Zitterbewegung and the uncertainty in velocity and acceleration in the Dirac theory

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A review is given of the results obtained by the authors concerning the uncertainties in the velocity and acceleration of a freely moving particle in the Dirac theory. It is shown that these uncertainties are an inherent consequence of the relativistic quantum mechanics of spin-1/2 particles. The magnetic analogy of Zitterbewegung is discussed. © *1997 American Institute of Physics.* [S1063-7796(97)00401-4]

1. INTRODUCTION

The Dirac equation[1] has led to results which have determined the further development of relativistic quantum theory. In addition, the Dirac Hamiltonian has led to a conclusion which is usually characterized as "quite unexpected,"[2] "physically untenable,"[3] or "a shortcoming of the Dirac theory."[4] This conclusion is that, in contrast to the classical theory and also nonrelativistic quantum mechanics, the velocity is not proportional to the momentum. The velocity operator **u** turns out to be related not to the momentum, but to the Dirac matrices $\boldsymbol{\alpha}$ as

$$\mathbf{u}=c\boldsymbol{\alpha}, \tag{1}$$

where c is the speed of light. It follows from (1) that the projection of the velocity on any direction has value $\pm c$. At first glance, this contradicts the observation that electrons have velocity smaller than the speed of light.

In the opinion of Dirac,[2] this contradiction is only apparent and not real, because the eigenvalues of the operator **u** from (1) are "theoretical" ones at a fixed instant of time. In contrast, the observed velocities are averages defined by dividing the change of coordinate over a small time interval by this interval. To obtain a good approximation to the instantaneous velocity it is necessary to use an infinitesimal change of coordinate. According to the uncertainty principle, this leads to momentum which is unbounded and which, therefore, may be infinite. According to the special theory of relativity, this is possible for $u=c$. Therefore, according to (1) the instantaneous value of the velocity has magnitude c.

This interpretation has been criticized by Fock.[4] It was noted in Ref. 4 that in the absence of the usual relation between the velocity and momentum of a particle it is impossible to use the uncertainty relation between momentum and coordinate to determine the velocity. In addition, if the Dirac argument were applied to the Schrödinger theory, it would lead to the incorrect conclusion that the electron velocity is always infinite.

A different approach was suggested by Schrödinger in 1930 (Ref. 5). Integrating the equation of motion for u_z over time (the z axis is directed along the momentum), Schrödinger showed that the operator for the velocity parallel to the momentum can be written as the sum

$$u_z=\frac{1}{2}\, i\hbar(a_z)_0 e^{-2iHt/\hbar}\, H^{-1}+c^2p_zH^{-1}, \tag{2}$$

where the first term on the right-hand side oscillates in time. In (2), $(a_z)_0$ is the operator of the projection of the particle acceleration parallel to the momentum at $t=0$, $\hbar$ is Planck's constant, H is the Hamiltonian, and p is the particle momentum. Thus, Schrödinger introduced the concept of Zitterbewegung.

According to (2), in a stationary state the average value of the oscillating part of u_z is equal to zero, and the average value of the constant part of u_z is given by

$$\bar{u}_z=\frac{c^2p_z}{E}, \tag{3}$$

where the particle energy is

$$E=\pm[m_0^2c^4+c^2p^2]^{1/2}, \tag{4}$$

and m_0 is the particle rest mass.

Equation (3) agrees with the expression

$$u_z=\frac{c^2p_z}{E}, \tag{5}$$

following from the special theory of relativity.

In a nonstationary state which is a superposition of states with $E>0$ and $E<0$ the average value of the oscillating part of u_z from (2) is nonzero and contains the exponentials $\exp[\pm i2|E|t/\hbar]$ (see, for example, Ref. 6). This corresponds to Zitterbewegung with frequency

$$\omega_D=\frac{2|E|}{\hbar}, \tag{6}$$

which is superimposed on the constant part of u_z.

As Pauli has pointed out,[6] this Zitterbewegung is absent in states containing eigenfunctions corresponding to only one sign of the energy.

Nevertheless, various methods have been proposed for eliminating the Zitterbewegung, which is viewed as an undesirable consequence of the Dirac theory.

For example, it has been suggested that all physical quantities should be represented only by the even part of the corresponding operators (see, for example, Refs. 6 and 7).

This is based on the fact that the operator L for a physical quantity can be represented as the sum of its even part $[L]$ (transforming a state with a given sign of the energy into a state with the same sign of the energy) and its odd part $\{L\}$ (transforming a state with a given sign of the energy into a state with the opposite sign of the energy). Since states with opposite sign of the energy are orthogonal to each other, for all states with given sign of the energy the average value of $\{L\}$ vanishes.

This approach is attractive because in the relativistic quantum theory the relations between the operators for physical quantities turn out to be analogous to the relations between the corresponding physical quantities in the classical theory.

However, we think that this approach has disadvantages. In the calculation of the average $\bar{L}$ in a state with definite sign of the energy, only $[L]$ actually contributes. If it is necessary to find the average $\overline{L^2}$ (for example, when calculating the rms uncertainty $\overline{[\Delta L^2]}^{1/2}$, which is important for quantum mechanics), then, since $\{L\}^2$ is an even operator, we have

$$\overline{L^2}=\overline{[L]^2}+\overline{\{L\}^2}. \tag{7}$$

Therefore, $\overline{L^2}$ contains the contribution of not only the even part $[L]$ of the operator L, but also the odd part $\{L\}$. We therefore conclude that the requirement that only the even part be kept in the operators of all physical quantities is not always legitimate.

Another viewpoint is that Zitterbewegung can be eliminated from the Dirac theory by a unitary transformation. For example, the authors of Ref. 8 suggested a canonical transformation for eliminating from the Dirac Hamiltonian the matrix $\boldsymbol{\alpha}$ connecting the two upper and two lower components of the wave function ψ and making two of its components vanish (see, for example, Refs. 3, 7, 9, and so on). The Dirac Hamiltonian then takes the form

$$H_{\rm FW}=K\beta, \tag{8}$$

where

$$K=[m_0^2c^4+c^2p^2]^{1/2} \tag{9}$$

is the modulus of the energy and β is the fourth Dirac matrix. The elimination of the matrix $\boldsymbol{\alpha}$ from (8) is interpreted as the "elimination of the Schrödinger Zitterbewegung" (see, for example, Ref. 3 or Ref. 10).

However, it should be realized that in going to the Foldy–Wouthuysen representation,[8] not only the form of the Dirac Hamiltonian but also the velocity operator is changed. Instead of (1), the latter takes the following form (see Table I of Ref. 8):

$$u_{z,\rm FW}=\frac{c^2p_z}{K}\beta+\frac{m_0c^3}{K}\alpha_z. \tag{10}$$

Accordingly, in this representation the velocity operator contains the oscillatory part

$$\frac{m_0c^3}{K}(\alpha_z)_0e^{-i2H_{\rm FW}t/\hbar}.$$

Therefore, this representation does not eliminate the Zitterbewegung. This is consistent with the fact that physical consequences must be independent of the representation.[13]

It therefore appears that Zitterbewegung must be viewed as a phenomenon which, while perhaps unusual from the classical viewpoint, is nevertheless an inherent feature of the Dirac theory.

