The method of complex Coulomb coupling constant in nuclear physics

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The method based on analytic continuation of the Coulomb coupling constant to complex values is systematically reviewed, and a description is given of how this method can be used to study some problems in nuclear physics. © *1996 American Institute of Physics.* [S1063-7796(96)00604-3]

INTRODUCTION

The study of bound and resonance states using the Schrödinger equation is an important problem in theoretical physics. In solving many problems in nuclear and atomic physics it is necessary to take into account simultaneously short-range and Coulomb forces. In this review we study the properties of systems consisting of two particles interacting via such forces.

The inclusion of long-range Coulomb forces significantly complicates the problem. We shall therefore restrict ourselves to the consideration of only spherically symmetric potentials. For simplicity, we adopt the system of units in which $\hbar=2\mu=c=1$, where μ is the reduced mass of the system. Our study is based on the Schrödinger partial-wave equation

$$\psi''+\left[k^2-\frac{l(l+1)}{r^2}-\frac{a}{r}-V(r)\right]\psi=0, \tag{1}$$

where $k=\sqrt{E}$ is the momentum, E is the energy of the system, l is the orbital angular momentum, and $V(r)$ is a short-range potential (for example, the effective nuclear interaction potential) satisfying the condition

$$\int_0^\infty r|V(r)|\exp(\alpha r)dr<\infty. \tag{2}$$

The Coulomb coupling constant (CCC) is $a=zZa_0$ ($a_0=e^2$), where z and Z are the charges of the interacting particles. Physical values a_{ph} correspond to integer z and Z. Using the method of Ref. 1, it is easy to show that the class of potentials (2) includes not only the interaction of two charged point particles, but also the case where two particles interact in the form of spheres of different radii uniformly charged throughout their volume.

Let the regular solution $\varphi(l,a,k,r)$ and the Jost solution $f(l,a,\pm k,r)$ of Eq. (1) satisfy the boundary conditions

$$\lim_{r\to 0}\varphi(l,a,k,r)r^{-l-1}=1, \tag{3}$$

$$\lim_{r\to\infty}f(l,a,\pm k,r)\exp\left[\pm i\left(kr-\frac{a}{2k}\ln 2kr\right)\right]=1. \tag{4}$$

Then these solutions are related as

$$\varphi(l,a,k,r)=\frac{1}{2ik}[f(l,a,k)f(l,a,-k,r)-f(l,a,-k)f(l,a,k,r)], \tag{5}$$

where the Jost functions $f(l,a,\pm k)$ are defined in terms of the Wronskians

$$f(l,a,\pm k)=W[f(l,a,\pm k,r),\varphi(l,a,k,r)], \tag{6}$$

and the S matrix can be written as

$$S(l,a,k)=\exp(i\pi l)\,\frac{f(l,a,k)}{f(l,a,-k)}. \tag{7}$$

In the case of the Coulomb potential ($V\equiv 0$)

$$\varphi_0(l,a,k,r)=(2ik)^{-l-1}M_{ia/2k,l+1/2}(2ikr),$$

$$f_0(l,a,\pm k,r)=\exp(a\pi/4K)W_{\pm ia/2k,l+1/2}(\pm 2ikr),$$

$$f_0(l,a,\pm k)=\frac{\exp(a\pi/4k)\Gamma(2l+2)}{(\pm 2ik)^l\Gamma(l+1\mp ia/2k)},$$

$$S_0(l,a,k)=\Gamma(l+1+ia/2k)/\Gamma(l+1-ia/2k), \tag{8}$$

where $M_{ia/2k,l+1/2}$ and $W_{\pm ia/2k,l+1/2}$ are the Whittaker functions and Γ is the Euler gamma function.

We see that the energy, orbital angular momentum, CCC, and nuclear interaction constant g [when $V(r)$ in (1) is replaced by $gV(r)$] enter into the Schrödinger partial-wave equation as parameters whose physical values are real (and, moreover, discrete in the case of l and a). A number of important results have been obtained in scattering theory by analytic continuation of some of these parameters to complex values. It has been shown that the specific analytic properties of the scattering matrix for complex values of the parameter (for example, the nature and locations of the singularities) give direct information about the physical states of the system.

