Integrable systems

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An algebraic approach is proposed for the construction of soliton solutions of nonlinear integrable systems based on the structure of the algebras of their internal symmetry. Its universality is demonstrated by examples of explicit solutions of many known integrable evolution equations and hierarchies in (1+1) and (2+1) dimensions. [S1063-7796(96)00105-2]

1. GENERAL REMARKS[1–7]

In this paper we will consider dynamical systems which have solutions of soliton type. The origin of the concept of solitons is connected with physical problems, in particular, with the study of waves on the water in canals. These waves correspond to processes of propagation of a perturbation peak (soliton wave) which carries finite energy. This peak is stable with respect to external influences. The soliton wave differs from ordinary periodic waves which have many peaks.

From a mathematical point of view all these systems are unified by the properties of their algebra of internal symmetry. These algebras are infinite-dimensional but have finite-dimensional representations with a "spectral parameter," i.e., they are realized by finite-dimensional matrices whose elements are rational functions of a parameter λ, taking values in the complex plane. This case is rather different from the finite-dimensional algebras of internal symmetry of exactly integrable systems.

Now we have generalized the partial problems of the soliton-solution behavior to the following mathematical scheme. We first formulate the problem. It is necessary to find an element g which takes values in some group and depends on the parameter λ. This element is also a function of independent variables ξ and satisfies the relation

$$\frac{\partial g}{\partial \xi} g^{-1}=u. \tag{1.1}$$

The elements u, taking values in the corresponding algebra, are assumed to be rational functions of the spectral parameter. In the general case the parameters which determine the positions of the poles are functions of the independent variables ξ.

The Maurer–Cartan identities subject to (1.1) reduce to the system

$$\frac{\partial u_i}{\partial \xi_j}-\frac{\partial u_j}{\partial \xi_i}=[u_i,u_j]. \tag{1.2}$$

From this identity we can extract the equalities of the residues at all the poles of any order. We then have the system of equations under consideration, i.e., there is a multicomponent equation which is equivalent to the investigated dynamical system. One can say that the system (1.2) with respect to the spectral parameter is a generating expression (Laurent series) for the equations of the dynamical system.

Originally, for the solution of systems of the type (1.2) the widely known inverse scattering method was elaborated. By means of this method many equations of importance for physical applications (such as the Korteveg–de Vries, sine–Gordon, nonlinear Schrödinger, and so on) were integrated. This method is described in detail in many well known monographs.

In further work the solution of the systems (1.2) was connected with the matrix Riemann problem (the so-called Zakharov–Shabat dressing method). This method provides a possibility of finding the solution of an integrable system when the solution of the Riemann problem is known from independent considerations.

In this paper we shall use a purely algebraic construction for finding a system of equations possessing soliton-type solutions together with their explicit form, bypassing the stage of investigating and exploring their internal symmetry algebra. The problem of the need for such an approach and the existence of systems having soliton solutions that do not fall within the scope of our construction will not be considered here. In any case all the systems that are integrable by the inverse scattering method fall within the construction which follows below. In all cases it yields explicit formulas for soliton-type solutions even when traditional methods are so cumbersome that it becomes impossible from the purely technical standpoint to obtain the result.

The initial (input) elements of the construction are specially coded data of the structure of the internal symmetry algebra of the system, which are used to express, by several algebraic operations, soliton-type solutions together with the system of equations that they satisfy. Here, the solution of the sine–Gordon, Korteveg–de Vries, nonlinear Schrödinger, and other wave and evolution equations is described by common formulas distinguished only by the parameters related to the internal symmetry algebra.

2. INFINITE-DIMENSIONAL RATIONAL-FUNCTION ALGEBRA[8,9]

In this section we will describe the construction of a special type of infinite-dimensional algebra, which is an essential point in the following approach to the consideration of the dynamical systems in question. We will call these the algebras of rational functions, because so far there is no terminology on this subject.

In order to explain the construction of these algebras we will examine simple algebraic identities for the decomposition into partial fractions, which are well known:

$$\frac{1}{(\lambda-a)^m}\frac{1}{(\lambda-b)^n}=\sum_{i=1}^{m}\frac{f_i(a,b,m,n)}{(\lambda-a)^i}+\sum_{j=1}^{n}\frac{g_j(a,b,m,n)}{(\lambda-b)^j}.$$

The explicit form of the functions f_i, g_j can be determined by many independent methods and can be found in any book on this subject.

Consider some Lie algebra with its generators L_i. Let the finite ambiguity of the arbitrary parameters $a=(a_1,a_2,\ldots,a_r)$ be denoted by one symbol. The generators of some infinite-dimensional algebra $L_i^{a,k}$ are determined by the relations

$$L_i^{a,k}=(\lambda-a)^{-k}L_i.$$

The new generators are labeled with an integer index k and a continuous complex number a; λ is the complex parameter. In the case of negative k we introduce additional generators $L_i^s=\lambda^s L_i$. The most significant fact is that the variety of these generators is a closed infinite-dimensional Lie algebra. Indeed, we calculate the commutator

$$[L_i^{a,k},L_j^{b,l}]=(\lambda-a)^{-k}(\lambda-b)^{l}\sum_m C_{i,j}^m L_m.$$

By virtue of the previous identity we represent the right-hand side of the last equality in the form of a linear combination of the generators constructed above. Continuing the last equality, we have

$$\left(\sum_{k'=1}^{k}\frac{f_{k'}(a,b,k,l)}{(\lambda-a)^{k'}}+\sum_{l'=1}^{l}\frac{g_{k'}(a,b,k,l)}{(\lambda-b)^{l'}}\right)\sum_m C_{ij}^m L_m,$$

or, retaining the first and last terms of the written equality, we obtain

$$[L_i^{a,k},L_j^{b,l}]=\sum_m C_{i,j}^m\left(\sum_{k'=1}^{k}\left(f_{k'}(a,b,k,l)L_m^{a,k'}+\sum_{l'=1}^{l}g_{k'}(a,b,k,l)L_m^{b,l'}\right)\right).$$

These can be considered as the commutation relations of an abstract infinite-dimensional algebra. We note that in the last sum there is only a finite number of generators. This may be connected with the filtration properties of the constructed algebras. We will call them the algebras of rational functions.

The algebras of the inner symmetry of the integrable systems have a direct connection to the subject of this section.

3. STATEMENT OF THE PROBLEM AND ITS NONLINEAR SYMMETRIES

Here, the problem considered briefly in Sec. 1 will be formulated more carefully, and its symmetry properties will be described. With the help of these symmetries it will be possible to construct the whole hierarchy of solutions of the problem if some solution of it is known.

The formulation of the problem is the following: It is required to find an element g, taking values in some group, which depends on a complex parameter λ and on arguments ξ: $(\xi_1,\ldots,\xi_n)$, constructed from g elements $u=(\partial g/\partial\xi)g^{-1}$ taking values in the corresponding algebra, such that it is a rational function in the complex λ plane.

The known data under this condition are the positions of the poles and their multiplicity for each of the elements u_i. In what follows we shall call the totality of these data the spectral structure of the element u_i. The unknown quantities are the residues at all the poles of the elements u_i as functions of ξ or, equivalently, the matrix elements of g as functions of λ and ξ.

The problem possesses remarkable symmetry properties, which we shall now describe.

Let g_0 be some solution. For definiteness, let g_0 belong to the group $SL(k,c)$. This means that g_0 is a $(k+1,k+1)$ matrix with determinant equal to 1. Let us introduce the matrix of the polynomials P, which has the following structure:

$$\begin{pmatrix} \tilde{P}_{n+1}^{11} & P_n^{12} & P_n^{13} & \cdots & P_n^{1,k+1} \\ P_n^{21} & \tilde{P}_{n+1}^{22} & P_n^{23} & \cdots & P_n^{2,k+1} \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ P_n^{k+1,1} & P_n^{k+1,2} & \cdots & \cdots & \tilde{P}_n^{k+1,k+1} \end{pmatrix}, \tag{3.1}$$

where $P_n^{\alpha,\beta}$ is a polynomial of degree n in λ, and $\tilde{P}_{n+1}^{\alpha,\alpha}$ is a polynomial of degree $n+1$ with coefficient 1 at λ^{n+1}. The equal degrees of the polynomials are chosen for simplicity, and in what follows this restriction will be replaced by some other restriction which holds in the definition of the polynomial matrix.

The coefficients of the elements of the polynomial matrix are defined by the requirement that at $(k+1)(n+1)$ different points of the λ plane there is a linear dependence in column $(k+1)$ of the matrix Pg^0. This means that

$$\sum_{s=1}^{k+1}(P_n(\lambda_i)g_0(\lambda_i))_{\alpha,\beta}c_\beta(\lambda_i)=0,$$

$$\alpha,\beta=1,2,\ldots,(k+1),\quad i=1,2,\ldots,(k+1)(n+1), \tag{3.2}$$

where $c_\beta(\lambda_i)$ are the totalities of the arbitrary c-number parameters. These conditions determine all the coefficient functions of the elements of the matrix P (3.2).

Indeed, let us take $\alpha=1$ in the last equality. We have

$$\sum_{s=1}^{k+1}P_n^{1\beta}(\lambda_i)(g_0(\lambda_i)c(\lambda_i))_\beta=0,\quad i=1,2,\ldots,(k+1)(n+1). \tag{3.3}$$

The last equality is a system of $(n+1)(k+1)$ linear algebraic equations, in which the unknowns are the $(n+1)(k+1)$ coefficient functions of the polynomials of the first row of the matrix P. (Each polynomial has exactly $n+1$ coefficients, and the number of polynomials is $k+1$).

From the definition of the matrix P it follows that its determinant is a polynomial of degree $(n+1)(k+1)$ with coefficient 1 at the highest power of λ. From (3.2) we conclude that the determinant of P vanishes at exactly $(n+1)(k+1)$ points of the λ plane. At these points, the columns of the matrix Pg_0 ($\text{Det}\, g_0=1$) are linearly dependent. Thus, we obtain

$$\text{Det}\, P=\prod_{1}^{(n+1)(k+1)}(\lambda-\lambda_i). \tag{3.4}$$

From the definition of the elements of the inverse matrix g^{-1} via the ratio of the k-order minors to the determinant of the matrix g, we have, for the elements of the matrix u,

$$u_{\alpha,\beta}=\left(\frac{\partial g}{\partial \xi}g^{-1}\right)_{\alpha,\beta}=\frac{\text{Det}\|(\beta\to\dot{\alpha})\|}{\text{Det}(\|g\|)}, \tag{3.5}$$

where by the symbol $\|\beta\to\dot{\alpha}\|$ we denote the matrix arising from the matrix g when its row β changes over the derivatives with respect to ξ from its row α.

Let us illustrate these formulas by the examples of the second-order matrix

$$u_{11}=\frac{\text{Det}\begin{pmatrix}\dot{g}_{11} & \dot{g}_{12}\\ g_{21} & g_{22}\end{pmatrix}}{\text{Det}\, g},\quad u_{21}=\frac{\text{Det}\begin{pmatrix}\dot{g}_{21} & \dot{g}_{22}\\ g_{21} & g_{22}\end{pmatrix}}{\text{Det}\, g},$$

$$u_{12}=\frac{\text{Det}\begin{pmatrix}g_{11} & g_{12}\\ \dot{g}_{11} & \dot{g}_{12}\end{pmatrix}}{\text{Det}\, g}, \tag{3.6}$$

where $\dot{g}=\partial g/\partial\xi$. To determine the analytic properties of the elements of u as functions of the parameter λ we use the general formulas for the case $g=Pg_0$. For the matrix $\dot{g}$ we have

$$\dot{g}=\dot{P}g_0+P\dot{g}_0=(\dot{P}+P\dot{g}_0g_0^{-1})g_0.$$

For the elements of u we obtain

$$u_{\alpha,\beta}=\frac{\text{Det}\|(P)_\beta\to(\dot{P}+Pu_0)_\alpha\|}{\Pi_1^{(k+1)(n+1)}(\lambda-\lambda_i)}. \tag{3.7}$$

The coefficients of $c_\beta(\lambda_i)$ in (3.2) are independent of ξ. Thus, the columns of the matrix (Pg_0) are linearly dependent. For this reason the numerator in the expression for the matrix elements of u (the matrix u_0 has a rational dependence on λ) will contain a factor which cancels with the denominator, and the analytic properties of the matrix u repeat those of u_0. The positions and maximal multiplicities of the poles are the same for the matrices u and u_0.

A separate treatment is needed to understand the behavior of u in the limit $\lambda\to\infty$. Let us first consider this situation for the example of the second-order matrix. The numerator of the element u_{11} is the determinant of the matrix:

$$\begin{pmatrix}\dot{P}^{11}_{n+1}+P^{11}_{n+1}u^0_{11}+P^{12}_n u^0_{21} & \dot{P}^{12}_{n+1}+P^{11}_{n+1}u^0_{12}+P^{12}_n u^0_{22}\\ P^{21}_n & P^{22}_{n+1}\end{pmatrix}.$$

As $\lambda\to\infty$, the denominator of the matrix element u_{11} has the asymptotic behavior $\lambda^{2(n+1)}$. The maximal power in λ in the numerator may be contained in the product $P^{11}_{n+1}P^{22}_{n+1}U^0_{11}$. This power is not more than $2(n+1)+s$, where by s we denote the maximal power of the matrix elements of u_0 at infinity.

Let us give a summary. It has been shown that if there is some solution g_0 of the problem, then the element Pg_0 constructed by the rules of this section is also a solution of the same problem. Thus, we obtain a whole hierarchy of solutions (n is arbitrary). The problem possesses some nonlinear symmetry. This symmetry is usually regarded as a Bäcklund transformation. The Bäcklund transformation plays the most important role in the theory of integrable systems. It will become clear from the next sections how these transformations are used for the construction of the exact solutions of the integrable system.

4. THE SPECTRAL EQUATION[8,11]

In this section the connection between the matrix equation (1.1) and the theory of ordinary differential equations will be considered. Equation (1.1), written in the form

$$\dot{g}=ug, \tag{4.1}$$

is the equation for the unknown g, taking values in some group, under the assumption that the element u, taking values in the corresponding algebra, is known. This equation is a system of equations in the parameters of the group element g (it is assumed that they are functions of an independent argument, and differentiation is carried out with respect to it), and in this sense it is invariant with respect to the choice of any representation of an algebra (or group). On the other hand, we can consider it in some fixed representation when g and u are certain finite-dimensional matrices. We shall use the Dirac notation for the basis vectors $\|\alpha\rangle,\langle\beta\|$ (α,β range from 1 to N, where N is the dimension of the representation). Equation (4.1) makes it possible to calculate the successive derivatives of the element g:

$$\ddot{g}\equiv g^{[.2]}=(\dot{u}+u^2)g\equiv u_2g,\quad g^{[.s]}=u_sg,$$

$$u_{s+1}=\dot{u}_s+u_su_1,\quad u_0\equiv 1.$$

Writing the matrix elements for the first N derivatives by using the basis vectors $\langle 1\|$ and $\|\alpha\rangle$, we have

$$\langle 1\|g_{[.s]}\|\alpha\rangle=\langle 1\|u_s\|1\rangle\langle 1\|g\|\alpha\rangle+\sum_2^N\langle 1\|u_s\|\beta\rangle$$

$$\times\langle\beta\|g\|\alpha\rangle. \tag{4.2}$$

Eliminating the $N-1$ matrix elements $\langle\beta\|g\|\alpha\rangle$ ($\beta=2,\ldots,k+1$) from the last system of N equations, we derive an ordinary Nth-order differential equation for the function $\psi_\alpha\equiv\langle 1\|g\|\alpha\rangle$:

$$\text{Det}\|\psi_\alpha^{[.s]}-\langle 1\|u_s\|1\rangle\psi_\alpha,\langle 1\|u_s\|2\rangle,\ldots,\langle 1\|u_s\|\beta\rangle,\ldots\|=0, \tag{4.3}$$

where the index s labels the rows and takes values from 1 to N; the index β labels the columns $2,\ldots,N$, except the first one. Thus, all the elements in the "first" row $\langle 1\|g\|\alpha\rangle$ satisfy the same Nth-order differential equation (i.e., they are its fundamental solutions), whose coefficients are expressed explicitly in terms of the elements of the matrix u and its derivatives up to the $(N-1)$th order. We shall call this equation a spectral equation. The matrix elements $\langle\beta\|g\|\alpha\rangle$ can be found from a linear system of $N-1$ equations (4.2) $(1\leq s\leq N)$, and in this manner the matrix g is explicitly expressed in terms of N fundamental solutions of the spectral equation, the matrix elements of u, and their derivatives up to the $(N-1)$th order, inclusive. The coefficient functions of the spectral equation can be expressed in terms of the set of its fundamental solutions by known relations. Let us rewrite the spectral equation in the form

$$\psi^{[.N]}-\ln\dot{V}\psi^{[.N-1]}+\sum_{k=0}^{N-2}(-1)^k a_k\psi^{[.k]}=0. \qquad (4.4)$$

We introduce the notation $\|\psi^{[.s_1]},\psi^{[.s_2]},\ldots,\psi^{[.s_N]}\|$ for the determinant of the matrix whose first column consists of the derivatives of the s_1 order of fundamental solutions of the spectral equation, the second column consists of the derivatives of the s_2 order of the fundamental solutions, and so on. In this notation we have

$$V=\|\psi,\psi^{[.1]},\ldots,\psi^{[.(N-1)]}\|,$$
$$Va_k=-\|\psi,\ldots,\psi^{[.k-1]},\psi^{[.N]},\psi^{[.k+1]},\ldots,\psi^{[.(N-1)]}\|. \qquad (4.5)$$

Comparing these expressions with the coefficients of the spectral equation, we see that to obtain them it is necessary to make the substitution $\psi_\alpha^{[.S]}\rightarrow\langle 1\|u_s\|\alpha\rangle$ in the last formulas. This relation will become useful in constructing the matrix elements via the known set of the fundamental solutions of the spectral equation.

5. CONSTRUCTION OF THE SOLUTIONS IF THE ELEMENT g_0 BELONGS TO A DIAGONAL (COMMUTATIVE) SUBGROUP[8,10–13]

The general construction of Sec. 3 will be used now to get the whole class of solutions of integrable systems for the algebra $SL(k,c)$. Let us consider the case in which the background equation has a trivial solution if one assumes that the element g takes values in a commutative (Cartan in the semisimple case) subgroup

$$g=\exp\sum_{s=1}^{r}h_s\tau_s,\quad [h_i,h_j]=0, \qquad (5.1)$$

where r is the dimension of the commutative subgroup. From the background equation (in our case $(\partial g/\partial\xi)g^{-1}=\Sigma_{s=1}^{r}h_s(\partial\tau_s/\partial\xi)$) it follows that the functions τ_s must be rational functions of the argument λ, whose analytic properties are determined by the spectral structure of the elements u. This means that the residues at the poles of τ and the coefficient functions of its Laurent expansion near the infinite point of the λ plane must be functions of the single argument ξ_i. Thus,

$$\tau_s=\sum_{i=1}^{r}\tau_s^i(\xi_i,\lambda)+\tau_s^0,$$

i.e., τ_s is the sum of rational functions of one argument, the number of which is equal to the number of independent parameters ξ in the problem, and some function τ_s^0 which depends on all the parameters except λ. We will call this function the null node of τ, and the whole function τ_s the source function. Let us take this solution in the capacity g_0 of the general construction of Sec. 3, and use the quadratic $(k+1,k+1)$ matrix of the polynomials P,

$$\begin{pmatrix} \tilde{P}^{11}_{n_1+1} & \tilde{P}^{12}_{n} & \cdots & \tilde{P}^{1,k+1}_{n_{k+1}} \\ P^{21}_{n_1} & \tilde{P}^{22}_{n+1} & \cdots & P^{2,k+1}_{n_{k+1}} \\ \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots \\ P^{k+1,1}_{n_1} & P^{k+1,2}_{n} & \cdots & \tilde{P}^{k+1,k+1}_{n} \end{pmatrix}, \qquad (5.2)$$

containing as before the notation $P_n(\lambda)$ for an arbitrary polynomial of degree n and $\tilde{P}_n(\lambda)$ for the same polynomial with coefficient at the highest power of λ.

The difference from the general scheme consists first in distinguishing the degrees of the polynomials of the columns, and second, all polynomials of the first row have sign $\tilde{P}$. The last modification is necessary for determination of the null modes of the functions τ_s, as will become clear from what follows. It is assumed, as before, that there is a linear dependence between the columns of the matrix $g=Pg_0=P\exp\Sigma_{s=1}^{r}h_s\tau_s$ at $\Sigma_{\alpha=1}^{k+1}(n_\alpha+1)$ different points of the λ plane,

$$\sum_{\beta=1}^{k+1}P^{\alpha,\beta}_{n_\beta+\delta_{\alpha,\beta}}(\lambda_i)c_\beta(\lambda_i)\exp(\tau_\beta-\tau_{\beta-1})=0,$$

$$\alpha=1,2,\ldots,(k+1),\quad i=1,2,\ldots,\sum_{\alpha=1}^{k+1}(n_\alpha+1), \qquad (5.3)$$

where $c_\beta(\lambda_i)$ is the totality of arbitrary c-number parameters, and $\tau_0=\tau_{k+1}=0$. The last equality is a system of $\Sigma_{\alpha=1}^{k+1}(n_\alpha+1)$ linear algebraic equations, where the coefficient functions of the polynomial matrix P and the null-mode components of the τ functions are unknown. The number of equations is equal to the number of unknown variables. Indeed, let us take $\alpha=1$ in (5.3). The number of coefficient functions of the first column of the polynomial matrix P is equal to $\Sigma_{\alpha=1}^{k+1}n_\alpha+1$, and the k null-mode components of the τ functions are unknown. Thus, the total number of unknown variables is $\Sigma(n_\alpha+1)$, i.e., exactly the number of equations. The same situation holds for the other columns, and consequently the system of equations (5.3) explicitly determines all the parameters of the polynomial matrix and the null-mode components of the τ functions.

We have $\operatorname{Det}P=\operatorname{Det}(Pg_0)$, and from the definition of the polynomial matrix P it follows that its determinant is a polynomial of degree $\Sigma(n_\alpha+1)$ with coefficient 1 at the highest power of λ. From (5.3) we know that $\operatorname{Det}P$ vanishes at $\Sigma(n_\alpha+1)$ points of the complex plane. Thus, we get

$$\operatorname{Det}P=\prod_{i=1}^{\Sigma_{\alpha=1}^{k+1}(n_\alpha+1)}(\lambda-\lambda_i). \qquad (5.4)$$

The general formulas of Sec. 3 for the elements of the matrix u remain valid after the obvious substitution $(u_0)_{\alpha,\beta}=\delta_{\alpha,\beta}(\tau_\beta-\tau_{\beta-1})$:

$$u_{\alpha,\beta}=\frac{\mathrm{Det}\|P_\beta\to\dot P_\alpha+\dot P_\alpha(\tau_\alpha-\tau_{\alpha-1})\|}{\Pi(\lambda-\lambda_i)}. \tag{5.5}$$

From the last expression we see, as in Sec. 3, that all the features of the elements of the matrix u, including the infinite point, are determined by the analytic properties of the source functions.

6. THE CASE OF THE ALGEBRA $SL(2,C)$ (REFS. 10 AND 13–15)

The results of Secs. 3 and 5 will be specified here for the case of an algebra $SL(2,C)$ which has many physical applications. In that case, it is possible to write all the expressions in a form convenient for practical calculations.

Let the polynomial matrix P (5.2) be rewritten in the form

$$P=\begin{pmatrix}\prod_{i=1}^{n_1+1}(\lambda-a_i) & \prod_{j=1}^{n_2}(\lambda-b_j)\\ P_{n_1} & P_{n_2+1}\end{pmatrix}. \tag{6.1}$$

The polynomials of the first row $\tilde P_{n_1+1},\tilde P_{n_2}$ are decomposed on the systems of their roots (a_i,b_j). In our case, $\tau_1=-\tau_2=\Sigma\tau_s(\xi_s,\lambda)+\tau_0\equiv\tau$. The coefficient functions of the polynomials and null mode τ_0 are determined by the linear system of algebraic equations

$$\exp 2\tau(\lambda_s)\exp 2\tau_0 P_{n_1+1}(\lambda_s)+c(\lambda_s)P_{n_2}(\lambda_s)=0,$$
$$s=1,2,\ldots,(n_1+n_2+1), \tag{6.2}$$
$$\exp 2\tau(\lambda_s)\exp 2\tau_0 P_{n_1}(\lambda_s)+c(\lambda_s)P_{n_2+1}(\lambda_s)=0.$$

The number of unknown quantities in the first system of equations is equal to n_1+1 symmetrical combinations composed of the roots a_i (coefficient functions of the polynomials P_{n_1+1}), n_2 symmetrical combinations composed of the roots b_j (coefficient functions of the polynomials P_{n_1}), and null-mode component in the form $\exp 2\tau_0$. This is exactly equal to the number of equations. The same situation holds for the second system (6.2). As in (5.5), for $u_{12}=u_+$ we obtain

$$u_+=\left(\frac{\Pi_{i=1}^{n_1+1}(\lambda-a_i)\Pi_{j=1}^{n_2}(\lambda-b_j)}{\Pi_{s=1}^{n_1+n_2+2}(\lambda-\lambda_s)}\right)\left(-\sum_{j=1}^{n2}\frac{\dot b_j}{\lambda-b_j}+\sum_{i=1}^{n_1+1}\frac{\dot a_i}{\lambda-a_i}+2\dot\tau\right). \tag{6.3}$$

It follows from the explicit expression for u_+ that the analytic dependence of $\dot\tau$ (the positions of the poles, their multiplicity, and the behavior at infinity) is the same for u_+ as for $\dot\tau$. Rewriting (6.3) in an equivalent form, we have

$$\left(-\sum_{j=1}^{n_2}\frac{\dot b_j}{\lambda-b_j}+\sum_{i=1}^{n_1+1}\frac{\dot a_i}{\lambda-a_i}+2\dot\tau\right)=\frac{\Pi_{s=1}^{n_1+n_2+2}(\lambda-\lambda_s)}{\Pi_{i=1}^{n_1+1}(\lambda-a_i)\Pi_{j=1}^{n_2}(\lambda-b_j)}u_+. \tag{6.4}$$

This equality is the definition of u_+, and the examples of the next section will show how it can be used. By decomposition of the right-hand side of the last equation in simple fractions we obtain expressions for the derivatives:

$$\dot b_j=-u_+(b_j)\frac{\pi(b_j)}{P_{n_1+1}(b_j)\hat P_{n_2}(b_j)},$$
$$\dot a_i=u_+(a_i)\frac{\pi(a_i)}{\hat P_{n_1+1}(a_i)P_{n_2}(a_i)}. \tag{6.5}$$

In these expressions the caret over a polynomial denotes this polynomial without the multiplier, which goes to zero at the actual value of its argument, $\pi(\lambda)=\Pi_{s=1}^{n_1+n_2+2}(\lambda-\lambda_s)$.

