Quantum theory of the spinor field in four-dimensional Riemannian spacetime

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The spinor field in four-dimensional Riemannian spacetime is the subject of this review. This field obeys the Dirac–Fock–Ivanenko equation. The principles of quantization of the spinor field in Riemannian spacetime are formulated. In the special case of flat spacetime these conditions are equivalent to the canonical quantization rules. The formulated principles are tested for the example of de Sitter spacetime. The study of quantum field theory in de Sitter spacetime is interesting because it leads to the method of an invariant well in flat spacetime. Study of the quantum theory of the spinor field in an arbitrary Riemannian spacetime allows the inclusion of the effect of an external gravitational field on the quantized spinor field.
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INTRODUCTION

In the literature we encounter authoritative statements that the general theory of relativity can be used to find a solution to the difficulties of the theory of quantized fields and elementary particles.[1–12] The full implementation of this idea in the theory of quantized fields obviously presupposes the quantization of the gravitational field along with the other fields,[13–18] but this very general formulation of the problem is apparently far from being solved. It is therefore interesting to study even a partial formulation of this problem, when fields different from the gravitational field are considered in four-dimensional spacetime with fixed Riemannian geometry. According to the general theory of relativity, this makes it possible to include the effect of an external gravitational field on the behavior of nongravitational fields.

The need to include the effect of an external gravitational field on other fields was first encountered in electrodynamics in writing down the Maxwell equations in general relativity. Here it was only necessary to replace the partial derivatives of the bivector electromagnetic field by the covariant derivatives.[19] The situation with the other fields proved more complicated. For example, it was shown that in the Klein–Fock equation for a scalar field, replacement of the partial derivatives by covariant ones is insufficient for going from flat-spacetime geometry to Riemannian geometry: it is necessary to add an extra term proportional to the scalar curvature of spacetime.[20–22] On the other hand, in making the same transition in the Dirac equation for a spinor field there was no need for a new term; instead, it was necessary to introduce a new geometrical concept: that of the covariant derivative of a spinor field, which itself is an enormous step forward. This was done by Fock and Ivanenko.[23–30] Owing to the importance of their result, we shall refer to the Dirac equation in Riemannian spacetime as the Dirac–Fock–Ivanenko equation. A comparative analysis of the description of the spinor and scalar fields in Riemannian spacetime was made in Ref. 31. In the present review we shall instead focus all our attention on the description of the spinor field.

In fact, the spinor field merits special attention. It contains information about the behavior of electrons and neutrinos (and other spin-1/2 particles) in an external gravitational field, and, therefore, about their behavior in invisible regions of the universe.

We shall write the Dirac–Fock–Ivanenko equation in Cartan form. According to Cartan,[32] a spinor has a metrical rather than an affine nature, and therefore "*it is necessary to introduce spinor fields into the classical methodology of studies in Riemannian geometry; that is, for an arbitrary system of coordinates x^i of the space it is impossible to define a spinor using a finite number N of components u_α such that the u_α admit covariant derivatives of the form*

$$u_{\alpha,i}=\frac{\partial u_\alpha}{\partial x^i}+\Lambda^\beta_{\alpha i}u_\beta,$$

where the $\Lambda^\beta_{\alpha i}$ are definite functions of x^i" (Ref. 32; the emphasis is Cartan's).

In order to introduce the spinor field in Riemannian spacetime it is necessary, as was done by Fock and Ivanenko, to specify on this spacetime a field of metrically congruent tangent bases. Transformation from one such field of bases to another is done by a generalized Lorentz transformation. The Dirac–Fock–Ivanenko equation is invariant under generalized Lorentz transformations. The latter are the subject of much attention.[33–35] In the theory of the spinor field they are simply essential.

The Dirac–Fock–Ivanenko equation is studied in the first part of this review.

In the second part we study the general principles of the quantum theory of the spinor field in Riemannian geometry.

In the third, auxiliary, part of the review these principles are realized for the example of two-dimensional Riemannian spacetimes in the F basis.

In the fourth part these principles are realized for the example of two-dimensional spherical de Sitter spacetime in the dX basis.

The F and dX bases are defined in the course of the review.

Interestingly, in the 1930s Dirac studied the consequences of replacing Poincaré–Minkowski spacetime by de Sitter spacetime in order to determine how theoretical physics and the mathematical theory of groups are related. In Ref. 36 he proposed a special equation for the spinor field in de Sitter spacetime. In Ref. 37 the present author was able to show that this Dirac equation is a special form of the spinor equation written down by Fock and Ivanenko for Riemannian spacetimes. It is obtained in going from the F basis to the dX basis.

De Sitter spacetime is worthy of attention because, being a space of constant curvature, it admits a ten-parameter isometry group. In the limit of infinitely large radius of curvature this group becomes the Poincaré group, and de Sitter spacetime itself becomes Poincaré–Minkowski spacetime. Instead of the Fourier integrals with which one deals in flat spacetime, series arise in de Sitter spacetime. In the above limit these series become integrals. Thus, the study of quantum field theory is justified by the fact that it leads to the method of an invariant well for flat spacetime. In Ref. 38 it was suggested that this method be used for the spinor field. The quantum theory of the spinor field in four-dimensional de Sitter spacetime in the dX basis was studied in Ref. 39.

However, in the general case of Riemannian spacetime the theory of the spinor field can have an applied nature. For example, Witten[40] has used the Dirac–Fock–Ivanenko equation to prove the theorem about the positivity of the energy of the gravitational field and matter. Bagrov and Obukhov[41] have solved the problem of the complete classification of spacetime manifolds in which the Dirac–Fock–Ivanenko equation admits complete separation of variables. Since the early 1960s spinors and the related isotropic tetrads have been widely used to study a number of problems in general relativity.[42,43]

In this review the displayed equations inside each section are numbered separately from those in the others. Equations from a different section are referred to with the section number preceding the equation number. There are not many such references. The section number is omitted when referring to equations in the section where they occur.

I. THE SPINOR FIELD IN RIEMANNIAN SPACETIME

1. The Dirac equations

In 1928 Dirac wrote down the following system of four first-order differential equations for describing the behavior of a relativistic electron (Ref. 44; see also Ref. 29):

$$\begin{aligned}
&(E-mc^2)\psi_1-c(p_x-ip_y)\psi_4-cp_z\psi_3=0,\\
&(E-mc^2)\psi_2-c(p_x+ip_y)\psi_3+cp_z\psi_4=0,\\
&(E+mc^2)\psi_3-c(p_x-ip_y)\psi_2-cp_z\psi_1=0,\\
&(E+mc^2)\psi_4-c(p_x+ip_y)\psi_1+cp_z\psi_2=0,
\end{aligned} \tag{1}$$

where c is the speed of light, m is the electron mass (and also the positron mass), $\hbar$ is Planck's constant,

$$E=i\hbar\frac{\partial}{\partial t},\quad p_x=-i\hbar\frac{\partial}{\partial x},\quad p_y=-i\hbar\frac{\partial}{\partial y},$$

$$p_z=-i\hbar\frac{\partial}{\partial z}.$$

Following Dirac, we write the system (1) in matrix form:

$$i\hbar\frac{\partial\psi}{\partial t}=\hat{H}\psi, \tag{2}$$

arranging the unknown functions ψ_1, ψ_2, ψ_3, and ψ_4 in the form of a column vector ψ and thereby bringing the system to the canonical Schrödinger form, where the operator $\hat{H}$ is called the Hamiltonian. It is equal to (see Ref. 45)

$$\hat{H}=\alpha_0mc^2+c\alpha_1p_x+c\alpha_2p_y+c\alpha_3p_z, \tag{3}$$

where α_ν are the Dirac matrices:

$$\alpha_0=\begin{pmatrix}1&0&0&0\\0&1&0&0\\0&0&-1&0\\0&0&0&-1\end{pmatrix},\quad \alpha_1=\begin{pmatrix}0&0&0&1\\0&0&1&0\\0&1&0&0\\1&0&0&0\end{pmatrix},$$

$$\alpha_2=\begin{pmatrix}0&0&0&-i\\0&0&i&0\\0&-i&0&0\\i&0&0&0\end{pmatrix},\quad \alpha_3=\begin{pmatrix}0&0&1&0\\0&0&0&-1\\1&0&0&0\\0&-1&0&0\end{pmatrix}. \tag{4}$$

These matrices satisfy the relations

$$\alpha_\mu\alpha_\nu+\alpha_\nu\alpha_\mu=2e\delta_{\mu\nu}, \tag{5}$$

where e is the unit matrix. Therefore,

$$\hat{H}\hat{H}=\{m^2c^4+c^2(p_xp_x+p_yp_y+p_zp_z)\}e, \tag{6}$$

so that the Hamiltonian $\hat{H}$ is the square root of the operator (6).

Of course, the discovery of the system of equations (1) proceeded in the opposite direction. It was known that a nonrelativistic electron is described by two (not four) functions because the electron spin is $\frac{1}{2}\hbar$, and that the electron spin is described by the following matrices:

$$\sigma_1=\begin{pmatrix}0&1\\1&0\end{pmatrix},\quad \sigma_2=\begin{pmatrix}0&-i\\i&0\end{pmatrix},\quad \sigma_3=\begin{pmatrix}1&0\\0&-1\end{pmatrix}. \tag{7}$$

These are called the Pauli matrices.[46] They satisfy the commutation relations

$$\begin{aligned}
&\sigma_2\sigma_3-\sigma_3\sigma_2=2i\sigma_1,\\
&\sigma_3\sigma_1-\sigma_1\sigma_3=2i\sigma_2,\\
&\sigma_1\sigma_2-\sigma_2\sigma_1=2i\sigma_3.
\end{aligned} \tag{8}$$

The Pauli matrices were also known to satisfy relations of the type (5), i.e.,

$$\begin{aligned}
&\sigma_1\sigma_1=\sigma_2\sigma_2=\sigma_3\sigma_3=e,\quad \sigma_2\sigma_3+\sigma_3\sigma_2=0,\\
&\sigma_3\sigma_1+\sigma_1\sigma_3=0,\quad \sigma_1\sigma_2+\sigma_2\sigma_1=0.
\end{aligned} \tag{9}$$

In contrast to (5), here e is the unit matrix consisting of two rows and two columns. However, as a rule in what follows we shall use 1 to denote the unit matrix of any order and any unit vector. Similarly, the zero matrix of any order, any zero operator, and also any zero vector will be denoted by 0. Then the Dirac matrices (4), as noted, for example, in Ref. 47, can be expressed in terms of the Pauli matrices (7) as

$$\alpha_0=\begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix},\quad \alpha_1=\begin{pmatrix}0 & \sigma_1\\ \sigma_1 & 0\end{pmatrix},$$

$$\alpha_2=\begin{pmatrix}0 & \sigma_2\\ \sigma_2 & 0\end{pmatrix},\quad \alpha_3=\begin{pmatrix}0 & \sigma_3\\ \sigma_3 & 0\end{pmatrix}.$$

In trying to solve the problem of constructing the Schrödinger equation for a relativistic electron, Dirac based his work on the Klein–Fock equation

$$E^2\psi=\{m^2c^4+c^2(p_xp_x+p_yp_y+p_zp_z)\}\psi. \tag{10}$$

The search for the square root of the operator on the right-hand side of (10) was successful. Using the Pauli matrices (4), Dirac established Eq. (6) and then also the Schrödinger equation (2), which is written out in full as the system of equations (1).

The fact that the description of the relativistic electron required not two but four functions led Dirac to the discovery of the positron.

It should be noted that the choice of the matrices α_ν is not unique. For example, Fock[46] suggested the following two substitutions:

$$\eta_1=\frac{\psi_1+\psi_3}{\sqrt{2}},\quad \eta_2=\frac{\psi_2+\psi_4}{\sqrt{2}},$$

$$\eta_3=\frac{\psi_2-\psi_4}{\sqrt{2}},\quad \eta_4=\frac{\psi_3-\psi_1}{\sqrt{2}}; \tag{11}$$

$$\zeta_1=\frac{\psi_1-i\psi_3}{\sqrt{2}},\quad \zeta_2=\frac{\psi_2-i\psi_4}{\sqrt{2}},$$

$$\zeta_2=\frac{-\psi_3+i\psi_1}{\sqrt{2}},\quad \zeta_4=\frac{-\psi_4+i\psi_2}{\sqrt{2}}. \tag{12}$$

They respectively transform the Dirac equations (1) into the equations

$$\begin{aligned}
&-i\hbar c\left(\frac{\partial\eta_2}{\partial x}-i\frac{\partial\eta_2}{\partial y}+\frac{\partial\eta_1}{\partial z}\right)-mc^2\eta_4=i\hbar\frac{\partial\eta_1}{\partial t},\\
&-i\hbar c\left(\frac{\partial\eta_1}{\partial x}+i\frac{\partial\eta_1}{\partial y}-\frac{\partial\eta_2}{\partial z}\right)+mc^2\eta_3=i\hbar\frac{\partial\eta_2}{\partial t},\\
&-i\hbar c\left(\frac{\partial\eta_4}{\partial x}+i\frac{\partial\eta_4}{\partial y}+\frac{\partial\eta_3}{\partial z}\right)+mc^2\eta_2=i\hbar\frac{\partial\eta_3}{\partial t},\\
&-i\hbar c\left(\frac{\partial\eta_3}{\partial x}-i\frac{\partial\eta_3}{\partial y}-\frac{\partial\eta_4}{\partial z}\right)-mc^2\eta_1=i\hbar\frac{\partial\eta_4}{\partial t},
\end{aligned} \tag{13}$$

and the equations

$$\begin{aligned}
&\frac{\partial\zeta_2}{\partial x}-\frac{\partial\zeta_4}{\partial y}+\frac{\partial\zeta_1}{\partial z}+\frac{1}{c}\frac{\partial\zeta_1}{\partial t}+\frac{mc}{\hbar}\zeta_2=0,\\
&\frac{\partial\zeta_1}{\partial x}+\frac{\partial\zeta_3}{\partial y}-\frac{\partial\zeta_2}{\partial z}+\frac{1}{c}\frac{\partial\zeta_2}{\partial t}-\frac{mc}{\hbar}\zeta_1=0,\\
&\frac{\partial\zeta_4}{\partial x}+\frac{\partial\zeta_2}{\partial y}+\frac{\partial\zeta_3}{\partial z}+\frac{1}{c}\frac{\partial\zeta_3}{\partial t}-\frac{mc}{\hbar}\zeta_4=0,\\
&\frac{\partial\zeta_3}{\partial x}-\frac{\partial\zeta_1}{\partial y}-\frac{\partial\zeta_4}{\partial z}+\frac{1}{c}\frac{\partial\zeta_4}{\partial t}+\frac{mc}{\hbar}\zeta_3=0.
\end{aligned} \tag{14}$$

It is remarkable that in the latter system of equations all the coefficients are real numbers, in contrast to the systems (1) and (13).

In turn, Cartan suggested the following substitution:

$$\psi_1^c=\xi_0=\frac{\psi_2-\psi_4}{\sqrt{2}},\quad \psi_2^c=\xi_{12}=\frac{\psi_1-\psi_3}{\sqrt{2}},$$

$$\psi_3^c=\xi_1=\frac{\psi_1+\psi_3}{\sqrt{2}},\quad \psi_4^c=\xi_2=\frac{-\psi_2-\psi_4}{\sqrt{2}} \tag{15}$$

(see Ref. 32). These take the system of equations (1) into the system

$$\begin{aligned}
&c(p_x+ip_y)\xi_1+(E+cp_z)\xi_2=-mc^2\xi_0,\\
&(cp_z-E)\xi_1-c(p_x-ip_y)\xi_2=-mc^2\xi_{12},\\
&c(p_x-ip_y)\xi_0+(E+cp_z)\xi_{12}=mc^2\xi_1,\\
&(cp_z-E)\xi_0-c(p_x+ip_y)\xi_{12}=mc^2\xi_2.
\end{aligned} \tag{16}$$

The somewhat unexpected set of indices has a deep meaning.

The invariance of the Dirac equation (2) under the Lorentz group is usually proved by introducing the γ matrices,[45,47–49] writing the equation as

$$\left(i\gamma^n\frac{\partial}{\partial x^n}-m\right)\psi(x)=0 \tag{17}$$

(see Ref. 49), although this can be avoided (see Ref. 46). We shall not dwell on this, because for this purpose and others it is more convenient to write out the Dirac equation in the Cartan form given in Ref. 32.

2. The Dirac equation in Cartan form

Cartan combined the system of equations (16) into a single equation:

$$\left(\frac{\hbar}{i}\frac{\partial}{\partial x}-mcK\right)\xi=0, \tag{18}$$

which in the more detailed notation proposed in Ref. 37 implies

$$\left(i\hbar\sum_{\nu=0}^{3}H^\nu\frac{\partial}{\partial x^\nu}+mcH^4\right)\xi=0. \tag{19}$$

Here

$$x^0=ct,\quad x^1=x,\quad x^2=y,\quad x^3=z, \tag{20}$$

and the matrices H are

$$H^0=\begin{pmatrix}0&0&0&-1\\0&0&1&0\\0&-1&0&0\\1&0&0&0\end{pmatrix},$$

$$H^1=\begin{pmatrix}0&0&1&0\\0&0&0&-1\\1&0&0&0\\0&-1&0&0\end{pmatrix},$$

$$H^2=\begin{pmatrix}0&0&i&0\\0&0&0&i\\-i&0&0&0\\0&-i&0&0\end{pmatrix},$$

$$H^3=\begin{pmatrix}0&0&0&1\\0&0&1&0\\0&1&0&0\\1&0&0&0\end{pmatrix},$$

$$H^4=K=\begin{pmatrix}-1&0&0&0\\0&-1&0&0\\0&0&1&0\\0&0&0&1\end{pmatrix}. \tag{21}$$

We shall also use the matrices $H_0=-H^0$, $H_a=H^a$, where $a=1, 2, 3, 4$. The matrices H satisfy the following relations:

$$H^AH^B+H^BH^A=2\eta^{AB}, \tag{22}$$

where $\eta^{00}=-1$, $\eta^{aa}=1$, $\eta^{AB}=0$ for $A\neq B$. In addition,

$$H_0H_1H_2H_3H_4=i. \tag{23}$$

We have encountered three types of index. Greek indices take the values 0, 1, 2, 3. Upper-case Latin indices take the values 0, 1, 2, 3, 4. Lower-case Latin indices take the values 1, 2, 3, 4. Identical super- and subscripts are understood to be summed over the range of values that they take, for example,

$$\sum_{\nu=0}^{3} H^\nu \frac{\partial}{\partial x^\nu}=H^\nu \frac{\partial}{\partial x^\nu}, \quad \sum_{B=0}^{4} \eta^{AB}H_B=\eta^{AB}H_B=H^A. \tag{24}$$

We shall raise and lower indices by using the tensor η^{AB} and its inverse η_{AB}. The interpretation is obvious: $\eta_{\alpha\beta}dx^\alpha dx^\beta$ is the metric form of the four-dimensional Poincaré–Minkowski spacetime, and $\eta_{AB}dx^A dx^B$ is the metric form of the five-dimensional Poincaré–Minkowski spacetime.