For example, Regge assumed that the orbital angular momentum can take complex values.[2] Then with each pole of the S matrix it is possible to associate a function $l=l(E)$, which for real values of E is called a Regge trajectory. It has been proved that a single pole moving in the l plane as the energy varies (a Regge trajectory) corresponds to several resonances and bound states having different values of l. Together, these form a family of states with identical internal quantum numbers, so that bound states and resonances can be described in a unified manner using the trajectory of an S-matrix pole in the complex l plane. Here it is important that in the Schrödinger equation all quantities (the CCC, reduced mass, and potential V) except the orbital angular momentum and energy are conserved, so that in the interaction of two fixed particles x and y the system $x+y$ has bound and resonance states located on a Regge trajectory.

The analytic properties of the scattering matrix in the complex angular-momentum plane are much simpler than those in the energy or momentum plane. For example, in the case of the Yukawa potential the S matrix is a meromorphic function in the region Re $\lambda>0$ ($\lambda=l+\frac{1}{2}$). The study of the analytic properties of the nonrelativistic scattering amplitude and the S matrix in the complex l plane carried out by Regge has been used to construct phenomenological models for describing collisions in high-energy physics. However, the initial high hopes regarding this theory proved to be unjustified.

Although the method of complex orbital angular momentum has been rigorously justified only for potential scattering, its applications to nonrelativistic problems in nuclear and atomic collisions are very limited. This is the case because the use of the Regge method requires solution of the Schrödinger equation and determination of the S matrix for a known potential. However, unfortunately, the exact form of the potential $V(r)$ is still unknown. Nevertheless, the Regge method has found some applications in nuclear and atomic scattering. In particular, the energy levels of compound nuclei have been used to construct Regge trajectories and to describe the light nuclear systems $\alpha-\alpha$, $\alpha-{}^{12}\mathrm{C}$, $\alpha-{}^{16}\mathrm{O}$, $p-{}^{12}\mathrm{C}$, $n-{}^{12}\mathrm{C}$, $p-{}^{16}\mathrm{O}$, and $n-{}^{16}\mathrm{O}$ (Ref. 3).

The analytic properties of the S matrix in the complex plane of the nuclear interaction constant have been studied in Refs. 4 and 5. The introduction of the Coulomb potential into the Schrödinger equation leads to difficulties in the study of the analytic properties of the S matrix, because in the integral equations used for this the free solutions and the kernels are constructed from Coulomb solutions, which have no universal representations. The analytic properties of the nuclear–Coulomb S matrix in the complex k plane for $l=0$, $a>0$ for a superposition of Yukawa potentials have been determined in Ref. 6. However, the most thorough and mathematically rigorous treatment of this problem for $l\geqslant 0$ and $a>0$ for short-range potentials is that of Ref. 1.

Thus, until recently the consequences of analytic continuation to complex values of all the parameters of the Schrödinger equation except for the CCC have been studied. It is therefore important and interesting to extend this work to the CCC. In this review we systematically describe a method based on analytic continuation of the CCC to complex values and on the principle of charge-independence of the nuclear forces, and we use it to study several problems in nuclear physics.

Useful physical results can be obtained by analytic continuation of the parameter a to complex values only when (by analogy with the Regge theory) all the quantities in the Schrödinger equation (the orbital angular momentum, reduced mass, and potential) are conserved, except for the CCC and the energy. In other words, there must exist *a single potential* $V(r)$ which for fixed orbital angular momentum determines the S matrix for different physical values of the CCC and the energy, and *a single reduced mass*. Clearly, in this case it is necessary to consider not the interaction of two fixed charged particles x and y, but rather the interaction of two sets of charged particles X and Y, such that the interaction potential $V(r)$ of any particle of the set X with any particle of the set Y and the reduced mass of the system consisting of these particles are identical, while the values of the CCCs are different. We stress that it is also possible to take a fixed charged particle x (or y) as X (or Y).

