Darboux transformation of the Schrödinger equation

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The recent developments in the theory of the generation of potentials for which the Schrödinger equation has an exact solution are discussed. The generalization of the Darboux transformation to the nonstationary Schrödinger equation is studied in detail. The supersymmetric generalization of the nonstationary Schrödinger equation is formulated. Versions corresponding to exact and spontaneously broken supersymmetry are discussed. New, exactly solvable nonstationary potentials are obtained as examples. The stationary Darboux transformation is viewed as a special case of the new transformation. Families of isospectral potentials with the spectra of the harmonic oscillator and the hydrogen-like atom are obtained. The effectiveness of these methods for describing the coherent states of the transformed Hamiltonians is demonstrated. © *1997 American Institute of Physics.* [S1063-7796(97)00304-5]

1. INTRODUCTION

One of the basic techniques for solving differential equations, that of reducing them to equations whose solutions are known, has a long history. It probably began with the work of Kummer[1] and Liouville,[2] who solved the problem of reducing a second-order differential equation with variable coefficients to an equation of previously specified form. The transformation which they used, which from that time has been known as the Kummer–Liouville transformation, is the most general transformation which preserves the linearity and order of the equation[3] (see also Ref. 4). A modern, systematic discussion of the results in this area can be found in Ref. 4. The application of this transformation to the Schrödinger equation involves some special features. The most complete study of Schrödinger equations reduced to the hypergeometric equation can be found in Ref. 5.

Transformations which raise the order of an equation have been studied by Moutard[6] and Imshenetskiĭ.[7] The first systematic study of such transformations was undertaken by Darboux,[8] and so these transformations have been named after him. They were subsequently rediscovered many times. For example, the factorization method, which became well known after the studies by Schrödinger[9] and has been studied systematically by Infeld and Hull,[10,11] is essentially another formulation of the Darboux transformation (see the discussion in Ref. 12). Supersymmetric quantum mechanics, first introduced by Witten,[13] uses transformations which intertwine the components of the super-Hamiltonian, and these are Darboux transformations of the original Schrödinger equation.[14] Mention should also be made of the Abraham–Moses–Pursey[15–17] integral transformations, which represent an application of the methods of inverse-scattering theory[18] for generating exactly solvable potentials of the one-dimensional stationary Schrödinger equation. In the most important and almost unique case where it is possible to obtain a closed-form solution of the Gel'fand–Levitan–Marchenko equations[19,20] (the case of degenerate kernel), these transformations are equivalent to a special case of the generalized Darboux transformations.[12,21,22]

Owing to this fact, a number of the results obtained by the inverse-scattering method can be reproduced by using Darboux transformations. This approach has certain advantages arising from the differential nature of the transformation. Especially attractive is the idea of using qualitative methods of controlling spectra, developed in Ref. 23 on the basis of inverse-scattering methods, together with Darboux transformations, which allow solutions of the transformed Schrödinger equation to be obtained fairly simply.

The method of Darboux transformations is undergoing active development at the present time. A hyperbolic equation with two independent variables is studied in Ref. 24, and a second-order Darboux transformation for the one- and two-dimensional stationary Schrödinger equations is studied in detail in Refs. 25 and 26. References 27–29 are devoted to the Darboux transformations associated with the scattering problem in quantum mechanics. References 30–32 study the development of this method for the one-dimensional nonstationary Schrödinger equation, which was first introduced in Ref. 33 in connection with the solution of the Kadomtsev–Petviashvili equation (see also Ref. 34). Let us consider these questions in more detail.

2. DARBOUX TRANSFORMATION OF THE NONSTATIONARY SCHRÖDINGER EQUATION

Let us consider the Schrödinger equation for a particle with potential energy $-V_0(x,t)$:

$$(i\partial_t - h_0)\psi(x,t)=0, \quad h_0=-\partial_x^2-V_0(x,t),$$
$$\partial_t=\partial/\partial t, \quad \partial_x^2=\partial_x\cdot\partial_x, \quad x\in[a,b]. \tag{1}$$

We shall say that the operator $\hat{L}$ defined on solutions of Eq. (1) is a transformation operator for this equation if it satisfies the operator equation

$$\hat{L}(i\partial_t - h_0)=(i\partial_t - h_1)\hat{L}, \tag{2}$$

where $h_1=-\partial_x^2-V_1(x,t)$ is the transformed Hamiltonian. The operator $\hat{L}$ defined by (2) transforms every solution $\psi(x,t)$ of Eq. (1) into a solution $\varphi(x,t)=\hat{L}\psi(x,t)$ of the transformed Schrödinger equation:

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$$(i\partial_t - h_1)\varphi(x,t) = 0, \quad x \in [a,b]. \tag{3}$$

We use T_0 and T_1 to denote the linear spaces of solutions of Eqs. (1) and (3), respectively. If the conjugation operator [denoted by the symbol $^+$], which possesses the property $(AB)^+ = B^+A^+$ and under which the Schrödinger operators $i\partial_t - h_{0,1}$ are self-adjoint, $(i\partial_t - h_{0,1})^+ = i\partial_t - h_{0,1}$, is defined for the operators acting in the spaces T_0 and T_1, then it follows from (2) that

$$(i\partial_t - h_0)\hat{L}^+ = \hat{L}^+(i\partial_t - h_1). \tag{4}$$

In this case the operator $\hat{L}^+$ effects a transformation in the reverse direction, i.e., from solutions of (3) to solutions of (1). The product of operators $\hat{L}^+\hat{L}$ will obviously transform the solutions of (1) into other solutions of (1) and will therefore be a symmetry operator of this equation. In complete analogy, $\hat{L}\hat{L}^+$ will be a symmetry operator of Eq. (3).

If $\hat{L}$ is a differential operator in the variables x and t, then by solving (2) in the space T_0 we can replace $i\partial_t$ in it by h_0. Therefore, the operator $\hat{L}$ can safely be assumed to be a differential operator in the variable x whose coefficients are functions of the variables x and t.

A differential operator of order N in the variable x with coefficients depending on x and t which satisfies Eq. (2) will be called an *Nth-order Darboux transformation operator*. For $N=1$ we shall simply call it a *Darboux transformation operator*.

The class of operators defined only by Eq. (2) is larger than the class of operators transforming one solution of the Kadomtsev–Petviashvili equation

$$\partial_x(V_t + 6VV_x + V_{xxx}) = 3V_{yy} \tag{5}$$

into another solution, which were studied earlier in Refs. 33 and 34. This fact can be explained as follows. Equation (5) is the compatibility condition for two linear differential equations (the Lax representation; see, for example, Ref. 35),

$$B^{(0)}\psi(x,y,t) = 0, \quad B^{(0)} = i\partial_y + \partial_x^2 + V^{(0)},$$
$$A^{(0)}\psi(x,y,t) = 0, \quad A^{(0)} = \partial_t + 4\partial_x^3 + 6V^{(0)}\partial_x + 3V^{(0)} + 3iw^{(0)}, \tag{6}$$

imposed on the function $V^{(0)} = V^{(0)}(x,y,t)$. Equation (6) is a nonstationary Schrödinger equation in which the variable y plays the role of the time. If now, following Zakharov and Shabat,[36] we find the operator L, which is involved simultaneously in two coupled equations

$$LB^{(0)} = B^{(1)}L,$$
$$LA^{(0)} = A^{(1)}L, \tag{7}$$

where $B^{(1)}$ and $A^{(1)}$ differ from $B^{(0)}$ and $A^{(0)}$ only by the replacement of $V^{(0)}$ by $V^{(1)}$ and $w^{(0)}$ by $w^{(1)}$, then the system of equations

$$B^{(1)}\varphi = 0, \quad A^{(1)}\varphi = 0, \quad \varphi = L\psi$$

will be compatible and the function $V = V^{(1)}(x,y,t)$ will be a solution of (5). From this it is clear that if we reject the condition (7), we enlarge the class of operators L.

2.1. The first-order transformation

Let us first consider the case $N=1$: $\hat{L} = L_0(x,t) + L_1(x,t)\partial_x$. Assuming that differentiation operators of different orders are linearly independent, from Eq. (2) we obtain a system of equations for the function L_0, L_1 and the potential difference $A(x,t) = V_1(x,t) - V_0(x,t)$:

$$L_{1x} = 0, \quad iL_{1t} 2L_{0x} = -AL_1,$$
$$L_1V_{0x} - iL_{0t} - L_{0xx} = AL_0. \tag{8}$$

When there is no possibility of confusion, the arguments of the functions will not be indicated in order to simplify the notation.

Introducing the new function $u(x,t)$ by means of the equation $u_x/u = -L_0/L_1$, we can integrate the system (8) one time. For the function u we then obtain the equation

$$iu_t + u_{xx} + (V_0 - C(t))u = 0.$$

The function $L_1 = L_1(t)$ remains arbitrary, and the ratio $r = L_0/L_1$ is independent of $C(t)$ and, therefore, can be calculated for $C(t) \equiv 0$. In this case the function u will be a solution of Eq. (1). For the potential difference, the system (8) leads to the expression $A = i(\ln L_1)_t + 2(\ln u)_{xx}$. The arbitrariness in the choice of the function $L_1(t)$ can be used to ensure reality of the potential difference A. The reality condition, written as $i(\ln|L_1|^2)_t = 2(\bar{r}_x - r_x)$ (the bar denotes complex conjugation), leads to the following condition for the function u:

$$(\ln u/\bar{u})_{xxx} = 0. \tag{9}$$

If the function L_1 is taken to be real, it is determined by the function u:

$$L_1 = \exp\left[2\int dt\ \mathrm{Im}(\ln u)_{xx}\right], \tag{10}$$

and for the potential difference we obtain the expression usual for a Darboux transformation:

$$A = (\ln|u|^2)_{xx}. \tag{11}$$

Finally, for the Darboux transformation operator we find

$$\hat{L} = L_1(-u_x/u + \partial_x) = L_1 u^{-1}\begin{vmatrix} u & 1 \\ u_x & \partial_x \end{vmatrix}. \tag{12}$$

Here and below, operator determinants will be treated as differential operators which are obtained in the expansion of the determinant with respect to the last column with the functional coefficients written in front of the differentiation operators. We stress the fact that the function u satisfying the original Schrödinger equation completely determines the transformation operator (12) and the potential difference (11). We shall call the system of functions completely determining the transformation operator and the transformed Schrödinger equation the *transformation functions*. The function u is therefore the transformation function for the first-order Darboux transformation operator.

It follows from (12) that $\hat{L}u = 0$. Nevertheless, the solution $\psi = u$ of the original Schrödinger equation can be associated with a nontrivial solution of the new equation. By

direct verification we find that when the condition (9) is satisfied and the function $L_1(t)$ is chosen in the form (10), the solution of (3) will be

$$v=1/[L_1(t)\bar{u}]. \tag{13}$$

The condition (9) represents an auxiliary condition on the solution $\psi=u$ of the Schrödinger equation (1). Therefore, in general there may exist potentials $V^0(x,t)$ for which the Schrödinger equation does not have any solution satisfying this condition. However, it is possible to write down the general form of the potential $V=V^{(0)}$ for which the Schrödinger equation has at least one solution satisfying the condition (9). For this we seek the function u in the form $u=R\exp(i\Phi)$, $\Phi=x^2\dot{f}/(2f)+x\ddot{s}/(2f^2)+\alpha$, where the arbitrary real-valued functions f, s, and α are independent of the variable x and the dot denotes a derivative with respect to t. Substituting $\psi=u$ into the original equation (1), we obtain a system of equations for the functions R and Φ. The first one determines the general form of the function R: $R=fP(\xi)$, $\xi=f^2x+s$, where $P(\xi)$ is an arbitrary real-valued function of a single variable, and the second determines the potential V:

$$V=\frac{3\dot{f}^2-f\ddot{f}}{2f^2}x^2+\frac{4\dot{f}\dot{s}-f\ddot{s}}{2f^3}x+\frac{\dot{s}^2}{4f^4}-\dot{\alpha}-f^4\frac{P''}{P}, \tag{14}$$

where $P''=d^2P/d\xi^2$.

In the special case where the arbitrary functions are chosen as

$$s=-x_0f^2,\quad f^2=[\gamma_0^2(t-t_0)^2+1]^{-1/2},$$

$$\dot{\alpha}=(\log f)_{tt}x_0^2/2,$$

where x_0, t_0, and γ_0 are arbitrary real constants, and the function P is taken to be the real solution of the equation

$$P''+(U(\xi)-\gamma_0^2\xi^2/4)P=0,$$

where $U(\xi)$ is an arbitrary real-valued function of a single variable, the potential V takes the form

$$V=\frac{1}{\gamma^2(t-t_0)^2+1}U\left(\frac{x-x_0}{[\gamma^2(t-t_0)^2+1]^{1/2}}\right). \tag{15}$$

This is the potential that was obtained in Ref. 37 as the potential admitting the existence of new real solutions of the Kadomtsev–Petviashvili equation (5). We stress the fact that the Schrödinger equation with a potential of the form (14) or (15) can have solutions which do not satisfy the condition (9).

2.2. Higher-order transformations

The corresponding system of equations for the second-order Darboux transformation operator has a more complicated form. In contrast to the previous case, here the system cannot be integrated completely. It was shown in Ref. 30 that the equations for the coefficients of the transformation operator admit a solution which can be written compactly as

$$\hat{L}_{02}=L_{02}(t)W^{-1}(u_1,u_2)\begin{vmatrix} u_1 & u_2 & 1\\ u_{1x} & u_{2x} & \partial_x\\ u_{1xx} & u_{2xx} & \partial_x^2\end{vmatrix}, \tag{16}$$

where u_1 and u_2 are arbitrary linearly independent solutions of the original Schrödinger equation. We use $W(u_1,u_2,\ldots,u_N)$ to denote the Wronskian of the functions u_1, $u_2,\ldots,u_N$.

When the condition

$$\left[\ln\frac{W(u_1,u_2)}{\bar{W}(u_1,u_2)}\right]_{xxx}=0 \tag{17}$$

is satisfied, the function $L_{02}(t)$ can be chosen to be real,

$$L_{02}(t)=\exp\left(2\int dt\ \mathrm{Im}[\ln W(u_1,u_2)]_{xx}\right), \tag{18}$$

and the difference between the potentials of the new and the original Schrödinger equation will be a real-valued function:

$$A_{02}=[\ln|W(u_1,u_2)|^2]_{xx}. \tag{19}$$

Using the transformation functions u_1 and u_2, we can perform two successive Darboux transformations determined by the operators $\hat{L}_{01}$ and $\hat{L}_{12}$. The transformation function for the second transformation will be the function into which u_2 is transformed after the first transformation, i.e.,

$$v_2=\hat{L}_{01}u_2=L_{01}(t)(-u_2u_{1x}/u_1+u_{2x}). \tag{20}$$

We use v_1 to denote the solution of the new Schrödinger equation corresponding to the transformation function u_1: $v_1=1/[L_{01}(t)\bar{u}_1]$. We assume that the transformation function u_1 satisfies the condition (9), and the function $L_{01}(t)$ is calculated using (10). The solutions of the intermediate Schrödinger equation, $\chi(x,t)$, with potential differing from that of (1) by the function A_{01} calculated from (11) with u replaced by u_1, will be found by using the transformation operator (12):

$$\chi=\hat{L}_{01}\psi=L_{01}(t)(-\psi u_{1x}/u_1+\psi_x). \tag{21}$$

The repeated Darboux transformation $\hat{L}_{12}$ performed on the solutions χ (21) has the same form as (12). We obtain

$$\varphi=\hat{L}_{12}\chi=L_{12}(t)(-\chi v_{2x}/v_2+\chi_x). \tag{22}$$

The function (22) is a solution of the Schrödinger equation with potential differing from that of the preceding equation by the function

$$A_{12}=-i(\ln L_{12})_t+2(\ln v_2)_{xx}.$$

Using the fact that the solutions v_2 and χ are expressed in terms of the solutions u_2 and ψ of the original equation by Eqs. (20) and (21), we can completely eliminate the solutions of the intermediate Schrödinger equation and express the solution φ (22) of the new equation in terms of the solution of Eq. (1). Then for φ we obtain the expression

$$\varphi=\hat{L}_{02}\psi=\hat{L}_{12}\hat{L}_{01}\psi=L_{02}(t)W(u_1,u_2,\psi)/W(u_1,u_2), \tag{23}$$

where $L_{02}(t)=L_{01}(t)L_{12}(t)$. The potential difference for the final and original equations will have the form

$$A_{02}=A_{01}+A_{12}=-i(\ln L_{02})_t+2[\ln W(u_1,u_2)]_{xx}. \quad (24)$$

The arbitrariness in the choice of the function $L_{02}(t)$ can be used to require that the function A_{02} be real. The reality condition has the form (17), and if the function L_{02} is chosen according to (18), the expression for the potential difference (24) coincides with (19), and the function φ (23) is obtained by applying the operator (16) to the solution ψ of Eq. (1). From this it is clear that Eqs. (16) and (19) give a realization of the second-order Darboux transformation operator in the form of a product of first-order Darboux transformation operators.

It follows from (23) that for $\psi=u_1$, u_2 the function φ vanishes. However, it can be shown by direct calculation that if the function L_{02} is calculated using (18), there are two linearly independent solutions φ_1 and φ_2 of the Schrödinger equation with the potential

$$V_2=V_0+A_{02}=V_0+i(\ln L_{02})_t+2[\ln \bar{W}(u_1,u_2)]_{xx}$$
$$=V_0+[\ln|W(u_1,u_2)|^2]_{xx},$$

$$\varphi_1=\frac{\bar{u}_1}{L_{02}\bar{W}(u_1,u_2)}, \quad \varphi_2=\frac{\bar{u}_2}{L_{02}\bar{W}(u_1,u_2)}. \quad (25)$$

We note that the solutions u_1 and u_2 of the original equation (1) completely determine the transformation operator (16) and the potential difference A_{02}, and are therefore the transformation functions for the second-order Darboux transformation operator $\hat{L}_{02}=\hat{L}_{12}\hat{L}_{01}$.

If we are not interested in the intermediate Schrödinger equation, it is meaningless to require that its potential be real. The only restriction on the transformation functions u_1 and u_2 will be the condition (17). If we do need the intermediate Schrödinger equation determined by the transformation function u_1, then this function must satisfy the condition (9).

It is important that, since the solution found for the coefficients of the second-order transformation operator is not a general solution of the corresponding system of differential equations, the question of whether or not every differential operator of a second-order transformation admits factorization by first-order transformation operators remains open. The analogous problem can be solved completely for the stationary Schrödinger equation. We shall discuss this later in Sec. 3.1.

