqslant 0$, and we see that for $\alpha \in (iK', iK'+K)$ and $\beta \in (K', 4K')$ we find $u_0 \geqslant 0$, and the variables u_1, u_2, u_3 change sign in the eight domains; i.e., $\beta \in (0,K')$, $\beta \in (K',2K')$, $\beta \in (2K',3K')$, and $\beta \in (3K',4K')$. We then have, for $\alpha \neq 0$,

$$\mathrm{sn}(0,k') = \mathrm{sn}(2K',k') = \mathrm{sn}(4K',k') = 0,$$

$$\mathrm{cn}(K',k') = \mathrm{cn}(3K',k') = 0, \tag{3.52}$$

and $\mathrm{dn}(\beta,k') > 0$, $\beta \in [0,4K')$. For convenience, we have made the choice $\beta \in (0,K')$, and the same is true in all the following systems. The situation is similar in the hyperbolic system, where we choose $\mu \in (iK', iK'+K)$, $\eta \in (0,K')$. In the sphero-elliptic system we must choose, for the same reasons, $\tilde{\alpha} \in (0,K)$ and $\tilde{\beta} \in (0,K')$.

3.2. The Coulomb potential

We consider the Coulomb potential on the three-dimensional hyperboloid $(k_{1,2} > 0)$,

$$V_2(\mathbf{u}) = -\frac{\alpha}{R}\left(\frac{u_0}{\sqrt{u_1^2+u_2^2+u_3^2}} - 1\right) + \frac{\hbar^2}{2M}\left(\frac{k_1^2 - \frac{1}{4}}{u_1^2} + \frac{k_2^2 - \frac{1}{4}}{u_2^2}\right), \tag{3.53}$$

which in the five separating coordinate systems has the following form:

Sphero-elliptic ($\tau > 0$, $\tilde{\alpha} \in (0,K)$, $\tilde{\beta} \in (0,K')$):

$$V_2(\mathbf{u}) = -\frac{\alpha}{R}(\coth\tau - 1) + \frac{\hbar^2}{2MR^2 \sinh^2\tau} \times \left(\frac{k_1^2 - \frac{1}{4}}{\mathrm{sn}^2\tilde{\alpha}\,\mathrm{dn}^2\tilde{\beta}} + \frac{k_2^2 - \frac{1}{4}}{\mathrm{cn}^2\tilde{\alpha}\,\mathrm{cn}^2\tilde{\beta}}\right). \tag{3.54}$$

Spherical ($\tau > 0$, $\vartheta \in (0,\pi)$, $\varphi \in (0,\pi/2)$):

$$= -\frac{\alpha}{R}(\coth\tau - 1) + \frac{\hbar^2}{2MR^2 \sinh^2\tau \sin^2\vartheta} \times \left(\frac{k_1^2 - \frac{1}{4}}{\cos^2\varphi} + \frac{k_2^2 - \frac{1}{4}}{\sin^2\varphi}\right). \tag{3.55}$$

Prolate elliptic II ($\alpha \in (iK', iK'+K)$, $\beta \in (0,K')$, $\varphi \in (0,\pi/2)$):

$$= -\frac{\alpha}{R}\left(\frac{k^2\,\mathrm{sn}\,\alpha\,\mathrm{cn}\,\beta - k'\,\mathrm{cn}\,\beta\,\mathrm{dn}\,\beta}{k^2\,\mathrm{cn}^2\alpha + k'^2\,\mathrm{cn}^2\beta} - 1\right) + \frac{\hbar^2}{2MR^2\,\mathrm{dn}^2\alpha\,\mathrm{sn}^2\beta}\left(\frac{k_1^2 - \frac{1}{4}}{\cos^2\varphi} + \frac{k_2^2 - \frac{1}{4}}{\sin^2\varphi}\right). \tag{3.56}$$

Semihyperbolic ($\mu_{1,2} > 0$, $\varphi \in (0,\pi/2)$):

$$= -\frac{\alpha}{R}\left(\frac{\sqrt{1+\mu_1^2} + \sqrt{1+\mu_2^2}}{\mu_1 + \mu_2} - 1\right) + \frac{\hbar^2}{2MR^2\mu_1\mu_2}\left(\frac{k_1^2 - \frac{1}{4}}{\cos^2\varphi} + \frac{k_2^2 - \frac{1}{4}}{\sin^2\varphi}\right). \tag{3.57}$$

Elliptic-parabolic 2 ($a > 0$, $\vartheta \in (0,\pi)$, $\varphi \in (0,\pi/2)$):

$$= -\frac{\alpha}{R}\left(\frac{\cosh^2 a + \cos^2\vartheta}{\cosh^2 a - \cos^2\vartheta} - 1\right) + \frac{\hbar^2}{2MR^2}\coth^2 a\,\cot^2\vartheta\left(\frac{k_1^2 - \frac{1}{4}}{\cos^2\varphi} + \frac{k_2^2 - \frac{1}{4}}{\sin^2\varphi}\right). \tag{3.58}$$

Here Λ_3 denotes the third component of the Pauli–Lenz–Runge vector on the hyperboloid,[77] i.e.,

$$\mathbf{A} = \frac{1}{2R}(\mathbf{L}\times\mathbf{K} - \mathbf{K}\times\mathbf{L}) - \frac{\alpha\mathbf{u}}{\sqrt{u_1^2+u_2^2+u_3^2}}, \quad \mathbf{u} = (u_1, u_2, u_3). \tag{3.59}$$

The path integral for the Coulomb potential on $\Lambda^{(3)}$ can be evaluated explicitly in three coordinate systems; we will discuss this point below. In the prolate-elliptic II and the semihyperbolic system no explicit solution is known.

3.2.1. Spherical and sphero-elliptic coordinates

The separation of the Coulomb problem in spherical coordinates is similar to the sphero-elliptic one, and we have

$$K^{(V_2)}(\tau'', \tau', \vartheta'', \vartheta', \varphi'', \varphi'; T) = R^{-3}\exp\left[-\frac{i}{\hbar}T\left(\frac{\hbar^2}{2MR^2} + \frac{\alpha}{R}\right)\right]\int_{\tau(t')=\tau'}^{\tau(t'')=\tau''}\mathscr{D}\tau(t) \times \sinh^2\tau\int_{\vartheta(t')=\vartheta'}^{\vartheta(t'')=\vartheta''}\mathscr{D}\vartheta(t)\sin\vartheta\int_{\varphi(t')=\varphi'}^{\varphi(t'')=\varphi''}\mathscr{D}\varphi(t) \times \exp\left\{\frac{i}{\hbar}\int_{t'}^{t''}\left[\frac{M}{2}R^2(\dot{\tau}^2 + \sinh^2\tau(\dot{\vartheta}^2 + \sin^2\vartheta\,\dot{\varphi}^2))\right.\right.$$

$$+\frac{\alpha}{R}\coth\tau-\frac{\hbar^2}{2MR^2\sinh^2\tau}\left(\frac{1}{\sin^2\vartheta}\right.$$

$$\left.\left.\left.\times\left(\frac{k_1^2-\frac{1}{4}}{\cos^2\varphi}+\frac{k_2^2-\frac{1}{4}}{\sin^2\varphi}-\frac{1}{4}\right)-\frac{1}{4}\right)\right]dt\right\} \tag{3.60}$$

$$=\sum_{m=0}^{\infty}\sum_{l=0}^{\infty}\left\{\sum_{N=0}^{N_{\max}}e^{-iE_NT/\hbar}\Psi_{Nlm}^{(V_2)}(\tau'',\vartheta'',\varphi'';R)\right.$$

$$\times\Psi_{Nlm}^{(V_2)*}(\tau',\vartheta',\varphi';R)+\int_0^\infty dp e^{-iE_pT/\hbar}\Psi_{plm}^{(V_2)}$$

$$\left.\times(\tau'',\vartheta'',\varphi'';R)\Psi_{plm}^{(V_2)*}(\tau',\vartheta',\varphi';R)\right\}. \tag{3.61}$$

The bound-state and continuum wave functions are given by

$$\Psi_{Nlm}^{(V_2)}(\tau,\vartheta,\varphi;R)=(\sinh\tau)^{-1}S_N^{(V_2)}(\tau;R)\times\sqrt{\left(l+\lambda_1+\tfrac{1}{2}\right)\frac{\Gamma(l+\lambda_1+1)}{l!}}\times P_{l+\lambda_1}^{-\lambda_1}(\cos\vartheta)\phi_m^{(\pm k_2,\pm k_1)}(\varphi), \tag{3.62}$$

$$\Psi_{plm}^{(V_2)}(\tau,\vartheta,\varphi;R)=(\sinh\tau)^{-1}S_p^{(V_2)}(\tau;R)\times\sqrt{\left(l+\lambda_1+\tfrac{1}{2}\right)\frac{\Gamma(l+\lambda_1+1)}{l!}}\times P_{l+\lambda_1}^{-\lambda_1}(\cos\vartheta)\phi_m^{(\pm k_2,\pm k_1)}(\varphi), \tag{3.63}$$

with $\lambda_1=2m\mp k_1\mp k_2+1$, $\lambda_2=l+\lambda_1+1/2$, $N=n+l+2m\mp k_1\mp k_2+2$, and the wave functions (3.64) and (3.66), with the same wave functions $\phi_m^{(\pm k_2,\pm k_1)}(\varphi)$ as in (3.21). The bound-state spherical wave functions and the energy spectrum are given by

$$S_N^{(V_2)}(\tau;R)=\frac{2^{\lambda_2+1/2}}{\Gamma\left(2\lambda_2+\frac{1}{2}\right)}\times\left[\frac{\sigma_N^2-\tilde N^2}{R^3\tilde N}\frac{\Gamma\left(\tilde N+\lambda_2+\frac{1}{2}\right)\Gamma\left(\sigma_N+\lambda_2+\frac{1}{2}\right)}{\Gamma(\tilde N-\lambda_2)\Gamma(\sigma_N-\lambda_2)}\right]^{1/2}\times(\sinh\tau)^{\lambda_2+1/2}e^{-\tau(\sigma_N-n)}{}_2F_1\left(-n,\lambda_2+\frac{1}{2}+\sigma_N;2\lambda_2+1;\frac{2}{1+\coth\tau}\right), \tag{3.64}$$

$$E_N=\frac{\alpha}{R}-\hbar^2\frac{\tilde N^2-1}{2MR^2}-\frac{M\alpha^2}{2\hbar^2\tilde N^2}. \tag{3.65}$$

Here we have abbreviated $a=\hbar^2/M\alpha$ (the Bohr radius), $\sigma_N=aR/\tilde N$, $\tilde N=N\mp k_1\mp k_2+2$, $N=n+l+2m$, $N=0,1,2,\ldots,N_{\max}<\sqrt{R/a}$. The continuous spectrum has the form

$$S_p^{(V_2)}(\tau;R)=\frac{2^{(i/2)(p-\tilde p)+\lambda_2+1/2}}{\pi\Gamma(2\lambda_2+1)}\sqrt{\frac{p\sinh\pi p}{2R^3}}\Gamma\left(\lambda_2+\frac{1}{2}+\frac{i}{2}(\tilde p-p)\right)\Gamma\left(\lambda_2+\frac{1}{2}-\frac{i}{2}(\tilde p+p)\right)\times(\sinh\tau)^{\lambda_2+1/2}\exp\left[\tau\left(\frac{i}{2}(\tilde p+p)-\lambda_2-\frac{1}{2}\right)\right]\times{}_2F_1\left(\lambda_2+\frac{1}{2}+\frac{i}{2}(\tilde p-p),\lambda_2+\frac{1}{2}-\frac{i}{2}(\tilde p+p);2\lambda_2+1;\frac{2}{1+\coth\tau}\right), \tag{3.66}$$

$\tilde p=\sqrt{2MR^2(E_p-\alpha\hbar^2/R)}/\hbar$, and E_p is the same as in (3.27).

In the case of the pure Coulomb problem the angular wave functions are just the spherical harmonics Y_l^m on $S^{(2)}$; i.e., we obtain for the wave functions in this case

$$\Psi_{nlm}^{(V_2)}(\tau,\vartheta,\varphi;R)=(\sinh\tau)^{-1}S_N^{(V_2)}(\tau;R)Y_l^m(\vartheta,\varphi), \tag{3.67}$$

$$\Psi_{plm}^{(V_2)}(\tau,\vartheta,\varphi;R)=(\sinh\tau)^{-1}S_p^{(V_2)}(\tau;R)Y_l^m(\vartheta,\varphi), \tag{3.68}$$

together with the principal quantum number $N=n+l+1=0,\ldots,N_{\max}<\sqrt{R/a}-\lambda_1-1/2$. In Ref. 25 it was shown that $S_N^{(V_2)}(\tau;R)$ and $S_p^{(V_2)}(\tau;R)$ yield the correct radial wave functions in $\mathbb{R}^3$ as $R\to\infty$.

The complete wave functions of the generalized Coulomb problem on the three-dimensional pseudosphere in sphero-elliptic coordinates are given by

$$\Psi_{nlh}^{(V_2)}(\tau,\tilde\alpha,\tilde\beta;R)=(\sinh^2\tau\,\mathrm{sn}\tilde\alpha\,\mathrm{cn}\tilde\alpha\,\mathrm{dn}\tilde\alpha\,\mathrm{sn}\tilde\beta\,\mathrm{cn}\tilde\beta\,\mathrm{dn}\tilde\beta)^{-1/2}\times S_N^{(V_2)}(\tau;R)\Xi_{lm}^{(\pm k_1,\pm k_2,\pm 1/2)}(\tilde\alpha,\tilde\beta), \tag{3.69}$$

$$\Psi_{plh}^{(V_2)}(\tau,\tilde\alpha,\tilde\beta;R)=(\sinh^2\tau\,\mathrm{sn}\tilde\alpha\,\mathrm{cn}\tilde\alpha\,\mathrm{dn}\tilde\alpha\,\mathrm{sn}\tilde\beta\,\mathrm{cn}\tilde\beta\,\mathrm{dn}\tilde\beta)^{-1/2}\times S_p^{(V_2)}(\tau;R)\Xi_{lm}^{(\pm k_1,\pm k_2,\pm 1/2)}(\tilde\alpha,\tilde\beta). \tag{3.70}$$

We note that in the pure Coulomb case the path-integral evaluation is almost the same, with only minor differences. The wave functions $\Xi_{lm}^{(\pm k_1,\pm k_2,\pm 1/2)}$ are replaced by the wave functions of the free motion on the sphere $S^{(2)}$, i.e., together with the notation $k,q=\pm 1$, $h+\tilde h=l(l+1)$,

$$\Xi_{lm}^{(\pm k_1,\pm k_2,\pm 1/2)}(\tilde\alpha,\tilde\beta)\to\Lambda_{lh}^k(\tilde\alpha)\Lambda_{lh}{}^q(\tilde\beta). \tag{3.71}$$

The quantum number λ_2 yields the usual angular-momentum number $l\in\mathbb{N}_0$. The discrete spectrum has the same form as in (3.65), but with the principal quantum number N now given by $N=n+l+1$, thus giving degeneracies with respect to the quantum number m. Everything else remains the same.