For u_0 we have the equivalent representations

$$u_0=\pi^{-1}(\lambda)\mathrm{Det}\begin{pmatrix}\dot P_{n_1+1}+\dot\tau P_{n_1+1} & \dot P_{n_2}-\dot\tau P_{n_2}\\ P_{n_1} & \tilde P_{n_2+1}\end{pmatrix}$$
$$=\frac{P_{n_1+1}P_{n_2}}{\pi(\lambda)}\mathrm{Det}\begin{pmatrix}-\sum\frac{\dot a_i}{\lambda-a_i}+\dot\tau & -\sum\frac{\dot b_j}{\lambda-b_j}-\dot\tau\\ \frac{P_{n_1}}{\tilde P_{n_1+1}} & \frac{\tilde P_{n_2+1}}{\tilde P_{n_2}}\end{pmatrix}$$
$$=\frac{\tilde P_{n_1+1}\tilde P_{n_2}}{\pi(\lambda)}\mathrm{Det}\begin{pmatrix}-\sum\frac{\dot a_i}{\lambda-a_i}+\dot\tau & \frac{\pi(\lambda)}{\tilde P_{n_1+1}\tilde P_{n_2}}u_+\\ \frac{P_{n_1}}{\tilde P_{n_1+1}} & \frac{\pi(\lambda)}{\tilde P_{n_1+1}\tilde P_{n_2}}\end{pmatrix}$$
$$=\mathrm{Det}\begin{pmatrix}-\sum\frac{\dot a_i}{\lambda-a_i}+\dot\tau & u_+\\ \frac{P_{n_1}}{\tilde P_{n_1+1}} & 1\end{pmatrix}. \tag{6.6}$$

In the last transformation we have used the definitions of $\pi(\lambda)$ and u_+. Then

$$\pi(\lambda)=\mathrm{Det}\begin{pmatrix}\tilde P_{n_1+1} & \tilde P_{n_2}\\ P_{n_1} & \tilde P_{n_2+1}\end{pmatrix}$$
$$=\tilde P_{n_1+1}\tilde P_{n_2}\left(\frac{\tilde P_{n_2+1}}{\tilde P_{n_2}}-\frac{P_{n_1}}{\tilde P_{n_1+1}}\right) \tag{6.7}$$

or

$$\frac{\pi(\lambda)}{\tilde P_{n_1+1}\tilde P_{n_2}}=\lambda-\delta+\sum\frac{B_j}{(\lambda-b_j)}-\sum\frac{A_i}{(\lambda-a_i)}.$$

Comparing the residues at the poles $\lambda=a_i$ and $\lambda=b_j$ on both sides of the last equality, we obtain

$$B_j=\frac{\pi(b_j)}{\tilde P_{n_1+1}(b_j)\tilde P_{n_2}(b_j)}=\frac{\dot b_j}{u_+(b_j)},$$

$$A_i=\frac{\pi(a_i)}{\tilde{P}_{n_1+1}(a_i)\tilde{P}_{n_2}(a_i)}=\frac{\dot{a}_i}{u_+(a_i)}.$$

Now we continue the interrupted calculation (6.7):

$$u_0=\mathrm{Det}\begin{pmatrix} -\sum \frac{\dot{a}_i}{\lambda-a_i}+\dot{\tau} & u_+ \\ -\sum \frac{a_i}{(\lambda-a_i)u_+(a_i)} & 1 \end{pmatrix}$$

$$=\dot{\tau}+\sum_{i=1}^{n_1+1}\frac{\dot{a}_i}{u_+(a_i)}\frac{u_+(\lambda)-u_+(a_i)}{\lambda-a_i}. \quad (6.8)$$

Let us finally calculate $u_-(\lambda)\equiv u_{12}$:

$$u_-=\pi^{-1}(\lambda)\mathrm{Det}\begin{pmatrix} \dot{g}_{21} & \dot{g}_{22} \\ g_{21} & g_{22}\end{pmatrix}$$

$$=\pi^{-1}(\lambda)g_{11}g_{12}\,\mathrm{Det}\begin{pmatrix} \frac{\dot{g}_{21}}{g_{11}} & \frac{\dot{g}_{22}}{g_{12}} \\ \frac{g_{21}}{g_{11}} & \frac{g_{22}}{g_{12}}\end{pmatrix}$$

$$=\pi^{-1}(\lambda)g_{11}g_{12}$$

$$\times\mathrm{Det}\begin{pmatrix} \frac{\dot{g}_{21}}{g_{11}} & \left(\frac{g_{22}}{g_{12}}-\frac{g_{21}}{g_{11}}\right)^{\cdot}+\frac{g_{22}}{g_{12}}\frac{\dot{g}_{12}}{g_{12}}-\frac{g_{21}}{g_{11}}\frac{\dot{g}_{11}}{g_{11}} \\ \frac{g_{21}}{g_{11}} & \frac{g_{22}}{g_{12}}-\frac{g_{21}}{g_{11}}\end{pmatrix}$$

$$=g_{11}g_{12}$$

$$\times\mathrm{Det}\begin{pmatrix} \frac{\dot{g}_{21}}{g_{11}} & \left(\frac{1}{g_{12}g_{11}}\right)^{\cdot}+\frac{\dot{g}_{12}}{g_{12}^2g_{11}}+\frac{g_{21}}{g_{11}^2g_{12}}u_+(\lambda) \\ \frac{g_{21}}{g_{11}} & \frac{1}{g_{12}g_{11}}\end{pmatrix}$$

$$=\mathrm{Det}\begin{pmatrix} \left(\frac{g_{21}}{g_{11}}\right)^{\cdot}+\dot{g}_{11}\frac{g_{21}}{g_{11}^2} & -\frac{\dot{g}_{11}}{g_{11}}+\frac{g_{21}}{g_{11}}u_+(\lambda) \\ \frac{g_{21}}{g_{11}} & 1\end{pmatrix}$$

$$=\left(\frac{g_{21}}{g_{11}}\right)^{\cdot}+2\frac{\dot{g}_{11}}{g_{11}}\frac{g_{21}}{g_{11}}-u_+(\lambda)\left(\frac{g_{21}}{g_{11}}\right)^2$$

$$=\left(\frac{g_{21}}{g_{11}}\right)^{\cdot}+2\frac{1}{u_0}\frac{g_{21}}{g_{11}}+u_+(\lambda)\left(\frac{g_{21}}{g_{11}}\right)^2. \quad (6.9)$$

In the last transformations we have used several times the equalities (6.4) and (6.7), which in the notation of the previous calculations have the form

$$\frac{g_{22}}{g_{12}}-\frac{g_{21}}{g_{11}}=\frac{\pi(\lambda)}{g_{11}g_{12}}\frac{\dot{g}_{12}}{g_{12}}-\frac{\dot{g}_{11}}{g_{11}}=\frac{\pi(\lambda)}{g_{11}g_{12}}u_+(\lambda).$$

We can proceed to the expression (6.9) for u_- more directly by virtue of the spectral equation of Sec. 4. The matrix elements $g_{11},g_{12}=\psi$ satisfy this equation, and so we have

$$\left(\frac{\dot{\psi}}{u_+}\right)^{\cdot}=\left(u_-+\frac{u_0^2}{u_+}\right)\psi.$$

Substituting the expression for u_0 (6.8) into this equation, we once more arrive at the previous formula (6.9). We limit ourselves to the first inference with the sole aim of preserving the uniformity of the computational scheme. Using (6.4) and (6.8), we rewrite (6.9) in the form

$$u_-=\left(\sum\frac{\dot{a}_i}{u_+(a_i)}\frac{1}{(a_i-\lambda)}\right)^{\cdot}+\left(\sum\frac{\dot{a}_i}{u_+(a_i)}\frac{1}{(a_i-\lambda)}\right)^2$$

$$+2\left(\sum\frac{\dot{a}_i}{u_+(a_i)}\frac{1}{(a_i-\lambda)}\right)$$

$$\times\left(\dot{\tau}+\sum\frac{\dot{a}_i}{u_+(a_i)}\frac{u_+(a_i)-u_+(\lambda)}{(a_i-\lambda)}\right). \quad (6.10)$$

Now we proceed to the remaining transformations of the last expression. As we know, the matrix element u_- has no singularities at the points $\lambda=a_i$. For this reason the residues at these poles must vanish. This results in a system of equations of second order for the functions a_i. We shall simplify these equations somewhat later. Note that their first integrals are contained in (6.5), where the constants λ_i play the role of constants of integration. The terms without singularities in (6.10) give rise to the final expression for u_-:

$$u_-=-2\sum\frac{\dot{a}_i}{u_+(a_i)}\frac{\dot{\tau}(a_i)-\dot{\tau}(\lambda)}{(a_i-\lambda)}$$

$$+\sum\sum\frac{\dot{a}_i}{u_+(a_i)}\frac{\dot{a}_j}{u_+(a_j)}\frac{1}{(a_i-\lambda)}\left(\frac{u_+(a_i)-u_+(\lambda)}{a_i-\lambda}\right.$$

$$\left.-\frac{u_+(a_i)-u_+(a_j)}{a_i-a_j}\right). \quad (6.11)$$

Equations (6.4), (6.8), and (6.11) solve the problem posed at the beginning of this section. The elements of the matrix u for every rational $\tau(\lambda)$ are expressed uniformly and allow us to avoid the hard operation of division by the polynomial $\pi(\lambda)$ in the general expressions of the previous section. Each of the elements $u_\pm,u_0$ is expressed in the form of derivatives of some combinations of symmetrical functions constructed from the a_i, i.e., the coefficient functions of the polynomial $\tilde{P}_{n+1}$. All these coefficient functions are solutions of the linear system of algebraic equations (6.2). The form of the matrix u essentially depends on the form of the background function $\tau(\lambda)$, and in each concrete case they can be obtained only by direct calculations with the help of the formulas of this section.

To conclude, we write down the second-order equations for the functions a_i,b_j. These equations make it possible to establish numerous recurrence relations among the symmetrical combinations composed of the "roots" a_i,b_j and their derivatives. These formulas will play an essential role in the next section when we proceed to concrete examples of integrable systems and their solutions. The equations for the functions a_i,b_j are as follows:

$$\ddot{a}_i+2\dot{\tau}(a_i)\dot{a}_i+2\sum\frac{\dot{a}_i\dot{a}_k}{a_k-a_i}=\frac{\dot{u}_+(a_i)}{u_+(a_i)}\dot{a}_i,$$

$$\ddot{b}_j-2\dot{\tau}(b_j)\dot{b}_j+2\sum \frac{\dot{b}_j\dot{b}_k}{b_k-b_j}=\frac{\dot{u}_+(b_j)}{u_+(b_j)}\dot{b}_j. \qquad (6.12)$$

In the last equations $\dot{\tau}(a_i)\equiv\dot{\tau}(\lambda)$, and only after this $\lambda=a_i$, and so on. The systems of ordinary differential equations (6.12) are of some special interest. They are exactly integrable, and the result of their integration can be found from the solution of the system of algebraic linear equations (6.2), where a_i (b_j) are the roots of a polynomial with known coefficients. The first integrals of the systems are known and are contained in (6.5). Their independent integration can be performed on the background of the solutions of integrable systems. An example connected with the sine–Gordon equation will be considered in a later section.

7. CONCRETE EXAMPLES[8,10,13–16]

Now we will examine the most familiar systems and equations related to the algebra of rational functions of second-order matrices. The aim is to demonstrate the general formulas of the previous section and their application to concrete examples.

For simplicity, in the first examples we assume that our problem is invariant under the transformation $\lambda\to-\lambda$ of the spectral parameter. This means that the second solution of the spectral equation is obtained from the first one by the same transformation, i.e.,

$$\tau(-\lambda)=-\tau(\lambda), \quad N_1=N_2=N, \quad a_i=-b_i.$$

7.1. $\tau=\lambda z+\lambda^3\bar{z}$—the Korteveg–de Vries system

By z, $\bar{z}$ in this section we understand two independent variables of our problem. Differentiation with respect to the variable z will be denoted by a dot, and that with respect to the variable $\bar{z}$ by a prime. Equation (6.4) under these conditions takes the form

$$2\lambda\left(1+\sum \frac{\dot{a}_i}{a_i^2-\lambda^2}\right)=\frac{\Pi(\lambda^2-\lambda_i^2)}{\Pi(\lambda^2-a_i^2)}u_+(\lambda). \qquad (7.1)$$

Comparing the highest degree of λ in the last equality, we conclude that $u_+=2\lambda+c$. Taking $\lambda=0$, we have $c=0$, and so $u_+=2\lambda$. From the definition (6.8) we now have $u_0=\lambda+\Sigma(\dot{a}_i/a_i)=\lambda+\partial\ln\Pi a_i/\partial z$. From (6.11), $u_-=-\partial\ln\Pi a_i/\partial z$. Finally, for the matrix u we have

$$u=\begin{pmatrix}\lambda+\dot{\rho} & 2\lambda\\ -\dot{\rho} & -(\lambda+\dot{\rho})\end{pmatrix}, \qquad (7.2)$$

where $\exp\rho=\Pi a_i/\lambda_i$.

Now we proceed to the same calculation for the variable $\bar{z}$. We shall denote the corresponding matrix by v. Equation (6.4) in this case has the form

$$2\lambda\left(\lambda^2+\sum \frac{\dot{a}_i}{a_i^2-\lambda^2}\right)=\frac{\Pi(\lambda^2-\lambda_i^2)}{\Pi(\lambda^2-a_i^2)}v_+(\lambda).$$

Comparing, as above, the coefficients of the highest power of λ, we conclude that $v_+=2\lambda^3+c\lambda$. A more convenient way to calculate c is through the substitution of the ratio of the two polynomials in the last equality which follows from (7.1). We have

$$\left(\lambda^2+\sum \frac{\dot{a}_i}{a_i^2-\lambda^2}\right)=\left(\lambda^2+\frac{c}{2}\lambda\right)\left(1+\sum \frac{\dot{a}_i}{a_i^2-\lambda^2}\right),$$

from which we conclude that $c=2\Sigma\dot{a}_i$ and $v_+=2\lambda^3+2s_0$. In what follows, $s_n=\Sigma a_i^n\dot{a}_i$. For the other calculations we recall Eq. (6.5), from which it follows that

$$\frac{\dot{a}_i}{u_+(a_i)}=\frac{a_i'}{v_+(a_i)}.$$

Bearing these equations in mind, from (6.7) and (6.11) we obtain

$$v_0=\lambda^3+\lambda^2 s_{-1}+\lambda s_0+s_1+s_0 s_{-1},$$

$$v_-=-\left(\lambda^2 s_{-1}+\lambda\left(\frac{s_{-1}^2}{2}+s_0\right)+s_1+s_0 s_{-1}\right).$$

The second-order equations (6.12) in the case under consideration are

$$\ddot{a}_i+2a_i\dot{a}_i+2\sum \frac{\dot{a}_i\dot{a}_k}{a_k-a_i}=0.$$

Multiplying each equation by a_i and summing the result, after some calculations we find a recurrence relation for the functions s_n. For the case $n>0$ we have

$$\dot{s}_n+2s_{n+1}-\sum_{k=0}^{n-1} s_k s_{n-1-k}=0. \qquad (7.3)$$

From the same equations there follow recurrence relations also for the case when $n<-1$. At the boundary we have $2s_0+\dot{s}_{-1}+s_{-1}^2=0$. The recurrence relations allow one to express all the functions s_n in terms of the function s_{-1} and its derivatives up to the order $n+1$. For the matrix v we have

$$\begin{pmatrix}\lambda^3+\lambda^2 s_{-1}+\lambda s_0+s_1+s_0 s_{-1} & 2\lambda^3+2s_0\\ -\left(\lambda^2 s_{-1}+\lambda\left(\frac{s_{-1}^2}{2}+s_0\right)+s_1+s_0 s_{-1}\right) & -(\lambda^3+\lambda^2 s_{-1}+\lambda s_0+s_1+s_0 s_{-1}\end{pmatrix}.$$

We know that the Cartan–Maurer identity or the condition of compatibility is satisfied. This gives us the equation for the function $Q=s_{-1}$:

$$Q_t-\left(\frac{Q_{xx}}{4}-\frac{Q^3}{2}\right)_x=0 \quad (t\equiv\bar{z},\ x\equiv z). \qquad (7.4)$$

This is a modified Korteveg–de Vries equation. We know from our construction that its solution is given by the expression $Q=\partial\ln\Pi a_i/\partial z$. To find it in explicit form, it is necessary to solve a system of linear algebraic equations. In the case of the Korteveg–de Vries equation this system is as follows:

$$\exp 2(\lambda_s z+\lambda_s^3\bar{z})\prod (a_i-\lambda_s)+c(\lambda_s)\prod (a_i+\lambda_s)=0.$$

An explicit solution of this system leads to the solution of the modified Korteveg–de Vries equation in the form of the derivative of the logarithm of the ratio of two determinants of order n.

In all cases, when τ is odd and has a polynomial structure there are equations for only one function. These are (modified) Korteveg–de Vries equations of the highest order.

The next example is connected with the simplest case of such an equation. Let $\tau=\lambda z+(\nu\lambda^3+\mu\lambda^5)\bar{z}$. The matrix u remains the same as above. For the calculation of the matrix v we need some modifications. From (6.4) we conclude that $v_+=A\lambda^5+B\lambda^3+C\lambda$, and the parameters A,B,C occur in the equality

$$2\left(\nu\lambda^4+\mu\lambda^2+\sum\frac{a_i'}{a_i^2-\lambda^2}\right)$$
$$=(A\lambda^4+B\lambda^2+C\lambda)\left(1+\sum\frac{\dot{a}_i}{a_i^2-\lambda^2}\right),$$

which allows one to determine them. We get the elements of v in the form

$$v_+=2\nu\lambda^5+2(\mu+\nu s_0)\lambda^3+2(\nu(s_2+s_0^2)+\mu s_0)\lambda,$$
$$v_0=\nu\lambda^5+\nu s_{-1}\lambda^4+(\nu s_0+\mu)\lambda^3+(\nu(s_1+s_0s_{-1})+\mu s_{-1})\lambda^2+(\nu(s_2+s_0^2)+\mu s_0)\lambda+\mu(s_1+s_0s_{-1})+\nu(s_3+s_0s_1+s_{-1}s_2+s_0^2s_{-1}),$$
$$-v_-=\nu s_{-1}\lambda^4+\nu\left(s_0+\frac{s_{-1}^2}{2}\right)\lambda^3+\nu(s_1+s_0s_{-1})+\mu s_1)\lambda^2+\left(\nu\left(s_2+s_1s_{-1}+\frac{s_0^2}{2}+\frac{s_0s_{-1}^2}{2}\right)+\mu\left(s_0+\frac{s_{-1}^2}{2}\right)\right)\lambda+\nu(s_3+s_2s_{-1}+s_1s_0+s_0^2s_{-1})+\mu(s_1+s_0s_{-1}).$$

We need the following recurrence relations:

$$s_0=-\frac{\dot{s}_{-1}+s_{-1}^2}{2},\quad s_1=-\frac{\dot{s}_0}{2},\quad s_2=\frac{s_0^2-\dot{s}_1}{2},$$
$$s_3=-\frac{\dot{s}_2}{2}+s_0s_1.$$

Thus, we conclude that all the s_n can be expressed in terms of the function $Q\equiv s_{-1}$ and its derivatives. The condition of compatibility leads to the equation

$$Q_t+Q_{xxxxx}-10Q^2Q_{xxx}-40QQ_xQ_{xx}-10(Q_x)^2+30Q^4Q_x=0, \qquad (7.5)$$

which is a modified Korteveg–de Vries equation of fifth order (in our general formulas we put $\nu=1$, $\mu=0$).

When τ takes null values at the point $\lambda=0$, there is always a second possibility of constructing the background element g and, as a consequence, some other integrable system. We can assume that the two fundamental solutions of the spectral equation coincide when $\lambda=0$. This means that the Wronskian of the spectral equation vanishes at this point, and the background equation (6.4) becomes

$$2\lambda\left(1+\sum\frac{\dot{a}_i}{a_i^2-\lambda^2}\right)=\lambda\frac{\Pi(\lambda^2-\lambda_i^2)}{\Pi(\lambda^2-a_i^2)}u_+(\lambda).$$

From this we conclude that $u_+(\lambda)=2$. From the equalities (6.8) and (6.11) we find $u_0=\lambda$, $u_+=-s_0$, and thus the matrix u has the form

$$u=\begin{pmatrix}\lambda & 2\\ -s_0 & -\lambda\end{pmatrix},$$

where, as above, $s_0=\Sigma_{k=1}^n\dot{a}_k$. The system of second-order equations is the same as in the preceding case. All the recurrence relations remain unchanged, and for the elements of the matrix v we obtain

$$\begin{pmatrix}\lambda^3+\lambda s_0-\dfrac{\dot{s}_0}{2} & 2\lambda^2+2s_0\\ -\left(\lambda^2s_0+\lambda\dfrac{\dot{s}_0}{2}+s_0^2-\dfrac{\ddot{s}_0}{4}\right) & -\left(\lambda^3+\lambda s_0-\dfrac{\dot{s}_0}{2}\right)\end{pmatrix}.$$

The Maurer–Cartan identity leads to an equation for the function $U\equiv s_0/2$:

$$-U_t+U_{xxx}+6UU_x=0\quad\left(t=\frac{\bar{z}}{4}\right). \qquad (7.6)$$

This is the Korteveg–de Vries equation in its original form.

For any odd polynomial in λ, τ_0, we obtain higher-order Korteveg–de Vries equations. It is not difficult to write down the explicit form for these equations. We consider only the Korteveg–de Vries equation of fifth order.

Let $\tau=\lambda z+\lambda^5)\bar{z}$.

The matrix u is the same as before. We obtain the elements of the matrix v by the same technique as in the modified case:

$$u_+=2\lambda^4+2s_0\lambda^{+}2(s_2+s_0^2),$$
$$v_0=\lambda^5+s_0\lambda^3+s_1\lambda^2+(s_2+s_0^2)\lambda+s_3+s_0s_1,$$
$$-v_-=s_0\lambda^4+s_1\lambda^3+\left(s_2+\frac{s_0^2}{2}\lambda^2+(s_3+s_1s_0)\lambda+\left(s_4+s_2s_0+\frac{s_1^2}{2}+\frac{s_0^3}{2}\right)\right).$$

The condition of consistency gives the Korteveg–de Vries equation for the function $U\equiv-2s_0$:

$$-U_t+U_{xxxxx}-10UU_{xxx}-20U_xU_{xx}+30U^2U_x=0\quad\left(t=\frac{\bar{z}}{4}\right). \qquad (7.7)$$

It follows from our construction that the connection between solutions of the Korteveg–de Vries equation and its modified version is given by the equality $U=\dot{Q}+Q^2$, as a consequence of the recurrence relations for s_0,s_{-1} (7.3) of this section.

7.2. $\tau=\lambda z+\lambda^{-1}\bar{z}$—the sine–Gordon equation

The calculations of the matrix u do not change, and, as before we have

$$u=\begin{pmatrix}\lambda+\dot\rho & 2\lambda\\ -\dot\rho & -(\lambda+\dot\rho)\end{pmatrix},$$

where $\dot\rho=s_{-1}$. For the calculation of v_+ the general equation (6.4) gives

$$2\left(\lambda^{-1}+\lambda\sum\frac{a_i'}{a_i^2-\lambda^2}\right)=\frac{\Pi(\lambda^2-\lambda_i^2)}{\Pi(\lambda^2-a_i^2)}v_+(\lambda).$$

From the last equality it follows that v_+ has a simple pole at the point $\lambda=0$, with residue equal to $2\exp 2\rho$, where $\exp\rho=\Pi a_i/\lambda_i$. Thus, $v_+=2\lambda^{-1}\exp 2\rho$. With the help of Eqs. (6.4) and (6.8) we find $v_0=\lambda^{-1}\exp 2\rho$, $v_-=-\lambda^{-1}\sinh\rho$. It should be noted that in these calculations we use the relation $1+s_2=\exp(-\rho)$, which follows from the equation for u_+ if its two sides are divided by 2λ before setting $\lambda=0$, and we have

$$v=\lambda^{-1}\begin{pmatrix}\exp 2\rho & 2\exp 2\rho\\ \sinh 2\rho & -\exp 2\rho\end{pmatrix}.$$

The Cartan–Maurer identity gives the sine–Gordon equation for the function ρ:

$$\frac{\partial^2\rho}{\partial z\partial\bar z}=2\sinh 2\rho. \tag{7.8}$$

As in the case of the Korteveg–de Vries equation, the assumption that τ is an odd-order polynomial in λ^{-1} results in higher-order sine–Gordon equations. This is related to the fact that all moments of negative degree $s_{-n}=\Sigma a_i^{-n}\dot a_i$ are related to each other by the system of recurrence relations which follow from the second-order equations for the functions a_i.

7.3. $\tau=\lambda z+(\mu\lambda^{-1}+\lambda^3)\bar z$

This example is connected with an equation which in a sense is "intermediate" between the sine–Gordon and Korteveg–de Vries equations.

The matrix u is the same as in the previous examples. For v_+, Eq. (6.4) holds:

$$2\lambda\left(\lambda^2+\mu\lambda^{-2}+\sum\frac{a_i'}{a_i^2-\lambda^2}\right)=+\frac{\Pi(\lambda^2-\lambda_i^2)}{\Pi(\lambda^2-a_i^2)}v_+(\lambda).$$

We conclude that the maximal degree of v_+ at infinity is 3, it has a pole at the point $\lambda=0$, and its residue is $\mu\exp 2\rho$. Using the technique of the previous examples, we readily find the other parameters. Finally, we have

$$v_+=2(\lambda^3+s_0\lambda+\mu\exp 2\rho\lambda^{-1}),$$

$$v_0=\lambda^3+s_{-1}\lambda^2+s_0\lambda+s_1+s_0s_{-1}+\mu\exp 2\rho\lambda^{-1},$$

$$v_-=-\left(s_1\lambda^2+\left(\frac{s_{-1}^2}{2}+s_0\right)\lambda+s_1+s_0s_{-1}\right.$$
$$\left.+\mu\sinh 2\rho\lambda^{-1}\right).$$

It should be noted that in this case no calculations are necessary. It is sufficient to take linear combinations of the matrices v of the Korteveg–de Vries and sine–Gordon equations. The Cartan–Maurer identity leads to an equation for the function ρ:

$$4\frac{\partial^2\rho}{\partial z\partial\bar z}=\frac{\partial^4\rho}{\partial z^4}-6\frac{\partial^2\rho}{\partial z^2}\left(\frac{\partial\rho}{\partial z}\right)^2+8\mu\sinh 2\rho=0. \tag{7.9}$$

7.4. $-\tau=z\lambda+\bar z\lambda_{-1}+f$—the Lund–Pohlmeyer–Regge system

This is the first example of the general case in which τ has a null mode. The general equation (6.4) for u_+ now has the form

$$\left(-\sum_{j=1}^{n_2}\frac{\dot b_j}{\lambda-b_j}+\sum_{i=1}^{n_1+1}\frac{\dot a_i}{\lambda-a_i}+2\dot f+2\lambda\right)$$
$$=\left(\frac{\Pi_{s=11}^{n_1n_2+2}(\lambda-\lambda_s)}{\Pi_{i=1}^{n_1+1}(\lambda-a_i)\Pi_{j=1}^{n_2}(\lambda-b_j)}\right)u_+.$$

Comparing the highest powers of λ, we conclude that $u_+=2$. Equations (6.7) and (6.8) give $u_0=\lambda+\dot f$, $u_-=-\Sigma\dot a_i=-s_0$. The equations of second order in the case under consideration,

$$\ddot a_i+2(a_i+\dot f)\dot a_i+2\sum\frac{\dot a_i\dot a_k}{a_k-a_i}=0,$$

enable us to find recurrence relations for s_n. It will be sufficient to multiply each equation by a_i^{-1} and sum the result. In this way we obtain

$$\dot s_{-1}+2s_0+2\dot f s_{-1}+s_{-1}^2=0.$$

Equation (6.4), i.e.,

$$\left(-\sum_{j=1}^{n_2}\frac{b_j'}{\lambda-b_j}+\sum_{i=1}^{n_1+1}\frac{\dot a_i}{\lambda-a_i}+2f'+2\lambda^{-1}\right)$$
$$=\left(\frac{\Pi_{s=1}^{n_1+n_2+2}(\lambda-\lambda_s)}{\Pi_{i=1}^{n_1+1}(\lambda-a_i)\Pi_{j=1}^{n_2}(\lambda-b_j)}\right)v_+,$$

determines $v_+=2f'\lambda^{-1}$.