Taking

$$\Xi=\xi \exp\left\{-\frac{imc}{\hbar}x^4\right\}, \tag{25}$$

we can write Eq. (19) in the form

$$H^A \frac{\partial}{\partial x^A}\Xi=0. \tag{26}$$

We note that the words "spin" and "spinor" are of English origin: the original sense of the English verb "to spin" is to "whirl around" (see Ref. 46). This was used to describe the intrinsic angular momentum of the electron, which is independent of its spatial motion. This word soon came to be used to refer to the intrinsic angular momentum of any elementary particle. The word "spinor" is used to describe geometrical objects whose behavior fundamentally differs from that of vectors and tensors. Physicists first encountered spinors after Dirac's discovery. More precisely, they encountered them somewhat earlier, in the Pauli theory, but they became interested in their behavior only after Dirac's discovery, when it became necessary to understand the invariance of the Dirac equation under Lorentz transformations. However, spinors were actually discovered by Cartan in 1913, a fact of which not only physicists but also mathematicians were apparently unaware. We have ventured to say this because spinors were rediscovered by van der Waerden[50] after 1928. Therefore, the Cartan lectures on spinor theory[32] written in 1938 are exceptionally interesting. Reading them, one understands the great amount of work Shirokov devoted to translating this book into Russian and supplementing it with a remarkable commentary. [Petr Alekseevich Shirokov (1895–1944) was a distinguished geometer and professor at Kazan University.[51]]

Commutation relations of the type (5), (9), and (22) were studied even earlier, in the last century.[52] The algebra that they generate is referred to as the Clifford algebra in the contemporary literature.

In a Clifford algebra with any number of independent elements H, the following relations hold for any signature of the metric tensor and in any basis:

$$H^\nu H^\alpha H^\mu=[H^\nu H^\alpha H^\mu]+\eta^{\alpha\mu}H^\nu+\eta^{\alpha\nu}H^\mu-\eta^{\mu\nu}H^\alpha, \tag{27}$$

$$\frac{1}{2}\{[H^\alpha H^\beta][H^\mu H^\nu]-[H^\mu H^\nu][H^\alpha H^\beta]\}$$
$$=\eta^{\mu\beta}[H^\alpha H^\nu]-\eta^{\alpha\mu}[H^\beta H^\nu]+\eta^{\nu\beta}[H^\mu H^\alpha]-\eta^{\nu\alpha}[H^\mu H^\beta]. \tag{28}$$

Here the square brackets stand for the alternating product of matrices.

3. The Dirac–Fock–Ivanenko equation

The transformation to the Dirac equation in Riemannian spacetime (i.e., to the Dirac–Fock–Ivanenko equation) is accomplished as follows.[37] The metric form of spacetime $g_{\alpha\beta}dx^\alpha dx^\beta$ can, by means of suitably chosen linear differential forms

$$f^\alpha=f^\alpha_\beta(x)dx^\beta, \tag{29}$$

be written in the following canonical form:

$$g_{\alpha\beta}dx^\alpha dx^\beta=\eta_{\alpha\beta}f^\alpha f^\beta. \tag{30}$$

Solving Eq. (29) for dx, we obtain

$$dx^\alpha=e^\alpha_\beta f^\beta,$$

where $f^{\alpha}_{\sigma}e^{\sigma}_{\beta}=\delta^{\alpha}_{\beta}$, so that $e^{\alpha}_{\sigma}f^{\sigma}_{\beta}=\delta^{\alpha}_{\beta}$, and δ^{α}_{β} is the unit affinor. The basis dual to f^{α} consists of the vector fields

$$e_{\alpha}=e^{\beta}_{\alpha}\partial_{\beta}. \tag{31}$$

We also have

$$\partial_{\alpha}=f^{\beta}_{\alpha}e_{\beta}.$$

Any covector field a_{α} can be written both in a coordinate basis $d=dx$ and in an orthogonal basis f: $a_{\alpha}d^{\alpha}=A_{\alpha}f^{\alpha}$, from which we find $A_{\alpha}=a_{\beta}e^{\beta}_{\alpha}$ and $a_{\alpha}=A_{\beta}f^{\beta}_{\alpha}$.

Similarly, any vector field a^{α} can be written both in a coordinate basis $\partial=\partial/\partial x$ and in an orthogonal basis e: $a^{\alpha}\partial_{\alpha}=A^{\alpha}e_{\alpha}$, from which we have $A^{\alpha}=a^{\beta}f^{\alpha}_{\beta}$ and $a^{\alpha}=A^{\beta}e^{\alpha}_{\beta}$.

The covariant differential of a vector field is

$$DA^{\alpha}=dA^{\alpha}+\omega^{\alpha}_{\mu}A^{\mu}, \tag{32}$$

and the covariant differential of a covector field is

$$DA_{\alpha}=dA_{\alpha}-\omega^{\mu}_{\alpha}A_{\mu}, \tag{33}$$

where $\omega^{\alpha}_{\mu}=\omega^{\alpha}_{\beta\mu}f^{\beta}$; in turn, $\omega^{\alpha}_{\beta\mu}$ are the components of the affine connection in the orthogonal basis given by the metric (30). From this we obtain the covariant deriatives of a vector and a covector:

$$D_{\beta}A^{\alpha}=e_{\beta}A^{\alpha}+\omega^{\alpha}_{\beta\mu}A^{\mu}, \tag{34}$$

$$D_{\beta}A_{\alpha}=e_{\beta}A_{\alpha}-\omega^{\mu}_{\beta\alpha}A_{\mu}. \tag{35}$$

The components of the connection are found by using two conditions. The first is the absence of torsion. For a scalar function F this implies

$$D_{\alpha}e_{\beta}F=D_{\beta}e_{\alpha}F.$$

From this we find the equation

$$\omega^{\gamma}_{\alpha\beta}-\omega^{\gamma}_{\beta\alpha}=C^{\gamma}_{\alpha\beta}, \tag{36}$$

where the components $C^{\gamma}_{\alpha\beta}$ are defined by Lie operations:

$$e_{\alpha}e_{\beta}-e_{\beta}e_{\alpha}=C^{\gamma}_{\alpha\beta}e_{\gamma}. \tag{37}$$

They are equal to

$$C^{\gamma}_{\alpha\beta}=e^{\mu}_{\alpha}e^{\nu}_{\beta}\left(\frac{\partial f^{\gamma}_{\mu}}{\partial x^{\nu}}-\frac{\partial f^{\gamma}_{\nu}}{\partial x^{\mu}}\right)=(e^{\mu}_{\alpha}e_{\beta}-e^{\mu}_{\beta}e_{\alpha})f^{\gamma}_{\mu}. \tag{38}$$

The second condition determining the components of the connection is conservation of the metric tensor in parallel transport:

$$D_{\alpha}\eta_{\beta\gamma}=-\omega^{\mu}_{\alpha\beta}\eta_{\mu\gamma}-\omega^{\mu}_{\alpha\gamma}\eta_{\beta\mu}=0. \tag{39}$$

Solving Eqs. (36) and (39) for ω, we find

$$\omega_{\nu\beta\alpha}=\frac{1}{2}(C_{\nu\beta\alpha}+C_{\nu\alpha\beta}-C_{\beta\alpha\nu}), \tag{40}$$

where we have written

$$\omega_{\nu\beta\alpha}=\omega^{\mu}_{\alpha\beta}\eta_{\nu\mu},\quad C_{\nu\beta\alpha}=C^{\mu}_{\alpha\beta}\eta_{\nu\mu}. \tag{41}$$

Like the metric tensor, the matrix H does not change in parallel transport. Therefore, for a vector matrix $A=A^{\alpha}H_{\alpha}$ we have

$$DA=dA+\omega^{\alpha}_{\mu}A^{\mu}H_{\alpha}=dA+\Omega A-A\Omega, \tag{42}$$

where

$$\Omega=\frac{1}{4}\omega_{\alpha\mu\nu}f^{\nu}H^{\alpha}H^{\mu}. \tag{43}$$

Since for a vector matrix Eq. (42) gives, up to infinitesimal quantities of higher order, the equation

$$DA-dA+A=(1+\Omega)A(1+\Omega)^{-1}, \tag{44}$$

a spinor must satisfy the equation

$$D\xi-d\xi+\xi=(1+\Omega)\xi. \tag{45}$$

Therefore, the covariant differential of a spinor ξ is

$$D\xi=d\xi+\frac{1}{4}\omega_{\alpha\mu\nu}f^{\nu}H^{\alpha}H^{\mu}\xi, \tag{46}$$

and the covariant derivative is

$$D_{\nu}\xi=e_{\nu}\xi+\frac{1}{4}\omega_{\alpha\mu\nu}H^{\alpha}H^{\mu}\xi. \tag{47}$$

The latter conclusion follows from the fact that if a vector matrix is transformed, according to the rule, from A into SAS^{-1}, then a spinor is transformed from ξ into $S\xi$.

The Dirac–Fock–Ivanenko equation is obtained from (19) by replacing the partial derivative of the spinor by the covariant derivative, so that it becomes

$$(i\hbar H^{\nu}D_{\nu}+mcH^{4})\xi=0. \tag{48}$$

For brevity we shall refer to this as the DFI equation.

The DFI equation can obviously be written as

$$H^{\nu}D_{\nu}\xi=\frac{imc}{\hbar}H^{4}\xi. \tag{49}$$

Therefore, for the Dirac-conjugate spinor

$$\bar{\xi}=\xi^{+}H_{0} \tag{50}$$

the DFI equation takes the form

$$D_{\nu}\bar{\xi}H^{\nu}=-\frac{imc}{\hbar}\bar{\xi}H^{4}, \tag{51}$$

where

$$D_{\nu}\bar{\xi}=e_{\nu}\bar{\xi}-\bar{\xi}\frac{1}{4}\omega_{\alpha\mu\nu}H^{\alpha}H^{\mu}. \tag{52}$$

Finally, we note that the solution of (40) can be found by using the equations

$$\omega_{\nu\beta\alpha}=\frac{1}{2}(\omega_{\nu\beta\alpha}+\omega_{\beta\nu\alpha})+\frac{1}{2}(\omega_{\alpha\nu\beta}+\omega_{\nu\alpha\beta})-\frac{1}{2}(\omega_{\alpha\beta\nu}+\omega_{\beta\alpha\nu})+\frac{1}{2}(\omega_{\alpha\beta\nu}-\omega_{\alpha\nu\beta})+\frac{1}{2}(\omega_{\nu\beta\alpha}-\omega_{\nu\alpha\beta})-\frac{1}{2}(\omega_{\beta\nu\alpha}-\omega_{\beta\alpha\nu}).$$

4. The Dirac–Fock–Ivanenko equation in the Lamé basis

The DFI equation is greatly simplified in the case of orthogonal coordinates (if they exist), when the Lamé basis can be constructed, i.e., when it is possible to take $f^{\alpha}_{\beta}=h^{\alpha}\delta^{\alpha}_{\beta}$. Using Eq. (27), we obtain

$$\frac{1}{4}\,\omega_{\alpha\mu\nu}H^{\nu}H^{\alpha}H^{\mu}=\frac{1}{4}\,\omega_{[\alpha\mu\nu]}H^{\nu}H^{\alpha}H^{\mu}+\frac{1}{2}\,\eta^{\alpha\nu}\omega_{\alpha\mu\nu}H^{\mu}.$$

However,

$$\omega_{[\alpha\mu\nu]}=\frac{1}{2}\,C_{[\alpha\mu\nu]},\quad \eta^{\alpha\nu}\omega_{\alpha\mu\nu}=C^{\alpha}_{\alpha\mu},$$

while in the Lamé basis

$$C^{\gamma}_{\alpha\beta}=\frac{1}{h^{\alpha}h^{\beta}}\left[\delta^{\gamma}_{\alpha}\,\frac{\partial h^{\gamma}}{\partial x^{\beta}}-\delta^{\gamma}_{\beta}\,\frac{\partial h^{\gamma}}{\partial x^{\alpha}}\right]$$

and, therefore,

$$C_{[\alpha\mu\nu]}=0,\quad \frac{1}{2}\,C^{\alpha}_{\alpha\mu}=\frac{1}{\sqrt{hh^{\mu}}}\,\frac{\partial}{\partial x^{\mu}}\,\sqrt{\frac{h}{h^{\mu}}},$$

where $h=h^0h^1h^2h^3$. Therefore, in the Lamé basis

$$\frac{1}{4}\,\omega_{\alpha\mu\nu}H^{\nu}H^{\alpha}H^{\mu}=\sum_{\mu=0}^{3}\frac{H^{\mu}}{\sqrt{hh^{\mu}}}\,\frac{\partial}{\partial x^{\mu}}\,\sqrt{\frac{h}{h^{\mu}}}\,,\tag{53}$$

and the DFI equation in this basis is written as

$$\sum_{\mu=0}^{3}\frac{H^{\mu}}{\sqrt{hh^{\mu}}}\,\frac{\partial}{\partial x^{\mu}}\left(\sqrt{\frac{h}{h^{\mu}}}\,\xi\right)=\frac{imc}{\hbar}\,H^4\xi.\tag{54}$$

The conjugate DFI equation in the Lamé basis is

$$\sum_{\mu=0}^{3}\frac{1}{\sqrt{hh^{\mu}}}\,\frac{\partial}{\partial x^{\mu}}\left(\sqrt{\frac{h}{h^{\mu}}}\,\bar{\xi}\right)H^{\mu}=-\frac{imc}{\hbar}\,\bar{\xi}H^4.\tag{55}$$

5. Transformation from one orthogonal basis to another

The metric form $ds^2=\eta_{\alpha\beta}f^{\alpha}f^{\beta}$ determines the basis f only up to an orthogonal transformation: $f^{\alpha}=L^{\alpha}_{\beta}f'^{\beta}$ and, conversely, $f'^{\alpha}=\widetilde{L}^{\alpha}_{\beta}f^{\beta}$. From $\eta_{\alpha\beta}f'^{\alpha}f'^{\beta}=\eta_{\alpha\beta}f^{\alpha}f^{\beta}$ we find $\eta_{\alpha\sigma}\widetilde{L}^{\sigma}_{\beta}=\eta_{\beta\sigma}\widetilde{L}^{\sigma}_{\alpha}$. We also have

$$e'_{\alpha}=L^{\beta}_{\alpha}e_{\beta},\quad e_{\alpha}=\widetilde{L}^{\beta}_{\alpha}e'_{\beta},\quad e^{\alpha}_{\beta}=e'^{\alpha}_{\gamma}L^{\gamma}_{\beta},\quad f^{\beta}_{\alpha}=f'^{\gamma}_{\alpha}L^{\beta}_{\gamma}.$$

Substituting the last two equations into (38), we find

$$C^{\gamma}_{\alpha\beta}=C'^{\sigma}_{\mu\nu}\widetilde{L}^{\mu}_{\alpha}\widetilde{L}^{\nu}_{\beta}L^{\gamma}_{\sigma}+\widetilde{L}^{\sigma}_{\alpha}e_{\beta}L^{\gamma}_{\sigma}-\widetilde{L}^{\sigma}_{\beta}e_{\alpha}L^{\gamma}_{\sigma}.\tag{56}$$

Therefore,

$$\omega_{\alpha\beta\gamma}=\omega'_{\mu\nu\sigma}\widetilde{L}^{\mu}_{\alpha}\widetilde{L}^{\nu}_{\beta}L^{\sigma}_{\gamma}+\eta_{\mu\nu}\widetilde{L}^{\mu}_{\alpha}e_{\gamma}\widetilde{L}^{\nu}_{\beta}.\tag{57}$$

From this we easily obtain the expressions

$$D_{\beta}A^{\alpha}=\widetilde{L}^{\mu}_{\beta}L^{\alpha}_{\nu}D'_{\mu}A'^{\nu},\quad D_{\beta}A_{\alpha}=\widetilde{L}^{\mu}_{\beta}\widetilde{L}^{\nu}_{\alpha}D'_{\mu}A_{\nu}$$

and the analogous ones for any tensors.

Let us now see how the covariant differential of a spinor transforms under orthochronous Lorentz transformations. Every such transformation $f'^{\alpha}=\widetilde{L}^{\alpha}_{\beta}f^{\beta}$ can be decomposed into the product of some number p of spatial symmetries. The number p is even if the determinant of the transformation matrix is 1 and odd if the determinant is -1, so that $(-1)^p=\det(L^{\alpha}_{\beta})$. The symmetry relative to the hyperplane P orthogonal to the unit vector a^{α} is expressed as $f'^{\alpha}=f^{\alpha}-2a^{\alpha}a_{\beta}f^{\beta}$. Since

$$-AH^{\alpha}A=H^{\alpha}-2a^{\alpha}A,$$

where $A=a_{\alpha}H^{\alpha}$, under any orthochronous Lorentz transformation

$$(-1)^pS^{-1}H^{\alpha}S=\widetilde{L}^{\alpha}_{\beta}H^{\beta},\quad (-1)^pSH^{\alpha}S^{-1}=L^{\alpha}_{\beta}H^{\beta},$$
$$(-1)^pSH_{\alpha}S^{-1}=\widetilde{L}^{\beta}_{\alpha}H_{\beta},\quad (-1)^pS^{-1}H_{\alpha}S=L^{\beta}_{\alpha}H_{\beta},\tag{58}$$

where $S=A_p\ldots A_1$ and $S^{-1}=A_1\ldots A_p$.

Therefore, according to (57) the matrix (43) transforms as

$$\Omega=S\Omega'S^{-1}+\frac{1}{4}\,S^{-1}H_{\mu}Sd(S^{-1}H^{\mu}S).\tag{59}$$

We shall show that

$$\frac{1}{4}\,S^{-1}H_{\mu}Sd(S^{-1}H^{\mu}S)=S^{-1}dS.\tag{60}$$

For $p=1$ this equation is easily verified. In fact,

$$\frac{1}{4}\,AH_{\mu}Ad(AH^{\mu}A)=\frac{1}{4}\,(2a_{\mu}A-H_{\mu})d(2a^{\mu}A-H^{\mu})$$
$$=\frac{1}{2}\,(2a_{\mu}A-H_{\mu})(Ada^{\mu}+a^{\mu}dA)$$
$$=AdA.\tag{61}$$

Here we have used the fact that $2a_{\mu}da^{\mu}=d(a_{\mu}a^{\mu})=0$, $dA^2=A\cdot dA+dA\cdot A=d(a_{\mu}a^{\mu})=0$, since we have $A^2=a_{\mu}a^{\mu}=1$. Now we shall show that if Eq. (60) is satisfied for any p, it is also satisfied for $p+1$. We therefore need to show that (60) leads to the equation

$$\frac{1}{4}\,S^{-1}AH_{\mu}ASd(S^{-1}AH^{\mu}AS)=S^{-1}Ad(AS).\tag{62}$$

Here for brevity we have dropped the subscript $p+1$ on the matrix A. We have

$$\frac{1}{4}\,S^{-1}AH_{\mu}ASd(S^{-1}AH^{\mu}AS)$$
$$=\frac{1}{4}\,S^{-1}AH_{\mu}H^{\mu}AdS$$
$$+\frac{1}{4}\,S^{-1}AH_{\mu}Ad(AH^{\mu}A)S$$
$$+\frac{1}{4}\,S^{-1}AH_{\mu}AS(dS^{-1})AH^{\mu}AS.$$

Since $H_{\mu}H^{\mu}$ is equal to a number—the dimensionality of spacetime—the first term is

$$\frac{1}{4}\,S^{-1}AH_{\mu}H^{\mu}AdS=\frac{1}{4}\,S^{-1}H_{\mu}H^{\mu}dS.$$

According to (61), the second term is

$$\frac{1}{4}S^{-1}AH_\mu Ad(AH^\mu A)S=S^{-1}A(dA)S.$$

The third term is

$$\frac{1}{4}S^{-1}AH_\mu AS(dS^{-1})AH^\mu AS$$

$$=\frac{1}{4}S^{-1}(2a_\mu A-H_\mu)S(dS^{-1})(2a^\mu A-H^\mu)S$$

$$=\frac{1}{4}(2a_\mu a^\nu-\delta^\nu_\mu)(2a^\mu a_\sigma-\delta^\mu_\sigma)S^{-1}H_\nu S(dS^{-1})H^\sigma S$$

$$=\frac{1}{4}S^{-1}H_\mu S(dS^{-1})H^\mu S.$$

Therefore,

$$\frac{1}{4}S^{-1}AH_\mu ASd(S^{-1}AH^\mu AS)$$

$$=\frac{1}{4}S^{-1}H_\mu Sd(S^{-1}H^\mu S)+S^{-1}A(dA)S. \qquad (63)$$

We have thus found that (62) follows from (60). Equation (60) is proved by induction, and so, according to (59),

$$\Omega=S\Omega'S^{-1}+S^{-1}dS. \qquad (64)$$

Setting

$$\xi'=S\xi, \qquad (65)$$

we find

$$D\xi=d\xi+\Omega\xi=S^{-1}(d\xi'+\Omega'\xi')=S^{-1}D'\xi'. \qquad (66)$$

Therefore, ξ and $D\xi$ transform according to the same rule.