3.2.2. Elliptic-parabolic 2 coordinates

In order to evaluate the path integral in elliptic-parabolic 2 coordinates, one first separates the φ path integration and then performs a time transformation. This gives ($\lambda_1=2m \mp k_1 \mp k_2+1$)

$$
\begin{aligned}
&K^{(V_2)}(a'',a',\vartheta'',\vartheta',\varphi'',\varphi';T)\\
&\doteq R^{-3}\exp\left[-\frac{i}{\hbar}T\left(\frac{\hbar^2}{2MR^2}+\frac{\alpha}{R}\right)\right]\int_{a(t')=a'}^{a(t'')=a''}\mathscr{D}a(t)\\
&\times\int_{\vartheta(t')=\vartheta'}^{\vartheta(t'')=\vartheta''}\mathscr{D}\vartheta(t)\tanh a\tan\vartheta\,\frac{\cosh^2 a-\cos^2\vartheta}{\cosh^2 a\cos^2\vartheta}\\
&\times\int_{\varphi(t')=\varphi'}^{\varphi(t'')=\varphi''}\mathscr{D}\varphi(t)\exp\left\{\frac{i}{\hbar}\times\int_{t'}^{t''}\left[\frac{M}{2}R^2\right.\right.\\
&\times\frac{(\cosh^2 a-\cos^2\vartheta)(\dot a^2+\dot\vartheta^2)+\sinh^2 a\sin^2\vartheta\dot\varphi^2}{\cosh^2 a\cos^2\vartheta}\\
&-\frac{\alpha}{R}\frac{\cosh^2 a+\cos^2\vartheta}{\cosh^2 a-\cos^2\vartheta}+\frac{\hbar^2}{2MR^2}\coth^2 a\cot^2\\
&\times\vartheta\left(\frac{k_1^2-\frac14}{\cos^2\varphi}+\frac{k_2^2-\frac14}{\sin^2\varphi}\right)\\
&\left.\left.+\frac{\hbar^2}{8MR^2}\frac{\cosh^2 a+\cos^2\vartheta-1}{\sinh^2 a\sin^2\vartheta}\right]dt\right\} \qquad (3.72)
\end{aligned}
$$

$$
\begin{aligned}
&=\frac{e^{i\hbar T/2MR^2}}{R^3}(\coth a'\coth a''\cot\vartheta'\cot\vartheta'')^{1/2}\\
&\times\sum_{m=0}^{\infty}\phi_m^{(\pm k_2,\pm k_1)}(\varphi'')\phi_m^{(\pm k_2,\pm k_1)}\\
&\times(\varphi')\int_{a(t')=a'}^{a(t'')=a''}\mathscr{D}a(t)\int_{\vartheta(t')=\vartheta'}^{\vartheta(t'')=\vartheta''}\\
&\times\mathscr{D}\vartheta(t)\frac{\cosh^2 a-\cos^2\vartheta}{\cosh^2 a\cos^2\vartheta}\exp\left\{\frac{i}{\hbar}\right.\\
&\times\int_{t'}^{t''}\left[\frac{M}{2R^2}\frac{\cosh^2 a\quad-\cos^2\vartheta}{\cosh^2 a\cos^2\vartheta}\right.\\
&\times(\dot a^2+\dot\vartheta^2)-\frac{\alpha}{R}\frac{\cosh^2 a+\cos^2\vartheta}{\cosh^2 a-\cos^2\vartheta}\\
&+\frac{\hbar^2\lambda_1^2}{2MR^2}\coth^2 a\cot^2\vartheta\\
&\left.\left.+\frac{\hbar^2}{8MR^2}\frac{\cosh^2 a+\cos^2\vartheta-1}{\cosh^2 a-\cos^2\vartheta}\right]dt\right\} \qquad (3.73)
\end{aligned}
$$

$$
\begin{aligned}
&=\frac{e^{-i\hbar T/2MR^2}}{R^3}(\coth a'\coth a''\cot\vartheta'\cot\vartheta'')^{1/2}\\
&\times\sum_{m=0}^{\infty}\phi_m^{(\pm k_2,\pm k_1)}(\varphi'')\phi_m^{(\pm k_2,\pm k_1)}\\
&\times(\varphi')\int_{\mathbb{R}}\frac{dE}{2\pi\hbar}e^{-iET/\hbar}\int_0^\infty ds''\int_{a(0)=a'}^{a(s'')=a''}\mathscr{D}a(s)\\
&\times\int_{\vartheta(0)=\vartheta'}^{\vartheta(s'')=\vartheta''}\mathscr{D}\vartheta(s)\exp\left\{\frac{i}{\hbar}\int_0^{s''}\left[\frac{M}{2}(\dot a^2+\dot\vartheta^2)\right.\right.\\
&\left.\left.-\frac{\hbar^2}{2M}\left(\frac{\lambda_1^2-\frac14}{\sin^2\vartheta}+\frac{\beta^2-\frac14}{\cos^2\vartheta}+\frac{\lambda_1^2-\frac14}{\sinh^2 a}-\frac{\nu^2-\frac14}{\cosh^2 a}\right)\right]ds\right\}, \qquad (3.74)
\end{aligned}
$$

where $\beta^2=1/4-2MER^2/\hbar^2$, and $\nu^2=1/4+2MR^2(2\alpha/R-E)/h^2$. The analysis of this path integral is rather involved and requires the same Green's function analysis as in the corresponding two-dimensional case,[31,35] which will not be repeated here in detail.

Let us note first that in the case of the pure Coulomb problem we must set $\lambda_1=|j|$, $j\in\mathbb{Z}$, and the wave functions in φ are circular waves, i.e., $\phi_j(\varphi)=e^{ij\varphi}/\sqrt{2\pi}$, $\varphi\in[0,2\pi)$. Everything else remains the same.

To analyze the general case we proceed in exactly the same way as in Ref. 35. For the discrete spectrum we expand the ϑ path integration in Pöschl–Teller wave functions $\Phi_{n_1}^{(\lambda_1,\beta)}(\vartheta)$, and the a path integration in the bound-state contribution of the modified Pöschl–Teller wave functions $\psi_{n_2}^{(\lambda_1,\nu)}(a)$ of (B.6). The resulting Green's function representation $G_{\rm disc}^{(V_2)}(E)$ of $K_{\rm disc}^{(V_2)}(T)$ has poles determined by the equation

$$(2n_1+\lambda_1+\beta+1)=-(2n_2+\lambda_1-\nu+1). \qquad (3.75)$$

Solving this equation for $E_{n_1n_2}$, we obtain exactly the energy spectrum (3.65), with the principal quantum number $N=n_1+n_2+\lambda_1+1=1,\dots,N_{\max}$, as before. The residue gives the bound-state wave functions, which are therefore given by

$$
\begin{aligned}
\Psi_{mn_1n_2}^{(V_2)}(a,\vartheta,\varphi;R)=(\coth a\cot\vartheta)^{1/2}\sqrt{\frac{\sigma_N^2-\tilde N^2}{2R^3\tilde N}}\\
\times\psi_{n_1}^{(\lambda_1,\nu)}(a)\phi_{n_2}^{(\lambda_1,\beta)}(\vartheta)\phi_m^{(\pm k_2,\pm k_1)}(\varphi),
\end{aligned} \qquad (3.76)
$$

where [$\phi_m^{(\pm k_2,\pm k_1)}(\varphi)$ is the same as in (3.21)]

$$
\begin{aligned}
&\psi_{n_1}^{(\lambda_1,\nu)}(a) \qquad (3.77)\\
&=\frac{1}{\Gamma(1+\lambda_1)}\\
&\times\left[\frac{2(\nu-\lambda_1-2n_1-1)\Gamma(n_1+1+\lambda_1)\Gamma(\nu-n_1)}{n_1!\Gamma(\nu\mp k_2-n_1)}\right]^{1/2}\\
&\times(\sinh a)^{1/2+\lambda_1}(\cosh a)^{2n_1+1/2-\nu_2}\\
&\times F_2(-n_1,\nu-n_1;1+\lambda_1;\tanh^2 a), \qquad (3.78)
\end{aligned}
$$

$$\phi_{n_2}^{(\lambda_1,\beta)}(\vartheta)=\left[2(\beta+\lambda_1+2n_2+1)\right.$$

$$\times\frac{n_2!\Gamma(\beta+\lambda_1+n_2+1)}{\Gamma(n_2+\lambda_1+1)\Gamma(n_2+\beta+1)}^{1/2}(\sin\vartheta)^{1/2+\lambda_1}$$
$$\times(\cos\vartheta)^{\beta+1/2}P_{n_2}^{(\lambda_1,\beta)}(\cos 2\vartheta). \tag{3.79}$$

For the analysis of the continuous spectrum we must insert the entire Green's functions of the Pöschl–Teller potential (A.5) and the modified Pöschl–Teller potential (B.11). We then find the Green's function $G^{(V_2)}(E)$ in elliptic-parabolic coordinates by considering the ds'' integration in (3.74) with the solutions of the Pöschl–Teller and modified Pöschl–Teller potentials, respectively. The result can be put in the following form (cf. also Ref. 31 for more details concerning the proper Green's function analysis):

$$G^{(V_2)}(a'',a',\vartheta'',\vartheta',\varphi'',\varphi';E)$$
$$=(R^2\tanh a'\tanh a''\tan\vartheta'\tan\vartheta'')^{-1/2}$$
$$\times\sum_{m=0}^{\infty}\phi_m^{(\pm k_2,\pm k_1)}(\varphi')\phi_m^{(\pm k_2,\pm k_1)}(\varphi'')$$
$$\times\Bigg\{\frac{1}{2}\sum_{n_2}\psi_{n_2}^{(\lambda_1,\nu)}(a'')\psi_{n_2}^{(\lambda_1,\nu)}(a')G_{PT}^{(\lambda_1,\beta)}$$
$$\times(\vartheta'',\vartheta';E')\Big|_{E'=\hbar^2(2n_1+\lambda_1+\beta+1)^2/2MR^2}$$
$$+\frac{1}{2}\int_0^\infty dk\psi_k^{(\lambda_1,\nu)}(a'')\psi_k^{(\lambda_1,\nu)*}(a')G_{PT}^{(\lambda_1,\beta)}$$
$$\times(\vartheta'',\vartheta';E')\Big|_{E'=-\hbar^2k^2/2MR^2}$$
$$+[\text{appropriate term with } a \text{ and } \vartheta \text{ interchanged}]\Bigg\}, \tag{3.80}$$

in the notation of (A.1), (A.5), (B.6), and (B.11), respectively. Equation (3.80) also represents the Green's function, which corresponds to the path integral (3.72). Analysis of the cuts gives the continuum states, which have the form ($\tilde{p}^2=\frac{1}{4}-2MR^2(2\alpha/R-E)/\hbar^2$)

$$\Psi_{pkm}^{(V_2)}(a,\vartheta,\varphi;R)=(R^3\tanh a\tan\vartheta)^{-1/2}\psi_k^{(\lambda_1,\tilde{p})}$$
$$\times(a)\phi_k^{(\lambda_1,p)}(\vartheta)\phi_m^{(\pm k_2,\pm k_1)}(\varphi), \tag{3.81}$$

where

$$\psi_k^{(\lambda_1,\tilde{p})}(a)$$
$$=\frac{\Gamma\left[\frac{1}{2}(1+\lambda_1+i\tilde{p}+ik)\right]\Gamma\left[\frac{1}{2}(1+\lambda_1+i\tilde{p}-ik)\right]}{\Gamma(1+\lambda_1)}$$
$$\times\sqrt{\frac{k\sinh\pi k}{2\pi^2}}(\tanh a)^{\lambda_1-1/2}$$
$$\times(\cosh a)^{ik}{}_2F_1\left(\frac{1+\lambda_1+i\tilde{p}+ik}{2},\frac{1+\lambda_1-i\tilde{p}+ik}{2};\right.$$
$$\left.1+\lambda_1;\tanh^2 a\right), \tag{3.82}$$

$$\phi_k^{(\lambda_1,p)}(\vartheta)$$
$$=\frac{\Gamma\left[\frac{1}{2}(1+\lambda_1+ip+ik)\right]\Gamma\left[\frac{1}{2}(1+\lambda_1+ip-ik)\right]}{\Gamma(1+\lambda_1)}$$
$$\times\sqrt{\frac{k\sinh\pi k}{2\pi^2}}(\tan\vartheta)^{\lambda_1-1/2}(\cos\vartheta)^{ip+1+\lambda_1}{}_2F_1$$
$$\times\left(\frac{1+\lambda_1+ip+ik}{2},\frac{1+\lambda_1-ip+ik}{2};\right.$$
$$\left.1+\lambda_1;-\sin^2\vartheta\right). \tag{3.83}$$

The energy spectra are the same as in (3.65) and (3.27), respectively. Combining the two results, we obtain the path-integral solution of the Coulomb problem on $\Lambda^{(3)}$ in elliptic parabolic 2 coordinates

$$K^{(V_2)}(a'',a',\vartheta'',\vartheta',\varphi'',\varphi';T)$$
$$=\sum_{m=0}^{\infty}\Bigg\{\sum_{n_1,n_2}e^{-iE_NT/\hbar}\Psi_{mn_1n_2}^{(V_2)}(a'',\vartheta'',\varphi'';R)\Psi_{mn_1n_2}^{(V_2)}$$
$$\times(a',\vartheta',\varphi';R)+\int_0^\infty dk\int_0^\infty dp e^{-iE_pT/\hbar}\Psi_{pkm}^{(V_2)}$$
$$\times(a'',\vartheta'',\varphi'';R)\Psi_{pkm}^{(V_2)*}(a',\vartheta',\varphi';R)\Bigg\}. \tag{3.84}$$

3.3. A radial scattering potential

We consider the potential in its five separating coordinate systems ($k_{1,2,3}>0$):

$$V_3(\mathbf{u})=\frac{\hbar^2}{2MR^2}\left[-\frac{k_0^2-\frac{1}{4}}{u_0^2}+\frac{1}{\sqrt{u_1^2+u_2^2}}\left(\frac{k_1^2-\frac{1}{4}}{\sqrt{u_1^2+u_2^2}+u_1}\right.\right.$$
$$\left.\left.+\frac{k_2^2-\frac{1}{4}}{\sqrt{u_1^2+u_2^2}-u_1}\right)+\frac{k_3^2-\frac{1}{4}}{u_3^2}\right]. \tag{3.85}$$

Spherical [$\tau>0$, $\vartheta\in(0,\pi/2)$, $\varphi\in(0,\pi)$]:

$$=\frac{\hbar^2}{2MR^2}\left[-\frac{k_0^2-\frac{1}{4}}{\cosh^2\tau}+\frac{1}{\sinh^2\tau}\left(\frac{1}{4\sin^2\vartheta}\left(\frac{k_1^2-\frac{1}{4}}{\cos^2(\varphi/2)}\right.\right.\right.$$
$$\left.\left.\left.+\frac{k_2^2-\frac{1}{4}}{\sin^2(\varphi/2)}\right)+\frac{k_3^2-\frac{1}{4}}{\cos^2\vartheta}\right)\right]. \tag{3.86}$$

Equidistant-cylindrical [$\tau_{1,2}>0$, $\varphi\in(0,\pi)$]:

$$=\frac{\hbar^2}{2MR^2}\left(\frac{k_3^2-\frac{1}{4}}{\sinh^2\tau_1}+\frac{1}{\cosh^2\tau_1}\left(-\frac{k_0^2-\frac{1}{4}}{\cosh^2\tau_2}\right.\right.$$
$$\left.\left.+\frac{1}{4\sinh^2\tau_2}\left(\frac{k_1^2-\frac{1}{4}}{\cos^2(\varphi/2)}+\frac{k_2^2-\frac{1}{4}}{\sin^2(\varphi/2)}\right)\right)\right). \tag{3.87}$$

Prolate elliptic $[\alpha\in(iK',iK'+K),\quad \beta\in(0,K'),\ \varphi\in(0,\pi)]$:

$$=-\frac{\hbar^2}{2MR^2}\left(-\frac{k_0^2-\frac14}{\operatorname{sn}^2\alpha\,\operatorname{dn}^2\beta}+\frac{1}{4\,\operatorname{dn}^2\alpha\,\operatorname{sn}^2\beta}\left(\frac{k_1^2-\frac14}{\cos^2(\varphi/2)}+\frac{k_2^2-\frac14}{\sin^2(\varphi/2)}\right)+\frac{k_3^2-\frac14}{\operatorname{cn}^2\alpha\,\operatorname{cn}^2\beta}\right). \tag{3.88}$$

Oblate elliptic $[\alpha\in(iK',iK'+K),\quad \beta\in(0,K'),\ \varphi\in(0,\pi)]$:

$$=-\frac{\hbar^2}{2MR^2}\left(-\frac{k_0^2-\frac14}{\operatorname{sn}^2\alpha\,\operatorname{dn}^2\beta}+\frac{1}{4\,\operatorname{cn}^2\alpha\,\operatorname{cn}^2\beta}\left(\frac{k_1^2-\frac14}{\cos^2(\varphi/2)}+\frac{k_2^2-\frac14}{\sin^2(\varphi/2)}\right)+\frac{k_3^2-\frac14}{\operatorname{dn}^2\alpha\,\operatorname{sn}^2\beta}\right). \tag{3.89}$$

Hyperbolic-cylindrical $2[\mu\in(iK',iK'+K),\ \eta\in(0,K'),\ \varphi\in(0,\pi)]$:

$$=-\frac{\hbar^2}{2MR^2}\left(-\frac{k_0^2-\frac14}{\operatorname{cn}^2\mu\,\operatorname{cn}^2\nu}+\frac{1}{4\,\operatorname{sn}^2\mu\,\operatorname{dn}^2\nu}\left(\frac{k_1^2-\frac14}{\cos^2(\varphi/2)}+\frac{k_2^2-\frac14}{\sin^2(\varphi/2)}\right)+\frac{k_3^2-\frac14}{\operatorname{dn}^2\mu\,\operatorname{sn}^2\nu}\right). \tag{3.90}$$

An explicit solution is possible only in the first two coordinate systems.