It follows directly from (6.7) and (6.8) that

$$v_0=f'+(1-f's_{-1})\lambda^{-1}\quad v_-=\frac{1}{2}(2s_{-1}-f's_{-1}^2)\lambda^{-1}.$$

Let us now perform a gauge transformation (see Sec. 4) with $g_0=\exp Hf$. The aim of such a transformation is to remove the derivatives of f from the elements of u_0,v_0. After the introduction of the new variables

$$x=\exp(-2f),\quad y=s_{-1}\exp 2f$$

the matrices u,v acquire a simple and suitable form:

$$u=\begin{pmatrix}\lambda & 2x\\ \dfrac{\dot y+y^2x}{2} & -\lambda\end{pmatrix},$$

$$v=\lambda^{-1}\begin{pmatrix} 1+\frac{yx'}{2} & -x' \\ y+\frac{y^2x'}{2} & -\left(1+\frac{yx'}{2}\right) \end{pmatrix}.$$

The consistency condition leads to the integrable system

$$\frac{\partial^2 x}{\partial z\partial\bar z}-4x-2(xy)\frac{\partial x}{\partial\bar z}=0,\quad \frac{\partial^2 y}{\partial z\partial\bar z}-4y+2(xy)\frac{\partial y}{\partial\bar z}=0. \tag{7.10}$$

If one makes the transformation (motivated by the form of the matrix v)

$$1+\frac{yx'}{2}=\cos\alpha,\quad x'=-\sin\alpha\exp\theta,$$

i.e., introduces the new pair of variables $(x,y)\to(\alpha,\theta)$ and then the pair (α,β) in accordance with the formulas

$$\theta_z=\frac{\beta_z\cos\alpha}{1+\cos\alpha},\quad \theta_{\bar z}=\frac{\beta_{\bar z}}{1+\cos\alpha},$$

then in the variables (α,β) the system under investigation takes the form

$$\alpha_{z\bar z}+4\sin\alpha-\frac{\sin\frac{\alpha}{2}}{2\left(\cos\frac{\alpha}{2}\right)^3}\beta_z\beta_{\bar z}=0,$$

$$\beta_{z\bar z}+\frac{\alpha_z\beta_{\bar z}+\alpha_{\bar z}\beta_z}{\sin\alpha}=0.$$

This is the Lund–Pohlmeyer–Regge system in its canonical form. It is connected with a certain geometrical construction.

7.5. $\tau=-(\lambda z+\lambda^2\bar z+f)$—the nonlinear Schrödinger equation

The matrix u is evidently the same as in the last case. From the general equation (6.4) we conclude (from its behavior at infinity) that $v_+=2\lambda+c$. To find c, it is suitable to take the ratio of two polynomials in the expression for v_+ from the definition of u_+, i.e.,

$$\left(-\sum\frac{b_j'}{\lambda-b_j}+\sum\frac{a_i'}{\lambda-a_i}+2f'+2\lambda^2\right)=$$

$$\left(-\sum\frac{\dot b_j}{\lambda-b_j}+\sum\frac{\dot a_i}{\lambda-a_i}+2\dot f+2\lambda\right)\left(\lambda+\frac{c}{2}\right).$$

Comparing the highest powers of λ, we conclude that $c=-2f'$. No difficulties arise in the calculations of the elements of v_0,v_-, and finally we have

$$v_+=2\lambda-2\dot f,\quad v_0=\lambda^2+f'+s_0,$$

$$v_-=-\lambda s_0+\frac{\dot s_0}{2}+\dot f s_0.$$

Making a gauge transformation which removes the derivatives of f from the elements of v_0,u_0, and introducing the new variables

$$r=2\exp(-2f),\quad q=-s_0\exp(2f),$$

we bring the matrices u,v to the form

$$u=\begin{pmatrix}\lambda & r\\ q & -\lambda\end{pmatrix},\quad v=\begin{pmatrix}\lambda^2-\frac{qr}{2} & \lambda r+\frac{\dot r}{2}\\ \lambda q-\frac{\dot q}{2} & -\left(\lambda^2-\frac{qr}{2}\right)\end{pmatrix}.$$

The consistency condition gives the integrable system

$$r'-\frac{\ddot r}{2}+(qr)r=0,\quad q'-\frac{\ddot q}{2}-(qr)q=0. \tag{7.11}$$

This is a nonlinear Schrödinger equation without derivatives. In the next section we will need an explicit form of its solutions.

Following the rules of Kramer, from the system of linear algebraic equations we get

$$\exp(-2f)=\frac{(\exp 2\tau,\ \exp 2\tau\lambda,\ldots,\exp 2\tau\lambda^{n_1+1};1,\lambda,\ldots,\lambda^{n_2-1})}{(\exp 2\tau,\exp 2\tau\lambda,\ldots,\exp 2\tau\lambda^{n_1};1,\lambda,\ldots,\lambda^{n_2})},$$

$$\sum a_i=\frac{(\exp 2\tau,\ldots,\exp 2\tau\lambda^{n_1-1},\exp 2\tau\lambda^{n_1+1};1,\lambda,\ldots,\lambda^{n_2})}{(\exp 2\tau,\exp 2\tau\lambda,\ldots,\exp 2\tau\lambda^{n_1};1,\lambda,\ldots,\lambda^{n_2})}.$$

From the last equality, by induction, we obtain an expression for $s_0=\Sigma\dot a_i$:

$$s_0=-2\frac{(\exp 2\tau,\exp 2\tau\lambda,\ldots,\exp 2\tau\lambda^{n_1-1};1,\lambda,\ldots,\lambda^{n_2+1})}{(\exp 2\tau,\exp 2\tau\lambda,\ldots,\exp 2\tau\lambda^{n_1};1,\lambda,\ldots,\lambda^{n_2})}\times\exp(-2f).$$

By $(a,b,\ldots,c)$ in the previous expressions (as in Sec. 4) we denote the nth-order determinant (n is the number of elements $a,b,\ldots,c$) whose sth row consists of the elements $(a_s,b_s,\ldots,c_s)$. The index s labels the points of the λ plane at which the columns of the polynomial matrix are linearly dependent. Finally, for the quantities of interest we obtain

$$r=2\frac{(\exp 2\tau,\exp 2\tau\lambda,\ldots,\exp 2\tau\lambda^{n_1+1};1,\lambda,\ldots,\lambda^{n_2-1})}{(\exp 2\tau,\exp 2\tau\lambda,\ldots,\exp 2\tau\lambda^{n_1};1,\lambda,\ldots,\lambda^{n_2})},$$

$$q=2\frac{(\exp 2\tau,\exp 2\tau\lambda,\ldots,\exp 2\tau\lambda^{n_1-1};1,\lambda,\ldots,\lambda^{n_2+1})}{(\exp 2\tau,\exp 2\tau\lambda,\ldots,\exp 2\tau\lambda^{n_1};1,\lambda,\ldots,\lambda^{n_2})}.$$

7.6. $\tau=\Sigma\xi_k/(\lambda-\theta_k)$—the principal chiral-field problem in n dimensions

The notation in the heading of this subsection should be understood as follows: ξ_k are the coordinates of the nth-order space, and θ_k is the totality of arbitrary c-number parameters ($\theta_k\neq\theta_s$ if $k\neq s$). In this case $\tau\to 0$ when $\lambda\to\infty$, and so τ has no null mode. The matrix of the polynomials in this case must be taken in the form

$$P=\begin{pmatrix}\tilde P_{n_1+1} & \tilde P_{n_2+1}\\ P_{n_1} & \tilde Q_{n_2+1}\end{pmatrix}.$$

Equation (6.4) can be written in the form

$$-\sum_{j=1}^{n_2+1} \frac{\dot b_j}{\lambda-b_j}+\sum_{i=1}^{n_1+1} \frac{\dot a_i}{\lambda-a_i}+\frac{2}{\lambda-\theta_k})$$
$$=\left(\frac{\Pi_{s=1}^{n_1+n_2+2}(\lambda-\lambda_s)}{\Pi_{i=1}^{n_1+1}(\lambda-a_i)\Pi_{j=1}^{n_2}(\lambda-b_j)}\right)u_+$$

$(\dot f\equiv\partial f/\partial\xi_k)$. From the last equation we conclude that $u_+^k=A_k/(\lambda-\theta_k)$. In the limit $\lambda\to\infty$, one finds

$$A_k=\sum \dot a_i-\sum \dot b_j+2=\frac{\partial}{\partial\xi_k}\left(\sum a_i-\sum b_j\right.$$
$$\left.+2\sum \xi_r\right).$$

The calculation of u_0^k,u_-^k is carried out according to the general scheme and results in the expressions

$$u_0^k=\frac{1-s_0}{\lambda-\theta_k}=\frac{\frac{\partial}{\partial\xi_k}(\Sigma\xi_r-\Sigma a_i)}{\lambda-\theta_k},$$

$$u_-^k=\frac{2s_0-s_0^2}{A_k(\lambda-\theta_k)}=\frac{\frac{\partial}{\partial\xi_k}\left(\frac{\theta_k s_0-s_1}{A_k}\right)}{\lambda-\theta_k}.$$

The last equality needs some explanation. In the case under investigation, the system of second-order equations for the functions a_i is as follows:

$$\ddot a_i+2\frac{\dot a_i}{a_i-\theta_k}+2\sum_{i\neq k}\frac{\dot a_i\dot a_k}{a_k-a_i}=\frac{\dot A_k}{A_k}\dot a_i.$$

Multiplying each equation of the system by $(a_i-\theta_i)$ and summing the results, we have

$$\dot s_1-\frac{\dot A_k}{A_k}s_1-\theta_k\left(\dot s_0-\frac{\dot A_k}{A_k}s_0\right)=s_0^2-2s_0$$

or

$$\frac{\partial}{\partial\xi_k}\left(\frac{s_1-\theta_k s_0}{A_k}\right)=\frac{s_0^2-2s_0}{A_k}.$$

This is just the same equation as the one used in the transformation of the expression for u_-.

We propose also another deduction, which permits us to obtain expressions for u_+,u_-,u_0 in a different form. From the direct definition of the matrix u, for the element u_0 we have

$$u_0^k=\frac{\mathrm{Det}\begin{pmatrix}\dot P_{n_1+1}+\frac{P_{n_1+1}}{\lambda-\theta_k} & \dot P_{n_2+1}-\frac{P_{n_2+1}}{\lambda-\theta_k}\\ P_{n_1} & Q_{n_2+1}\end{pmatrix}}{\Pi_{s=1}^{n_1+n_2+2}(\lambda-\lambda_s)}=\frac{1}{\lambda-\theta_k}$$
$$\times\frac{\mathrm{Det}\begin{pmatrix}(\lambda-\theta_k)\dot P_{n_1+1}+P_{n_1+1} & (\lambda-\theta_k)\dot P_{n_2+1}-P_{n_2+1}\\ P_{n_1} & Q_{n_2+1}\end{pmatrix}}{\Pi_{s=1}^{n_1+n_2+2}(\lambda-\lambda_s)}.$$

We know that in the last equation the numerator is divided by the denominator, and so it is only necessary to find the coefficient of $\lambda^{n_1+n_2+2}$ in the determinant of the numerator. This can be done easily. The result is

$$u_0=\frac{1+(\tilde P_{n_1+1}^{n_1})_{\xi_k}}{\lambda-\theta_k}.$$

The upper index on the symbol for the polynomial indicates the coefficient function of the corresponding power of λ. Here $\tilde P_{n_1+1}^{n_1}=-\Sigma a_i$, and so we again obtain the expression derived above. In the same manner,

$$u_+^k=\frac{(\tilde P_{n_2+1}^{n_2})_{\xi_k}-(\tilde P_{n_1+1}^{n_1})_{\xi_k}+2}{\lambda-\theta_k},\quad u_-^k=\frac{(P^{n_1})_{\xi_k}}{\lambda-\theta_k}.$$

If the first expression is the same as before, then in the second one the coefficient $P_{n_1}^{n_1}$ arises from the solution of a system of linear algebraic equations for the second row of the polynomial matrix. This is the explicit result of summation in the previous expression for u_-^k. Finally, we have $u^k=\partial F/\partial\xi_k$, and the condition of compatibility gives us an equation for the matrix-valued function F:

$$(\theta_i-\theta_j)\frac{\partial^2F}{\partial\xi_i\partial\xi_j}=\left[\frac{\partial^2F}{\partial\xi_i},\frac{\partial^2F}{\partial\xi_j}\right].$$

This is the system of equations of the principal chiral-field problem in the n-dimensional space, which plays an important role in the general theory of self-dual equations.

8. SOLUTION OF THE GENERAL EQUATION IN THE SOLVABLE CASE AND THE DISCRETE TRANSFORMATION[17,18]

Here we shall show that the equation for the element g has a regular exact solution not only if, as is proposed, the element g belongs to a commutative group, but also if it belongs to a solvable one [diagonal plus upper (lower) triangular matrices].

We shall consider this situation for the example the $SL(2,R)$ group (algebra):

$$\frac{\partial g}{\partial x_i}=u^ig,\quad g=\exp x^+\alpha\exp H\tau,$$

or

$$u^i=\tau_{\xi_i}H+(\alpha_{\xi_i}-2\alpha\tau_{\xi_i})X^+,$$

where the element u^i belongs to a solvable algebra and must have definite analytic properties as a function of the spectral parameter λ in the complex plane. The solution of this problem for τ is the same as in the diagonal case: $\tau(\lambda)$ is a rational function, and the residue at each of its poles is a function of only one variable ξ_i. For α we obtain

$$\alpha=P(\lambda)\int_C d\lambda'\,\frac{\alpha(\lambda')}{\lambda-\lambda'}\exp 2\tau(\lambda),$$

where $\alpha(\lambda)$ is an arbitrary function of one variable, and C is some circle in the λ plane, on which the integral is defined; $P(\lambda)$ is some polynomial. Indeed,

$$u^i=\alpha_{\xi_i}-2\alpha\tau_{\xi_i}$$
$$=P(\lambda)\int d\lambda'\alpha(\lambda')\frac{\tau_{\xi_i}(\lambda)-\tau_{\xi_i}(\lambda')}{\lambda-\lambda'}\exp 2\tau(\lambda').$$

It follows from the last expression that u_+ has the same singularities in the finite λ plane as $\tau(\lambda)$. At infinity, if $\tau(\lambda)\sim\lambda^s$ and $P(\lambda)\sim\lambda^t$, then $u_+\sim\lambda^{s+t-1}$. The situation for solvable groups of higher dimensions is the same, and it is not difficult to obtain a solution and explicit formulas in that case. Yet this is not so important for our purposes.

Now we take the solution for the solvable case as the element g_0 in our general construction of Sec. 5. In this way we obtain solutions which depend on an arbitrary function, the definite choice of which allows us to find solutions with definite boundary conditions, to solve the reduction problem, and so on.

Let us consider this question more carefully. The matrix of the polynomials is taken in the usual form:

$$P=\begin{pmatrix}\tilde{P}_{n_1+1} & \tilde{P}_{n_2}\\ P_{n_1} & \tilde{P}_{n_2+1}\end{pmatrix}.$$

The condition of linear dependence of the columns of the matrix $g=Pg_0$ has the form (we write it only for the first row)

$$\tilde{P}_{n_1+1}\exp\tau+c\exp-\tau[\tilde{P}_{n_2}+\alpha\tilde{P}_{n_1+1}]=0$$

or

$$\tilde{P}_{n_1+1}\left(\exp 2\tau(\lambda)c^{-1}(\lambda)+P(\lambda)\int d\lambda'\frac{\alpha(\lambda')}{\lambda-\lambda'}\exp 2\tau(\lambda')\right)+\tilde{P}_{n_2}=0.$$

From the last equation we see that there is only one difference as compared with the diagonal case (α=0). That is a formal substitution in all the formulas:

$$\exp 2\tau(\lambda)c^{-1}(\lambda)\to\exp 2\tilde{\tau}(\lambda)\equiv\exp 2\tau(\lambda)c^{-1}(\lambda)+P(\lambda)\int d\lambda'\frac{\alpha(\lambda')}{\lambda-\lambda'}\exp 2\tau(\lambda')\equiv\exp 2\tau'(\lambda). \tag{8.1}$$

To have the correct behavior of u^i at infinity one must require that $n_2>n_1+1$. All the formulas of Secs. 6 and 7 remain correct under the substitution (8.1). We have a hierarchy of solutions of the investigated system, which depend on the arbitrary function $\alpha(\lambda)$.

These solutions can be related to the solutions of the diagonal case by a certain limiting process. We explain this for the example of the nonlinear Schrödinger equation. From the results of the corresponding subsection we have the explicit form of its solutions:

$$r_{n_1,n_2}=\frac{(\exp 2\tau,\exp 2\tau\lambda,...,\exp 2\tau\lambda^{n_1+1};1,\lambda,...,\lambda^{n_2-1})}{(\exp 2\tau,\exp 2\tau\lambda,...,\exp 2\tau\lambda^{n_1};1,\lambda,...,\lambda^{n_2})},$$

$$q_{n_1,n_2}=\frac{(\exp 2\tau,\exp 2\tau\lambda,...,\exp 2\tau\lambda^{n_1-1};1,\lambda,...,\lambda^{n_2+1})}{(\exp 2\tau,\exp 2\tau\lambda,...,\exp 2\tau\lambda^{n_1};1,\lambda,...,\lambda^{n_2})}. \tag{8.2}$$

Let n_1 have fixed values and $n_2\to\infty$ (more precisely, $n_2=n_1+1+N$, $N\to\infty$).

First of all, we prove the following equality:

$$(\exp 2\tau,\exp 2\tau\lambda,...,\exp 2\tau\lambda^{n_1};1,\lambda,...,\lambda^{n_2})=W(\lambda_1,\lambda_2,...,\lambda_{n_1+n_2})\sum_{i,j,..k}^{n_1}\Phi(\lambda_i)\Phi(\lambda_j)...\Phi(\lambda_k)\times W^2(\lambda_i,\lambda_j,...,\lambda_k), \tag{8.3}$$

where $W(\lambda_l\lambda_n...\lambda_m)$ is a Vandermond determinant and $\Phi(\lambda)$ is defined by the expression

$$\Phi(\lambda_i)=\exp 2\tau(\lambda_i)\prod_{k\neq i}^{n_1+n_2}(\lambda_k-\lambda_i)^{-1}.$$

Let n_1=1. Expanding the determinant with respect to the elements of the first column, we obtain

$$(\exp 2\tau;1,\lambda,...,\lambda^{n_2})=\sum_{s=1}^{n_2+1}\exp 2\tau(\lambda_s)(-1)^{s+1}W'(\lambda_l\lambda_n....\lambda_m)=W(\lambda_1,\lambda_2,...,\lambda_{n_2+1})\sum_{s=1}^{n_2+1}\Phi(\lambda_s),$$

where W' is the Vandermond determinant constructed from the n_2 values of λ_i except λ_s. Let n_1=2. Expanding the determinant with respect to the minors of the two first columns, we have

$$(\exp 2\tau,\exp 2\tau\lambda;1,\lambda,...,\lambda^{n_2})=\sum_{s,k}^{n_2+2}(-1)^{s+k}\exp 2\tau(\lambda_s)\exp 2\tau(\lambda_k)(\lambda_s-\lambda_k),$$

$$W''(\lambda_l\lambda_n...\lambda_m)=W(\lambda_1,\lambda_2,...,\lambda_{n_2+2})\sum_{s,k}^{n_2+2}\Phi(\lambda_s)\Phi(\lambda_k)\times(\lambda_s-\lambda_k)^2.$$

In the case of arbitrary n_1 the expansion of the determinant with respect to the minors of its first n_1 columns and some regrouping of the factors under the summation sign prove the validity of the proposition.[8,44]

Now let us return to the expressions (8.2), use Eq. (8), and take the limit $n_2\to\infty$. Then for the values

$$q_n=q_{n,\infty},\quad r_n=r_{n,\infty}$$

we obtain

$$q_n=\frac{\Theta_{n+1}}{\Theta_n},\quad r_n=\frac{\Theta_{n-1}}{\Theta_n}, \tag{8.4}$$

where

$$\Theta_n=\int...\int d\lambda_1...d\lambda_n$$

$$\times \exp 2\tau(\lambda_1)\ldots\exp 2\tau(\lambda_n)W^2(\lambda_1,\lambda_2,\ldots,\lambda_{n_2}).$$

This is just the same result as that obtained if for g_0 we select the element of the solvable group of the upper triangular matrix at the beginning of this section.

Now we shall draw some conclusions from the last expressions for the solutions of the system of nonlinear Schrödinger equations, which after some obvious transformations can be written in the form

$$r' - \ddot{r} + 2(qr)r = 0, \quad q' - \ddot{q} - 2(qr)q = 0.$$

From (8.4) we see that

$$r_{n+1} = \frac{\Theta_n}{\Theta_{n+1}} = \frac{1}{q_n}.$$

Let us assume that the system under investigation is invariant under the substitution $R = 1/q$, $Q = ?$. Then from the first equation we get $Q = q(qr - \ln q)$. By a direct check we see that the second equation is also satisfied. Thus, we conclude that our system is invariant under the transformation

$$R = \frac{1}{q}, \quad Q = q(qr - \ln q),$$

which we will call the discrete transformation for this system.

In the next sections we shall see that the integrable systems under consideration have their own discrete transformation, and by solving them one can find a large class of solutions of an integrable system, including solutions of soliton type. Thus, the independent construction of discrete transformations opens up a new, more direct method for solving integrable systems.

9. DISCRETE TRANSFORMATION FOR THE MAIN CHIRAL-FIELD PROBLEM[19]

In this section, for the example of the main chiral-field problem, we propose a direct method for the construction of the discrete transformation, using only the form of the equations of the integrable system.

As was mentioned in Sec. 7, the system of equations of the main chiral-field problem in n-dimensional space has the form

$$(\theta_i - \theta_j) \frac{\partial^2 f}{\partial \xi_i \partial \xi_j} = \left[\frac{\partial f}{\partial \xi_i}, \frac{\partial f}{\partial \xi_j} \right], \tag{9.1}$$

where the function f takes values in an arbitrary semisimple algebra, and θ_i are numerical parameters.

First of all, we describe in detail the calculations for the algebra A_1, which allow us to simplify the calculations for the general case. Let $f = X^+ f_+ + H f_0 + X^- f_-$ (here $[X^+, X^-] = H$, $[H, X^\pm] = \pm 2X^\pm$) be some solution of (9.1), and $F = X^+ F_+ + H F_0 + X^- F_-$ be a solution of (9.1) which is related to f via the discrete transformation. The explicit form of this transformation will be given below.

Let $F_- = 1/f_+$. This suggestion is not accidental, but comes from the explicit form of the soliton solutions to the main chiral-field problem (Sec. 7). From (9.1) we have the following equation for F_-:

$$(\theta_i - \theta_j) \frac{\partial^2 F_-}{\partial x_i \partial x_j} = -2 \left\{ \frac{\partial F_0}{\partial x_i} \frac{\partial F_-}{\partial x_j} - \frac{\partial F_0}{\partial x_j} \frac{\partial F_-}{\partial x_i} \right\}.$$

Substituting $F_- = 1/f_+$ into the last equation and using the equation for f_+ which follows from (9.1), we find for $F^* = F_0 f_+$ the equation

$$\frac{\partial F^*}{\partial x_i} = f_0 \frac{\partial f_+}{\partial x_i} - \frac{\partial f_0}{\partial x_i} f_+ + \theta_i \frac{\partial f_+}{\partial x_i}. \tag{9.2}$$

The second mixed derivatives calculated from (9.2) are equal by virtue of (9.1). Thus, for the derivatives of F_0 we have

$$\frac{\partial F_0}{\partial x_i} = (f_0 - F_0 + \theta_i) \frac{\partial}{\partial x_i} \ln f_+ - \frac{\partial f_0}{\partial x_i}. \tag{9.3}$$

Substituting (9.3) into the zeroth component of Eq. (9.1), we arrive at

$$\frac{\partial F_+}{\partial x_i} = (f_0 - F_0 + \theta_i)^2 \frac{\partial f_+}{\partial x_i} - 2 f_+ (f_0 - F_0 + \theta_i) \frac{\partial f_0}{\partial x_i} - f_+^2 \frac{\partial f_-}{\partial x_i}. \tag{9.4}$$

Finally, substituting (9.3) and (9.4) into the "positive" component of the system (9.1) for F, we conclude that the corresponding equations are satisfied identically. Let us rewrite the relations (9.3) and (9.4) in the matrix form

$$\frac{\partial F}{\partial x_i} = \exp[-X^+(f_0 - F_0 + \theta_i) f_+] \times \exp[H \ln f_+] r \frac{\partial f}{\partial x_i} r^{-1} = \exp[-H \ln f_+] \exp[X^+(f_0 - F_0 + \theta_i) f_+], \tag{9.5}$$

where r is an automorphism of the algebra A_1 with the properties $rX^\pm r^{-1} = -X^\mp$, $rHr^{-1} = -H$. In what follows rfr^{-1} will also be denoted by f for brevity. We define the element S with values in the $SL(2,R)$ group:

$$S = \exp[-X^+(f_0 - F_0) f_+] \exp H \ln f_+ .$$

By direct calculation we see that

$$S^{-1} \frac{\partial S}{\partial x_i} = \frac{1}{f_+} \left[\frac{\partial f}{\partial x_i}, X^+ \right] - \theta_i \frac{\partial}{\partial x_i} \frac{1}{f_+} X^+. \tag{9.6}$$

In terms of S, Eq. (9) can be rewritten in the form

$$\frac{\partial F}{\partial x_i} = S \frac{\partial f}{\partial x_i} S^{-1} + \theta_i \frac{\partial S}{\partial x_i} S^{-1}. \tag{9.7}$$

Equations (9.6) and (9.7), being equivalent to Eqs. (9.3) and (9.4), realize the discrete transformation for the system (9.1) in a form which can be generalized to the case of an arbitrary semisimple Lie algebra.