Equations (16)–(19) have an obvious generalization to a chain of N Darboux transformations. For this it is sufficient to have N linearly independent solutions of Eq. (1), u_1, u_2, ..., u_N, to replace the second-order Wronskian by the Nth-order one, $W(u_1,u_2)\to W(u_1,u_2,\ldots,u_N)$, and to replace the third-order operator determinant in (16) by the corresponding $(N+1)$th-order determinant. If, in addition, it is necessary to know all the intermediate Schrödinger equations and their solutions, then the condition (17) must be imposed on all the Wronskians $W(u_1,u_2,\ldots,u_k)$, $k=1,2,\ldots,N$, assuming that $W(u)\equiv u$.

We also note that in addition to the reality condition, it is reasonable to require that the potential differences $A_{pq}(x)$ $(p<q\leqslant N)$ satisfy the condition of regularity on the interval $R=[a,b]$, on which the potential $V_0(x,t)$ of the original equation (1) is discontinuous in the variable x. In this case the potential of the transformed equation (3) will be a continuous function on this interval R. From (19) we see that for this it is sufficient to require that the sign of the Wronskian $W(u_1,u_2,\ldots,u_N)$ be conserved, and for a chain of transformations this condition must be imposed on all the Wronskians $W(u_{p+1},u_{p+2},\ldots,u_q)$, $p<q\leqslant N$. We shall analyze this condition in more detail for some special cases.

2.3. Relation to the symmetry algebra

Let us return to the first-order transformation operator $\hat{L}$. An operator of the form (12) with function $L_1(t)$ calculated using (10) can be constructed only if the transformation function u satisfies the condition (9). In addition, the operator $\hat{L}^+\hat{L}$ of second order in ∂_x is a symmetry operator of Eq. (1). The symmetry operators of first order in ∂_x and ∂_t for Eq. (1) are well known. They form a (real) Lie algebra. The maximal real algebra $sch(1.1)$ corresponds to a potential which is a polynomial of second order in x with arbitrary constant coefficients. If this polynomial has nonzero coefficient of x^2, we will have a linear harmonic oscillator with constant frequency. The case where this coefficient is zero but the coefficient of x is nonzero corresponds to a particle in a uniform external field. If both of these coefficients are zero, the particle will be free. In all cases the symmetry algebra of the Schrödinger equation is the six-dimensional real algebra $sch(1.1)$ (Ref. 38). For other potentials the symmetry algebra of the Schrödinger equation can be a subalgebra of this algebra. We shall therefore consider the algebra $sch(1.1)$ in more detail, and pay special attention to its representation corresponding to a free particle.[38]

The following operators form a basis of the algebra $sch(1.1)$:

$$K_2=-t^2\partial_t-tx\partial_x-t/2+ix^2/4, \quad K^0=x\partial_x+2t\partial_t+1/2,$$
$$K_{-2}=\partial_t, \quad K_1=-t\partial_x+ix/2, \quad K_0=i, \quad K_{-1}=\partial_x. \quad (26)$$

Every nontrivial second-order symmetry operator of Eq. (1) with $V_0(x,t)=0$ belongs to the space of second-order operators of the enveloping algebra of the algebra $sch(1.1)$ [i.e., Eq. (1) is an equation of class I in the terminology of Ref. 38].

The algebra $sch(1.1)$ is the semidirect sum of the Weyl algebra $w_1=\mathrm{span}(K_1,K_0,K_{-1})$ (we use "span" to denote the linear envelope over the field C or R, depending on whether we want to obtain a complex or real linear space, which is usually clear from the context) and the algebra $sl(2,R)$ isomorphic to the algebra $su(1.1)$ $=\mathrm{span}(K_2,K^0,K_{-2})$. We note that the equation $\partial_t=i\partial_{xx}$ holds for solutions of Eq. (1) (i.e., in the space T_0). The algebra $su(1.1)$ must therefore be realized by differential operators of second order in the variable x, while the algebra w_1 remains an algebra of operators of order no higher than the first. In addition, the basis of operators of second order in ∂_x of the enveloping algebra of the algebra w_1 (i.e., operators of the form K_1^2, K_{-1}^2, and $K_1K_{-1}+K_{-1}K_1$) belongs to the algebra $su(1.1)$, as is easily verified by direct calculation.

From this it is clear that all the symmetry operators of second order in ∂_x acting in the space T_0 are restricted to the elements of the algebra $su(1.1)$.

The representation of the Weyl group W_1 in the space of functions analytic on the entire axis is given by[38]

$$T(u,v,\rho)\Phi(t,x)=\exp[i\rho+i(uv+2ux-u^2t)/4]\times\Phi(t,x+v-ut). \quad (27)$$

The corresponding representation of the group $SL(2,R)$ has the form

$$T(\alpha,\beta,\gamma,\delta)\Phi(t,x)=(\delta+t\beta)^{-1/2}\exp\left[\frac{ix^2\beta}{4(\delta+t\beta)}\right]\times\Phi\left(\frac{\gamma+t\alpha}{\delta+t\beta},\frac{x}{\delta+t\beta}\right), \quad (28)$$

where u, v, ρ, and α, β, γ, δ ($\alpha\delta-\gamma\beta=1$) are the group parameters.

It follows from these expressions that if the function $\Phi(t,x)$ satisfies the condition (9), the transformed functions (27) and (28) will also satisfy this condition. In other words, Eq. (9) is invariant under group transformations from the Schrödinger group $SCH(1.1)$. This property can also be used to classify the space $sch(1.1)$ under the adjoint representation Ad of the group $SCH(1.1)$. The space $sch(1.1)$ is divided into five orbits with representatives[38] $O_1=K_{-2}-K_2$, $O_2=K^0$, $O_3=K_{-2}-K_1$, $O_4=K_{-2}$, and $O_5=K_{-1}$. The representative O_5 is an operator of first order in ∂_x and, in addition, in T_0 the equation $iK_{-1}^2=K_2$ is satisfied. Therefore, we shall not consider this representative.

We find the eigenfunctions of the operators $O_1,\ldots,O_4$ by the method of R-separation of variables.[38] The solution of the equation

$$(K_{-2}-K_2)\psi_\lambda=i\lambda\psi_\lambda$$

is the function

$$\psi_\lambda(x,t)=(1+t^2)^{-1/4}\exp\left[\frac{i}{4}x^2t(1+t^2)^{-1}+i\lambda\arctan t\right]Q_\lambda(z),\quad z=x/\sqrt{1+t^2}, \quad (29)$$

where $Q_\lambda(z)$ is a parabolic-cylinder function satisfying the equation $Q_\lambda''(z)-(z^2/4+\lambda)Q_\lambda(z)=0$. The function (29) satisfies the condition (9) for all real λ. In addition, choosing the transformation function in the form

$$u=\psi_{-1/2}(x,t)=(1+t^2)^{-1/4}\exp\left[\frac{x^2(it-1)}{4(1+t^2)}-\frac{i}{2}\arctan t\right],$$

we find

$$\hat{L}=L_1(t)[x(1-it)(2+2t^2)^{-1}+\partial_x],$$

$$L_1(t)=(1+t^2)^{1/2}$$

and $O_1=i(\hat{L}^+\hat{L}-1/2)$.

The eigenfunctions of the operator K^0,

$$\psi_\lambda(x,t)=t^{i\lambda/2-1/4}\exp\left(\frac{i}{8t}x^2\right)Q_{i\lambda}\left(\frac{x}{2\sqrt{t}}e^{-i\pi/4}\right),$$

$$K^0\psi_\lambda=i\lambda\psi_\lambda,$$

do not satisfy the condition (9) for all values of λ. In those special cases where this function satisfies the condition (9), it is also an eigenfunction of some other symmetry operator which is also factorized by the transformation operators constructed using this function. For example, taking the transformation function in the form $u=t^{-1/2}\exp(ix^2/4t)$, $K^0u=0$, we find $\hat{L}=ix/2+t\partial_x$. From this we have $\hat{L}^+\hat{L}=iK_2$ and $K_2u=0$. The eigenfunctions of the operator K_2, $\psi_\alpha(x,t)=t^{-1/2}\exp[(i/4t)(x^2-\alpha^2)-\alpha x/2t]$, satisy the condition (9) for all real values of α. The transformation operators constructed using these functions factorize the operator $K_2=-i(\hat{L}^+\hat{L}+\alpha^2/4)$ and are its eigenfunctions: $K_2\psi_\alpha(x,t)=-i\alpha^2/4\cdot\psi_\alpha(x,t)$.

Let us now consider the operator O_3. Its eigenfunctions

$$\psi_\lambda(x,t)=\exp[it(x/2-t^2/6-\lambda)]\mathrm{Ai}(2^{-1/3}x-2^{-4/3}t^2-2^{2/3}\lambda),$$

where $\mathrm{Ai}(z)$ is the Airy function satisfying the equation $\mathrm{Ai}''(z)=z\mathrm{Ai}(z)$, also satisfy the condition (9) for all real λ. The Darboux transformation operator constructed using this function ensures the factorization of the operator $O_3=K_{-2}-K_1=-i(\hat{L}^+\hat{L}+\lambda)$.

The last operator $K_{-2}=\partial_t=i\partial_x^2$ has the eigenfunctions

$$\psi_\lambda(x,t)=\exp(i\lambda^2t+\lambda x),\quad K_{-2}\psi_\lambda=i\lambda^2\psi_\lambda,$$

which also satisfy the condition (9) for all values of the parameter λ and factorize this operator $K_{-2}=-i(\hat{L}^+\hat{L}-\lambda^2)$.

Thus, by direct calculation we have established the validity of the following statement. Every operator $g\in sch(1.1)$, which when restricted to the space T_0 is an operator of second order in ∂_x (∂_t is eliminated) and for which all the eigenfunctions, corresponding to purely imaginary eigenvalues, satisfy the condition (9), admits a factorization by Darboux transformation operators $\hat{L}$ of the form $g=-i(\hat{L}^+\hat{L}+\alpha)$ with transformation function u satisfying the equation $igu=\alpha u$, $\alpha\in\mathbb{R}$. The operator L is given by (12) and (10).

In addition, since in each of the cases considered we were dealing with solutions of Eq. (1) in R-separated variables, it can be stated that only such solutions can be used as transformation functions. The analogous result for the stationary Schrödinger equation in two-dimensional space was obtained in Refs. 24, 39, and 43.

We should also mention Ref. 40, where the possibility of factorization of the symmetry operators for the harmonic oscillator with time-dependent frequency was studied. We shall see that this approach is practically equivalent to the nonstationary Darboux transformation. An advantage of the approach developed here is the fact that Eqs. (9)–(12) give a clear, constructive method of finding the operators effecting the factorization.

2.4. Invertibility questions

If we restrict ourselves to higher-order transformation operators which are products of first-order operators, it is sufficient to consider the invertibility of the first-order operators.

The first-order operator $\hat{L}$ (12) has nontrivial kernel $\ker\hat{L}=\mathrm{span}(u)$. Therefore, to find the inverse of this operator it is necessary to restrict the action of the operator $\hat{L}$ to some subspace. In addition, we shall seek the inverse operator in the class of linear operators independent of the functions on which they act. This condition can also lead to a restriction of the space on which the action of the operator $\hat{L}$ must be considered.

It is intuitively clear that an operator which is the inverse of a differential operator must be an integral operator. Considering the action of the operator $\hat{L}$ (12) on some arbitrary function $f(x)$ [temporarily we take $L_1(t)=1$],

$$F(x)=\hat{L}f(x)=-f(x)u_x/u+f_x(x),$$

as a differential equation for $f(x)$, we find f as

$$f(x)=u\left(C+\int_{x_0}^{x}u^{-1}F(x)dx\right). \tag{30}$$

The function u appearing in this expression must be a solution of Eq. (1), and we want the function F to satisfy Eq. (3). For the right-hand side of (30) to be expressed in terms of solutions of the same equation (3), it is necessary, according to (13), to make the replacement $u\to v=(L_1\bar{u})^{-1}$ in it. The derivative with respect to x of the expression in parentheses in (30) is proportional to the product vF in this case. In addition, in determining the inverse operator it is necessary to take into account the time dependence of all the functions.

For each v and $\varphi\in T_1$ we determine the function $w=w(\bar{v},\varphi)=w(x,t)$ from the equations

$$w_x(\bar{v},\varphi)=\bar{v}\varphi,$$

$$w_t(\bar{v},\varphi)=i(\bar{v}\varphi_x-\bar{v}_x\varphi).$$

Using (3), we can show that these equations are consistent: $w_{xt}=w_{tx}$. Therefore, the function w can be calculated in two different ways:

$$w=i\int_{t_0}^{t}(\bar{v}\varphi_x-\bar{v}_x\varphi)dt+C_1(x), \tag{31}$$

$$w=\int_{x_0}^{x}\bar{v}\varphi dx+C_2(t), \tag{32}$$

where the functions C_1 and C_2 are given by

$$C_1'(x)=(\bar{v}\varphi)_{t=t_0},$$

$$C_2'(t)=i(\bar{v}\varphi_x-\bar{v}_x\varphi)_{x=x_0},$$

with the bar denoting the derivative of the function with respect to its argument. We fix the function $v\in T_1$ and, assuming that it satisfies the condition (9), we call it the *transformation function.*

Clearly, the set of functions φ such that $C_1'(x)=0$ [or $C_2'(t)=0$] forms a linear space. We write $T_{11}=\{\varphi:C_1'(x)=0\}$ and $T_{12}=\{\varphi:C_2'(t)=0\}$. Now we define the linear operator M by the equation

$$\psi(x,t)=M\varphi(x,t)=[L_1(t)\bar{v}]^{-1}w(\bar{v},\varphi), \tag{33}$$

in which the function w is given by (31) when $\varphi\in T_{11}$ and by (32) when $\varphi\in T_{12}$, and the function $L_1(t)$ is calculated using (10), in which we must make the substitution $u\to\bar{v}^{-1}$. It can be verified directly that under these conditions the function (33) satisfies (1). The function $u=(L_1\bar{v})^{-1}$ will also be a solution of this equation.

If the transformation functions u and v of Eqs. (1) and (3) used in Eqs. (12) and (13) satisfy the condition $v=(L_1\bar{u})^{-1}$, the function $L_1(t)$ involved in these expressions will be the same. Moreover, it is obvious that the image T_{01} of the space T_{11} (or T_{12}) induced by the operator M, i.e., $T_{01}=\{\psi:\ \psi=M\varphi,\ \varphi\in T_{11}\}$ (or $T_{02}=\{\psi:\ \psi=M\varphi,\ \varphi\in T_{12}\}$), will, owing to the linearity of the operator M, be a linear subspace in T_0. Restricting the action of the operator $\hat{L}$ to this subspace, it can be shown that $M\hat{L}\psi=\psi$, $\forall\psi\in T_{01}$. In addition, we also establish the validity of the equation $\hat{L}M\varphi=\varphi$, $\forall\varphi\in T_{11}$ (or T_{12}). The operators $\hat{L}$ and M are therefore the inverses of each other.

2.5. Transformation of the Hilbert spaces

If the space T_0 contains a subspace $H_0(R)$ of functions which are Lebesgue-integrable in the squared modulus on the interval $R=[a,b]$ in the variable x, then in $H_0(R)$ it is possible to define the scalar product in the usual manner:

$$\langle\psi_1|\psi_2\rangle=\int_a^b\bar{\psi}_1(x,t)\psi_2(x,t)dx. \tag{34}$$

The structure of a separable Hilbert space is therefore defined on the space T_0 (Ref. 38). The restriction of the operator $\hat{L}$ to the space $H_0(R)$ will be denoted by the same symbol $\hat{L}$.

Since we assume that $H_0(R)\subset T_0$, in the operators $g\in al$ [in particular, $g\in sch(1.1)$] we can replace $i\partial_t$ by h_0. The resulting operators will be skew-symmetric on the space of infinitely differentiable functions with compact support, and the operators ig will be self-adjoint.[38] We shall assume that al contains a semibounded operator g_0 with a purely discrete spectrum, and its eigenfunctions belonging to $H_0(R)$ will be labeled according to the number of their zeros, $n=0,1,2,\ldots$. We shall not discuss the exact domain of definition of the operators g here, but refer the interested reader to Ref. 38, where the corresponding references are given. If, in addition, g_0 satisfies the factorization condition discussed in Sec. 1.3, its eigenfunctions can be used as transformation functions. The only (up to a constant) transformation function from the space $H_0(R)$ leading to a regular potential difference (11) will be $u=\psi_0(x,t)$. There are richer possibilities for choosing the transformation functions outside the space $H_0(R)$. Assuming that the spectrum of the operator ig_0 is bounded, for example, from below by the value ε_0, we must choose the transformation functions to be the eigen-

functions ψ_α of the operator ig_0 ($ig_0\psi_\alpha=\alpha\psi_\alpha$) with $\alpha<\varepsilon_0$. The function $|\psi_\alpha(x,t)|$ will become infinite at one or both ends of the interval R.

Let us choose $u=\psi_\alpha(x,t)$ as the transformation function. Then, according to the factorization property, $g_0=-i(\hat{L}^+\hat{L}-\alpha)$ or $\hat{L}^+\hat{L}=ig_0+\alpha$. If we now consider the scalar product of the vectors ψ and $\hat{L}^+\hat{L}\psi'\neq 0$ from $H_0(R)$, then, using the skew-Hermiticity of the operator ∂_x $[(\partial_x)^\dagger=-\partial_x]$ with respect to the scalar product (34), which causes the conjugation operation introduced above for $\hat{L}$ to coincide with the operation of Hermitian conjugation relative to the scalar product (34), i.e., $\hat{L}^+=\hat{L}^\dagger$, we can write $\langle\psi|\hat{L}^+\hat{L}\psi'\rangle=\langle\hat{L}\psi|\hat{L}\psi'\rangle=\langle\varphi|\varphi'\rangle$. From this we see that if as ψ and ψ' we choose eigenfunctions of the operator ig_0, which are also eigenfunctions of the operator $\hat{L}^+\hat{L}$, the system of functions $\varphi=\hat{L}\psi$ will be orthogonal and normalizable, and the space $H_1(R)=\{\varphi:\ \varphi=\hat{L}\psi,\ \psi\in H_0(R)\}$ will be a subspace of functions integrable in the squared modulus on R. Defining the scalar product on it in the same manner as on $H_0(R)$ [i.e., using (34)], we obtain a Hilbert space contained in T_1.

A very important question is that of the completeness of the space $H_1(R)$, i.e., whether or not every square-integrable solution of Eq. (3) can be obtained by acting with the operator $\hat{L}$ on a square-integrable solution of Eq. (1). This question is closely related to that of the one-to-one correspondence of the vectors from T_0 and T_1.