In this review we shall present the results of investigations which we have carried out from this point of view. These investigations are based on our earlier study,[11] where we showed that the, at first sight paradoxical, equality of the moduli of the eigenvalues of the particle-velocity projection operators to the speed of light should not be viewed in isolation from the uncertainty in the velocity of a spin-1/2 particle characteristic of the Dirac theory. In the stationary state of a particle with $m_0\neq 0$ the wave function is a superposition of two states,[11] in one of which the velocity projection parallel to the momentum is c and in the other it is $-c$. The average value of $\mathbf{u}_z$ is then given by an equation of the form (3), and the uncertainty in the projection parallel to the momentum is given by

$$\sqrt{\overline{(\Delta u_z)^2}}=\frac{m_0c^3}{K}. \tag{11}$$

Therefore, in contrast to the momentum, which is determined in a stationary state of a free particle, the particle velocity in such a state is not determined. The above-mentioned difficulty associated with Eq. (1) is resolved in the sense that the squared eigenvalue of the velocity projection operator, equal to the squared speed of light, is, according to (7), given by the sum of the squared average of the velocity projection and its squared uncertainty. The observed velocity, which is equal to the average velocity, is then smaller than the speed of light for $m_0\neq 0$.

2. THE UNCERTAINTY IN THE VELOCITY PROJECTIONS AND THE PHYSICAL MEANING OF THE FOURTH DIRAC MATRIX

The Dirac Hamiltonian for free motion is

$$H=c(\boldsymbol{\alpha}\mathbf{p})+\beta m_0c^2. \tag{12}$$

According to the Hellmann–Feynman theorem,[12] we have

$$\overline{\frac{\partial H}{\partial p_x}}=\frac{\partial E}{\partial p_x},\quad \overline{\frac{\partial H}{\partial p_y}}=\frac{\partial E}{\partial p_y},\quad \overline{\frac{\partial H}{\partial p_z}}=\frac{\partial E}{\partial p_z},\quad \overline{\frac{\partial H}{\partial m_0}}=\frac{\partial E}{\partial m_0}. \tag{13}$$

From this taking, into account (12) and (4), we find

$$\bar{\alpha}_x=\frac{cp_x}{E},\quad \bar{\alpha}_y=\frac{cp_y}{E},\quad \bar{\alpha}_z=\frac{cp_z}{E}, \tag{14}$$

$$\bar{\beta}=\frac{m_0c^2}{E}. \tag{15}$$

According to (1) and (14), the average values of the velocity projection are given by

$$\bar{u}_x=\frac{c^2p_x}{E},\quad \bar{u}_y=\frac{c^2p_y}{E},\quad \bar{u}_z=\frac{c^2p_z}{E}. \tag{16}$$

Since $\alpha_x^2=\alpha_y^2=\alpha_z^2=1$, from (1) we have

$$u_x^2=u_y^2=u_z^2=c^2. \tag{17}$$

Equations (17) and (16) give expressions for the rms uncertainties of the velocity projections:

$$\sqrt{\overline{(\Delta u_x)^2}}=\frac{c^2}{K}[m_0^2c^2+p_y^2+p_z^2]^{1/2},$$

$$\sqrt{\overline{(\Delta u_y)^2}}=\frac{c^2}{K}[m_0^2c^2+p_x^2+p_z^2]^{1/2},$$

$$\sqrt{\overline{(\Delta u_z)^2}}=\frac{c^2}{K}[m_0^2c^2+p_y^2+p_z^2]^{1/2}. \tag{18}$$

In the special case where the momentum is directed along the z axis (i.e., $p_x=p_y=0$, $p_z=p$), from (18) we find

$$\sqrt{\overline{(\Delta u_x)^2}}=\sqrt{\overline{(\Delta u_y)^2}}=c \tag{19}$$

and Eq. (11). We see from (19) that the uncertainties in the velocity projections transverse to the momentum are Lorentz-invariant.

In the rest frame (RF), in which $p=0$, Eqs. (19) and (11) give

$$\sqrt{\overline{(\Delta u_x)^2}}|_{\mathrm{RF}}=\sqrt{\overline{(\Delta u_y)^2}}|_{\mathrm{RF}}=\sqrt{\overline{(\Delta u_z)^2}}|_{\mathrm{RF}}=c. \tag{20}$$

In an inertial reference frame moving relative to the rest frame, according to (11) the uncertainty of the velocity projection parallel to the momentum decreases with increasing p [and therefore, according to (4), with increasing energy modulus].

According to (1), the average values of the first three Dirac matrices $\bar{\alpha}_x$, $\bar{\alpha}_y$, and $\bar{\alpha}_z$ define the average values of the three velocity projections u_x, u_y, and u_z in a stationary state. The physical meaning of the fourth Dirac matrix β remains unclear (see, for example, Ref. 13). From (15) and (11) we find

$$\bar{\beta}=\pm\sqrt{\overline{(\Delta u_z)^2}}. \tag{21}$$

This explains the physical meaning of the fourth Dirac matrix. Its average value determines the relative (to c) rms uncertainty of the velocity projection parallel to the momentum. Since $\beta^2=1$, using (15) the rms uncertainty of β is given by

$$\sqrt{\overline{(\Delta\beta)^2}}=c\left|\frac{p_z}{E}\right|. \tag{22}$$

Using the third equation in (16), we can write (22) as

$$\sqrt{\overline{(\Delta\beta)^2}}=\frac{1}{c}|u_z|. \tag{22'}$$

Therefore, the rms uncertainty of β is equal to the modulus of the relative (in the sense indicated above) average value of the velocity projection parallel to the momentum. Regarding the uncertainties (11) and (19), it should be mentioned that they arise from the odd parts of the operators $\boldsymbol{\alpha}$ and β. Actually, the even part of the operator L is given by $[L]=\frac{1}{2}(L+\Lambda L\Lambda)$, where $\Lambda=H/K$ is the sign operator [see, for example, Eq. (62) of Ref. 27]. Here $[\boldsymbol{\alpha}]=(c\mathbf{p}/K)\Lambda$ and $[\beta]=(m_0c^2/K)\Lambda$. Since $\Lambda^2=1$ [see, for example, Eq. (20.22) of Ref. 6] and $\Lambda=\pm1$ (see, for example, Ref. 7), the rms uncertainties of $[\boldsymbol{\alpha}]$ and $[\beta]$ are equal to zero. The odd part of the operator is given by $\{L\}=\frac{1}{2}(L-\Lambda L\Lambda)$ [see, for example, Eq. (63) of Ref. 27)]. Here $\{\boldsymbol{\alpha}\}=\boldsymbol{\alpha}-(c\mathbf{p}/K)\Lambda$ and $\{\beta\}=\beta-(m_0c^2/K)\Lambda$. Therefore, we have

$$\{\boldsymbol{\alpha}\}=\frac{1}{K^2}[K^2\boldsymbol{\alpha}-c^2\mathbf{p}(\boldsymbol{\alpha}\mathbf{p})-m_0c^3\mathbf{p}\beta],$$

$$\{\beta\}=\frac{c^2}{K^2}[p^2\beta-m_0c(\boldsymbol{\alpha}\mathbf{p})]. \tag{23}$$

Accordingly, the rms uncertainties of the odd parts of $\boldsymbol{\alpha}$ and β are given by

$$\sqrt{\overline{(\Delta\{\alpha_x\})^2}}=\frac{1}{K}\sqrt{E^2-c^2p_x^2},$$

$$\sqrt{\overline{(\Delta\{\alpha_y\})^2}}=\frac{1}{K}\sqrt{E^2-c^2p_y^2},$$

$$\sqrt{\overline{(\Delta\{\alpha_z\})^2}}=\frac{1}{K}\sqrt{E^2-c^2p_z^2},\quad \sqrt{\overline{(\Delta\{\beta\})^2}}=\frac{cp}{K}, \tag{24}$$

which agrees with (18) and (22).