Spherical coordinates. In the first separating coordinate system we have the following path-integral representation, together with its solution $[\lambda_1=m+1/2(1\mp k_1\mp k_2),\ \lambda_2=2l\mp k_3+\lambda_1+1]$:

$$K^{(V_3)}(\mathbf{u}'',\mathbf{u}';T)$$
$$=\frac{e^{-i\hbar T/2MR^2}}{R^3}\int_{\tau(t')=\tau'}^{\tau(t'')=\tau''}\mathscr{D}\tau(t)\sinh^2\tau\int_{\vartheta(t')=\vartheta'}^{\vartheta(t'')=\vartheta''}\mathscr{D}\vartheta(t)\times\sin\vartheta\int_{\varphi(t')=\varphi'}^{\varphi(t'')=\varphi''}\mathscr{D}\varphi(t)\exp\left\{\frac{i}{\hbar}\int_{t'}^{t''}\left[\frac{M}{2}R^2(\dot\tau^2+\sinh^2\tau(\dot\vartheta^2+\sin^2\vartheta\dot\varphi^2))-\frac{\hbar^2}{2MR^2}\times\left(-\frac{k_0^2-\frac14}{\cosh^2\tau}+\frac{1}{\sinh^2\tau}\left(\frac{1}{4\sin^2\vartheta}\left(\frac{k_1^2-\frac14}{\cos^2(\varphi/2)}+\frac{k_2^2-\frac14}{\sin^2(\varphi/2)}-1\right)+\frac{k_3^2-\frac14}{\cos^2\vartheta}-\frac14\right)\right)\right]dt\right\} \tag{3.91}$$

$$=\sum_{m=0}^{\infty}\sum_{l=0}^{\infty}\int_0^\infty dp\,e^{-iE_p/\hbar}\Psi_{plm}^{(V_3)}(\tau'',\vartheta'',\varphi'';R)\Psi_{plm}^{(V_3)*}(t',\vartheta',\varphi';R). \tag{3.92}$$

The path-integral evaluation is performed by applying the path-integral solution of the Pöschl–Teller potential in φ and ϑ, and for the $1/\sinh^2 r$ potential in τ. The spectrum is purely continuous, the wave functions have the form

$$\Psi_{plm}^{(V_3)}(\tau,\vartheta,\varphi;R)=S_p^{(\lambda_2,k_0)}(\tau;R)\phi_l^{(\lambda_1,\pm k_3)}(\vartheta)\times\phi_m^{(\pm k_2,\pm k_1)}\left(\frac{\varphi}{2}\right), \tag{3.93}$$

and E_p is the same as in (3.27). Here the wave functions $S_p^{(\lambda_2,k_0)}$ are the usual continuous modified Pöschl–Teller wave functions analogous to (3.25) with the parameters (λ_2,k_0), the $\phi_l^{(\lambda_1,\pm k_3)}(\vartheta)$ are the same as in (3.34), and the wave functions $\phi_m^{(\pm k_2,\pm k_1)}(\varphi/2)$ are the same as in (3.21) with $\varphi\to\varphi/2$ and an additional factor $1/\sqrt{2}$.

Equidistant-cylindrical coordinates. The solution in second coordinate system has the form $(\lambda_1=2m\mp k_1\mp k_2+1)$

$$K^{(V_3)}(\mathbf{u}'',\mathbf{u}';T)$$
$$=\frac{e^{-i\hbar T/2MR^2}}{R^3}\int_{\tau_1(t')=\tau_1'}^{\tau_1(t'')=\tau_1''}\mathscr{D}\tau_1(t)\cosh^2\tau_1\int_{\tau_2(t')=\tau_2'}^{\tau_2(t'')=\tau_2''}\times\mathscr{D}\tau_2(t)\sinh\tau_2\int_{\varphi(t')=\varphi'}^{\varphi(t'')=\varphi''}\mathscr{D}\varphi(t)\exp\left\{\frac{i}{\hbar}\int_{t'}^{t''}\left[\frac{M}{2}R^2\times\left(\dot\tau_1^2+\cosh^2\tau_1(\dot\tau_2^2+\sinh^2\tau_2\dot\varphi^2)-\frac{\hbar^2}{2MR^2}\times\left(\frac{k_3^2-\frac14}{\sinh^2\tau_1}+\frac{1}{\cosh^2\tau_1}\left(-\frac{k_0^2-\frac14}{\cosh^2\tau_2}+\frac{1}{4\sinh^2\tau_2}\times\left(\frac{k_1^2-\frac14}{\cos^2\frac{\varphi}{2}}+\frac{k_2^2-\frac14}{\sin^2\frac{\varphi}{2}}-1\right)+1\right)\right)\right)\right]dt\right\} \tag{3.94}$$

$$=\sum_{m=0}^{\infty}\int_0^\infty dk\int_0^\infty dp\,e^{-iE_P/\hbar}\Psi_{pkm}^{(V_3)}(\tau_1'',\tau_2'',\varphi'';R)\Psi_{pkm}^{(V_3)*}(\tau_1',\tau_2',\varphi';R). \tag{3.95}$$

The path-integral evaluation is performed by means of the path-integral solution for the Pöschl–Teller potential in φ, for the $1/\sinh^2 r$ potential in τ_2 and for the modified Pöschl–Teller potential in τ_1. The continuum wave functions have the form (λ_1 is the same as before)

$$\Psi_{pkm}^{(V_3)}(\tau_1,\tau_2,\varphi;R)=S_p^{(\pm k_3,ik)}(\tau_1;R)\psi_k^{(\lambda_1,\pm k_0)}$$

$$\times(\tau_2)\phi_m^{(\pm k_2,\pm k_1)}\left(\frac{\varphi}{2}\right) \quad (3.96)$$

with $\phi_m^{(\pm k_2,\pm k_1)}(\varphi/2)$ as before, $S_p^{(\pm k_3,ik)}(\tau_1;R)$ the same as in (3.44), and $\psi_p^{(\lambda_1,k_0)}$ the same as in (3.93) with $\lambda_2\to\lambda_1$, $\tau\to\tau_2$, and $R=1$.

3.4. A Stark-effect potential

We consider the potential $(k_{1,2}>0)$

$$V_4(\mathbf{u})=\frac{\hbar^2}{4M\sqrt{u_1^2+u_2^2}}\left(\frac{k_1^2-\frac14}{\sqrt{u_1^2+u_2^2}+u_1}+\frac{k_2^2-\frac14}{\sqrt{u_1^2+u_2^2}-u_1}\right)+k_3u_3. \quad (3.97)$$

In its four separating coordinate systems it has the following form:

Equidistant-elliptic II $[\alpha\in(iK',iK'+K),\ \beta\in(0,K'),\ \tau>0]$:

$$V_4(\mathbf{u})=\frac{1}{\cosh^2\tau}\frac{\hbar^2}{4MR^2}\left[\frac{k_1^2+k_2^2-\frac12}{k^2\,\mathrm{cn}^2\,\alpha+k'^2\,\mathrm{cn}^2\,\beta}\times\left(\frac{k'^2}{\mathrm{dn}^2\,\alpha}-\frac{1}{\mathrm{sn}^2\,\beta}\right)+(k_1^2-k_2^2)\frac{k'}{k}\frac{k^2\,\mathrm{sn}\,\alpha\,\mathrm{cu}\,\alpha+k'\,\mathrm{cn}\,\beta\,\mathrm{dn}\,\beta}{k^2\,\mathrm{cn}^2\,\alpha+k'^2\,\mathrm{cn}^2\,\beta}\right]+k_3R\sinh\tau. \quad (3.98)$$

Equidistant-semihyperbolic $(\tau\in\mathbb{R},\ \mu_{1,2}>0)$:

$$=\frac{\hbar^2}{4MR^2\cosh^2\tau}\frac{1}{\mu_1+\mu_2}\left[\left(k_1^2+k_2^2-\frac12\right)\left(\frac{1}{\mu_1}+\frac{1}{\mu_2}\right)+(k_1^2-k_2^2)\left(\frac{\sqrt{1+\mu_1^2}}{\mu_1}-\frac{\sqrt{1+\mu_2^2}}{\mu_2}\right)\right]+k_3R\sinh\tau. \quad (3.99)$$

Equidistant-elliptic-parabolic $[\tau\in\mathbb{R},\ a>0,\ \vartheta\in(0,\pi/2)]$:

$$=\frac{\hbar^2}{2MR^2\cosh^2\tau}\frac{\cosh^2 a\cos^2\vartheta}{\cosh^2 a-\cos^2\vartheta}\left(\frac{k_1^2-\frac14}{\sin^2\vartheta}+\frac{k_2^2-\frac14}{\sinh^2 a}\right)+k_3R\sinh\tau. \quad (3.100)$$

Equidistant-cylindrical $[\tau_1\in\mathbb{R},\ \tau_2>0,\ \varphi\in(0,\pi)]$:

$$=\frac{\hbar^2}{8MR^2\cosh^2\tau_1\sinh^2\tau_2}\left(\frac{k_1^2-\frac14}{\sin^2(\varphi/2)}+\frac{k_2^2-\frac14}{\cos^2(\varphi/2)}\right)+k_3R\sinh\tau_1. \quad (3.101)$$

Because there are only four observables, the potential V_4 is only minimally and not maximally superintegrable on the hyperboloid. However, its flat-space analog $V_5(\mathbf{x})$ is maximally superintegrable, and we have nevertheless included V_4 in this section. Unfortunately, the path integrals in all coordinate systems are not solvable, and we give no further details.

4. PATH-INTEGRAL FORMULATION OF THE MINIMALLY SUPERINTEGRABLE POTENTIALS ON $\Lambda^{(3)}$

In this section we list our findings for the minimally superintegrable potentials on $\Lambda^{(3)}$. They include the following:

1. The class of potentials which are the analogs of the minimally superintegrable potentials in $\mathbb{R}^3$ (Refs. 16 and 33). For instance, the potentials V_5 and V_6 correspond to the double-ring-shaped oscillator and the Hartmann potential in $\mathbb{R}^3$, respectively. The four potentials that have been found are discussed in some detail.
2. The class of potentials which correspond to the group reduction $SO(3,1)\supset E(2)$, i.e., which are superintegrable in $\mathbb{R}^2$ (Ref. 33). Here the results of Ref. 33 will be used, and the problem of self-adjoint extensions for Hamiltonians unbounded from below is briefly mentioned.
3. The class of potentials which correspond to the group reduction $SO(3,1)\supset SO(3)$, i.e., which are superintegrable on $S^{(2)}$ (Ref. 34). In our list we have chosen for convenience a dependence according to $1/u_0^2$, but any function $F=F(u_0)$ admits separation of variables.
4. The class of potentials which correspond to the group reduction $SO(3,1)\supset SO(2,1)$, i.e., which are superintegrable on $\Lambda^{(2)}$ (Ref. 35). In our list we have chosen for convenience a dependence according to $1/u_3^2$, but any function $F=F(u_3)$ admits separation of variables. Because the features are repeating themselves, all those potentials can be treated simultaneously.

4.1. Analogs of the minimally superintegrable potentials in $\mathbb{R}^3$

4.1.1. Double-ring-shaped oscillator

We consider the minimally superintegrable double-ring-shaped potential V_5 on $\Lambda^{(3)}$ $(k_3>0)$,

$$V_5(\mathbf{u})=\frac{M}{2}\omega^2R^2\frac{u_1^2+u_2^2+u_3^2}{u_0^2}+\frac{\hbar^2}{2M}\times\left(\frac{k_3^2-\frac14}{u_3^2}+\frac{F(u_2/u_1)}{u_1^2+u_2^2}\right), \quad (4.1)$$

which in the five separating coordinate systems has the following form (φ with the appropriate range):

Spherical $[\tau>0,\ \vartheta\in(0,\pi/2)]$:

$$V_5(\mathbf{u})=\frac{M}{2}\omega^2R^2\tanh^2\tau+\frac{\hbar^2}{2MR^2\sinh^2\tau}\times\left(\frac{k_3^2-\frac14}{\cos^2\vartheta}+\frac{F(\tan\varphi)}{\sin^2\vartheta}\right). \quad (4.2)$$

Equidistant-cylindrical $(\tau_{1,2}>0)$:

$$=\frac{M}{2}\omega^2R^2\left(1-\frac{1}{\cosh^2\tau_1\cosh^2\tau_2}\right)+\frac{\hbar^2}{2MR^2}$$

$$\times\left(\frac{k_3^2-\frac{1}{4}}{\sinh^2\tau_1}+\frac{F(\tan\varphi)}{\cosh^2\tau_1\sinh^2\tau_2}\right). \tag{4.3}$$

Prolate elliptic $[\alpha\in(iK',iK'+K),\ \beta\in(0,K')]$:

$$=\frac{M}{2}\omega^2R^2\left(1-\frac{1}{\mathrm{sn}^2\alpha\,\mathrm{dn}^2\beta}\right)-\frac{\hbar^2}{2MR^2}$$

$$\times\left(\frac{k_3^2-\frac{1}{4}}{\mathrm{cn}^2\alpha\,\mathrm{cn}^2\beta}+\frac{F(\tan\varphi)}{\mathrm{dn}^2\alpha\,\mathrm{sn}^2\beta}\right). \tag{4.4}$$

Oblate elliptic $[\alpha\in(iK',iK'+K),\ \beta\in(0,K')]$:

$$=\frac{M}{2}\omega^2R^2\left(1-\frac{1}{\mathrm{sn}^2\alpha\,\mathrm{dn}^2\beta}\right)-\frac{\hbar^2}{2MR^2}$$

$$\times\left(\frac{k_3^2-\frac{1}{4}}{\mathrm{dn}^2\alpha\,\mathrm{sn}^2\beta}+\frac{F(\tan\varphi)}{\mathrm{cn}^2\alpha\mathrm{cn}^2\beta}\right). \tag{4.5}$$

Hyperbolic-cylindrical 2 $[\mu\in(iK',iK'+K),\ \eta\in(0,K')]$:

$$=\frac{M}{2}\omega^2R^2\left(1-\frac{1}{k^2\mathrm{cn}^2\mu\,\mathrm{cn}^2\nu}\right)-\frac{\hbar^2}{2MR^2}$$

$$\times\left(\frac{k_3^2-\frac{1}{4}}{\mathrm{dn}^2\mu\,\mathrm{sn}^2\nu}+\frac{F(\tan\varphi)}{\mathrm{sn}^2\mu\,\mathrm{dn}^2\nu}\right). \tag{4.6}$$

An explicit solution is available in two coordinate systems, and we have the following path-integral representations, together with their solutions:

Spherical coordinates. In spherical coordinates the solution is not very different from the solution of the generalized oscillator on $\Lambda^{(3)}$, the only difference being the φ dependence. Hence we obtain $[\lambda_2=2l\pm k_3+\lambda_F+1,\ \nu^2=M^2\omega^2R^4/\hbar^2+1/4,\ n=0,\dots,N_n<(\nu-\lambda_2-1)/2]$

$$K^{(V_5)}(\mathbf{u}'',\mathbf{u}';T)$$

$$=\frac{e^{-i\hbar T/2MR^2}}{R^3}\int_{\tau(t')=\tau'}^{\tau(t'')=\tau''}\mathcal{D}\tau(t)\sinh^2\tau\int_{\vartheta(t')=\vartheta'}^{\vartheta(t'')=\vartheta''}\mathcal{D}\vartheta(t)$$

$$\times\sin\vartheta\int_{\varphi(t')=\varphi'}^{\varphi(t'')=\varphi''}\mathcal{D}\varphi(t)\exp\left\{\frac{i}{\hbar}\int_{t'}^{t''}\left[\frac{M}{2}R^2(\dot\tau^2\right.\right.$$

$$+\sinh^2\tau(\dot\vartheta^2+\sin^2\vartheta\dot\varphi^2)-\omega^2\tanh^2\tau)$$

$$\left.\left.-\frac{\hbar^2}{2MR^2\sinh^2\tau}\left(\frac{k_3^2-\frac{1}{4}}{\cos^2\vartheta}+\frac{F(\tan\varphi)-\frac{1}{4}}{\sin^2\vartheta}-\frac{1}{4}\right)\right]dt\right\} \tag{4.7}$$

$$=\int dE_\lambda\sum_{l=0}^{\infty}\left\{\sum_{n=0}^{N_n}e^{-iE_NT/\hbar}\Psi_{\lambda ln}^{(V_5)}(\tau'',\vartheta'',\varphi'';R)\Psi_{\lambda ln}^{(V_5)}\right.$$

$$\times(\tau',\vartheta',\varphi';R)+\int_0^\infty dp e^{-iE_pT/\hbar}\Psi_{pl\lambda}^{(V_5)}$$

$$\left.\times(\tau'',\vartheta'',\varphi';R)\Psi_{pl\lambda}^{(V_5)*}(\tau',\vartheta',\varphi';R)\right\}. \tag{4.8}$$

The bound-state wave functions and energy spectrum are given by $(N=l+n)$

$$\Psi_{\lambda ln}^{(V_5)}(\tau,\vartheta,\varphi;R)=(\sinh^2\tau\sin\vartheta)^{-1/2}S_n^{(\lambda_2,\nu)}$$

$$\times(\tau;R)\phi_l^{(\lambda_F,\pm k_3)}(\vartheta)\phi_\lambda^{(F)}(\varphi), \tag{4.9}$$

$$E_N=-\frac{\hbar^2}{2MR^2}[(2(N+1)\pm k_3+\lambda_F-\nu)^2-1]$$

$$+\frac{M}{2}\omega^2R^2. \tag{4.10}$$

The scattering states and continuous spectrum are

$$\Psi_{pl\lambda}^{(V_5)}(\tau,\vartheta,\varphi;R)=(\sinh^2\tau\sin\vartheta)^{-1/2}S_p^{(\lambda_2,\nu)}$$

$$\times(\tau;R)\phi_l^{(\lambda_F,\pm k_3)}(\vartheta)\phi_\lambda^{(F)}(\varphi), \tag{4.11}$$

$$E_p=\frac{\hbar^2}{2MR^2}(p^2+1)+\frac{M}{2}\omega^2R^2. \tag{4.12}$$

Here the wave functions $\phi_\lambda^{(F)}(\varphi)$ are the eigenfunctions corresponding to the potential term $F(\tan\varphi)$ with eigenvalues $E_\lambda=\hbar^2\lambda^2/2M$, the $\phi_l^{(\lambda_F,\pm k_3)}(\vartheta)$ are the same as in (3.34) with $\lambda_1\to\lambda_F$, and the wave functions $S_n^{(\lambda_2,\nu)}(\tau;R)$ and $S_p^{(\lambda_2,\nu)}(\tau;R)$ are the same as in (3.35) and (3.37), respectively.

Equidistant-cylindrical coordinates. Here $(\lambda_1=2l+\lambda_F-\nu+1,\ \nu^2=M^2\omega^2R^4/\hbar^2+1/4)$

$$K^{(V_5)}(\mathbf{u}'',\mathbf{u}';T)$$

$$=R^{-3}\exp\left[-\frac{i}{\hbar}T\left(\frac{\hbar^2}{2MR^2}+\frac{M}{2}R^2\omega^2\right)\right]\int_{\tau_1(t')=\tau_1'}^{\tau_1(t'')=\tau_1''}$$

$$\times\mathcal{D}\tau_1(t)\cosh^2\tau_1\int_{\tau_2(t')=\tau_2'}^{\tau_2(t'')=\tau_2''}\mathcal{D}\tau_2(t)\sinh\tau_2$$

$$\times\int_{\varphi(t')=\varphi'}^{\varphi(t'')=\varphi''}\mathcal{D}\varphi(t)\exp\left\{\frac{i}{\hbar}\int_{t'}^{t''}\left[\frac{M}{2}R^2\left(\dot\tau_1^2+\cosh^2\tau_1\right.\right.\right.$$

$$\left.\times\left(\dot\tau_2^2+\sinh^2\tau_1\dot\varphi^2+\frac{\omega^2}{\cosh^2\tau_1\cosh^2\tau_2}\right)-\frac{\hbar^2}{2MR^2}\right.$$

$$\left.\left.\times\left(\frac{1}{\cosh^2\tau_1}\left(\frac{F(\tan\varphi)-\frac{1}{4}}{\sinh^2\tau_2}+\frac{1}{4}\right)+\frac{k_3^2-\frac{1}{4}}{\sinh^2\tau_1}\right)\right]dt\right\} \tag{4.13}$$

$$=\int dE_\lambda\left\{\sum_{l=0}^{N_l}\left[\sum_{n=0}^{N_n}e^{-i\hbar E_NT/\hbar}\Psi_{\lambda ln}^{(V_5)}(\tau_1'',\tau_2'',\varphi'';R)\Psi_{\lambda ln}^{(V_5)}\right.\right.$$

$$\times(\tau_1',\tau_2',\varphi';R)+\int_0^\infty dp e^{-iE_pT/\hbar}\Psi_{pl\lambda}^{(V_5)}(\tau_1'',\tau_2'',\varphi'';R)$$

$$\left.\times\Psi_{pl\lambda}^{(V_5)*}(\tau_1',\tau_2',\varphi';R)\right]+\int_0^\infty dk\int_0^\infty dp e^{-iE_pT/\hbar}\Psi_{pk\lambda}^{(V_5)}$$

$$\left.\times(\tau_1'',\tau_2'',\varphi'';R)\Psi_{pk\lambda}^{(V_5)*}(\tau_1',\tau_2',\varphi';R)\right\}. \tag{4.14}$$

Again, we have similar features as for the general oscillator, and therefore we have one set of bound-state wave functions

TABLE III. Minimally superintegrable potentials on $\Lambda^{(3)}$: analogs of three-dimensional flat space.

Potential $V(\mathbf{u})$	Coordinate systems	Observables
$V_5(\mathbf{u})=\frac{M}{2}\omega^2R^2\frac{u_1^2+u_2^2+u_3^2}{u_0^2}+\frac{\hbar^2}{2M}\left(\frac{k_3^2-\frac14}{u_3^2}+\frac{F(u_2/u_1)}{u_1^2+u_2^2}\right)$	Spherical Equidistant-cylindrical	$I_1=\frac{1}{2MR^2}(\mathbf{K}^2-\mathbf{L}^2)+V_5(\mathbf{u}),\quad I_2=\frac{1}{2M}L_3^2+F(\tan\varphi)$
	Prolate elliptic Oblate elliptic	$I_3=\frac{1}{2M}\mathbf{L}^2+\frac{\hbar^2}{2M}\left(\frac{k_3^2-\frac14}{\cos^2\vartheta}+\frac{F(\tan\varphi)}{\sin^2\vartheta}\right)$
	Hyperbolic-cylindrical 2	$I_4=\frac{1}{2M}(K_1^2+K_2^2-L_3)^2-\frac{M}{2}\frac{\omega^2}{\cosh^2\tau_2}+\frac{\hbar^2}{2M}\frac{F(\tan\varphi)}{\tau_1\sinh^2\tau_2}$
$V_6(\mathbf{u})=-\frac{\alpha}{R}\left(\frac{u_0}{\sqrt{u_1^2+u_2^2+u_3^2}}-1\right)$	Spherical	$I_1=\frac{1}{2MR^2}(\mathbf{K}^2-\mathbf{L}^2)+V_6(\mathbf{u}),\quad I_2=\frac{1}{2M}L_3^2+F(\tan\varphi)$
$+\frac{\hbar^2}{2M(u_1^2+u_2^2)}\left(\frac{\beta u_3}{\sqrt{u_1^2+u_2^2+u_3^2}}+F\left(\frac{u_2}{u_1}\right)\right)$	Prolate elliptic II	$I_3=\frac{1}{2M}\mathbf{L}^2+\frac{\hbar^2}{2M}\frac{F(\tan\varphi)+\beta\cos\vartheta}{\sin^2\vartheta}$
$\sinh^2 f=k'^2/k^2$	Semihyperbolic	$I_4=\frac{1}{2M}\Big[\Big(\cosh 2f\mathbf{L}^2-\frac12\sinh 2f(\{K_2,L_1\}-\{K_1,L_2\})\Big]$ $-\alpha R\frac{k^2\mathrm{sn}\alpha\,\mathrm{cn}\alpha-k'\mathrm{cn}\beta\,\mathrm{dn}\beta}{k^2\mathrm{cn}^2\alpha+k'^2\mathrm{cn}^2\beta}$ $+\frac{\hbar^2}{4M}\left(\frac{F(\tan(\varphi)+1}{k^2\mathrm{cn}^2\alpha+k'^2\mathrm{cn}^2\beta}\left(\frac{k'^2}{\mathrm{dn}^2\alpha}-\frac{1}{\mathrm{sn}^2\beta}\right)\right)$ $-\beta\frac{k'}{k}\frac{k^2\mathrm{sn}\alpha\,\mathrm{cn}\alpha+k'\mathrm{cn}\beta\,\mathrm{dn}\beta}{k^2\mathrm{cn}^2\alpha+k'^2\mathrm{cn}^2\beta}$
$V_7(\mathbf{u})=F(u_1^2+u_2^2+u_3^2)$	Sphero-elliptic	$I_1=\frac{1}{2MR^2}(\mathbf{K}^2-\mathbf{L}^2)+V_7(\mathbf{u})$
$+\frac{\hbar^2}{2M}\left(-\frac{k_0^2-\frac14}{u_0^2}+\frac{k_1^2-\frac14}{u_1^2}+\frac{k_2^2-\frac14}{u_2^2}+\frac{k_3^2-\frac14}{u_3^2}\right)$	Spherical	$I_2=\frac{1}{2M}L_3^2+\frac{\hbar^2}{2M}\left(\frac{k_1^2-\frac14}{\cos^2\varphi}+\frac{k_2^2-\frac14}{\sin^2\varphi}\right)$ $I_3=\frac{1}{2M}\mathbf{L}^2+\frac{\hbar^2}{2M}\left(\frac{1}{\sin^2\vartheta}\left(\frac{k_1^2-\frac14}{\cos^2\varphi}+\frac{k_2^2-\frac14}{\sin^2\varphi}\right)+\frac{k_3^2-\frac14}{\cos^2\vartheta}\right)$ $I_4=\frac{1}{4M}(L_1^2+k'^2L_2^2)-\frac{\hbar^2}{2M(k^2\mathrm{cn}^2\tilde\alpha+k'^2\mathrm{cn}^2\tilde\beta)}\left(\left(k_1^2-\frac14\right)\left(\frac{1}{\mathrm{sn}^2\tilde\alpha}-\frac{k^2}{\mathrm{dn}^2\tilde\beta}\right)\right.$ $\left.+\left(k_2^2-\frac14\right)\left(\frac{k'^2}{\mathrm{cn}^2\tilde\alpha}-\frac{k^2}{\mathrm{dn}^2\tilde\beta}\right)-\left(k_3^2-\frac14\right)\left(\frac{k'^2}{\mathrm{dn}^2\tilde\alpha}-\frac{1}{\mathrm{sn}^2\tilde\beta}\right)\right)$
$V_8(\mathbf{u})=\frac{\hbar^2}{2M}\frac{k_2^2-\frac14}{u_2^2}$	Horicyclic	$I_1=\frac{1}{2MR^2}(\mathbf{K}^2-\mathbf{L}^2)+V_8(\mathbf{u}),\quad I_2=\frac{1}{2M}P_{x_1}^2+2M\omega^2x_1^2-\lambda x_1$
$+\frac{M}{2}\left(\frac{\alpha}{(u_0-u_3)^2}+\omega^2\frac{R^2+4u_1^2+u_2^2}{(u_0-u_3)^4}-\frac{\lambda u_1}{(u_0-u_3)^3}\right)$	Semicircular-parabolic	$I_3=\frac{1}{2M}P_{x_2}^2+\frac{M}{2}\omega^2x_2^2+\frac{\hbar^2}{2M}\frac{k_2^2-\frac14}{x_2^2}$
$P_{x_i}=\frac{\hbar}{\mathrm{i}}\frac{\partial}{\partial x_i},\quad i=1,2$		$I_4=\frac{1}{2M}(\{L_3,K_1+L_2\}+\{K_3,K_2-L_1\})$ $+\frac{\xi^2\eta^2}{2}\frac{2\alpha(\xi^2+\eta^2)+\lambda(\xi^4-\eta^4)+M\omega^2(\xi^6+\eta^6)}{\xi^2+\eta^2}$

and two sets of scattering states. They are given by [$n=0,\ldots,N_n<(\lambda_2-k_3-1)/2$, $l=0,\ldots,N_l<(\nu-\lambda_F-1)/2$]

$$\Psi_{\lambda ln}^{(V_5)}(\tau_1,\tau_2,\varphi;R)=(\cosh^2\tau_1\sinh\tau_2)^{-1/2}S_n^{(\pm k_3,\lambda_2)}(\tau_1;R)\psi_l^{(\lambda_F,\nu)}(\tau_2)\phi_\lambda^{(F)}(\varphi), \tag{4.15}$$

$$\Psi_{pk\lambda}^{(V_5)}(\tau_1,\tau_2,\varphi;R)=(\cosh^2\tau_1\sinh\tau_2)^{-1/2}S_p^{(\pm k_3,\lambda_2)}(\tau_1;R)\psi_l^{(\lambda_F,\nu)}(\tau_2)\phi_\lambda^{(F)}(\varphi), \tag{4.16}$$

$$\Psi_{pl\lambda}^{(V_5)}(\tau_1,\tau_2,\varphi;R)=(\cosh^2\tau_1\sinh\tau_2)^{-1/2}S_p^{(\pm k_3,\mathrm{i}k)}(\tau_1;R)\psi_k^{(\lambda_F,\nu)}(\tau_2)\phi_\lambda^{(F)}(\varphi). \tag{4.17}$$

The wave functions $\psi_{l,k}^{(\lambda_F,\nu)}(\tau_2)$ are the same as in (3.20) and (3.30) with $\pm k_1\to\lambda_F$ and $m\to l$, respectively. The energy spectra E_N and E_p are the same as in the previous paragraph.