The transformation function $u\in T_0$ is a solution of a second-order differential equation in x, $\hat{L}^+\hat{L}u=0$, and the function $v=(L_1\bar{u})^{-1}\in T_1$ satisfies the analogous equation $\hat{L}\hat{L}^+v=0$. Each of these equations has another linearly independent solution. In particular, the function

$$v'=\frac{1}{L_1\bar{u}}\int u\bar{u}dx=vL_1^{-2}\int\frac{dx}{v\bar{v}}, \tag{35}$$

possessing the property $\hat{L}^+v'=u$, also obviously satisfies Eq. (3).

For Eq. (1) the analogous property holds for the function

$$u'=uL_1^{-2}\int\frac{dx}{u\bar{u}}, \tag{36}$$

$\hat{L}u'=v$ and $\hat{L}^+\hat{L}u'=0$. It is somewhat more complicated to check that it is a solution of Eq. (1). For this we must use the property $i(u\bar{u})_t=W_x(u,\bar{u})$ and the expression for the function $L_1(t)$: $i(\ln L_1)_t=[\ln(u/\bar{u})]_{xx}$. From this we obtain an expression for the integral:

$$i\int\frac{(u\bar{u})_t}{(u\bar{u})^2}dx=-\frac{W(u,\bar{u})}{(u\bar{u})^2}-\frac{2iL_{1t}}{L_1}\int\frac{dx}{u\bar{u}}. \tag{37}$$

Using (37), we can verify directly that the function (36) is a solution of Eq. (1).

Let us now represent each of the spaces T_0 and T_1 as a direct sum $T_{0,1}=T^0_{0,1}\oplus T^1_{0,1}$, where $T^1_0=\mathrm{span}(u,u')=\ker(\hat{L}^+\hat{L})$ and $T^1_1=\mathrm{span}(v,v')=\ker(\hat{L}\hat{L}^+)$. The functions u, u' and v, v' form bases of the spaces T^1_0 and T^1_1. It is clear from this construction that the operators $\hat{L}$ and $\hat{L}^+$ effect a one-to-one correspondence between the spaces T^0_0 and T^0_1, and by using them we establish a correspondence between the bases of the spaces T^1_0 and T^1_1: $u'\to v$, $v'\to u$.

Let us return to the Hilbert space $H_0(R)$. Let the transformation function be $u=\psi_0\in H_0(R)$. Then, obviously, $u'\notin H_0(R)$. Since u in this case vanishes at the boundaries of R, u^{-1} becomes infinite at the boundaries of R and thus is not measurable on R. Therefore, neither v nor v' belongs to the Hilbert space $H_1(R)\subset T_1$. From this it is clear that the operator $\hat{L}$ maps the space $H^0_0(R)=\mathrm{span}\{\psi_n,n=1,2,\ldots\}$ onto the entire space $H_1(R)$, and, therefore, the set of vectors $\{\varphi_n=\hat{L}\psi_n,\ n=1,2,\ldots\}$ forms a basis of the space $H_1(R)$. The function $u=\psi_0$ does not have an image in $H_1(R)$.

Now let $u\notin H_0(R)$ and u become infinite only at one of the ends of the interval R and be zero at the other end. In this case u' also does not belong to $H_0(R)$, because its behavior at the ends of the interval R coincides with the behavior of the function u apart from interchange of the boundary points, $a\leftrightarrow b$. Obviously, the function u^{-1} here will also not be measurable on R, so that $v,v'\notin H_1(R)$. The operators $\hat{L}$ and $\hat{L}^+$ effect a one-to-one correspondence between the spaces $H_0(R)$ and $H_1(R)$.

Finally, let us consider the case where $u\notin H_0(R)$ and u becomes infinite at both ends of the interval R, so that u^{-1} will be a measurable function. Then $v=(L_1\bar{u})^{-1}\in H_1(R)$ and $v'\notin H_1(R)$. In this case the operator $\hat{L}$ maps the direct sum $H_0(R)\oplus\mathrm{span}(u')$ onto $H_1(R)$, and the operator $\hat{L}^+$ maps the space $H_1(R)$ onto $H_0(R)$. This case is obviously equivalent to the first case that we considered if Eq. (3) is taken as the original Schrödinger equation and Eq. (1) is taken as the transformed equation.

We again stress that we have always assumed that the sign of the function u on R is conserved.

Let us now discuss some of the features which can arise in studying higher-order transformations.

So far we have not learned of any Nth-order ($N>1$) transformation which cannot be represented as a product of N first-order transformations. There is also no proof that such transformations cannot exist. In fact, every known Nth-order transformation can be viewed as a chain of first-order transformations. The remarkable feature of this chain is that some of the intermediate Schrödinger equations may not have a physical meaning (the potential becomes a complex-valued function or has singularities inside the interval R), while the final solution is physically meaningful. If we are not interested in the intermediate Schrödinger equation, we can allow this type of transformation.

Let us consider a solution of Eq. (1) with R-separated variables,[38] $u_\lambda=R(x,t)u_{1\lambda}u_{2\lambda}$, assuming that the function $u_{2\lambda}$ is an eigenfunction of some symmetry operator, $igu_{2\lambda}=\lambda u_{2\lambda}$. From the condition $(ig)^+=ig$ we obtain $u_{2\bar{\lambda}}=\bar{u}_{2\lambda}$. Use of the functions u_λ and $u_{\bar{\lambda}}$ for complex values of λ as the transformation functions in some transformations leads to complex potentials. However, it is possible to formulate conditions on the functions R and $u_{1\lambda}$ for which the potential resulting from a chain of such transformations will be real. For this it is necessary to calculate the Wronskian $W(u_\lambda,u_{\bar{\lambda}})$ and use Eq. (17). After this we can see that it

is sufficient to impose the following conditions on the functions $u_{1\lambda}$ and $R(x,t)$:

$$u_{1\lambda x}=0, \quad [\ln(R(x,t)/\bar{R}(x,t))]_{xxx}=0. \tag{38}$$

Using the solutions studied in Sec. 1.3, corresponding to the operators K_2-K_{-2}, K^0, and $K_{-2}-K_1$ of the algebra $sch(1.1)$, it can be seen that they all satisfy the conditions (38).

A single transformation using a function $u_k\neq\psi_0$, $u_k\in H_0(R)$ leads to a potential difference singular on R, because the function u in this case has zeros inside R. In a repeated transformation the additional possibility arises of using the transformation function $u_{k+1}\in H_0(R)$, leading to a regular potential difference. This possibility is related to the fact that the Wronskian $W(u_{k_1},u_{k_2},\ldots,u_{k_N})$ determining the potential difference resulting from a chain of N Darboux transformations conserves the sign on R if the functions $u_{k_i}\in H_0(R)$, where k_i coincides with the number of zeroes of these functions, are chosen in such a way that the inequality $(k-k_1)(k-k_2)\ldots(k-k_N)>0$ is satisfied for all $k=0,1,2,\ldots$. This property was first formulated for stationary transformation functions by Kreĭn.[41] It is satisfied, in particular, if the functions u_{k_i} are pairwise adjacent. This condition has recently been discussed in Refs. 12 and 42 for the stationary Schrödinger equation. In this case $\mathrm{Ker}\hat{L}_{0N}=\mathrm{span}\{u_{k_i},\ i=1,2,\ldots,N\}$, and the space $H_1(R)$ is completely defined by the operator $\hat{L}_{0N}$: $H_1(R)=\{\varphi:\ \varphi=\hat{L}_{0N}\psi,\ \psi\in H_0(R)\}$.

2.6. Supersymmetry of the nonstationary Schrödinger equation

Supersymmetric quantum mechanics first appeared in the studies by Witten[12] as a model illustrating spontaneous supersymmetry breaking at the quantum level. The overwhelming majority of studies in this area (see the reviews of Refs. 43 and 44) concern stationary supersymmetric quantum mechanics. Its nonstationary generalization has hardly been developed at all. The authors know of only one study devoted to supersymmetry of the nonstationary Schrödinger equation.[32]

The relation between the supersymmetric quantum mechanics of Witten and the Darboux trnsformation is pointed out in Ref. 14. The Darboux transformation of the nonstationary Schrödinger equation can be used to construct its supersymmetric generalization.

Let two Schrödinger equations with Hamiltonians h_0 and h_1 be related to each other via the operators $\hat{L}$ and $\hat{L}^+$. We consider the matrix Hamiltonian (super-Hamiltonian) $\mathscr{H}=\mathrm{diag}(h_0,h_1)$ and the matrices of the mutually conjugate supercharges

$$\mathscr{Q}_0=\hat{L}^+\sigma^+, \quad \mathscr{Q}_0^+=\hat{L}\sigma^-, \quad \sigma^+=\begin{pmatrix}0&1\\0&0\end{pmatrix},$$

$$\sigma^-=\begin{pmatrix}0&0\\1&0\end{pmatrix}. \tag{39}$$

The Schrödinger equations (1) and (3) can now be rewritten as a single matrix equation

$$(i/\partial_t-\mathscr{H})\Psi(x,t)=0, \tag{40}$$

where I is the second-order unit matrix and the space of column vectors $\Psi(x,t)$ for each function $\psi(x,t)$ is defined as

$$\Psi(x,t)=\mathrm{span}(\Psi_+,\Psi_-), \quad \Psi_+(x,t)=\psi(x,t)e_+,$$

$$\Psi_-(x,t)=\varphi(x,t)e_-, \quad \varphi(x,t)=\hat{L}\psi(x,t),$$

where $e_\pm$ are column matrices: $e_+=(1,0)^t$, $e_-=(0,1)^t$ (the symbol t denotes the transpose).

The condition for the supercharges (39) to commute with the Schrödinger superoperator $iI\partial_t-\mathscr{H}$ is equivalent to the intertwining relations (2) and (4). Using the symmetry operators $g_0^0=-i(\hat{L}^+\hat{L}+\alpha)$ and $g_0^1=-i(\hat{L}\hat{L}^++\alpha)=g_0^0+iL_1^2(t)A$ and Eqs. (1) and (3), we construct the symmetry operator of Eq. (40): $\mathscr{G}_0=\mathrm{diag}(g_0^0,g_0^1)$. The operators $\mathscr{G}_0$, $\mathscr{Q}_0^+$, and $\mathscr{Q}_0$ form the simplest superalgebra $[\mathscr{Q}_0,\mathscr{G}_0]=[\mathscr{Q}_0^+,\mathscr{G}_0]=0$, $\{\mathscr{Q}_0,\mathscr{Q}_0^+\}=i\mathscr{G}_0-\alpha I$, $\mathscr{Q}_0^2=(\mathscr{Q}_0^+)^2=0$ and transform one solution of Eq. (40) into another solution.

We can use the operator $M=\hat{L}^{-1}$ to construct other symmetry operators of Eq. (3): $g^1=\hat{L}g^0M$, $g^0\in al$. These clearly form an algebra isomorphic to the algebra al of Eq. (1) and can be used to construct the matrix symmetry operators $\mathscr{G}_g=\mathrm{diag}(g^0,g^1)$ of Eq. (40), which also form an algebra isomorphic to al.

The odd sector of the superalgebra can be extended by nilpotent operators $\mathscr{Q}_g=g^0M\sigma^+$, provided that they satisfy the relations $\{\mathscr{Q}_g,\mathscr{Q}_0^+\}=\mathscr{G}_g$, $[\mathscr{G}_{g_1},\mathscr{Q}_{g_2}]=\mathscr{Q}_{g_{12}}$, where $g_{12}=[g_1^0,g_2^0]$ and $\{\mathscr{Q}_{g_1},\mathscr{Q}_{g_2}\}=0$, $[\mathscr{G}_g,\mathscr{Q}_0^+]=0$. Here the element $\mathscr{Q}_0$ defined in (39) will be $\mathscr{Q}_0=(ig_0^0-\alpha)M\sigma^+$. In addition, it can be shown that the generalized Jacobi identities are valid for the set of odd $(\mathscr{Q}_g,\mathscr{Q}_0^+)$ and even $(\mathscr{G}_g)$ elements, i.e., these elements form a Lie superalgebra.

In this approach it is not completely clear how the super-Hilbert space should be defined to ensure that exponentiation of these operators gives a supergroup representation. This difficulty does not arise if we restrict ourselves to the subalgebra formed by the operators $\mathscr{Q}_0$, $\mathscr{Q}_0^+$, and $\mathscr{G}_0$.

Let us now consider the chain of transformations generated by the set of N linearly independent solutions u_1, u_2, ..., u_N of Eq. (1) with potential V_0 regular on R such that $ig^0u_k=\alpha_ku_k$ and the Wronskians $W(u_q,u_{q+1},\ldots,u_p)$, $q\leq p=1,\ldots,N$, preserve the sign on R and satisfy the reality condition (17). Such transformations generate a chain of Hermitian Hamiltonians $h_0\to h_1\to\ldots\to h_N$, the potentials of which are regular on R, and a chain of symmetry operators $g^0\to g^1\to\ldots\to g^N$. Every two adjacent Schrödinger equations and every two adjacent symmetry operators g^p, g^{p+1} are intertwined by the first-order Darboux transformation operators $\hat{L}_{p,p+1}$, $\hat{L}_{p,p+1}^+$, $p=0,1,\ldots,N-1$, calculated according to Eqs. (10) and (12). The nth-order ($\leq N$) transformation operators $\hat{L}_{p,p+n}=\hat{L}_{p+n-1,p+n},\ldots,\hat{L}_{p+1,p+2}\hat{L}_{p,p+1}$ and their conjugates intertwine the symmetry operators g^p and g^{p+n} and the Schrödinger equations with Hamiltonians h_p and

h_{p+n}, and are involved in the factorization $\hat{L}^+_{p,p+n}\hat{L}_{p,p+n}=\Pi^n_{k=1}(ig^p-\alpha_{p+k})$, $\hat{L}_{p,p+n}\hat{L}^+_{p,p+n}=\Pi^n_{k=1}(ig^{p+n}-\alpha_{p+k})$. Such a chain of transformations will be termed *completely reducible*. The concept of reducibility of a chain of Darboux transformations for the stationary Schrödinger equation was first introduced in Ref. 45. The second-order Darboux transformation, which is termed irreducible in Ref. 45, corresponds to complex-valued intermediate potential and real-valued final potential.

We shall assume that transformations giving singular intermediate potentials but final potential which is regular on a given interval R are also irreducible. It is therefore quite natural to term a chain of transformations in which each Hamiltonian is Hermitian and the potential is regular on R completely reducible.

This chain can be used to link the super-Hamiltonian $\mathscr{H}=i\mathrm{diag}(h_0,h_1,\ldots,h_N)$, the symmetry operator $\mathscr{G}_0=i\mathrm{diag}(g^0,g^1,\ldots,g^N)$, and the supercharges $Q^+_{p,q}=\hat{L}_{p,q}e_{p,q}$, $Q_{p,q}=\hat{L}^+_{p,q}e_{q,p}$, where $e_{p,q}$ is a matrix of dimension $N+1$ in which the only nonzero element, equal to unity, stands at the intersection of the pth column and the qth row. The chain of $N+1$ Schrödinger equations reduces to a single matrix equation (40) in which I is a unit matrix of dimension $N+1$. The equations intertwining the Schrödinger operators $iI\partial_t-h_p$ and $iI\partial_t-h_q$ via the operators $\hat{L}_{pq}$ and $\hat{L}^+_{pq}$ are equivalent to the commutators of the supercharges Q_{pq} and the Schrödinger superoperator $iI\partial_t-\mathscr{H}$. Therefore, all the operators Q_{pq} are integrals of the motion of the system with super-Hamiltonian $\mathscr{H}$. The condition that the chain of transformations be completely reducible leads to the following nonlinear algebra:

$$
\begin{aligned}
&Q_{s,p}Q_{p,q}=Q_{s,q}, \quad N+1\geqslant q>p>s,\\
&Q^+_{p,p+n}Q_{p,p+n+m}=\prod_{i=1}^{n}(\mathscr{G}_0-\alpha_{p+i})Q_{p+n,p+n+m},\\
&p+n+m\leqslant N+1,\\
&Q_{p-n-m,p}Q^+_{p-n,p}=\prod_{i=1}^{n}(\mathscr{G}_0-\alpha_{p+i-1})Q_{p-n-m,p-n},\\
&p-n-m\geqslant 0, \quad p\leqslant N+1,\\
&Q_{p,p+n}Q^+_{p,p+n}Q_{p,p+n}=\prod_{i=1}^{n}(\mathscr{G}_0-\alpha_{p+i})Q_{p,p+n},\\
&p+n\leqslant N+1, \quad n,m=1,2,\ldots,
\end{aligned}
\tag{41}
$$

and the Hermitian conjugates of these relations. All other products of any two supercharges are equal to zero. Equations (41) are characteristic of parasuperalgebras (see, for example, Refs. 46 and 47), which are known for stationary Darboux transformations.[48]

2.7. Nonstationary, exactly solvable potentials

Here we shall give some very simple applications of our technique which illustrate the possibility of generating nonstationary, exactly solvable potentials.