In the case of any semisimple Lie algebra for an element f, which takes values in it and obeys the system (9.1), the following statement holds:

There exists an element S taking the values in the gauge group such that

$$S^{-1}\frac{\partial S}{\partial x_i}=\frac{1}{f_-}\left[\frac{\partial f}{\partial x_i},X_M^+\right]-\theta_i\frac{\partial}{\partial x_i}\frac{1}{f_-}X_M^+. \tag{9.8}$$

Here X_M^+ is the element of the algebra corresponding to its maximal root divided by its norm, i.e., $[X_M^+,X^-]=H$, $[H,X^\pm]=\pm 2X^\pm$; f_- is the coefficient function in the decomposition of f for the element corresponding to the minimal root of the algebra. To prove the statement it is necessary to verify that the Cartan–Maurer identity is satisfied. After substituting (9.8) into this identity, bearing in mind the definitions of f_- and X_M^+ given above, we obtain the following expression, which should vanish:

$$\left[\frac{\partial f_-}{\partial x_j}\frac{\partial f}{\partial x_i}-\frac{\partial f_-}{\partial x_i}\frac{\partial f}{\partial x_j},X_M^+\right]+(\theta_i-\theta_j)\frac{\partial^2 f_-}{\partial x_i\partial x_j}X_M^+$$
$$-\left[\left[\frac{\partial f}{\partial x_j},X_M^+\right],\left[\frac{\partial f}{\partial x_i},X_M^+\right]\right].$$

This fact becomes obvious if one commutes Eq. (9.1) twice with the element X_M^+. Now we define the element F taking values in the algebra by the following relations:

$$\frac{\partial F}{\partial x_i}=S\frac{\partial f}{\partial x_i}S^{-1}+\theta_i\frac{\partial S}{\partial x_i}S^{-1}. \tag{9.9}$$

To prove the consistency of (9.9), we compare the second derivatives of F:

$$S^{-1}\left(\frac{\partial^2 F}{\partial x_i\partial x_j}-\frac{\partial^2 F}{\partial x_j\partial x_i}\right)S$$
$$=\left[S^{-1}\frac{\partial S}{\partial x_j},\frac{\partial f}{\partial x_i}\right]-\left[S^{-1}\frac{\partial S}{\partial x_i},\frac{\partial f}{\partial x_j}\right]$$
$$+\theta_i\frac{\partial}{\partial x_i}\left(S^{-1}\frac{\partial S}{\partial x_j}\right)$$
$$+\frac{\partial}{\partial xj}\left(S^{-1}\frac{\partial S}{\partial xi}\right)$$
$$=\frac{1}{f_-}\left[\left\{(\theta_i-\theta_j)\frac{\partial^2 f}{\partial x_i\partial x_j}-\left[\frac{\partial f}{\partial x_i},\frac{\partial f}{\partial x_j}\right]\right\},X_M^+\right]$$
$$=0.$$

In the same way, using (9.9), we arrive at the following equality:

$$(\theta_i-\theta_j)\frac{\partial^2 F}{\partial x_i\partial x_j}-\left[\frac{\partial F}{\partial x_i},\frac{\partial F}{\partial x_j}\right]$$
$$=S\left\{(\theta_i-\theta_j)\frac{\partial^2 f}{\partial x_i\partial x_j}-\left[\frac{\partial f}{\partial x_i},\frac{\partial f}{\partial x_j}\right]\right\}S^{-1}.$$

This means that (9.8) and (9.9) realize the discrete transformation for the main chiral-field problem in the case of an arbitrary semisimple Lie algebra.

As a direct consequence of (9.8) and (9.9) one gets the discrete transformation for the two-dimensional main chiral-problem with moving poles:

$$(\xi-\bar\xi)\frac{\partial^2}{\partial\xi\partial\bar\xi}f=\left[\frac{\partial}{\partial\xi}f,\frac{\partial}{\partial\bar\xi}f\right]. \tag{9.10}$$

In this case the relations (9.8) and (9.9) realizing the discrete transformation are changed as follows:

$$S^{-1}\frac{\partial}{\partial\xi}S=\frac{1}{f_-}\left[\frac{\partial}{\partial\xi}f,X_M^+\right]-\xi\frac{\partial}{\partial\xi}\frac{1}{f_-}X_M^+,$$
$$S^{-1}\frac{\partial}{\partial\bar\xi}S=\frac{1}{f_-}\left[\frac{\partial}{\partial\bar\xi}f,X_M^+\right]-\bar\xi\frac{\partial}{\partial\bar\xi}\frac{1}{f_-}X_M^+, \tag{9.11}$$

and the discrete transformation itself takes the form

$$\frac{\partial}{\partial\xi}F=S\left(\frac{\partial}{\partial\xi}f\right)S^{-1}-\xi\frac{\partial S}{\partial\xi}S^{-1},$$
$$\frac{\partial}{\partial\bar\xi}F=S\left(\frac{\partial}{\partial\bar\xi}f\right)S^{-1}-\bar\xi\frac{\partial S}{\partial\bar\xi}S^{-1}. \tag{9.12}$$

10. LIST OF DISCRETE TRANSFORMATIONS FOR INTEGRABLE SYSTEMS[20]

Now we present the discrete transformation and its integration (in some manner) for the most widely known and applicable integrable systems. Here, the discrete transformation plays the role of a nonlinear mapping which translates any given solution into another one. However, we do not investigate the properties of the transformation, its geometrical interpretation (if any), etc. So far, we have not used the general method for the construction of the transformation in question. General properties of the discrete transformation together with the system of equations which determine it will be considered in a later section. To prove the validity of all formulas of this section, one can make a direct check which uses only one operation—differentiation.

As a hint for obtaining the discrete transformation it is possible to use a purely algebraic method for the construction of the soliton-type solutions, which is given in Secs. 6–8.

The starting point of the construction given below uses the following two facts. The integrable systems under consideration admit the transformation s:

$$\theta\Rightarrow\tilde\theta\equiv s\tilde\theta=F(\theta,\theta_i^{(1)}\ldots\theta_i^n),\quad S^N\neq 1.$$

Here θ and $\tilde\theta$ are unknown functions (variables) satisfying the corresponding partial differential equations $\theta_i^s\equiv\partial^s\theta/\partial x_i^s$.

There is an obvious solution of the nonlinear system in question, which depends on a set of arbitrary functions. The soliton-type solutions, reductions related to discrete groups, and solutions with definite boundary conditions are defined by a special choice of the arbitrary functions mentioned above. Let us note that θ_0 is a solution of a system of linear partial differential equations and can be represented as a parametric integral in the plane of the complex variable λ.

In the case of integrable systems this circumstance is just the main reason for applying the methods of the theory of functions of complex variables, the technique of the Riemann problem, and the methods of the inverse scattering problem. The results of this section reduce the inverse scattering method to a simple technical rule.

Here we give a list of integrable systems together with the discrete transformations for them and the corresponding solutions.

10.1. Hirota equation

$$\dot v+\alpha(v'''-6uvv')-i\beta(v''-2v^2u)+\gamma v'+i\delta v=0,$$

$$\dot u+\alpha(u'''-6uvu')+i\beta(u'-2u^2v)+\gamma u'-i\delta u=0;$$

$$\cdot\equiv\frac{\partial}{\partial t},\quad '\equiv\frac{\partial}{\partial x};$$

$$\tilde v\equiv sv=\frac{1}{u},\quad \tilde u\equiv su=u(uv-(\ln u))'',\quad v_0=0,$$

$$\dot u_0+\alpha u_0'''+i\beta u_0''+\gamma u_0'-i\delta u_0=0. \tag{10.1}$$

In this and in the other cases the main role will be played by the principal minors of the following matrix:

$$\begin{pmatrix} \phi^s & \phi^{s+1} & \phi^{s+2} & \dots \\ \phi^{s+1} & \phi^{s+2} & \phi^{s+3} & \dots \\ \phi^{s+2} & \phi^{s+3} & \phi^{s+4} & \dots \\ \dots & \dots & \dots & \dots \end{pmatrix}.$$

The principal minors of these matrices will be denoted by the symbol D_r^n. Here n is the rank of the matrix, and r is the symbol of its element at the left upper corner. For the solution of the discrete transformation in the case of the Hirota equation we obtain

$$v_n=(-1)^n\frac{D_0^{n-1}}{D_0^n},\quad u_n=(-1)^{n+1}\frac{D_0^{n+1}}{D_0^{n+1}}. \tag{10.2}$$

The methods of the theory of functions of complex variables lead to the same expression, with D_0^n given by the nonlocal integral

$$D_0^n=\int d\lambda_1\dots d\lambda_n c(\lambda_1)\dots c(\lambda_n)W_n^2(\lambda_1,\dots,\lambda_n), \tag{10.3}$$

where $W_n(\lambda)$ is the Vandermond determinant and $c(\lambda)$ is the integral in the representation for u_0.

10.2. Nonlinear Schrödinger equations

10.2.1. Simple nonlinear Schrödinger equation

$$\dot q+q''-2rq^2=0,\quad \tilde q=\frac{1}{r},\quad \tilde r=r[rq-(\ln r)''];$$

$$-\dot r+r''-2qr^2=0,\quad q_0=0,\quad \dot r_0=r_0''. \tag{10.4}$$

The solution of the discrete transformation is the same as in the last subsection.

10.2.2. Modified nonlinear Schrödinger equation

$$\dot q+q''+2(rq)q'=0,\quad \tilde q=\frac{1}{r},\quad \tilde r=r\left[(rq)+\left(\ln\frac{r}{r'}\right)'\right];$$

$$-\dot r+r''-2(rq)r'=0,\quad q_0=0,\quad \dot r_0=r_0''. \tag{10.5}$$

The solution of the discrete transformation is as follows:

$$q_n=(-1)^n\frac{D_1^{n-1}}{D_0^n},\quad r_n=(-1)^{n+1}\frac{D_0^{n+1}}{D_1^{n+1}}. \tag{10.6}$$

10.2.3. Nonlinear Schrödinger equation with derivative

$$\dot q+q''-2(rq^2)'=0,\quad \tilde q=r,\quad \tilde r=q-\left(\frac{1}{r}\right)';$$

$$-\dot r+r''+2(r^2q)'=0,\quad q_0=0,\quad \dot r_0=r_0''. \tag{10.7}$$

The solution of the discrete transformation is as follows:

$$q_{2n}=\frac{D_1^{n-1}D_1^n}{(D_0^n)^2},\quad r_{2n}=\frac{D_0^{n-1}D_0^n}{(D_1^n)^2},$$

$$q_{2n+1}=\frac{D_0^{n-1}D_0^n}{(D_0^n)^2},\quad r_{2n+1}=\frac{D_1^{n-1}D_1^n}{(D_0^{n+1})^2}. \tag{10.8}$$

10.2.4. Nonlinear Schrödinger equation with cubic nonlinearity

$$\dot q+q''-2q^2(r'+r^2q)=0,\quad -\dot r+r''+2r^2(q'-q^2r)=0.$$

The discrete transformation in this case is a little more complicated:

$$\tilde q=(r'+qr^2)^{-1},\quad \tilde r=-(r'+qr^2)'+r^{-1}(r'+qr^2)^2.$$

As in the last cases, $q_0=0$, $-\dot r_0+r_0''=0$, and the solution of the discrete transformation under these boundary conditions has the form

$$q_n=\frac{D_{n-1}^1}{D_n^1},\quad r_n=\frac{D_{n+1}^0}{D_n^0},\quad D_{-1}^\alpha=0,\quad D_0^\alpha=0.$$

10.3. One-dimensional Heisenberg ferromagnet in the classical region (XXX model)

$$\dot S=[S,S''],\quad S=(S_-,S_0,S_+),\quad S_0^2+S_-S_+=1;$$

$$\tilde S_-=S_-+2\left(\frac{1}{\left(\frac{S_+}{1+S_0}\right)'}\right)',\quad \tilde S_+=S_++2\left(\frac{1}{\left(\frac{S_-}{1-S_0}\right)'}\right)',$$

$$\tilde S_0+1=-\tilde S_-\frac{S_+}{1+S_0},\quad S_-^0=0,\quad S_0^0=1,\quad \dot S_+=2S_+'',$$

$$S_-^n=\frac{D_2^{n-1}D_2^n}{(D_1^n)^2},\quad S_0^n+1=2\frac{D_0^nD_2^n}{(D_1^n)^2},$$

$$S_0^n-1=2\frac{D_2^{n-1}D_0^{n+1}}{(D_1^n)^2},\quad S_+^n=-4\frac{D_0^{n+1}D_0^n}{(D_1^n)^2}. \tag{10.9}$$

10.4. XYZ model in the classical region. The Landau–Lifshitz equation

$$(\dot{\mathbf{S}})=\mathbf{S}\times\mathbf{S}''+\mathbf{S}\times(\mathbf{J}\mathbf{S}),$$

$$\mathbf{S}=(S_1,S_2,S_3),\quad (\mathbf{S})^2=1,\quad J=\mathrm{diag}(J_1,J_2,J_3).$$

Under the stereographic projection

$$u=\frac{S_1+iS_2}{1+S_3},\quad v=\frac{S_1-iS_2}{1+S_3}$$

and the substitution $-it\to t$ we obtain the following system of equations:

$$\dot{u}+u''-2v\,\frac{(u')^2+R(u)}{1+uv}+\frac{1}{2}\frac{\partial}{\partial u}R(u)=0,$$

$$-\dot{v}+v''-2u\,\frac{(v')^2+R(v)}{1+uv}+\frac{1}{2}\frac{\partial}{\partial v}R(v)=0,\qquad(10.10)$$

where $R(x)=\alpha x^4+\gamma x^2+\alpha$, $\partial R/\partial x=4\alpha x^3+2\gamma x=2[R+\alpha(x^4-1)]/x$, $\alpha=(J_2-J_1)/4$, $\gamma=(J_1+J_2)/2-J_3$. The system (10.10) is invariant under the transformation $u\to U$, $v\to V$:

$$U=\frac{1}{v},\quad \frac{1}{1+VU}-\frac{1}{1+uv}=\frac{vv''-(v')^2+\alpha(v^4-1)}{(v')^2+R(v)},\qquad(10.11)$$

which is the discrete transformation for this system. The reader can find the corresponding solution in Refs. 40 and 41.

10.5. Lund–Pohlmeyer–Regge model

$$\dot{y}'-4y+2(xy)\dot{y}=0,\quad \tilde{x}=(y'+xy^2)^{-1},$$

$$\dot{x}'-4x-2(xy)\dot{x}=0,\quad \tilde{y}=-(y'+xy^2)'+\frac{(y'+xy^2)^2}{y};\qquad(10.12)$$

$$x_0=0,\quad \dot{y}_0'=4y_0.$$

$$x_n=(-1)^{n+1}\frac{D_1^{n-1}}{D_1^n},\quad y_n=(-1)^n\frac{D_0^{n+1}}{D_0^n}.\qquad(10.13)$$

It is interesting to note that the discrete transformation of the LPR system is identical to the nonlinear Schrödinger equation (10.2.4). Indeed, these two systems belong to the same hierarchy.[42]

10.6. The main chiral-field problem in a space of *n* dimensions (the case of the algebra A_1)

The main chiral-field problem in n-dimensional space is described by the system of equations

$$(\theta_i-\theta_j)\frac{\partial^2 f}{\partial x_i\partial x_j}=\left[\frac{\partial f}{\partial x_i},\frac{\partial f}{\partial x_j}\right],\qquad(10.14)$$

where the function f takes values in the algebra A_1, and θ_i are numerical parameters.

This system is invariant under the transformation[10]

$$F_-=\frac{1}{f_+},$$

$$\frac{\partial F_0}{\partial x_i}=(f_0-F_0+\theta_i)\frac{\partial}{\partial x_i}\ln f_+-\frac{\partial f_0}{\partial x_i},$$

$$\frac{\partial F_+}{\partial x_i}=(f_0-F_0+\theta_i)^2\frac{\partial f_+}{\partial x_i}-2f_+(f_0-F_0+\theta_i)\frac{\partial f_0}{\partial x_i}$$

$$-f_+^2\frac{\partial f_-}{\partial x_i}.\qquad(10.15)$$

These equations can be rewritten in the matrix form

$$\frac{\partial F}{\partial x_i}=\exp[-X^+(f_0-F_0+\theta_i)f_+]$$

$$\times\exp[H\ln f_+]r\frac{\partial f}{\partial x_i}r^{-1}$$

$$=\exp[-H\ln f_+]\exp[X^+(f_0-F_0+\theta_i)f_+],\qquad(10.16)$$

where r is an automorphism of the algebra A_1 with the properties

$$rX^{\pm}r^{-1}=-X^{\mp},\quad rHr^{-1}=-H;$$

$$f_-^0=0,\quad f_0^0=\tau,\quad f_+^0=\alpha^0,$$

where

$$\frac{\partial^2\tau}{\partial x_i\partial x_j}=0,\quad (\theta_i-\theta_j)\frac{\partial^2\alpha^0}{\partial x_i\partial x_j}=2\left[\frac{\partial\tau}{\partial x_i}\frac{\partial\alpha^0}{\partial x_j}-\frac{\partial\tau}{\partial x_j}\frac{\partial\alpha^0}{\partial x_i}\right].\qquad(10.17)$$

To solve the discrete transformation, let us consider the system of linear equations

$$\theta_i\frac{\partial\alpha^l}{\partial x_i}-2\frac{\partial\tau}{\partial x_i}\alpha^l=\frac{\partial\alpha^{l+1}}{\partial x_i}.\qquad(10.18)$$

From these equations it follows that each function α^l is a solution of the equation for α^0. We have an explicit expression for α^s:

$$\tau=\sum\phi_i(x_i),\quad \alpha^s=\int d\lambda(\lambda)^s c(\lambda)\exp\left(\sum\frac{\phi_i(x_i)}{\lambda-\theta_i}\right).\qquad(10.19)$$

In terms of the α^l the discrete transformation has the solution

$$f_0^n=\frac{D_0^{n-1}}{D_0^n},\quad f_0^n=\tau-\frac{\dot{D}_0^n}{D_0^n},\quad f_+^n=\frac{D_0^{n+1}}{D_0^n}.\qquad(10.20)$$

In the determinant $\dot{D}_0^n$ the number of indices of the last row is increased by one. Applications of the results of this subsection to the problem of constructing multisoliton solutions of sigma–chiral models can be found in Ref. 43.

10.7. The main-chiral field problem for an arbitrary semisimple Lie algebra

For a semisimple Lie algebra and for an element f which is a solution of (12), the following statement holds:[10] *There exists an element S taking values in a gauge group such that*

$$S^{-1}\frac{\partial S}{\partial x_i}=\frac{1}{f_-}\left[\frac{\partial f}{\partial x_i},X_M^+\right]-\theta_i\frac{\partial}{\partial x_i}\frac{1}{f_-}X_M^+. \tag{10.21}$$

Here X_M^+ is the element of the algebra corresponding to its maximal root, divided by its norm, i.e.,

$$[X_M^+,X^-]=H,\quad [H,X^\pm]=\pm 2X^\pm,$$

and f_- is the coefficient function in the decomposition of f for the element corresponding to the minimal root of the algebra. In these terms the discrete transformation becomes

$$\frac{\partial F}{\partial x_i}=S\frac{\partial f}{\partial x_i}S^{-1}+\theta_i\frac{\partial S}{\partial x_i}S^{-1}. \tag{10.22}$$

The system of equations in the case under consideration can be written as an equality between the group g and the algebra of f-valued functions as

$$g_{x_i}g^{-1}=\theta_i f_{x_i}.$$

The discrete transformation for a group-valued function takes the form

$$G=Sg,$$

where the group element is determined by the above relations. The explicit expression for the group element g_n after n applications of the discrete transformation can be found in Ref. 7.

10.8. The system of self-dual equations in four-dimensional space (the case of the algebra A_1)

The self-dual equations for an element f with values in a semisimple Lie algebra have the following form:

$$\frac{\partial^2 f}{\partial y\partial\bar{y}}+\frac{\partial^2 f}{\partial z\partial\bar{z}}=\left[\frac{\partial f}{\partial y},\frac{\partial f}{\partial z}\right]. \tag{10.23}$$

The discrete transformation for this system is[44,45]

$$F_-=\frac{1}{f_-},$$

$$\frac{\partial}{\partial y}F_0=\frac{\partial}{\partial\bar{z}}\ln f_- -\frac{\partial}{\partial y}f_0+(f_0-F_0)\frac{\partial}{\partial y}\ln f_-,$$

$$\frac{\partial}{\partial z}F_0=-\frac{\partial}{\partial\bar{y}}\ln f_- -\frac{\partial}{\partial z}f_0+(f_0-F_0)\frac{\partial}{\partial z}\ln f_-,$$

$$\frac{\partial}{\partial y}F_+=-f_-\left\{(f_0-F_0)\frac{\partial}{\partial y}(f_0-F_0)+\frac{\partial}{\partial\bar{z}}(f_0-F_0)\right\}-f_-^2\frac{\partial}{\partial y}f_+,$$

$$\frac{\partial}{\partial z}F_+=-f_-\left\{(f_0-F_0)\frac{\partial}{\partial z}(f_0-F_0)-\frac{\partial}{\partial\bar{y}}(f_0-F_0)\right\}-f_-^2\frac{\partial}{\partial z}f_+. \tag{10.24}$$

Substitution of (12.20) into the density of topological charge yields

$$Q_F=q_f+\Box\Box\ \ln f_-.$$

For the integration of the discrete transformation we have the system of linear equations

$$\frac{\partial\alpha^l}{\partial\bar{y}}+2\frac{\partial\tau}{\partial z}\alpha^l=\frac{\partial\alpha^{l+1}}{\partial z},\quad \frac{\partial\alpha^l}{\partial\bar{z}}-2\frac{\partial\tau}{\partial y}\alpha=-\frac{\partial\alpha^{l+1}}{\partial y}. \tag{10.25}$$

In these terms the solution of the self-dual system is given by

$$f_-^n=\frac{D_0^{n-1}}{D_0^n},\quad f_0^n=\frac{\dot{D}_0^n}{D_0^n}+\tau,\quad f_+^0=\frac{D_0^{n+1}}{D_0^n}. \tag{10.26}$$

10.9. The system of self-dual equations for an arbitrary semisimple algebra

The following statement holds:[44,45]

There exists an element S taking values in the gauge group such that

$$S^{-1}\frac{\partial S}{\partial y}=\frac{1}{f_-}\left[\frac{\partial f}{\partial y},X_M^+\right]-\frac{\partial}{\partial\bar{z}}\left(\frac{1}{f_-}\right)X_M^+,$$

$$S^{-1}\frac{\partial S}{\partial z}=\frac{1}{f_-}\left[\frac{\partial f}{\partial z},X_M^+\right]+\frac{\partial}{\partial\bar{y}}\left(\frac{1}{f_-}\right)X_M^+. \tag{10.27}$$

Here X_M^+ is the element of the algebra corresponding to its maximal root, divided by its norm, i.e.,

$$[X_M^+,X^-]=H,\quad [H,X^\pm]=\pm 2X^\pm,$$

and f_- is the coefficient function in the decomposition of f for the element corresponding to the minimal root of the algebra. The discrete transformation has the form

$$\frac{\partial F}{\partial y}=S\frac{\partial f}{\partial y}S^{-1}+\frac{\partial S}{\partial\bar{z}}S^{-1},\quad \frac{\partial F}{\partial z}=S\frac{\partial f}{\partial z}S^{-1}-\frac{\partial S}{\partial\bar{y}}S^{-1}. \tag{10.28}$$

10.10. The main chiral-field problem with moving poles

Many integrable systems arise from the equations (10.23) by imposing symmetry requirements on their solutions. The cylindrically symmetric condition in four-dimensional space restricts the form of the function f:

$$f=\frac{1}{y}f(\xi,\bar{\xi}),\quad \xi=\frac{z-\bar{z}}{2}+\left[\left(\frac{z+\bar{z}}{2}\right)^2+y\bar{y}\right]^{1/2},\quad \bar{\xi}=-\xi^*,$$

$$(\xi-\bar{\xi})\frac{\partial^2}{\partial\xi\partial\bar{\xi}}f=\left[\frac{\partial}{\partial\xi}f,\frac{\partial}{\partial\bar{\xi}}f\right]. \tag{10.29}$$

This is the equation for the main chiral-field problem with moving poles.