4.1.2. Hartmann potential

The next potential represents the analog of the Hartmann potential V_6 in $\mathbb{R}^3$ (Ref. 33). We consider

$$V_6(\mathbf{u})=-\frac{\alpha}{R}\left(\frac{u_0}{\sqrt{u_1^2+u_2^2+u_3^2}}-1\right)+\frac{\hbar^2}{2M(u_1^2+u_2^2)}\left(\frac{\beta u_3}{\sqrt{u_1^2+u_2^2+u_3^2}}+F\left(\frac{u_2}{u_1}\right)\right), \tag{4.18}$$

which in the two separating coordinate systems has the following form (φ with the appropriate range):

Spherical $[\tau>0,\ \vartheta\in(0,\pi/2)]$:

$$V_6(\mathbf{u})=-\frac{\alpha}{R}(\coth\tau-1)+\frac{\hbar^2}{2MR^2\sinh^2\tau}\frac{F(\tan\varphi)+\beta\cos\vartheta}{\sin^2\vartheta}. \tag{4.19}$$

Prolate elliptic II $(\alpha\in[iK',iK'+K),\ \beta\in(0,K')]$:

$$-\frac{\alpha}{R}\left(\frac{k^2\text{sn}\alpha\,\text{cn}\alpha-k'\text{cn}\beta\,\text{dn}\beta}{k^2\text{cn}^2\alpha+k'^2\text{cn}^2\beta}-1\right)+\frac{\hbar^2}{4MR^2}\left(\frac{F(\tan\varphi)+1}{k^2\text{cn}^2\alpha+k'^2\text{cn}^2\beta}\left(\frac{k'^2}{\text{dn}^2\alpha}-\frac{1}{\text{sn}^2\beta}\right)-\beta\frac{k'}{k}\frac{k^2\text{sn}\alpha\,\text{cn}\alpha+k'\text{cn}\beta\,\text{dn}\beta}{k^2\text{cn}^2\alpha+k'^2\text{cn}^2\beta}\right)\bigg]. \tag{4.20}$$

We treat only the spherical case with $F(\tan\varphi)=\gamma$. Then we obtain the following expression, together with $\gamma\geq|\beta|$, $\lambda_\pm^2=n^2+\gamma\pm\beta$, $\lambda_2=m+(\lambda_++\lambda_-+1)/2$, $\varphi\in[0,2\pi)$:

$$\begin{aligned}K^{(V_6)}&(\tau'',\tau',\vartheta'',\vartheta',\varphi'',\varphi';T)\\
&=R^{-3}\exp\left[-\frac{\mathrm{i}}{\hbar}T\left(\frac{\hbar^2}{2MR^2}+\frac{\alpha}{R}\right)\right]\int_{\tau(t')=\tau'}^{\tau(t'')=\tau''}\mathscr{D}\tau(t)\sinh^2\tau\int_{\vartheta(t')=\vartheta'}^{\vartheta(t'')=\vartheta''}\mathscr{D}\vartheta(t)\sin\vartheta\int_{\varphi(t')=\varphi'}^{\varphi(t'')=\varphi''}\mathscr{D}\varphi(t)\exp\Bigg\{\frac{\mathrm{i}}{\hbar}\\
&\times\int_{t'}^{t''}\Bigg[\frac{M}{2}R^2(\dot\tau^2+\sinh^2\tau(\dot\vartheta^2+\sin^2\vartheta\dot\varphi^2))+\frac{\alpha}{R}\coth\tau-\frac{\hbar^2}{8MR^2\sinh^2\tau}\left(\frac{\gamma+\beta-\frac14}{\sin^2(\vartheta/2)}+\frac{\gamma-\beta-\frac14}{\cos^2(\vartheta/2)}-\frac14\right)\Bigg]dt\Bigg\}
\end{aligned} \tag{4.21}$$

$$\begin{aligned}&=\sum_{m=0}^{\infty}\sum_{l=0}^{\infty}\Bigg\{\sum_{n=0}^{N_n}\mathrm{e}^{-iE_NT/\hbar}\Psi_{nlm}^{(V_6)}(\tau'',\vartheta'',\varphi'';R)\\
&\times\Psi_{nlm}^{(V_6)*}(\tau',\vartheta',\varphi';R)+\int_0^\infty dp\,\mathrm{e}^{-iE_pT/\hbar}\Psi_{plm}^{(V_6)}(\tau'',\vartheta'',\varphi'';R)\Psi_{plm}^{(V_6)*}(\tau',\vartheta',\varphi';R)\Bigg\}.\end{aligned} \tag{4.22}$$

The path integration in ϑ is of the Pöschl–Teller type, whereas the path integration in τ is essentially the same as for the Coulomb potential. Therefore, the wave functions for the bound-state and continuous spectra are given by ($n=0,\dots,N_n<\sqrt{R/a}-\lambda_2-1/2$; $a=\hbar^2/M\alpha$ is the Bohr radius)

$$\Psi_{nlm}^{(V_6)}(\tau,\vartheta,\varphi;R)=(\sinh^2\tau\sin\vartheta)^{-1/2}S_N(\tau;R)\times\phi_l^{(\lambda_+,\lambda_-)}(\vartheta)\frac{\mathrm{e}^{im\varphi}}{\sqrt{2\pi}}, \tag{4.23}$$

$$\Psi_{plm}^{(V_6)}(\tau,\vartheta,\varphi;R)=(\sinh^2\tau\sin\vartheta)^{-1/2}S_p(\tau;R)\times\phi_l^{(\lambda_+,\lambda_-)}(\vartheta)\frac{\mathrm{e}^{im\varphi}}{\sqrt{2\pi}}, \tag{4.24}$$

$$\phi_l^{(\lambda_+,\lambda_-)}(\vartheta)=\left[(2l+\lambda_++\lambda_-+1)\frac{l!\Gamma(l+\lambda_++\lambda_-+1)}{\Gamma(l+\lambda_++1)\Gamma(l+\lambda_-+1)}\right]^{1/2}\times\left(\sin\frac{\vartheta}{2}\right)^{1/2+\lambda_+}\left(\cos\frac{\vartheta}{2}\right)^{1/2+\lambda_-}P_l^{(\lambda_+,\lambda_-)}\times(\cos\vartheta), \tag{4.25}$$

with the Coulomb wave functions $S_N(\tau;R)$, $S_p(\tau;R)$ as in (3.64) and (3.66) and the energy spectra (3.65) and (3.27), respectively. In the limit $R\to\infty$ the flat-space limit is recovered.[33,56,57]

4.1.3. Generalized radial potential

We consider the potential $(k_{0,1,2,3}>0)$

$$V_7(\mathbf{u})=F(u_1^2+u_2^2+u_3^2)+\frac{\hbar^2}{2M}\left(-\frac{k_0^2-\frac14}{u_0^2}+\frac{k_1^2-\frac14}{u_1^2}+\frac{k_2^2-\frac14}{u_2^2}+\frac{k_3^2-\frac14}{u_3^2}\right), \tag{4.26}$$

which in the two separating coordinate systems has the form

Spherical $(\tau>0,\ \vartheta\in(0,\pi/2),\ \varphi\in(0,\pi/2))$:

$$V_7(\mathbf{u})=F(\sinh^2\tau)+\frac{\hbar^2}{2MR^2}\left[\frac{1}{\sinh^2\tau}\left(\frac{1}{\sin^2\vartheta}\left(\frac{k_1^2-\frac14}{\cos^2\varphi}+\frac{k_2^2-\frac14}{\sin^2\varphi}\right)+\frac{k_3^2-\frac14}{\cos^2\vartheta}\right)-\frac{k_0^2-\frac14}{\cosh^2\tau}\right]. \tag{4.27}$$

Sphero-elliptic $(\tau>0,\ \tilde\alpha\in(0,K),\ \tilde\beta\in(0,K'))$:

$$=F(\sinh^2\tau)+\frac{\hbar^2}{2MR^2}\left(\frac{1}{\sinh^2\tau}\left(\frac{k_1^2-\frac14}{\text{sn}^2\tilde\alpha\text{dn}^2\tilde\beta}+\frac{k_2^2-\frac14}{\text{cn}^2\tilde\alpha\text{cn}^2\tilde\beta}+\frac{k_3^2-\frac14}{\text{dn}^2\tilde\alpha\text{sn}^2\tilde\beta}\right)-\frac{k_0^2-\frac14}{\cosh^2\tau}\right). \tag{4.28}$$

This potential is the analog of the minimally superintegrable potential $V_1(\mathbf{x})$ in $\mathbb{R}^3$ (Ref. 33). We have the following two path-integral representations for $K^{(V_7)}(\mathbf{u}'',\mathbf{u}';T)$:

Sphero-elliptic [$\lambda_2=2(m+l)\pm k_3+\lambda_1+2$ (l,h as in Sec. 3.1.2)]:

$$=\frac{e^{i\hbar T/2MR^2}}{R^3}\int_{\tau(t')=\tau'}^{\tau(t'')=\tau''}\mathscr{D}\tau(t)\sinh^2\tau\int_{\tilde{\alpha}(t')=\tilde{\alpha}'}^{\tilde{\alpha}(t'')=\tilde{\alpha}''}\mathscr{D}\tilde{\alpha}(t)$$

$$\times\int_{\tilde{\beta}(t')=\tilde{\beta}'}^{\tilde{\beta}(t'')=\tilde{\beta}''}\mathscr{D}\tilde{\beta}(t)(k^2\,\mathrm{cn}^2\tilde{\alpha}$$

$$+k'^2\,\mathrm{cn}^2\tilde{\beta})\exp\left\{\frac{i}{\hbar}\int_{t'}^{t''}\left[\frac{M}{2}R^2(\dot{\tau}^2+\sinh^2\tau(k^2\,\mathrm{cn}^2\tilde{\alpha}\right.\right.$$

$$+k'^2\,\mathrm{cn}^2\tilde{\beta})(\dot{\tilde{\alpha}}^2+\dot{\tilde{\beta}}^2))-F(\tau)-\frac{\hbar^2}{2MR^2}\left(-\frac{k_0^2-\frac{1}{4}}{\cosh^2\tau}\right.$$

$$\left.\left.\left.+\frac{1}{\sinh^2\tau}\left(\frac{k_1^2-\frac{1}{4}}{\mathrm{sn}^2\tilde{\alpha}\,\mathrm{dn}^2\tilde{\beta}}+\frac{k_2^2-\frac{1}{4}}{\mathrm{cn}^2\tilde{\alpha}\,\mathrm{cn}^2\tilde{\beta}}+\frac{k_3^2-\frac{1}{4}}{\mathrm{dn}^2\tilde{\alpha}\,\mathrm{sn}^2\tilde{\beta}}\right)\right)\right]\mathrm{d}t\right\}$$

$$=(R^2\sinh^2\tau'\sinh^2\tau''\,\mathrm{sn}\tilde{\alpha}'\,\mathrm{cn}\tilde{\alpha}'\,\mathrm{dn}\tilde{\alpha}'\,\mathrm{sn}\tilde{\beta}'\,\mathrm{cn}\tilde{\beta}'$$

$$\times\mathrm{dn}\tilde{\beta}'\,\mathrm{sn}\tilde{\alpha}''\,\mathrm{cn}\tilde{\alpha}''\,\mathrm{dn}\tilde{\alpha}''\,\mathrm{sn}\tilde{\beta}''\,\mathrm{cn}\tilde{\beta}''\,\mathrm{dn}\tilde{\beta}'')^{-1/2} \tag{4.29}$$

$$\times\sum_{lm}\Xi_{lm}^{(\pm k_1,\pm k_2,\pm k_3)}(\tilde{\alpha}'',\tilde{\beta}')\Xi_{lm}^{(\pm k_1,\pm k_2,\pm k_3)*}$$

$$\times(\tilde{\alpha}',\tilde{\beta}')K_{lm}^{(V_7)}(\tau'',\tau';T). \tag{4.30}$$

Spherical ($\lambda_1=2m\mp k_1\mp k_2+1,\ \lambda_2=2l\mp k_3+\lambda_1+1$):

$$=\frac{e^{-i\hbar T/2MR^2}}{R^3}\int_{\tau(t')=\tau'}^{\tau(t'')=\tau''}\mathscr{D}\tau(t)\sinh^2\tau\int_{\vartheta(t')=\vartheta'}^{\vartheta(t'')=\vartheta''}\mathscr{D}\vartheta(t)$$

$$\times\sin\vartheta\int_{\varphi(t')=\varphi'}^{\varphi(t'')=\varphi''}\mathscr{D}\varphi(t)\exp\left\{\frac{i}{\hbar}\int_{t'}^{t''}\left[\frac{M}{2}R^2(\dot{\tau}^2\right.\right.$$

$$+\sinh^2\tau(\dot{\vartheta}^2+\sin^2\vartheta\dot{\varphi}^2))-F(\tau)$$

$$-\frac{\hbar^2}{2MR^2}\left(\frac{1}{\sinh^2\tau}\left(\frac{1}{\sin^2\vartheta}\left(\frac{k_1^2-\frac{1}{4}}{\cos^2\varphi}+\frac{k_2^2-\frac{1}{4}}{\sin^2\varphi}-\frac{1}{4}\right)\right.\right.$$

$$\left.\left.\left.\left.+\frac{k_3^2-\frac{1}{4}}{\cos^2\vartheta}-\frac{1}{4}\right)-\frac{k_0^2-\frac{1}{4}}{\cosh^2\tau}\right)\right]dt\right\} \tag{4.31}$$

$$=\frac{e^{-i\hbar T/2MR^2}}{R}(\sinh^2\tau'\sinh^2\tau''\sin\vartheta'\sin\vartheta'')^{-1/2}$$

$$\times\sum_{m=0}^{\infty}\phi_m^{(\pm k_2,\pm k_1)}(\varphi'')\phi_m^{(\pm k_2,\pm k_1)}(\varphi')\sum_{l=0}^{\infty}\phi_l^{(\lambda_1,\pm k_3)}$$

$$\times(\vartheta'')\phi_l^{(\lambda_1,\pm k_3)}(\vartheta')K_{lm}^{(V_7)}(\tau'',\tau';T), \tag{4.32}$$

with the remaining path integral $K_{lm}^{(V_7)}(T)$,

$$K_{lm}^{(V_7)}(\tau'',\tau';T)=\int_{\tau(t')=\tau'}^{\tau(t'')=\tau''}\mathscr{D}\tau(t)\exp\left\{\frac{i}{\hbar}\int_{t'}^{t''}\left[\frac{M}{2}R^2\dot{\tau}^2\right.\right.$$

$$-F(\tau)-\frac{\hbar^2}{2MR^2}\left(\frac{\lambda_2^2-\frac{1}{4}}{\sinh^2\tau}\right.$$

$$\left.\left.\left.-\frac{k_0^2-\frac{1}{4}}{\cosh^2\tau}\right)\right]dt\right\}. \tag{4.33}$$

The wave functions $\phi_m^{(\pm k_2,\pm k_1)}(\varphi)$ and $\phi_l^{(\lambda_1,\pm k_3)}(\vartheta)$ are the same as in (3.21) and (3.34), respectively. This path integral cannot be further specified until $F(\tau)\equiv F(\sinh^2\tau)$ is known. The special case $F\equiv 0$ is trivial.

4.1.4. Analog of the Holt potential

The potential V_8 can be considered as an analog of the minimally superintegrable Holt potential $V_6(\mathbf{x})$ in $\mathbb{R}^3$ (Refs. 33 and 34) ($\alpha,\lambda,\omega>0$):

$$V_8(\mathbf{u})=\frac{\hbar^2}{2M}\frac{k_2^2-\frac{1}{4}}{u_2^2}+\frac{\alpha}{(u_0-u_3)^2}+\frac{M}{2}\omega^2\frac{R^2+4u_1^2+u_2^2}{(u_0-u_3)^4}$$

$$-\frac{\lambda u_1}{(u_0-u_3)^2}. \tag{4.34}$$

In the two separating coordinate systems it has the form

Horicyclic ($x_2,y>0,\ x_1\in\mathbb{R}$):

$$V_8(\mathbf{u})=\frac{y^2}{R^2}\left[\alpha+\frac{M}{2}\omega^2(4x_1^2+x_2^2+y^2)-\lambda x_1\right]$$

$$+y^2\frac{\hbar^2}{2MR^2}\frac{k_2^2-\frac{1}{4}}{x_2^2}. \tag{4.35}$$

Semicircular-parabolic ($\xi,\eta,\varrho>0$):

$$=\frac{\xi^2\eta^2}{R^2}\left[\frac{\alpha(\xi^2+\eta^2)-\frac{\lambda}{2}(\eta^4-\xi^4)+\frac{M}{2}\omega^2(\xi^6+\eta^6)}{\xi^2+\eta^2}\right.$$

$$\left.+\left(\frac{M}{2}\omega^2\varrho^2+\frac{\hbar^2}{2M}\frac{k_2^2-\frac{1}{4}}{\varrho^2}\right)\right]. \tag{4.36}$$