Let us see how the conjugate spinor (50) transforms. It is easily verified that if the matrix A corresponds to a unit vector, then

$$A^+H_0=-H_0A^{-1}, \qquad (67)$$

where the + sign, as in (50), indicates Hermitian conjugation. Therefore,

$$S^+H_0=-(-1)^pH_0S^{-1}. \qquad (68)$$

The conjugate spinor thus transforms according to the rule

$$\bar{\xi}'=(-)^p\bar{\xi}S^{-1}. \qquad (69)$$

The covariant differential of the conjugate spinor transforms according to the same rule:

$$D\bar{\xi}=d\bar{\xi}-\bar{\xi}\Omega=(-1)^p(d\bar{\xi}'-\Omega'\bar{\xi}')S$$

$$=(-1)^p(D'\bar{\xi}')S. \qquad (70)$$

Now it is easy to show that Eqs. (49) and (51) are covariant under transformations from one orthogonal basis to another. In fact, (66) and (70) lead to the transformation rules for the covariant derivatives of a spinor:

$$D_\nu\xi=\tilde{L}^\mu_\nu S^{-1}D'_\mu\xi',$$

$$D_\nu\bar{\xi}=(-1)^p\tilde{L}^\mu_\nu(D'_\mu\bar{\xi}')S. \qquad (71)$$

According to (58), we obtain

$$H^\nu D_\nu\xi=(-1)^pS^{-1}H^\mu D'_\mu\bar{\xi}',$$

$$D_\nu\bar{\xi}H^\nu=D'_\mu\bar{\xi}'H^\mu S. \qquad (72)$$

In addition, we have

$$(-1)^pSH^4S^{-1}=H^4. \qquad (73)$$

Therefore,

$$H^\nu D_\nu\xi-\frac{imc}{\hbar}H^4\xi=(-1)^pS^{-1}\left(H^\nu D'_\nu\xi'-\frac{imc}{\hbar}H^4\xi'\right),$$

$$D_\nu\bar{\xi}H^\nu+\frac{imc}{\hbar}\bar{\xi}H^4=\left(D'_\nu\bar{\xi}H^\nu+\frac{imc}{\hbar}\bar{\xi}H^4\right)S, \qquad (74)$$

so that Eqs. (49) and (51) are actually covariant.

6. The second derivative of a spinor and the curvature tensor

The second covariant derivative of a spinor leads us to the definition of the Riemann–Christoffel curvature tensor:

$$R_{\mu\nu,\alpha\beta}=\eta_{\nu\sigma}R^\sigma_{\mu,\alpha\beta}, \qquad (75)$$

where

$$R^\sigma_{\mu,\alpha\beta}=e_\beta\omega^\sigma_{\alpha\mu}-e_\alpha\omega^\sigma_{\beta\mu}+C^\gamma_{\alpha\beta}\omega^\sigma_{\gamma\mu}+\omega^\gamma_{\alpha\mu}\omega^\sigma_{\beta\gamma}$$

$$-\omega^\gamma_{\beta\mu}\omega^\sigma_{\alpha\gamma}. \qquad (76)$$

Actually, the covariant derivative (47) of the spinor ξ behaves both as a spinor and as a covector. Therefore, the second covariant derivative of a spinor ξ also behaves as a spinor and as a tensor. According to (35) and (47), it is

$$D_\alpha D_\beta\xi=\left(e_\alpha+\frac{1}{4}\omega_{\mu\nu\alpha}H^\mu H^\nu\right)D_\beta\xi-\omega^\sigma_{\alpha\beta}D_\sigma\xi. \qquad (77)$$

From this it follows that

$$D_\alpha D_\beta\xi-D_\beta D_\alpha\xi=\frac{1}{4}H^\mu H^\nu R_{\mu\nu,\alpha\beta}. \qquad (78)$$

For the proof we must use Eqs. (28) and (37).

7. Quadrature of the Dirac–Fock–Ivanenko equation

The so-called quadrature of the DFI equation is accomplished by the following trick. We have

$$\left(H^\alpha D_\alpha-\frac{imc}{\hbar}H^4\right)\left(H^\beta D_\beta-\frac{imc}{\hbar}H^4\right)$$

$$=H^\alpha H^\beta D_\alpha D_\beta-\left(\frac{mc}{\hbar}\right)^2$$

and, furthermore,

$$H^\alpha H^\beta D_\alpha D_\beta=\frac{1}{2}H^\alpha H^\beta(D_\alpha D_\beta+D_\beta D_\alpha)$$

$$+\frac{1}{2}H^\alpha H^\beta(D_\alpha D_\beta-D_\beta D_\alpha).$$

In the last sum the first term is

$$\frac{1}{2} H^{\alpha}H^{\beta}(D_{\alpha}D_{\beta}+D_{\beta}D_{\alpha})$$

$$=\frac{1}{4}(H^{\alpha}H^{\beta}+H^{\beta}H^{\alpha})(D_{\alpha}D_{\beta}+D_{\beta}D_{\alpha})$$

$$=\frac{1}{2}\eta^{\alpha\beta}(D_{\alpha}D_{\beta}+D_{\beta}D_{\alpha})=\eta^{\alpha\beta}D_{\alpha}D_{\beta},$$

and according to (79) the second term is

$$\frac{1}{2} H^{\alpha}H^{\beta}(D_{\alpha}D_{\beta}-D_{\beta}D_{\alpha})=\frac{1}{8} H^{\alpha}H^{\beta}H^{\mu}H^{\nu}R_{\mu\nu,\alpha\beta}.$$

Furthermore, since the alternation of the Riemann–Cristoffel tensor over three indices gives zero, according to (27)

$$H^{\alpha}H^{\beta}H^{\mu}R_{\mu\nu,\alpha\beta}=(\eta^{\beta\mu}H^{\alpha}+\eta^{\beta\alpha}H^{\mu}-\eta^{\alpha\mu}H^{\beta})R_{\mu\nu,\alpha\beta}$$

$$=2H^{\alpha}\eta^{\beta\mu}R_{\mu\nu,\alpha\beta}=2H^{\alpha}R_{\nu\alpha}.$$

Therefore, the second term is

$$\frac{1}{2} H^{\alpha}H^{\beta}(D_{\alpha}D_{\beta}-D_{\beta}D_{\alpha})=\frac{1}{4} H^{\alpha}H^{\nu}R_{\nu\alpha}=\frac{1}{4} R.$$

We have thereby obtained the squared DFI equation:

$$\left\{\eta^{\alpha\beta}D_{\alpha}D_{\beta}+\frac{1}{4} R-\left(\frac{mc}{\hbar}\right)^{2}\right\}\xi=0. \tag{79}$$

In general, this system of second-order equations does not split up into four separate equations for each component of the spinor field. Nevertheless, we shall find this system useful in the second part of this review.

II. PRINCIPLES OF THE QUANTUM THEORY OF THE SPINOR FIELD

1. The anticommutator of the spinor field

As we are trying to preserve the basic concepts of quantum field theory, we shall consider only Riemannian spacetimes in which there are spacelike hypersurfaces separating the spacetime into two parts. One of these will be the past and the other will be the future, and the hypersurface itself will be the present. We shall term such hypersurfaces complete and denote them by Σ. We shall assume that the solution of the DFI equation in the entire Riemannian spacetime is uniquely specified by the values of the spinor field on the complete hypersurface Σ. The spinor field can be specified arbitrarily on this hypersurface.

Let us consider the system of equations

$$H^{\nu}D_{\nu}u=\frac{imc}{\hbar} H^{4}u,$$
$$D_{\nu}\bar{v}H^{\nu}=-\frac{imc}{\hbar}\bar{v}H^{4}, \tag{1}$$

consisting of the DFI equation and its conjugate. Let u, $\bar{v}$ be the solution of this system. The divergence of a vector

$$S^{\nu}=-\bar{v}H^{\nu}u, \tag{2}$$

equal to

$$D_{\nu}S^{\nu}=-(D_{\nu}\bar{v}H^{\nu})u-\bar{v}(H^{\nu}D_{\nu}),$$

is equal to zero according to (1). Therefore, according to the Gauss theorem, the integral

$$\int_{\Sigma} S_{\mu}d\sigma^{\mu} \tag{3}$$

is independent of the choice of the complete hypersurface Σ. In the integral (3), $d\sigma^{\alpha}$ is the vector corresponding to an elementary surface element of Σ. If we use $\varepsilon_{\alpha\beta\gamma\nu}$ to denote the completely antisymmetric tensor with the condition $\varepsilon_{0123}=1$, then

$$S_{\mu}d\sigma^{\mu}=\varepsilon_{\alpha\beta\gamma\mu}q_1^{\alpha}q_2^{\beta}q_3^{\gamma}S^{\mu}, \tag{4}$$

where q_1^{α}, q_2^{α}, and q_3^{α} are the vectors of elementary displacements along Σ. If, for example, the hypersurface Σ is specified by the equations

$$x^{\alpha}=T^{\alpha}(q^1,q^2,q^3),$$

then

$$q_1^{\alpha}=f_{\mu}^{\alpha}\frac{\partial T^{\mu}}{\partial q^1}dq^1,\quad q_2^{\alpha}=f_{\mu}^{\alpha}\frac{\partial T^{\mu}}{\partial q^2}dq^2,$$

$$q_3^{\alpha}=f_{\mu}^{\alpha}\frac{\partial T^{\mu}}{\partial q^3}dq^3.$$

Otherwise, the sum (4) is written in the form of the determinant

$$S_{\mu}d\sigma^{\mu}=\begin{vmatrix} q_1^0 & q_1^1 & q_1^2 & q_1^3 \\ q_2^0 & q_2^1 & q_2^2 & q_2^3 \\ q_3^0 & q_3^1 & q_3^2 & q_3^3 \\ S^0 & S^1 & S^2 & S^3 \end{vmatrix}. \tag{5}$$

Furthermore, since

$$\varepsilon_{\alpha\beta\gamma\nu}H^{\mu}=-iH_4[H_{\alpha}H_{\beta}H_{\gamma}]. \tag{6}$$

then, according to (2),

$$S_{\mu}d\sigma^{\mu}=i\bar{v}H_4[Q_1Q_2Q_3]u, \tag{7}$$

where

$$Q_1=H_{\alpha}q_1^{\alpha},\quad Q_2=H_{\alpha}q_2^{\alpha},\quad Q_3=H_{\alpha}q_3^{\alpha}. \tag{8}$$

The integral (3) specifies the scalar quadrature in the space of solutions of the system of equations (1). Here and below, the spinor fields u and $\bar{v}$ are assumed to be classical, not operator, fields.

On the other hand, the spinor field ξ is quantized, and it is quantized according to Fermi statistics. Here it is necessary that the pair ξ and $\bar{\xi}$ satisfy Eq. (1) and generate an (infinite-dimensional) Clifford algebra. The generators of this algebra, that is, ξ and $\bar{\xi}$, on the complete hypersurface Σ are assumed to be linearly independent.

Let us consider the following vectors from the linear envelope of this algebra:

$$U=i\int_{\Sigma}\bar{\xi}H_4[Q_1Q_2Q_3]u,$$
$$V^*=i\int_{\Sigma}\bar{v}H_4[Q_1Q_2Q_3]\xi. \tag{9}$$

The integrals (9) do not depend on the choice of complete hypersurface Σ, because the pairs $u,\bar{\xi}$ and $\xi,\bar{v}$ satisfy the system of equations (1).

The sum

$$U+V^* \tag{10}$$

is by definition the general element of the linear envelope of generators of the Clifford algebra that we want to construct. Setting

$$(U+V^*)^2=i\int_\Sigma \bar{v}H_4[Q_1Q_2Q_3]u, \tag{11}$$

we introduce the symmetric scalar product in the envelope of generators, which is the exact expression of the principle of quantization in accordance with Fermi statistics.

Since the pairs u, 0 and 0, $\bar{v}$ satisfy the system of equations (1), from (11) it follows that

$$U^2=0, \quad V^{*2}=0. \tag{12}$$

Therefore, the anticommutator

$$\{UV^*\}=UV^*+V^*U \tag{13}$$

is equal to

$$\{UV^*\}=i\int_\Sigma \bar{v}H_4[Q_1Q_2Q_3]u. \tag{14}$$

Since the spinor field $\bar{v}$ can be specified arbitrarily on Σ, we find that on Σ

$$\{\xi(x)U\}=u(x), \tag{15}$$

and since the spinor field u can also be specified arbitrarily on Σ, we find that on Σ

$$\{V^*\bar{\xi}(x)\}=\bar{v}(x). \tag{16}$$

Furthermore, we see that since the pair ξ, $\bar{\xi}$ satisfies the system (1), the pair $\{\xi U\}$, $\{V^*\bar{\xi}\}$ also satisfies the system (1). However, the pair u, $\bar{v}$ also satisfies this system of equations. Since the two latter pairs coincide on the complete surface Σ, then owing to the assumed uniqueness of the Cauchy problem for the system (1), Eqs. (15) and (16) are valid not only on Σ, but throughout spacetime.

Similarly, from (12) we find that throughout spacetime

$$\{\xi(x)V^*\}=0, \quad \{U\bar{\xi}(x)\}=0. \tag{17}$$

Owing to the arbitrariness of u, $\bar{v}$ on Σ, from (17) we find that for any point y lying on Σ

$$\{\xi_p(x)\xi_q(y)\}=0, \quad \{\bar{\xi}_q(y)\bar{\xi}_p(x)\}=0, \tag{18}$$

where p and q label the components of the spinors ξ and $\bar{\xi}$. Owing to the uniqueness of the Cauchy problem for the system of equations (1), the equalities (18) are valid for any spacetime point y.

We use

$$\{\xi(x)\bar{\xi}(y)\} \tag{19}$$

to denote the matrix with components $\{\xi_p(x)\bar{\xi}_q(y)\}$. It satisfies the following condition:

$$\{\xi(x)\bar{\xi}(y)\}H^0=H_0\{\xi(x)\bar{\xi}(y)\}^+. \tag{20}$$

As before, the symbol + stands for Hermitian conjugation.

It follows from (9) and (13) that if a complete hypersurface Σ containing the two points x and y can be constructed, then $\{\xi(x)\bar{\xi}(y)\}=0$. In particular, $\{\xi(x)\bar{\xi}(x)\}=0$.

Writing (15) and (16) in expanded form

$$\begin{aligned} u(x)&=i\int_\Sigma \{\xi(x)\bar{\xi}(y)\}H_4[Q_1Q_2Q_3]u(y),\\ \bar{v}(x)&=i\int_\Sigma \bar{v}(y)H_4[Q_1Q_2Q_3]\{\xi(y)\bar{\xi}(x)\}, \end{aligned} \tag{21}$$

we notice that the anticommutator $\{\xi(x)\bar{\xi}(y)\}$ gives the solution of the Cauchy problem for the system of equations (1).

Since the anticommutator $\{\xi(x)\bar{\xi}(y)\}$ itself also satisfies this system, according to (21) we have

$$\{\xi(x)\bar{\xi}(y)\}=i\int_\Sigma \{\xi(x)\bar{\xi}(z)\}H_4[Q_1Q_2Q_3]\{\xi(z)\bar{\xi}(y)\}. \tag{22}$$

Now for some complete hypersurface Σ let the Cauchy problem for the system of equations (1) be somehow solved and the solution represented as

$$\begin{aligned} u(x)&=i\int_\Sigma \bar{S}(x,y)H_4[Q_1Q_2Q_3]u(y),\\ \bar{v}(x)&=i\int_\Sigma \bar{v}(y)H_4[Q_1Q_2Q_3]S(y,x). \end{aligned} \tag{23}$$

Comparing (21) with (23), we see that on the direct product $M\times\Sigma$, where M is the entire spacetime,

$$\begin{aligned} \{\xi(x)\bar{\xi}(y)\}&=\bar{S}(x,y),\\ \{\xi(y)\bar{\xi}(x)\}&=S(y,x). \end{aligned} \tag{24}$$

According to (20), the functions $\bar{S}$ and S are related by the condition

$$\bar{S}(x,y)H^0=H_0S^+(y,x). \tag{25}$$

From (22) we find the anticommutator in the form

$$\{\xi(x)\bar{\xi}(y)\}=i\int \bar{S}(x,z)H_4[Q_1Q_2Q_3]S(z,y) \tag{26}$$

everywhere on $M\times M$.

We note that if the pair $(u,\bar{v})$ is a solution of the system (1), then the pair $(u,\bar{v})^*=(v,\bar{u})$ is also a solution of the system (1). We shall term a solution real if $(u,\bar{v})^*=(u,\bar{v})$, that is, if $u=v$. The element

$$U+U^* \tag{27}$$

of the envelope of generators corresponding to a real solution is termed the real or Hermitian element.

2. The current vector and the charge operator

In the spinor case the current vector is given by Eq. (2) and is

$$J^\mu=e\bar{\xi}H^\mu\xi, \tag{28}$$

where e is the positron charge.