The odd part $\{L\}$ of the operator L oscillates (see, for example, Ref. 27) in time with the frequency (6). This suggests that the uncertainties in $\mathbf{u}$ and β that we have found and the Zitterbewegung of the operators corresponding to them are due to the same cause: the presence of an odd part in these operators.

Using (1), (7), (14), and (23), we find

$$(\bar{u}_x)^2+\overline{(\Delta u_x)^2}=c^2,\quad (\bar{u}_y)^2+\overline{(\Delta u_y)^2}=c^2,$$

$$(\bar{u}_z)^2+\overline{(\Delta u_z)^2}=c^2. \tag{7'}$$

These equations follow from (16) and (18). According to (7′) and (17), the squared eigenvalue of the velocity projection operator, equal to the squared speed of light, is the sum of the squared average value of the velocity projection, determined by the even part of the velocity projection operator, and the squared uncertainty of the velocity projection. Therefore, both the average of the velocity projection and its uncertainty contribute to the eigenvalues of the velocity operator (1), which have magnitude c.

3. THE RELATION BETWEEN THE PARTICLE ENERGY AND THE UNCERTAINTY OF THE VELOCITY PROJECTION PARALLEL TO THE MOMENTUM

From (11), using (21), we find the following relation between the particle energy and the rms uncertainty of the velocity projection parallel to the momentum, and also the average value of the fourth Dirac matrix β:

$$|E|=\frac{m_0c^3}{\sqrt{\overline{(\Delta u_z)^2}}}=\frac{m_0c^2}{\bar{\beta}}. \tag{25}$$

According to (25), in the rest frame, in which (20) holds, the particle energy has its smallest absolute value, $|E_0|=m_0c^2$.

In another inertial frame moving relative to the rest frame, as p increases the uncertainty of the velocity projection parallel to the momentum decreases as described in Sec. 2, and according to (25) the absolute value of the energy increases.

Comparing (25) with the relation between the particle energy, rest mass, and velocity known from the special theory of relativity,

$$E=\frac{m_0c^2}{\sqrt{1-v^2/c^2}}, \tag{26}$$

we see that (25) can be viewed as the quantum-mechanical analog of (26). Here we have the equation

$$\frac{1}{c}\sqrt{\overline{(\Delta u_z)^2}}=\bar{\beta}=\sqrt{1-v^2/c^2}. \tag{27}$$

This is consistent with the association of the scalar $\sqrt{1-v^2/c^2}$ characterizing a Lorentz contraction with the fourth Dirac matrix β (see, for example, Ref. 3).

According to (12), (1), and (21), the particle energy in a stationary state can be written as a sum:

$$E=\bar{u}_z p\pm m_0 c\sqrt{\overline{(\Delta u_z)^2}}. \tag{4'}$$

This equation also demonstrates the importance of including not only the average velocity, but also its rms uncertainty in the Dirac theory.

4. THE PARTICLE ACCELERATION OPERATOR AND ITS EIGENVALUES

The presence of the acceleration operator in the oscillatory part of Schrödinger's expression[5] (2) suggests that this operator should be studied in more detail. According to (1) and the expression for $d\alpha/dt$ given in Eq. (60.30) of Ref. 7, the acceleration operator $\mathbf{a}$ is given by

$$\mathbf{a}=\frac{2c^2}{\hbar}[\mathbf{p}\boldsymbol{\Sigma}]+\frac{i2m_0c^3}{\hbar}\boldsymbol{\gamma}, \tag{28}$$

where $\boldsymbol{\Sigma}$ is the spin operator and

$$\boldsymbol{\gamma}=\beta\boldsymbol{\alpha}. \tag{29}$$

In the case where the z axis is directed along the momentum, from (28) we find

$$a_x=-\frac{2c^2p}{\hbar}\Sigma_y+\frac{i2m_0c^3}{\hbar}\gamma_x,\quad a_z=\frac{i2m_0c^3}{\hbar}\gamma_z,$$

$$a_y=\frac{2c^2p}{\hbar}\Sigma_x+\frac{i2m_0c^3}{\hbar}\gamma_y,$$

$$a_xa_y-a_ya_x=i2c^2\omega_D^2\Sigma_z. \tag{30}$$

Since $\gamma_x^2=\gamma_y^2=\gamma_z^2=-1$ and

$$\Sigma_x\gamma_y+\gamma_y\Sigma_x=\Sigma_y\gamma_x+\gamma_x\Sigma_y=0,$$

from (29) we find the equations

$$a_x^2=a_y^2=\left(\frac{2cE}{\hbar}\right)^2, \tag{31}$$

$$a_z^2=\left(\frac{2m_0c^3}{\hbar}\right)^2. \tag{32}$$

According to (31) and (32), the acceleration projection operators have the eigenvalues

$$a_x=\pm\frac{2c|E|}{\hbar},\quad a_y=\pm\frac{2c|E|}{\hbar}, \tag{33}$$

$$a_z=\pm\frac{2m_0c^3}{\hbar}. \tag{34}$$

We therefore conclude that in the Dirac theory the doubling of states characteristic of spin-1/2 particles extends not only to the eigenvalues of the velocity projection, but also to the eigenvalues of the acceleration projection.

The eigenvalues (33) of the acceleration projections transverse to the momentum differ quantitatively from those of the velocity projections transverse to the momentum u_x and u_y by a factor equal to the frequency ω_D in (6). The eigenvalues (34) of the acceleration projection parallel to the momentum differ from those for the velocity projection parallel to the momentum u_z by a factor equal to the frequency

$$\omega_0=\frac{2m_0c^2}{\hbar}. \tag{35}$$

According to (34) and (35), we have

$$a_z=\pm\omega_0 c. \tag{36}$$

The qualitative difference between the eigenvalues of the acceleration projection operators and those of the velocity projection operators is that Planck's constant enters into the right-hand sides of (33) and (34). We therefore conclude that the eigenvalues of the acceleration projection operators are not only relativistic, but also quantum quantities. The commutation relation for the acceleration projections transverse to the momentum from (30) also depends on Planck's constant in addition to c and Σ_z, in contrast to the commutation relation for the velocity projections transverse to the momentum [see, for example, Eq. (4.197) in Ref. 10],

$$u_xu_y-u_yu_x=\mathrm{i}2c^2\Sigma_z. \tag{37}$$

According to (34), the modulus of the eigenvalues a_z increases in direct proportion to the particle rest mass. For example, for the electron $|a_z|\sim5\times10^{31}$ cm/sec^2, for the muon $|a_z|\sim10^{34}$ cm/sec^2, and for the proton $|a_z|\sim10^{35}$ cm/sec^2. According to (34), the modulus of the eigenvalues a_z is related to the critical electric field strength