The result of integration of Eq. (10.9) is given by

$$S=S(\xi,\bar{\xi})\exp X_M^+\frac{z}{f_-};$$

$$S^{-1}\frac{\partial}{\partial\xi}S=\frac{1}{f_-}\left[\frac{\partial}{\partial\xi}f,X_M^+\right]-\xi\frac{\partial}{\partial\xi}\frac{1}{f_-}X_M^+,$$

$$S^{-1}\frac{\partial}{\partial\bar{\xi}}S=\frac{1}{f_-}\left[\frac{\partial}{\partial\bar{\xi}}f,X_M^+\right]-\bar{\xi}\frac{\partial}{\partial\bar{\xi}}\frac{1}{f_-}X_M^+, \tag{10.30}$$

and the discrete transformation has the following form:

$$\frac{\partial}{\partial \xi} F = S\left(\frac{\partial}{\partial \xi} f\right) S^{-1} - \xi \frac{\partial S}{\partial \xi} S^{-1},$$

$$\frac{\partial}{\partial \bar{\xi}} F = S\left(\frac{\partial}{\partial \bar{\xi}} f\right) S^{-1} - \bar{\xi} \frac{\partial S}{\partial \bar{\xi}} S^{-1}. \tag{10.31}$$

The relations (10.10) and (10.31) describe the discrete transformation for the main chiral-field problem with moving poles.[20]

10.11. The self-dual equation with cylindrical symmetry in three-dimensional space

The condition of cylindrical symmetry in three-dimensional space leads to the following form of the solution:

$$f = \frac{1}{\bar{y}} f(\xi, \bar{\xi}), \quad \xi = \frac{z+\bar{z}}{2} + i(y\bar{y})^{1/2}, \quad \bar{\xi} = -\xi^*;$$

$$(\xi - \bar{\xi}) \frac{\partial^2 f}{\partial \xi \partial \bar{\xi}} = \frac{1}{2}\left(\frac{\partial f}{\partial \bar{\xi}} - \frac{\partial f}{\partial \xi}\right) + \left[\frac{\partial f}{\partial \bar{\xi}} \frac{\partial f}{\partial \xi}\right]. \tag{10.32}$$

The discrete transformation for Eq. (10.32) arising from (10.9) and (10.28) has the form[20]

$$S^{-1} \frac{\partial}{\partial \xi} S = \frac{1}{f_-}\left[\frac{\partial}{\partial \xi} f, X_M^+\right] + \left(\frac{1}{f_-} - \frac{\xi - \bar{\xi}}{2} \frac{\partial}{\partial \xi} \frac{1}{f_-}\right) X_M^+,$$

$$S^{-1} \frac{\partial}{\partial \bar{\xi}} S = \frac{1}{f_-}\left[\frac{\partial}{\partial \bar{\xi}} f, X_M^+\right] + \left(\frac{1}{f_-} - \frac{\xi - \bar{\xi}}{2} \frac{\partial}{\partial \bar{\xi}} \frac{1}{f_-}\right) X_M^+,$$

$$\frac{\partial}{\partial \xi} F = S\left(\frac{\partial}{\partial \xi} f\right) S^{-1} + \frac{\bar{\xi} - \xi}{2} \frac{\partial S}{\partial \xi} S^{-1},$$

$$\frac{\partial}{\partial \bar{\xi}} F = S\left(\frac{\partial}{\partial \bar{\xi}} f\right) S^{-1} - \frac{\bar{\xi} - \xi}{2} \frac{\partial S}{\partial \bar{\xi}} S^{-1}.$$

In the case of the algebra A_1 the system (10.32) arises in the integration problem of general relativity with two commuting Killing vectors.[13]

10.12. The cylindrically symmetric solution invariant under two orthogonal four-dimensional axes

$$x_1 \frac{\partial^2 F}{\partial x_1^2} + x_2 \frac{\partial^2 F}{\partial x_2^2} = \left[\frac{\partial F}{\partial x_1}, \frac{\partial F}{\partial x_2}\right]. \tag{10.33}$$

In the case of the algebra A_1 the explicit form of the discrete transformation is[20]

$$F_- = \frac{1}{f_-},$$

$$\frac{\partial F_0}{\partial x_1} = -1 + (f_0 - F_0) \frac{\partial \ln f_-}{\partial x_1} + x_2 \frac{\partial \ln f_-}{\partial x_2} - \frac{\partial f_0}{\partial x_1},$$

$$\frac{\partial F_0}{\partial x_2} = 1 + (f_0 - F_0) \frac{\partial \ln f_-}{\partial x_2} - x_1 \frac{\partial \ln f_-}{\partial x_1} - \frac{\partial f_0}{\partial x_2},$$

$$\frac{\partial F_+}{\partial x_1} = -f_-\left[(f_0 - F_0) \frac{\partial (f_0 - F_0)}{\partial x_1} + x_2 \frac{\partial (f_0 - F_0)}{\partial x_2}\right] - f_-^2 \frac{\partial f_+}{\partial x_1},$$

$$\frac{\partial F_+}{\partial x_2} = -f_-\left[(f_0 - F_0) \frac{\partial (f_0 - F_0)}{\partial x_2} - x_1 \frac{\partial (f_0 - F_0)}{\partial x_1}\right] - f_-^2 \frac{\partial f_+}{\partial x_2}. \tag{10.34}$$

The system of linear equations has the form

$$x_1 \frac{\partial \alpha^l}{\partial x_1} - \alpha^l + 2\alpha^l \frac{\partial \tau^l}{\partial x_2} = \frac{\partial \alpha^{l+1}}{\partial x_2},$$

$$x_2 \frac{\partial \alpha^l}{\partial x_2} - \alpha^l - 2\alpha^l \frac{\partial \tau^l}{\partial x_1} = -\frac{\partial \alpha^{l+1}}{\partial x_1},$$

$$\tau^{l+1} = \tau^l + \frac{x_1 - x_2}{2}. \tag{10.35}$$

The solution of the discrete transformation is identical to the result in the self-dual case [see (10.25) and (10.26)].

10.13. The (2+1) matrix Davey–Stewartson system

The matrix Davey–Stewartson equation is the following system of two equations for two unknown $s \times s$ matrix functions u, v:

$$u_t + a u_{xx} + b u_{yy} - 2au \int dy (uv)_x - 2b \int dx (uv)_y u = 0,$$

$$-v_t + a v_{xx} + b v_{yy} - 2a \int dy (uv)_x v - 2bv \int dx (uv)_y = 0, \tag{10.36}$$

where a, b are arbitrary numerical parameters, and x, y are the coordinates of two-dimensional space. For $s=1$, the order of the factors is not essential and (1.1) is the usual Davey–Stewartson equation in its original form.[15]

By direct but not very simple computations one finds that the system (10.36) is invariant under the following change of the unknown functions:

$$\tilde{u} = \frac{1}{v}, \quad \tilde{v} = [vu - (v_x v^{-1})_y] v \equiv v[uv - (v^{-1} v_y)_x]. \tag{10.37}$$

The substitution (10.37) is the discrete transformation under which all the equations of the matrix Davey–Stewartson hierarchy are invariant.[46] In the case of one-dimensional space this substitution was mentioned in Ref. 47.

The substitution (10.37) can also be rewritten in the form of an infinite chain

$$((v_n)_x v_n^{-1})_y = v_n v_{n-1}^{-1} - v_{n+1} v_n^{-1} \quad \left(u_{n+1} = \frac{1}{v_n}\right), \tag{10.38}$$

where (v_{n-1}, u_{n-1}) denotes the result of n applications of (10.37) to given matrix functions (v_0, u_0).

Under the boundary condition $v_{-1}^{-1}=v_N=0$ (the so-called matrix Toda chain with fixed ends) the general solution of (10.38) takes the form[16]

$$v_0=\sum_{t=0}^{N}\phi_t(x)\bar{\phi}_t(y),$$

where $\phi_t(x),\bar{\phi}_t(y)$ are arbitrary $s\times s$ matrix functions of the corresponding arguments.

In the scalar case $s=1$ the general solution of the Toda chain with fixed ends was found in Ref. 48 for all series of semisimple algebras except E_7,E_8. In Ref. 49 this result was reproduced in terms of the invariant root technique applicable to all semisimple series.

11. THEORY OF INTEGRABLE SYSTEMS FROM THE POINT OF VIEW OF REPRESENTATION THEORY OF THE DISCRETE GROUP OF INTEGRABLE MAPPINGS[50–53]

Here we consider the results of the preceding section from a more general point of view. We will start with a short historical discussion. Our aim is to find a place for the discrete transformation in the usual, more traditional ways of investigating the theory of integrable systems.

Liouville introduced the term "integrability" for dynamical systems. He proved that if a dynamical system possesses a sufficiently large number of integrals of the motion in involution, then the system is integrable. But neither general methods for the construction of a solution in explicit form nor any mention of the symmetry of the system under consideration are contained within Liouville's criterion.

In the case of Lie symmetries the theorem of Noether fills this gap. It teaches us that the number of conservation laws coincides with the dimension of the Lie group and makes it possible (in the case of Lagrange theory) to obtain explicit expressions for the integrals of the motion.

Roughly speaking, the modern theory of integrable systems up to now has maintained the Liouville definition (an integrable system must contain an infinite number of integrals of the motion in involution), and many people have found various consequences which follow from this fact.

The aim of this section is to show in a deductive way that the theory of integrable systems can be understood as a theory of the linear representation of the discrete group of integrable mappings. This does not mean that we can propose at present a complete theory of this connection from the mathematical point of view. We only demonstrate that all known results of the theory of integrable systems are consistent with this hypothesis.

11.1. Discrete transformation of integrable systems

Let us consider a local invertible transformation described by the substitution

$$\tilde{u}=\phi(u,u',u'',\dots,u^{(r)})=\phi(u), \tag{11.1}$$

where u is an s-dimensional vector function, and $u',u'',\dots$ are its derivatives of the corresponding orders with respect to the "space" coordinates (the dimension of the space can be arbitrary).

The property of invertibility means that Eq. (11.1) can be solved, and the "old" function u can be expressed locally in terms of new functions $\tilde{u}$ and their derivatives. The connection (11.1) with integrable systems is illustrated by the examples of the preceding section.

The Fréchet derivative $\phi'(u)$ corresponding to the substitution (11.1) is the $s\times s$ matrix operator defined as

$$\phi'(u)=\phi_u+\phi_{u'}D+\phi_{u''}D^2+\dots, \tag{11.2}$$

where D^m is the operator of differentiation m times with respect to the corresponding space coordinates $u,u',u'',\dots,d$. More detailed information about this construction can be found in Refs. 47 and 50.

Let us consider the equation which in another notation first appeared in Ref. 51:

$$F_n(\phi(u))=\phi'(u)F_n(u), \tag{11.3}$$

where $F_n(u)$ is an unknown s-component vector function, each component of which depends on u and its derivatives up to a maximal order n.

For each substitution Eq. (11.3) possesses one obvious trivial solution, $F_n(u)=u'$. To prove this, it is sufficient to differentiate Eq. (11.1) once with respect to one of its space coordinates.

If Eq. (11.2) possesses some solution [for given $\phi(u)$] other than the trivial one, then we will call such a substitution an integrable substitution or mapping.

We specially emphasize that Eq. (11.3) contains two unknown functions $\phi(u)$ and $F_n(u)$, and only for a narrow class of integrable substitutions does it possess nontrivial solutions for the function $F_n(u)$.

Each nontrivial solution of (11.2) generates an equation (system) of evolution type,

$$u_t=F_n(u), \tag{11.4}$$

which is obviously invariant under the substitution $u\rightarrow\phi(u)$. (In this connection, we emphasize that the equation $u_t=u'$ is indeed invariant under an arbitrary substitution.)

Let us now compare Eq. (11.3) with the definition of a linear representation $T(g)$ of some group (for definiteness one can have in mind a Lie group):

$$\Phi(gx)=T(g)\Phi(x), \tag{11.5}$$

where g is the group element, $T(g)$ is the group operator for some representation, and $\Phi(x)$ is the basis of the corresponding representation space.

An obvious correspondence occurs whenever the defining relationship (11.5) is compared with Eq. (11.3):

$$\Phi(x)\rightarrow F_n(u),\quad T(g)\rightarrow\phi'(u).$$

Using this correspondence, let us interpret Eq. (11.3) at the group-theoretical level. We have some discrete group of transformations, the group element of which is exactly the substitution $u\rightarrow\phi(u)$; $\phi'(u)$ (the Fréchet derivative) is a linear representation of the group element; and $F_n(u)$ (the equations of the hierarchy) is a basis in the representation space. If this representation is irreducible (it is necessary to verify this by independent methods), then all possible bases of this representation [solutions of Eq. (11.3) with different n] must be related by some operator $W_{n,n'}$:

$$F_n(u)=W_{n,n'}F_{n'}. \tag{11.6}$$

Indeed, the same situation occurs in the theory of (1+1) integrable systems. All equations of the same hierarchy are connected by the "raising" operators constructed from the skew-symmetrical (nonlocal) Hamiltonian operators $J_n=-J_n^T$:

$$W_{n,n'}=J_nJ_{n'}^{-1}. \tag{11.7}$$

11.2. Equations for "raising" and Hamiltonian operators

Two equations will be important for our further considerations:

$$\phi'(u)W(u)\phi'(u)^{-1}=W(\phi(u)),$$

$$\phi'(u)J(u)\phi'(u)^T=J(\phi(u)), \tag{11.8}$$

where $\phi'(u)^T=\phi_u^T-D\phi_{u'}^T+D^2\phi_{u''}^T-...$, and $W(u)$, $J(u)$ are unknown $s\times s$ matrix operators, whose elements are polynomials of some finite order in the operator of differentiation D (of both positive and negative degrees).

From (11.3) and (11.8) it follows immediately that if $F_n(u)$ is some solution of the principal equation (11.3), then $W^p(u)F_n(u)$ (p is an arbitrary natural number) will be another solution of the same equation.

The solution of the second equation of (11.8) under the additional restriction of skew symmetry may be connected (interpreted) as a Poisson structure which is invariant under the transformation of discrete symmetry. Skew-symmetrical operators $J(u)$ are known as Hamiltonian ones. Two different solutions of the second equation of (11.8) (if it is possible to find them), say, $J_1(u)$ and $J_2(u)$ in the combination $J_1J_2^{-1}$, obey the first equation of (11.8). The operator $J_1J_2^{-1}J_1(u)$ is again a solution of the second equation of (11.8), and so on. This is the way in which Hamiltonian operators usually arise in the theory of integrable systems. It is necessary, from some independent assumptions, to find two different Poisson structures. All other objects can then be constructed by the above scheme.

In the problem of the construction of Hamiltonian operators for integrable systems, the equations (11.8) were used for the first time in Ref. 52.

11.3. Conditions under which the evolution equation can be rewritten in Hamiltonian form

Let us consider some scalar function $H(u)$ locally dependent on u and its derivatives, obeying the equality (equation)

$$H(\phi(u))-H(u)=\text{Ker}\in\frac{\delta}{\delta u}. \tag{11.9}$$

In other words, the difference between the function after one application of the discrete transformation and its original value is a divergence with respect to the space coordinates. Let us compare the variational derivatives $H(u)$ before and after the discrete transformation. We have

$$\frac{\delta H(u)}{\delta u}=\phi'^T(u)\,\frac{\delta H(\phi(u))}{\delta\phi(u)}. \tag{11.10}$$

This equality is a direct corollary of (11.9) and the obvious fact that the variational derivative of a divergence vanishes identically.

Let $J(u)$ be any solution of (11.8). Then we have

$$\phi'(u)J(u)\,\frac{\delta H(u)}{\delta u}=\phi'(u)J(u)\phi'^T(u)\,\frac{\delta H(\phi(u))}{\delta\phi(u)}$$

$$=J(\phi(u))\,\frac{\delta H(\phi(u))}{\delta\phi(u)}. \tag{11.11}$$

Thus, we see that the function $F(u)=J(u)\delta H(u)/\delta u$ is just a solution of our main equation (11.3), and the corresponding evolution equation (11.4) takes a Hamiltonian form (cf. Ref. 5).

11.4. Conservation laws

All known integrable substitutions in one-dimensional space [(1+1) integrable systems] obey all the conditions of the preceding section. This means that it is possible to find in explicit form an infinite number of Hamiltonian functions $H_n(u)$ and an infinite number of Hamiltonian operators $J_n(u)$. Thus, in the (1+1)-dimensional case all integrable systems of the evolution type (11.4) can be written in Hamiltonian form. As a consequence, it is possible to determine the Poisson brackets between two local functions by the rule

$$\{N(u),M(u)\}\equiv\left(\frac{\delta N(u)}{\delta u}\,J(u)\,\frac{\delta M(u)}{\delta u}\right) \tag{11.12}$$

and to prove, using technical manipulations, that all conserved integrals are in involution:

$$\{H_n(u),H_{n'}(u)\}=\text{Ker}\in\frac{\delta}{\delta u}. \tag{11.13}$$

This result is usually interpreted as fulfillment of the Liouville criterion of integrability.

In the case of (1+2)-dimensional integrable systems it is impossible to write the investigated systems in Hamiltonian form (except for some trivial cases). But whenever in the (1+2)-dimensional case functions obeying Eq. (11.9) can be found, they are in general nonlocal, the number of them is infinite, and they are invariant under time evolution in the sense that

$$(H_n^0(u))_t=\sum_s (H_n^s(u))_{x_s},$$

where x_s are the independent space coordinates of the problem.

It is true that we can present an infinite number of concrete examples of the validity of the last propositions, but also it is true that at present we have no idea how to prove them at the group-theoretical level in the general case.

11.5. The general hypothesis

To conclude our discussion, we are able to formulate the following general hypothesis about the structure of a future theory of integrable systems: The problem of integrable systems is equivalent to the theory of representations of the discrete group of integrable mappings.

Indeed, if from independent considerations it turns out to be possible to obtain a solution of our main equation (11.3), then we automatically produce an integrable equation of evolution type (11.4), and each space of irreducible representations of (11.5) will give us the exact solution of it. We are well aware that the form of our main equation (11.3) is not very suitable for obtaining direct conclusions from it. In this connection we can note that, by analogy with the difference between the original definition of semisimple algebras (in the sense of the absence of nontrivial ideals) and the Cartan classification into A,B,C,D,E,F,G, and E, it may be of comparable magnitude to the problem of classification of the solutions of our main equation.

We hope that something of this kind will be achieved in the case of the representation theory for discrete groups of integrable mappings.

11.6. Conclusion

The main result of the present section is contained in the new equation (11.3). Its solution will provide the answers to two of the most important questions of the theory of integrable systems. The first question can be regarded as the "quantization" of substitutions, i.e., the choice from among the infinite number of invertible substitutions of only the one that is integrable in the above sense. Except for the obvious remark that this will depend essentially on the dimensions of the spaces involved, the author knows almost nothing about how to solve this problem and concludes that it is not going to be resolved quickly.

The second, more tractable problem from our point of view is the question of the solution of the main equation (11.3) for a given (*ad hoc*) integrable substitution $\phi(u)$ (in this connection, see the next section). It is possible to suppose that the solution to this problem is closely connected with the theory of representations of the discrete group of integrable mappings. From known examples of integrable systems it follows that the discrete group of integrable mapping possesses a rich store of different irreducible representations. Each of these representations may be connected with a definite class of exact solutions of the corresponding integrable system. In a sense the soliton-like solutions (which will be discussed below) correspond to finite-dimensional representations of such groups.

12. TWO-DIMENSIONAL INTEGRABLE MAPPINGS AND EXPLICIT FORM OF EQUATIONS OF (1+2)-DIMENSIONAL HIERARCHIES OF INTEGRABLE SYSTEMS[54–57]

Here we shall complete the second part of the general program of the last section: we shall find the explicit form of the solution of our main equation (11.3) for an *ad hoc* given integrable mapping. The equations of (1+2) integrable systems belonging to the Darboux–Toda, Heisenberg, and Lotka–Volterra hierarchies which are invariant with respect to discrete transformations of the corresponding integrable mappings will be presented in explicit form.

12.1. Two-dimensional integrable mappings

Below, we will discuss three concrete examples of two-dimensional integrable mappings which can be considered by similar methods.

12.1.1. Darboux–Toda substitution

The explicit form of the direct and inverse DT integrable substitution is as follows:

$$\overleftarrow{u}=\frac{1}{v}, \quad \overleftarrow{v}=v(uv-(\ln v)_{xy}),$$
$$\overrightarrow{v}=\frac{1}{u}, \quad \overrightarrow{u}(vu-(\ln u)_{xy}). \tag{12.1}$$

The result obtained from the function $f(u,v)$ after s applications of the direct transformation will be denoted by $\overset{\leftarrow s}{f}$, and that after s applications of the inverse transformation by $\overset{s\rightarrow}{f}$, with the following convention:

$$\overset{\leftarrow(-m)}{f} \equiv \overset{m\rightarrow}{f} \quad \text{for } m\geqslant 0.$$

As a direct corollary of (12.1) one finds a Toda-like recurrence relation for the function $T_0=uv$, which will be important for our further considerations:

$$(\ln T_0)_{xy}=-\overleftarrow{T}_0+2T_0-\overrightarrow{T}_0. \tag{12.2}$$

The Fréchet derivative corresponding to (12.1) has the form

$$\phi'(u)=\begin{pmatrix} 0 & -\dfrac{1}{v^2} \\ v^2 & 2(uv)-\dfrac{v_x v_y}{v^2}+\dfrac{v_x}{v}D_y+\dfrac{v_y}{v}D_x-D_{xy} \end{pmatrix}, \tag{12.3}$$

where $D_y\equiv\partial/\partial y$, $D_x\equiv\partial/\partial x$.

The system (11.3) in the concrete case of the DT substitution can be rewritten as

$$\overleftarrow{F}_1=-\frac{1}{v^2},F_2, \quad \overleftarrow{F}_2=v^2F_1+\left(2(uv)-\frac{v_x v_y}{v^2}+\frac{v_x}{v}D_y\right.$$
$$\left.+\frac{v_y}{v}D_x-D_{xy}\right)F_2. \tag{12.4}$$

It is not difficult to verify by direct calculation that $F_0=(u,-v)$ is a solution of this equation, and so the substitution (12.1) is integrable in the sense of Ref. 51.

After the introduction of the new functions $F_1=uf_1$, $F_2=vf_2$, the system (12.4) takes the form of a single equation for only one unknown function f_2:

$$(\overleftarrow{u}\overleftarrow{v})(\overleftarrow{f}_2-f_2)-(uv)(f_2-\overrightarrow{f}_2)=-D_{xy}f_2, \quad f_1=-\overleftarrow{f}_2. \tag{12.5}$$

The notation in this equation is explained after Eq. (12.1).

In further transformations of (12.5) we will use the fact that the condition of invariance of a function with respect to the discrete transformation $\overleftarrow{F}=F$ is equivalent to the condition $F\equiv$const. This is in a sense analogous to Liouville's theorem in the theory of analytic functions. Using this fact for the function T $[f_2=\int dy(\overleftarrow{T}-T)]$, we obtain the Toda chain-like equation

$$-T_x=T_0\int dy(\overleftarrow{T}-2T+\overrightarrow{T}),\quad T_0=uv. \tag{12.6}$$

In terms of the solution of (12.6) the evolution-type equation (11.4) [invariant with respect to the DT substitution (12.1)] takes the form

$$v_t=v\int dy(\overleftarrow{T}-T),\quad u_t=u\int dy(\overrightarrow{T}-T). \tag{12.7}$$

12.1.2. Two-dimensional Heisenberg substitution

By this term we will mean the direct and inverse transformations of two functions (u,v) of the form

$$\overleftarrow{u}=v^{-1},\quad \frac{1}{1+\overleftarrow{uv}}=\frac{1}{1+uv}+\frac{\phi_{xy}}{\phi_x\phi_y},\quad \phi=\ln v,$$

$$\overrightarrow{v}=u^{-1},\quad \frac{1}{1+\overrightarrow{uv}}=\frac{1}{1+uv}+\frac{\psi_{xy}}{\psi_x\psi_y},\quad \psi=\ln u. \tag{12.8}$$

One can check that the functions t_m $[t_1=u_yv_x/(1+uv)^2=-(\overrightarrow{v})_yv_x/(\overrightarrow{v}+v)^2,\quad t_2=v_yu_x/(1+uv)^2=-(\overrightarrow{v})_xv_y/(\overrightarrow{v}+v)^2$ obey the Toda-like recurrence relations

$$(t_m)_x=t_m\int dy\Delta_m\quad (m=1,2), \tag{12.9}$$

where $\Delta_m=\overleftarrow{t_m}-2t_m+\overrightarrow{t_m}$.

The explicit form of the Fréchet-derivative operator reads

$$\phi'(u)=\left(\begin{array}{l} 0-v^{-2}\left(\dfrac{v\overset{\leftarrow -1}{R}}{R}\right)^2 \\ -\left(1+(vu\overset{\leftarrow -1}{R})/R^2\right. \\ \left.+(\overset{\leftarrow -1}{R})^2\delta\left(\phi_x^{-1}D_x+\phi_y^{-1}D_y-\dfrac{v}{v_{xy}}D_{xy}\right)\right)\end{array}\right.$$

$$\delta=\frac{vv_{xy}}{v_xv_y},\quad R=1+uv,\quad \overleftarrow{R}=1+\overleftarrow{uv}. \tag{12.10}$$

By a short calculation it is possible to show that Eq. (11.3) possesses a nontrivial solution $F_1=u$, $F_2=-v$, so that the Heisenberg substitution is by definition integrable.

Now we can rewrite Eq. (11.3) in a more transparent form. Let us introduce the notation $F_1=uB$, $F_2=vA$. From the first equation of (11.3) we immediately obtain $B=-\overrightarrow{A}$. The second equation, after some transformations, can be rewritten in the form of a single equation for the function A:

$$\left(\frac{\overleftarrow{uv}}{(1+uv)^2}\right)(\overleftarrow{A}-A)-\frac{uv}{(1+uv)^2}(A-\overrightarrow{A})$$
$$=(\phi_x\phi_y)^{-1}\left(\frac{\phi_{xy}}{\phi_x}A_x+\frac{\phi_{xy}}{\phi_y}A_y-A_{xy}\right). \tag{12.11}$$

As we know, the main equation (11.3) always possesses the trivial solution $F_1=u_x,(u_y)$; $F_2=v_x,(v_y)$ or $A=\phi_x,(\phi_y)$. Let us seek a solution of (12.10) in the form $A=\phi_x\alpha$. Instead of (12.10) we obtain the following equation for α:

$$\left(\frac{\overleftarrow{u_xv_x}}{(1+uv)^2}\right)(\overleftarrow{\alpha}-\alpha)-\frac{u_xv_x}{(1+uv)^2}(\alpha-\overset{1\rightarrow}{\alpha})=\left(\frac{\alpha_y}{\theta}\right)_x,$$

$$\theta=\frac{\phi_y}{\phi_x}. \tag{12.12}$$

Making in (12.12) the substitution

$$\left(\frac{\alpha_y}{\theta}\right)_x=\overleftarrow{T}-T,$$

we arrive at the following equation for T:

$$T_x=T_0\int dy[\theta(\overleftarrow{T}-T)-\overrightarrow{\theta}(T-\overrightarrow{T})], \tag{12.13}$$

where

$$T_0=\frac{u_xv_x}{(1+uv)^2}.$$

12.1.3. Lotka–Volterra substitution

In this case the direct and inverse transformations have the form

$$\overleftarrow{u}=u+(\ln v)_x,\quad \overleftarrow{v}=v+(\ln\overleftarrow{u})_y,$$
$$\overrightarrow{u}=u-(\ln\overrightarrow{v})_x,\quad \overrightarrow{v}=v-(\ln u)_y. \tag{12.14}$$

As in the previous case, the functions $t_1=uv$, $t_2=\overleftarrow{u}\overrightarrow{v}$ obey Toda-like recurrence relations (12.9).

The Fréchet operator in this case has the form

$$\phi'(u)=\begin{pmatrix} 1 & D_x v^{-1} \\ D_y(\overleftarrow{u})^{-1} & 1+D_y(\overleftarrow{u})^{-1}D_x v^{-1}\end{pmatrix}. \qquad (12.15)$$

By the same technique as in the previous subsections we obtain a single equation for the unknown function T and expressions for the equations of the hierarchy in terms of this solution:

$$T_y=v\int dx[\overset{\leftarrow 1}{u}(\overleftarrow{T}-T)+u(T-\vec{T})], \qquad (12.16)$$

and, finally,

$$u_t=u(T-\vec{T}), \quad v_t=D_yT.$$

12.2. Solution of the main equation

In spite of essential differences in the form of the Fréchet operators in the three cases considered above, the main equations of the problems, (12.6), (12.14), and (12.16), have the same structure and can be solved by similar methods. We shall demonstrate these methods for the more complicated example of the Heisenberg substitution and present the results of calculations for the other cases.