The charge operator is given by the integral

$$\hat{e}=\int_{\Sigma}J_{\mu}d\sigma^{\mu}=-ie\int_{\Sigma}\bar{\xi}H_4[Q_1Q_2Q_3]\xi \tag{29}$$

over the complete hypersurface Σ. It is independent of the choice of Σ, because

$$D_{\mu}J^{\mu}=0. \tag{30}$$

3. The energy–momentum tensor

The components of the energy–momentum tensor in Riemannian spacetime in any orthogonal basis have the same form as in flat spacetime in Cartesian coordinates, namely:

$$T_{\mu\nu}=\frac{i\hbar}{4}\{S_{\mu\nu}+S_{\nu\mu}\}, \tag{31}$$

where

$$S_{\mu\nu}=\bar{\xi}H_{\mu}\xi_{\nu}-\bar{\xi}_{\nu}H_{\mu}\xi. \tag{32}$$

In turn,

$$\xi_{\mu}=D_{\mu}\xi,\quad \bar{\xi}_{\mu}=D_{\mu}\bar{\xi}. \tag{33}$$

We shall show that the two divergences of the tensor (32) are equal to zero. Actually, one of them is

$$D^{\nu}S_{\mu\nu}=\bar{\xi}^{\nu}H_{\mu}\xi_{\nu}+\bar{\xi}H_{\mu}(D^{\nu}\xi_{\nu})-(D^{\nu}\bar{\xi}_{\nu})H_{\mu}\xi-\bar{\xi}_{\nu}H_{\mu}\xi^{\nu}$$
$$=\bar{\xi}H_{\mu}(D^{\nu}\xi_{\nu})-(D^{\nu}\bar{\xi}_{\nu})H_{\mu}\xi,$$

and the latter difference vanishes because the spinor field obeys the squared DFI equation (I.79). Therefore,

$$D^{\nu}S_{\mu\nu}=0. \tag{34}$$

Turning to the other divergence of the tensor (32), we note that if the pair $\xi,\bar{\xi}$ satisfies the system of equations (1), then

$$S_{\nu\mu}-S_{\mu\nu}=D_{\alpha}\{\bar{\xi}[H_{\mu}H_{\nu}H^{\alpha}]\xi\}. \tag{35}$$

This is verified by using Eq. (I.22). From (34) and (35) we find

$$D^{\nu}S_{\nu\mu}=D_{\nu}D_{\alpha}\{\bar{\xi}[H_{\mu}H^{\nu}H^{\alpha}]\xi\}$$
$$=\frac{1}{2}(D_{\nu}D_{\alpha}-D_{\alpha}D_{\nu})\{\bar{\xi}[H_{\mu}H^{\nu}H^{\alpha}]\xi\}$$
$$=\frac{1}{2}R_{\mu\sigma,\nu\alpha}\{\bar{\xi}[H^{\sigma}H^{\nu}H^{\alpha}]\xi\}=0. \tag{36}$$

We therefore have

$$T_{\mu\nu}=T_{\nu\mu},\quad D^{\mu}T_{\mu\nu}=0. \tag{37}$$

In addition,

$$T=\eta^{\mu\nu}T_{\mu\nu}=-mc\bar{\xi}H^4\xi. \tag{38}$$

It follows from (31) and (35) that

$$T_{\mu\nu}=\frac{i\hbar}{4}(2S_{\mu\nu}+D_{\alpha}\{\bar{\xi}[H_{\mu}H_{\nu}H^{\alpha}]\xi). \tag{39}$$

We shall need the energy–momentum tensor in this form for what follows.

4. The second-quantized isometric and conformal momenta

According to the Gauss theorem, if a hypersurface ∂o bounds a simple region o, then for any tensor field $T_{\mu\nu}$ and any vector field K^{μ} we have

$$\int_{\partial o}K^{\mu}T_{\mu\nu}d\sigma^{\nu}=\int_{o}D^{\nu}(K^{\mu}T_{\mu\nu})dv. \tag{40}$$

If the tensor field $T_{\mu\nu}$ satisfies the conditions (37), then

$$D^{\nu}(K^{\mu}T_{\mu\nu})=K_{\mu\nu}T^{\mu\nu}+FT, \tag{41}$$

where T is the trace of the tensor $T_{\mu\nu}$, which in this case is given by (38), and

$$F=\frac{1}{4}D_{\alpha}K^{\alpha}, \tag{42}$$

$$K_{\mu\nu}=\frac{1}{2}(D_{\mu}K_{\nu}+D_{\nu}K_{\mu})-F\eta_{\mu\nu}. \tag{43}$$

The one-parameter group of spacetime transformations obtained by solving the system of ordinary differential equations

$$f^{\alpha}_{\beta}\frac{dx^{\beta}}{d\lambda}=\frac{f^{\alpha}}{d\lambda}=K^{\alpha} \tag{44}$$

is termed conformal when

$$K_{\mu\nu}=0. \tag{45}$$

In the more general case when

$$K_{\mu\nu}=0,\quad F=0, \tag{46}$$

it is termed isometric. The system of equations (46) is equivalent to the system

$$D_{\mu}K_{\nu}+D_{\nu}K_{\mu}=0. \tag{47}$$

In the case (46) the divergence (41) is equal to zero, and the integral (40) also vanishes. Therefore, in this case the integral

$$\hat{\mathscr{K}}=\int_{\Sigma}K^{\mu}T_{\mu\nu}d\sigma^{\nu} \tag{48}$$

is independent of the choice of the complete hypersurface Σ. We shall call this the second-quantized isometric momentum operator.

For the same reason as in the case (45), the integral (48) is also independent of the choice of the complete hypersurface Σ, but only with the condition $T=0$. According to (38), this is satisfied if the mass of the spinor particle is zero. In this case the integral (48) for $m=0$ will be called the second-quantized conformal momentum operator.

Let us give a general (in the spinor variant) definition of a second-quantized operator.

Let an operator $\hat{K}$ act in the space of solutions of the DFI equation. This means that if ξ satisfies the DFI equation, then $\hat{K}\xi$ also satisfies the DFI equation. To each such operator there corresponds an integral

$$\hat{\mathscr{K}}=(\bar{\xi},\hat{K}\xi)=i\int_{\Sigma}\bar{\xi}H_4[Q_1Q_2Q_3]\hat{K}\xi, \tag{49}$$

independent of the choice of the complete hypersurface Σ, which we refer to as a second-quantized operator.

In particular, the charge operator (29) is such a second-quantized operator. It is

$$\hat{e}=(\bar{\xi},-e\xi). \tag{50}$$

In this case $\hat{K}$ is an operator which multiplies by the number $-e$.

The integral (48) falls under the general definition of a second-quantized operator, because it is equal to the integral (49), where

$$\hat{K}=-i\hbar\left\{K^\mu D_\mu+\frac{1}{4}(D_\alpha K_\beta)H^\alpha H^\beta\right\} \tag{51}$$

in the isometric case and

$$\hat{K}=-i\hbar\left\{K^\mu D_\mu+\frac{1}{4}(D_\alpha K_\beta)H^\alpha H^\beta+\frac{1}{2}F\right\}$$
$$=-i\hbar\left\{K^\mu D_\mu+\frac{1}{4}(D_\alpha K_\beta-F\eta_{\alpha\beta})H^\alpha H^\beta+\frac{3}{2}F\right\} \tag{52}$$

in the conformal case. In fact, taking $T_{\mu\nu}$ in the form (39), we easily verify that the vector field

$$T^\alpha=K^\mu T_{\mu\nu}\eta^{\nu\alpha}+\bar{\xi}H^\alpha\hat{K}\xi$$

is the divergence $T^\alpha=D_\beta A^{\alpha\beta}$ of an antisymmetric tensor field:

$$A^{\alpha\beta}=i\hbar\left\{\frac{1}{4}\bar{\xi}[H^\alpha K H^\beta]\xi+\frac{1}{2}\bar{\xi}(K^\alpha H^\beta-K^\beta H^\alpha)\xi\right\},$$

where $K=K^\nu H_\nu$, so that, according to the Stokes theorem,

$$\int_\Sigma T^\alpha\eta_{\alpha\nu}d\sigma^\nu=0.$$

5. The isometric momentum operator

Equation (47) has a nontrivial solution if and only if the isometry group acts in the spacetime. We therefore have the equality

$$f^{\alpha'}_{\beta'}(x')=f^\alpha_\beta(x), \tag{53}$$

where the prime denotes the result of an operation from this group. For an infinitesimal transformation

$$x^{\alpha'}=x^\alpha+K^\mu e^\alpha_\mu\lambda \tag{54}$$

this implies that

$$K^\mu e_\mu f^\alpha_\beta=0. \tag{55}$$

Let us consider the corresponding transformation of the basis linear form. Owing to (53) and (54),

$$f^{\alpha'}=f^{\alpha'}_{\beta'}(x')dx^{\beta'}=f^\alpha_\beta(dx^\beta+dK^\mu e^\beta_\mu\lambda+K^\mu de^\beta_\mu\lambda)$$
$$=f^\alpha+(dK^\alpha-K^\mu e^\beta_\mu df^\alpha_\beta)\lambda.$$

According to Eq. (I.38),

$$C^\alpha_{\mu\nu}f^\nu=e^\beta_\mu df^\alpha_\beta-e^\beta_\nu e_\mu f^\alpha_\beta f^\nu.$$

Therefore, owing to (55), in this case

$$K^\mu C^\alpha_{\mu\nu}f^\nu=K^\mu e^\beta_\mu df^\alpha_\beta,$$

so that

$$f^{\alpha'}=f^\alpha+(dK^\alpha-K^\mu C^\alpha_{\mu\nu}f^\nu)\lambda.$$

Taking into account (I.32) and (I.36), we finally obtain

$$f^{\alpha'}=f^\alpha+(D_\nu K^\alpha-K^\mu\omega^\alpha_{\mu\nu})f^\nu\lambda. \tag{56}$$

Therefore, for an infinitesimal isometric transformation (54) the basis linear form undergoes an infinitesimal rotation (56) given by the antisymmetric matrix

$$D_\nu K_\alpha+K^\mu\omega_{\nu\alpha\mu}. \tag{57}$$

According to (I.42) and (I.47), it follows from this that the addition to the spinor field in the transformation (54) is

$$\xi'(x')-\xi(x)=\lambda\left\{K^\mu e_\mu+\frac{1}{4}K^\mu\omega_{\nu\alpha\mu}H^\nu H^\alpha+\frac{1}{4}(D_\nu K_\alpha)H^\nu H^\alpha\right\}\xi=\lambda\left\{K^\mu D_\mu+\frac{1}{4}(D_\nu K_\alpha)H^\nu H^\alpha\right\}\xi. \tag{58}$$

We therefore arrive at the isometric momentum operator (51) of the spinor field.

Let us show that this operator commutes with the operator $H^\nu D_\nu$ from the left-hand side of the DFI equation. We have

$$\frac{i}{\hbar}[H^\nu D_\nu\hat{K}-\hat{K}H^\nu D_\nu]=K^\mu H^\nu(D_\nu D_\mu-D_\mu D_\nu)+\frac{1}{4}(D_\nu D_\alpha K_\beta)H^\nu H^\alpha H^\beta+\hat{O}, \tag{59}$$

where

$$\hat{O}=(D_\nu K^\mu)H^\nu D_\mu+\frac{1}{4}(D_\alpha K_\beta)(H^\nu H^\alpha H^\beta-H^\alpha H^\beta H^\nu)D_\nu.$$

Since

$$H^\nu H^\alpha H^\beta-H^\alpha H^\beta H^\nu=2\eta^{\alpha\nu}H^\beta-2\eta^{\beta\nu}H^\alpha,$$

then $\hat{O}=0$. Furthermore, using (I.78) we find that the commutator (59) is

$$\frac{1}{4}\{K^\mu R_{\alpha\beta,\nu\mu}+(D_\nu D_\alpha K_\beta)\}H^\nu H^\alpha H^\beta.$$

It is easily verified that for a field obeying Eq. (47),

$$K^\mu R_{\alpha\beta,\nu\mu}+(D_\nu D_\alpha K_\beta)=0. \tag{60}$$

Therefore, the commutator (59) is zero. However, it is obvious that $\hat{K}H^4=H^4\hat{K}$.

Consequently, if the spinor field ξ satisfies the DFI equation, the spinor field $\hat{K}\xi$ satisfies the same equation.

6. Conformal invariance of the neutrino behavior

If all fermion interactions except that with the gravitational field are neglected, the behavior of the fermions in this approximation is completely determined by the DFI equation (I.49), the scalar product

$$(\bar{v},u)=i\int_{\Sigma}\bar{v}H_4[Q_1Q_2Q_3]u, \tag{61}$$

determining the algebra of the field operators, and the energy–momentum tensor (31). For the problems discussed here, which were studied in Ref. 49, it is not important whether the neutrino field is described by a four-component spinor or a two-component half-spinor. We shall therefore describe it by the equation

$$H^{\nu}\xi_{\nu}=0, \tag{62}$$

simply setting $m=0$ in the DFI equation.

We shall show that the behavior of the neutrino is conformally invariant. This means that in a spacetime with metric ds^2 the neutrino behaves as in a spacetime with metric $ds'^2=B^2ds^2$. Two gravitational fields, one corresponding to metric ds^2 and the other to metric ds'^2, are identical as seen by the neutrino.

Let us explain this for an important example, when one of such a pair of fields is static. It corresponds to the metric reduced to the form

$$ds^2=\sum_{i=1}^{3}\sum_{k=1}^{3}g_{ik}(x^1,x^2,x^3)dx^idx^k-c^2dt^2. \tag{63}$$

Clearly, a static field cannot generate neutrinos, just as for other fermions or bosons. The conformal invariance of the neutrino behavior implies, in particular, that the gravitational field with metric $ds'^2=B^2ds^2$ proportional to (63) also cannot generate neutrinos.

An example of such a field is the gravitational field in Friedmannian spacetime. In this case the factor B depends only on time. The spatial part in (63) is the metric form of Euclidean space, spherical space, or Lobachevsky space.

To prove our statement about the conformal invariance of the neutrino behavior it is sufficient to prove the conformal invariance of Eq. (62), the scalar product (61), and the integral (48), where K^{μ} is a vector field.

We shall construct the proof for n-dimensional spacetime. For the DFI equation it is important that the number n be even. Of course, in the real world $n=4$. However, in the next (the third) part of this review we shall consider the model case $n=2$.

Let there be an orthogonal basis linear form f^{α} such that in a spacetime with metric ds^2 we have $ds^2=\eta_{\alpha\beta}f^{\alpha}f^{\beta}$. In a spacetime with metric $ds'^2=B^2ds^2$ the basis linear form $f'^{\alpha}=Bf^{\alpha}$ satisfies the analogous condition $ds'^2=\eta_{\alpha\beta}f'^{\alpha}f'^{\beta}$. We also have $e_{\alpha}=Be'_{\alpha}$. Therefore, we take

$$f'^{\alpha}_{\beta}=Bf^{\alpha}_{\beta},\quad e^{\alpha}_{\beta}=Be'^{\alpha}_{\beta}. \tag{64}$$

From this it follows that

$$[Q'_1\dots Q'_{n-1}]=B^{n-1}[Q_1\dots Q_{n-1}].$$

Therefore, if we make the substitution

$$u=B^{(n-1)/2}u',\quad \bar{v}=B^{(n-1)/2}\bar{v}', \tag{65}$$

we obtain the equality

$$\bar{v}'H_n[Q'_1\dots Q'_{n-1}]u'=\bar{v}H_n[Q_1\dots Q_{n-1}]u, \tag{66}$$

needed to prove the conformal invariance of the scalar product determining the algebra of the field operators.

According to (65), we must use the further substitution

$$\xi=B^{(n-1)/2}\xi'. \tag{67}$$

Then from (64) we find the relation between the nonholonomy coefficients,

$$C^{\gamma}_{\alpha\beta}=BC'^{\gamma}_{\alpha\beta}+\delta^{\gamma}_{\beta}e'_{\alpha}B-\delta^{\gamma}_{\alpha}e'_{\beta}B,$$

and the rotation coefficients of the basis linear forms,

$$\omega_{\alpha\beta\gamma}=B\omega'_{\alpha\beta\gamma}+\eta_{\beta\nu}e'_{\alpha}B-\eta_{\alpha\nu}e'_{\beta}B.$$

From this we obtain

$$\begin{aligned}\xi_{\nu}&=B^{(n+1)/2}\xi'_{\nu}+\frac{1}{2}B^{(n-1)/2}(ne'_{\nu}B-H_{\nu}H^{\alpha}e'_{\alpha}B)\xi',\\ \bar{\xi}_{\nu}&=B^{(n+1)/2}\bar{\xi}'_{\nu}+\frac{1}{2}B^{(n-1)/2}\bar{\xi}'(ne'_{\nu}B-e'_{\alpha}BH^{\alpha}H_{\nu}).\end{aligned} \tag{68}$$

Therefore,

$$H^{\nu}\xi_{\nu}=B^{(n+1)/2}H^{\nu}\xi'_{\nu}; \tag{69}$$

$$T_{\mu\nu}=B^nT'_{\mu\nu}. \tag{70}$$

Owing to (69), Eq. (62) is conformally invariant.

Finally, the vector field K^{μ} in one spacetime corresponds to the vector field $K'^{\mu}=BK^{\mu}$ in the other. In addition, $d\sigma'^{\nu}=B^{n-1}d\sigma^{\nu}$. Therefore, owing to (71),

$$K'^{\mu}T'_{\mu\nu}d\sigma'^{\nu}=K^{\mu}T_{\mu\nu}d\sigma^{\nu}. \tag{71}$$

The last equation is needed to prove the conformal invariance of the integral (48).

We have almost proved our assertion that the neutrino behavior is conformally invariant. To complete the proof, we need to assume that the functions B and B^{-1} are known everywhere and do not vanish anywhere.

However, the latter condition can be weakened. We can instead require that these functions are known and do not vanish in some region containing the complete hypersurface Σ. In this region the neutrino behavior will again be conformally invariant.

This weakened condition is particularly interesting in the case of Friedmannian spacetime, when the function B is independent of the spatial coordinates, and hypersurfaces on which the time coordinate is given can be taken as complete. We see that the neutrino behavior in Friedmannian spacetime is determined only by the sign of the curvature of the hypersurface $t=\text{const}$. Therefore, to study the neutrino behavior in Friedmannian spacetime it is sufficient to study flat spacetime with the metric

$$d\rho^2+\rho^2(d\theta^2+\sin^2\theta d\varphi^2)-c^2dt^2,$$

spherical spacetime with the metric

$$d\rho^2+R^2\sin^2\frac{\rho}{R}(d\theta^2+\sin^2\theta d\varphi^2)-c^2dt^2,$$

and Lobachevsky spacetime with the metric

$$d\rho^2+k^2\sinh^2\frac{\rho}{k}(d\theta^2+\sin^2\theta d\varphi^2)-c^2dt^2.$$

If no constraints are imposed on the functions B and B^{-1}, then in the region where they are both nonzero some of the features of the conformally invariant neutrino behavior will not be preserved. Features like (61) and (48) described by integral characteristics will not be preserved, and only those like (62), (66), and (71) described by differential characteristics will be. We shall use this observation below to derive the conformal momentum operator (52).