Let us consider a free particle, $V_0(x,t)=0$, $R=(-\infty,+\infty)$. The discrete basis of the space $H_0(R)$ is formed by the functions (29) for $\lambda=\lambda_n=-n-1/2$, $n=0,1,2,\ldots$, $\psi_n\equiv\psi_{\lambda_n}$. In addition, for $\lambda=\lambda_m=m+1/2$, $m=0,2,4,\ldots$, these functions preserve the sign on R for all t and can be chosen as the transformation functions $u^{(m)}_1=\psi_{\lambda_m}$. The transformation operator $\hat{L}$ is found from (10) and (12):

$$
\begin{aligned}
&\hat{L}=\hat{L}_{0,1}=L_1(t)\partial_x-\frac{x}{2}\sqrt{\frac{1+it}{1-it}}-im\,\frac{He_{m-1}(iz)}{He_m(iz)},\\
&L_1(t)=\sqrt{1+t^2},
\end{aligned}
\tag{42}
$$

where $He_m(z)=2^{-m/2}H_m(z/\sqrt{2})$ are the Hermite polynomials. The potential of the new Schrödinger equation is found from (11):

$$
V_1=A_{0,1}=(1+t^2)^{-1}\left[1-2m(m-1)\,\frac{He_{m-2}(iz)}{He_m(iz)}+2m^2\,\frac{He^2_{m-1}(iz)}{He^2_m(iz)}\right].
\tag{43}
$$

This potential difference detemines the symmetry operator $g^1=g^0-iL^2_1(t)A_{0,1}$. We obtain the orthonormal basis of the space $H_1(R)$ by using the operator $\hat{L}$ (42) and the solution (13):

$$
\begin{aligned}
&\varphi_{-1}=\sqrt{m!}(2\pi)^{-1/4}[L_1(t)\bar{u}_m(x,t)]^{-1},\\
&g^1\varphi_{-1}=i(m+1/2)\varphi_{-1},\\
&\varphi_n=[n!(n+m+1)(1+it)\sqrt{2\pi}]^{-1/2}\exp[-x^2/(4+4it)-in\arctan l]\times\left[He_{n+1}(z)+imHe_n(z)\,\frac{He_{m-1}(iz)}{He_m(iz)}\right],\\
&ig^1\varphi_n=(n+1/2)\varphi_n, \quad n=0,1,2,\ldots.
\end{aligned}
$$

Acting with the operator (42) on the function (29) with $\lambda=\lambda_l=l+1/2$, $l=m+1$, $m+3,\ldots$, we obtain solutions of the Schrödinger equation with the potential (43) which are nonzero everywhere:

$$
\begin{aligned}
&v^{(l)}(x,t)=\hat{L}_{0,1}\varphi_{\lambda_l}(x,t)=(1-it)^{-1/2}\exp[x^2/(4-4it)+il\arctan t]\,\frac{f_{m,l}(z)}{He_m(iz)},\\
&f_{m,l}(z)=i[He_l(iz)He_{m+1}(iz)-He_m(iz)He_{l+1}(iz)],
\end{aligned}
$$

to which a repeat transformation can be applied. The transformation operator is again constructed from Eqs. (10) and (12), using the transformation function $u=v^{(l)}(x,t)$. The resulting potential is

$$
V_{0,2}=2(1+t^2)^{-1}\left[1+\frac{f''_{ml}(z)}{f_{ml}(z)}-\left(\frac{f'_{ml}(z)}{f_{ml}(z)}\right)^2\right].
\tag{44}
$$

The corresponding symmetry operator has the form $g^{(2)}=g^{(0)}-iL_1^2(t)V_{0,2}$. We find the discrete basis of the space $H_2(R)$ by using the operator $\hat{L}_{0,2}=\hat{L}_{1,2}\hat{L}_{0,1}$ and Eq. (25):

$$\chi_{-2}=[(2\pi)^{-1/2}l!(l-m)(1-it)^{-1}]^{1/2}\exp[-x^2/(4+4it)+il\arctan l]\frac{He_m(iz)}{f_{ml}(z)},$$

$$\chi_{-1}=[(2\pi)^{-1/2}m!(l-m)(1-it)^{-1}]^{1/2}\exp[-x^2/(4+4it)+im\arctan t]\frac{He_l(iz)}{f_{ml}(z)},$$

$$\chi_n=L_{1,2}\varphi_n=L_{1,2}L_{0,1}\psi_n=[(n+l+1)(n+m+1)]^{-1/2}\times\left[-(l+n+1)\psi_n+(l-m)u_m\frac{W(\psi_n,u_l)}{W(u_m,u_l)}\right],$$

$n=0,1,2,...,$

where

$$W(\psi_n,u_l)=-(n!\sqrt{2\pi})^{-1/2}(1+t^2)^{-1}\exp[itx^2/(2+2t^2)+i(l-n)\arctan t]\times[nHe_{n-1}(z)He_l(iz)+iHe_n(z)He_{l+1}(iz)]$$

and

$$W(u_m,u_l)=((1-it)\sqrt{1+t^2})^{-1}f_{ml}(z)\exp[x^2/(2-2it)+i(m+l)\arctan t].$$

The transformation operators $\hat{L}_{0,2}$ and $\hat{L}_{0,2}^+$ intertwine the initial and final Schrödinger equations and participate in the factorization $\hat{L}_{0,2}^+\hat{L}_{0,2}=(ig^0+m+1/2)(ig^0+l+1/2)$ and $\hat{L}_{0,2}\hat{L}_{0,2}^+=(ig^2+m+1/2)(ig^2+l+1/2)$. Using the transformation operators $\hat{L}_{p,q}$, $p,q=0,1,2$ $(p<q)$, we construct the supercharge operators $Q_{p,q}=\hat{L}_{p,q}e_{p,q}$ and $Q_{p,q}^+=(Q_{p,q})^\dagger$ and the symmetry superoperator $\mathcal{G}_0=i\,\mathrm{diag}\{g^0,g^1,g^2\}$, which form a nonlinear algebra of parasupersymmetric structure,[49]

$$[\mathcal{G}_0,Q_{p,q}]=0,\quad Q_{1,2}Q_{0,1}=Q_{0,2},$$

$$Q_{0,1}Q_{0,1}^+Q_{0,1}=(\mathcal{G}_0+m+1/2)Q_{0,1},$$

$$Q_{1,2}Q_{1,2}^+Q_{1,2}=(S+l+1/2)Q_{1,2},$$

$$Q_{0,1}Q_{0,2}^+=(\mathcal{G}_0+m+/2)Q_{1,2}^+,$$

$$Q_{0,2}^+Q_{1,2}=(\mathcal{G}_0+l+1/2)Q_{0,1}^+,$$

$$Q_{0,2}Q_{0,2}^+Q_{0,2}=(\mathcal{G}_0+m+1/2)(\mathcal{G}_0+l+1/2)Q_{0,2},$$

and the Hermitian-conjugate expressions. All other products of any two supercharges are equal to zero. The lowest eigenvalue of the superoperator $\mathcal{G}_0$, equal to $-l-1/2$, is nondegenerate. It corresponds to the function $\Psi_{-2}=(0,0,\chi_{-2})^t$. The next eigenvalue, equal to $-m-1/2$, is doubly degenerate and corresponds to the two eigenfunctions $\Psi_{-1}^1=(0,\varphi_{-1},0)^t$ and $\Psi_{-1}^2=(0,0,\chi_{-1})^t$. The rest of the discrete spectrum of the superoperator $\mathcal{G}_0$ coincides with the harmonic-oscillator spectrum and is threefold degenerate. The corresponding eigenfunctions are constructed by using the functions ψ_n, φ_n, and χ_n.

A single transformation using the function $u_1=\psi_n$ gives a potential difference with n singularities. However, a second transformation using the function $u_2=\psi_{n+1}$ removes all the singularities and leads to the regular potential difference

$$V^{(n+n+1)}=\frac{2}{1+t^2}\left[\frac{J_n''(z)}{J_n(z)}-\left(\frac{J_n'(z)}{J_n(z)}\right)^2-1\right],$$

$$J_n(z)=\sum_{s=0}^{n}\frac{\Gamma(n+1)}{\Gamma(s+1)}He_s^2(z)=nJ_{n-1}(z)+He_n^2(z),$$

$$J_0(z)=1,\quad J_1(z)=z^2+1,\quad J_2(z)=z^4+3,\ldots. \tag{45}$$

The second-order operator $\hat{L}_{0,2}$ and its conjugate $\hat{L}_{0,2}^+$ participate in the factorization of the operator $\mathcal{G}_0$: $\hat{L}_{0,2}^+\hat{L}_{0,2}=(i\mathcal{G}_0-n-1/2)(i\mathcal{G}_0-n-3/2)$. The supercharges Q, Q^+ constructed using it and the operator $\mathcal{G}_0=i\,\mathrm{diag}\{g^0,g^1\}$, $g^1=g^0-i(1+t^2)V^{(n,n+1)}$ form a quadratic subalgebra $[\mathcal{G}_0,Q]=[\mathcal{G}_0,Q^+]=0$, $Q^2=(Q^+)^2=0$, $\{Q,Q^+\}=(\mathcal{G}_0-n-1/2)(\mathcal{G}_0-n-3/2)$. The ground state of the operator $\mathcal{G}_0$ is doubly degenerate, and the corresponding eigenfunctions $\Psi_0^+=(\psi_0,0)^t$ and $\Psi_0^-=(0,\hat{L}_{0,2}\psi_0)^t$ are transformed into each other by the supercharge operators Q and Q^+. The states $\Psi_n=(\psi_n,0)^t$ and $\Psi_{n+1}=(\psi_{n+1},0)^t$ annihilated by the two supercharges are nondegenerate and correspond to the eigenvalues $n+1/2$ and $n+3/2$ of the operator $\mathcal{G}_0$. We therefore obtain a supersymmetric quantum-mechanical model in which the lowest-lying state of the symmetry operator is not annihilated by the two supercharges, and the states which they do annihilate lie inside the discrete spectrum. This situation was first discussed for the stationary Schrödinger equation in Ref. 50. It can arise only in supersymmetric models with supercharges containing higher derivatives with respect to x.

Our next example will be the harmonic oscillator with variable frequency:

$$h_0=-\partial_x^2+\omega^2(t)x^2, \tag{46}$$

where $\omega(t)$ is an arbitrary real function. The solution of the Schrödinger equation with this Hamiltonian is well known (see, for example, Refs. 51 and 52). We shall need not only the functions of the space $H_0(R)$, but also the solutions $\psi(x,t)\notin H_0(R)$. Let us consider this problem in more detail.

We consider annihilation and creation operators of the form[53]

$$a=\varepsilon\partial_x-\frac{i}{2}\dot{\varepsilon}x,\quad a^+=-\bar{\varepsilon}\partial_x+\frac{i}{2}\dot{\bar{\varepsilon}}x,\quad \varepsilon=\varepsilon(t). \tag{47}$$

The dot denotes the derivative with respect to t. The requirement that the operators (47) be integrals of the motion of the system with Hamiltonian (46) leads to the following equation for the parameter ε:

$$\ddot{\varepsilon}+4\omega^2\varepsilon=0. \tag{48}$$

Owing to the reality of the function ω^2, it follows from (48) that $(\dot{\varepsilon}\bar{\varepsilon}-\varepsilon\dot{\bar{\varepsilon}})_t=0$. Equation (48) determines the function ε

up to a constant. This arbitrariness can be used to satisfy the condition $[a,a^+]=1$. For this it is sufficient to require that

$$\dot{\varepsilon}\bar{\varepsilon}-\varepsilon\dot{\bar{\varepsilon}}=2i,$$

The operators (47) in this case realize a representation of the Heisenberg–Weyl algebra w_1, and the quadratic combinations of them

$$K_-=\frac{1}{2}a^2, \quad K_+=\frac{1}{2}(a^+)^2, \quad K_0=\frac{1}{4}\{a,a^+\} \tag{49}$$

form a representation of the algebra $su(1.1)$:

$$[K_0,K_\pm]=\pm K_\pm, \quad [K_-,K_+]=2K_0.$$

The semidirect sum of these representations gives a representation of the Schrödinger algebra $sch(1.1)$ different from that studied in Sec. 2.3, and the set of operators a, a^+, I, K_0, and $K_\pm$ realize the Cartan–Weyl basis of this algebra:

$$[a,K_0]=\frac{1}{2}a, \quad [a^+,K_0]=-\frac{1}{2}a^+.$$

The Cartan subalgebra is span(K_0,I). The representation of the discrete series is therefore determined by the quantization equation for the operator K_0, which can also be obtained by acting with the creation operator a^+ on the vacuum state $\psi_0(x,t)$. The latter, in turn, is determined by the condition $a\psi_0=0$, $\psi_0\in H_0(R)$.

Since we shall be interested not only in the solutions of Eq. (1) from the space $H_0(R)$, we use the technique of R-separation of variables[38] to find them. The solutions of Eq. (1) in R-separated variables,

$$\psi(x,t)=R(x,t)P(\tau)Q(\xi), \tag{50}$$

are defined as the common solutions of the Schrödinger equation (1) and the eigenfunction equation for the operator K_0. Substituting the explicit form of the operators a and a^+ (47) into the expression for K_0 (49), we find

$$4K_0=-2\gamma\partial_x^2+i\dot{\gamma}x\partial_x+\frac{i}{2}\dot{\gamma}+\frac{1}{4}\ddot{\gamma}x^2+2\omega^2\gamma x^2, \tag{51}$$

where $\gamma=\varepsilon\bar{\varepsilon}$. Since the equation $i\partial_t+\partial_x^2-\omega^2(t)x^2=0$ is satisfied on the space T_0, the operator (51) can be rewritten as a first-order operator in ∂_x and ∂_t:

$$4K_0=2i\gamma\partial_t+i\dot{\gamma}x\partial_x+\frac{i}{2}\dot{\gamma}+\frac{1}{4}\ddot{\gamma}x^2. \tag{52}$$

The eigenvalue equations of the operators (51) and (52) in the coordinates $\xi=x/\sqrt{\gamma}$ and $\tau=t$, with the function $R(x,t)$ chosen in the form

$$R(x,t)=\exp\left(i\frac{\dot{\gamma}x^2}{8\gamma}\right), \tag{53}$$

are equations in separated variables for the functions $Q(\xi)$ and $P(\tau)$. The first of these equations has the form

$$Q_{\xi\xi}(\xi)-(\xi^2/4+\lambda)Q(\xi)=0, \tag{54}$$

and the second is easily integrated:

$$P(t)=\gamma^{-1/4}\exp\left(i\lambda\int\gamma^{-1}dt\right), \tag{55}$$

where λ is the separation constant and is expressed in terms of the phase of the function $\varepsilon(t)$ (Ref. 52).

The discrete basis $\psi_n(x,t)$ of the space $H_0(R)$ corresponds to the choice $\lambda=\lambda_n=-n-1/2$:

$$\psi_n(x,t)=N_n\gamma^{-1/4}\exp\left(\frac{(i\dot{\gamma}-2)x^2}{8\gamma}+i\lambda\int\frac{dt}{\gamma}\right)He_n(x/\sqrt{\gamma}), \tag{56}$$

where N_n is a normalization constant.

It follows from (53)–(55) that the function (50) satisfies the reality condition (9) for all real λ. In complete analogy with the preceding case, we can use the pairwise adjacent functions (56) as the transformation functions for the Nth-order Darboux transformation, or for $N=1$ we can use the functions $u=\psi_\lambda$ for $\lambda>0$; in particular, the values $\lambda=n+1/2$ correspond to transformation functions of elementary form. The resulting exactly solvable potentials will be expressed in terms of Eqs. (43)–(45) with the replacement $1+t^2\to\gamma$.

In concluding this section, we note that the use of other orbit representatives in the Schrödinger algebra to obtain solutions of the Schrödinger equation with the Hamiltonian (46) allows other exactly solvable nonstationary Hamiltonians to be obtained.

3. THE STATIONARY SCHRÖDINGER EQUATION

When the original potential V_0 is independent of time, only stationary solutions of the Schrödinger equation (1) are of interest. It is well known that such solutions are constructed by using the solutions of the stationary Schrödinger equation

$$h_0\psi(x)=E\psi(x), \quad h_0=-\partial_x^2+V_0(x). \tag{57}$$

For stationary potentials we use the usual sign of the potential energy $V_0(x)$. The time-dependent part of the wave function will be denoted by the same symbol as the complete function: $\psi(x,t)=\exp(-iEt)\psi(x)$. The reality condition (9) is satisfied for all real functions $\psi(x)$. We shall therefore consider solutions of Eq. (57) for real E. Under these conditions, the nonstationary transformation introduced in Sec. 2.1 becomes the well known Darboux transformation.[8] Many of the known properties of this transformation can be obtained as special cases of the properties that we have derived for the nonstationary transformation. On the other hand, in the stationary case it is possible to establish properties which cannot be proved for the more general nonstationary equation. We shall consider these problems in more detail.

3.1. Factorization of the *N*th-order Darboux transformation

1. The effect of a chain of N Darboux transformations is equivalent to a single Nth-order transformation. The question of whether every Nth-order transformation can be represented as a chain of first-order Darboux transformations seems to us to require further study. The second-order Darboux transformation was studied in Ref. 24, where it was shown that there are two types of such transformation: re-

ducible and irreducible. Both types amount to a superposition of first-order transformations, but in the first case the intermediate Hamiltonian is Hermitian, while in the second it is not. (In our opinion, it can only be stated that, in general, in the first case the intermediate potential is a real-valued function, while in the second it is a complex-valued one; the final potential in both cases is a real-valued function.) In Ref. 54 (Theorem 5) for the special case of potentials whose Darboux transformation leads to a new potential differing from the original one by a constant term, it is stated that an Nth-order transformation can always be represented as a superposition of N first-order transformations. This statement was generalized in Ref. 12 to arbitrary potentials, but the arguments of that study are obscure. We think that this statement can be rigorously proved for an arbitrary potential and made more precise by excluding from the chain transformations whose composition is a symmetry operator of the original Schrödinger equation.

2. Let us consider the Schrödinger equation (57). In this case let T_0 be the functional (if necessary, topological) space of solutions of Eq. (57). In this section we shall use the notation $\partial_x \equiv D$ for the differentiation operator.

Definition. *A linear differential operator $\hat{L}^{(N)}$ of order N, with coefficient of D^N equal to unity, acting from T_0 to $T_{N_1}=\{\varphi\colon\ \varphi=\hat{L}\psi,\ \psi\in T_0\}$, will be called an order-N Darboux transformation operator if*

$$\hat{L}^{(N)}h_0-h_0\hat{L}^{(N)}=A_N(x)\hat{L}^{(N)}, \tag{58}$$

where $A_N(x)$ is a sufficiently smooth function. For $A_N(x)\equiv 0$ the operator $\hat{L}^{(N)}$ will be termed trivial.

It immediately follows from this definition that the function $\varphi=\hat{L}^{(N)}\psi$ satisfies the Schrödinger equation with potential $V_N(x)=V_0(x)+A_N(x)$, and $T_{N_1}\subset T_N$, where T_N is the functional space of solutions of the Schrödinger equation with potential $V_N(x)$.

For $N=1$, Eq. (58) defines an ordinary first-order Darboux transformation [see Eqs. (11) and (12) for $L_1=1$]. The function u_α entering into these expressions is called the transformation function and is determined by the initial Hamiltonian h_0: $h_0u_\alpha=\alpha u_\alpha$, $\alpha\in\mathbb{R}$, $\operatorname{Im}u_\alpha=0$. The operator $\hat{L}$ has nontrivial kernel $\ker\hat{L}=\operatorname{span}\{u_\alpha\}$, where span denotes the linear envelope over the field $\mathbb{C}$.