First of all, we note that Eq. (12.14) has the particular solution

$$T=T_0,$$

as can be seen with the help of the following equality, which is a direct corollary of (12.8) and (12.9):

$$\overleftarrow{T_0}-T_0=2\phi_x\left(\frac{1}{1+uv}\right)_x+2\phi_{xy}\frac{\phi_x}{\phi_y}\frac{1}{1+uv}$$
$$+\phi_x\left(\frac{\phi_{xy}}{\phi_x\phi_y}\right)_x-\phi_{xy}\frac{\phi_x}{\phi_y}+\frac{\phi_{xy}^2}{\phi_y^2}.$$

Let us now seek a solution of (12.14) as $T=T_0\int dy\,\alpha_0$. Instead of (12.14) we obtain an equation for the function α_0:

$$(\alpha_0)_x+\alpha_0\int dy[\overleftarrow{t_1}-t_1+\vec{t_2}-t_2]$$
$$=\overleftarrow{t_1}\int dy(\overleftarrow{\alpha_0}-\alpha_0)+\vec{t_2}\int dy(\vec{\alpha_0}-\alpha_0). \qquad (12.17)$$

As will be shown later, this equation will arise many times, and it will be important for us to consider two possibilities for its further evolution. Let us use the following ansatz:

$$\alpha_0=\overleftarrow{t_1}\alpha_1+\vec{t_2}\beta_1.$$

After substituting this expression into (12.17) and equating to zero the coefficients in front of the terms $\overleftarrow{t_1},\vec{t_2}$ (this is an additional assumption), we find equations for the unknown functions α_1, β_1:

$$(\alpha_1)_x+\alpha_1\int dy[\overset{\leftarrow 2}{t_1}-\overleftarrow{t_1}+\vec{t_2}-t_2]=\int dy(\overleftarrow{\alpha_0}-\alpha_0),$$

$$(\beta_1)_x+\beta_1\int dy[\overleftarrow{t_1}-t_1+\overset{2\rightarrow}{t_2}-\vec{t_2}]=\int dy(\vec{\alpha_0}-\alpha_0). \qquad (12.18)$$

Summing the directly transformed second equation of (12.18) with the first one, we obtain

$$(\alpha_1+\overleftarrow{\beta_1})_x+(\alpha_1+\overleftarrow{\beta_1})\int dy[\overset{\leftarrow 2}{t_1}-\overleftarrow{t_1}+\vec{t_2}-t_2]=0.$$

This means that the system (12.18) has a particular solution $\alpha_1+\overleftarrow{\beta_1}=0$. We will use this solution in what follows.

For this solution the system (12.18) is equivalent to a single equation for the unknown function α_1:

$$(\alpha_1)_x+\alpha_1\int dy[\overset{\leftarrow 2}{t_1}-\overleftarrow{t_1}+\vec{t_2}-t_2]$$
$$=\int dy[(\overset{\leftarrow 2}{t_1}\overleftarrow{\alpha_1}-t_2\alpha_1)-(\overset{\leftarrow 1}{t_1}\alpha_1-\vec{t_2}\vec{\alpha_1})].$$

This equation has the obvious solution $\alpha_1=1$. As a corollary, we obtain a second particular solution of our main equation:

$$T_1=T_0\int dy(\overleftarrow{t_1}-\vec{t_2}).$$

Further evolution of the equation for α_1 is connected with the representation of the unknown function in an integral form $\alpha_1\rightarrow\int dy\,\alpha_1$ (we retain the same symbol for the unknown function because this cannot lead to misunderstanding),

$$(\alpha_1)_x+\alpha_1\int dy[\overset{\leftarrow 2}{t_1}-\overleftarrow{t_1}+\vec{t_2}-t_2]$$
$$=\overset{\leftarrow 2}{t_1}\int dy(\overleftarrow{\alpha_1}-\alpha_1)+\vec{t_2}\int dy(\vec{\alpha_1}-\alpha_1). \qquad (12.19)$$

Apart from the obvious replacement $\overleftarrow{t_1}\rightarrow\overset{\leftarrow 2}{t_1}$, this coincides with Eq. (12.17) for α_0.

We can repeat the same trick for this equation as for the equation for α_0, and after k steps we will come to the substitution

$$\alpha_k=\overset{\leftarrow(k+1)}{t_1}\alpha_{k+1}-\vec{t_2}\overrightarrow{\alpha_{k+1}}$$

and the equation for α_{k+1},

$$(\alpha_{k+1})_x+\alpha_{k+1}\int dy[\overset{\leftarrow k+2}{t_1}-\overset{\leftarrow k+1}{t_1}+\vec{t_2}-t_2]$$
$$=\int dy[(\overset{\leftarrow k+2}{t_1}\overleftarrow{\alpha_{k+1}}-t_2\alpha_1)-(\overset{\leftarrow k+1}{t_1}\alpha_1-\vec{t_2}\vec{\alpha_1})],$$

with the obvious solution $\alpha_{k+1}=1$.

Collecting all the results together, we obtain a particular solution of the main equation in the formula

$$T_n=T_0\prod_{i=1}^{n}\left(1-L_i\exp\left[-(i+1)d_i-\sum_{k=i+1}^{n}d_k\right]\right)\int dy\overset{\leftarrow 1}{t_1}\int dy\overset{\leftarrow 2}{t_1}\ldots\int dy\overset{\leftarrow n}{t_1}, \qquad (12.20)$$

where the symbol $\exp d_s$ denotes the shift of the argument of the s-fold integral $(\ldots\int dy\overset{h\rightarrow}{t_1}\ldots\rightarrow\ldots\int dy\overset{h+1\rightarrow}{t_1}\ldots)$ in (12.13) by 1, and the symbol L_p indicates the exchange of $\overset{r\rightarrow}{t_1}$ with $\overset{r\rightarrow}{t_2}$ in the p-fold integral $\ldots\int dy\, \overset{r\rightarrow}{t_1}\ldots\rightarrow\ldots\int dy t_2 r\ldots$.

The expression (3.20) is directly applicable to the Heisenberg and Lotka–Volterra integrable hierarchies. In the case of the DT hierarchy it is necessary to put all the operators $L_i=1$ and use the equality $t_1=t_2=T_0$.

12.3. Examples

In this subsection we present the simplest integrable systems for the usual unknown functions u,v corresponding to the lowest solutions T_n of the main equation for the DT, Heisenberg, and LV substitutions.

12.3.1. Darboux–Toda substitution

$n=0$:

$$T_0=uv, \quad u_t=au_x+bu_y, \quad v_t=av_x+bv_y.$$

In the examples below we shall choose $a=1$, $b=0$, bearing in mind that it is always possible to add a term (with an arbitrary numerical coefficient) in which x is replaced by y and vice versa.

$n=1$:

$$T_1=vu_x-v_xu,$$

$$u_t=u_{xx}-u\int dy(uv)_x, \quad -v_t=v_{xx}-v\int dy(uv)_x.$$

This is the Davey–Stewartson equation in its original form.

$n=2$:

$$T_2=(uv)_{xx}-3u_xv_x-3uv\int dy(uv)_x,$$

$$u_t=u_{xxx}-3u_x\int dy(uv)_x-3u\int dy(u_xv)_x,$$

$$v_t=v_{xxx}-3v_x\int dy(uv)_x-3v\int dy(v_xu)_x.$$

This is the Veselov–Novikov equation.

$n=3$:

$$T_3=-(T_1)_{xx}-2(u_xv_{xx}-v_xu_{xx})+2uv\int dy(T_1)_x$$
$$+4T_1\int dy(uv)_x,$$

$$v_t=-v_{xxxx}+4v_{xx}\int dy(uv)_x-2v_x\left(\int dy(T_1)_x\right.$$
$$\left.-2\int dy(uv)_{xx}\right)+2v\left(\int dy(uv)_{xxx}\int dy(u_xv_x)_x\right.$$
$$+\int (uv_{xx})_x-\left(\left[\int dy(uv)\right]^2\right)_{xx}$$
$$\left.-\left[\int dy(uv)_x\right]^2\right).$$

The equation for u can be obtained from the one for v by the substitutions $u\rightarrow v$, $v\rightarrow u$, $t\rightarrow -t$.

12.3.2. Heisenberg substitution

$n=0$:

$$v_t=-v_{xx}+2v_x\int dy\left(\frac{uv_y}{1+uv}\right)_x,$$

$$-u_t=-u_{xx}+2u_x\int dy\left(\frac{vu_y}{1+uv}\right)_x.$$

$n=1$:

$$v_t+v_{xxx}-3v_{xx}\int dy\left(\frac{uv_y}{1+uv}\right)_x$$
$$+3v_x\left[\int dy\left(\frac{uv_y}{1+uv}\right)_x\right]^2+3v_x\int dy\left(\frac{u_xv_y}{(1+uv)^2}\right)_x$$
$$-3v_x\int dy\left(\frac{uv_y}{1+uv}\right)_{xx},$$

$$u_t+u_{xxx}-3u_{xx}\int dy\left(\frac{vu_y}{1+uv}\right)_x$$
$$+3u_x\left[\int dy\left(\frac{vu_y}{1+uv}\right)_x\right]^2+3u_x\int dy\left(\frac{v_xu_y}{(1+uv)^2}\right)_x$$
$$-3u_x\int dy\left(\frac{vu_y}{1+uv}\right)_{xx}.$$

12.3.3. Lotka–Volterra substitution

$n=0$:

In the case $T_0=v$ we obtain a trivial system with the help of (12.2):

$$u_t=u_y, \quad v_t=v_y.$$

$n=1$:

In this case,

$$S_1=v\int dx(\overleftarrow{t_1}-\overrightarrow{t_2})=v_y+v^2+2v\int dx(u_y).$$

The corresponding integrable system has the form

$$u_t=-u_{yy}+2(uv)_y+2u_y\int dx(u_y),$$

$$v_t=\left(v^2+v_y+2v\int dx(u_y)\right)_y.$$

In the one-dimensional case $D_x=D_y$ this system is a particular case of the more general integrable system described in Ref. 25.

$n=2$:

In this case,

$$S_2=v^3+3vv_y+v_{yy}+3vD_x^{-1}(uv)_y+3(v_y$$
$$+v^2)D_x^{-1}(u_y)+3v(D_x^{-1}(u_y))^2.$$

The corresponding integrable system can be written as

$$u_t=D_y(u_{yy}-3(vu_y)+3v^2u-3(u_y-uv)D_x^{-1}(u)_y)$$

$$+D_x(3D_x^{-1}(u_y)D_x^{-1}(uv)_y+(D_x^{-1}(u_y))^3),$$

$$v_t=D_y(v^3+3vv_y+v_{yy}+3vD_x^{-1}(uv)_y+3(v_y+v^2)D_x^{-1}(u_y)+3v(D_x^{-1}(u_y))^2).$$

13. FORMALISM OF THE SCALAR LA PAIR APPLIED TO PERIODIC TODA LATTICES[11,12,35]

Now we consider concrete realizations of the general results of Sec. 4 and apply them to the case of the system of equations of the periodic Toda lattice related to the classical A_n series. The case of the algebra A_1 (the sine–Gordon equation) has been considered in detail in the preceding subsection.

We use the following formulation of the equations for the generalized Toda lattice in two-dimensional space:

$$\text{(a)}\quad \frac{\partial^2 x_i}{\partial z\partial\bar{z}}=\exp(\tilde{K}x)_i;$$

$$\text{(b)}\quad \frac{\partial^2 \rho_\alpha}{\partial z\partial\bar{z}}=\exp\,\delta_\alpha-t_\alpha W_\alpha^{-1}\,\exp\left(-\sum\,\delta_\nu t_\nu\right),$$

$$\delta_\alpha=(k\rho)_\alpha, \tag{13.1}$$

where in case (a) the index i takes the values $1,2,...,r+1$; here $r+1$ is the rank of the simple infinite-dimensional algebra of finite growth with the generalized Cartan matrix $\tilde{K}$. In case (b) the number of values taken by the index α is one less than in case (a); k is the Cartan matrix (corresponding to $\tilde{K}$) of the finite-dimensional semisimple algebra, and the t_ν are the coefficients of the expansion of the maximal root of the algebra over the set of its simple roots. The system (a) admits the transformation $x_i\to x_i+(\Theta(z)+\bar{\Theta}(\bar{z}))N_i$, where $\vec{N}$ is the null vector of the generalized Cartan matrix: $(\tilde{K}N)=0$. Equation (b) is equivalent to (a) after eliminating the trivial solution of the homogeneous Laplace equation by means of a conformal transformation. Equation (b) is a direct consequence of the Lax representation

$$A_z=(h\rho_z)+\lambda\left(\sum_{\alpha=1}^{r}\,X_\alpha^+ + X_M^-\right),$$

$$A_{\bar{z}}=\lambda^{-1}\left(\sum_{\alpha=1}^{r}\,\exp(-\delta_\alpha X_\alpha^-)+\exp(M\rho)X_M^+\right),$$

$$[\partial_z-A_z,\partial_{\bar{z}}-A_{\bar{z}}]=0, \tag{13.2}$$

where $X_\alpha^\pm$ and $X_M^\pm$ are the root vectors of the simple and maximal roots of the algebra; $(M\rho)\equiv\Sigma t_\nu\delta_\nu$, and λ is the spectral parameter. For $X_M^\pm$ we take the normalization

$$[X_M^+,X_M^-]=\sum\,t_\nu W_\nu^{-1}h_\nu,\quad Wk=(kW)^T.$$

The algebra whose local part consists of the subspaces

$$g_{-1}=(\lambda^{-1}X_\alpha^-,\lambda^{-1}X_M^+),\quad g_1=(\lambda X_\alpha^+,\lambda X_M^-),\quad g_0=(h_\alpha)$$

is an infinite-dimensional semisimple algebra of finite growth. The degree of the parameter λ distinguishes the identical elements of the finite-dimensional algebra, relating them to the subspaces with whose grading index they are compared. Thus, they eliminate the degeneracy of the representations of algebras realized by finite-dimensional matrices. The Cartan elements of the finite-dimensional algebra appear in the subspaces whose grading index is the product of some integer and the height of the maximal root $m=\Sigma t_\nu$ increased by 1. The only element that is not distinguished by the degree of λ is the element H of the null subspace, $H=[X_M^+,X_M^-]=\Sigma t_{nu}W_\nu^{-1}h_\nu$. This circumstance explains why the number of unknown functions in (b) is smaller than in (a). Thus, the operators of the LA pair (13.2) should be treated as operator-valued functions of the finite-dimensional representation of an infinite-dimensional algebra of finite growth. The spectral parameter λ here plays the role of a grading parameter. The algebra of internal symmetry of the equations of the generalized Toda lattice is infinite-dimensional and coincides with the solvable part of such an infinite-dimensional semisimple algebra.

Equations (13.2) imply a "gradientness" of the LA pair operators:

$$g_z g^{-1}=(h\rho_z)-\sum_{\alpha=1}^{r}\,X_\alpha^+ +\lambda^{n+1}X_M^-,$$

$$g_{\bar{z}}g^{-1}=\sum_{\alpha=1}^{r}\,\exp(-\delta_\alpha X_\alpha^-)+\lambda^{-n-1}\,\exp(M\rho)X_M^+ \tag{13.3}$$

[(13.3) differs from (13.2) by a gauge transformation $g\to\exp\frac{1}{2}\ln H$, which results in the change $\lambda\to\lambda-1$ for the generators of the simple roots); g is an element of the complex hull of the group spanned by the elements of the semisimple algebra. Equation (13.3) can be regarded as a system of equations for the parameters of the element g. This system is naturally invariant under the choice of the concrete representation of the algebraic elements $X_\alpha^\pm$ in (13.3). We parameterize the element g by the Gauss decomposition $g=Z^-\exp(h\tau)Z^+$. Then (13.3) has as a consequence an essentially nonlinear system of equations relating the parameters of the elements Z^+, Z^-, τ. It seems quite remarkable that the equations for the parameters τ split from the general system, remaining essentially nonlinear. However, they split, in their turn, to a linear differential equation for the functions $\Psi_l=\exp\Sigma\tau_\alpha l_\alpha$, which are equal to the matrix elements of the group element g between the highest states of the representation $(l_1,l_2,\dots,l_r)$. However, in order to obtain the scalar LA pair equations of the given representation, one does not need to write the complete system of equations for the parameters of the element g and then single out the linear system for Ψ_l from it. It would be sufficient to calculate the derivatives of the Ψ_l up to the order N_l-1, using (13.3), and then to express them in terms of linear combinations of the matrix elements $\langle\alpha\|g\|l\rangle$ as follows (see Sec. 4):

$$\dot{\Psi}_l=\langle l\|\dot{g}\|l\rangle=\left\langle l\left\|\left[(h\rho_z)-\sum_{\alpha=1}^{r}\,X_\alpha^+ +\lambda^{n+1}X_M^-\right]\right\|l\right\rangle$$

$$=(l\rho_z)\Psi_l+\sum\,f_\alpha^1\langle\alpha\|g\|l\rangle.$$

Analogously, for the derivative of order s we obtain

$$\Psi_l^{[.s]}=\sum_{\alpha=1}^{N_l} f_\alpha^s\langle\alpha\|g\|l\rangle.$$

Inverting this equation, we find the equality

$$\langle\alpha\|g\|l\rangle=\sum_0^{N_l-1} F_s^\alpha\Psi_l^{[.s]}=\sum_0^{N_l-1}(\bar{F})_t^\alpha\Psi_l^{['t]}.$$

The matrix elements $\langle\alpha\|g\|l\rangle$ can be calculated by means of (13.3) in two forms, i.e., they can be expressed in terms of the derivatives with respect to the argument z or with respect to the argument $\bar{z}$. As a result, one obtains two forms of the matrix element $\langle\alpha\|g\|l\rangle$, which are N_l-1 equations for the scalar LA pair in the representation l. Two missing equations appear if one excludes the N_l matrix elements $\langle\alpha\|g\|l\rangle$ from the N_l+1 linear relations connecting them with the derivatives of Ψ_l up to the order N_l with respect to both arguments. These are two spectral equations of the representation l. In the general case, the structure of the spectral equations is as follows:

$$\Theta_{N_l}(D)\Psi_l=\lambda^{m+1}\Theta_{N_{l-m-1}}(D)\Psi_l,$$

where $\Theta_n(D)$ denotes the differential operator of nth order whose coefficient functions are homogeneous (with respect to differentiation) polynomials in ρ_α (13.3). For $N_l=m+1$, the right-hand side of the spectral equation does not contain the differentiation at all. Such a situation occurs only in the case of the simplest representations (of the lowest dimensions) of the algebras A_k, C_k, $(AB)_k$. For the classical series B_k and D_k the degrees of the differential operator on the right-hand side of the spectral equation are one and two, respectively.

14. SOLUTION OF THE SINE–GORDON EQUATION IN A FORM INVARIANT UNDER THE CHOICE OF THE REPRESENTATION[10]

Now, for the example of the sine–Gordon equation, we demonstrate the method of constructing the solutions without using a concrete realization of the algebra. This is a particular case of the general construction of the preceding section.

The sine–Gordon equation results from the compatibility of the linear system (the Lax pair)

$$\dot{g}g^{-1}=h\dot{\rho}+\lambda(X^++X^-),$$

$$g'g^{-1}=\lambda^{-1}(X^+\exp 2\rho+X^-\exp(-2\rho))$$

with g being an element of the $SL(2,C)$ group, and where X^+,X^-,h are the elements of its algebra, $[X^+,X^-]=h$, $[h,X^{+-}]=2X^\pm$. The internal symmetry algebra of the sine–Gordon equation is connected with the graded algebra of finite growth $SL(2,C)\times Z_2=\tilde{A}_1$ that has the background elements $X_{1,2}^\pm,h_{1,2}$ from which the whole algebra is constructed. The commutation relations are

$$[X_\alpha^+,X_\beta^-]=\delta_{\alpha,\beta}h_\alpha,\quad [h_\alpha,X_\beta^{+-}]=k_{\alpha\beta}X_\beta^{+-},$$

$$\alpha,\beta=1,2,\quad k=\begin{pmatrix}2 & -2\\ -2 & 2\end{pmatrix}.$$

By a direct check we confirm that the algebra $\tilde{A}_1$ has a multidimensional representation with the spectral parameter and generators

$$X_1^+=\lambda X^+,\quad X_2^+=\lambda X^-,\quad X_1^-=\lambda^{-1}X^-,$$

$$X_2^-=\lambda^{-1}X^+,\quad h_1=-h_2=h,$$

where $X^\pm,h$ are the elements of the algebra $SL(2,R)$ introduced above. Here the algebra as a whole consists of the sets of elements

$$R^-=(\lambda^{-2s-1}X^\pm,\lambda^{-2s}h),\quad R^0=(h),$$

$$R^+=(\lambda^{2s+1}X^\pm,\lambda^{2s}h),\quad s>0,$$

i.e., the positive, negative, and zero subalgebras of the initial algebra. Going back to the LA pair representation, we see that the element g can be considered as belonging not to the group $SL(2,C)$ but to some degenerate representation of the $\tilde{A}_1$ group. An arbitrary element of the $SL(2,C)$ group can be represented as the Gauss decomposition $g=\exp X^+\alpha\exp h\tau\exp X^-\beta$. Consequently, from the Lax representation, there appears a system of equations connecting the functions α, β, τ, which is obviously invariant under the choice of a certain representation of the algebra $SL(2,C)$. Thus, we have

$$\dot{g}g^{-1}=h\dot{\rho}+\lambda(X^++X^-)=(\dot{\alpha}-2\alpha\dot{\tau}-\alpha^2\dot{\beta}\exp(-2\tau))X^+$$
$$+(\dot{\tau}+\alpha\dot{\beta}\exp(-2\tau))h+\dot{\beta}\exp(-2\tau X^-),$$

where

$$\dot{\beta}\exp(-2\tau)=\lambda,\quad \alpha=\frac{\dot{\rho}-\dot{\tau}}{\lambda},\quad -\ddot{\tau}+(\dot{\tau})^2=\lambda^2-\ddot{\rho}+(\dot{\rho})^2.$$

Similarly,

$$\beta'\exp(-2\tau)=\lambda^{-1}\exp 2\rho,\quad \alpha=-\tau'\exp(-2\rho),$$
$$-\ddot{\tau}+2\dot{\tau}\dot{\rho}+(\dot{\rho})^2=\lambda^{-2}.$$

As a result of these equalities, we have a system of three equations

$$\ddot{\Psi}=(\lambda^2-\ddot{\rho}+(\dot{\rho})^2)\Psi,$$

$$(\Psi\exp\rho)''=(\lambda^{-2}-\rho''+(\rho')^2)\Psi,$$

$$\Psi'\exp(-2\rho)=\lambda^{-2}(\dot{\Psi}+\dot{\rho}\Psi), \tag{14.1}$$

where $\Psi=\exp(-\tau)$, $\alpha\exp(-\tau)$. Now we seek a solution of this system in the form

$$\Psi=\exp(\lambda z+\lambda^{-1}\bar{z})\prod_{k=1}^n(\lambda-a_k),$$

where the a_k are unknown functions defined from the system of equations which arise after substituting Ψ into the previous equations and comparing the terms with the same powers of λ:

$$\ddot{a}_i+2a_i\dot{a}_i+2\sum\frac{\dot{a}_i\dot{a}_k}{a_k-a_i}=0,$$

$$-\ddot{\rho}+(\dot{\rho})^2=2\sum\dot{a}_k. \tag{14.2}$$

The last equality can be satisfied by the substitution $\exp(-\rho)=\Pi a_k/\Pi\lambda_k$ [see the recurrence relations which follow from the system (7.3)]. The system (14.2) contains n first integrals, which give expressions for the first derivatives of a_k:

$$\dot{a}_k=\frac{P_n(a_k^2)}{\Pi_{l\neq k}(a_k^2-a_l^2)},\quad P_n(a_k^2)=\prod_{l=1}^{n}(a_k^2-a_l^2),\quad \dot{x}_l=0.$$

The parameters x_l independent of z in fact represent the first integrals of the system (14.2). Calculating $\ddot{a}_k$ from the last expression for $\dot{a}_k$, we confirm that (14.2) is valid. Further integration of the system is connected with the following identity from the theory of the symmetric functions of n arguments:

$$\Theta^s(x_j)=\sum_{i=1}^{n}\frac{x_i^s}{\Pi_{i\neq j}(x_i-x_j)}=0,$$

if $1<s<n-2$ and $\Theta^{n-1}(x_j)=1$. In fact, when reduced to a common denominator, $\Theta^s(x_j)$ represents a ratio of two homogeneous symmetric functions. Here the denominator is the Vandermond determinant, which is antisymmetric under permutation of all its arguments. Hence, the numerator should be characterized by the same property, which is possible only when s is not less than $n-1$. In this way we can carry out a further integration, with the result

$$\sum_{k=1}^{n}\int\frac{da_k}{a_k^2-x_s^2}=z+\frac{y_s}{x_s},\quad \prod_{k=1}^{n}\frac{a_k-x_s}{a_k+x_s}=\exp[2(x_sz+y_s)].$$

Thus, the last n relations mainly determine the general solution of the system (14.2), depending on exactly $2n$ parameters. So far only the first equation of the system (14.1) has been integrated, and to obtain the explicit dependence of the parameters x_s, y_s on the argument $\bar{z}$, one must use the remaining two equations. As far as $\exp(-\rho)=\Pi a_k/\Pi\lambda_k$ the function $\Psi\exp\rho$ has the form

$$\Psi\exp\rho=\lambda^n\exp(\lambda z+\lambda^{-1}\bar{z})\prod_{k=1}^{n}(\lambda^{-1}-a_k^{-1}),$$

i.e., with respect to the argument $\bar{z}$ the function $\Psi\exp\rho$ has the same structure as Ψ with respect to z, with the obvious substitution $a_k\to a_k^{-1}$, $\lambda\to\lambda^{-1}$. Thus, the second equation of (14.1) entails the system of equations involving differentiation with respect to the argument $\bar{z}$:

$$(a_k^{-1})'=\frac{\tilde{P}_n(a_k^{-2})}{\Pi_{l\neq k}(a_k^{-2}-a_l^{-2})},$$

$$\tilde{P}_n(a_k^{-2})=\prod_{l=1}^{n}(a_k^{-2}-a_l^{-2}),\quad (x_l)'=0,$$

or

$$a_k'=(-1)^n a_k^{2n}\frac{\tilde{P}_n(a_k^{-2})}{\Pi_{l\neq k}(a_k^2-a_l^2)}\prod_{l\neq k}a_l^2.$$

We have not yet used the third equation of the system (14.1). After simple transformations it can be written as

$$\dot{a}_k=\exp 2\rho a_k^2 a_k',\quad \exp 2\rho\left(1+\sum a_k'\right)=1,$$

$$\dot{\rho}=\exp 2\rho\sum a_k' a_k,$$

which in turn, after a corresponding substitution, lead to

$$P_n(a_k^2)=(-1)^n a_k^{2n}\tilde{P}_n(a_k^{-2})\exp 2\rho\prod a_l^2.$$

This means that $\dot{a}_k$ and a_k' are defined by the same polynomial, whose roots x_l^2 depend neither on z nor on $\bar{z}$, i.e.,

$$\tilde{x}_l=x_l,\quad \exp(-\rho)=\frac{\Pi a_l}{\Pi x_l}.$$

It is easy to see that the system of equations is invariant under the substitution $z\to\bar{z}$, $a_k\to a_k^{-1}$, $x_k\to x_k^{-1}$. Then from our previous results we find

$$\prod_{k=1}^{n}\frac{a_k-x_s}{a_k+x_s}=\exp[2(x_sz+x_s^{-1}\bar{z}+y_s)]\equiv\exp 2z_s,$$

where the parameters x_k, y_k are independent of both z and $\bar{z}$. From the last equality we obtain a linear algebraic system for the homogeneous symmetric functions $s_r=\Sigma_{i\neq j\neq\ldots\neq k}a_ia_j\ldots a_k$ in the form

$$\sinh z_l s_n-x_k\cosh z_k s_{n-1}+x_k^2\sinh z_k s_{n-2}-\ldots=0,$$

$$s=1,2,\ldots,n.$$

Solution of this system for $s_n=\Pi a_l$ yields the well-known n-soliton solution of the sine–Gordon equation in the form of the ratio of two determinants of order n.