7. The conformal momentum operator

The conformal momentum operator of the spinor field has been derived in Refs. 53 and 54. For two spacetimes in conformal correspondence with each other we have

$$BD'_\alpha K'_\beta=BD_\alpha K_\beta+\eta_{\alpha\beta}K^\nu e_\nu B+K_\beta e_\alpha B-K_\alpha e_\beta B. \quad (72)$$

Considering the tensor (43), where in the n-dimensional case F is

$$F=\frac{1}{n}D_\alpha K^\alpha, \quad (73)$$

rather than (42), we find

$$K'_{\alpha\beta}=K_{\alpha\beta},\quad BF'=BF+K^\nu e_\nu B. \quad (74)$$

Now let the condition (44) be satisfied for the field K^α. In this case the same condition also holds for the field K'^α, no matter what the function B is. However, in some neighborhood of a spacetime point it is always possible to find a function B such that the condition $F'=0$ is satisfied. Choosing such a function B, that is, solving the equation

$$BF+K^\nu e_\nu B=0, \quad (75)$$

we will obtain the isometric momentum operator taking the spinor ξ' into the spinor

$$-i\hbar\left\{K'^\mu\xi'_\mu+\frac{1}{4}(D'_\alpha K'_\beta)H^\alpha H^\beta\xi'\right\}.$$

Using the substitution (67) and Eqs. (68), (72), and (75), we easily obtain the conformal momentum operator:

$$\hat{K}=-i\hbar\left\{K^\mu D_\mu+\frac{1}{4}(D_\alpha K_\beta-F\eta_{\alpha\beta})H^\alpha H^\beta+\frac{n-1}{2}F\right\}$$
$$=-i\hbar\left\{K^\mu D_\mu+\frac{1}{4}(D_\alpha K_\beta)H^\alpha H^\beta+\frac{n-2}{4}F\right\}. \quad (76)$$

For an arbitrary vector field K^α we obtain the commutator

$$\frac{i}{\hbar}[H^\nu D_\nu\hat{K}-\hat{K}H^\nu D_\nu]=\frac{1}{2}H^\alpha(D^\beta K_{\alpha\beta})+H^\alpha K_{\alpha\beta}D^\beta+FH^\alpha D_\alpha. \quad (77)$$

In fact, we have

$$[H^\nu D_\nu,K^\mu D_\mu]=K^\mu H^\nu(D_\nu D_\mu-D_\mu D_\nu)+H^\nu(D_\nu K^\mu)D_\mu,$$

$$\left[H^\nu D_\nu,\frac{1}{4}(D_\alpha K_\beta)H^\alpha H^\beta\right]$$
$$=\frac{1}{4}(D_\nu D_\alpha K_\beta)H^\nu H^\alpha H^\beta+\frac{1}{4}(D_\alpha K_\beta)(H^\nu H^\alpha H^\beta-H^\alpha H^\beta H^\nu)D_\nu,$$

$$\left[H^\nu D_\nu,\frac{n-2}{4}F\right]=\frac{n-2}{4}F_\nu H^\nu.$$

Furthermore, from the identity

$$D_\nu D_\alpha K_\beta=D_\nu(K_{\alpha\beta}+F\eta_{\alpha\beta})+D_\alpha(K_{\beta\nu}+F\eta_{\beta\nu})-D_\beta(K_{\alpha\nu}+F\eta_{\alpha\nu})+\frac{1}{2}\{(D_\beta D_\alpha-D_\alpha D_\beta)K_\nu+(D_\nu D_\alpha-D_\alpha D_\nu)K_\beta-(D_\nu D_\beta-D_\beta D_\nu)K_\alpha\}$$

we find the equation

$$D_\nu D_\alpha K_\beta=D_\nu K_{\alpha\beta}+D_\alpha K_{\beta\nu}-D_\beta K_{\alpha\nu}+F_\nu\eta_{\alpha\beta}+F_\alpha\eta_{\beta\nu}-F_\beta\eta_{\alpha\nu}+R^\sigma_{\nu,\beta\alpha}K_\sigma.$$

Using (I.78), we obtain (77).

Therefore, if the spinor field ξ satisfies Eq. (62), the field $\hat{K}\xi$ also satisfies this equation if $K_{\alpha\beta}=0$.

III. QUANTUM THEORY OF THE SPINOR FIELD IN TWO-DIMENSIONAL SPACETIME

1. Solution of the DFI equation in two-dimensional spacetime

This part of the review is based on Refs. 55–57. The metric of two-dimensional spacetime can locally be written as

$$ds^2=a^2(dz^2-c^2dt^2), \quad (1)$$

where a is some function of z and t. The spinor analysis can be performed directly in two-dimensional spacetime, but here it is preferable to embed the two-dimensional spacetime into four-dimensional spacetime with the metric

$$ds^2=dx^2+dy^2+a^2(dz^2-c^2dt^2)$$

and to use the results obtained above.

If in the original Dirac equations (I.1) we set

$$\psi_1=0,\quad \psi_2=\psi_2(z,t),\quad \psi_3=0,\quad \psi_4=\psi_4(z,t), \quad (2)$$

we obtain the Dirac equations in two-dimensional Minkowski spacetime:

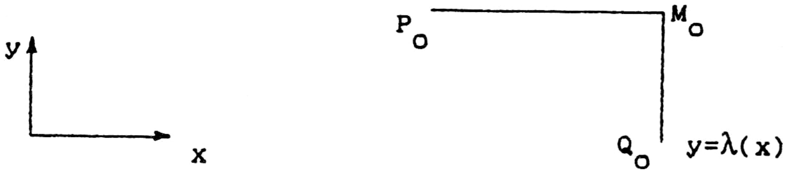

FIG. 1.

$$(E-mc^2)\psi_2+cp_z\psi_4=0,$$
$$(E+mc^2)\psi_4+cp_z\psi_2=0. \tag{3}$$

The Cartan substitution (I.15) with the conditions (2) brings Eq. (I.16) and also (3) to the form

$$(E+cp_z)\xi_2=-mc^2\xi_0,$$
$$(cp_z-E)\xi_0=mc^2\xi_2. \tag{4}$$

According to (I.53), the DFI equations in Cartan form take the form (4) not only in two-dimensional Minkowski spacetime, but also in the spacetime with the metric (1) if we take

$$E\xi=\frac{i\hbar}{a\sqrt{a}}\frac{\partial}{\partial t}(\sqrt{a}\xi),\quad P_z\xi=-\frac{i\hbar}{a\sqrt{a}}\frac{\partial}{\partial z}(\sqrt{a}\xi).$$

This leads to the following equations for $u=\sqrt{a}\xi$:

$$\left(\frac{\partial}{\partial z}-\frac{1}{c}\frac{\partial}{\partial t}\right)u_2+i\frac{mc}{\hbar}au_0=0,$$
$$\left(\frac{\partial}{\partial z}+\frac{1}{c}\frac{\partial}{\partial t}\right)u_0-i\frac{mc}{\hbar}au_2=0. \tag{5}$$

In this part of the review the entire discussion is about two-dimensional spacetime, so that the following notation for the isotropic coordinates should not lead to confusion:

$$x=\frac{ct+z}{2},\quad y=\frac{ct-z}{2}. \tag{6}$$

In the isotropic coordinates (6), Eq. (5) can be rewritten in the following canonical form:

$$\frac{\partial}{\partial x}u_0=i\frac{mc}{\hbar}au_2,$$
$$\frac{\partial}{\partial y}u_2=i\frac{mc}{\hbar}au_0. \tag{7}$$

We shall solve the Cauchy problem for this system of equations: we must find the solution of the system (7) if the functions u_0 and u_2 are arbitrarily specified on an arbitrary spacelike curve $y=\lambda(x)$. This implies that the derivative $\lambda'(x)$ is negative.

The classical Riemann method for a single second-order equation of the hyperbolic type can be extended to the system of equations (7).

If the pair of functions u_0, u_2 satisfies the system of equations (7) and the pair of functions v_0, v_2 satisfies the conjugate system of equations

$$\frac{\partial}{\partial x}v_0=-i\frac{mc}{\hbar}av_2,$$
$$\frac{\partial}{\partial y}v_2=-i\frac{mc}{\hbar}av_0, \tag{8}$$

then

$$\frac{\partial}{\partial x}(v_0u_0)+\frac{\partial}{\partial y}(v_2u_2)=0. \tag{9}$$

Therefore, the linear form

$$v_2u_2dx-v_0u_0dy \tag{10}$$

is a total differential, and the integral

$$\oint(v_2u_2dx-v_0u_0dy)=0 \tag{11}$$

over any closed contour is equal to zero.

Let us choose the contour $P_0M_0Q_0$ (Fig. 1) formed by the lines $x=x_0$, $y=y_0$, and the curve $y=\lambda(x)$, where $M_0=(x_0,y_0)$ is the point at which we want to find the value of the unknown pair of functions. The point Q_0 has coordinates x_0, $\lambda(x_0)$; the point P_0 has coordinates $\lambda^{-1}(y_0)$, y_0; $x=\lambda^{-1}(y)$ is the function which is the inverse of the function $y=\lambda(x)$. For this contour the condition (11) implies

$$\int_{Q_0}^{M_0}v_0(M)u_0(M)\Big|_{x=x_0}dy+\int_{P_0}^{M_0}v_2(M)u_2(M)\Big|_{y=y_0}dx$$
$$=\int_{P_0}^{Q_0}[v_2(M)u_2(M)-\lambda'(x)v_0(M)u_0(M)]\Big|_{y=\lambda(x)}dx. \tag{12}$$

Let us choose the functions $v_0(M)=v_{00}(M;M_0)$ and $v_2(M)=v_{20}(M;M_0)$ such that

$$v_{00}(M;M_0)|_{x=x_0}=0,$$
$$v_{20}(M;M_0)|_{y=y_0}=i\frac{mc}{\hbar}a(x,y_0). \tag{13}$$

The first of Eqs. (7) and Eq. (12) give

$$u_0(M_0)=u_0(P_0)+\int_{P_0}^{Q_0}[v_{20}(M;M_0)u_2(M)$$
$$-\lambda'(x)v_{00}(M;M_0)u_0(M)]\Big|_{y=\lambda(x)}dx. \tag{14}$$

Now let us choose the functions $v_0(M)=v_{02}(M;M_0)$ and $v_2(M)=v_{22}(M;M_0)$ such that

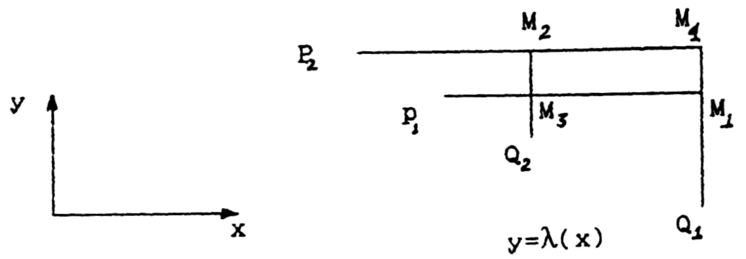

FIG. 2.

$$v_{02}(M;M_0)|_{x=x_0}=i\,\frac{mc}{\hbar}\,a(x_0,y),$$

$$v_{22}(M;M_0)|_{y=y_0}=0. \qquad (15)$$

The second of Eqs. (7) and Eq. (12) give

$$u_2(M_0)=u_2(Q_0)+\int_{P_0}^{Q_0}[v_{22}(M;M_0)u_2(M) - \lambda'(x)v_{02}(M;M_0)u_0(M)]\Big|_{y=\lambda(x)} dx. \qquad (16)$$

Equations (14) and (16) reduce the Cauchy problem to that of seeking the Riemann (matrix) function $v_{ik}(M;M_0)$. The latter can be expressed in terms of a matrix function $w_{ik}(M;M_0)$ which satisfies the same equations as $v_{ik}(M;M_0)$, i.e., the equations

$$\begin{aligned}\frac{\partial}{\partial x}\,w_{00}&=-i\,\frac{mc}{\hbar}\,aw_{20}, & \frac{\partial}{\partial x}\,w_{02}&=-i\,\frac{mc}{\hbar}\,aw_{22},\\ \frac{\partial}{\partial y}\,w_{20}&=-i\,\frac{mc}{\hbar}\,aw_{00}, & \frac{\partial}{\partial y}\,w_{22}&=-i\,\frac{mc}{\hbar}\,aw_{02},\end{aligned} \qquad (17)$$

but is subject to simpler characteristic conditions, namely,

$$\begin{aligned}w_{00}|_{x=x_0}&=1, & w_{02}|_{x=x_0}&=0,\\ w_{20}|_{y=y_0}&=0, & w_{22}|_{y=y_0}&=1.\end{aligned} \qquad (18)$$

The differential equations (17) with the conditions (18) are equivalent to the following integral equations:

$$w_{00}(x,y;x_0,y_0)=1-i\,\frac{mc}{\hbar}\int_{x_0}^{x}a(\xi,y)w_{20}(\xi,y;x_0,y_0)d\xi,$$

$$w_{20}(x,y;x_0,y_0)=-i\,\frac{mc}{\hbar}\int_{y_0}^{y}a(x,\eta)w_{00}(x,\eta;x_0,y_0)d\eta,$$

$$w_{22}(x,y;x_0,y_0)=1-i\,\frac{mc}{\hbar}\int_{y_0}^{y}a(x,\eta)w_{02}(x,\eta;x_0,y_0)d\eta,$$

$$w_{02}(x,y;x_0,y_0)=-i\,\frac{mc}{\hbar}\int_{x_0}^{x}a(\xi,y)w_{22}(\xi,y;x_0,y_0)d\xi. \qquad (19)$$

The Riemann function $v_{ik}(M;M_0)$ is obtained from the function $w_{ik}(M;M_0)$ by differentiation with respect to the parameters x_0 and y_0:

$$\begin{aligned}\frac{\partial w_{00}}{\partial y_0}&=v_{00}, & \frac{\partial w_{02}}{\partial x_0}&=v_{02},\\ \frac{\partial w_{20}}{\partial y_0}&=v_{20}, & \frac{\partial w_{22}}{\partial x_0}&=v_{22},\end{aligned} \qquad (20)$$

as is easily verified.

The function $w_{ik}(M;M_0)$ can be found by the method of successive approximations, reducing the system of equations (19) to the following equations:

$$w_{00}(x,y;x_0,y_0) =1-\frac{m^2c^2}{\hbar^2}\int_{x_0}^{x}a(\xi,y)d\xi\int_{y_0}^{y}a(\xi,\eta)w_{00}(\xi,\eta;x_0,y_0)d\eta,$$

$$w_{22}(x,y;x_0,y_0) =1 -\frac{m^2c^2}{\hbar^2}\int_{y_0}^{y}a(x,\eta)d\eta\int_{x_0}^{x}a(\xi,\eta)w_{22}(\xi,\eta;x_0,y_0)d\xi. \qquad (21)$$

It is easily seen that the first of Eqs. (21) is obtained by substituting the second of Eqs. (19) into the first. The second of Eqs. (21) is obtained by substituting the fourth of Eqs. (19) into the third.

2. Antisymmetry of the Riemann matrix function

Antisymmetry of the Riemann matrix function implies

$$v_{ik}^*(M_1;M_2)=-v_{ki}(M_2;M_1). \qquad (22)$$

Let us prove this.

We arbitrarily choose two points $M_1=(x_1,y_1)$ and $M_2=(x_2,y_2)$, and from them we construct the point $M_3=(x_3,y_3)$, lying at the intersection of the lines P_1M_1 and Q_2M_2, and the point $M_4=(x_4,y_4)$, lying at the intersection of the lines P_2M_2 and Q_1M_1 (Fig. 2). Now we draw four contours $P_kM_kQ_k$, $k=1, 2, 3, 4$, like the contour $P_0M_0Q_0$ shown in Fig. 1, where $P_3=P_1$, $Q_3=Q_2$, $P_4=P_2$, and $Q_4=Q_1$. Applying Eq. (11) to them, we obtain four expressions of the type (12).

In each of the four expressions of the type (12) we first take

$$\begin{aligned}u_0(M)&=v_{00}^*(M;M_1), & u_2(M)&=v_{20}^*(M;M_1),\\ v_0(M)&=v_{00}(M;M_2), & v_2(M)&=v_{20}(M;M_2),\end{aligned}$$

and then

$$u_0(M)=v^*_{00}(M;M_1),\quad u_2(M)=v^*_{20}(M;M_1),$$
$$v_0(M)=v_{02}(M;M_2),\quad v_2(M)=v_{22}(M;M_2),$$

and

$$u_0(M)=v^*_{02}(M;M_1),\quad u_2(M)=v^*_{22}(M;M_1),$$
$$v_0(M)=v_{00}(M;M_2),\quad v_2(M)=v_{20}(M;M_2),$$

and finally

$$u_0(M)=v^*_{02}(M;M_1),\quad u_2(M)=v^*_{22}(M;M_1),$$
$$v_0(M)=v_{02}(M;M_2),\quad v_2(M)=v_{22}(M;M_2).$$

As a result, we find

$$\begin{aligned}
\int_{P_1}^{Q_1}\Lambda_{ik}\Big|_{y=\lambda(x)}dx&=\int_{Q_1}^{M_1}X_{ik}\Big|_{x=x_1}dy+\int_{P_1}^{M_1}Y_{ik}\Big|_{y=y_1}dx,\\
\int_{Q_1}^{P_2}\Lambda_{ik}\Big|_{y=\lambda(x)}dx&=\int_{M_4}^{Q_1}X_{ik}\Big|_{x=x_1}dy+\int_{M_4}^{P_2}Y_{ik}\Big|_{y=y_2}dx,\\
\int_{P_2}^{Q_2}\Lambda_{ik}\Big|_{y=\lambda(x)}dx&=\int_{Q_2}^{M_2}X_{ik}\Big|_{x=x_2}dy+\int_{P_2}^{M_2}Y_{ik}\Big|_{y=y_2}dx,\\
\int_{Q_2}^{P_1}\Lambda_{ik}\Big|_{y=\lambda(x)}dx&=\int_{M_3}^{Q_2}X_{ik}\Big|_{x=x_2}dy+\int_{M_3}^{P_1}Y_{ik}\Big|_{y=y_1}dx
\end{aligned}\tag{23}$$

[these expressions are written in the order 1, 4, 2, 3; in the expressions pertaining to the vertices M_3 and M_4 we have interchanged the upper and lower limits of integration relative to (12)], where

$$\Lambda_{ik}=Y_{ik}-\lambda'(x)X_{ik},$$

$$\begin{aligned}
X_{00}&=v_{00}(M;M_2)v^*_{00}(M;M_1),\\
X_{20}&=v_{02}(M;M_2)v^*_{00}(M;M_1),\\
X_{02}&=v_{00}(M;M_2)v^*_{02}(M;M_1),\\
X_{22}&=v_{02}(M;M_2)v^*_{02}(M;M_1),\\
Y_{00}&=v_{20}(M;M_2)v^*_{20}(M;M_1),\\
Y_{20}&=v_{22}(M;M_2)v^*_{20}(M;M_1),\\
Y_{02}&=v_{20}(M;M_2)v^*_{22}(M;M_1),\\
Y_{22}&=v_{22}(M;M_2)v^*_{22}(M;M_1),
\end{aligned}$$

Adding to the left-hand side of (23) the four integrals over the curve $y=\lambda(x)$, we obtain zero. Therefore, the sum of all the integrals on the right-hand side of (23) is also zero. The sum of the two integrals along the line $x=x_1$ is equal to the integral from M_4 to M_1, and the sum along the line $x=x_2$ is equal to the integral from M_3 to M_2. The sum of the two integrals along the line $y=y_1$ is equal to the integral from M_3 to M_1, and the sum along the line $y=y_2$ is equal to the integral from M_4 to M_2. Therefore,

$$\int_{M_4}^{M_1}X_{ik}\Big|_{x=x_1}dy+\int_{M_3}^{M_2}X_{ik}\Big|_{x=x_2}dy+\int_{M_3}^{M_1}Y_{ik}\Big|_{y=y_1}dx+\int_{M_4}^{M_2}Y_{ik}\Big|_{y=y_2}dx=0.\tag{24}$$