If we choose $\tilde{u}_\alpha$ such that $W(u_\alpha,\tilde{u}_\alpha)=1$, where W denotes the Wronskian, then $\hat{L}\tilde{u}_\alpha=u_\alpha^{-1}=v_\alpha$ and $h_1v_\alpha=\alpha v_\alpha$, $h_1=h_0+A_1(x)$. In addition, by direct calculation we can show that $\lim_{E\to\alpha}R^{-1}(E)\hat{L}\psi_E(x)=\tilde{v}_\alpha(x)$, $R(E)=E-\alpha$, under the condition that $\psi_E(x)\to u_\alpha(x)$ for $E\to\alpha$. In this case, $h_1\tilde{v}_\alpha=\alpha\tilde{v}_\alpha$ and $W(v_\alpha,\tilde{v}_\alpha)=W(u_\alpha,\tilde{u}_\alpha)=1$. Therefore, we can always define a linear operator L on the space T_0 by setting $\varphi_E=L\psi_E=R^{-1/2}(E)\hat{L}\psi_E$, $\forall E\neq\alpha$, $L\tilde{u}_\alpha=\hat{L}\tilde{u}_\alpha=v_\alpha=u_\alpha^{-1}$, and $Lu_\alpha=\tilde{v}_\alpha$. The operator L associates a unique element $\varphi\in T_1$, where T_1 is the functional space of solutions of the Schrödinger equation with Hamiltonian h_1, with each element $\psi\in T_0$. Here $W(\varphi_E,\tilde{\varphi}_E)=W(\psi_E,\tilde{\psi}_E)$, $\forall\psi_E,\tilde{\psi}_E\in T_0$.

It follows from (58) that if $A_N(x)$ is a real-valued function, then $\hat{L}^{(N)+}\hat{L}^{(N)}$, where $\hat{L}^{(N)+}$ is the formal conjugate of the operator $\hat{L}^{(N)}$, will be a differential symmetry operator of order $2N$ for Eq. (57), and so it will be a polynomial of order N in h_0.

Before turning to the proof of the basic theorem, let us consider three lemmas.

Lemma 1. *The operator $\hat{L}\equiv\hat{L}^{(1)}$ is a Darboux transformation operator if and only if $\hat{L}^+\hat{L}=h_0-\alpha$, $\alpha\in\mathbb{R}$.*

This lemma is fairly easy to prove by direct calculation, and we shall not dwell on this. We only note that this statement forms the basis of the well known factorization method in quantum mechanics (see, for example, Ref. 11).

Since $\ker\hat{L}^+=\operatorname{span}\{v_\alpha=u_\alpha^{-1}\}$, we can define the operator L^+ on the space T_1, setting $L^+\varphi_E=R^{-1/2}(E)\hat{L}^+\varphi_E$, $\forall E\neq\alpha$, and $L^+\tilde{v}_\alpha=\hat{L}^+\tilde{v}_\alpha=u_\alpha=v_\alpha^{-1}$, $L^+v_\alpha=\tilde{u}_\alpha$. The operators L and L^+ effect a one-to-one mapping of the spaces T_0 and T_1. In addition, $T_0=T_{01}\cup\operatorname{span}\{\tilde{u}_\alpha\}$, $T_1=T_{11}\cup\operatorname{span}\{\tilde{v}_\alpha\}$, $T_{01}=\{\psi\colon\ \psi=\hat{L}^+\varphi,\ \varphi\in T_1\}$.

Lemma 2. *The operator $\hat{L}\equiv\hat{L}^{(2)}$ can always be written in the form $\hat{L}=\hat{L}_2\hat{L}_1$, where $\hat{L}_1=-u_1'/u_1+D$, $\hat{L}_2=-v'/v+D$ are first-order Darboux transformation operators, u_1 is a transformation function satisfying Eq. (57) for some eigenvalue C_1, and v is the transformation function for the next first-order Darboux transformation satisfying the Schrödinger equation with intermediate potential V_1 obtained from V_0 as the result of a Darboux transformation with operator $\hat{L}_1$ and corresponding to the eigenvalue C_2. If C_1 and $C_2\in\mathbb{R}$, then they are arbitrary and the functions u_1 and v are real-valued. If C_1 and $C_2\in\mathbb{C}$, then $C_2=\bar{C}_1$ and $v=\hat{L}_1\bar{u}_1$. The potential difference $A_2(x)$ is a real-valued function.*

First of all, we note that an analogous statement is made in Ref. 24, but some details of the proof will be useful for what follows, and we shall present it in its entirety.

Let us consider $\hat{L}=a_0(x)+a_1(x)D+a_2(x)D^2$. From (58) we obtain a system of equations for the functions $a_i(x)$, $i=0,1,2$, and $A(x)\equiv A_2(x)$. It follows from this system that $a_2=\text{const}$, and we can safely set $a_2=1$. In addition, $A=2a_1'$. Eliminating a_0 and A from the system, we obtain a differential equation for the function a_1, which can easily be integrated twice with the constants $2\alpha_1$ and $\alpha_2\in\mathbb{R}$. As a result, we obtain the following differential equation for a_1:

$$a_1^2V_0+a_1^2a_1'-\frac{1}{2}a_1a_1''+\frac{1}{4}a_1'^2-\frac{1}{4}a_1^4-\alpha_1a_1^2-\alpha_2=0. \tag{59}$$

We introduce the new variable u_1, setting

$$u_1'/u_1=\frac{1}{2}a_1'/a_1-\frac{1}{2}a_1-\sqrt{\alpha_2}/a_1. \tag{60}$$

Here Eq. (59) takes the form $-D^2u_1+(V_0-C_1)u_1=0$, where $C_1=\alpha_1-\sqrt{\alpha_2}$, i.e., the function u_1 is a solution of the original Schrödinger equation. By determining this function we solve (60). We introduce the new function v, setting $a_1=-[\ln(vu_1)]'$. From Eq. (60) we obtain the equation for the function v:

$$-D^2v+(V_1-C_2)v=0,$$

where $C_2=\alpha_1+\sqrt{\alpha_2}$ and $V_1=V_0-2(\ln u_1)_{xx}$. From this we see that the function v is a solution of the Schrödinger equation obtained from (57) by using the transformation operator $\hat{L}_1$ with the transformation function u_1. Using the fact that $a_1=-[\ln(vu_1)]'$ and $a_0=u_1'v'/(u_1v)-(\ln u_1)''$, we obtain the expression for the operator $\hat{L}$ stated by the lemma. In addition, for $C_2\neq C_1$ we have $v=\hat{L}_1u_2=u_1^{-1}W(u_1,u_2)$, where $h_0u_2=C_2u_2$. For the potential difference we obtain the expression

$$A=-2[\ln W(u_1,u_2)]''. \tag{61}$$

For $C_2=C_1=C$ we have $v=\beta_1u_1^{-1}+\beta_2\tilde{v}$, where β_1, $\beta_2\in\mathbb{R}$, and u_1^{-1} and $\tilde{v}$ are linearly independent solutions of the Schrödinger equation with potential V_1 for $E=C$. In this case the Wronskian W in (61) must be replaced: $W\to\beta_1+\beta_2u_1\tilde{v}$.

Corollary 1. *It follows directly from Lemmas 1 and 2 that* $\hat{L}^+\hat{L}=(h_0-C_1)(h_0-C_2)$.

Remark 1. *For* $C_1=C_2=C\in\mathbb{R}$ *and* $v=u_1^{-1}$ $(\beta_2=0)$, *the operator* $\hat{L}=-\hat{L}_1^+\hat{L}_1=C-h_0$ *is the trivial transformation operator.*

Remark 2. *In the case* $C_2\neq C_1$ *the intermediate function v can be eliminated. Here we obtain a well known expression (Ref. 41) for* $\hat{L}$, *which for a chain of N transformations has the form*

$$\hat{L}^{(N)}=\hat{L}_N\hat{L}_{N-1}\ldots\hat{L}_1=W^{-1}(u_1,\ldots,u_N)\times\begin{vmatrix} u_1 & u_2 & \ldots & 1\\ u_1' & u_2' & \ldots & D\\ \ldots & \ldots & \ldots & \ldots\\ u_1^{(N)} & u_2^{(N)} & \ldots & D^N\end{vmatrix}. \tag{62}$$

Moreover, $\hat{L}^{(N)+}\hat{L}^{(N)}=P(h_0)=\Pi_{i=1}^N(h_0-C_i)$, *where* $h_0u_i=C_iu_i$ *and all the* C_i *are distinct. If the coefficients of the polynomial* $P(x)$ *belong to the field* $\mathbb{R}$, *then the intermediate potentials can be complex-valued [when* $P(x)$ *contains complex zeros], but the final potential will be a real-valued function for the appropriate choice of transformation functions.*

We use Q to denote the following polynomial of degree N with real coefficients: $Q(h_0)=\hat{L}^{(N)+}\hat{L}^{(N)}$. Let $C_1,\ldots,C_N$ be the zeros of the polynomial $Q(x)$ of arbitrary multiplicity.

Lemma 3. *If* $\hat{L}^{(N)}$ *is an Nth-order Darboux transformation operator, then* $\ker\hat{L}^{(N)}\cap\cup_{i=1}^N\ker(h_0-C_i)\neq\emptyset$.

Let us consider one of the zeros of the polynomial $Q(x)$, for example, C_1. If $\ker\hat{L}^{(N)}\cap\ker(h_0-C_1)\neq\emptyset$, the lemma is proved. Let

$$\ker\hat{L}^{(N)}\cap\ker(h_0-C_1)=\emptyset. \tag{63}$$

We write $v_1=\hat{L}^{(N)}u_1$, $\tilde{v}_1=\hat{L}^{(N)}\tilde{u}_1$, where u_1 and $\tilde{u}_1$ form a basis in the space $\ker(h_0-C_1)$. Owing to the linearity of the operator $\hat{L}^{(N)}$ and the assumption (63), $\operatorname{span}\{v_1,\tilde{v}_1\}$ cannot be a one-dimensional space. Then

$$\operatorname{span}\{v_1,\tilde{v}_1\}=\ker(h_N-C_1)\subset\ker\hat{L}^{(N)+}.$$

Using Proposition 2.1 of Ref. 4, the operator $\hat{L}^{(N)+}$ can be written in the form $\hat{L}^{(N)+}=\hat{L}^{(N-2)+}\hat{L}_2^+\hat{L}_1^+$, where $\hat{L}_1^+=(d/dx)\ln v_1-D$, $\hat{L}_2^+=(d/dx)\ln(W(v_1,\tilde{v}_1)/v_1)-D$. Using the fact that the functions v_1 and $\tilde{v}_1$ are linearly independent solutions of the Schrödinger equation for a given E, we obtain

$$\hat{L}^{(N)+}=-\hat{L}^{(N-2)+}(h_N-C_1).$$

Here $\hat{L}^{(N-2)+}$ is the $(N-2)$th-order Darboux transformation operator from solutions of the Schrödinger equation with Hamiltonian h_N to solutions of the same equation with Hamiltonian h_0. However, then $\hat{L}^{(N)}=-\hat{L}^{(N-2)}(h_0-\bar{C}_1)$, which for $C_1\in\mathbb{R}$ contradicts (63), and for $C_1\in\mathbb{C}$ leads to the statement of the lemma, because in this case $\bar{C}_1$ is also a zero of the polynomial $Q(x)$.

Now we can state and prove the basic theorem.

Theorem. *The action of every nontrivial operator* $\hat{L}^{(N)}$ *is equivalent to the action resulting from a chain of k* $(k\leqslant N)$ *first-order Darboux transformations.*

Using Lemmas 2 and 3 and arguing by induction, the operator $\hat{L}^{(N)}$ can always be written in the form $\hat{L}^{(N)}=\hat{L}_N\hat{L}_{N-1}\ldots\hat{L}_1$, which corresponds to a chain of N first-order Darboux transformations. If the conditions of Remark 1 are satisfied, some operator products in the chain will be trivial transformation operators. In this case $\hat{L}^{(N)}=\hat{L}^{(k)}P(h_0)$, where $\hat{L}^{(k)}=\hat{L}_{t+k}\hat{L}_{t+k-1}\ldots\hat{L}_t$ and $P(x)$ is a polynomial. The transformation operators $\hat{L}^{(N)}$ and $\hat{L}^{(k)}$ lead to the same potential difference $A_N(x)$, which proves the theorem.

Remark 3. *We have pointed out the possibility of trivial transformation operators, which can be realized for an arbitrary potential* $V_0(x)$. *There are known examples of potentials (Ref. 54) for which longer chains reduce to the trivial transformation operator.*

3. In conclusion, we note that the operator $\hat{L}^{(k)+}$ effecting the transformation from the solutions of the Schrödinger equation with potential V_N to solutions of Eq. (57) can be used to construct the operator $\hat{L}^{(k)^{-1}}$. In addition, if $C_1,\ldots,C_q$ are distinct zeros of the polynomial $P(h_0)=\hat{L}^{(k)+}\hat{L}^{(k)}$, then for the space T_0 we have the expansion $T_0=T_{01}\cup\cup_{i=1}^q\operatorname{span}\{\tilde{u}_i\}$, $\ker(h_0-C_i)=\operatorname{span}\{u_i,\tilde{u}_i\}$, $i=1,\ldots,q$. The functions u_i are the transformation functions for the intermediate transformation operator $\hat{L}^{(q)}=\hat{L}_q\hat{L}_{q-1}\ldots\hat{L}_1$. An analogous expansion can be written down for the space T_N.

3.2. Covering chains

Let us see what constructions lead to the "covering chains" introduced in Refs. 55 and 56 and studied in detail in Ref. 54. Let a chain of Hamiltonians $h_0,h_1,\ldots,h_i=-\partial_x^2+V_i(x)$ be generated by successive application of first-order Darboux transformations. Then, according to (58), they are intertwined by the Darboux transformation operators $\hat{L}_i=-L_{0i}(x)+\partial_x$:

$$h_{i+1}\hat{L}_i=\hat{L}_ih_i. \tag{64}$$

It follows from (64) that $V_{i+1}-V_i=-2L_{0i}'(x)$ and

$$L_{0i}''+(L_{0i}^2)'-V_i'=-L_{0i}''+(L_{0i}^2)'-V_{i+1}'=0.$$

From this we see that

$$L'_{0i}+L^2_{0i}-V_i=-L'_{0i}+L^2_{0i}-V_{i+1}=-\alpha_i, \tag{65}$$

where α_i is an integration constant. From this we find the usual relations for Darboux transformations:

$$L_{0i}=\ln u_i, \quad A_i=V_{i+1}-V_i=-2(\ln u_i)'',$$

$$h_i u_i=\alpha_i u_i. \tag{66}$$

Let us consider the difference of two successive relations (65):

$$(L_{0i}+L_{0i+1})'-L^2_{0i}+L^2_{0i+1}=\beta_i, \quad \beta_i=\alpha_i-\alpha_{i+1}. \tag{67}$$

The system of equations (67) is called a covering chain in Ref. 54. It is also clear that the Hamiltonian nature of this chain, which is studied in detail in Ref. 54, ultimately derives from the relations (66) and, in particular, from the possibility of integrating it using transformation functions u_i satisfying the Schrödinger equation with Hamiltonian h_i.

The properties of closed chains have been studied in detail in Ref. 54. The closure condition in terms of the chain of Hamiltonians is $h_N=h_0+\alpha$, where α is an arbitrary real constant. The Nth-order operator $\hat{L}^{(N)}=\hat{L}_{N-1}\hat{L}_{N-2}\ldots\hat{L}_0$ will effect the transformation from the Hamiltonian h_0 directly to h_N. According to (58), it will be defined by the relation

$$[\hat{L}^{(N)},h_0]=\alpha\hat{L}^{(N)}. \tag{68}$$

Equation (68) can be viewed as an equation determining the operators h_0 and $\hat{L}^{(N)}$. It turns out[54] that the properties of the operator h_0 depend strongly on whether or not α is equal to zero. For $\alpha=0$ the operator $\hat{L}^{(N)}$ is a symmetry operator for the Schrödinger equation with Hamiltonian h_0. If $N=2n+1$ is an odd number, the operator h_0 has no more than n zones in its spectrum, and all the operators of the chain, $h_0,h_1,\ldots,h_{N-1}$, are finite-zone operators. Such operators have been studied in Ref. 57. In the simplest nontrivial case $N=3$ we obtain a one-zone potential of the form $V_0(x)=2\wp(x-x_0)+\text{const}$, where $\wp$ is the Weierstrass elliptic $\wp$ function satisfying the differential equation

$$(\wp')^2=4(\wp-e_1)(\wp-e_2)(\wp-e_3),$$

where e_i are the potential parameters. The solutions of the Schrödinger equation with this potential are expressed in terms of elliptic functions. The case of even N reduces to the previous one. All the elements of the theory of finite-zone potentials can be reproduced by using the properties of the covering chain.[54]

For $\alpha\neq 0$, Eq. (68) gives an overdetermined system of equations for the coefficients of the operator $\hat{L}^{(N)}$. The condition for its solvability leads to a nonlinear equation for the potential $V_0(x)$ (Ref. 58). In Refs. 58 and 59 several elementary solutions of this equation were obtained which lead to potentials whose spectrum consists of an equidistant part and an isolated ground-state level. It was shown in Ref. 12 that these solutions can be obtained from the harmonic-oscillator potential by an ordinary Darboux transformation. Later on we will give other potentials of elementary form which can be obtained in this manner.

It was shown in Ref. 54 that the spectrum of the operator h_0 satisfying Eq. (68) for odd N consists of N arithmetic progressions with the first terms $0,\varepsilon_1,\varepsilon_1+\varepsilon_2,\ldots,\varepsilon_1+\ldots+\varepsilon_{N-1}$ and the difference $\varepsilon=\varepsilon_1+\varepsilon_2+\ldots+\varepsilon_N$, where $\varepsilon_i=-\beta_i>0$. In particular, for $N=3$ the general solution of the chain and the explicit form of the potential $V_0(x)$ can be expressed in terms of the solutions of the fourth Panlevé equation (PIV). The following expression was obtained for $V_0(x)$ in Ref. 54:

$$V_0(x)=-2Z'+\alpha^2x^2/4+\text{const}, \tag{69}$$

where the function Z is a solution of the equation

$$(Z'')^2-\alpha^2(Z-xZ')^2+4P(Z')=0,$$

$$P(t)=t(t+\beta_2)(t-\beta_3).$$

The function $L_{01}(x)$ determining the first operator in the transformation chain is expressed in terms of the solutions of the PIV equation:

$$y''=\frac{1}{2y}(y')^2+\frac{3}{2}y^3+4xy^2+2(x^2-a)y+\frac{b}{y},$$

$$L_{01}=y-x.$$

The form of the potential $V_0(x)$ suggests that it can be obtained from the harmonic-oscillator potential $\alpha^2x^2/4$ by using a Darboux transformation or a chain of such transformations with a suitable choice of transformation functions. This approach could establish a connection between the PIV equation and the Schrödinger equation for the harmonic oscillator. In particular, the elementary solution of Eq. (68) obtained in Ref. 58 corresponds to one of the rational solutions of the PIV equation.[60] We think that the results of Ref. 12 can be used to establish an analogous correspondence between other known polynomial solutions of the PIV equation[61] and solutions of the Schrödinger equation for the harmonic oscillator.