15. GENERALIZED BARGMANN POTENTIALS[13]

In this section we establish a condition for the nth-order ordinary differential equation

$$\Psi^{[.k+1]}+\sum_{i=0}^{k-1}u_i\Psi^{[.i]}=\lambda^{k+1}\Psi \tag{15.1}$$

to have a solution with the following analytic dependence on λ:

$$\Psi=\exp\lambda z\prod_{k=1}^{n}(a_k-\lambda).$$

A problem of this type, applied to the quantum-mechanical one-dimensional Schrödinger equation, was first considered by Bargmann. For this reason, the coefficient functions u_i of the last equation will be called generalized Bargmann potentials. To solve the Bargmann problem, we need an expression for the coefficient functions of an ordinary differential equation in terms of the full set of its linearly independent solutions. The following statement generalizing the Wiett and Gauss theorems for the case of polynomials holds: The equation

$$\Psi^{[.k+1]}+\sum_{i=0}^{k-1}u_i\Psi^{[.i]}=0$$

can be represented in the form

$$V_k^{-1}(V_k^2V_{k-1}^{-1}(V_k^{-1}V_{k-1}^2V_{k-2}^{-1}(\ldots(V_1^2V_2^{-1}(V_1^{-1}\Psi)^\cdot)^\cdot\ldots)^\cdot=0,$$

where the V_i are the principal minors of the matrix of the Wronskian $V_\alpha^\beta=\Psi_\beta^{[\alpha-1]}$ $(1<\alpha,\ \beta<k+1)$ and generate the full set of $k+1$ linearly independent solutions of the equation. This is the Frobenius theorem. The condition that the Wronskian is a constant, $V_{k+1}=1$, is solved as follows:

$$\Psi_1=\varphi_1,\quad \Psi_1=\varphi_1\int^z dz_1\varphi_2,$$

$$\Psi_s=\varphi_1\int^z dz_1\varphi_2\int^{z_1}dz_2\varphi_3\ldots\int^{z_{s-1}}dz_{s-1}\varphi_s, \tag{15.2}$$

where the functions φ_l obey the single condition $\Pi_{l=1}^{k+1}\varphi^{k+2-l}=V_{k+1}=1$. The set of $(k+1)$ functions Ψ_l manifestly obeys the equation

$$(\varphi_{k+1}^{-1}(\varphi_k^{-1}(\ldots(\varphi_2^{-1}(\varphi_1^{-1}\Psi)^\cdot)^\cdot)\ldots)^\cdot=0.$$

All that remains is to express φ_l in terms of Ψ_k. As a consequence of the definition of the matrix V and of Ψ, we find

$$V_s=\prod_{l=1}^{s}\varphi_l^{s-l+1},\quad \varphi_1=V_1,\quad \varphi_2=V_1^{-2}V_2,\ldots,$$

$$\varphi_{l+1}=V_{l-1}V_l^{-2}V_{l+1}.$$

The substitution of the expressions obtained for φ into the previous equation completes the proof of the theorem.

The problem concerning the generalized Bargmann potentials is solved according to the following theorem:

The solution of Eq. (15.1) has an analytic dependence on the parameter λ of the form $\Psi=\exp\lambda z\Pi_{c=1}^n(a_c-\lambda)$ if the functions a_c are defined from the condition of the vanishing of the function

$$\tilde{\Psi}=\sum_{\alpha=1}^{k+1}c(\lambda_\alpha)\exp\lambda_\alpha z\prod_{c=1}^{n}(a_c-\lambda_\alpha),\quad \lambda_\alpha^{k+1}=\lambda^{k+1}$$

at n different points of the λ^{k+1} plane, λ_b^{k+1} $(1<b<n)$. The generalized Bargmann potentials u_i are expressed in terms of symmetric combinations constructed from a_c and their derivatives via quantities B_s^i, which are defined from the expressions for the derivative of order s of the function Ψ,

$$\Psi^{[.s]}=\left(\lambda^s+\sum_{i=0}^{s-2}B_s^i\lambda^i+\sum_{c=1}^{n}\frac{A_c^s}{a_c-\lambda}\right)\Psi,$$

as follows:

$$u_i=-\widetilde{B}_{k+1}^i\equiv-\left(B_{k+1}^i-\sum_{l=0}^{i-2}B_{k+1}^l\widetilde{B}_l^i,\right.$$

$$B_j^i=0,\quad j<i+2. \tag{15.3}$$

Equation (15.1) can be represented in the form

$$\exp\delta_k(\exp\delta_{k-1}(\ldots(\exp\delta_1(\exp(-\rho_1\Psi)^\cdot)^\cdot\ldots)^\cdot$$
$$=\lambda^{k+1}\exp\rho_k\Psi, \tag{15.4}$$

where $\exp\rho_s=J_{s-1}^0\Pi_{c=1}^n a_c^s$, the J_b^0 are the principal minors of the matrix of the conserved integrals, and $\delta_s=-\rho_{s-1}+2\rho_s-\rho_{s+1}$, $\rho_0=\rho_{k+1}=0$,

$$J_{i,j}=\delta_{i,j}+B_i^j+\sum_{c=1}^{n}\frac{A_c^ia_c^{k-j}}{a_c^{k+1}-\lambda^{k+1}} \tag{15.5}$$

for the null value of the parameter λ.

After calculating the logarithmic derivative of Ψ, we obtain

$$\dot{\Psi}=\left(\lambda+\sum_{c=1}^{n}\frac{\dot{a}_c}{a_c-\lambda}\right)\Psi\equiv\varphi^1\Psi.$$

For the sth derivative we have, by induction,

$$\Psi^{[.s]}=\varphi^s\Psi,\quad \varphi^{s+1}=\dot{\varphi}^s+\varphi^s\varphi^1.$$

With the help of these equalities we find recurrence relations for A_c^s and B_i^s.

Substituting the proposed form of Ψ into the general equation (15.1) and equating the quantities for different powers of λ, we obtain expressions for the Bargmann potentials according to the conditions of the theorem.

From the condition of the vanishing of the residues at the poles at the points $\lambda=a_c$ we obtain a system of nonlinear differential equations for the functions a_c:

$$\widetilde{A}_c^{k+1}\equiv A_c^{k+1}-\sum_{i=1}^{k-1}\widetilde{B}_{k+1}^iA_c^i=A_c^{k+1}-\sum_{i=1}^{k-1}B_{k+1}^i\widetilde{A}_c^i=0. \tag{15.6}$$

We shall show that the a_c, as defined by the conditions of the theorem, obey the relations (15.6). For this purpose, we consider the Wronskian constructed from the functions Ψ_α. In the notation of the preceding sections, we obtain

$$V_{k+1}=\|\Psi,\dot{\Psi},\ldots,\Psi^{[.k]}\|=\prod_{c=1}^{n}(a_c^{k+1}-\lambda^{k+1}),$$

$$\left\|1,\lambda+\sum_{c=1}^{n}\frac{A_c^s}{a_c-\lambda}\ldots\lambda^k+\sum_{i=0}^{k-2}B_k^i\lambda^i+\sum_{c=1}^{n}\frac{A_c^k}{a_c-\lambda}\right\| \tag{15.7}$$

$$=\prod_{c=1}^{n}(a_c^{k+1}-\lambda^{k+1})W(\lambda_1,\lambda_1,\ldots,\lambda_{k+1})\det_k J,$$

where W is the Vandermond determinant. The calculations in (15.7) were performed by the standard procedure, i.e., by subtracting the first column from the remaining one and removing the factor $\Pi_{\alpha=1}^{k+1}(\lambda_\alpha-\lambda_1)$, etc. It follows from (15.7) that, up to W_{k+1}, V_{k+1} is a polynomial of order n in the argument λ^{k+1} that vanishes as a result of the linear dependence of Ψ_α, in accordance with the conditions of the theorem, at n points λ_b^{k+1}, i.e.,

$$V_{k+1}=W_{k+1}\prod_{b=1}^{n}(\lambda_b^{k+1}-\lambda^{k+1}).$$

Consequently, $\dot{V}_{k+1}=\|\Psi,\dot{\Psi},\ldots,\Psi^{[.k-1]}\Psi^{[.k+1]}\|=0$. Calculating this determinant in the same way as (15.7), we verify that it has (15.4) as its consequence. To prove (15.5), we make use of the fact that neither the Bargmann potentials u_i nor the equations (15.6) for A_c^i depend on the parameter λ. According to the Frobenius theorem, we have

$$\varphi_1=\Psi_1(\lambda=0)=\prod_{c=1}^{n} a_c,$$

$$\varphi_1^2\varphi_2=\lim_{\lambda\to 0}(\lambda_2-\lambda_1)^{-1}\|\Psi,\dot\Psi\|=\lim_{\lambda\to 0}(\lambda_2-\lambda_1)\left\| 1,\lambda + \sum_{c=1}^{n}\frac{A_c^1}{a_c-\lambda}\right\|\prod_{c=1}^{n}(a_c-\lambda_1)(a_c-\lambda_2)$$

$$=\lim_{\lambda\to 0}\prod_{c=1}^{n}(a_c-\lambda_1)(a_c-\lambda_2)\times\left(1+\sum_{c=1}^{n}\frac{A_c^1}{(a_c-\lambda_1)(a_c-\lambda_2)}\right)$$

$$=\prod_{c=1}^{n}a_c^2\left(1+\sum_{c=1}^{n}\frac{A_c^1}{a_c^2}\right)=\prod_{c=1}^{n}a_c^2J_1^0.$$

Continuing the reduction procedure, we find

$$\prod_{\alpha=1}^{s}\varphi_\alpha^{s-\alpha+1}=\lim_{\lambda\to 0}W_s^{-1}(\|\Psi,\dot\Psi,\ldots,\Psi^{[.s]}\|)=\prod_{c=1}^{n}a_c^sJ_{s-1}^0.$$

From this we see that all the statements of the theorem are fulfilled, and the other form of the Bargmann potentials can be found after the differentiation in Eq. (15.4).

16. SOLUTION OF THE PERIODIC TODA LATTICE FOR THE A_k SERIES[11,12,35]

In this section a system of equations is constructed for the scalar LA pair of the first fundamental representation ($k+1$ dimension) of the algebra A_k. Using the results of the preceding section, its "wave function" and the solutions of the periodic Toda lattice are obtained.

The highest vector of the first fundamental representation $\|l\rangle$ ($\langle l\|$) satisfies the conditions

$$X_\alpha^+\|l\rangle=0,\quad \langle l\|X_\alpha^-=0h_\alpha\|l\rangle=\delta_{\alpha,1},\quad \langle l\|h_\alpha=\delta_{\alpha,1}.$$

The set of its basis vectors is as follows:

$$\|l\rangle,\quad X_1^-\|l\rangle,\quad X_2^-X_1^-\|l\rangle,\quad X_k^-\ldots X_2^-X_1^-\|l\rangle;$$

$$\langle l\|,\quad \langle l\|X_1^+,\quad \langle l\|X_1^+X_2^+,\quad \langle l\|X_1^+X_2^+,\ldots X_k^+.$$

We introduce the wave function $\langle l\|g\|l\rangle$ and, using (13.3), calculate its derivatives with respect to z:

$$\dot\Psi=\langle l\|\dot g\|l\rangle=\langle l\|\left(h\dot\rho+\sum_{\alpha=1}^{k}X_\alpha^++\lambda^{k+1}X_M^-\right)g\|l\rangle$$

$$=\dot\rho_1\Psi+\langle l\|X_1^+g\|l\rangle$$

or

$$\exp\rho_1(\exp(-\rho_1\Psi))=\langle l\|X_1^+g\|l\rangle.$$

Next,

$$(\exp\rho_1(\exp(-\rho_1\Psi)^{\cdot})^{\cdot}=\langle l\|X_1^+\dot g\|l\rangle=\langle l\|X_1^+(h\dot\rho)g\|1\rangle+\langle l\|X_1^+X_2^+g\|l\rangle$$

has as its consequence

$$\exp\rho_2-\rho_1(\exp\delta_1(\exp(-\rho_1\Psi))^{\cdot})^{\cdot}=\langle l\|X_1^+X_2^+g\|l\rangle.$$

Continuing the reduction procedure up to the sth step, we obtain

$$\exp\rho_s-\rho_{s-1}(\exp\delta_{s-1}(\exp\delta_{s-2}\ldots(\exp\delta_1(\exp(-\rho_1\Psi))^{\cdot})^{\cdot}\ldots)^{\cdot})^{\cdot})=\langle l\|X_1^+X_2^+\ldots X_s^+g\|l\rangle.\quad(16.1)$$

Finally, the $(k+1)$th step

$$\langle 1\|X_1^+X_2^+\ldots X_k^+(h\dot\rho+\lambda^{k+1}X_M^-)g\|l\rangle=-\dot\rho_k\langle 1\|X_1^+X_2^+\ldots X_k^+g\|l\rangle+\lambda^{k+1}\Psi$$

leads to the spectral equation

$$(\exp\delta_k(\exp\delta_{k-1}\ldots(\exp\delta_1(\exp(-\rho_1\Psi))^{\cdot})^{\cdot}\ldots)^{\cdot})^{\cdot}=\lambda^{k+1}\exp\rho_k\Psi.$$

Similarly, by differentiation with respect to $\bar z$ we obtain

$$\exp\delta_{s+1}(\exp\delta_{s+2}\ldots(\exp\delta_k(\exp(-(\rho_1+\rho_k)\Psi))')'\ldots)'=\lambda^{-(k+1)}\langle 1\|X_1^+X_2^+\ldots X_s^+g\|l\rangle.\quad(16.2)$$

Eliminating the matrix elements of the element g from (16.1) and (16.2), we obtain

$$\exp\rho_s-\rho_{s-1}(\exp\delta_{s-1}(\exp\delta_{s-2}\ldots(\exp\delta_1\times(\exp(-\rho_1\Psi))^{\cdot})^{\cdot}\ldots)^{\cdot})^{\cdot}=\lambda^{k+1}\exp\delta_{s+1}(\exp\delta_{s+2}\ldots(\exp\delta_k(\exp(-(\rho_1+\rho_k)\Psi))')'\ldots)',$$

$$(\exp\delta_k(\exp\delta_{k-1}\ldots(\exp\delta_1(\exp(-\rho_1\Psi))^{\cdot})^{\cdot}\ldots)^{\cdot})^{\cdot}=\lambda^{k+1}\exp\rho_k\Psi,$$

$$(\exp\delta_2(\exp\delta_3\ldots(\exp\delta_k(\exp(-(\rho_1+\rho_k)\Psi))')'\ldots)')'=\lambda^{-(k+1)}\exp(-\delta_1\Psi).\quad(16.3)$$

The system of $k+2$ equations (16.3) is, in fact, a scalar LA pair of the first (vector) fundamental representation of the algebra A_k. The system (16.3) is invariant under the substitution

$$z\to\bar z,\quad \Psi\to\exp(-\rho_1\Psi),\quad \lambda\to\lambda_{-1},$$

$$\rho_1\to-\rho_1,\quad \rho_{k+2-s}\to\rho_s-\rho_1$$

($1<s<k+1$, $\rho_{k+1}=0$), i.e., under the Weyl reflection of the first simple root of the algebra A_k.

We shall seek the wave function of the system in the "soliton" form

$$\Psi=\exp(\lambda z+\lambda^{-1}\bar z)\prod_{c=1}^{n}(a_c-\lambda)\lambda_\alpha^{k+1}.$$

Eliminating $\Psi^{[.k]}$ from the Wronskian $\|\Psi,\dot\Psi,\ldots,\Psi^{[.k]}\|$, using the equation relating $\Psi,\dot\Psi,\ldots,\Psi^{[.k]}$ and Ψ' [the equation with $s=k$ in (16.3)], we obtain the equality

$$\|\Psi,\dot\Psi,\ldots,\Psi^{[.k]}\|=\lambda^{k+1}\exp(-(\rho_1+\rho_k))\|\Psi,\dot\Psi,\ldots,\Psi^{[.k-1]},\Psi^{[.]}\|.$$

Continuing the procedure of further elimination of the derivatives with the help of (16.3), we get the following chain of equations for the determinants:

$$\|\Psi,\dot{\Psi},\ldots,\Psi^{[.k]}\|=\lambda^{k+1}\exp(-(\rho_1+\rho_k))\|\Psi,\dot{\Psi},\ldots,\Psi^{[.k-1]},\Psi^{[']}\|$$
$$=\lambda^{2(k+1)}\exp(-(2\rho_1+\rho_{k-1}))\|\Psi,\dot{\Psi},\ldots,\Psi^{[.k-2]},\Psi^{['']},\Psi^{[']}\|=\ldots$$
$$=\lambda^{s(k+1)}\exp(-(s\rho_1+\rho_{k+1-s}))\|\Psi,\dot{\Psi},\ldots,\Psi^{[.k-s]},\Psi^{['s]},\ldots,\Psi'\|$$
$$=\lambda^{k(k+1)}\exp(-(k+1)\rho_1)\|\Psi,\Psi^{['k]},\ldots,\Psi'\|\lambda^{k(k+1)}\times(-1)^{k(k+1)}\|\bar{\Psi},\bar{\Psi}',\ldots,\bar{\Psi}^{['k]}\|, \tag{16.4}$$

where $\bar{\Psi}\equiv\exp(-\rho_1\Psi)$. The chain of equations (16.4), completed with two spectral equations, is completely equivalent to the system of equations of the scalar LA pair (16.3). It follows from the explicit form of the spectral equation with respect to the argument z that the first term in the equality chain (16.4) does not depend on z, and the last one does not depend on $\bar{z}$, and therefore each term of the chain is equal to some constant.

As for the equations of the scalar LA pair in the form (16.3), the following theorem, which generalizes the results of the preceding section in a natural way, is valid.

The solution of the system of equations for the scalar LA pair (16.4) is the wave function $\Psi=\exp(\lambda z+\lambda^{-1}\bar{z})\Pi_{c=1}^{n}(a_c-\lambda)$, where the $a_c(z,\bar{z})$ are defined by the condition of vanishing of the function

$$\tilde{\Psi}=\sum_{\alpha=1}^{k+1}c(\lambda_\alpha)\exp(\lambda_\alpha z+\lambda_\alpha^{-1})\prod_{c=1}^{n}(a_c-\lambda_\alpha)\equiv\sum_{\alpha=1}^{k+1}c_\alpha\Psi_\alpha,\quad \lambda_\alpha^{k+1}=\lambda^{k+1},$$

at n different points λ_b^{k+1} of the λ^{k+1} plane $(1<b<n)$. The solutions of the equations of the periodic Toda lattice are given by the relations

$$\exp\rho_s=\prod_{c=1}^{n}\left(\frac{a_c}{\lambda_c}\right)^{s}J_{s-1}^{0}=\prod_{c=1}^{n}\left(\frac{a_c}{\lambda_c}\right)^{s-k-1}\tilde{J}_{s-k-1}^{\infty},$$

where the J_s^0 are the principal minors of the conserved integral matrix (see the preceding section) when $\lambda=0$, and $\tilde{J}_{s-k-1}^{\infty}$ are the principal minors of the matrix $\tilde{J}=\sigma J(a^{-1},\lambda^{-1}\sigma)$ for an infinite value of λ [σ is the constant $(k+1)\times(k+1)$ matrix with 1 on its antidiagonal and 0 in the other positions].

It follows from the results of the preceding section that the first spectral equation (with respect to the argument z) (16.3) is satisfied when the first expression for $\exp\rho_s$ from the formulation of the theorem is used; the second spectral equation, which is obtained from the first one by the Weyl transformation, is satisfied if the second expression for ρ is used. Thus, all that remains for us to do is to prove the consistency of these identifications. For this purpose we calculate the determinants in the equality chain. The first determinant was calculated in the preceding section, with the result

$$\|\Psi,\dot{\Psi},\ldots,\Psi^{[.k]}\|=W_{k+1}\prod_{b=1}^{n}(\lambda_b^{k+1}-\lambda^{k+1}).$$

For the simplification of all the following formulas we propose $\Pi_{c=1}^{n}\lambda_c^{k+1}=1$.

For the function $\tilde{\Psi}=\exp(-\rho_1\Psi)$, we have

$$\tilde{\Psi}=\prod_{c=1}^{n}a_c^{-1}\exp(\lambda z+\lambda^{-1}\bar{z})\prod_{c=1}^{n}(a_c-\lambda)$$
$$=(-1)^n\lambda^n\exp(\lambda z+\lambda^{-1}\bar{z})\prod_{c=1}^{n}(a_c^{-1}-\lambda^{-1})$$
$$=(-1)^n\lambda^{-1}\Psi(z\to\bar{z},\lambda\to\lambda^{-1},a\to a^{-1}).$$

A common term in (16.4), rewritten in the adopted notation, is calculated by the general scheme and leads to the result

$$\exp(-\rho_{k+1-s}\times(-1)^{ns}\lambda^{s(k+1)})\|\Psi,\dot{\Psi},\ldots,\Psi^{[.k-s]},\Psi^{[,s]},\ldots,\Psi'\|$$
$$=W_{k+1}\exp(-\rho_{k+1-s})\prod_{b=1}^{n}(\lambda_b^{k+1}-\lambda^{k+1})\det J^{k,s},$$

where the first $k+1-s$ rows of the matrix $J^{k,s}$ coincide with those of the integral of the motion matrix J, with elements

$$J_{i,j}=\delta_{i,j}+B_i^j+\sum_{c=1}^{n}\frac{A_c^i a_c^{k-j}}{a_c^{k+1}-\lambda^{k+1}},$$

$$\tilde{J}_{i,j}=\delta_{i,j}+\tilde{B}_{k+1-i}^{k+1-j}+\sum_{c=1}^{n}\frac{\tilde{A}_c^{k+1-i}a_c^{-k-1+j}}{a_c^{k+1}-\lambda^{k+1}}.$$

The quantities $\tilde{A},\tilde{B}$ are obtained from the corresponding quantities A,B by a Weyl transformation. By virtue of the condition of the theorem, the determinant $\|\Psi,\dot{\Psi},\ldots,\Psi^{[.k-s]},\Psi^{[,s]},\ldots,\Psi'\|$ vanishes at n points of the λ^{k+1} plane. Thus, the determinant (which is a polynomial of the nth degree in λ^{-1}) can differ from the Vandermond determinant only by some factor. Finally, we have

$$\exp\rho_{k+1-s}=\prod_{c=1}^{n}\frac{a_c^{k+1}-\lambda^{k+1}}{\lambda_c^{k+1}-\lambda^{k+1}}\det J^{k,s}. \tag{16.5}$$

This expression does not depend on λ, and it is convenient to calculate it when $\lambda=0$, $\lambda=\infty$. In the first case, the matrix $\tilde{J}$ becomes an upper triangular matrix with 1 on the principal diagonal; for this reason, from the last equality we obtain

$$\exp\rho_{k+1-s}=\prod_{c=1}^{n}a_c^{k+1-s}J_{k-s}^{0}.$$

In the second case, J becomes a lower triangular matrix, and so (16.5) results in

$$\exp\rho_{k+1-s}=\prod_{c=1}^{n}a_c^{-s}\tilde{J}_s^{\infty}.$$

Thus, the theorem is proved, and one more expression (16.5) is obtained for the solution of the periodic Toda-lattice equations for the series A_k.

Let us now write some relations that are useful for concrete calculations. Introducing the function $F=\sum_{a=1}^{k+1} c(\lambda_\alpha)\exp(\lambda_\alpha z+\lambda_\alpha^{-1})$ and the notation s_l for the elementary symmetric functions constructed from a_c, $s_l=\sum_{c\neq b\neq\ldots\neq d} a_c a_b \ldots a_d$, we rewrite the expression for the function $\tilde{\Psi}$, which appeared in the formulation of the theorem, in the form

$$\tilde{\Psi}=\sum_{c=0}^{n}(-1)^c s_{n-s}F^{[.c]}, \quad s_0=1,$$

$$F^{.(k+1)}=\lambda^{k+1}F, \quad F^{,(k+1)}=\lambda^{-k-1}F.$$

The system of equations for s_l can be written in the form

$$\sum_{c=0}^{n}(-1)^c s_{n-s}F_b^{[.c]}=0, \quad 1<b<n,$$

where each of the n functions F_b satisfies the equations

$$F_b^{.(k+1)}=\lambda_b^{k+1}F_b, \quad F_b^{,(k+1)}=\lambda_b^{-k-1}F_b.$$

For the matrix $J_{i,j}^0$, we have a recurrence relation that relates every row to the preceding ones and thus allows us to reconstruct the matrix as a whole, using only the elements of its first row. To do this, we take into consideration the fact that in accordance with the definition of J^0 [see (15.5) and the following formulas] the matrix elements J_{si}^0 appear in the expansions of the functions φ^s in powers of λ. That is,

$$\varphi^s=\lambda^s+\sum_{i=0}^{s-2}B_s^i\lambda^i+\sum_{c=1}^{n}\frac{A_c^s}{a_c-\lambda}=\lambda^s+\sum_{i=0}^{s-2}B_s^i\lambda^i$$

$$+\sum_{c=1}^{n}\frac{A_c^s}{a_c}+\sum_{j=1}^{\infty}\lambda^j\sum_{c=1}^{n}\frac{A_c^s}{a_c^{j+1}}=B_s^0+\sum_{c=1}^{n}\frac{A_c^s}{a_c^{-1}}$$

$$+\sum_1^{\infty}J_{si}^0\lambda^i=\varphi_0^s+\sum_1^{\infty}J_{si}^0\lambda^i.$$

The recurrence relations connecting the functions φ^s make it possible to establish the dependence of interest,

$$\varphi^{s+1}=\varphi_0^s+\sum_1^{\infty}J_{si}^0\lambda^i=\dot{\varphi}^s+\varphi^s\varphi^1=\dot{\varphi}_0^s+\varphi_0^s\varphi_0^1$$

$$+\varphi_0^1\sum_1^{\infty}J_{si}^0\lambda^i+\varphi_0^s\sum_1^{\infty}J_{1i}^0\lambda^i$$

$$+\sum_1^{\infty}\lambda^i\sum_{k=1}^{i-1}J_{sk}^0J_{k,i-k}^0+\sum_1^{\infty}\dot{J}_{si}^0\lambda^i,$$

$$\dot{J}_{si}^0=\varphi_0^sJ_{1,i}^0+\varphi_0^1J_{s,i}^0+\dot{J}_{si}^0+\sum_{k=1}^{i-1}J_{sk}^0J_{k,i-k}^0.$$

In the last equation, the first two terms, being proportional to the elements of the first and the sth rows, do not contribute to the principal minors (one can show that they may be omitted in the recurrence procedure as well)

Finally, we obtain

$$J_{s+1,i}^0=\dot{J}_{si}^0+\sum_{k=1}^{i-1}J_{sk}^0J_{k,i-k}^0.$$

As in the case of the Toda lattice with fixed end-points, from the solutions of the periodic Toda chain for the A_k series it is possible to construct solutions for some other series. The system of equations of the periodic Toda lattice is invariant under the substitution $\rho_\alpha\rightarrow\rho_{k+1-\alpha}$, and, consequently, among its solutions there are those such that $\rho_\alpha=\rho_{k+1-\alpha}$. A direct check shows that in the case $k=2n+1$ the system of equations of the periodic Toda-lattice series A_{2n+1} goes over into the system of equations of the periodic Toda-lattice series C_n; and in the case $k=2n$, to the series $(AB)_k$.