Furthermore, according to (13) and (15) we have

$$\begin{aligned}
&X_{00}|_{x=x_1}=0,\quad X_{00}|_{x=x_2}=0,\\
&X_{20}|_{x=x_1}=0,\quad Y_{20}|_{y=y_2}=0,\\
&X_{02}|_{x=x_2}=0,\quad Y_{02}|_{y=y_1}=0,\\
&Y_{22}|_{y=y_1}=0,\quad Y_{22}|_{y=y_2}=0.
\end{aligned}\tag{25}$$

Therefore, (24) is equivalent to the following equations:

$$\begin{aligned}
&\int_{M_3}^{M_1}Y_{00}\Big|_{y=y_1}dx+\int_{M_4}^{M_2}Y_{00}\Big|_{y=y_2}dx=0.\\
&\int_{M_3}^{M_2}X_{20}\Big|_{x=x_2}dy+\int_{M_3}^{M_1}Y_{20}\Big|_{y=y_1}dx=0.\\
&\int_{M_4}^{M_1}X_{02}\Big|_{x=x_1}dy+\int_{M_4}^{M_2}Y_{02}\Big|_{y=y_2}dx=0.\\
&\int_{M_4}^{M_1}X_{22}\Big|_{x=x_1}dy+\int_{M_2}^{M_2}X_{22}\Big|_{x=x_2}dy=0.
\end{aligned}\tag{26}$$

Now, according to (13) and (15), we have

$$\begin{aligned}
Y_{00}|_{y=y_1}&=-i\frac{mc}{\hbar}a(x,y_1)v_{20}(x,y_1;M_2),\\
Y_{00}|_{y=y_2}&=+i\frac{mc}{\hbar}a(x,y_2)v^*_{20}(x,y_2;M_1),\\
Y_{20}|_{y=y_1}&=-i\frac{mc}{\hbar}a(x,y_1)v_{22}(x,y_1;M_2),\\
Y_{02}|_{y=y_2}&=+i\frac{mc}{\hbar}a(x,y_2)v^*_{22}(x,y_2;M_1),\\
X_{20}|_{x=x_2}&=+i\frac{mc}{\hbar}a(x_2,y)v^*_{00}(x_2,y;M_1),\\
X_{02}|_{x=x_1}&=-i\frac{mc}{\hbar}a(x_1,y)v_{00}(x_1,y;M_2),\\
X_{22}|_{x=x_1}&=-i\frac{mc}{\hbar}a(x_1,y)v_{02}(x_1,y;M_2),\\
X_{22}|_{x=x_2}&=+i\frac{mc}{\hbar}a(x_2,y)v^*_{02}(x_2,y:M_1),
\end{aligned}\tag{27}$$

and since the function v_{ik} satisfies (17), we have

$$Y_{00}|_{y=y_1}=\frac{\partial}{\partial x}v_{00}(x,y_1;M_2),$$
$$X_{20}|_{x=x_2}=\frac{\partial}{\partial y}v_{20}^*(x_2,y;M_1),$$
$$X_{02}|_{x=x_1}=\frac{\partial}{\partial y}v_{20}(x_1,y;M_2),$$
$$X_{22}|_{x=x_1}=\frac{\partial}{\partial y}v_{22}(x_1,y;M_2),$$
$$Y_{00}|_{y=y_2}=\frac{\partial}{\partial x}v_{00}^*(x,y_2;M_1),$$
$$Y_{20}|_{y=y_1}=\frac{\partial}{\partial x}v_{02}(x,y_1;M_2),$$
$$Y_{02}|_{y=y_2}=\frac{\partial}{\partial x}v_{02}^*(x,y_2;M_1),$$
$$X_{22}|_{x=x_2}=\frac{\partial}{\partial y}v_{22}^*(x_2,y;M_1). \quad (28)$$

Substituting (28) into (26), we find

$$\begin{aligned}&v_{00}(M_1;M_2)-v_{00}(x_3,y_1;M_2)+v_{00}^*(M_2;M_1)\\&\quad-v_{00}^*(x_4,y_2;M_1)=0,\\&v_{20}^*(M_2;M_1)-v_{20}^*(x_2,y_2;M_1)+v_{02}(M_1;M_2)\\&\quad-v_{02}(x_3,y_1;M_2)=0,\\&v_{20}(M_1;M_2)-v_{20}(x_1,y_4;M_2)+v_{02}^*(M_2;M_1)\\&\quad-v_{02}^*(x_4,y_2;M_1)=0,\\&v_{22}(M_1;M_2)-v_{22}(x_1,y_4;M_2)+v_{22}^*(M_2;M_1)\\&\quad-v_{22}^*(x_2,y_3;M_1)=0.\end{aligned} \quad (29)$$

Since $x_3=x_2$, $x_4=x_1$, $y_3=y_1$, and $y_4=y_2$, then

$$\begin{aligned}&v_{00}(x_3,y_1;M_2)=v_{00}(M_3;M_2)=v_{00}(x_2,y_1;M_2),\\&v_{00}^*(x_4,y_2;M_1)=v_{00}^*(M_4;M_1)=v_{00}^*(x_1,y_2;M_1),\\&v_{20}^*(x_2,y_3;M_1)=v_{20}^*(M_3;M_1)=v_{20}^*(x_2,y_1;M_1),\\&v_{02}(x_3,y_1;M_2)=v_{02}(M_3;M_2)=v_{02}(x_2,y_1;M_2),\\&v_{20}(x_1,y_4;M_2)=v_{20}(M_4;M_2)=v_{20}(x_1,y_2;M_2),\\&v_{02}^*(x_4,y_2;M_1)=v_{02}^*(M_4;M_1)=v_{02}^*(x_1,y_2;M_1),\\&v_{22}(x_1,y_4;M_2)=v_{22}(M_4;M_2)=v_{22}(x_1,y_2;M_2),\\&v_{22}^*(x_2,y_3;M_1)=v_{22}^*(M_3;M_1)=v_{22}^*(x_2,y_1;M_1).\end{aligned} \quad (30)$$

From the conditions (13) and (15) we obtain the following equations:

$$v_{00}(x_3,y_1;M_2)=0,\quad v_{00}^*(x_4,y_2;M_1)=0,$$
$$v_{20}^*(x_2,y_3;M_1)=-i\frac{mc}{\hbar}a(x_2,y_1),$$
$$v_{02}(x_3,y_1;M_2)=+i\frac{mc}{\hbar}a(x_2,y_1),$$
$$v_{20}(x_1,y_4;M_2)=+i\frac{mc}{\hbar}a(x_1,y_2),$$
$$v_{02}^*(x_4,y_2;M_1)=-i\frac{mc}{\hbar}a(x_1,y_2),$$
$$v_{22}(x_1,y_4;M_2)=0,\quad v_{22}^*(x_2,y_3;M_1)=0. \quad (31)$$

Substituting these equations into (29), we obtain (22).

3. The anticommutator of the spinor field in two-dimensional spacetime

As can be seen from Eqs. (14) and (16), the anticommutator of the spinor field is defined throughout two-dimensional spacetime by the commutation relations on some spacelike curve $y=\lambda(x)$. Let us write down the commutation relations on this curve, introducing operators of the form

$$F=\int_{-\infty}^{\infty}\{f_2u_2^*(x)+\tilde{f}_2u_2(x)-\lambda'(x)[f_0u_0^*(x)+\tilde{f}_0u_0(x)]\}dx, \quad (32)$$

where (f_0,f_2) and $(\tilde{f}_0,\tilde{f}_2)$ are "trial" spinor functions of x, and

$$u_k(x)=u_k(x,\lambda(x)),\quad u_k^*(x)=u_k^*(x,\lambda(x)),\quad k=0,\ 2.$$

The operator (32) will be Hermitian if $\tilde{f}_k=f_k^*$.

For any two operators of the form (32) we take

$$\{FG\}=FG+GF=2(f,g), \quad (33)$$

where

$$(f,g)=\frac{1}{2}\int_{-\infty}^{\infty}\{f_2\tilde{g}_2+\tilde{f}_2g_2-\lambda'(x)[f_0\tilde{g}_0+\tilde{f}_0g_0]\}dx. \quad (34)$$

The commutation relations on the curve $y=\lambda(x)$ are thereby specified.

According to (14) and (16), the operators $u_k(M_0)$ and $u_k^*(M_0)$ have the form (32), so that Eqs. (33) and (34) are valid for them. Obviously,

$$\{u_j(M_1)u_k(M_2)\}=0,\quad \{u_j^*(M_1)u_k^*(M_2)\}=0. \quad (35)$$

Using (23) for the Riemann function, it is easy to show that

$$\begin{aligned}&\{[u_0(M_1)u_0^*(M_2)]\}=\varepsilon(M_1,M_2)v_{00}^*(M_1;M_2)+\delta(y_2-y_1),\\&\{u_0(M_1)u_2^*(M_2)\}=\varepsilon(M_1,M_2)v_{02}^*(M_1;M_2),\\&\{[u_2(M_1)u_0^*(M_2)]\}=\varepsilon(M_1,M_2)v_{20}^*(M_1;M_2),\\&\{u_2(M_1)u_2^*(M_2)\}=\varepsilon(M_1,M_2)v_{22}^*(M_1;M_2)+\delta(x_2-x_1),\end{aligned} \quad (36)$$

where

$$\varepsilon(M_1,M_2)=\left|\begin{array}{l}1,\ \text{if } M_2 \text{ is in the ``future'' relative to } M_1,\\0,\ \text{if } M_1 \text{ and } M_2 \text{ are spatially similar},\\-1,\ \text{if } M_1 \text{ is in the future'' relative to } M_2,\end{array}\right.,$$

The function $\varepsilon(M_1,M_2)$ can be written as

$$\varepsilon(M_1,M_2)=\theta(x_2-x_1)\theta(y_2-y_1) - \theta(x_1-x_2)\theta(y_1-y_2), \quad (37)$$

where

$$\theta(x)=\begin{vmatrix} 1, & \text{if } x>0, \\ 0, & \text{if } x<0. \end{vmatrix} \quad (38)$$

It follows from (36) that the anticommutator does not depend on the choice of curve $\lambda(x)$. The fact that the anticommutator is equal to zero for spacelike separations of the points M_1 and M_2 is an expression of causality.

4. The Riemann function in two-dimensional flat spacetime

In this case $a=1$. Let us find the function $w_{ik}(M;M_0)$. According to (21) for $a=1$, it is obvious that

$$w_{00}(x,y;x_0,y_0)=w_{22}(x,y;x_0,y_0)=w(x,y;x_0,y_0). \quad (39)$$

For the function $w(x,y;x_0,y_0)$ we have the equation

$$w(x,y;x_0,y_0)=1-\frac{m^2c^2}{\hbar^2}\int_{x_0}^{x}d\xi\int_{y_0}^{y}w(\xi,\eta;x_0,y_0)d\eta, \quad (40)$$

which is easily solved by the method of successive approximations. The solution can be expressed in terms of the Bessel function $J_0(z)$:

$$w(x,y;x_0,y_0)=\sum_{k=0}^{\infty}\frac{(-1)^k}{k!k!}\left(\frac{z}{2}\right)^{2k}=J_0(z), \quad (41)$$

where

$$z=2\frac{mc}{\hbar}\sqrt{(x-x_0)(y-y_0)}.$$

Therefore, the function $w(M;M_0)$ depends only on the separation of the points M and M_0.

From (19) we find

$$\begin{aligned} w_{20}&=-i\frac{mc}{\hbar}(y-y_0)\sum_{k=0}^{\infty}\frac{(-1)^k}{k!(k+1)!}\left(\frac{z}{2}\right)^{2k} = \sqrt{\frac{y-y_0}{x-x_0}}J_1(z), \\ w_{02}&=-i\frac{mc}{\hbar}(x-x_0)\sum_{k=0}^{\infty}\frac{(-1)^k}{k!(k+1)!}\left(\frac{z}{2}\right)^{2k} = \sqrt{\frac{x-x_0}{y-y_0}}J_1(z). \end{aligned} \quad (42)$$

The Riemann function is then found from (20).

5. The Riemann function in two-dimensional de Sitter spacetime

Two-dimensional de Sitter spacetime is represented as a hyperboloid of one sheet $(x^1)^2+(x^2)^2-(x^0)^2=r^2$ in three-dimensional flat spacetime with the metric $(dx^1)^2+(dx^2)^2-(dx^0)^2$. Convenient coordinates on the hyperboloid are the angles θ and φ:

$$\begin{aligned} &x^0=r\tan\theta, \quad x^1=r\frac{\cos\varphi}{\cos\theta}, \quad x^2=r\frac{\sin\varphi}{\cos\theta}, \\ &-\frac{\pi}{2}<\theta<\frac{\pi}{2}, \quad 0\leqslant\varphi\leqslant 2\pi. \end{aligned} \quad (43)$$

The metric on the hyperboloid is

$$(dx^1)^2+(dx^2)^2-(dx^0)^2=a^2(d\varphi^2-d\theta^2),$$

where

$$a=\frac{r}{\cos\theta}. \quad (44)$$

Introducing the isotropic coordinates

$$x=\frac{\theta+\varphi}{2}, \quad y=\frac{\theta-\varphi}{2}, \quad (45)$$

we find that in this case the Riemann function obeys the equations

$$\begin{aligned} &\frac{\partial}{\partial x}v_{0k}=-\frac{im}{\cos(x+y)}v_{2k}, \\ &\frac{\partial}{\partial y}v_{2k}=-\frac{im}{\cos(x+y)}v_{0k} \quad (k=0,2) \end{aligned} \quad (46)$$

and satisfies the following characteristic conditions:

$$\begin{aligned} &v_{00}|_{x=x_0}=0, \quad v_{02}|_{x=x_0}=\frac{im}{\cos(x_0+y)}, \\ &v_{20}|_{y=y_0}=\frac{im}{\cos(x+y_0)}, \quad v_{22}|_{y=y_0}=0, \end{aligned} \quad (47)$$

where $m=mcr/\hbar$ is a dimensionless parameter.

Now let us introduce the coordinates $\gamma=\cosh\Gamma$ and $\beta=\tanh B$ associated with the point M_0, where Γ is the geodesic distance between the points M and M_0 and B is the angle between the segment M_0M and the coordinate line $\varphi=\varphi_0$:

$$\begin{aligned} &\gamma=\frac{\cos(x-x_0+y_0-y)-\sin(x+y)\sin(x_0+y_0)}{\cos(x+y)\cos(x_0+y_0)}, \\ &\beta=\frac{\cos(x_0+y_0)\sin(x-x_0+y_0-y)}{\sin(x+y)-\sin(x_0+y_0)\cos(x-x_0+y_0-y)}. \end{aligned} \quad (48)$$

The angle f of the Lorentz rotation from the local basis linear form $(d\theta,d\varphi)$ to the local basis linear form $(d\gamma,d\beta)$ is found from the expression

$$e^{f/2}=\sqrt{\frac{\sin(x-x_0)\cos(x+y_0)}{\sin(y-y_0)\cos(x_0+y)}}. \quad (49)$$

The general procedure for spinor transformation in going from one local basis linear form to another leads to the substitution

$$\begin{aligned} &\tilde{v}_{00}=v_{00}\sqrt{\frac{\cos(x+y)}{r}}e^{-f/2}, \\ &\hat{v}_{02}=v_{02}\sqrt{\frac{\cos(x+y)}{r}}e^{-f/2}, \end{aligned}$$

(50)

$$\tilde{v}_{20}=v_{20}\sqrt{\frac{\cos(x+y)}{r}}\,e^{f/2},$$

$$\tilde{v}_{22}=v_{22}\sqrt{\frac{\cos(x+y)}{r}}\,e^{f/2},$$

which significantly simplifies the system of equations (46).

Actually, the $\tilde{v}_{jk}$ satisfy a system of equations with separated variables, namely,

$$\left(\frac{1-\beta^2}{\sqrt{\gamma^2-1}}\frac{\partial}{\partial\beta}+\sqrt{\gamma^2-1}\frac{\partial}{\partial\gamma}+\frac{1}{2}\frac{\gamma}{\sqrt{\gamma^2-1}}\right)\tilde{v}_{0k}=-im\tilde{v}_{2k},$$
$$\left(\frac{\beta^2-1}{\sqrt{\gamma^2-1}}\frac{\partial}{\partial\beta}+\sqrt{\gamma^2-1}\frac{\partial}{\partial\gamma}+\frac{1}{2}\frac{\gamma}{\sqrt{\gamma^2-1}}\right)\tilde{v}_{2k}=-im\tilde{v}_{0k}. \tag{51}$$

To satisfy the characteristic conditions (47) and to ensure that the v_{jk} do not have singularities, it is necessary to set

$$\tilde{v}_{j1}=\frac{im}{\sqrt{r}\,\cos(x_0+y_0)}\,G_{j1}(\gamma)e^{B/2},$$
$$\hat{v}_{j2}=\frac{im}{\sqrt{r}\,\cos(x_0+y_0)}\,G_{j2}(\gamma)e^{-B/2} \tag{52}$$

and to require that the following equations be satisfied:

$$G_{02}(\gamma)=G_{20}(\gamma)=G_0(\gamma),\quad G_0(1)=1.$$
$$G_{22}(\gamma)=G_{00}(\gamma)=G_2(\gamma). \tag{53}$$

Here the functions G_0 and G_2 satisfy the following system of ordinary differential equations:

$$\sqrt{\gamma^2-1}\,\frac{d}{d\gamma}\,G_0+\frac{1}{2}\sqrt{\frac{\gamma-1}{\gamma+1}}\,G_0=imG_2,$$
$$\sqrt{\gamma^2-1}\,\frac{d}{d\gamma}\,G_2+\frac{1}{2}\sqrt{\frac{\gamma+1}{\gamma-1}}\,G_2=imG_0, \tag{54}$$

the solution of which is known:[58]

$$G_0(\gamma)=\sqrt{\frac{\gamma+1}{2}}\,F\left(1-im,1+im;1;\,\frac{1-\gamma}{2}\right),$$
$$G_2(\gamma)=-im\sqrt{\frac{\gamma-1}{2}}\,F\left(1-im,1+im;2;\,\frac{1-\gamma}{2}\right), \tag{55}$$

where F is the hypergeometric function.

Collecting the results, we obtain the Riemann function:

$$v_{00}=\frac{m^2}{\cos(x+y)}\frac{\sin(x-x_0)}{\cos(x_0+y_0)}\,F\left(1-im,1+im;2;\,\frac{1-\gamma}{2}\right),$$

$$v_{20}=\frac{im}{\cos(x+y)}\frac{\cos(x_0+y)}{\cos(x_0+y_0)}\,F\left(1-im,1+im;1;\,\frac{1-\gamma}{2}\right), \tag{56}$$

$$v_{02}=\frac{im}{\cos(x+y)}\frac{\cos(x+y_0)}{\cos(x_0+y_0)}\,F\left(1-im,1+im;1;\,\frac{1-\gamma}{2}\right),$$

$$v_{22}=\frac{m^2}{\cos(x+y)}\frac{\sin(y-y_0)}{\cos(x_0+y_0)}\,F\left(1-im,1+im;2;\,\frac{1-\gamma}{2}\right).$$

6. Creation and annihilation operators in two-dimensional de Sitter spacetime

To introduce the particle creation and annihilation operators in two-dimensional de Sitter spacetime, we write the DFI equations in the following matrix form:

$$\frac{\partial u}{\partial\theta}=\frac{im}{\cos\theta}\,Lu+K\,\frac{\partial u}{\partial\varphi}, \tag{57}$$

where u is a column matrix, and K and L are the matrices

$$u=\begin{pmatrix}u_0\\u_2\end{pmatrix},\quad K=\begin{pmatrix}-1&0\\0&1\end{pmatrix},\quad L=\begin{pmatrix}0&1\\1&0\end{pmatrix}. \tag{58}$$

We shall seek the solution of (57) in the form of a Fourier series:

$$u=\frac{1}{\sqrt{2\pi}}\sum_{s=-\infty}^{\infty}u_{s+1/2}\exp\left\{i\left(s+\frac{1}{2}\right)\varphi\right\}. \tag{59}$$

The expansion in half-integer harmonics is determined by the condition

$$u(\theta,2\pi)=-u(\theta,0). \tag{60}$$

The Riemann function, determined locally, also satisfies this global condition.