3.3. Examples of exactly solvable potentials

3.3.1. Potentials with equidistant and quasiequidistant spectra

Let $h_0=-\partial_x^2+x^2/4-1/2$. First we consider the case with a transformation function $u(x)\notin H_0(R)$. In this case there is a series of elementary solutions of the Schrödinger equation of the form

$$u_m(x)=\psi_{-m}(x)=\exp(x^2/4)H_m(ix/\sqrt{2}),$$

$$h_0u_m=-(m+1)u_m, \quad m=0,1,2,..., \tag{70}$$

where $H_m(z)$ are the Hermite polynomials.[62] A single Darboux transformation with the functions (70) for even $m=2k$ generates a family of potentials regular on the entire axis:

$$V_1^{(2k)}=x^2/4-3/2+8k^2[q_{2k-1}(x)/q_{2k}(x)]^2-4k(2k-1)q_{2k-2}(x)/q_{2k}(x),$$

$$q_k(x)=(-i)^kHe_k(ix), \quad He_k(x)=2^{-k/2}H_k(x/\sqrt{2}),$$

$$q_{k+1}(x)=xq_k(x)+kq_{k-1}(x), \quad q_0(x)=1, \quad q_1(x)=x. \tag{71}$$

The potential (71) for $k=1$ was studied in detail in Ref. 59. According to Ref. 54, these potentials can be interpreted as potentials whose spectrum is made up of $2k+1$ segments with equidistant spectrum. The discrete spectrum of the new Hamiltonians $h_1^{(m)}=-\partial_x^2+V_1^{(m)}(x)$ has one extra ground-state level $E_0=-(m+1)$ in addition to the equidistant spectrum of the original Hamiltonian h_0. All the potentials (71) have a sharp minimum near $x=0$, and as x increases their behavior approaches the parabolic law $x^2/4$ more and more closely. The wave functions of the discrete spectrum have the form

$$\begin{aligned}\varphi_0(x)&=\sqrt{(2k)!}(2\pi)^{-1/4}\exp(-x^2/4)/q_{2k}(x),\\ \varphi_{n+1}(x)&=(2\pi)^{-1/4}(n!)^{-1/2}(n+2k+1)^{-1/2}\\ &\quad\times[He_n(x)q_{2k+1}(x)/q_{2k}(x)\\ &\quad-nHe_{n-1}(x)]\exp(-x^2/4),\\ n&=0,1,2,\ldots.\end{aligned} \tag{72}$$

For odd $m=2k+1$ the functions (70) vanish only at $x=0$ and can be used as the transformation functions for the spectral problem on the semiaxis $x\geqslant 0$. We then obtain the following family of new potentials:

$$\begin{aligned}V_1^{(2k+1)}(x)&=x^2/4-3/2+2(2k+1)^2[q_{2k}(x)/q_{2k+1}(x)]^2\\ &\quad-4k(2k+1)q_{2k-1}(x)/q_{2k+1}(x).\end{aligned}$$

As the solutions of the new Schrödinger equation for $E=n$ $(n=0,1,2,\ldots)$ we obtain the functions

$$\begin{aligned}\varphi_n(x)&=\sqrt{2}(2\pi)^{-1/4}(n!)^{-1/2}(n+2k+2)^{-1/2}\\ &\quad\times[He_n(x)q_{2k+2}(x)/q_{2k+1}(x)\\ &\quad-nHe_{n-1}(x)]\exp(-x^2/4).\end{aligned} \tag{73}$$

For odd n the functions (73) vanish at $x=0,\infty$, and for even n they are singular at the origin. Therefore, odd values of n correspond to the discrete spectrum of the new potentials. The functions (73) are normalized to unity on the semiaxis. For $n=1$ we obtain the ground-state function.

Now we use two functions of the discrete spectrum, $\psi_k(x)=\exp(-x^2/4)He_k(x)$ and $\psi_{k+1}(x)$, as the transformation functions for a twofold Darboux transformation. As a result, we obtain another family of potentials:

$$V_2^{(k,k+1)}(x)=\frac{x^2}{4}+\frac{3}{2}-2\frac{J_k''(x)}{J_k(x)}+2\left[\frac{J_k'(x)}{J_k(x)}\right]^2,$$

$$J_k(x)=\sum_{i=0}^{k}\frac{\Gamma(k+1)}{\Gamma(i+1)}He_i^2(x).$$

These potentials are even functions of x and are similar to the parabola $x^2/4$, at the bottom of which are located k small minima at the same depth. In Fig. 1 we give graphs of the potentials $V=V_2^{(k,k+1)}+5k$, shifted for convenience, where k corresponds to the number of the curve. The discrete spectrum of these potentials differs from the harmonic-oscillator

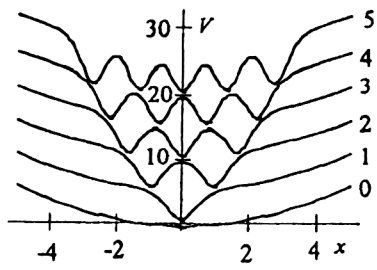

FIG. 1. k-well potentials with quasi-equidistant spectrum.

spectrum by the presence of the levels $E=k$ and $E=k+1$. The functions of the discrete spectrum, normalized to unity, have the form

$$\begin{aligned}\varphi_n(x)&=(2\pi)^{-1/4}(n!)^{-1/2}[(n-k)(n-k-1)]^{-1/2}\\ &\quad\times\exp(-x^2/4)[(n-k)He_n(x)+(He_k(x)\\ &\quad\times He_{n+1}(x)-He_n(x)He_{k+1}(x))\\ &\quad\times He_{k+1}/J_k(x)],\\ n&\neq k,\ k+1.\end{aligned}$$

Let us now consider the properties of the second-order Wronskian constructed from the functions (70) not belonging to the discrete spectrum of the original problem:

$$\begin{aligned}W_{m,l}(x)&\equiv W(u_m,u_l)=f_{m,l}(x)\exp(x^2/2),\\ f_{m,l}(x)&=q_m(x)q_{l+1}(x)-q_l(x)q_{m+1}(x).\end{aligned}$$

Using the expression for the derivative of this Wronskian, $W'_{m,l}(x)=(l-m)q_m(x)q_l(x)\exp(x^2/2)$, and the definition of the polynomials $q_m(x)$ in (71), we can show that the function $W_{m,l}(x)$ is monotonic. The parity of the polynomials $q_m(x)$ is the same as the parity of the number m. The coefficients of all powers of x are integers. For even m, $q_m(x)>0$, and for odd m, $q_m(x)=0$ only for $x=0$, and the zero is simple. Moreover, $W_{m,l}(0)=l!(m-1)!>0$. Therefore, for positive m and l of opposite parity the function $W_{m,l}(x)$ for $l>m$ falls off for $x<0$, increases for $x>0$, has a single minimum at $x=0$, and is always positive. For $m>l$, $W_{m,l}(x)$ grows for $x<0$, decreases for $x>0$, and has two symmetrically located simple zeros. When m and l have the same parity, assuming for definiteness $l>m$, we find that for even m,l, $W'_{m,l}(x)\geqslant 0$. The equality holds only for $x=0$, and the zero is of second order. For odd m,l, $W'_{m,l}>0$, and in both cases $W'_{m,l}(x)=W'_{m,l}(-x)$. In this case the function $W_{m,l}(x)$ increases monotonically and has a single simple zero at $x=0$ for even m and l, and a triple zero at $x=0$ for odd m and l. Therefore, for $m=0,2,4,\ldots$ and $l=m+1,\ m+3,\ m+5,\ldots$ the functions (70) are suitable for a twofold Darboux transformation.

For the new potential we obtain the expression

$$\begin{aligned}V_2^{(m,l)}(x)&=x^2/4-5/2-2f''_{m,l}(x)/f_{m,l}(x)\\ &\quad+2[f'_{m,l}(x)/f_{m,l}(x)]^2.\end{aligned} \tag{74}$$

The potential (74) is a regular function on the entire real axis. The Hamiltonians $h_2^{(m,l)}$ have two additional levels in

the discrete spectrum, $E_0=-l-1$ and $E_1=-m-1$, relative to h_0. The first level corresponds to the ground state, and the second one to the first excited state. The wave functions of these states, normalized to unity, are

$$\varphi_0^{(m,l)}(x)=(2\pi)^{-1/4}\sqrt{l!(l-m)}\times\exp(-x^2/4)q_m(x)/f_{m,l}(x),$$

$$\varphi_1^{(m,l)}(x)=(2\pi)^{-1/4}\sqrt{m!(l-m)}\times\exp(-x^2/4)q_l(x)/f_{m,l}(x).$$

For the other functions of the discrete spectrum we obtain

$$\varphi_{n+2}(x)=(2\pi)^{-1/4}(n!)^{-1/2}[(n+l+1)(n+m+1)]^{-1/2}\exp(-x^2/4)[(n+1)He_n(x)+((m-l)q_m(x)q_l(x)He_{n+1}(x)-mlHe_n(x)f_{m-1,l-1}(x))/f_{m,l}(x)],$$

$n=0,1,2,\ldots$.

Higher-order transformations can be treated in a completely analogous fashion. For example, the fourth-order Darboux transformation with two functions of the discrete spectrum with $n=2$ and $n=3$ and two functions (70) with $m=2$ and $m=3$ generates a potential of the form

$$V_4(x)=x^2/4-1/2-24(1467x^2+6x^6-x^{10})Q^{-1}(x)+82944(105x^2+140x^6+3x^{10})Q^{-2}(x),$$

$$Q(x)=315+315x^4+9x^8+x^{12}.$$

Let us consider another interesting potential obtained by a single Darboux transformation. We choose the transformation function in the form $u=\psi_{-1/2}(x)={}_0F_1(3/4, x^4/64)$, where ${}_0F_1$ is the hypergeometric function ${}_pF_q$. This function is a solution of the Schrödinger equation for the harmonic oscillator with $E=-1/2$. As a result, we obtain a double-well potential. The minima are located symmetrically at the points $x=\pm 1.68$. The corresponding minimum value of the potential energy is $V_1(x_{\min})=-0.94$. The potential energy has a local minimum at $x=0$, and $V_1(0)=-1/2$. An additional level of the discrete spectrum is located at $E=-1/2$, i.e., the energy level is tangent to the potential curve at its maximum. The ground-state wave function is determined by the transformation function, up to a normalization: $\varphi_0(x)=\psi_{-1/2}^{-1}(x)$.

Let us now consider families of isospectral potentials. The simplest one is generated by using the following solution of the original Schrödinger equation:

$$u(x)=\exp(x^2/4)(C+\operatorname{erf}(x/\sqrt{2}),\quad h_0u(x)=-u(x). \tag{75}$$

The resulting potential

$$V_1(x)=x^2/4-3/2+2xQ_1^{-1}(x)\exp(-x^2/2)+2Q_1^{-2}(x)\exp(-x^2),$$

$$Q_1(x)=\sqrt{\frac{\pi}{2}}\,(C+\operatorname{erf}(x/\sqrt{2})) \tag{76}$$

was generated by using an integral transformation in Ref. 15 and later by using a Darboux transformation in Ref. 63. However, the complete expressions for the wave functions, including normalization factors, are apparently not to be found in the accessible literature. For $|C|>1$ we obtain the following set of wave functions, normalized to unity:

$$\varphi_0(x)=(2\pi)^{-1/4}\sqrt{C^2-1}\,u^{-1}(x),$$

$$\varphi_{n+1}(x)=(2\pi)^{-1/4}[(n+1)!]^{-1/2}(He_{n+1}(x)\times\exp(-x^2/4)+He_n(x)Q_1^{-1}(x)\times\exp(-3x^2/4)),\quad n=0,1,2,\ldots . \tag{77}$$

Using the general solution of the original Schrödinger equation for $E=-2$,

$$u(x)=\exp(-x^2/4)+x\,\exp(x^4/4)\times\left(-C+\sqrt{\frac{\pi}{2}}\operatorname{erf}(x/\sqrt{2})\right),$$

for $|C|<\sqrt{\pi/2}$ we obtain another isospectral potential:

$$V_1(x)=-\frac{x^2}{4}-\frac{7}{2}+\frac{1}{2}\left(\frac{x+(2+x^2)\left(\sqrt{\frac{\pi}{2}}\operatorname{erf}(x/\sqrt{2})-C\right)\exp(x^2/2)}{1-x\left(C-\sqrt{\frac{\pi}{2}}\operatorname{erf}(x/\sqrt{2})\right)\exp(x^2/2)}\right)^2.$$

3.3.2. The effective Coulomb potential

As our next example we consider the effective Coulomb potential

$$V_0(x)=-2z/x+l(l+1)/x^2,\quad E_n=-z^2/n^2,$$

$n=1,2,3,\ldots$,

$$\psi_{nl}(x)=N_{nl}^0x^{l+1}\exp(-zx/l)L_{n-l-1}^{2l+1}(2zx/n),$$

$$N_{nl}^0=(2/n^2)(2/n)^lz^{l+3/2}\sqrt{(n-l-1)!/(n+l)!}.$$

Here $L_n^\alpha(x)$ are the generalized Laguerre polynomials,[62] and the functions ψ_{nl} are normalized by the condition $\int_0^\infty\psi_{nl}^2(x)dx=1$.

A twofold Darboux transformation with functions of the discrete spectrum for $n=k$ and $n=k+1$ as the transformation functions generates the following exactly solvable potential:

$$V_2^{(l,k,k+1)}(x)=-2z/x+(l+1)(l+4)/x^2-2w_0''(x)/w_0(x)+2[w_0'(x)/w_0(x)]^2,$$

where

$$w_0(x)=2(k+1)L_{k-l-2}^{2l+2}(2zx/k)L_{k-l}^{2l+1}(2zx/(k+1))+L_{k-l-1}^{2l+1}(2zx/k)[L_{k-l}^{2l+1}(2zx/(k+1))-2kL_{k-l-1}^{2l+2}(2zx/(k+1))].$$

The functions of the discrete spectrum, normalized to unity, have the form

$$\varphi_{nl}^{(k)}(x)=N_{nk}\left[\left(\frac{1}{n^2}+\frac{w_1(x)}{nk(k+1)w_0(x)}\right)\psi_{nl}(x)\right.$$
$$-N_{nl}^0\frac{2(2k+1)}{nk(k+1)w_0(x)}L_{k-l-1}^{2l+1}(2zx/k)$$
$$\times L_{k-l}^{2l+1}(2zx/(k+1))x^{l+1}L_{n-l-2}^{2l+2}(2zx/n)$$
$$\left.\times\exp(-zx/n)\right],$$

where

$$w_1(x)=2n(k+1)L_{k-l-1}^{2l+1}(2zx/k)L_{k-l-1}^{2l+2}(2zx/(k+1))$$
$$+L_{k-l}^{2l+1}(2zx/(k+1))[(n-1)L_{k-l-1}^{2l+1}(2zx/k)$$
$$-2k(nL_{k-l-2}^{2l+2}(2zx/k)+L_{k-l-1}^{2l+1}(2zx/k))]$$

and

$$N_{nk}=n^2k(k+1)[(k^2-n^2)((k+1)^2-n^2)]^{-1/2}.$$

The simplest nontrivial potential corresponds to the case $l=0$, $k=2$:

$$V_2^{(0,2,3)}(x)=-2z/x+10/x^2+40Q_0^{-1}(xz)(2-xz)/x^2$$
$$-100Q_0^{-2}(xz)(2xz-3)/x^2,$$
$$Q_0(x)=-15+10x-2x^2.$$

Its discrete spectrum does not contain levels with $n=2$ and $n=3$. The functions of the discrete spectrum, normalized to unity, are determined by a system of new polynomials $P_n(x)$ orthogonal on the interval $(0,\infty)$:

$$\varphi_{nl}(x)=\frac{1}{6}N_{nl}^0N_{n2}n^{-2}z^2x^3P_n(xz)Q_0^{-1}(xz)$$
$$\times\exp(-xz/n).$$

These polynomials are defined in terms of Laguerre polynomials as

$$x^3P_n(x)=10n(-54+63x-22x^2+2x^3)L_{n-2}^2(2x/n)$$
$$+L_{n-1}^1(2x/n)[270n(n-1)-45(2-7n$$
$$+5n^2)x+10(6-11n+5n^2)x^2-2(6-5n$$
$$+n^2)x^3].$$

The first five polynomials have the form

$$P_1(x)=-4,\quad P_2(x)=P_3(x)=0,$$
$$P_4(x)=\frac{1}{24}(-84+63x-18x^2+2x^3),$$
$$P_5(x)=\frac{8}{625}(-875+700x-220x^2+30x^3-x^4).$$

Let us also give the expression for the potential obtained by using a fourfold Darboux transformation with transformation functions with $l=0$, $n_1=3$, $n_2=4$, $n_3=5$, and $n_4=6$:

$$V_4^{(0,3,4,5,6)}(x)=-2z/x+38/x^2-144x^{-2}Q_2^{-1}(xz)(492$$
$$-228xz+27x^2z^2-x^3z^3)$$
$$-31104x^{-2}Q_2^{-2}(xz)(9240-2640xz$$
$$+255x^2z^2-8x^3z^3),$$
$$Q_2(x)=11880-3960x+540x^2-36x^3+x^4.$$

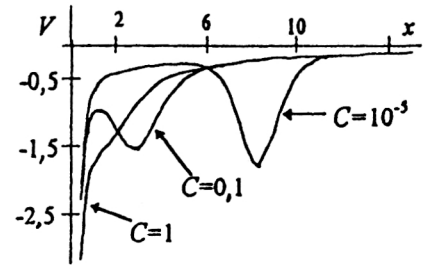

FIG. 2. Isospectral potentials with the spectrum of the hydrogen atom.