$$\mathscr{E}_k=\frac{m_0c^3}{e\hbar}$$

(see, for exmple, Refs. 27, 40, and 41), at which the vacuum becomes unstable to electron–positron pair production, as

$$2e\mathscr{E}_k=m_0|a_z|. \tag{38}$$

At first glance, Eqs. (33) and (34) contradict the first law of Newtonian dynamics, according to which a free particle moves due to inertia without acceleration. However, it should be remembered that the acceleration operator **a** defined in (28) is odd. In fact, it is easily verified that for $p_z=p$ we have

$$\Lambda\gamma_x\Lambda=\frac{1}{K^2}[(c^2p^2-m_0^2c^4)\gamma_x-2im_0c^3p\Sigma_y],$$

$$\Lambda\gamma_y\Lambda=\frac{1}{K^2}[(c^2p^2-m_0^2c^4)\gamma_y+2im_0c^3p\Sigma_x],$$

$$\Lambda\gamma_z\Lambda=-\gamma_z,$$

$$\Lambda\Sigma_x\Lambda=\frac{1}{K^2}[(-c^2p^2+m_0^2c^4)\Sigma_x-2im_0c^3p\gamma_y],$$

$$\Lambda\Sigma_y\Lambda=\frac{1}{K^2}[(-c^2p^2+m_0^2c^4)\Sigma_y+2im_0c^3p\gamma_x]. \quad (39)$$

Since $[\gamma_z]=\frac{1}{2}(\gamma_z+\Lambda\gamma_z\Lambda)$ (see Sec. 2), the condition $[\gamma_z]=0$ is satisfied when the third of the equations in (39) is taken into account. Therefore, according to (30),

$$a_z=\{a_z\}. \quad (40)$$

Similarly, from (39) and (30) we find the equations

$$a_x=\{a_x\}, \quad a_y=\{a_y\}. \quad (40')$$

Therefore, the acceleration operator is odd.

Since in a stationary state with definite sign of the energy the expectation value of the odd part $\{L\}$ of the operator L vanishes (see Sec. 1), using (40) and (40′) we have

$$\bar{a}_x=\bar{a}_y=\bar{a}_z=0. \quad (41)$$

It follows from these equations that in the Dirac theory the first law of Newtonian dynamics is satisfied on the average.

Taking into account the fourth and fifth equations in (39), the odd parts of the operators Σ_x and Σ_y are defined by the expressions

$$\{\Sigma_x\}=\frac{p}{K^2}(c^2p\Sigma_x+im_0c^3\gamma_y),$$

$$\{\Sigma_y\}=\frac{p}{K^2}(c^2p\Sigma_y-im_0c^3\gamma_z). \quad (42)$$

It is easily verified that these expressions satisfy Eq. (64) from Ref. 27 for the Zitterbewegung of a spin.

According to (42) and (30), a_x, a_y and $\{\Sigma_x\}$, $\{\Sigma_y\}$ are related as

$$\{\Sigma_x\}=\frac{\hbar p}{2K^2}a_y, \quad \{\Sigma_y\}=-\frac{\hbar p}{2K^2}a_x. \quad (43)$$

It follows from (33), (34), and (41) that the rms uncertainties of the acceleration projections are given by

$$\sqrt{\overline{(\Delta a_x)^2}}=\sqrt{\overline{(\Delta a_y)^2}}=\frac{2cK}{\hbar}, \quad \sqrt{\overline{(\Delta a_z)^2}}=\frac{2m_0c^3}{\hbar}. \quad (44)$$

This leads to the conclusion that, although in a stationary state there is no acceleration of a free particle on the average, the rms uncertainty of the acceleration is nonzero. According to (44), in the rest frame we have

$$m_0=\frac{\hbar}{2c^3}\sqrt{\overline{(\Delta a_x)^2}}\Big|_{\rm RF}=\frac{\hbar}{2c^3}\sqrt{\overline{(\Delta a_y)^2}}\Big|_{\rm RF}=\frac{\hbar}{2c^3}\sqrt{\overline{(\Delta a_z)^2}}\Big|_{\rm RF}. \quad (45)$$

It seems to us that these equations reflect a relation between the rest mass of a particle and its intrinsic motion.

Beginning from the first studies on relativistic quantum mechanics,[1] Dirac repeatedly suggested that the matrices $\boldsymbol{\alpha}$ and β describe some intrinsic motion of a point charged electron associated with its spin, the existence of which had been postulated by earlier theories (see, for example, Refs. 28 and 2). Feynman[38] noted that Dirac at first suggested that his equation shows that a key role is played by the spin (or internal angular momentum), but negative energies and the existence of antiparticles proved to be very important. The question of internal degrees of freedom of the electron associated with its spin and negative energies has also been discussed in other studies (see, for example, Refs. 4 and 36. The uncertainties of the velocity and acceleration that we have found make it possible to associate the intrinsic motion of a point electron with virtual transitions from an initial state to an intermediate state and vice versa, accompanied by a change of sign of the rest energy. In fact, according to (12), in the rest frame stationary states are eigenstates of the matrix β. They correspond to energy eigenvalues $\pm m_0c^2$. In the rest frame the even part of the operator $\boldsymbol{\alpha}$, equal to $(c\mathbf{p}/K)\Lambda$ (see Ref. 27, for example), vanishes. Therefore, in the rest frame the operator $\boldsymbol{\alpha}$ is odd and converts an eigenfunction of the operator β, corresponding to definite sign of the rest energy, into another eigenfunction of the operator β, corresponding to the opposite sign of the rest energy. For example, the uncertainty of u_z satisfies the equation

$$\overline{(\Delta u_z)^2}\big|_n=c\langle n|\alpha_z|k\rangle\langle k|\alpha_z|n\rangle,$$

where in the intermediate state $|k\rangle$ the sign of the rest energy is opposite to the sign of the rest energy in the initial and final stationary state $|n\rangle$. For the uncertainties of u_x and u_y, not only the signs of the energy, but also the spin projections are opposite in the intermediate state $|k\rangle$ and initial state $|n\rangle$. We note that analogous matrix elements describing virtual transitions in which the momentum is conserved while the sign of the energy changes are considered, for example, in the derivation of the Klein–Nishina formula.[39]

According to (28) and (29), in the rest frame the acceleration operator is proportional to $\beta\boldsymbol{\alpha}$. Therefore, the same sort of "intrinsic motion" corresponds to the rms uncertainty of the acceleration in the rest frame. From (45), using (20) and (35), we obtain

$$\sqrt{\overline{(\Delta a_x)^2}}\big|_{\rm RF}=\omega_0\sqrt{\overline{(\Delta u_x)^2}}\big|_{\rm RF},$$

$$\sqrt{\overline{(\Delta a_y)^2}}\big|_{\rm RF}=\omega_0\sqrt{\overline{(\Delta u_y)^2}}\big|_{\rm RF},$$

$$\sqrt{\overline{(\Delta a_z)^2}}\big|_{\rm RF}=\omega_0\sqrt{\overline{(\Delta u_z)^2}}\big|_{\rm RF}. \quad (46)$$

Therefore, the rms uncertainties of the velocity and acceleration projections and the frequency of the Zitterbewegung in the rest frame are all related.