For the coefficients of the series (59) we obtain the equation

$$\frac{d}{d\theta}\,u_{s+1/2}=i\left(s+\frac{1}{2}\right)Ku_{s+1/2}+\frac{im}{\cos\theta}\,Lu_{s+1/2}. \tag{61}$$

Since the sum Σ in the expansion (59) can be written as

$$\sum_{p=0}^{\infty}\left[u_{p+1/2}\exp\left\{i\left(p+\frac{1}{2}\right)\varphi\right\}+u_{-p-1/2}\times\exp\left\{-i\left(p+\frac{1}{2}\right)\varphi\right\}\right], \tag{62}$$

to determine the expansion coefficients it is necessary to solve the following two systems of equations for all $p>0$:

$$\frac{df}{d\theta}+i\left(p+\frac{1}{2}\right)f=\frac{im}{\cos\theta}\,g,$$
$$\frac{dg}{d\theta}-i\left(p+\frac{1}{2}\right)g=\frac{im}{\cos\theta}\,f \tag{63}$$

and

$$\frac{df}{d\theta}-i\left(p+\frac{1}{2}\right)f=\frac{im}{\cos\theta}g,$$
$$\frac{dg}{d\theta}+i\left(p+\frac{1}{2}\right)g=\frac{im}{\cos\theta}f. \tag{64}$$

The first system has the following two solutions:

1) $f=f_p(\theta),\quad g=g_p(\theta);$

2) $f=-g_p^*(\theta)=-g_p(-\theta),\quad g=f_p^*(\theta)=f_p(-\theta).$

They are linearly independent, because their determinant is

$$\begin{vmatrix} f_p(\theta) & g_p(\theta) \\ -g_p^*(\theta) & f_p^*(\theta)\end{vmatrix}=|f_p(\theta)|^2+|g_p(\theta)|^2 \tag{65}$$

and in view of (63) is independent of the time θ.

The second system has the following two solutions:

1) $f=-g_p(\theta),\quad g=-f_p(\theta);$

2) $f=f_p^*(\theta),\quad g=-g_p^*(\theta).$

They are also linearly independent for the same reason.

Therefore, the coefficients of the Fourier series (62) can be written as

$$u_{p+1/2}(\theta)=\begin{pmatrix} f_p(\theta) \\ g_p(\theta)\end{pmatrix}A_{p+1/2}+\begin{pmatrix} -g_p^*(\theta) \\ f_p^*(\theta)\end{pmatrix}B^+_{-p-1/2},$$
$$u_{-p-1/2}(\theta)=\begin{pmatrix} -g_p(\theta) \\ -f_p(\theta)\end{pmatrix}A_{-p-1/2}+\begin{pmatrix} f_p^*(\theta) \\ -g_p^*(\theta)\end{pmatrix}B^+_{p+1/2}, \tag{66}$$

where A and B are operator constants.

The explicit form of the functions $f_p(\theta)$ and $g_p(\theta)$ is

$$f_p(\theta)=N_p e^{-i(p+1/2)\theta}F\left(-im,im;p+1;\frac{e^{-i\theta}}{2\cos\theta}\right),$$
$$g_p(\theta)=-N_p\frac{m}{2(p+1)\cos\theta}e^{-i(p+1/2)\theta}F\left(1-im,1+im;p+2;\frac{e^{-i\theta}}{2\cos\theta}\right), \tag{67}$$

where N_p is a normalization factor. Assuming that the determinant (65) is equal to 1, we find

$$N_p=\frac{\sqrt{\Gamma(p+1+im)\Gamma(p+1-im)}}{p!}. \tag{68}$$

According to (36), the field operator has the following commutation relations:

$$\{u_k(\theta,\varphi_2)u_j(\theta,\varphi_1)\}=0,\quad \{u_k^*(\theta,\varphi_2)u_j^*(\theta,\varphi_1)\}=0,$$
$$\{u_k(\theta,\varphi_2)u_j^*(\theta,\varphi_1)\}=\delta_{kj}\delta(\varphi_2-\varphi_1). \tag{69}$$

From this it follows that the operators A and B satisfy the commutation relations

$$\{A_{r+1/2},A_{s+1/2}\}=0,\quad \{B_{r+1/2},B_{s+1/2}\}=0,$$
$$\{A^+_{r+1/2},A^+_{s+1/2}\}=0,\quad \{B^+_{r+1/2},B^+_{s+1/2}\}=0, \tag{70}$$
$$[A_{r+1/2},A^+_{s+1/2}]=\delta_{rs},\quad [B_{r+1/2},B^+_{s+1/2}]=\delta_{rs}.$$

In addition, operators denoted by different letters (A and B) anticommute, no matter what their indices are. The operators A^+ and A can be viewed as electron creation and annihilation operators, and B^+ and B can be viewed as positron creation and annihilation operators.

7. The canonical method

For the metric

$$ds^2=a^2(\varphi,\theta)(d\varphi^2-d\theta^2) \tag{71}$$

the DFI equation

$$\frac{\partial u}{\partial\theta}=\frac{imc}{\hbar}aLu+K\frac{\partial u}{\partial\varphi}, \tag{72}$$

a special case of which is Eq. (57), can be obtained canonically from the action integral

$$S=\int\int\Lambda d\varphi d\theta, \tag{73}$$

where

$$\Lambda=\frac{i\hbar}{2}\left(u^*\frac{\partial u}{\partial\theta}-\frac{\partial u^*}{\partial\theta}u-u^*K\frac{\partial u}{\partial\theta}+\frac{\partial u^*}{\partial\theta}Ku-\frac{2imc}{\hbar}au^*Lu\right). \tag{74}$$

Canonically, from this we obtain the current vector

$$J_\theta=-u^*u,\quad J_\varphi=-u^*Ku \tag{75}$$

and the energy–momentum tensor

$$T_{\theta\theta}=\frac{i\hbar}{2}\left(u^*\frac{\partial u}{\partial\theta}-\frac{\partial u^*}{\partial\theta}u\right),$$
$$T_{\theta\varphi}=T_{\varphi\theta}=\frac{i\hbar}{2}\left(u^*K\frac{\partial u}{\partial\theta}-\frac{\partial u^*}{\partial\theta}Ku\right)=\frac{i\hbar}{2}\left(u^*\frac{\partial u}{\partial\varphi}-\frac{\partial u^*}{\partial\varphi}u\right), \tag{76}$$
$$T_{\varphi\varphi}=\frac{i\hbar}{2}\left(u^*K\frac{\partial u}{\partial\varphi}-\frac{\partial u^*}{\partial\varphi}Ku\right).$$

The trace of the energy–momentum tensor is

$$T=a^{-2}(T_{\varphi\varphi}-T_{\theta\theta})=\frac{mc}{a}u^*Lu. \tag{77}$$

The covariant divergences of any vector J_α and any symmetric tensor $T_{\alpha\beta}$ in the spacetime with metric (71) are

$$g^{\mu\alpha}\nabla_\mu J_\alpha=g^{\mu\alpha}\partial_\mu J_\alpha, \tag{78}$$
$$g^{\mu\alpha}\nabla_\mu T_{\alpha\beta}=g^{\mu\alpha}\partial_\mu T_{\alpha\beta}-Ta^{-1}\partial_\beta a. \tag{79}$$

Substituting (75)–(77) into this and taking into account (72), we find that the covariant divergences of the current vector and the energy–momentum tensor are equal to zero.

Accordingly, the integral

$$\hat{e}=\int j_\alpha d\sigma^\alpha \tag{80}$$

over Σ, where Σ is a spacelike curve separating the spacetime into two parts, is independent of the choice of Σ. This is the charge operator. If there exists a vector field ζ^α generating an isometric transformation, then the integral

$$M=\int T_{\alpha\beta}\zeta^\alpha d\sigma^\beta \tag{81}$$

along the curve Σ is also independent of the choice of Σ. On the curve $\theta=\text{const}$, which is one of the curves of the type Σ,

$$j_\alpha d\sigma^\alpha=j_\theta d\varphi,\quad T_{\alpha\beta}\zeta^\alpha d\sigma^\beta=T_{\alpha\theta}\zeta^\alpha d\varphi. \tag{82}$$

In the special case (44) the three vector fields

$$-\frac{i}{\hbar}\underset{(01)}{X}=\cos\theta\cos\varphi\frac{\partial}{\partial\theta}-\sin\theta\sin\varphi\frac{\partial}{\partial\varphi},$$

$$-\frac{i}{\hbar}\underset{(02)}{X}=\cos\theta\sin\varphi\frac{\partial}{\partial\theta}+\sin\theta\cos\varphi\frac{\partial}{\partial\varphi}, \tag{83}$$

$$-\frac{i}{\hbar}\underset{(12)}{X}=-\frac{\partial}{\partial\varphi}$$

generate the group of isometric transformations of the hyperboloid (43). Here it is convenient to combine the two real fields $\underset{(01)}{X}$ and $\underset{(02)}{X}$ into a single complex field:

$$-\frac{i}{\hbar}(\underset{(01)}{X}-i\underset{(02)}{X})=e^{i\varphi}\left(\cos\theta\frac{\partial}{\partial\theta}+i\sin\theta\frac{\partial}{\partial\varphi}\right). \tag{84}$$

Using the expansion (59), we obtain the integral (80) in the form

$$\hat{e}=-e\int_0^{2\pi}u^*u\,d\varphi=-e\sum_{s=-\infty}^{\infty}u^*_{s+1/2}u_{s+1/2} \tag{85}$$

and the two integrals of the type (81) in the form

$$\underset{(12)}{M}=-\int_0^{2\pi}T_{\theta\varphi}d\varphi=\hbar\sum_{s=-\infty}^{\infty}\left(s+\frac{1}{2}\right)u^*_{s+1/2}u_{s+1/2}, \tag{86}$$

$$\underset{(01)}{M}+i\underset{(02)}{M}=\int_0^{2\pi}(\cos\theta T_{\theta\theta}+i\sin\theta T_{\theta\varphi})e^{i\varphi}d\varphi=$$

$$-\hbar\sum_{s=-\infty}^{\infty}\mu_s, \tag{87}$$

where

$$\mu_s=u^*_{s+1/2}(mL+sKe^{iK\theta})u_{s-1/2}. \tag{88}$$

All these integrals are independent of θ, which can be verified by using the fact that Eq. (61) gives

$$\frac{d}{d\theta}u^*_{s+1/2}u_{s+1/2}=0,\quad\frac{d}{d\theta}\mu_s=0. \tag{89}$$

From (66) we find

$$u^*_{p+1/2}u_{p+1/2}=A^+_{p+1/2}A_{p+1/2}+B_{-p-1/2}B^+_{-p-1/2},$$

$$u^*_{-p-1/2}u_{-p-1/2}=A^+_{-p-1/2}A_{-p-1/2}+B_{p+1/2}B^+_{p+1/2}; \tag{90}$$

$$\mu_0=m(B_{-1/2}B^+_{1/2}-A^+_{1/2}A_{-1/2}),$$

$$\mu_{p+1}=\sqrt{(p+1)^2+m^2}(B_{p-3/2}B^+_{-p-1/2}-A^+_{p+3/2}A_{p+1/2}), \tag{91}$$

$$\mu_{-p-1}=\sqrt{(p+1)^2+m^2}(B_{p+1/2}B^+_{p+3/2}-A^+_{-p-1/2}A_{-p-3/2}).$$

Therefore,

$$\hat{e}=e\sum_{s=-\infty}^{\infty}(B^+_{s+1/2}B_{s+1/2}-A^+_{s+1/2}A_{s+1/2}),$$

$$\underset{(12)}{M}=\hbar\sum_{s=-\infty}^{\infty}\left(s+\frac{1}{2}\right)(A^+_{s+1/2}A_{s+1/2}+B^+_{s+1/2}B_{s+1/2}), \tag{92}$$

$$\underset{(01)}{M}+i\underset{(02)}{M}=\hbar\sum_{s=-\infty}^{\infty}\sqrt{s^2+m^2}(A^+_{s+1/2}A_{s-1/2}+B^+_{s+1/2}B_{s-1/2}).$$

We see from the first two expressions that $A^+_{s+1/2}$ and $B^+_{s+\frac{1}{2}}$ are the creation operators for an electron and positron in the state with angular momentum $\hbar(s+\frac{1}{2})$, and that $A_{s+1/2}$and $B_{s+1/2}$ are the annihilation operators for an electron and positron in the state with angular momentum $\hbar(s+\frac{1}{2})$.

It is remarkable that the operators (92) are invariant under the substitution

$$A_{s+1/2}\to\frac{e^{-i\alpha}}{\sqrt{1+|\lambda|^2}}[A_{s+1/2}+i(-1)^sB^+_{-s-1/2}],$$

$$B_{s+1/2}\to\frac{e^{i\beta}}{\sqrt{1+|\lambda|^2}}[B_{s+1/2}+i(-1)^sA^+_{-s-1/2}], \tag{93}$$

where α and β are any real numbers, and λ is any complex number. This substitution is canonical in the sense that it preserves the commutation relations (70). It is analogous to the Bogolyubov substitution in the theory of superconductivity.[59]

Therefore, the continuous group of isometries in de Sitter spacetime does not determine the vacuum state uniquely. Spatial reflection $\varphi\to-\varphi$ also does not decrease the arbitrariness in the choice of vacuum. On the other hand, time reflection $\theta\to-\theta$ imposes the following constraints:

$$\alpha=\beta;\quad\lambda=\lambda^*. \tag{94}$$

8. Expansion of the anticommutator in a Fourier series

In Secs. 3 and 5 we obtained the anticommutator of the spinor field in two-dimensional de Sitter spacetime by the Riemann method, but it can also be found directly from the conditions (70). For the series (59) with coefficients (66) the anticommutator is

$$\{u_0(M_1)u_0^*(M_2)\}=\frac{1}{2\pi}\sum_{s=-\infty}^{\infty}F_{s+1/2}\times\exp\left\{i\left(s+\frac{1}{2}\right)\varphi_0\right\},$$

$$\{u_2(M_1)u_0^*(M_2)\}=\frac{1}{2\pi}\sum_{s=-\infty}^{\infty}G_{s+1/2}\times\exp\left\{i\left(s+\frac{1}{2}\right)\varphi_0\right\},$$

$$\{u_0(M_1)u_2^*(M_2)\}=\frac{1}{2\pi}\sum_{s=-\infty}^{\infty}G_{-s-1/2}\times\exp\left\{i\left(s+\frac{1}{2}\right)\varphi_0\right\},\tag{95}$$

$$\{u_2(M_1)u_2^*(M_2)\}=\frac{1}{2\pi}\sum_{s=-\infty}^{\infty}F_{-s-1/2}\times\exp\left\{i\left(s+\frac{1}{2}\right)\varphi_0\right\},$$

where $\varphi_0=\varphi_1-\varphi_2$.

The expansion coefficients are

$$\begin{aligned}F_{p+1/2}&=f_p(\theta_1)f_p^*(\theta_2)+g_p^*(\theta_1)g_p(\theta_2),\\ F_{-p-1/2}&=f_p(\theta_1)f_p^*(\theta_2)+g_p^*(\theta_1)g_p(\theta_2),\\ G_{p+1/2}&=g_p(\theta_1)f_p^*(\theta_2)-f_p^*(\theta_1)g_p(\theta_2),\\ G_{-p-1/2}&=f_p(\theta_1)g_p^*(\theta_2)-g_p^*(\theta_1)f_p(\theta_2),\end{aligned}\tag{96}$$

where $p=0,\ 1,\ 2,\ \dots$. They satisfy the system of differential equations

$$\begin{aligned}&\frac{\partial F}{\partial\theta_1}+i\left(s+\frac{1}{2}\right)F=\frac{im}{\cos\theta_1}G,\\ &\frac{\partial G}{\partial\theta_1}-i\left(s+\frac{1}{2}\right)G=\frac{im}{\cos\theta_1}F,\end{aligned}\tag{97}$$

with initial data $F=1,\ G=0$ for $\theta_1=\theta_2$.

However, the following pair of functions satisfies the same system of differential equations with the same initial data:

$$\begin{aligned}I_{s+1/2}&=\exp\left\{-i\left(s+\frac{1}{2}\right)\theta_0\right\}\\ &-\frac{m^2}{2}\int_{-\theta_0}^{\theta_0}\frac{\sin\dfrac{\theta_0+\varphi_0}{2}}{\cos\theta_1\cos\theta_2}F\left(1-im,1+im;2;\frac{1-\gamma}{2}\right)\exp\left\{-i\left(s+\frac{1}{2}\right)\varphi_0\right\}d\varphi_0,\\ J_{s+1/2}&=\frac{im}{2}\int_{-\theta_0}^{\theta_0}\frac{\cos\dfrac{\theta_0+\varphi_0}{2}}{\cos\theta_1\cos\theta_2}F\left(1-im,1+im;2;\frac{1-\gamma}{2}\right)\exp\left\{-i\left(s+\frac{1}{2}\right)\varphi_0\right\}d\varphi_0,\end{aligned}\tag{98}$$

where

$$\theta_0=\theta_1-\theta_2,\quad \gamma=\frac{\cos\varphi_0-\sin\theta_1\sin\theta_2}{\cos\theta_1\cos\theta_2}.\tag{99}$$

Therefore,

$$F_{s+1/2}=I_{s+1/2},\quad G_{s+1/2}=J_{s+1/2}.\tag{100}$$

This allows the series (95) to be summed, and we obtain the anticommutator of the spinor field in the familiar form (36).

IV. QUANTUM THEORY OF THE SPINOR FIELD IN THE *dX* BASIS

1. The special Dirac equation in de Sitter spacetime

The transformation from the DFI equation to the special equation written down by Dirac for de Sitter spacetime in Ref. 36 is described in Ref. 37. The $2n$-dimensional case is studied in Ref. 60.

The $2n$-dimensional de Sitter spacetime can be represented as a hyperboloid $\eta_{AB}x^Ax^B=r^2$ in $(2n+1)$-dimensional Minkowski spacetime. Here $\eta_{aa}=-\eta_{00}=1$, $\eta_{AB}=0$ for $A\neq B$. The upper-case Latin letters take the values 0, 1, 2, ..., $2n$, and the lower-case Latin letters take the values 1, 2, ..., $2n$. In this general case the spinor has 2^n components,[32] and we can choose one anti-Hermitian matrix H^0 and $2n$ Hermitian matrices H^a satisfying the conditions $H^AH^B+H^BH^A=2\eta^{AB}$.