The approach considered here is of not only mathematical but also physical interest, because in nuclear and atomic physics there are real systems satisfying these conditions. In fact, if we take X and Y to be isobaric multiplets, the reduced mass of the system consisting of any member of X and any member of Y is practically constant, and, according to the principle of the charge-independence of nuclear forces, the interaction potential between any members of X and Y is the same. These arguments remain valid if X (or Y) is not a multiplet, but a fixed charged cluster x (or y). It is also possible to take X (or Y) to be isobars with identical spins and parities J^{π}, if they do not contain magic numbers of neutrons or protons. In atomic physics these conditions are satisfied by isoelectronic atoms and ions.

Here we should make two remarks regarding the method of complex values of the CCC. First, intensive work on constructing the theory of complex angular momentum was carried out for 15 years beginning in 1959 by a large number of authors, and there is extensive literature on this subject. However, no one has noticed the fact that analogous studies (at least in the mathematical sense) can be performed for the CCC. This is apparently because the physical meaning of the analytic continuation of the CCC to the complex plane has not been fully understood. Actually, in the Regge theory the physical interpretation is quite clear, because one is dealing with the interaction potential of two fixed particles x and y leading to several bound and resonance states of the composite system $x+y$ with different orbital angular momenta belonging to a single trajectory in the l plane. The psychological barrier probably arises because in the case of the complex a plane the values of the CCC must be variable, so that the Schrödinger equation can no longer be regarded as describing two fixed particles x and y (as is usual in quantum mechanics), but instead describes two groups of different charged particles X and Y. It has also been difficult to identify real physical systems (isobaric multiplets, nuclear isobars, isoelectronic atoms and ions) whose properties can be described by the method of complex CCC. Second, study of the complex a plane began after 1980, when the output of publications on "Reggeism" had practically ceased, and almost all the fundamental results of the theory of complex CCC have been obtained by the author of the present review.

Let us briefly outline this review. In Sec. 1 we consider several questions related to the analytic structure of the scattering matrix in the complex CCC plane. In Sec. 2 we formulate the inverse problem of quantum scattering theory in a new way and describe the procedure for reconstructing the interaction potential for two isobaric multiplets or for a charged cluster and a multiplet in terms of the phase shifts at fixed energy and orbital angular momentum, but different values of the CCC. In Sec. 3 the method of complex CCC is used to find regularities in the separation (decay) of even–even isobars into fragments, and semiempirical expressions are obtained for estimating the half-lives of α-active nuclei. In Sec. 4 we obtain rules for the separation or decay of members of isobaric multiplets and derive expressions for calculating some of the characteristics of nuclear states.

1. COMPLEX VALUES OF THE COULOMB COUPLING CONSTANT

In this section we consider the consequences of the analytic continuation of the CCC to complex values. In particular, we study the analytic and asymptotic properties of the regular and Jost solutions, the Jost functions, and the S matrix in the complex CCC plane. We investigate the properties of the poles of the scattering matrix in the a plane and give their physical interpretation. An upper bound is obtained on the number of bound states of a system of two particles for a given superposition of centrifugal, repulsive Coulomb, and nuclear potentials. The effect of a charge-independent potential on the behavior of poles in the a plane is studied, along with the threshold behavior of the poles. A modification of the Levinson theorem for the complex a plane is proposed. The method of complex CCC is used to solve the problem of separating the "nuclear" phase shift into purely nuclear and nuclear–Coulomb interference phase shifts. The behavior of the poles of the scattering matrix in the a plane for the case of the Coulomb potential is also studied.