5. UNCERTAINTIES OF THE VELOCITY AND ACCELERATION OF A PARTICLE WITH ZERO REST MASS

For $m_0=0$, from Eqs. (16), (19), (20), (30), and (44) we obtain

$$\bar{u}_x=\bar{u}_y=0, \quad \bar{u}_z=\pm c,$$

$$\sqrt{\overline{(\Delta u_\perp)^2}}=c, \quad \sqrt{\overline{(\Delta u_\parallel)^2}}=0,$$

and

$$a_x=-\frac{2c^2p}{\hbar}\Sigma_y, \quad a_y=\frac{2c^2p}{\hbar}\Sigma_x, \quad a_z=0,$$

$$(a_x)_{c3}=\pm\frac{2c^2p}{\hbar}, \quad (a_y)_{c3}=\pm\frac{2c^2p}{\hbar}, \quad (a_z)_{c3}=0,$$

$$\sqrt{\overline{(\Delta a_x)^2}}=\frac{2c^2p}{\hbar}, \quad \sqrt{\overline{(\Delta a_y)^2}}=\frac{2c^2p}{\hbar}, \quad \sqrt{\overline{(\Delta a_z)^2}}=0, \tag{47}$$

where we have used the fact that, according to (4), for $m_0=0$ we have

$$|E|=cp.$$

We see from (47) that for $m_0=0$ the uncertainties of the velocity and acceleration do not completely vanish in a stationary state. The velocity projections transverse to the momentum fluctuate about zero average value. Their rms uncertainties are equal to the speed of light. According to (47), the acceleration projections transverse to the momentum also fluctuate. An equation analogous to (46) is satisfied:

$$\sqrt{\overline{(\Delta a_\perp)^2}}=\omega_D\sqrt{\overline{(\Delta u_\perp)^2}}. \tag{46'}$$

This suggests that in the Dirac theory, uncertainties of the velocity and acceleration exist in a stationary state not only for electrons or other spin-1/2 particles with nonzero rest mass, but also for spin-1/2 particles with $m_0=0$, in particular, the neutrino and antineutrino.

6. THE RELATION BETWEEN THE VELOCITY UNCERTAINTY AND THE POLARIZATION PROPERTIES OF PARTICLES IN THE DIRAC THEORY

It is well known that in the Dirac theory, the polarization states of a particle[4] are characterized by a polarization pseudovector with components [see Eq. (29.8) of Ref. 14, for example]

$$a_0=\frac{p}{m_0c}\zeta_\parallel, \quad a_\perp=\zeta_\perp, \quad a_\parallel=\frac{E}{m_0c^2}\zeta_\parallel, \tag{48}$$

where $\boldsymbol{\zeta}$ is the three-dimensional polarization vector in the rest frame, and the subscripts $\parallel$ and $\perp$ denote the vector components parallel and perpendicular to the momentum **p**. From (48), using (11), we obtain

$$a_\parallel=\frac{c}{\sqrt{\overline{(\Delta u_\parallel)^2}}}\zeta_\parallel=\frac{\zeta_\parallel}{\beta}. \tag{49}$$

According to (49), the relation between $a_\parallel$ and $\zeta_\parallel$ is determined by the relative rms uncertainty of the velocity projection parallel to the momentum or to the average value of the fourth Dirac matrix. This is a manifestation of the relation between the polarization properties of a moving particle and the uncertainty of its velocity.

For Coulomb scattering, the polarization arising in the scattering of initially completely right-polarized electrons is given by the well known expression [see, for example, Eq. (7.97) of Ref. 9]

$$p_R=1-\frac{2m_0^2c^4}{E^2\cot^2\frac{\theta}{2}+m_0^2c^4}. \tag{50}$$

Using (11), this can be rewritten as

$$p_R=1-\frac{2\overline{(\Delta u_\parallel)^2}}{\overline{(\Delta u_\parallel)^2}+c^2\cot^2\frac{\theta}{2}}. \tag{51}$$

Therefore, in this case also the polarization properties turn out to be related to the uncertainty of the velocity projection parallel to the momentum. It follows from (49) and (51) that the velocity uncertainty can be determined experimentally by studying the polarization properties.

7. TRANSFORMATION OF THE VELOCITY PROJECTIONS AND THEIR UNCERTAINTIES IN THE DIRAC THEORY

The relativistic transformation of the current–charge density four-vector (see, for example, Ref. 29) following from the Lorentz covariance of the continuity equation leads to the following theorem for the addition of the average values of the velocity projections in the Dirac theory:

$$u_x=\frac{u_{x,0}\sqrt{1-v^2/c^2}}{1+\frac{v}{c^2}u_{z,0}}, \quad u_y=\frac{u_{y,0}\sqrt{1-v^2/c^2}}{1+\frac{v}{c^2}u_{z,0}}, \tag{52}$$

$$u_z=\frac{u_{z,0}+v}{1+\frac{v}{c^2}u_{z,0}}, \tag{53}$$

where the subscript 0 labels the values of the velocity projections in an inertial frame K_0 moving relative to the inertial frame K with constant speed v along the z axis. Equations (52) and (53) are consistent with (1) and (14) and with the relativistic transformation of the energy–momentum four-vector.

According to (52), (53), and (17), the uncertainties of the velocity projections transform as

$$\sqrt{\overline{(\Delta u_x)^2}}=\sqrt{\overline{(\Delta u_{x,0})^2}}=c, \quad \sqrt{\overline{(\Delta u_y)^2}}=\sqrt{\overline{(\Delta u_{y,0})^2}}=c \tag{54}$$

$$\sqrt{\overline{(\Delta u_z)^2}}=\frac{\sqrt{1+v^2/c^2}}{1+\frac{v}{c^2}\bar{u}_{z,0}}\sqrt{\overline{(\Delta u_{z,0})^2}}. \tag{55}$$

Therefore, the uncertainties of the velocity projections transverse to the momentum are Lorentz-invariant. The uncertainty of the velocity projection parallel to the momentum transforms like the average of the velocity projection transverse to the momentum.

8. TRANSFORMATION OF THE PROBABILITIES OF STATES WITH VELOCITY PROJECTION PARALLEL TO THE MOMENTUM EQUAL TO THE SPEED OF LIGHT

The probabilistic interpretation characteristic of modern quantum mechanics makes it reasonable to study transformations of the probabilities $W(c)$ and $W(-c)$ of states with velocity projection parallel to the momentum equal to c and $-c$, respectively.