The effect of the x_2 and ϱ path integration in both cases ($x_2,\varrho>0$) is that in separating the corresponding variable the quantity α is shifted by the additional quantum numbers. The resulting path integrals in the variables (y,x_1) and (ξ,η) separate, but only the former can be evaluated. Indeed, we have already solved almost the same path-integral problem in Ref. 35. The solution in horicyclic coordinates then has the following structure ($z=x_1-\lambda/4M\omega^2$):

$$K^{(V_8)}(\mathbf{u}'',\mathbf{u}';T)$$

$$=\frac{1}{R^3}\int_{y(t')=y'}^{y(t'')=y''}\frac{\mathscr{D}y(t)}{y^3}\int_{x(t')=x'}^{x(t'')=x''}\mathscr{D}x(t)$$

$$\times\exp\left\{\frac{i}{\hbar}\int_{t'}^{t''}\left[\frac{M}{2}R^2\frac{\dot{\mathbf{x}}^2+\dot{y}^2}{y^2}-\frac{y^2}{R^2}\left(\alpha+\frac{M}{2}\omega^2\right.\right.\right.$$

$$\left.\left.\left.\times(4x_1^2+x_2^2+y^2)-\lambda x_1\right)-\frac{y^2\hbar^2}{2MR^2}\frac{k_2^2-\frac{1}{4}}{x_2^2}\right]dt\right\} \tag{4.37}$$

$$=\frac{2M\omega}{\hbar R}\sqrt{x_2'x_2''}\sum_{l\in\mathbb{N}_0}\frac{l!}{\Gamma(l\pm k_2+1)}$$

$$\times\left(\frac{M\omega}{\hbar}x_2'x_2''\right)^{\pm k_2}\exp\left(-\frac{M\omega}{2\hbar}(x_2'^2+x_2''^2)\right)L_l^{(\pm k_2)}$$

$$\times\left(\frac{M\omega}{\hbar}x_2'^2\right)L_n^{(\pm k_2)}\left(\frac{M\omega}{\hbar}x_2''^2\right)\sum_{m\in\mathbb{N}_0}\left(\frac{2M\omega}{\pi\hbar}\right)^{1/2}$$

$$\times\frac{1}{2^m m!}H_m\left(\sqrt{\frac{2M\omega}{\hbar}}z'\right)H_m\left(\sqrt{\frac{2M\omega}{\hbar}}z''\right)$$

$$\times\exp\left[-\frac{M\omega}{\hbar}(z'^2+z''^2)\right]\int_{y(t')=y'}^{y(t'')=y''}\frac{\mathscr{D}y(t)}{y}$$

$$\times\exp\left[\frac{iM}{2\hbar}\int_{t'}^{t''}\left(R^2\frac{\dot{x}^2+\dot{y}^2}{y^2}-\frac{y^2}{R^2}\right.\right.$$

$$\left.\left.\times(E_{\alpha,\omega,\lambda}+\omega^2y^2)\right)dt\right],\tag{4.38}$$

where $E_{\alpha,\omega,\lambda}$ is given by

$$E_{\alpha,\omega,\lambda}=\alpha+\hbar\omega(2m+2l\pm k_2+2)-\frac{\lambda^2}{8M\omega^2}.\tag{4.39}$$

A path integral like this was calculated in Ref. 26; here we must distinguish two cases, first, where $E_{\alpha,\omega,\lambda}>0$ and, second, where $E_{\alpha,\omega,\lambda}<0$. In the first case only a continuous spectrum occurs, whereas in the second case bound states can exist, with the number of levels given by $n=0,1,\ldots,N_n=[E_{\alpha,\omega,\lambda}/2\hbar\omega-1/2]$. According to Ref. 35, we therefore obtain the following path-integral solution for $V_8(\mathbf{u})$ in horicyclic coordinates ($\nu=-i\sqrt{2MR^2E/\hbar^2-1/4}$):

$$K^{(V_8)}(\mathbf{u}'',\mathbf{u}';T)$$

$$=\frac{1}{R}\sum_{n=0}^{\infty}\psi_n(x_2')\psi_n(x_2'')\sum_{m=0}^{\infty}\psi_m(x_1')\psi_m(x_1'')$$

$$\times\int_{\mathbb{R}}\frac{dE}{2\pi i}\,e^{-iET/\hbar}\frac{\Gamma\left[\frac{1}{2}(1+\nu+E_{\alpha,\omega,\lambda}/\hbar\omega)\right]}{\sqrt{y'y''}\hbar\omega\Gamma(1+\nu)}$$

$$\times\times W_{-E_{\alpha,\omega,\lambda}/2\hbar\omega,\nu/2}\left(\frac{M\omega}{\hbar}y_>^2\right)M_{-E_{\alpha,\omega,\lambda}/2\hbar\omega,\nu/2}$$

$$\times\left(\frac{M\omega}{\hbar}y_<^2\right)\tag{4.40}$$

$$=\sum_{l,m=0}^{\infty}\left[\sum_{n=0}^{N_n}\Psi_{nlm}^{(V_8)}(x_1'',x_2'',y'';R)\Psi_{nlm}^{(V_8)}\right.$$

$$\times(x_1',x_2',y';R)e^{-iE_nT/\hbar}+\int_0^\infty dp\,\Psi_{plm}^{(V_8)}$$

$$\left.\times(x_1'',x_2'',y'';R)\Psi_{plm}^{(V_8)*}(x_1',x_2',y';R)e^{-iE_pT/\hbar}\right].\tag{4.41}$$

The bound-state wave functions have the form

$$\Psi_{nmq}^{(V_8)}(x_1,x_2,y;R)=\psi_n(y;R)\psi_m(x_1)\psi_l(x_2),\tag{4.42}$$

where

$$\psi_n(y;R)=\sqrt{\frac{2n!(|E_{\alpha,\omega,\lambda}|/\hbar\omega-2n-1)y}{R^3\Gamma(|E_{\alpha,\omega,\lambda}|/\hbar\omega-n)}}$$

$$\times\left(\frac{M\omega}{\hbar}y^2\right)^{|E_{\alpha,\omega,\lambda}|/2\hbar\omega-n-1/2}$$

$$\times\exp\left(-\frac{M\omega}{2\hbar}y^2\right)L_n^{(|E_{\alpha,\omega,\lambda}|/\hbar\omega-2n-1)}$$

$$\times\left(\frac{M\omega}{\hbar}y^2\right),\tag{4.43}$$

$$\psi_m(x_1)=\left(\frac{2M\omega}{\pi\hbar 2^{2m}(m!)^2}\right)^{1/4}H_m\left(\sqrt{\frac{2M\omega}{\hbar}}\right.$$

$$\left.\times\left(x_1-\frac{\lambda}{8\omega^2}\right)\right)\exp\left(-\frac{M\omega}{\hbar}\left(x_1-\frac{\lambda}{8\omega^2}\right)^2\right),\tag{4.44}$$

$$\psi_l(x_2)=\sqrt{\frac{2M\omega}{\hbar}\frac{l!}{\Gamma(l\pm k_2+1)}}x_2\left(\frac{M\omega}{\hbar}x_2^2\right)^{\pm k_2/2}$$

$$\times\exp\left(-\frac{M\omega}{2\hbar}x_2^2\right)L_l^{(\pm k_2)}\left(\frac{M\omega}{\hbar}x_2^2\right),\tag{4.45}$$

with the discrete energy spectrum

$$E_n=\frac{\hbar^2}{8MR^2}-\frac{\hbar^2}{2MR^2}\left(\frac{|E_{\alpha,\omega,\lambda}|}{\hbar\omega}-2n-1\right)^2.\tag{4.46}$$

The continuum wave functions and the energy spectrum have the form

$$\Psi_{plm}^{(V_3)}(x_1,x_2,y;R)=\psi_p(y;R)\psi_m(x_1)\psi_l(x_2),\tag{4.47}$$

$$\psi_p(y;R)=\sqrt{\frac{\hbar}{M\omega}\frac{p\sinh\pi p}{2\pi^2R^3y}}\Gamma\left[\frac{1}{2}\left(1+ip\right.\right.$$

$$\left.\left.+\frac{E_{\alpha,\omega,\lambda}}{\hbar\omega}\right)\right]W_{-E_{\alpha,\omega,\lambda}/2\hbar\omega,ip/2}\left(\frac{M\omega}{\hbar}y^2\right),\tag{4.48}$$

$$E_p=\frac{\hbar^2}{2MR^2}\left(p^2+\frac{1}{4}\right),\tag{4.49}$$

and the $\psi_m(x_1),\psi_l(x_2)$ are the same as in (4.44) and (4.45).

4.2. Minimally superintegrable potentials from the group chains SO(3,1)⊃E(2), SO(3,1)⊃SO(3), and SO(3,1)⊃SO(2,1)

While in the previous section we presented the minimally superintegrable potentials which are the analogs of the $\mathbb{R}^3$ case, there are several potentials which emerge from the group structure of the three-dimensional hyperboloid. They are as follows:

1. There are four potentials which emerge from SO(3,1)⊃E(2), i.e., the four maximally superintegrable potentials in $\mathbb{R}^2$ are contained as minimally superintegrable potentials on $\Lambda^{(3)}$. The four potentials in $\mathbb{R}^2$ are the oscillator, the Holt potential, the Coulomb potential, and a modified Coulomb potential.

2. There are two potentials which emerge from the group chain SO(3,1)⊃SO(3), i.e., the Higgs oscillator and the Coulomb potential on the two-dimensional sphere. However, the case of the Higgs oscillator is already contained in the potential V_7, i.e., a generalized radial potential.

3. There are five potentials which emerge from the group chain SO(3,1)⊃SO(2,1), and all superintegrable potentials

TABLE IV. Minimally superintegrable potentials on $\Lambda^{(3)}$ from the group chain $SO(3,1)\supset E(2)$.

Potential $V(\mathbf{u})$	Coordinate systems	Observables
$V_9(\mathbf{u})=F(u_0-u_3)+\frac{M}{2}\omega^2\frac{u_1^2+u_2^2}{(u_0-u_3)^4}+\frac{\hbar^2}{2M}\left(\frac{k_1^2-\frac14}{u_1^2}+\frac{k_2^2-\frac14}{u_2^2}\right)$ $P_{x_i}=\frac{\hbar}{\mathrm{i}}\frac{\partial}{\partial x_i},\quad i=1,2$	Horicyclic Horicyclic-cylindrical Horicyclic-elliptic	$I_1=\frac{1}{2MR^2}(\mathbf{K}^2-\mathbf{L}^2)+V_9(\mathbf{u})$ $I_2=\frac{1}{2M}P_{x_1}^2+\frac{M}{2}\omega^2x_1^2+\frac{\hbar^2}{2M}\frac{k_1^2-\frac14}{x_1^2}$ $I_3=\frac{1}{2M}P_{x_2}^2+\frac{M}{2}\omega^2x_2^2+\frac{\hbar^2}{2M}\frac{k_2^2-\frac14}{x_2^2}$ $I_4=\frac{1}{2M}L_3^2+\frac{\hbar^2}{2M}\left(\frac{k_1^2-\frac14}{\cos^2\varphi}+\frac{k_2^2-\frac14}{\sin^2\varphi}\right)$
$V_{10}(\mathbf{u})=F(u_0-u_3)+\frac{M}{2}\omega^2\frac{4u_1^2+u_2^2}{(u_0-u_3)^4}+\frac{k_1u_1}{(u_0-u_3)^3}+\frac{\hbar^2}{2M}\frac{k_2^2-\frac14}{u_2^2}$ $P_{x_i}=\frac{\hbar}{\mathrm{i}}\frac{\partial}{\partial x_i},\quad i=1,2$	Horicyclic Horicyclic-parabolic	$I_1=\frac{1}{2MR^2}(\mathbf{K}^2-\mathbf{L}^2)+V_{10}(\mathbf{u})$ $I_2=\frac{1}{2M}P_{x_1}^2+2M\omega^2x_1^2+k_1x_1$ $I_3=\frac{1}{2M}P_{x_2}^2+\frac{M}{2}\omega^2x_2^2+\frac{\hbar^2}{2M}\frac{k_2^2-\frac14}{x_2^2}$ $I_4=\frac{1}{4M}\{L_3,K_1+L_2\}+\frac{M\omega^2(\xi^6+\eta^6)+2k_1(\xi^4-\eta^4)+\hbar^2\left(k_2^2-\frac14\right)(1/\xi^2+1/\eta^2)/M}{2(\xi^2+\eta^2)}$
$V_{11}(\mathbf{u})=F(u_0-u_3)-\frac{\alpha}{\sqrt{u_1^2+u_2^2}}\frac{1}{u_0-u_3}+\frac{R^2\hbar^2}{4M}\frac{1}{(u_0-u_3)^2}\frac{1}{\sqrt{u_1^2+u_2^2}}\times\left(\frac{k_1^2-\frac14}{\sqrt{u_1^2+u_2^2}+u_1}+\frac{k_2^2-\frac14}{\sqrt{u_1^2+u_2^2}-u_1}\right)$	Horicyclic-cylindrical Horicyclic-elliptic II Horicyclic-parabolic	$I_1=\frac{1}{2MR^2}(\mathbf{K}^2-\mathbf{L}^2)+V_{11}(\mathbf{u})$ $I_2=\frac{1}{2M}L_3^2+\frac{\hbar^2}{8M}\left(\frac{k_1^2-\frac14}{\cos^2(\varphi/2)}+\frac{k_2^2-\frac14}{\sin^2(\varphi/2)}\right)$ $I_3=\frac{1}{2M}[(K_1+L_2)^2+(K_2-L_1)^2]-\frac{\alpha}{\varrho}+\frac{\hbar^2}{8M\varrho^2}\left(\frac{k_1^2-\frac14}{\cos^2(\varphi/2)}+\frac{k_2^2-\frac14}{\sin^2(\varphi/2)}\right)$ $I_4=\frac{1}{4M}\{L_3,P_2\}+\frac{1}{\xi\eta}\left[-\alpha(\xi-\eta)+\left(\frac12-2k_1^2\right)\frac{\eta}{\xi}+\left(2k_2^2-\frac12\right)\frac{\xi}{\eta}\right]$
$V_{12}(\mathbf{u})=F(u_0-u_3)-\frac{\alpha}{\sqrt{u_1^2+u_2^2}}\frac{1}{u_0-u_3}+\frac{\beta_1\sqrt{\sqrt{u_1^2+u_2^2}+u_1}+\beta_2\sqrt{\sqrt{u_1^2+u_2^2}-u_1}}{2(u_0-u_3)^{3/2}\sqrt{u_1^2+u_2^2}}$ $P_y=\frac{\hbar}{\mathrm{i}}\frac{\partial}{\partial y}$	Horicyclic-mutually parabolic	$I_1=\frac{1}{2MR^2}(\mathbf{K}^2-\mathbf{L}^2)+V_{12}(\mathbf{u})$ $I_2=\frac{1}{2M}P_y^2+F(R/y)$ $I_3=\frac{1}{4M}\{L_3,P_1\}-\frac{\alpha(\lambda-\mu)+\beta_1\mu\sqrt{\lambda}-\beta_2\lambda\sqrt{\mu}}{\lambda+\mu}$ $I_4=\frac{1}{4M}\{L_3,P_2\}-\frac{\alpha(\xi-\eta)+(\beta_1+\beta_2)\eta\sqrt{\xi/2}-(\beta_1+\beta_2)\xi\sqrt{\eta/2}}{\xi+\eta}$

on $\Lambda^{(2)}$ are minimally superintegrable on $\Lambda^{(3)}$; these include the Higgs oscillator and the Coulomb potential.

In all cases the (path-integral) solution is very easy. First, one can separate the underlying, two-dimensional, superintegrable, potential term. In the first case, the remaining path integral is a path integral in the horicyclic variable y in the Poincaré upper half-plane $\mathscr{H}$, say, and in the second and third cases one is left, for instance, with a modified Pöschl–Teller or Rosen–Morse path integral. The specific form depends, of course, on the remaining "hyperbolic radial" potential, which can be chosen arbitrarily. It must be noted that in the cases of the Coulomb potential on $S^{(2)}$ and $\Lambda^{(2)}$ the problem of self-adjoint continuation arises from the negative bound states; this problem will be discussed in detail elsewhere.

The following tables summarize our determination of the minimally superintegrable potentials on $\Lambda^{(3)}$ due to the group structure of SO(3,1). We omit any details concerning the solution, and the interested reader is invited to consult Refs. 33–35 to check the solutions of the corresponding two-dimensional systems, in order to obtain the solution on $\Lambda^{(3)}$.

TABLE V. Minimally superintegrable potentials on $\Lambda^{(3)}$ from the group chain SO(3,1)⊃SO(3).