1.1. The analytic properties of the Jost functions and the *S* matrix in the complex CCC plane

For studying the analytic properties of the Jost functions and the S matrix it is necessary to transform from the Schrödinger equation (1) with the boundary conditions (3) and (4) to the corresponding integral equations for $\varphi(l,a,k,r)$ and $f(l,a,-k,r)$. Use of the estimates of Ref. 7 for $\varphi_0(l,a,k,r)$ and $f_0(l,a,-k,r)$ and solution of the integral equations by iteration leads to the following estimates for $\varphi(l,a,k,r)$ and $f(l,a,-k,r)$:

$$|\varphi(l,a,k,r)|<A\left(\frac{r}{1+|k|r}\right)^{l+1}\exp\left[\left|\operatorname{Im} k\right|r+\frac{1}{2}\left|\operatorname{Im}\left(\frac{a}{k}\right)\right|\ln(1+|k|r)+AP(r)\right],$$

$$P(r)=\int_0^r\frac{r'|V(r')|}{1+|k|r'}\,dr',$$

$$|f(l,a,-k,r)|<B\left(\frac{|k|r}{1+|k|r}\right)^{-l}\exp\left[-(\operatorname{Im} k)r+\frac{1}{2}\operatorname{Im}\left(\frac{a}{k}\right)\ln(1+|k|r)+BQ(r)\right],$$

$$|\arg(-ik)|<\pi,$$

$$Q(r)=\int_r^\infty\frac{r'|V(r')|}{1+|k|r'}\exp\left\{(|\operatorname{Im} k|-\operatorname{Im} k)r'+\frac{1}{2}\left[\left|\operatorname{Im}\left(\frac{a}{k}\right)\right|+\operatorname{Im}\left(\frac{a}{k}\right)\right]\ln(1+|k|r')\right\}dr',\qquad(9)$$

where $|k|\geqslant k_0$, k_0 is an arbitrarily small positive number, and A and B are positive constants depending on l and k_0 (Ref. 8).

Then from (9), using (6) and (7), we conclude the following. If the potential satisfies the condition (2), the Jost function $f(l,a,-k)$ is analytic in the entire complex a plane for values $k\neq 0$ from the region $\operatorname{Im} k>-\alpha/2$ with a cut along the negative imaginary k semiaxis from $k=-i\alpha/2$ to $k=0$, and the scattering matrix $S(l,a,k)$ for physical values of l is meromorphic in the direct product of the a plane and the strip $|\operatorname{Im} k|<\alpha/2$ (excluding the essential singularity at $k=0$) with a cut along the imaginary k axis from $k=-i\alpha/2$ to $k=i\alpha/2$. This result shows that the analytic structure of the S matrix in the complex CCC plane is much simpler than in the momentum (energy) plane or in the orbital angular-momentum plane.

1.2. Poles of the *S* matrix in the complex CCC plane

It is well known that the nuclear–Coulomb S matrix can be represented as

$$\begin{aligned}S(l,a,k)&=S_0(l,a,k)S_{\mathrm{n}}(l,a,k)=S_0(l,a,k)\,\frac{f_{\mathrm{n}}(l,a,k)}{f_{\mathrm{n}}(l,a,-k)}\\&=S_0(l,a,k)\exp[2i\delta_l(a,k)]\\&=S_0(l,a,k)\,\frac{\cot\,\delta_l(a,k)+i}{\cot\,\delta_l(a,k)-i},\end{aligned}\qquad(10)$$

where $S_0(l,a,k)$ is the scattering matrix for the Coulomb potential, $f_{\mathrm{n}}(l,a,\pm k)$ and $S_{\mathrm{n}}(l,a,k)$ are the "nuclear" Jost functions and S matrix, and $\delta_l(a,k)$ is the "nuclear" phase shift added to the Coulomb phase shift and arising from the potential $V(r)$ (Refs. 7 and 9). Therefore, the S matrix in the complex a plane has "nuclear" poles [poles of the meromorphic function $S_{\mathrm{n}}(l,a,k)$] and an infinite denumerable set of Coulomb poles. For the time being we restrict ourselves to the study of the behavior of the "nuclear" poles. We consider a real potential $V(r)$. For real k, Eq. (1), the boundary conditions (3) and (4), and also (7) lead to the conjugation conditions

$$\varphi^*(l,a^*,k,r)=\varphi(l,a,k,r),\quad f^*(l,a^*,-k,r)=f(l,a,k,r),$$

$$f^*(l,a^*,-k)=f(l,a,k),\quad S^*(l,a^*,k)=S^{-1}(l,a,k).\qquad(11)$$

This implies that if for real k the scattering matrix has a pole at the point a, it also has a zero at the point a^*.