According to (9) and (10) from Ref. 11, we have

$$W(\pm c)=\frac{1}{2}\left(1\pm\frac{cp_z}{E}\right). \tag{56}$$

Taking into account (16), Eq. (56) can be written as

$$W(\pm c)=\frac{1}{2}\left(1\pm\frac{\bar{u}_z}{c}\right). \tag{57}$$

From (57), taking into account (53), we find the following for the probability transformation:

$$W_K(\pm c)=\frac{1\pm\frac{v}{c}}{1+\frac{v}{c^2}\bar{u}_{z,0}}\,W_{K_0}(\pm c), \tag{58}$$

where, according to (57), the subscripts K_0 and K correspond to the rest frame and an inertial frame, and

$$W_{K_0}(\pm c)=\frac{1}{2}\left(1\pm\frac{\bar{u}_{z,0}}{c}\right). \tag{59}$$

According to (59), the probabilities $W_{K_0}(c)$ and $W_{K_0}(-c)$ satisfy the normalization condition

$$W_{K_0}(c)+W_{K_0}(-c)=1. \tag{60}$$

When (60) holds, the probabilities $W_K(c)$ and $W_K(-c)$ from (58) also satisfy the following normalization condition in the inertial frame K:

$$W_K(c)+W_K(-c)=1. \tag{61}$$

According to (58) and (59), for $v=c$ the equations $W_K(c)=1$ and $W_K(-c)=0$ are satisfied, and for $v=-c$ the equations $W_K(c)=0$ and $W_K(-c)=1$ are satisfied. The equations $W_K(c)=1$ and $W_K(-c)=0$ also hold for $\bar{u}_{z,0}=c$, and $W_K(c)=0$ and $W_K(-c)=1$ hold for $\bar{u}_{z,0}=-c$. This is the quantum-mechanical probabilistic expression of the equation $u_z=c$ for $v=c$ or $u_{z,0}=c$ and the equation $u_z=-c$ for $v=-c$ or $u_{z,0}=-c$ following from the velocity-addition theorem of the special theory of relativity.

Therefore, the probability transformations (58) are consistent with the limiting nature of the speed of light.

9. THE RELATION BETWEEN THE TRANSFORMATION OF THE AVERAGE VALUES OF THE VELOCITY PROJECTIONS IN THE DIRAC THEORY AND THE LOBACHEVSKY GEOMETRY

When dealing with the limiting nature of the speed of light, it is convenient to use the rapidity rather than the velocity (see, for example, Refs. 15–18, and 37). Setting

$$u_z=c\tanh\varphi,\quad u_{z,0}=c\tanh\varphi_0,\quad v=c\tanh\psi, \tag{62}$$

where φ, φ_0, and ψ are the rapidities determining u_z, $u_{z,0}$, and v, the velocity-addition theorem from the special theory of relativity can be represented as the sum of the rapidities:

$$\varphi=\varphi_0+\psi. \tag{63}$$

Similarly, if in the Dirac theory we take

$$\bar{u}_z=c\tanh\varphi,\quad \bar{u}_{z,0}=c\tanh\varphi_0,\quad v=c\tanh\psi_0, \tag{64}$$

from (52) and (53) we obtain the transformations of the average values of the velocity projections in the form

$$\tanh\varphi\cdot\cos\alpha=\frac{\tanh\varphi_0\cos\alpha_0+\tanh\psi}{1+\tanh\psi\tanh\varphi_0\cos\alpha_0}, \tag{65}$$

$$\tanh\varphi\cdot\sin\alpha=\frac{\tanh\varphi_0\sin\alpha_0}{\cosh\psi+\sinh\psi\tanh\varphi_0\cos\alpha_0}, \tag{66}$$

where α is the angle between $\mathbf{p}$ and the z axis. Using (65) and (66), it is easily verified that the Lorentz transformation of the average values of the velocity projections (52) and (53) is consistent (like the velocity-addition theorem in the special theory of relativity) with the sine and cosine theorems in the Lobachevsky plane (see, for example, Ref. 19).

10. THE RELATION BETWEEN THE VELOCITY AND THE SPIN

It is well known that in the case of particles with zero rest mass the Dirac equation for the four-component wave function can be written as two Weyl equations[20]

$$H_L\psi_L=E\psi_L,\quad H_R\psi_R=E\psi_R, \tag{67}$$

for the two-component wave functions ψ_L and ψ_R, where

$$H_L=-c(\boldsymbol{\sigma}\mathbf{p}),\quad H_R=c(\boldsymbol{\sigma}\mathbf{p}), \tag{68}$$

and σ are the Pauli spin matrices [see Eqs. (7.140) and (7.141) of Ref. 21]. In the two-component theory of the neutrino,[22–24] the neutrino and antineutrino Hamiltonians H_ν and $H_{\bar{\nu}}$ are given by

$$H_\nu=H_L,\quad H_{\bar{\nu}}=H_R. \tag{69}$$

Taking into account the equation of motion

$$\mathbf{u}=\frac{1}{i\hbar}[\mathbf{r}H]_-, \tag{69'}$$

where r is the radius vector, in the case of H_L in (68) we obtain the following expression for the velocity operator:

$$\mathbf{u}=-c\boldsymbol{\sigma}. \tag{70}$$

In the case of H_R in (68) the velocity operator is found from (69′) to be

$$\mathbf{u}=c\boldsymbol{\sigma}. \quad (71)$$

We see from (70) and (71) that for particles with $m_0=0$, as for particles with $m_0\neq 0$ [see (1)], the velocity operator $\mathbf{u}$ (in contrast to the velocity in the special theory of relativity) is not related to the momentum. In the case of particles with $m_0=0$ the velocity operator is, in contrast to (1), directly proportional to the spin operator. From this it follows that the properties of the velocity projections u_x, u_y, and u_z are determined by the properties of the spin operators σ_x, σ_y, and σ_z. For example, from the equations $\sigma_x^2=\sigma_y^2=\sigma_z^2=1$ it follows that for particles with $m_0=0$ the equation (17) in which u_x, u_y, and u_z have eigenvalues equal to $\pm c$ is also satisfied. The commutation relations for σ_x, σ_y, and σ_z lead to analogous commutation relations for u_x, u_y, and u_z. Here the uncertainties [see (47)] of the velocity projections transverse to the momentum turn out to be directly related to the uncertainties of σ_x and σ_y in the case where the eigenvalue of σ_z is determined. From the equation of motion

$$\frac{d\boldsymbol{\sigma}}{dt}=\frac{1}{i\hbar}[\boldsymbol{\sigma}H]_- \quad (72)$$

and (68) we find

$$\frac{d\boldsymbol{\sigma}}{dt}=\pm\frac{2c}{\hbar}[\mathbf{p}\boldsymbol{\sigma}], \quad (73)$$

where the upper sign is obtained for the Hamiltonian H_L, and the lower one for the Hamiltonian H_R. Equation (73), taking into account the expression

$$\mathbf{a}=c\frac{d\boldsymbol{\sigma}}{dt} \quad (74)$$

following from (71), agrees with Eq. (28) for $m_0=0$.

If the z axis is directed along $\mathbf{p}$, then from (73) we obtain

$$\frac{d\sigma_x}{dt}=\pm\frac{2}{\hbar}cp\sigma_y,$$

$$\frac{d\sigma_y}{dt}=\mp\frac{2}{\hbar}cp\sigma_x,$$

$$\frac{d\sigma_z}{dt}=0. \quad (75)$$

According to (75), the projection σ_z has a definite value (which corresponds to the definite helicity of particles with $m_0=0$, in particular, for the neutrino and antineutrino). The commutation relations for the Pauli spin matrices give $\bar{\sigma}_x=\bar{\sigma}_y=0$. Therefore, according to the first two equations in (75) we have

$$\overline{\frac{d\sigma_x}{dt}}=\overline{\frac{d\sigma_y}{dt}}=0, \quad (76)$$

$$\left(\frac{d\sigma_\perp}{dt}\right)^2=\frac{4c^2}{\hbar^2}p^2. \quad (77)$$

From these two equations we find

$$\sqrt{\left(\Delta\frac{d\sigma_\perp}{dt}\right)^2}=\frac{2c}{\hbar}p, \quad (78)$$

which agrees with (47) when (74) is taken into account. Therefore, for $m_0=0$ the uncertainty of the acceleration projection transverse to the momentum turns out to be directly related to the uncertainty of the time derivative of the spin projection transverse to the momentum.