17. GENERAL SOLUTION OF THE PERIODIC TODA LATTICE[36,37]

Here we will consider the problem of constructing the general solution of the systems under consideration: the solution which possesses a sufficient set of arbitrary functions for the solution of the Cauchy or Goursat problems. We use the methods of construction of the general solution of the Toda chain with fixed end-points. As is well known, the algebra of the internal symmetry of the Toda lattice with fixed ends is finite, and therefore we have a finite number of terms in the expression for $\exp(-\rho)$ in its solution; in the periodic case, the algebra of the internal symmetry is infinite-dimensional and thus the number of terms in the corresponding expression is infinite. But it may be possible to prove that as a consequence of the properties of the semisimple infinite-dimensional algebras of finite growth these series converge absolutely.

From the beginning, for convenience we restrict ourselves to the case of the one-dimensional equations which arise from the general system of the Toda lattice:

$$\frac{\partial^2\rho_\alpha}{\partial z\partial\bar{z}}=\sum_{\beta=1}^{r}K_{\alpha,\beta}\exp\rho_\beta,$$

$$\frac{\partial^2 x_\alpha}{\partial z\partial\bar{z}}=\exp\sum_{\beta=1}^{r}K_{\alpha,\beta}x_\beta,$$

where $K_{\alpha,\beta}$ coincides with the generalized Cartan matrix of the semisimple infinite-dimensional algebra of restricted growth. The generalized Cartan matrix for the graded algebras of second rank brings this system of equations to the form

$$\frac{\partial^2x_1}{\partial z\partial\bar{z}}=\exp(2x_1-2x_2), \quad \frac{\partial^2x_2}{\partial z\partial\bar{z}}=\exp(-2x_1+2x_2),$$
$$\frac{\partial^2x_1}{\partial z\partial\bar{z}}=\exp(2x_1-x_2), \quad \frac{\partial^2x_2}{\partial z\partial\bar{z}}=\exp(-4x_1+2x_2), \tag{17.1}$$

which, if the variables x_1-x_2, $2x_1-x_2$ are introduced, yields the sine–Gordon equation $\partial^2x/\partial z\partial\bar{z}=\exp(2x)-\exp(-2x)$ and the Dodd–Boullow–Jeber–Schabat equation $\partial^2x/\partial z\partial\bar{z}=\exp x-\exp(-2x)$, respectively, in the first and second cases. Note that these equations together with the Liouville equation $\partial^2x/\partial z\partial\bar{z}=\exp 2x$ are exceptional among all equations of the form $\partial^2 x/\partial z \partial \bar{z} = f(x)$ because of the presence of the nontrivial internal symmetry group.

We know that in the case of the Toda chain with fixed end-points a solution for $\exp(-x_\alpha)$ can be expressed, apart from factors dependent only on z and $\bar{z}$, in terms of powers of repeated integrals of arbitrary functions. Let us assume that such a structure is also valid for the solutions in the contragradient case and rewrite the system (17.1) in the form (henceforth we will use only the first system)

$$\frac{\partial^2 x_1}{\partial z \partial \bar{z}} = \varphi_1 \bar{\varphi}_1 \exp(2x_1 - 2x_2),$$

$$\frac{\partial^2 x_2}{\partial z \partial \bar{z}} = \varphi_2 \bar{\varphi}_2 \exp(-2x_1 + 2x_2),$$

where φ_1, φ_2 are arbitrary functions of the argument z, and $\bar{\varphi}_1, \bar{\varphi}_2$ are the same for the argument $\bar{z}$. We have introduced arbitrary functions into (17.1) (which can be done by the substitution $x_\alpha \rightarrow x_\alpha + \ln \varphi_\alpha + \ln \bar{\varphi}_\alpha$ and a further conformal transformation), which play the role of inhomogeneities. After the substitution $\exp(-x_j) \rightarrow X_j$, the previous system becomes

$$X_1 \frac{\partial^2 X_1}{\partial z \partial \bar{z}} - \frac{\partial X_1}{\partial z} \frac{\partial X_1}{\partial \bar{z}} = \varphi_1 \bar{\varphi}_1 X_2^2 \quad (X_2),$$

$$X_2 \frac{\partial^2 X_2}{\partial z \partial \bar{z}} - \frac{\partial X_2}{\partial z} \frac{\partial X_2}{\partial \bar{z}} = \varphi_2 \bar{\varphi}_2 X_1^2 \quad (X_1^4).$$

In the brackets in these equations we give the right-hand side for the second system of (17.1). In the finite case, X_j are polynomials in the repeated integrals, the first term being equal to unity. Therefore, we assume that in the zeroth approximation in φ_l we have $X_1 = X_2 = 1$. Then the equations can be solved by iteration, where the small quantities are the corresponding powers of $\varphi_\alpha, \bar{\varphi}_\alpha$. The first-order approximation gives

$$X_1^1 = -\int dz \varphi_1 \int d\bar{z} \bar{\varphi}_1 \equiv -(1)(\bar{1}),$$

$$X_2^1 = -\int dz \varphi_2 \int d\bar{z} \bar{\varphi}_2 \equiv -(2)(\bar{2}).$$

The results of calculations up to the eighth order are listed below. It is worth noting that the proposed procedure for solving the systems with the exponential interaction is also applicable in the case of finite semisimple algebras. The only difference from the case of infinite graded algebras is the finiteness of the series in powers of the repeated integrals. The results are as follows:

$$X_1 = \sum_{k=0}^{\infty} (-1)^k \sum_{s=1} X_s^k \bar{X}_s^k = 1 - X_1^1 \bar{X}_1^1 + 2X_1^2 \bar{X}_1^2 - 4X_1^3 \bar{X}_1^3$$
$$-2X_2^3 \bar{X}_2^3 + 4X_1^4 \bar{X}_1^4 + 8X_2^4 \bar{X}_2^4 - 8X_1^5 \bar{X}_1^5 - 8X_2^5 \bar{X}_2^5$$
$$-16X_3^5 \bar{X}_3^5 + 8X_1^6 \bar{X}_1^6 + 16X_2^6 \bar{X}_2^6 + 16X_3^6 \bar{X}_3^6 + 32X_4^6 \bar{X}_4^6$$
$$-16X_1^7 \bar{X}_1^7 - 16X_2^7 \bar{X}_2^7 - 16X_3^7 \bar{X}_3^7 - 32X_4^7 \bar{X}_4^7 - 64X_5^7 \bar{X}_5^7$$
$$+32X_1^8 \bar{X}_1^8 + 32X_2^8 \bar{X}_2^8 + 64X_3^8 \bar{X}_3^8 + 64X_4^8 \bar{X}_4^8 + 64X_5^8 \bar{X}_5^8$$
$$+128X_6^8 \bar{X}_6^8.$$

The upper indices of X_s^k ($\bar{X}_s^k$) indicate the order of the approximation, while the lower ones stand for the order number in it. The values of X_s^k are as follows:

$$X_1^1 = (1), \quad X_1^2 = (12), \quad X_1^3 = (122), \quad X_2^3 = (121),$$

$$X_1^4 = (1212) + (1221), \quad X_2^4 = (1221),$$

$$X_1^5 = (12122) + (12212),$$

$$X_2^5 = (12121) + 2(12211),$$

$$X_3^5 = (12211),$$

$$X_1^6 = (122121) + (121221),$$

$$X_2^6 = (121211) + 3(122111),$$

$$X_3^6 = (121212) + (122121) + (21221) + 2(122112),$$

$$X_4^6 = (122112),$$

where $(ij\ldots k) = \int^z dz_1 \varphi_i \int^{z_1} dz_2 \varphi_j \ldots \int^{z_{n-1}} dz_n \varphi_k$ and the $\bar{X}_j^i$ are obtained from the X_j^i by the substitution $\varphi_s \rightarrow \varphi$, $z \rightarrow \bar{z}$. The term X_2 is obtained from X_1 by the replacement of indices $1 \rightarrow 2$ in the expressions for X_j^i. The number of terms in X_j^i will be called the length of X_j^i, $L(X_j^i)$. Thus, $L(X_1^2) = 1$, $L(X_2^5) = 3$, $L(X_3^6) = 5$, etc. Taking into account the fact that $(i,i,\ldots,i) = (i)^s/s!$ (where s is the number of repeated integrals) and the obvious solution $X_1 = X_2 = \exp(-(1)(\bar{1}))$ when $\varphi_1 = \varphi_2$, we find from the definition that

$$\sum_s c_s (L(X_s^k))^2 = k!,$$

from which it follows that for arbitrary functions $\varphi_{1,2}$ and $\bar{\varphi}_{1,2}$ bounded on the intervals (z_0, z) and $(\bar{z}_0, \bar{z})$ we can estimate the term of the kth approximation as

$$\sum_s c_s X_s^k \bar{X}_s^k \leqslant \frac{M^k \bar{M}^k}{k!} (z - z_0)^k (\bar{z} - \bar{z}_0)^k,$$

where M is the supremum of the functions $\varphi_{1,2}$ on the interval (z, z_0), and $\bar{M}$ is the same for the functions $\bar{\varphi}_{1,2}$ on the corresponding interval $(\bar{z}, \bar{z}_0)$. The series which gives the solutions $X_{1,2}$ converges absolutely. For this estimate it is essential that all the terms of the kth approximation, as well as all the terms in X_j^i, enter with the same sign. This is a direct consequence of the properties of the contragraded algebras of restricted growth. To obtain closed expressions for $X_{1,2}$, which would allow one, in particular, to calculate any term in the series, it is necessary to have some information about the representation theory for such algebras.

The set of simple roots of the graded algebras of finite growth $X_\alpha^\pm$ and its Cartan elements h_α satisfies the system of commutation relations

$$[X_\alpha^+, X_\beta^-] = \delta_{\alpha,\beta}, h_\beta, \quad [h_\beta, X_\alpha^{+-}] = \pm K_{\alpha,\beta} X_\alpha^\pm, \quad (17.2)$$

where K is the generalized Cartan matrix. Classification theorems and the explicit form of the matrix K for the algebras under consideration are well known. The Cartan matrix for the considered equations (a,b) has the form

$$\begin{pmatrix} 2 & -2 \\ -2 & 2 \end{pmatrix}\begin{pmatrix} 2 & -1 \\ -4 & 2 \end{pmatrix}.$$

In complete analogy with the finite case, the graded semisimple algebras possess a set of fundamental representations. Each of these representations is determined by its highest vector $\alpha\rangle$ with the properties

$$X^+_\beta \alpha\rangle = 0, \quad h_\beta \alpha\rangle = \delta_{\alpha,\beta}\alpha\rangle,$$

and all the other basis vectors of the representation, which is infinite-dimensional in this case, are constructed by successive applications of the generators of the negative simple roots to the highest vector. The properties of the graded semisimple algebras allow one to construct an invariant bilinear form in the representation space. To calculate the scalar products of basis vectors of the representations with the highest vector it is sufficient to know only the background commutation relations (17.2).

Now we describe the method of constructing the solutions for the case of arbitrary semisimple graded algebras of finite growth. First of all, two equations of the S-matrix type must be solved:

$$\frac{\partial M_+}{\partial z} = M_+ L^+(z), \quad \frac{\partial M_-}{\partial \bar{z}} = M_- L^-(\bar{z}),$$

where

$$L^+ = \sum_{\alpha=1}^{r} \varphi_\alpha(z) X^+_\alpha, \quad L^- = \sum_{\alpha=1}^{r} \bar{\varphi}_\alpha(\bar{z}) X^-_\alpha.$$

The functions $\varphi_\alpha(z)$, $\bar{\varphi}_\alpha(\bar{z})$ contained in the definition of the Lagrangians $L^\pm$ are arbitrary functions of their arguments. The solutions of the S-matrix equations can be represented in the form of ordered integrals, but the number of terms in this expansion will be infinite. With our notation and definitions for the X_α, for arbitrary semisimple algebras of finite growth we have

$$X_\alpha = \langle \alpha \| M_+^{-1} M_- \| \alpha \rangle. \tag{17.3}$$

The results of the beginning of this section obtained by the methods of perturbation theory are in fact special cases of the general formula (17.3).

Coming back to the beginning of this section, for the solution of the sine-Gordon equation we have

$$\exp x = \varphi_1^{1/2} \bar{\varphi}_1^{1/2} X_2 X_1^{-1},$$

where in the expressions for $X_{1,2}$ which follow from (17.3) we must put $\varphi_2 = \varphi_1^{-1}$, $\bar{\varphi}_2 = \bar{\varphi}_1^{-1}$. For the solution of the second equation we obtain

$$\exp x = \varphi_1 \bar{\varphi}_1 X_2 X_1^{-2}, \quad \varphi_2 = \varphi_1^{-2}, \quad \bar{\varphi}_2 = \bar{\varphi}_1^{-2}.$$

It should be noted that at present we have no proof of (17.1) except for the series expansion in powers of repeated integrals and a direct check of the validity of (17.3) in each order.

Thus, in the considered case the form of the general solution of the periodic Toda lattice is the same as for the Toda chain with fixed end-points. The main difference is that in the case of finite-dimensional algebras the series (given by perturbation theory) are finite, while in the case of infinite-dimensional algebras they are infinite. But the requirement of restricted growth guarantees their absolute convergence.

Now it is not known how to choose the "creating" functions $\varphi, \bar{\varphi}$ for constructing the soliton solution of the preceding section, and what is the criterion for summation of the series corresponding to this situation. This is an interesting unsolved problem.

18. SOLUTION OF THE MAIN CHIRAL PROBLEM WITH MOVING POLES BY THE METHODS OF THE RIEMANN PROBLEM[38,39]

In this section, by a special choice of the coefficient function of the homogeneous Riemann problem we show that its solution is connected with the main chiral-field problem with moving poles. This approach is by no means unique; the method of the Bäcklund transformation leads to the same results.

The main chiral-field problem with moving poles is described by the equation

$$(\xi - \bar{\xi}) \frac{\partial^2 F}{\partial \xi \partial \bar{\xi}} + \left[\frac{\partial F}{\partial \xi}, \frac{\partial F}{\partial \bar{\xi}}\right] = 0.$$

We illustrate the general scheme of its integration by the example of the simplest case of the $SL(2,C)$ algebra. Let the homogeneous Riemann problem on some contour have its usual form $\Omega_0 \Omega_+ = \Omega_-$, where $\Omega_\pm$ are the boundary values of two functions analytic outside and inside the contour, respectively. The element Ω_0 is chosen in the form

$$\begin{pmatrix} a & b\left(\dfrac{\lambda - \xi}{\lambda - \bar{\xi}}\right)^n \\ c\left(\dfrac{\lambda - \xi}{\lambda - \bar{\xi}}\right)^{-n} & d \end{pmatrix},$$

where a,b,c,d $(ad - cb = 1)$ are arbitrary functions of the argument λ with no singularities under analytic continuation inside the contour; the points $\lambda = \xi$, $\lambda = \bar{\xi}$, $\lambda = 0$ lie there. The condition in the neighborhood of infinity in the λ plane is $\Omega_+ \to 1 + f/\lambda$.

Let us rewrite the Riemann problem in the following form, which is more useful for our purpose:

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \tilde{\Omega}_+ = \exp\left[-n\left(\ln \frac{\lambda - \bar{\xi}}{\lambda - \xi}\right) \frac{H}{2} \Omega_-\right], \tag{18.1}$$

where $\tilde{\Omega}_+ = \exp[-n \ln[(\lambda - \bar{\xi})/(\lambda - \xi)](H/2)\Omega_+]$ with the new asymptotic condition $\tilde{\Omega}_+ \to 1 + [f + (n/2)(\xi - \bar{\xi})]/\lambda = 1 + F/\lambda$. The problem (18.1), rewritten in differential form, becomes

$$(\lambda - \xi)(\tilde{\Omega}_+)^{-1}(\tilde{\Omega}_+)_\xi = (\lambda - \xi)\Omega^{-1}(\Omega_-)_\xi + \Omega^{-1}\frac{nH}{2}\Omega_-,$$

$$(\lambda - \bar{\xi})(\tilde{\Omega}_+)^{-1}(\tilde{\Omega}_+)_{\bar{\xi}} = (\lambda - \bar{\xi})\Omega^{-1}(\Omega_-)_{\bar{\xi}} - \Omega^{-1}\frac{nH}{2}\Omega_-.$$

Taking into account the Liouville theorem and the properties of the Riemann problem, we conclude that both of these expressions are polynomials in the whole complex plane. The additional asymptotic condition is that the degrees of the polynomials are zeros. Calculating these polynomials in the neighborhood of infinity and at the point $\lambda=0$, which lies inside the contour, we have

$$F_\xi=-\xi G^{-1}G_\xi,\quad F_{\bar\xi}=-\bar\xi G^{-1}G_{\bar\xi},$$

$$G=\exp\left[-\frac{n}{2}\ln\frac{\bar\xi}{\xi}H\Omega(0)\right].$$

Making use of the Maurer–Cartan identity, we now conclude that F satisfies the equation of the main chiral field with moving poles.

Let us find the solution of the Riemann problem in the form

$$\Omega_+=\begin{pmatrix} 1+\sum_1^n \frac{e_s}{(\lambda-\xi)^s} & \sum_1^n \frac{f_s}{(\lambda-\xi)^s} \\ \sum_1^n \frac{g_s}{(\lambda-\bar\xi)^s} & 1+\sum_1^n \frac{h_s}{(\lambda-\bar\xi)^s} \end{pmatrix},$$

where e_s,f_s,g_s,h_s are $4n$ parameters, chosen so that the element Ω_- has no singularities at the points $\lambda=\xi$, $\lambda=\bar\xi$. We denote by $\varphi_s(y)$ the first s terms of its expansion in a Taylor series near the point $\lambda=y$, i.e.,

$$\varphi_s(y)=\varphi(y)+\frac{\lambda-y}{1!}\varphi(y)^{(1)}+\ldots+\frac{(\lambda-y)^s}{s!}\varphi(y)^{(s)}.$$

For the matrix element $(\Omega_-)_{1,1}$ we have

$$(\Omega_-)_{1,1}=a(\lambda)+\sum_1^n\frac{a(\lambda)e_s}{(\lambda-\xi)^2}+\sum_1^n\frac{b(\lambda)(\lambda-\bar\xi)^{n-s}g_s}{(\lambda-\xi)^n}$$

$$=a(\lambda)+\sum_1^n\frac{a(\lambda)-a_{s-1}(\xi)e_s}{(\lambda-\xi)^s}+\sum_1^n\frac{b(\lambda)(\lambda-\bar\xi)^{n-s}-(b(\lambda)(\lambda-\bar\xi)^{n-s})_{n-1}(\xi)g_s}{(\lambda-\xi)^n}$$

$$+\sum_1^n\frac{a_{s-1}(\xi)e_s}{(\lambda-\xi)^s}+\sum_1^n\frac{(b(\lambda)(\lambda-\bar\xi)^{n-s})_{n-1}(\xi)g_s}{(\lambda-\xi)^n}.$$

Within the contour C, the singularities may have only the terms of the last line of the previous equality. The absence of them in $(\Omega_-)_{1,1}$ is equivalent to the zero values of the residues up to the nth order in the above-mentioned expression. The same conditions on the matrix elements $(\Omega_-)_{1,2}$,$(\Omega_-)_{2,2}$,$(\Omega_-)_{2,1}$ lead to systems of linear algebraic equations which determine the unknowns e_s,g_s,f_s,h_s. In what follows we shall write them in the form of n-ordered columns. We have

$$\begin{pmatrix} \Gamma_+(\varphi^{-1})e+\Gamma_-(\xi,\bar\xi)g=0 \\ \Gamma_+(\varphi^{-1})f+\Gamma_-(\xi,\bar\xi)h=T \\ (\Gamma_-(\xi,\bar\xi))^{-1}e+\Gamma_+(\bar\varphi)g=-T \\ (\Gamma_-(\xi,\bar\xi))^{-1}f+\Gamma_+(\bar\varphi)h=0 \end{pmatrix},$$

where $\Gamma_+(\varphi)$ $(\Gamma_+(\bar\varphi))$ are upper triangular matrices all of whose elements parallel to the main diagonal are the same and equal to $(1/s!)\varphi^{(s)}(\xi)$ $((1/s!)\bar\varphi^{(s)}(\bar\xi))$, where s is the distance from the main diagonal. On the main diagonal there is the function $\varphi(\xi)$ $(\bar\varphi(\bar\xi))$ by itself; in the next place, its first derivative, and so on. The functions $\varphi(\xi)$, $\bar\varphi(\bar\xi)$ are related to the matrix elements of the homogeneous Riemann problem by the expressions

$$\varphi(\xi)=\frac{a}{b}(\lambda=\xi),\quad \bar\varphi(\bar\xi)=\frac{d}{c}(\lambda=\bar\xi),$$

$$\Gamma_+(\varphi_1)\Gamma_+(\varphi_2)=\Gamma_+(\varphi_1\varphi_2).$$

The sth row of the lower triangular matrix, $\Gamma_-(\xi,\bar\xi)=\Gamma_-(\xi-\bar\xi)\equiv\Gamma_-(x)$, consists of terms of the binomial expansion $(1+x)^s$ $[\Gamma_-(-x)=(\Gamma_-(x))^{-1}]$:

$$\Gamma_-(x)=\begin{pmatrix} 1 & 0 & \cdots & \ldots & 0 \\ x & 1 & 0 & \ldots & 0 \\ x^2 & 2x & 1 & \ldots & 0 \\ \ldots & \ldots & \ldots & \ldots & 0 \\ \ldots & \ldots & \ldots & \ldots & 0 \\ x^n & x^{n-1}C_n^1 & x^{n-2}C_n^2 & \ldots & 1 \end{pmatrix},$$

$$T=\begin{pmatrix} (-x)C_n^{n-1} \\ \ldots \\ \ldots \\ (-x)^{n-2}C_n^2 \\ (-x)^{n-1}C_n^1 \\ (-x)^n \end{pmatrix}.$$

In accordance with the previous results, the solution of the main chiral-field problem with moving poles is determined by the asymptotic form of the homogeneous Riemann problem, and its explicit form is

$$F=g_1X_-+\left(e_1+(\xi-\bar\xi)\frac{n}{2}\right)H+f_1X_+.$$

The explicit expressions for $e_1=-h_1,g_1,f_1$ as a solution of the system of linear algebraic equations can be represented as a sum of elements of an $n\times n$ matrix, which we denote by the letters corresponding to e,g,h,f:

$$G=(\varphi_--\varphi_+)^{-1},\quad E=-\frac{1}{2}(\varphi_-+\varphi_+)(\varphi_--\varphi_+)^{-1},$$

$$F=-\varphi_-(\varphi_--\varphi_+)^{-1}\varphi_+,$$

where φ_-, φ_+ are the lower and upper triangular matrices with equal elements at equal distances from the main diagonal. They can be written as

$$\varphi_-^s=-\frac{1}{(s-1)!}\frac{\partial^{s-1}}{\partial\xi^{s-1}}(\xi-\bar\xi)^s\frac{\partial\varphi}{\partial\xi},$$

$$\varphi_+^s=\frac{1}{(s-1)!}\frac{\partial^{s-1}}{\partial\bar{\xi}^{s-1}}(\xi-\bar{\xi})^s\frac{\partial\bar{\varphi}}{\partial\bar{\xi}},$$

$$s=1,2,\ldots,\varphi_-^0=\varphi(\xi),\quad \bar{\varphi}_+^0=\bar{\varphi}(\bar{\xi}).$$

Now we consider some more simple examples. Let $n=1$. In this case, all the matrices G,E,F are one-dimensional, and the solution of the main chiral-field problem takes the form

$$F=\frac{1}{\varphi-\bar{\varphi}}X_- - \frac{1}{2}\frac{\varphi+\bar{\varphi}}{\varphi-\bar{\varphi}}H-\frac{\varphi\bar{\varphi}}{\varphi-\bar{\varphi}}X_+.$$

This is no more than (apart from a gauge transformation) the 't Hooft solution in the spherically symmetric case.

Let $n=2$. All the matrices in the problem are two-dimensional. The matrices $\varphi_\pm$ have the form

$$\varphi_+=\begin{pmatrix}\bar{\varphi} & x\dot{\bar{\varphi}}\\ 0 & \bar{\varphi}\end{pmatrix},\quad \varphi_-=\begin{pmatrix}\varphi & 0\\ -x\varphi' & \varphi\end{pmatrix},$$

where the dot and the prime denote differentiation with respect to the independent arguments ξ and $\bar{\xi}$, respectively; as before, $x=(\varphi-\bar{\varphi})$. For the solution F we obtain

$$\frac{x}{D}[(2\delta+x(\varphi'+\dot{\bar{\varphi}}))X_+ +(\delta(\bar{\varphi}+\varphi)+x(\bar{\varphi}\varphi'+\varphi\dot{\bar{\varphi}}))H$$
$$-(2\bar{\varphi}\varphi\delta+x(\bar{\varphi}^2\varphi'+\varphi^2\dot{\bar{\varphi}}))X_+],$$

where $D=\delta^2-x^2\dot{\bar{\varphi}}\varphi'$ and $\delta=\varphi-\bar{\varphi}$. The expression for the "instanton" charge density for the main chiral-field problem with moving poles was established in Sec. 1. For arbitrary n, the solution of the present section becomes

$$q\infty\ln\frac{\mathrm{Det}(\varphi_- - \varphi_+)}{(\xi-\bar{\xi})^{n^2}}=\frac{1}{(\xi-\bar{\xi})^2}\left[\frac{\partial^2}{\partial\xi\partial\bar{\xi}}\frac{(\xi-\bar{\xi})^2}{2}\right.$$
$$\left.+1\right]\frac{\partial^2}{\partial\xi\partial\bar{\xi}}\ln\frac{\mathrm{Det}(\varphi_- - \varphi_+)}{(\xi-\bar{\xi})^{n^2}}.$$

If the functions $\bar{\varphi}$ and φ are chosen in the "pole" form

$$\varphi=\sum_1^N\frac{c_s}{\xi+ia_s},\quad \bar{\varphi}=\sum_1^N\frac{c_s}{\bar{\xi}+ia_s},$$

where c_s, a_s are real parameters, then after substitution into the charge density and integration over the invariant measure we find the whole charge equal to N $(N\geqslant n)$.

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