In contrast to the harmonic oscillator, here we can obtain families of isospectral potentials of elementary form. For this it is necessary to use solutions of the original equation singular at $x=0,\infty$ as the transformation functions. We construct the elementary solutions with the required properties by using the functions

$$u_{nl}(x)=x^{-1}L_n^{-2l-1}(2zx/(n-l))\exp(-zx/(n-l)),$$
$$h_0u_{nl}(x)=-z^2/(n-l)^2u_{nl}(x).$$

For $l=1$ and $n=2$ the general solution of the original Schrödinger equation has the form

$$u(x)=x^{-1}\exp(-xz)Q_3(xz),$$
$$Q_3(x)=1+2x+2x^2+C\exp(2x),$$
$$h_0u(x)=-z^2u(x).$$

Using this transformation function, we obtain a family of potentials whose spectrum coincides with that of the hydrogen-like atom:

$$V_1(x)=-2z/x-16z^3x(xz-1)Q_3^{-1}(xz)$$
$$+32z^6x^4Q_3^{-2}(xz),\quad C\in(-\infty,-1)\cup(0,\infty).$$

These potentials are shown in Fig. 2 for $z=1$ and several values of C. The functions of the discrete spectrum, normalized to unity, have the form

$$\varphi_1(x)=2\sqrt{z^3C(C+1)}u^{-1}(x),$$
$$\varphi_n(x)=N_{nl}^0nz^{-1}(n^2-1)^{-1/2}\exp(-xz/n)[x(2$$
$$-xz/n)L_{n-2}^3(2xz/n)-2x^2(z/n)L_{n-3}^4(2xz/n)$$
$$+x(1+xz+C\exp(2xz)(1-xz)$$
$$+2x^3z^3)L_{n-2}^3(2xz/n)Q_3^{-1}(xz)],$$

$n=2,3,\ldots$.

Potentials of the next order in complexity are generated for $n=4$ and $l=2$ by using the transformation function

$$u(x)=x^{-2}\exp(-xz/2)Q_4(xz),$$
$$Q_4(x)=24+24x+12x^2+4x^3+x^4+C\exp(x).$$

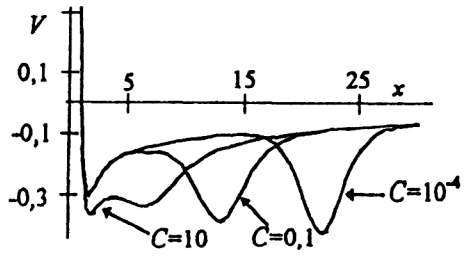

FIG. 3. Isospectral centrally symmetric potentials with centrifugal term.

The resulting potentials have a centrifugal term:

$$V_1(x)=-2z/x+2/x^2+2z^5x^3(4-xz)Q_4^{-1}(xz)+2z^{10}x^8Q_4^{-2}(xz).$$

The behavior of these potentials for $z=1$ is shown in Fig. 3. The functions of the discrete spectrum, normalized to unity, are calculated from

$$\varphi_2(x)=z^{5/2}\sqrt{C(C/4!+1)}u^{-1}(x),$$

$$C\in(-\infty,-4!)\cup(0,\infty),$$

$$\varphi_n(x)=2N_{n2}^0(n/z)\cdot(n^2-4)^{-1/2}\exp(-xz/n)\Big[x^2(3-xz/n)L_{n-3}^5(2xz/n)-2zx^3/n\cdot L_{n-4}^6(2xz/n)+\frac{1}{2}x^2Q_4^{-1}(xz)(96+72xz+24x^2z^2+4x^3z^3+x^5z^5+(4-xz)C\exp(xz))L_{n-3}^5(2xz/n)\Big],$$

$n=3,4,5\ldots$.

It is interesting to note that, as can be seen from Figs. 2 and 3, the graph of the Coulomb-like potentials is similar to that of a moving soliton, especially with the change of sign $V\to -V$.

3.3.3. The Morse potential

As the initial Morse potential we take

$$V_0(x)=\exp(-2\alpha x)-A\exp(-\alpha x),\quad A=\text{const},$$

$$E_n=-\frac{1}{4}[A-\alpha(2n+1)]^2,$$

$$\psi_n=z^\mu\exp(-z/2)L_n^{2\mu}(z),$$

$$z=\frac{2}{\alpha}\exp(-\alpha x),\quad \mu=\sqrt{|E_n|}/\alpha,$$

$$n\leqslant(A/\alpha-1)/2,\quad n=0,1,2,\ldots .$$

We shall assume that the variable x takes all real values. A twofold Darboux transformation with the functions ψ_1 and ψ_2 leads to the following exactly solvable potential:

$$V_2^{(1,2)}(x)=\exp(-2\alpha x)-\exp(-\alpha x)(A-4\alpha)+128(2\alpha-A)^{-1}\alpha^3Q_4^{-2}(x)[(A-3\alpha)\exp(\alpha x)-1]+8(2\alpha-A)^{-1}\alpha^2Q_4^{-1}(x)[4\alpha-(A-2\alpha)(A-3\alpha)\exp(\alpha x)],$$

$$Q_4(x)=4-4(A-3\alpha)\exp(\alpha x)+(A-2\alpha)(A-3\alpha)\exp(2\alpha x),\quad A\neq 2\alpha.$$

Its discrete spectrum does not contain levels with $n=1$ and $n=2$.

The Morse potential can be used as the basis for constructing a family of isospectral potentials of elementary form with a single level of the discrete spectrum. This is possible because for half-integer μ both particular solutions of the Schrödinger equation have an elementary form.

Let μ have a fixed value $\mu=-(k+1)/2$ and $A=-k\alpha$, $k=0,1,2,\ldots$. In this case the Morse potential

$$V_0(x)=\alpha^2z^2/4+\alpha^2kz/2$$

has no discrete spectrum at all. However, for $E_0=-\frac{1}{4}(k+1)^2\alpha^2$ the general solution of the Schrödinger equation has an elementary form:

$$u(x)=\exp(-z/2)z^{-(k+1)/2}Q_5(z),$$

$$Q_5(z)=1+C\exp(z)\left(z^k+k!\sum_{i=1}^{k}(-1)^iz^{k-i}/(k-i)!\right).$$

Using this function as the transformation function, we obtain a family of potentials with a single level of the discrete spectrum:

$$V_1(x)=\alpha^2z^2/4+\alpha^2(k/2+1)z-2\alpha^2zd\ln Q_5(z)/dz-2\alpha^2z^2d^2\ln Q_5(z)/dz^2,$$

with energy $E=E_0$ and wave function normalized to unity of the form

$$\varphi_0(x)=[\alpha C((-1)^kk!C+1)]^{1/2}u^{-1}(x).$$

It is clear from this expression that for even n the constant C must satisfy the condition $C\in(-\infty,-1/k!)\cup(0,\infty)$, while for odd n we must have $C\in(0,1/k!)$.

The simplest of these potentials corresponds to $k=0$:

$$V_1(x)=\exp(-2\alpha x)+2\alpha[1-C\exp(2e^{-\alpha x}/\alpha)]Q_6^{-1}(x)-8C\exp(2e^{-\alpha x}/\alpha)Q_6^{-2}(x),$$

$$Q_6(x)=\exp(\alpha x)[1+C\exp(2e^{-\alpha x}/\alpha)].$$

The behavior of these potentials for $k=2$ is shown in Fig. 4.

3.3.4. The singular oscillator

By singular oscillator we mean the quantum-mechanical system with the Hamiltonian

$$h_0=-\partial_x^2+x^2/4+b/x^2,\quad b\in\mathbb{R},\quad x\in R=[0,\infty).\qquad (78)$$

Many quantum-mechanical problems and some problems in quantum field theory reduce to solving the Schrödinger equation with the Hamiltonian (78). Examples are the well known problem of a particle moving in a Coulomb field and the two-body interaction of three particles in one-dimensional

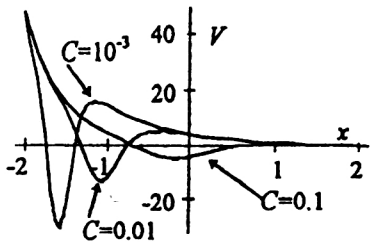

FIG. 4. Isospectral one-level potentials constructed on the basis of the Morse potential.

space, which was first solved by Calogero.[65] Interest in this Hamiltonian has revived recently in connection with its use in describing spin chains,[66] the quantum Hall effect,[67] and fractional statistics and anyons.[68]

The discrete basis of the space $H_0(R)$ is determined by the representation of the discrete series of the algebra $su(1.1)$ characterized by the value $k=1/2+(1/4)\sqrt{1+4b}=E_0/2$, determining the eigenvalue of the Casimir operator:

$$C=\frac{1}{2}\{K_+,K_-\}-K_0=3/16-b/4=k(k+1),$$

where

$$K_+=\frac{1}{2}[(a^+)^2-b/x^2],\quad K_-=\frac{1}{2}[a^2-b/x^2],$$

$$K_0=\frac{1}{2}h_0,$$

$$a=\partial_x+x/2,\quad a^+=-\partial_x+x/2,$$

and $K_\pm$ and K_0 are generators of the $su(1.1)$ algebra. The ground-state vector (vector of lowest weight) is determined by the conditions

$$K_0|0\rangle=k|0\rangle,\quad K_-|0\rangle=0.$$

The other basis vectors of this representation are determined by using the raising operator K_+:

$$|n\rangle=\sqrt{\frac{\Gamma(2k)}{n!\Gamma(2k+n)}}(K_+)^n|0\rangle,\quad K_0|n\rangle=(k+n)|n\rangle,$$

and in coordinate space they are expressed in terms of the Laguerre polynomials L_n^α:

$$\psi_n(x)=[n!2^{1-2k}\Gamma^{-1}(n+2k)]^{1/2}x^{2k-1/2}\times L_n^{2k-1}(x^2/2)\exp(-x^2/4). \quad (79)$$

In addition to the solutions $\psi_n(x)\in H_0(R)$, the Schrödinger equation with Hamiltonian (78) has other solutions of elementary form which are suitable as transformation functions. In particular, functions of the form

$$u_p(x)=x^{-2k+3/2}\exp(x^2/4)L_p^{1-2k}(y),\quad y=-x^2/2 \quad (80)$$

will be solutions of the original Schrödinger equation singular on the boundaries of the interval $[0,\infty)$:

$$h_0u_p(x)=2(k-p-1)u_p(x),\quad p=0,1,2,\ldots.$$

These functions generate the following exactly solvable potential:

$$V_1^{(p)}(x)=V_0(x)+A_p(x),$$

$$A_p(x)=-1+\frac{3-4k}{x^2}+2\left(\frac{xL_{p-1}^{2-2k}(y)}{L_p^{1-2k}(y)}\right)^2-2\frac{x^2L_{p-2}^{3-2k}(y)+L_{p-1}^{2-2k}(y)}{L_p^{1-2k}(y)}.$$

The operator transforming from solutions of the Schrödinger equation with the Hamiltonian (78) to solutions of the equation with the Hamiltonian $h_1^{(p)}=-\partial_x^2+V_1^{(p)}(x)$ has the form

$$\hat{L}=\frac{4k-3}{2x}-\frac{x}{2}-\frac{xL_{p-1}^{2-2k}(y)}{L_p^{1-2k}(y)}-\partial_x.$$

The function $u_p^{-1}(x)$ determining the kernel of the operator $\hat{L}^+$ is normalizable only for even p, because

$$\int_0^\infty\frac{dx}{u_p^2(x)}=(-1)^p2^{2k-2}p!\Gamma(2k-p-1).$$

For odd p the potential difference $A_p(x)$ has a singularity on the $(0,\infty)$ semiaxis. The functions (80) are suitable as transformation functions only for even p. The number of different values of p is limited by the condition $p<2k-1$. Since $k\geqslant 1$, it is always possible to take $p=0$. This case corresponds to a shape-invariant potential differing from the original one only by the value of the parameter b, $A_1=-1+(3-4k)/x^2$. The concept of shape invariance was introduced in Ref. 69. It is quite clear that all the shape-invariant potentials classified in that study have among the solutions of the Schrödinger equation solutions which, after the second logarithmic derivative is taken, give a functional dependence differing from that of the original potential only by values of the parameters. The Darboux transformation for them them does not change the form of the functional dependence when the transformation function is chosen appropriately.

For even p the functions (80) generate a supersymmetric quantum-mechanical model in which the supersymmetry is exact. The nondegenerate vacuum state of the super-Hamiltonian is constructed by using the function

$$\varphi_0(x)=2^{k-1}\sqrt{p!\Gamma(2k-p-1)}u_p^{-1}(x).$$

To construct a model with spontaneously broken supersymmetry, we consider a solution of the original Schrödinger equation which vanishes at the origin and becomes infinite for $x\to\infty$:

$$u_p(x)=\psi_p(ix),\quad h_0u_p=-2(k+p)u_p,\quad p=0,1,2,\ldots.$$

The normalization integral of the function u_p^{-1} has the form

$$\int_0^\infty u_p^{-2}(x)dx=(-1)^p2^{-2k}p!\Gamma(1-p-2k).$$

Therefore, the function u_p^{-1} is not normalizable for all p, but the potential difference generated by using it,

$$A_p(x)=-1+\frac{4k-1}{x^2}-2\frac{x^2L_{p-2}^{2k+1}(y)+L_{p-1}^{2k}(y)}{L_p^{2k-1}(y)}$$
$$+2x^2\left[\frac{L_{p-1}^{2k}(y)}{L_p^{2k-1}(y)}\right]^2,$$

is a regular function on the semiaxis $(0,\infty)$. The solutions of the new Schrödinger equation forming a basis of the space $H_0(R)$ are obtained by using the transformation operator

$$\hat{L}=\frac{1-4k}{2x}-\frac{x}{2}-\frac{xL_{p-1}^{2k}(y)}{L_p^{2k-1}(y)}-\partial_x.$$

In the simplest cases $p=0,1$ the potential differences are

$$A_0(x)=\frac{4k-1}{x^2}-1,\quad A_1(x)=A_0(x)+\frac{4}{4k+x^2}$$
$$-\frac{32k}{(4k+x^2)^2}.$$

3.4. Coherent states of transformed Hamiltonians

Coherent states as nonspreading wave packets were first introduced by Schrödinger for the harmonic oscillator.[70] Their properties were later studied by Klauder[71] and Glauber.[72] At the present time they are very widely used in diverse areas of physics and mathematics[41,73] such as quantum optics,[72,74,75] radiophysics,[76] and mathematical physics.[77,78] There are various definitions of coherent states[79] which lead to the same result for the harmonic oscillator and, as a rule, different results for other systems.

By now a rather large number of exactly solvable potentials are known (see the review of Ref. 44 and Sec. 3.3 of the present review). However, there are almost no studies devoted to the construction and study of systems of coherent states for the new Hamiltonians. Except for the harmonic oscillator, where in the ordinary supersymmetric approach the new potential differs from the original one by an additive constant and therefore has the same system of coherent states as the original potential, the authors know of only three studies on this topic. Two of them[80,81] are devoted to the study of the coherent states of isospectral Hamiltonians with equidistant spectrum, and one[60] is devoted to systems of coherent states of q-deformed harmonic-oscillator potentials and one class of self-dual potentials.

The method of Darboux transformation operators is a powerful tool for constructing and studying systems of coherent states of transformed Hamiltonians.

We shall define a coherent state as an eigenstate of the annihilation operator which is an integral of the motion of the system in question. By annihilation and creation operators $\hat{a}$ and $\hat{a}^+$ we mean any two integrals of the motion forming a Heisenberg–Weyl algebra w_1^0: $[\hat{a},\hat{a}^+]=1$. We shall assume that the representation of this algebra in the Hilbert space $H_0(R)$ is irreducible.

As noted in Sec. 2.5, we are faced with three possibilities. In the first case the transformation function is $u=\psi_0 \in H_0(R)$ and $H_1(R)=\{\varphi:\varphi=\hat{L}\psi,\ \psi\in H_0^0(R)\}$. In the second case, $u\notin H_0(R)$ and $H_1(R)=\{\varphi:\varphi=\hat{L}\psi,\ \psi\in H_0(R)\}$. In the third case, $u\notin H_0(R)$, $H_1^1(R)=\{\varphi:\varphi=\hat{L}\psi,\ \psi\in H_0(R)\}$, and $H_1(R)=H_1^1(R)\oplus\ker\hat{L}^+$.

We shall assume that the operator $M=\hat{L}^{-1}$ is defined on the entire space $H_1(R)$ in the first and second cases. In the third case the space $H_0(R)$ is mapped not onto the entire Hilbert space $H_1(R)$, but only onto its subspace $H_1^1(R)$. Therefore, the inverse operator M is defined on the space $H_1^1(R)$.

With these assumptions, on the space $H_1(R)$ [and in the third case on $H_1^1(R)$] we can define the operators $\tilde{a}=\hat{L}\hat{a}M$ and $\tilde{a}^+=\hat{L}\hat{a}^+M$, which realize a representation of the Heisenberg–Weyl algebra w_1^1. Moreover, if for the original Hamiltonian h_0 a system of coherent states $\psi_\alpha,\hat{a}\psi_\alpha=\alpha\psi_\alpha$, $\psi_\alpha\in H_0(R)$ is known, then the function $\varphi_\alpha=\hat{L}\psi_\alpha$ will be an eigenfunction of the annihilation operator $\tilde{a}$ and can be interpreted as a coherent state of the transformed Hamiltonian h_1. Owing to the irreducibility of the representation of the Heisenberg–Weyl algebra w_1^1, these states form a complete (more precisely, overcomplete) set of states in the corresponding irreducible spaces.

Now let there be in some region $D\subset\mathbb{C}$ of variation of the variable α for the states ψ_α a known measure $\mu(\alpha)$ which realizes the expansion of unity in the space $H_0(R)$:

$$1=\int_D|\psi_\alpha\rangle\langle\psi_\alpha|d\mu(\alpha).$$

Then the corresponding expansion for the space $H_1(R)$ [or $H_1^1(R)$] takes the form

$$1=(h_1-C)^{-1}\int_D|\varphi_\alpha\rangle\langle\varphi_\alpha|d\mu(\alpha),$$

where C is an eigenvalue of the operator h_1 corresponding to the transformation function $v=[L_1(t)\bar{u}]^{-1}$ and by construction $h_1\varphi\neq C\varphi$, $\forall\varphi\in H_1(R)$ [or $H_1^1(R)$].

3.4.1. Coherent states of simple quantum systems

In this section we present the well known systems of coherent states of quantum systems which we shall need below and for which we have constructed the exactly solvable transformed Hamiltonians.