This suggests that in the case of particles with $m_0=0$ the uncertainties of the velocities and accelerations, which are unusual from the classical viewpoint, are associated with the description of particles possessing spin in the Dirac theory.

For particles with $m_0\neq 0$, instead of (70) and (71) we have [see, for example, Eq. (24) from Sec. 69 and Eq. (9) from Sec. 67 in Ref. 2]

$$\mathbf{u}=c\rho_1\boldsymbol{\Sigma}, \quad (79)$$

where

$$\rho_1=\begin{pmatrix}0&0&1&0\\0&0&0&1\\1&0&0&0\\0&1&0&0\end{pmatrix}, \quad (80)$$

$$\boldsymbol{\Sigma}=\begin{pmatrix}\boldsymbol{\sigma}&0\\0&\boldsymbol{\sigma}\end{pmatrix}. \quad (81)$$

Therefore, here the velocity operator $\mathbf{u}$ is related not only to the spin operator $\boldsymbol{\Sigma}$, but also to the matrix ρ_1. Since ρ_1 commutes with the four-dimensional spin matrix $\boldsymbol{\Sigma}$ and, in addition, $\rho_1^2=1$, the commutation relations for the velocity projections u_x, u_y, and u_z are determined by the commutation relations for Σ_x, Σ_y, and Σ_z, as for particles with $m_0=0$. The eigenvalues of u_x, u_y, and u_z remain equal in magnitude to the speed of light c, as in the case $m_0=0$. However, since the matrix ρ_1 anticommutes with the fourth Dirac matrix β, for particles with $m_0\neq 0$, according to Eqs. (11), (30), and (44), m_0-dependent terms arise in the uncertainty of the velocity projection parallel to the momentum and in the acceleration operators and their uncertainties.

11. THE MAGNETIC ANALOG OF ZITTERBEWEGUNG

The Pauli operators were first applied to the microscopic theory of superconductivity by Bogolyubov[30] in 1958. These operators are related to the Pauli spin matrices. The method of the pseudospin and pseudomagnetic field was then used to study the BCS–Bogolyubov (Refs. 26 and 31, 32, respectively) theory of superconductivity in a number of other studies (see, for example, Refs. 25 and 33–36). In Ref. 11 we noted that this method is also applicable to the Dirac theory.

In this case the pseudospin state $|\uparrow\rangle$ is associated with the wave function $f_\uparrow(c)$ of a state with $u_z=c$, and the pseudospin state $|\downarrow\rangle$ is associated with the wave function $f_\downarrow(-c)$ of a state with $u_z=-c$. These functions are given by[11]

$$f_\uparrow(c)=\frac{1}{\sqrt{2}}\begin{pmatrix}1\\0\\1\\0\end{pmatrix},\quad f_\downarrow(-c)=\frac{1}{\sqrt{2}}\begin{pmatrix}1\\0\\-1\\0\end{pmatrix}. \quad (82)$$

The motion of a magnetic moment $\boldsymbol{\mu}=g\boldsymbol{\sigma}$ in a magnetic field $\mathbf{B}=\mathbf{i}B_x+\mathbf{k}B_z$, where B_x and B_z are independent of time, is determined by the Hamiltonian

$$H=-g(\boldsymbol{\sigma}\mathbf{B}) \tag{83}$$

in the (x,z) plane. In a stationary state the eigenfunctions and eigenvalues of the Hamiltonian (83) have the form

$$\psi_-=\frac{1}{\sqrt{2}}\left(\sqrt{1+\frac{\omega_1}{K_1}}\left|\uparrow\right\rangle+\right)\sqrt{1-\frac{\omega_2}{K_1}}\left|\downarrow\right\rangle\bigg),$$

$$E_-=-gB,$$

$$\psi_+=\frac{1}{\sqrt{2}}\left(\sqrt{1-\frac{\omega_1}{K_1}}\left|\uparrow\right\rangle+\sqrt{1+\frac{\omega_2}{K_1}}\left|\downarrow\right\rangle\right),$$

$$E_+=gB,$$

$$\omega_1=gB_z,\quad \omega_2=gB_x,\quad K_1=\sqrt{\omega_1^2+\omega_2^2}. \tag{84}$$

Using the sign operator $\Lambda=H/gB$, where H is the Hamiltonian from (83), we easily verify that the even and odd parts of σ_x, σ_y, and σ_z are given by

$$[\sigma_x]=-\frac{B_x}{B}\Lambda,\quad \{\sigma_x\}=\frac{B_z}{B^2}[\mathbf{B}\boldsymbol{\sigma}]_y,$$

$$[\sigma_y]=0,\quad \{\sigma_y\}=\sigma_y,$$

$$[\sigma_z]=-\frac{B_z}{B}\Lambda,\quad \{\sigma_z\}=-\frac{B_z}{B^2}[\mathbf{B}\boldsymbol{\sigma}]_y. \tag{85}$$

Accordingly, the averages of σ_x, σ_y, and σ_z and their rms uncertainties are given by

$$\overline{\sigma_x}|_{\mp}=\pm\frac{\omega_2}{K_1},\quad \overline{\sigma_y}=0,\quad \overline{\sigma_z}|_{\mp}=\pm\frac{\omega_1}{K_1},$$

$$\sqrt{\overline{(\Delta\sigma_x)^2}}=\frac{\omega_1}{K_1},\quad \sqrt{\overline{(\Delta\sigma_y)^2}}=1,\quad \sqrt{\overline{(\Delta\sigma_z)^2}}=\frac{\omega_2}{K_1}. \tag{86}$$

According to (86), in a stationary state there is no precession (Zitterbewegung) of the magnetic moment, but there are uncertainties in the spin projections associated with the odd parts of the corresponding operators. These possess nonzero off-diagonal matrix elements between the stationary states ψ_+ and ψ_- from (84) with energies equal in magnitude and opposite in sign.

Comparison of (83) and (12) shows that the changeover to the Dirac theory can be accomplished by the correspondence

$$gB_z\to-cp,\quad gB_x\to-m_0c^2,$$

$$\sigma_z\to\alpha_z,\quad \sigma_x\to\beta. \tag{87}$$

Here the wave functions (84) become the wave functions (7) from Ref. 11, the odd expressions $\{\sigma_x\}$ and $\{\sigma_z\}$ from (85(become $\{\beta\}$ and $\{\alpha_z\}$ from (23) for $p_z=p$, and the expressions in (86) become (15), (41), and (16) for $\bar{\beta}$, $\bar{a}_z$, and $\bar{u}_z$ and (22), (44), and (11) for $\sqrt{\overline{(\Delta\beta)^2}}$, $\sqrt{\overline{(\Delta a_z)^2}}$, and $\sqrt{\overline{(\Delta u_z)^2}}$.