The special Dirac equation in the $2n$-dimensional case is generalized as

$$H^Am_A\Xi=(n+im)\frac{X}{r}\Xi,\tag{1}$$

where Ξ is the spinor ξ in the basis dx^A, $X=x^AH_A$, and

$$m_A=r\frac{\partial}{\partial x^A}-\frac{x_Ax^B}{r}\frac{\partial}{\partial x^B}.\tag{2}$$

Since

$$\frac{1}{2}H^AH^Bm_{AB}=\frac{1}{2}(XH^A-H^AX)\frac{\partial}{\partial x^A}=\frac{X}{r}H^Am_A,\tag{3}$$

where

$$m_{AB}=x_A\frac{\partial}{\partial x^B}-x_B\frac{\partial}{\partial x^A},\tag{4}$$

Eq. (1) can be written as

$$\left(\frac{1}{2}H^AH^Bm_{AB}-n\right)\Xi=im\Xi.\tag{5}$$

The operator M, given by

$$M=\frac{1}{2}H^AH^Bm_{AB}-n,\tag{6}$$

anticommutes with the operator X, that is,

$$MX+XM=0.\tag{7}$$

From this we conclude that the operator equal to $-iH_0XM$ is Hermitian. Moreover, the operator H_0X is also Hermitian. Therefore, for real values of the parameter m, Eq. (1) is made Hermitian by multiplication by the operator $-iH_0$.

The Dirac-conjugate spinor

$$\bar{\Xi}=\Xi^*H_0\tag{8}$$

satisfies the equation

$$m_A \Xi H^A = (n - im)\Xi \frac{X}{r} \tag{9}$$

or, equivalently,

$$\frac{1}{2} m_{AB} \Xi H^A H^B = (n - im)\Xi. \tag{10}$$

Together with (7), the operator M has yet another important property, namely,

$$(M+n)(M+1-n) = -\frac{1}{2} m_{AB} m^{AB} = -m_A m^A. \tag{11}$$

Therefore, each of the 2^n components of the spinor Ξ separately satisfies the same equation

$$m_A m^A \Xi = (n + im)(n - 1 - im)\Xi \tag{12}$$

if the spinor Ξ itself satisfies Eq. (5). Here each component of the conjugate spinor $\bar{\Xi}$ satisfies the equation

$$m_A m^A \bar{\Xi} = (n - im)(n - 1 + im)\bar{\Xi}. \tag{13}$$

From the obvious identity

$$(M+n)(M+1-n) - (n+im)(im+1-n)$$
$$= (M - im)(M + 1 + im) \tag{14}$$

we obtain the lemma:

If the spinor Φ satisfies Eq. (12), then the spinor

$$\Xi = (M + 1 + im)\Phi \tag{15}$$

satisfies Eq. (1) [or, equivalently, Eq. (5)].

The associated lemma is proved similarly:

If the spinor $\bar{\Phi}$ satisfies Eq. (13), then the spinor

$$\bar{\Xi} = \bar{\Phi}(M + 1 - im) \tag{16}$$

satisfies Eq. (9) [or, equivalently, Eq. (10)].

Equations (1), (5), and (12) for the case $n=2$ were written down by Dirac himself in Ref. 36. Here we shall study the case $n=1$ in detail.

2. Transformation from the *F* basis to the *dX* basis

Equation (III.72) can be written as

$$H^0 \frac{\partial u}{\partial \theta} + H^1 \frac{\partial u}{\partial \varphi} = \frac{imc}{\hbar} a H^2 u, \tag{17}$$

where the matrices H^A are

$$H^0 = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \quad H^1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad H^2 = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}. \tag{18}$$

These matrices satisfy the following conditions:

$$H^0 H^1 = H^2, \quad H^0 H^2 = -H^1,$$
$$H^1 H^2 = -H^2, \quad H^0 H^1 H^2 = 1, \tag{19}$$
$$H^A H^B + H^B H^A = 2r^{AB}.$$

The column vector u is associated with the spinor ξ in the f basis ($f^0 = a\,d\theta$, $f^1 = a\,d\varphi$) as $u = \sqrt{a}\xi$.

The covariant derivative of the spinor in the f basis is

$$\xi_\mu = D_\mu \xi = e_\mu \xi + \frac{1}{4} \omega_{\alpha\beta\mu} H^\alpha H^\beta \xi, \tag{20}$$

where $e_0 = (1/a)\,\partial/\partial\theta$, $e_1 = (1/a)\,\partial/\partial\varphi$, $\omega_{\alpha\beta\mu} = \varepsilon_{\alpha\beta} C_\mu$, $\varepsilon_{\alpha\beta}$ is the antisymmetric tensor with $\varepsilon_{10} = 1$, $C_0 = a^{-2}\,\partial a/\partial\varphi$, and $C_1 = a^{-2}\,\partial a/\partial\theta$. The covariant derivative of the conjugate spinor $\bar{\xi} = \xi^* H_0$ in the f basis is

$$\bar{\xi}_\mu = D_\mu \bar{\xi} = e_\mu \bar{\xi} - \frac{1}{4} \bar{\xi} \omega_{\alpha\beta\mu} H^\alpha H^\beta. \tag{21}$$

In this notation Eq. (17) is written as

$$\left(H^\nu D_\nu - \frac{imc}{\hbar} H^2 \right) \xi = 0. \tag{22}$$

Let us return to the study of two-dimensional de Sitter spacetime, representing it as a hyperboloid (III.43). Here $a = r(\cos\theta)^{-1}$. To the differential forms f^0, f^1 we add the form $f^2 = dr$, and from the basis

$$F = \left\{ \frac{r}{\cos\theta} d\theta, \frac{r}{\cos\theta} d\varphi, dr \right\} \tag{23}$$

we transform to the basis

$$dX = \{dx^0, dx^1, dx^2\}, \tag{24}$$

constructed of differentials of the functions (III.43). Here it is convenient to make the replacement

$$\varphi = \frac{\pi}{2} - \varphi. \tag{25}$$

The transformation from the F basis to the dX basis is accomplished by a Lorentz transformation of the form $dx^A = \tilde{L}^A_B f^B$, $f^B = L^B_A dx^A$. We can choose a matrix S such that

$$S^- H^A S = \tilde{L}^A_B H^B, \quad S H^A S^{-1} = L^A_B H^B,$$
$$S H_A S^{-1} = \tilde{L}^B_A H_B, \quad S^{-1} H_A S = L^B_A H_B. \tag{26}$$

If we write $dX = H_A dx^A$ and $F = H_A f^A$, then $dX = SFS^{-1}$ and $F = S^{-1} dX S$. The transformation from the F basis to the dX basis is accompanied by the substitution

$$\Xi = S\xi. \tag{27}$$

In this case

$$S = \left(\cos\frac{\varphi}{2} + H_1 H_2 \sin\frac{\varphi}{2} \right) \frac{1}{\sqrt{\cos\theta}} \left(\cos\frac{\theta}{2} + H_0 H_2 \sin\frac{\theta}{2} \right),$$
$$S^{-1} = \frac{1}{\sqrt{\cos\theta}} \left(\cos\frac{\theta}{2} - H_0 H_2 \sin\frac{\theta}{2} \right) \times \left(\cos\frac{\varphi}{2} - H_1 H_2 \sin\frac{\varphi}{2} \right). \tag{28}$$

We have

$$S^{-1} H^0 S = H^0 \frac{1}{\cos\theta} + H^2 \tan\theta,$$

$$S^{-1} H^1 S = H^0 \tan\theta \sin\varphi + H^1 \cos\varphi + H^2 \frac{\sin\varphi}{\cos\theta}, \tag{29}$$

$$S^{-1}H^2S=H^0\tan\theta\cos\varphi-H^1\sin\varphi+H^2\frac{\cos\varphi}{\cos\theta}$$

and, inversely,

$$SH^0S^{-1}=H^0\frac{1}{\cos\theta}-H^1\tan\theta\sin\varphi-H^2\tan\theta\cos\varphi,$$

$$SH^1S^{-1}=H^1\cos\varphi-H^2\sin\varphi, \tag{30}$$

$$SH^2S^{-1}=-H^0\tan\theta+H^1\frac{\sin\varphi}{\cos\theta}+H^2\frac{\cos\varphi}{\cos\theta}=\frac{X}{r}.$$

It is easily verified that

$$\frac{1}{4}\omega_{\alpha\beta\nu}H^\alpha H^\beta=S^{-1}e_\nu S-\frac{1}{2r}H_\nu H^2. \tag{31}$$

Therefore,

$$D_\nu\xi=S^{-1}e_\nu\Xi-\frac{1}{2r}H_\nu H^2S^{-1}\Xi,$$

$$D_\nu\bar{\xi}=(e_\nu\bar{\Xi})S+\frac{1}{2r}\bar{\Xi}SH_\nu H^2. \tag{32}$$

We thus find

$$H^\nu D_\nu\xi=H^\nu S^{-1}e_\nu\Xi-\frac{1}{r}H^2S^{-1}\Xi,$$

$$D_\nu\bar{\xi}H^\nu=(e_\nu\bar{\Xi})SH^\nu-\frac{1}{r}\bar{\Xi}SH^2. \tag{33}$$

Furthermore, the vector fields (2) in this case are

$$m_0=\frac{\partial}{\partial\theta},\quad m_1=-\sin\theta\sin\varphi\frac{\partial}{\partial\theta}+\cos\theta\cos\varphi\frac{\partial}{\partial\varphi},$$

$$m_2=-\sin\theta\cos\varphi\frac{\partial}{\partial\theta}-\cos\theta\sin\varphi\frac{\partial}{\partial\varphi}. \tag{34}$$

Therefore, in the form (30),

$$rSH^\nu S^{-1}e_\nu=H^Am_A, \tag{35}$$

so that Eq. (22) in the dX basis takes the form (1) for $n=1$. The conjugate equation

$$D_\nu\bar{\xi}H^\nu=-\frac{imc}{\hbar}\bar{\xi}H^2 \tag{36}$$

takes the form (9) for $n=1$.

3. The current vector and energy–momentum tensor in the *dX* basis

In two-dimensional spacetime in the F basis the current vector is (II.28), and according to (II.31) and (II.35) the energy–momentum tensor is

$$T_{\mu\nu}=\frac{i\hbar}{2}(\bar{\xi}H_\mu\xi_\nu-\bar{\xi}_\nu H_\mu\xi)=\frac{i\hbar}{2}(\bar{\xi}H_\nu\xi_\mu-\bar{\xi}_\mu H_\nu,\xi), \tag{37}$$

since in the two-dimensional case $[H_\mu H_\nu H^\alpha]=0$.

To transform the current vector and the energy–momentum tensor to the dX basis, in addition to (27) and (32) we need to use the equations

$$\bar{\xi}=\bar{\Xi}S,\quad D_\mu\bar{\xi}=(e_\mu\bar{\Xi})-\frac{1}{2r}\bar{\Xi}SH_2H_\mu. \tag{38}$$

As a result, we find

$$J^A=\frac{e}{\nu}\bar{\Xi}X^A\Xi, \tag{39}$$

$$T_{AB}=\frac{i\hbar}{2r^2}\left[\bar{\Xi}X_A\Xi_B-\bar{\Xi}_BX_A\Xi+\varepsilon_{ABC}X^C\bar{\Xi}\Xi\right.$$

$$=\frac{i\hbar}{2r^2}[\bar{\Xi}X_A\Xi_B-\bar{\Xi}_BX_A\Xi+\bar{\Xi}X_B\Xi_A-\bar{\Xi}_AX_B\Xi], \tag{40}$$

where ε_{ABC} is the completely antisymmetric tensor, $\varepsilon_{012}=1$, $\Xi_A=m_A\Xi$, $\bar{\Xi}_A=m_A\bar{\Xi}$, and

$$X_A=m_AX=rH_A-\frac{X_A}{r}X=\frac{X}{2r}[XH_A-H_AX].$$

Since the divergences of the current vector and of the energy–momentum tensor are zero in the F basis, in the dX basis

$$m_AJ^A=0,\quad m_AT^{AB}=0. \tag{41}$$

4. The charge operator in the *dX* basis

The charge operator $\hat{e}$ in the F basis is equal to the integral of the determinant

$$\begin{vmatrix} f^0 & f^1 \\ J^0 & J^1 \end{vmatrix}$$

along the curve Σ. The same operator in the dX basis is equal to the integral along the curve Σ of the determinant

$$\frac{1}{r}\begin{vmatrix} J^0 & x^0 & dx^0 \\ J^1 & x^1 & dx^1 \\ J^2 & x^2 & dx^2 \end{vmatrix}. \tag{42}$$

Substituting the current vector (39) into this, we find

$$\hat{e}=-\frac{e}{r}\int\bar{\Xi}XdX\Xi. \tag{43}$$

This integral is independent of the choice of the curve Σ. Specifying it by the equation $\theta=0$, we find $XdX=r^2H^0d\varphi$ and

$$\hat{e}=-\hat{e}r\int_0^{2\pi}\Xi^*\Xi d\varphi. \tag{44}$$

5. The isometric momenta in the *dX* basis

In two-dimensional de Sitter spacetime there are three Killing vector fields of the type (4) with components

$$\underset{(PQ)}{Z^C}=X_P\delta_Q^C-X_Q\delta_P^C. \tag{45}$$

They correspond to the isometric operators

$$\underset{(PQ)}{K}=-i\hbar\left(m_{PQ}+\frac{1}{2}H_PH_Q\right) \tag{46}$$

and the second-quantized operators given by the integrals

$$\underset{(PQ)}{M}=-\frac{1}{r}\int\bar{\Xi}XdX\underset{(PQ)}{K}\Xi \tag{47}$$

along the curve Σ. The latter are determined by the integrals along the curve Σ of expressions of the type (42), in which the components J^A are replaced by contractions $T^A_C\underset{(PQ)}{Z^C}$ of the tensor (40) with the vector (45). Specifying Σ by the equation $\theta=0$, we find

$$\underset{(PQ)}{M}=-i\hbar r\int_0\Xi^*(m_{pq}+H_PH_Q)\Xi d\varphi. \tag{48}$$

6. Anticommutator of the spinor field in the dX basis

In the third part of this review we obtained the anticommutator of a field obeying Eq. (17). For this we had to solve the Cauchy problem. It was solved by the Riemann method, and the anticommutator was expressed in terms of the Riemann function for Eq. (17). There we found the explicit form of the Riemann function for Eq. (17). If the anticommutator obtained there is transformed to the dX basis, it can be expressed in either of the following two forms:

$$\{\bar{\Xi}(X)\Xi(y)\}=r^{-2}(\underset{x}{M}+1+im)D^{(+)}(x,y)Y$$

$$=r^{-2}XD^{(-)}(x,y)(\underset{y}{M}+1-im), \tag{49}$$

where $\{\Xi(x)\bar{\Xi}(y)\}$ is a matrix whose elements are $\Xi_p(x)\bar{\Xi}_q(y)+\bar{\Xi}_q(y)\Xi_p(x)$. The symbols x and y below M indicate that the operator M refers to the points x and y, respectively. The matrix Y is $Y=y^AH_A$.

The function $D^{(+)}$ is

$$D^{(+)}(x,y)=\varepsilon(x^0-y^0)\,\frac{1+\varepsilon(\Lambda)}{2}\,F\left(-im,1+im;\,\frac{\Lambda}{4}\right), \tag{50}$$

where $\Lambda=r^{-2}(x^A-y^A)(x_A-y_A)$, and $\varepsilon(\Lambda)$ is the sign of Λ. The function $D^{(+)}$ obeys Eq. (12) for $n=1$.

The function $D^{(-)}$ is obtained from $D^{(+)}$ by replacing m by $-m$. It obeys Eq. (13) for $n=1$.

The anticommutator (49) obeys the equation

$$(\underset{x}{M}-im)\{\Xi(x)\bar{\Xi}(y)\}=0. \tag{51}$$

7. Solution of the Cauchy problem in the dX basis

The anticommutator (49) can be obtained directly by solving the Cauchy problem for Eq. (1). For $n=1$ we have

$$m_0=\frac{\partial}{\partial\theta},\quad m_1=-\sin\theta\sin\varphi\,\frac{\partial}{\partial\theta}+\cos\theta\cos\varphi\,\frac{\partial}{\partial\varphi},$$

$$m_2=-\sin\theta\cos\varphi\,\frac{\partial}{\partial\theta}+\cos\theta\sin\varphi\,\frac{\partial}{\partial\varphi}. \tag{52}$$

From this we find

$$m_Am^A=\cos^2\theta\left[\frac{\partial^2}{\partial\theta^2}-\frac{\partial^2}{\partial\varphi^2}\right]. \tag{53}$$

According to (12), the spinor field in this case obeys the equation

$$\cos^2\theta\left[\frac{\partial^2\Xi}{\partial\theta^2}-\frac{\partial^2\Xi}{\partial\varphi^2}\right]=im(1+im)\Xi. \tag{54}$$

Let the Cauchy data refer to the curve Σ separating the hyperboloid (23) into two parts. It can be specified by the equation $\theta=f(\varphi)$, where $f(2\pi)=f(0)$ and $-1<d\theta/d\varphi<1$. The solution of the Cauchy problem for Eq. (54) is expressed in terms of the function $D^{(+)}$:

$$\Xi(x)=\int_0^{2\pi}[D^{(+)}(x,y)\Xi_{(o)}(y)-D^{(+)}_{(o)}(x,y)\Xi(y)]\Big|_{\tilde{\theta}=f(\tilde{\varphi})}d\tilde{\varphi}, \tag{55}$$

where (o) is the sign of the derivative normal to the curve Σ. For example,

$$\Xi_{(o)}(y)=\left[\frac{\partial}{\partial\tilde{\theta}}+\frac{df(\tilde{\varphi})}{d\tilde{\varphi}}\frac{\partial}{\partial\tilde{\varphi}}\right]\Xi(y). \tag{56}$$

Taking it from Eq. (1) for $n=1$, we can obtain the solution of the Cauchy problem[56] for this equation in the form (15), where

$$\Phi=\frac{1}{r}\int D^{(+)}(x,y)dY\Xi(y), \tag{57}$$

$$dY=\left[\frac{\partial Y}{\partial\tilde{\varphi}}+\frac{\partial Y}{\partial\tilde{\theta}}\frac{df(\tilde{\varphi})}{d\tilde{\varphi}}\right]d\tilde{\varphi} \tag{58}$$

is the elementary displacement of the matrix Y along the curve Σ.

This solution can be written in terms of the anticommutator (49):

$$\Xi(x)=\int\{\Xi(x)\bar{\Xi}(y)\}\,\frac{Y}{r}\,dY\Xi(y). \tag{59}$$

The solution of the Cauchy problem for the conjugate Dirac equation can be written in terms of the same anticommutator (49):

$$\bar{\Xi}(x)=\int d\bar{\Xi}(y)\,\frac{Y}{r}\,dY\{\Xi(x)\bar{\Xi}(y)\}. \tag{60}$$

The proposed method of solving the Cauchy problem is easily generalized to the $2n$-dimensional case.

CONCLUSION

Here we have considered only a few of the problems concerning the spinor field in Riemannian spacetime. For example, we have not discussed the quantum theory of the spinor field in four-dimensional de Sitter spacetime constructed in Refs. 37–39. We have also left out the general method of separating variables in the DFI equation. Apparently, the exact—not approximate—quantization of the spinor field is possible in just those spacetimes where separation of variables occurs. We have also neglected the interaction of the spinor field with other fields and, most importantly, with the electromagnetic field. The study of all these topics lies far outside the scope of a single review.

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