Potential $V(\mathbf{u})$	Coordinate systems	Observables
$V_7'(\mathbf{u})=F(u_0)+\frac{M}{2}\frac{u_1^2+u_2^2}{u_1^2+u_2^2+u_3^2}\frac{\omega^2}{u_3^2}+\frac{\hbar^2}{2M}\left(\frac{k_1^2-\frac{1}{4}}{u_1^2}+\frac{k_2^2-\frac{1}{4}}{u_2^2}\right)$	Sphero-elliptic	$I_1=\frac{1}{2MR^2}(\mathbf{K}^2-\mathbf{L}^2)+V_7'(\mathbf{u})$
$\lambda^2=\frac{M^2\omega^2}{\hbar^2}R^4+\frac{1}{4}$	Spherical	$I_2=\frac{1}{2M}\mathbf{L}^2+\frac{\hbar^2}{2M}\left(\frac{1}{\sin^2\vartheta}\left(\frac{k_1^2-\frac{1}{4}}{\cos^2\varphi}+\frac{k_2^2-\frac{1}{4}}{\sin^2\varphi}\right)+\frac{k_3^2-\frac{1}{4}}{\cos^2\vartheta}\right)$
		$I_3=\frac{1}{2M}L_3^2+\left(\frac{k_1^2-\frac{1}{4}}{\cos^2\varphi}+\frac{k_2^2-\frac{1}{4}}{\sin^2\varphi}\right)$
		$I_4=\frac{1}{2M}(L_1^2+k'^2L_2^2)-\frac{\hbar^2}{2M(k^2\mathrm{cn}^2\tilde{\alpha}+k'^2\mathrm{cn}^2\tilde{\beta})}\left[\left(k_1^2-\frac{1}{4}\right)\left(\frac{1}{\mathrm{sn}^2\tilde{\alpha}}-\frac{k^2}{\mathrm{dn}^2\tilde{\beta}}\right)+\left(k_2^2-\frac{1}{4}\right)\left(\frac{k'^2}{\mathrm{cn}^2\tilde{\alpha}}-\frac{k^2}{\mathrm{dn}^2\tilde{\beta}}\right)-\left(\lambda^2-\frac{1}{4}\right)\left(\frac{k'^2}{\mathrm{dn}^2\tilde{\alpha}}-\frac{1}{\mathrm{sn}^2\tilde{\beta}}\right)\right]$
$V_{13}(\mathbf{u})=F(u_0)-\frac{\alpha}{u_1^2+u_2^2+u_3^2}\frac{u_3}{\sqrt{u_1^2+u_2^2}}$	Sphero-elliptic*	$I_1=\frac{1}{2MR^2}(\mathbf{K}^2-\mathbf{L}^2)+V_{13}(\mathbf{u})$
$+\frac{\hbar^2}{4M\sqrt{u_1^2+u_2^2}}\frac{k_1^2-\frac{1}{4}}{\sqrt{u_1^2+u_2^2}+u_1}+\frac{k_2^2-\frac{1}{4}}{\sqrt{u_1^2+u_2^2}-u_1}$	Spherical	$I_2=\frac{1}{2M}L_3^2+\frac{\hbar^2}{8M}\left(\frac{k_1^2-\frac{1}{4}}{\sin^2(\varphi/2)}+\frac{k_2^2-\frac{1}{4}}{\cos^2(\varphi/2)}\right)$
		$I_3=\frac{1}{2M}\mathbf{L}^2-\alpha\cot\vartheta+\frac{\hbar^2}{8M\sin^2\vartheta}\left(\frac{k_1^2-\frac{1}{4}}{\sin^2(\varphi/2)}+\frac{k_2^2-\frac{1}{4}}{\cos^2(\varphi/2)}\right)$
		$I_4=\frac{1}{2M}\left(\frac{1}{2}\sin 2f\{L_1,L_3\}-\cos 2fL_3^2\right)-\alpha\frac{k^2k'\mathrm{sn}^2\tilde{\alpha}\,\mathrm{sn}\tilde{\beta}\,\mathrm{dn}\tilde{\beta}-(k^2+k'^2\mathrm{cn}^2\tilde{\beta})k\,\mathrm{sn}\,\tilde{\alpha}\,\mathrm{dn}\tilde{\alpha}}{k^2\mathrm{cn}^2\tilde{\alpha}+k'^2\mathrm{cn}^2\tilde{\beta}}+\frac{\hbar^2}{2M(k^2\mathrm{cn}^2\tilde{\alpha}+k'^2\mathrm{cn}^2\tilde{\beta})}\times\left(\frac{\left(k_1^2+k_2^2-\frac{1}{2}\right)k'^2(k_2^2-k_1^2)k'\,\mathrm{sn}\tilde{\alpha}\,\mathrm{dn}\tilde{\alpha}}{\mathrm{cn}^2\tilde{\alpha}}+\frac{\left(k_1^2+k_2^2-\frac{1}{2}\right)k^2+(k_2^2-k_1^2)k\,\mathrm{sn}\tilde{\beta}\,\mathrm{dn}\tilde{\beta}}{\mathrm{cn}^2\tilde{\beta}}\right)$

*After an appropriate rotation, $\sin^2 f=k^2$.

5. SUMMARY AND DISCUSSION

The purpose of this paper has been to present a comprehensive discussion of superintegrable potentials on the three-dimensional hyperboloid $\Lambda^{(3)}$. It has included an enumeration of the coordinate systems on $\Lambda^{(3)}$ as known from the literature, a systematic search for maximally and minimally superintegrable potentials by appropriate generalizations from Euclidean space, a statement of the constants of the motion (respectively, operators), and in the soluble cases an evaluation of the corresponding path-integral representation in order to find the quantum-mechanical propagators, the Green's functions, the discrete and continuum wave functions, and the energy spectra.

In the enumeration of the 34 coordinate systems in Sec. 2 we have followed Refs. 52 and 75, supplemented by the corresponding Hamiltonian and the form of the corresponding separable potential, and several rotated coordinate systems, i.e., the sphero-elliptic rotated, the equidistant-elliptic rotated, and the prolate-elliptic rotated. These rotated systems correspond in their respective flat-space limits to sphero-conical II, cylindrical-elliptic II, and prolate-spheroidal II coordinate systems, which in turn contain as additional degenerate systems the respective parabolic systems. However, for the complicated two-parameter systems XXIX–XXIV hardly any statement and usage could have been made.

In Sec. 3 we presented our results concerning the maximally superintegrable potentials on $\Lambda^{(3)}$. These are the (generalized) Higgs oscillator $V_1(\mathbf{u})$, the (generalized) Coulomb potential $V_2(\mathbf{u})$, and a specific scattering potential $V_3(\mathbf{u})$. The potential $V_4(\mathbf{u})$, which is only minimally superintegrable on $\Lambda^{(3)}$, has been included in this section because its flat-space analog in $\mathbb{R}^3$ is maximally superintegrable.

The Higgs oscillator and the Coulomb potential have been discussed in some detail, first for the pure oscillator and Coulomb case, and second with additional centrifugal terms

TABLE VI. Minimally superintegrable potentials on $\Lambda^{(3)}$ from the group chain SO(3,1)⊃SO(2,1).

Potential $V(\mathbf{u})$	Coordinate systems	Observables
$V_{14}(\mathbf{u})=F(u_3)+\frac{M}{2}\frac{\omega^2}{u_0^2-u_1^2-u_2^2}\frac{u_1^2+u_2^2}{u_0^2}$	Equidistant-elliptic Equidistant-hyperbolic	$I_1=\frac{1}{2MR^2}(\mathbf{K}^2-\mathbf{L}^2)+V_{14}(\mathbf{u}),\quad I_2=K_1^2+K_2^2-L_3^2$
$+\frac{\hbar^2}{2M}\left(\frac{k_1^2-\frac14}{u_1^2}+\frac{k_2^2-\frac14}{u_2^2}\right)$	Equidistant-cylindrical	$I_3=\frac{1}{2M}L_3^2+\frac{\hbar^2}{2M}\left(\frac{k_1^2-\frac14}{\cos^2\varphi}+\frac{k_2^2-\frac14}{\sin^2\varphi}\right)$
	Equidistant	$I_4=\frac{1}{2M}K_2^2-\frac{M}{2}\frac{\omega^2R^4}{\cosh^2\tau_3}+\frac{\hbar^2}{2M}\frac{k_1^2-\frac14}{\sinh^2\tau_3}$
$V_{15}(\mathbf{u})=F(u_3)=-\frac{\alpha}{u_0^2-u_1^2-u_2^2}\left(\frac{u_0}{\sqrt{u_1^2+u_2^2}}-1\right)$ $+\frac{\hbar^2}{4M\sqrt{u_1^2+u_2^2}}\left(\frac{k_1^2-\frac14}{\sqrt{u_1^2+u_2^2}+u_1}\right.$ $\left.+\frac{k_2^2-\frac14}{\sqrt{u_1^2+u_2^2}-u_1}\right)$	Equidistant-elliptic*	$I_1=\frac{1}{2MR^2}(\mathbf{K}^2-\mathbf{L}^2)+V_{15}(\mathbf{u}),\quad I_2=K_1^2+K_2^2-L_3^2$
	Equidistant-semihyperbolic	$I_3=\frac{1}{2M}L_3^2+\frac{\hbar^2}{8M}\left(\frac{k_1^2-\frac14}{\sin^2(\varphi/2)}+\frac{k_2^2-\frac14}{\cos^2(\varphi/2)}\right)$
	Equidistant-elliptic-parabolic Equidistant-cylindrical	$I_4=\frac{1}{4M}\{K_1,L_3\}-\alpha R\frac{\sqrt{1+\mu_1}+\sqrt{1+\mu_2}}{\mu_1+\mu_2}+\frac{\hbar^2}{4M}\frac{1}{\mu_1+\mu_2}$ $\times\left[\left(k_1^2+k_2^2-\frac12\right)\left(\frac{\mu_1}{\mu_2}+\frac{\mu_2}{\mu_1}\right)+(k_1^2-k_2^2)\left(\frac{\sqrt{1+\mu_1}}{\mu_1}-\frac{\sqrt{1+\mu_2}}{\mu_2}\right)\right]$
$V_{16}(\mathbf{u})=F(u_3)+\frac{\alpha}{(u_0-u_1)^2}$ $+\frac{M}{2}\omega^2\frac{u_0^2-u_1^2+3u_2^2}{(u_0-u_1)^4}-\lambda\frac{u_2}{(u_0-u_1)^3}$	Equidistant-semicircular-parabolic	$I_1=\frac{1}{2MR^2}(\mathbf{K}^2-\mathbf{L}^2)+V_{16}(\mathbf{u}),\quad I_2=K_1^2+K_2^2-L_3^2$
	Equidistant-horicyclic	$I_3=\frac{1}{2M}(K_1-L_3)^2+\alpha+2M\omega^2x^2-\lambda x.$ $I_4=\frac{1}{4M}(\{(K_1,K_2)-\{K_2,L_3\})$ $+\frac{\xi^2n^2}{\xi^2+\eta^2}\left[\alpha(\xi^2+\eta^2)+\frac{\lambda}{2}(\xi^2-\eta^4)+\frac{M}{2}\omega^2(\xi^6+\eta^6)\right]$
$V_{17}(\mathbf{u})=F(u_3)+\frac{M}{2}\frac{\omega^2}{(u_0-u_1)^2}+\frac{\hbar^2}{2M}\frac{k_2^2-\frac14}{u_2^2}$	Equidistant-elliptic-parabolic Equidistant-hyperbolic-parabolic	$I_1=\frac{1}{2MR^2}(\mathbf{K}^2-\mathbf{L}^2)+V_{17}(\mathbf{u}),\quad I_2=K_1^2+K_2^2-L_3^2$
	Equidistant-semicircular-parabolic	$I_3=\frac{1}{2M}(K_1-L_3)^2+\frac{\hbar^2}{2M}\frac{x^2-\frac14}{x^2}$
	Equidistant Equidistant-horicyclic	$I_4=\frac{1}{2M}K_2^2+\frac{M}{2}\omega^2e^{2\tau_3}$
$V_{18}(\mathbf{u})=F(u_3)+\frac{\alpha}{u_0^2-u_1^2-u_2^2}\frac{u_2}{\sqrt{u_0^2-u_1^2}}$	Equidistant-semicircular-parabolic	$I_1=\frac{1}{2MR^2}(\mathbf{K}^2-\mathbf{L}^2)+V_{18}(\mathbf{u}),\quad I_2=K_1^2+K_2^2-L_3^2$ $I_3=\frac{1}{4M}(\{K_1,K_2\}-\{K_2,L_3\})+\alpha R\frac{\xi^2\eta^2}{\xi^2+\eta^2}\left(\frac{1}{\xi^2}-\frac{1}{\eta^2}\right)$
	Equidistant	$I_4=K_2^2$

*After an appropriate rotation, $\sin^2 f=k^2$.

which do not spoil the property of maxima superintegrability, as in the corresponding cases in $\mathbb{R}^3$ and on $S^{(3)}$. The energy spectrum and degeneracy of the levels was also discussed.

In Sec. 4 we discussed the minimally superintegrable potentials on $\Lambda^{(3)}$. We have found the four analogs of the flat-space case, in particular, the ring-shaped oscillator, the Hartmann potential, a radial potential, and the Holt potential. The remaining minimally superintegrable potentials emerged from the subgroup structure of SO(3,1), i.e., we had to take into account the group chains SO(3,1)⊃$E(2)$, SO(3,1)⊃SO(3), SO(3,1)⊃SO(2,1), which gave rise to four, one, and five new minimally superintegrable potentials, respectively. In total, we have found 15 minimally superintegrable potentials on $\Lambda^{(3)}$. Whereas we have treated the ring-shaped oscillator and the Hartmann-potential in some detail, the discussion for the other potentials has been mostly rather sketchy because the underlying, superintegrable, two-dimensional systems have already been solved in previous publications.

We have therefore continued the study of superinte-

TABLE VII. Correspondence of the maximally superintegrable potentials in three dimensions.

$V_{\Lambda^{(3)}}(\mathbf{u})$	# Systems	$V_{\mathbb{R}^3}(\mathbf{x})$	# Systems	$V_{S^{(3)}}(\mathbf{s})$	# Ssytems
$V_1(\mathbf{u})$	14(8)	$V_1(\mathbf{x})$	8	$V_1(\mathbf{s})$	6(8)
$V_2(\mathbf{u})$	5(4)	$V_3(\mathbf{x})$	4	$V_2(\mathbf{s})$	3(4)
$V_3(\mathbf{u})$	5(4)	$V_4(\mathbf{x})$	4	$V_3(\mathbf{s})$	2(2) [3(4)]
$V_4(\mathbf{u})$	4(4)	$V_5(\mathbf{x})$	4	-	1(1)
$V_{19}(\mathbf{u})$	1(2)	$V_2(\mathbf{x})$	4	-	1(1)

grable systems in spaces of constant curvature. Furthermore, we would like to draw attention to the following observations:

1. Let us consider the potential V_{19} in semihyperbolic coordinates:

$$\begin{aligned} V_{19}(\mathbf{u}) &= \frac{M}{2}\,\omega^2\left(\frac{4u_0^2u_3^2}{R^2}+u_1^2+u_2^2\right)+2k_3u_0u_3 \\ &\quad+\frac{\hbar^2}{2M}\left(\frac{k_1^2-\frac{1}{4}}{u_1^2}+\frac{k_2^2-\frac{1}{4}}{u_2^2}\right) \\ &= \frac{R^2}{\mu_1+\mu_2}\left[\frac{M}{2}\,\omega^2(\mu_1^3+\mu_2^3)+k_3(\mu_1^2-\mu_2^2)\right] \\ &\quad+\frac{\hbar^2}{2MR^2}\frac{1}{\mu_1\mu_2}\left(\frac{k_1^2-\frac{1}{4}}{\cos^2\varphi}+\frac{k_2^2-\frac{1}{4}}{\sin^2\varphi}\right), \end{aligned} \tag{5.1}$$

and the potential is separable in this system. It has two flat-space limits in its full range of parameters, i.e., without the restrictions we made in Sec. 2, namely, circular polar and parabolic coordinates. While it was possible to simply state the potential $V_4(\mathbf{u})$, no explicit solution could have been found. This Stark-effect-like potential could be of some interest, in particular, in comparison with the potential $V_{19}(\mathbf{u})$. The potential in this limit corresponds to the second maximally superintegrable potential $V_2(\mathbf{x})$ of Ref. 33, and the limiting case separates in these two coordinate systems, and also in the Cartesian and circular-elliptic system.