1. Let us consider the coherent states of the harmonic oscillator: $V_0(x)=x^2/4$. The creation and annihilation operators, which are integrals of the motion for the system and form a Heisenberg–Weyl algebra w_1^0, have the form

$$\hat{a}(t)=\exp(it)(\partial_x+x/2),$$
$$\hat{a}^+(t)=\exp(-it)(-\partial_x+x/2).$$

The coherent states $|z,t\rangle$, which are the eigenstates of the annihilation operator, correspond to the solution of the Schrödinger equation in separated variables:

$$\psi_z(x,t)=(2\pi)^{-1/4}\exp\left(-\frac{1}{4}x^2-\frac{1}{2}it+zx-\frac{1}{2}z^2\right.$$
$$\left.-\frac{1}{2}z\bar{z}\right),$$
$$\hat{a}\psi_z(x,t)=\zeta\psi_z(x,t),\quad z=\zeta\exp(-it),\quad \zeta\in\mathbb{C},\qquad(81)$$

The vector $|z,t\rangle$ can be expanded in an orthonormal set of stationary states $|n,t\rangle=\exp(-i(n+1/2)t)|n\rangle$ which is complete in $H_0(R)$:

$$|z,t\rangle=\exp(-z\bar{z}/2-it/2)\sum_{n=0}^{\infty}\frac{z^n}{\sqrt{n!}}|n,t\rangle. \tag{82}$$

We shall also need the coordinate representation of the stationary states:

$$\psi_n(x,t)=(2\pi)^{-1/4}(n!)^{-1/2}\exp\left(-\frac{1}{4}x^2-i\left(n+\frac{1}{2}\right)t\right)He_n(x). \tag{83}$$

2. Let us consider the coherent states of a free particle. The creation and annihilation operators for this system are given by

$$\hat{a}(t)=(i-t)\partial_x+ix/2,\quad \hat{a}^+(t)=(i+t)\partial_x-ix/2.$$

We define the coherent states from the equation $\hat{a}\psi_z=z\psi_z$, $\psi_z\in H_0(R)$, $z\in\mathbb{C}$:

$$\psi_z(x,t)=(2\pi)^{-1/4}(1+it)^{1-2}\exp(-(z+\bar{z})^2/4-(1+it)^{-1}(x/2+iz)^2). \tag{84}$$

The functions of the discrete basis of the space $H_0(R)$ are the eigenfunctions of the operator $K_{-2}-K_2$ (see Ref. 38):

$$\psi_n(x,t)=(-i)^n(n!2^n\sqrt{2\pi})^{-1/2}(1+it)^{-1/2}\times\exp(-in\arctan t-(1+it)^{-1}x^2/4)\times H_n(x/\sqrt{2+2t^2}). \tag{85}$$

The operators $\hat{a}$ and $\hat{a}^+$ are ladder operators for these functions: $\hat{a}\psi_n=\sqrt{n}\psi_n$, $\hat{a}^+\psi_n=\sqrt{n+1}\psi_{n+1}$. The expansion of the function (84) in the basis (85) looks like

$$|z,t\rangle=\exp\left(\frac{1}{2}z^2-\frac{1}{4}(z+\bar{z})^2\right)\sum_{n=0}^{\infty}\frac{z^n}{\sqrt{n!}}|n,t\rangle. \tag{86}$$

3.4.2. Coherent states of anharmonic oscillators with quasi-equidistant spectra

By a quasi-equidistant spectrum we mean an equidistant spectrum with a finite number of lacunae, i.e., an equidistant spectrum with a finite number of levels removed. Such a spectrum is obtained, in particular, from the harmonic-oscillator spectrum by Darboux transformations. The coherent states of such systems are discussed in Ref. 31.

As the transformation functions we take the functions (70) for $m=2k$. Let us dwell in more detail on the integral transformation operator M, defined by Eqs. (31) and (33). The choice $t_0=-i\infty$ leads to the condition $C_1'(x)=0$, and we set $C_1=0$ in (31). Acting with the operator M on the stationary states (83) gives, up to a constant, the same functions (72) of stationary states of the Hamiltonian $h_1^{(2k)}=-\partial_x^2+V_1^{(2k)}(x)+1/2$, $\varphi_{n+1}(x)$, $n=0,1,2,\ldots$, as the differential transformation operator (12). Let us apply the operator M to the coherent state (81). As a result, we obtain the function

$$\varphi_z(x,t)=\exp\left(-\frac{x^2}{4}-\frac{z\bar{z}}{2}-i\left(2k+\frac{1}{2}\right)t\right)\left[\frac{P_{2k}'(x)}{P_{2k}(x)}+x-\frac{\partial}{\partial_x}\right]\frac{\partial^{2k}}{\partial x^{2k}}I(z,x),$$

$$I(z,x)=\sqrt{\frac{\pi}{2}}\exp\left(\frac{x^2}{2}\right)\operatorname{erf}\left(\frac{z-x}{\sqrt{2}}\right), \tag{87}$$

describing the coherent states of the Hamiltonian $h_1^{(2k)}$. The expansion (82) can be used to obtain an expansion of the functions (87) in the set of stationary states $\{\varphi_n(x,t)=\varphi_n(x)\exp(-i(n+1/2)t),\ n=1,2,\ldots\}$:

$$\varphi_z(x,t)=(2\pi)^{1/4}\exp(-z\bar{z}/2)\left[\sqrt{(2k)!}\psi_0+\exp(-it/2)\zeta^{2k+1}\sum_{s=0}^{\infty}\frac{z^s}{\sqrt{(2k+s+1)s!}}\psi_{s+1}\right],$$

$$\zeta\in\mathbb{C}. \tag{88}$$

The expansion coefficient for the function $\psi_0(x,t)=\psi_0(x)\exp(i(2k+1/2)t)$ must be calculated by direct integration.

We note that for $\zeta=0$ the function (88) coincides with the ground-state function φ_0. The expansion (88) is useful for calculating integrals involving the functions (87). For example, for the normalization factor of these functions and the expectation value of the energy we find

$$\langle z,t|z,t\rangle=\sqrt{2\pi}(2k)!\sum_{s=0}^{2k}\frac{(-1)^s}{s!}(z\bar{z})^s, \tag{89}$$

$$\langle h_1^{(2k)}\rangle=\left[(2k)!\sum_{s=0}^{2k}\frac{(-1)^s}{s!}(z\bar{z})^{s-2k-1}\right]^{-1}-2k-\frac{1}{2}. \tag{90}$$

The function (87) is expressed in terms of the error function erfc(z). However, the explicit expressions for the function $\varphi_z(x,t)$ contain only elementary functions. In particular, for $k=0$ we obtain the harmonic oscillator with the energy zero shifted upward by one unit. Equations (71), (72), and (87)–(90) become the corresponding oscillator expressions. For example, from (90) we obtain $\langle h_1^{(0)}\rangle=z\bar{z}-1/2$.

For $k=1$ the function (87) has the form

$$\varphi_z(x,t)=N_z\left[z^2+2\frac{1-xz}{1+x^2}\right]\exp\left(-\frac{x^2}{4}+xz-\frac{z^2}{2}-\frac{z\bar{z}}{2}+\frac{5}{2}it\right), \tag{91}$$

where $N_z=[(2\pi)^{1/2}(1+(1-z\bar{z})^2)]^{-1/2}$.

The expectation value of the coordinate and momentum in the states (91) is expressed in terms of the error function erfc(z), in particular,

$$\langle x\rangle = z+\bar{z}+\frac{\sqrt{2\pi}}{1+(1-z\bar{z})^2}\exp\left(\frac{1}{2}-\frac{1}{2}(z+\bar{z})^2\right)\mathrm{Re}\left[(i-z\right.$$

$$\left.-\bar{z}-iz\bar{z})\exp(-iz-i\bar{z})\mathrm{erfc}\,\frac{1-iz-i\bar{z}}{\sqrt{2}}\right].$$

The differential transformation leads to a different system of coherent states. For example, for $k=1$ we obtain the states

$$\tilde{\varphi}_z(x,t)=\hat{L}\psi_z(x,t)=(2\pi)^{-1/4}\left(z-x-\frac{2x}{1+x^2}\right)$$

$$\times\exp\left(-\frac{1}{4}x^2+zx-\frac{1}{2}z\bar{z}-\frac{1}{2}z^2-\frac{1}{2}it\right),$$

which for $\zeta=0$ coincide with the function of the first excited state of the Hamiltonian $h_1^{(2)}$ possessing the property $\tilde{a}\varphi_1=0$.

3.4.3. Anharmonic isospectral Hamiltonians with equidistant spectrum

The coherent states of isospectral anharmonic Hamiltonians with equidistant spectrum have been studied in Refs. 80–82. Our discussion will follow that of Ref. 82.

We consider the Hamiltonian $h_1=-\partial_x^2+V_1(x)+1/2$, where the potential $V_1(x)$ is obtained by using the transformation function (75) and is defined in (76). In fact, we have a family of potentials depending on the parameter C, the eigenvalue spectrum of which coincides with that of the harmonic-oscillator potential. The eigenfunctions of these potentials can be obtained by acting with either the integral transformation operator (31), (33) or the differential operator (12) on the stationary states (83) and are given by Eq. (77):

$$\varphi_0(x,t)=\varphi_0(x)\exp(it/2),$$

$$\varphi_n(x,t)=\varphi_n(x)\exp(-i(n+1/2)t),\quad n=1,2,\ldots .$$

Let us consider in more detail the action of the integral operator (31) and (33) on the coherent states of the harmonic oscillator (81). We obtain the coherent states of the transformed Hamiltonians:

$$\varphi_z(x,t)=M\psi_z(x,t)=(2\pi)^{-1/4}(z\zeta)^{1-4}\exp\left(-\frac{1}{2}z\bar{z}\right.$$

$$\left.-\frac{1}{4}x^2\right)\left[\exp\left(xz-\frac{1}{2}z^2\right)+\sqrt{\frac{\pi}{2}}\,\mathrm{erf}\left(\frac{z-x}{\sqrt{2}}\right)\right.$$

$$\left.\cdot Q_1^{-1}(x)\right],\quad \zeta\neq 0.$$

Direct calculation of the normalization factor of this function gives

$$\langle\varphi_z|\varphi_z\rangle=(z\bar{z})^{-1}\left(\frac{\exp(-z\bar{z})}{C^2-1}+1\right).$$

This function can be expanded in the set of functions $\{\varphi_i(x,t)\}$. The expansion coefficient for the function $\varphi_0(x,t)$ must be calculated by direct integration:

$$\langle\varphi_0|\varphi_z\rangle=(2\pi)^{-1/4}\zeta^{-1}C(C^2-1)^{-1/2}\exp\left(-\frac{1}{2}z\bar{z}\right),$$

while to calculate the other coefficients we use the expansion (82):

$$\varphi_z(x,t)=\exp\left(-\frac{1}{2}z\bar{z}\right)\left[\frac{C}{\sqrt{C^2-1}}\,\varphi_0(x,t)\right.$$

$$\left.+\sqrt{z\zeta}\sum_{n=0}^{\infty}\frac{z^n}{\sqrt{(n+1)!}}\,\varphi_{n+1}(x,t)\right].$$

Taking into account the normalization factor, we obtain the final expression for the coherent states of anharmonic oscillators with equidistant spectrum:

$$\Theta_z(x,t)=C[1+(C^2-1)\exp(z\bar{z})]^{-1/2}\left[\psi_0(x,t)\right.$$

$$\left.+\frac{\sqrt{C^2-1}}{C}\sqrt{\frac{\zeta}{z}}\sum_{n=1}^{\infty}\frac{z^n}{\sqrt{n!}}\,\psi_n(x,t)\right]. \qquad (92)$$

The function (92) admits the value $\zeta=0$ and coincides in this case with the ground-state function of the new Hamiltonian. Another limiting case, $C\to\infty$, corresponds to the harmonic oscillator with shifted energy zero. It is easy to see that the function $\Theta_z(x,t)$ tends to the coherent state of the harmonic oscillator (82) in this case.

Using the expansion (92), we can calculate the expectation value of the energy in the coherent state $\Theta_z(x,t)$:

$$\langle E\rangle=\frac{z\bar{z}(C^2-1)}{\exp(-z\bar{z})+C^2-1}-\frac{1}{2}.$$

The differential transformation operator $\hat{L}$ leads to a different system of coherent states:

$$\tilde{\varphi}_z(x,t)=(2\pi)^{-1/4}\exp\left(-\frac{1}{2}z\bar{z}-\frac{1}{2}z^2+zx-\frac{1}{4}x^2\right.$$

$$\left.-\frac{i}{2}t\right)\left(z-x-\sqrt{\frac{2}{\pi}}\frac{\exp(-x^2/2)}{C+\mathrm{erf}(x/\sqrt{2})}\right). \qquad (93)$$

For $\zeta=0$ this function coincides with that of the first excited state of the new Hamiltonian, which possesses the property $\tilde{a}\psi_1=0$.

The w-deformed Heisenberg–Weyl algebra was used in Ref. 81 to obtain the coherent states of this system, which for deformation parameter $w=2$ coincide with the states (93). The approach developed in that study does not allow the coherent states (92) to be obtained for such values of the deformation parameter.

3.4.4. Coherent states of the one-soliton potential

The coherent states of the one-soliton potential were studied in Ref. 83. The one-soliton potential of the KdV equation $V_1(x)=-2a^2\,\mathrm{sech}(ax+b)$, $a>0$, which has a single level in the discrete spectrum, $E=-a^2$, whose position is independent of the parameter b, is obtained by a single Darboux transformation from the Schrödinger equation for a free particle by using the transformation function

$$u(x,t)=\cosh(ax+b)\exp(ia^2t). \tag{94}$$

The parameter b effecting the isospectral deformation of the potential is not important for us, and we set $b=0$. The wave function of the discrete spectrum has the form

$$\varphi_0(x)=\sqrt{a/2}\ \mathrm{sech}(ax),$$

$$h_1\varphi_0=-a^2\varphi_0,\quad h_1=-\partial_x^2-2a^2\ \mathrm{sech}^2(ax). \tag{95}$$

The Darboux transformation operator taking us from solutions of the Schrödinger equation with zero potential to solutions with potential $V_1(x)$ has the form

$$\hat{L}=-a\ \tanh(ax)+\partial_x. \tag{96}$$

Acting with it on the basis functions (85) of the space $H_0(R)$, we find

$$\begin{aligned}\varphi_{n+1}(x,t)=&\frac{(-i)^n}{2}(2^n\sqrt{2\pi}n!)^{-1/2}(1+it)^{-3/2}\\&\times\exp(-(1+it)^{-1}x^2/4-in\ \arctan t)\\&\times[2\sqrt{2}n(1+it)(1+t^2)^{-1/2}\\&\times H_{n-1}(x/\sqrt{2+2t^2})-(x+2a\\&\times(1+it)\tanh(ax)H_n(x/\sqrt{2+2t^2}))],\end{aligned}$$

$$n=0,1,2,\ldots. \tag{97}$$

The system of functions (95) and (97) forms a basis of the space $H_1(R)$.

The symmetry operator $\hat{L}^+\hat{L}$ is expressed in terms of the free-particle Hamiltonian $h_0=-\partial_x^2$: $\hat{L}^+\hat{L}=h_0+a^2=\frac{1}{4}(\hat{a}+\hat{a}^+)^2+a^2$. This property can be used to calculate the normalization integral of the basis functions: $\langle\varphi_{n+1}|\varphi_{n+1}\rangle=\langle\psi_n|h_0+a^2|\psi_n\rangle=n/2+1/4+a^2$. Each basis function (97), except for $n=1,2$, is nonorthogonal to the other two: $\langle\varphi_{n+1}|\varphi_{n+3}\rangle=\frac{1}{4}\sqrt{(n+1)(n+2)}$.

Acting with the operator (96) on the coherent-state function of the free particle (84), we find the coherent-state function of the one-soliton potential:

$$\begin{aligned}\varphi_z(x,t)=&-\frac{1}{2}(2\pi)^{-1/4}(1+it)^{-3/2}[x+2iz+2a(1\\&+it)\tanh(ax)]\exp\left(-\frac{1}{4}(z+\bar{z})^2-(1\right.\\&\left.+it)^{-1}(x/2+iz)^2\right).\end{aligned} \tag{98}$$

The normalization integral for this function has the form $\langle\varphi_z|\varphi_z\rangle=1/4+(1/4)(z+\bar{z})^2+a^2$.

If we use Θ_n to denote the functions of the discrete basis of the space $H_1(R)$, normalized to unity, then by using Eq. (86) we can write down the expansion of the coherent-state function, normalized to unity, in this basis:

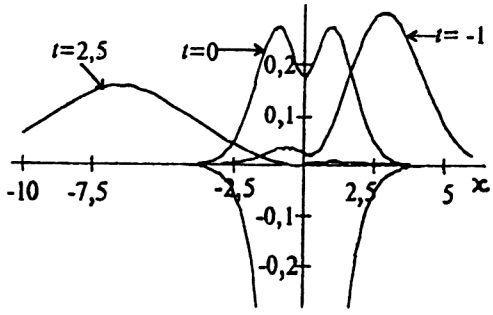

FIG. 5. Probability density of the coherent state of the one-soliton potential.

$$\begin{aligned}|z,t\rangle=&[1+4a^2+(z+\bar{z})^2]^{-1/2}\exp\left(\frac{1}{2}z^2-\frac{1}{4}(z\right.\\&\left.+\bar{z})^2\right)\sum_{n=0}^{\infty}\frac{z^n}{\sqrt{n!}}\sqrt{1+4a^2+2n}|\Theta_{n+1}\rangle.\end{aligned}$$

The orthogonality property $\langle\Theta_0|z,t\rangle=0$ can be established by direct integration.

Since the potential is reflectionless, we have $|\varphi_z(x,t)|^2=|\varphi_z(-x,-t)|$. In Fig. 5 we show the graph of the probability distribution $|\varphi_z(x,t)|^2$ for $a=1$ and $z=1$ at the times $t=-1$, 0, and 2.5. In the lower part of the graph we show the part of the potential well with depth equal to -2. The $E_0=-1$ level of the discrete spectrum is located exactly in the middle of the well. At times $t=\pm\infty$ the particle is maximally delocalized. For $t<0$ the particle is more localized in the region $x>0$, at $t=0$ the particle is localized symmetrically in the well region, and for $t>0$ the particle is more localized in the region $x<0$, i.e., the particle moves from right to left, and at $t=0$ it is maximally localized in the well. The integral operator M defined by (31) and (33) for $C_1=0$ leads to a different system of coherent states:

$$\begin{aligned}\tilde{\varphi}_z(z,t)=M\psi_z(x,t)=&\frac{-i}{4}\sqrt{\pi}(2\pi)^{-1/4}\cosh^{-1}(ax)\\&\times\exp\left(-\frac{1}{4}(z+\bar{z})^2+a^2(1+it)\right)\\&\times\left[\exp(2iaz)\mathrm{erfc}\left(a\sqrt{1+it}+\frac{x/2+iz}{\sqrt{1+it}}\right)\right.\\&\left.-\exp(-2iaz)\mathrm{erfc}\left(a\sqrt{1+it}-\frac{x/2+iz}{\sqrt{1+it}}\right)\right].\end{aligned}$$

We note that multisoliton potentials and their coherent states can be obtained by performing another Darboux transformation.

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