In nonstationary states there is precession (Zitterbewegung) of the magnetic moment about the resulting magnetic field B. If for definiteness we take the initial condition $\psi|_{t=0}=|\uparrow\rangle$, the wave function will be (see Ref. 42)

$$\psi=\left(\cos K_1t+\mathbf{i}\frac{\omega_1}{K_1}\sin K_1t\right)|\uparrow\rangle+\frac{\mathbf{i}\omega_2}{K_1}\sin K_1t|\downarrow\rangle. \tag{88}$$

In the state (88) the average values of the spin projection are given by

$$\sigma_z=1-\frac{2\omega_2^2}{K_1^2}\sin^2 K_1t,$$

$$\sigma_x=\frac{2\omega_1\omega_2}{K_1^2}\sin^2 K_1t,$$

$$\sigma_y=\frac{\omega_2}{K_1}\sin^2 K_1t. \tag{89}$$

By analogy with (89), taking into account (87), the wave function of a nonstationary state is determined, according to the Dirac theory, by the expression [with K from (9)]

$$\psi_\uparrow=\left(\cos\frac{K}{\hbar}t+\frac{icp}{K}\sin\frac{Kt}{\hbar}\right)f_\uparrow(c)$$

$$-i\left(\frac{m_0c^2}{K}\sin\frac{K}{\hbar}t\right)f_\uparrow(-c). \tag{90}$$

In this state we have

$$u_z=c\left(1-\frac{2m_0^2c^4}{K^2}\sin^2\frac{K}{\hbar}t\right),$$

$$\beta=\frac{2m_0c^3p}{K^2}\sin^2\frac{K}{\hbar}t,$$

$$\bar{a}_z=-\frac{2m_0^2c^5}{\hbar K}\sin^2\frac{K}{\hbar}t,$$

$$\bar{u}_x=\bar{u}_y=0,\quad \sqrt{\overline{(\Delta u_x)^2}}=\sqrt{\overline{(\Delta u_y)^2}}=c,$$

$$\bar{a}_x=\bar{a}_y=0,\quad \sqrt{\overline{(\Delta a_x)^2}}=\sqrt{\overline{(\Delta a_y)^2}}=\frac{2cK}{\hbar}. \tag{91}$$

Therefore, according to the Dirac theory, the Zitterbewegung in a nonstationary state corresponds to precession of the vector $\mathbf{i}\bar{\beta}+\mathbf{j}\bar{a}_z+\mathbf{k}\bar{u}_z$ in the space $(\bar{\beta},\bar{a}_z,\bar{u}_z)$.

In the nonstationary state $\psi_\uparrow$ in (90), the average value of the operator $c^2p_zH^{-1}$ is given by

$$\overline{c^2p_zH^{-1}}=\frac{1}{c}\left(\frac{c^2p}{E}\right)^2. \tag{92}$$

We therefore conclude that in a nonstationary state in which there is Zitterbewegung,[5] the similarity of the nonoscillating part of the velocity operator u_z in (2) to the corresponding expression (5) from the special theory of relativity is only apparent and does not hold for the average value of this operator.

12. CONCLUSION

We can conclude from the above discussion that the uncertainties of the velocity and acceleration are intrinsic features of the Dirac theory describing spin-1/2 particles. The uncertainties in α and β and the Zitterbewegung of the corresponding operators arise from the same source: the presence of an odd part in these operators which transforms a state with a given sign of the energy into a state with the opposite sign of the energy. The eigenvalues of the velocity projection operators, with magnitude equal to the speed of light, receive a contribution from both the average value of the velocity projection due to the even part of the velocity projection operator, and from the uncertainty of the velocity projection due to the odd part of the velocity projection operator.

The average values of the first three Dirac matrices in a stationary state determine the average values of the velocity projections parallel and transverse to the momentum. The average value of the fourth Dirac matrix determines the relative rms uncertainty of the velocity projection parallel to the momentum of a spin-1/2 particle of nonzero rest mass. It also determines the relation between the energy of this particle and its rest energy.

In the Dirac theory, the doubling of states characteristic of spin-1/2 particles extends not only to the eigenvalues of the velocity projections, which are either c or $-c$, but also to the eigenvalues of the acceleration projections. These eigenvalues turn out to be not only relativistic, but also quantum quantities.

In a stationary state the average values of the acceleration projections are zero. Accordingly, the first law of Newtonian dynamics is satisfied in the Dirac theory in an average sense in the case of free motion. In a stationary state the rms uncertainties in the acceleration projections are nonzero. They are related to the particle rest mass. The relation between the rest mass of a particle and its intrinsic motion, characterized by the uncertainty of its acceleration in the rest frame, is thereby explained. The modulus of the eigenvalues of the acceleration projection parallel to the momentum is related to the critical electric field strength at which the vacuum becomes unstable to electron–positron pair production.

The Lorentz transformation of the average values of the velocity projections in the Dirac theory has the same outward form as the velocity-addition theorem in the special theory of relativity. Accordingly, this transformation possesses the same properties as the addition of definite values of the velocity projection in the special theory of relativity.

The uncertainties of the velocity projections transverse to the momentum are Lorentz-invariant. The uncertainty of the velocity projection parallel to the momentum transforms like the average value of the velocity projection transverse to the momentum.

Spin-1/2 particles with zero rest mass also have uncertainties in the velocity and acceleration projections transverse to the momentum. This is due to the fact that such particles possess axial symmetry about the direction of the momentum (see, for example, Ref. 14). Here the spin projection on the direction of the momentum has a definite value, and the spin projections transverse to the momentum are (according to the commutation relations for the spin projection operators) indefinite. Owing to the proportionality of the velocity and spin projection operators for particles with zero rest mass, this leads to uncertainty of the velocity projections transverse to the momentum.

For spin-1/2 particles with nonzero rest mass there exists a rest frame in which there is rotational invariance (see, for example, Refs. 14 and 16). In this frame the velocity uncertainties are the same in all directions and equal in magnitude to the speed of light. In an inertial frame moving relative to the rest frame with velocity tending toward the speed of light, the particle velocity projection parallel to the momentum tends to the speed of light. Particles of zero rest mass move with this velocity and, as noted above, have uncertainties in only the velocity projections transverse to the momentum. Accordingly, as the momentum increases the uncertainty of the velocity projection parallel to the momentum of a spin-1/2 particle with nonzero rest mass decreases and tends to zero in the ultrarelativistic limit.

The Lorentz transformations of the average values of the velocity projections coincide with the sine and cosine theorems in the Lobachevsky plane, like the velocity-addition theorem in the special theory of relativity.

The Lorentz transformations of the probabilities of states with velocity projection parallel to the momentum equal in magnitude to the speed of light are consistent with the limiting nature of the speed of light.

It should also be noted that the uncertainties of the velocity and acceleration projections transverse to the momentum are stable. When we transform from a nonstationary state to a stationary state the Zitterbewegung vanishes, as described in Sec. 1. These uncertainties can remain unchanged, as can be seen by comparing Eq. (91) with (19) and (44). This suggests that the velocity and acceleration uncertainties that we have found are more closely related to the intrinsic motion of particles described by the Dirac theory than is Zitterbewegung.

In conclusion, we can state that the Zitterbewegung of a relativistic spin-1/2 particle discovered by Schrödinger in 1930 is a real property of the particle in nonstationary states. In stationary states the intrinsic uncertainty of the velocity is preserved, and, in contrast to the Heisenberg uncertainty relations, is not related to any uncertainty in the parameter conjugate to it.

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