2. Let us consider the potential V_{20} in semihyperbolic coordinates:

$$\begin{aligned} V_{20}(\mathbf{u}) &= \frac{M}{2}\,\omega^2\left(\frac{4u_0^2u_3^2}{R^2}+u_1^2+u_2^2\right)+\frac{\hbar^2}{2M}\frac{F(u_2/u_1)}{u_1^2+u_2^2} \\ &= \frac{R^2}{\mu_1+\mu_2}\frac{M}{2}\,\omega^2(\mu_1^3+\mu_2^3)+\frac{\hbar^2}{2MR^2}\frac{F(\tan\varphi)}{\mu_1\mu_2}, \end{aligned} \tag{5.2}$$

and the potential is separable in this system. The potential in the flat-space limit corresponds to the sixth minimally superintegrable potential $V_6(\mathbf{x})$ of Ref. 33, and the limiting case separates in circular-polar and parabolic coordinates.

3. The previous observations allow the following statement: We have found all five analogs of the maximally superintegrable potentials in $\mathbb{R}^3$, where we have the identification given in Table VII (the enumeration of the potentials in $\mathbb{R}^3$ is given according to Ref. 33, and the enumeration of the potentials on $S^{(3)}$ according to Ref. 34). In parentheses we have indicated the number of limiting coordinate systems as $R\to\infty$. Note that for $V_3(\mathbf{s})$ we have two separating coordinate systems. For $V_3(\mathbf{s})$ we have also indicated the additional coordinate system, which emerges and causes an additional observable if $k_3^2-1/4=0$. From the rotated sphero-elliptic system on $S^{(3)}$ two coordinate systems on $\mathbb{R}^3$ can be obtained by means of contraction as $R\to\infty$: the cylindrical-elliptic II and the cylindrical-parabolic systems. Note also that $V_4(\mathbf{u})$ is only minimally superintegrable, but not maximally superintegrable. Furthermore, in Ref. 34 several potential systems were overlooked. However, the additional systems turn out to be only integrable but not superintegrable.

4. The linear potential on the hyperboloid seems to have a structure according to u_0u_3, which turns out to be separable in an appropriately chosen parabolic coordinate system. However, on spaces of constant (nonvanishing) curvature, there seems to be no analog of a Cartesian coordinate system which separates these kinds of potentials as well.

5. The coordinate systems XXX–XXXIII separate a radial potential according to $V(u_2,u_3)\propto\alpha/u_2^2+\beta/u_3^2$, and XXXIV separate a potential according to $V(u_3)\propto\beta/u_3^2$, but these are trivial and not very interesting.

6. Let us finally note another application of the prolate elliptic coordinate system. It has the property that it separates the two-center Coulomb problem on the hyperboloid, just as the prolate elliptic system on the sphere separates the two-center Coulomb problem on $S^{(3)}$ (Refs. 6 and 76). Let us consider two point charges located at $u_{1,2}=(1,0,0,\pm k')/k$ on the hyperboloid. Then it is not difficult to show by means of the prolate elliptic coordinate system that one has in algebraic form ($Z_\pm=Z_1\pm Z_2$)

$$\begin{aligned} &V(u_1,u_2,u) \\ &\quad=-Z_1\frac{u_1\cdot u}{\sqrt{(u_1\cdot u)^2-1}}-Z_2\frac{u_2\cdot u}{\sqrt{(u_2\cdot u)^2-1}} \\ &\quad=-\frac{Z_+\sqrt{(\varrho_1-a_2)(\varrho_1-a_3)}-Z_-\sqrt{(\varrho_2-a_2)(\varrho_2-a_3)}}{\varrho_1-\varrho_2}. \end{aligned} \tag{5.3}$$

A detailed investigation of this problem will be presented elsewhere.[36]

We cannot say with certainty whether we have really found all possible superintegrable potentials on the hyperboloid. For a systematic search one must solve differential equations which emerge from the general form of a potential separable in a particular coordinate system, changing the variables. Because there are 34 coordinate systems on the hyperboloid which separate the Schrödinger equation, there are $33!\simeq 8.7\times10^{36}$ such differential equations. This is not tractable, and one must seek alternative procedures, in particular, physical arguments. In this respect, we have found the relevant potentials which matter from a physical point of view, and which are the analogs of the flat-space limit in $\mathbb{R}^3$, including the corresponding coordinate systems.

In summary, we have enumerated and classified superintegrable systems in spaces of constant (positive, zero, or negative) curvature. Further studies along these lines could include the investigation of the corresponding interbasis ex-

TABLE VIII. Correspondence of the minimally superintegrable potentials in three dimensions.

$V_{\Lambda^{(3)}}(\mathbf{u})$	# Systems	$V_{\mathbb{R}^3}(\mathbf{x})$	# Systems	$V_{S^{(3)}}(\mathbf{s})$	# Systems
Analogs of flat space					
$V_5(\mathbf{u})$	5(3)	$V_5(\mathbf{x})$	4	$V_4(\mathbf{s})$	4(4)
$V_6(\mathbf{u})$	3(4)	$V_7(\mathbf{x})$	3	$V_5(\mathbf{s})$	2(3)
$V_7(\mathbf{u})$	2(2)	$V_1(\mathbf{x})$	2	$V_6(\mathbf{s})$	2(2)
$V_8(\mathbf{u})$	2(1)	$V_3(\mathbf{x})$	2	-	1(1)
$V_{20}(\mathbf{u})$	1(2)	$V_6(\mathbf{x})$	2	-	1(2)
Potentials emerging from SO(3,1)$\supset E(2)$					
$V_9(\mathbf{u})$	3(3)	$V_2(\mathbf{x})$	3	-	-
V_9'	7(3)	$\frac{\hbar^2}{2M}\left(\frac{k_1^2-\frac{1}{4}}{x^2}+\frac{k_2^2-\frac{1}{4}}{y^2}\right)+F(z)$	3	-	-
$V_{10}(\mathbf{u})$	2(2)	$V_3(\mathbf{x})$	2	-	-
$V_{11}(\mathbf{u})$	3(3)	$V_4(\mathbf{x})$	3	-	-
$V_{12}(\mathbf{u})$	2(2)	$V_8(\mathbf{x})$	2	-	-
Potentials emerging from SO(3,1)$\supset$SO(3)					
$V_7'(\mathbf{u})$	2(2)	$V_1(\mathbf{x})$	2	$V_6(\mathbf{s})$	2(2)
$V_{13}(\mathbf{u})$	2(2)	$V_9(\mathbf{x})$	2	$V_7(\mathbf{s})$	2(2)
Potentials emerging from SO(3,1)$\supset$SO(2,1)					
$V_{14}(\mathbf{u})$	4(3)	$V_2(\mathbf{x})$	3	-	-
$V_{15}(\mathbf{u})$	4(3)	$V_4(\mathbf{x})$	3	-	-
$V_{16}(\mathbf{u})$	2(1)	$V_3(\mathbf{x})$	2	-	-
$V_{17}(\mathbf{u})$	5(2)	$\frac{\hbar^2}{2M}\frac{k_2^2-\frac{1}{4}}{x^2}+F(z)$	2	-	-
$V_{18}(\mathbf{u})$	2(1)	$\alpha x+F(z)$	2	-	-

pansions, the contraction of the wave functions in the curved spaces with respect to their Euclidean flat-space limit, their pseudo-Euclidean flat-space limit, and the solutions of the various superintegrable potentials in the generic (respectively, parametric) coordinate systems.[37] Among the latter, the most important cases are the Coulomb problems, for instance, the Coulomb problem in $\Lambda^{(2)}$ or $\Lambda^{(3)}$ in semihyperbolic coordinates, and the investigation of the Stark effect in spaces of constant curvature, which includes the solution of the corresponding Schrödinger equations. We hope to return to these issues in the future.

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APPENDIX A. PATH-INTEGRAL IDENTITY FOR THE PÖSCHL–TELLER POTENTIAL

As we shall see, we encounter, particularly in the case of the Higgs oscillator, the Pöschl–Teller and the modified Pöschl–Teller potential in our path-integral problems. The path-integral solution of the Pöschl–Teller potential reads as follows (Böhm and Junker,[7] Duru,[14] Refs. 31, 39, and 40, Fischer *et al.*,[18] Inomata *et al.*,[48] and Kleinert and Mustapic[60] for $0<x<\pi/2$):

$$\int_{x(t')=x'}^{x(t'')=x''}\mathscr{D}x(t)\exp\left\{\frac{i}{\hbar}\int_{t'}^{t''}\left[\frac{M}{2}\dot{x}^2-\frac{\hbar^2}{2M}\left(\frac{\alpha^2-\frac{1}{4}}{\sin^2 x}+\frac{\beta^2-\frac{1}{4}}{\cos^2 x}\right)\right]dt\right\}=\sum_{n\in\mathbb{N}_0}e^{-iE_nT/\hbar}\times\phi_n^{(\alpha,\beta)}(x')\phi_n^{(\alpha,\beta)}(x'') \tag{A1}$$

$$=\int_{\mathbb{R}}\frac{dE}{2\pi i}e^{-iET/\hbar}G_{\mathrm{PT}}^{(\alpha,\beta)}(x'',x';E). \tag{A2}$$

The bound-state wave functions and the energy spectrum are given by

$$\phi_n^{(\alpha,\beta)}(x)=\left[2(\alpha+\beta+2n+1)\frac{n!\Gamma(\alpha+\beta+n+1)}{\Gamma(\alpha+n+1)\Gamma(\beta+n+1)}\right]^{1/2}\times(\sin x)^{\alpha+1/2}(\cos x)^{\beta+1/2}P_n^{(\alpha,\beta)}(\cos 2x), \tag{A3}$$

$$E_n=\frac{\hbar^2}{2M}(2n+\alpha+\beta+1)^2. \tag{A4}$$

The $P_n^{(\alpha,\beta)}$ are Jacobi polynomials. The Pöschl–Teller wave functions $\phi^{(\alpha,\beta)}(x)$ are normalized to unity with respect to the scalar product $\int_0^{\pi/2}|\phi_n^{(\alpha,\beta)}(x)|^2dx=1$. The Green's function $G_{\mathrm{PT}}^{(\alpha,\beta)}(E)$ is

$$G_{\mathrm{PT}}^{(\alpha,\beta)}(x'',x';E)=\frac{M}{2\hbar^2}\sqrt{\sin x'\sin x''}\times\frac{\Gamma(m_1-L_E)\Gamma(L_E+m_1+1)}{\Gamma(m_1+m_2+1)\Gamma(m_1-m_2+1)}\times\left(\frac{1-\cos 2x'}{2}\cdot\frac{1-\cos 2x''}{2}\right)^{(m_1-m_2)/2}$$

$$\times\left(\frac{1+\cos 2x'}{2}\cdot\frac{1+\cos 2x''}{2}\right)^{(m_1+m_2)/2}$$

$$\times{}_2F_1\left(-L_E+m_1,L_E+m_1+1;m_1-m_2+1;\frac{1-\cos 2x_<}{2}\right){}_2F_1\left(-L_E+m_1,L_E+m_1+1;m_1+m_2+1;\frac{1+\cos 2x_>}{2}\right),\tag{A5}$$

where $m_{1,2}=\frac{1}{2}(\beta\pm\alpha)$, $L_E=-\frac{1}{2}+\frac{1}{2}\sqrt{2ME}/\hbar$, ${}_2F_1(a,b;c;z)$ is the hypergeometric function, and $x_>,x_<$ denote the larger (respectively, smaller) of x',x''.

APPENDIX B. PATH-INTEGRAL IDENTITY FOR THE MODIFIED PÖSCHL–TELLER POTENTIAL

The case of the modified Pöschl–Teller potential was considered in Refs. 7, 19, 31, 39, 40, 48, and 60:

$$\int_{r(t')=r'}^{r(t'')=r''}\mathcal{D}r(t)\exp\left\{\frac{i}{\hbar}\int_{t'}^{t''}\left[\frac{M}{2}\dot{r}^2-\frac{\hbar^2}{2M}\left(\frac{\kappa^2-\frac{1}{4}}{\sinh^2 r}-\frac{\lambda^2-\frac{1}{4}}{\cosh^2 r}\right)\right]dt\right\}=\sum_{n=0}^{N_{\max}}e^{-iE_nT/\hbar}\psi_n^{(\kappa,\lambda)*}(r')\psi_n^{(\kappa,\lambda)}(r'')+\int_0^\infty dp\,e^{-iE_pT/\hbar}\psi_p^{(\kappa,\lambda)*}(r')\psi_p^{(\kappa,\lambda)}(r'')\tag{B6}$$

$$=\int_{\mathbb{R}}\frac{dE}{2\pi i}e^{-iET/\hbar}G_{m\mathrm{PT}}^{(\kappa,\lambda)}(r'',r';E).\tag{B7}$$

The bound states have the form

$$\psi_n^{(\kappa,\lambda)}(r)=N_n^{(\kappa,\lambda)}(\sinh r)^{\kappa+1/2}(\cosh r)^{n-\lambda+1/2}{}_2F_1(-n,\lambda-n;1+\kappa;\tanh^2 r),\tag{B8}$$

$$N_n^{(\kappa,\lambda)}=\frac{1}{\Gamma(1+\kappa)}\left[\frac{2(\lambda-\kappa-2n-1)\Gamma(n+1+\kappa)\Gamma(\lambda-n)}{\Gamma(\lambda-\kappa-n)n!}\right]^{1/2},$$

$$E_n=-\frac{\hbar^2}{2M}(2n+\kappa-\lambda+1)^2.\tag{B9}$$

Here $n=0,1,\ldots,N_{\max}=[\frac{1}{2}(\lambda-\kappa-1)]\geq 0$, and only a finite number of bound states can exist, depending on the strength of the attractive potential trough and the repulsive centrifugal term as well. Here $[x]$ denotes the integer part of the real number x. The continuum states are

$$\psi_p^{(\kappa,\lambda)}(r)=N_p^{(\kappa,\lambda)}(\cosh r)^{ip}(\tanh r)^{\kappa+1/2}{}_2F_1\left(\frac{\lambda+\kappa+1-ip}{2},\frac{\kappa-\lambda+1-ip}{2};1+\kappa;\tanh^2 r\right),$$

$$N_p^{(\kappa,\lambda)}=\frac{1}{\Gamma(1+\kappa)}\sqrt{\frac{p\sinh\pi p}{2\pi^2}}\,\Gamma\left(\frac{\lambda+\kappa+1-ip}{2}\right)\times\Gamma\left(\frac{\kappa-\lambda+1-ip}{2}\right).\tag{B10}$$

The Green's function $G_{m\mathrm{PT}}^{(\kappa,\lambda)}(E)$ is

$$G_{m\mathrm{PT}}^{(\kappa,\lambda)}(r'',r';E)=\frac{M}{2\hbar^2}\frac{\Gamma(m_1-L_\lambda)\Gamma(L_\lambda+m_1+1)}{\Gamma(m_1+m_2+1)\Gamma(m_1-m_2+1)}\times(\cosh r'\cosh r'')^{-(m_1-m_2)}\times(\tanh r'\tanh r'')^{m_1+m_2+1/2}{}_2F_1\left(-L_\lambda+m_1,L_\lambda+m_1+1;m_1-m_2+1;\frac{1}{\cosh^2 r_<}\right){}_2F_1(-L_\lambda+m_1,L_\lambda+m_1+1;m_1+m_2+1;\tanh^2 r_>),\tag{B11}$$

where we have set $m_{1,2}=\frac{1}{2}(\kappa\pm\sqrt{-2ME}/\hbar)$, and $L_\lambda=\frac{1}{2}(\lambda-1)$. We make extensive use of the solutions of the Pöschl–Teller potential and the modified Pöschl–Teller potential.

[1)] However, they are periodic globally if the frequencies $\omega_{1,2,3}$ are commensurable, i.e., if their respective quotients are rational numbers.

[2)] In the sequel we use the notions of "energy levels" and "periodicity of closed orbits," "observables" and "constants of the motion," "Coulomb-" or "Kepler-problem," referring to quantum-mechanical or classical-mechanical properties, respectively, as synonymous.

[3)] The notion "quasiperiodic" means that they are periodic in each coordinate, but not necessarily periodic in a global way. They are periodic globally if the respective periods are commensurable.

[4)] The notion of minimally superintegrable systems in two dimensions does not make sense, because the number of integrals of motion is two, and is thus equal to the number of integrals of motion required for the system to be separable at all.

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