Since the poles of the S matrix are the zeros of the function $f(l,a,-k)$, from (4) and (5) at a pole we have

$$\varphi(l,a,k,r)\sim\frac{f(l,a,k)}{2ik}\exp\left[i\left(kr-\frac{a}{2k}\ln 2kr\right)\right]\quad(r\to\infty).\qquad(12)$$

If we multiply Eq. (1) for $\varphi(l,a,k,r)$ by $\varphi^*(l,a,k,r)$, subtract the complex conjugate from the result, and then integrate over r from 0 to ∞, for real $k^2\neq 0$ we find

$$2i\,\operatorname{Im} a\int_0^\infty\frac{|\varphi(l,a,k,r)|^2}{r}\,dr=\lim_{r\to\infty}(\varphi^*\varphi'-\varphi^{*\prime}\varphi).\qquad(13)$$

According to (12), for $k^2<0$ $(k=i\varkappa,\ \varkappa>0)$ the right-hand side of (13) is zero. Therefore, $\text{Im}\, a=0$, so that the poles are located on the real a axis. In Refs. 8 and 10 it was shown that these poles are simple.

To study the motion of the poles with varying $k^2<0$, we differentiate the left-hand side of (1) for $\varphi(l,a,k,r)$ with respect to k^2, multiply by $\varphi(l,a,k,r)$, and subtract (1) multiplied by $\partial\varphi(l,a,k,r)/\partial k^2$. Integration of the resulting expression over r from 0 to ∞ at the pole, taking into account (11), (12), and the k parity of the function $\varphi(l,a,k,r)$ gives

$$\frac{\partial a}{\partial k^2}=\int_0^\infty |\varphi(l,a,i\varkappa,r)|^2 dr\left(\int_0^\infty \frac{|\varphi(l,a,i\varkappa,r)|^2}{r}\, dr\right)^{-1}>0. \tag{14}$$

From this equation it follows that as $k^2<0$ increases, the poles move to the right along the real a axis. Since $\partial a/\partial k^2>0$ for any $k^2<0$, the direction of motion of the poles for $k^2\to -0$ cannot change during the motion. Here $a(k)$ takes physical values, that is, $a(k)=a_{\text{ph}}$. From this condition we find $k_0=i\varkappa_0$ and $E_0=-\varkappa_0^2$. Since the corresponding wave function $\varphi(l,a_{\text{ph}},i\varkappa_0,r)$ is square-integrable, at the energy E_0 there exists a bound state of the system of two interacting charged particles with CCC a_{ph} and orbital angular momentum l. For $k^2\to -\infty$, the "nuclear" poles apparently pass through the point $a=0$ to the left-hand a half-plane and move to infinity along the negative real a semiaxis.

Let us consider the case of real $k>0$. We shall show that $f(l,a,-k)\neq 0$ for real a. If $f(l,a,-k)=0$, then from (5) and (11) $\varphi(l,a,k,r)\equiv 0$, which contradicts the boundary condition (3). Therefore, for $k>0$ the poles must leave the real a axis and go into the complex a plane. Then $a(k)=\text{Re}\, a(k)+i\,\text{Im}\, a(k)$, where $\text{Im}\, a(k)\gtrless 0$. We assume that

$$\text{Re}\, a(k)=a_{\text{ph}},\quad |\text{Im}\, a(k)|\ll 1. \tag{15}$$

The pole (15) corresponds to a physical value of the momentum and a complex value of the CCC. Let us go to the complex k plane, choosing the complex value $k_0=\text{Re}\, k_0 + i\,\text{Im}\, k_0$ such that $\text{Re}\, a(k_0)=a_{\text{ph}}$ and $\text{Im}\, a(k_0)=0$. Expanding $a(k)$ in a series near k_0 and keeping only the linear term, we obtain

$$\text{Re}\, a(k)+i\,\text{Im}\, a(k)=a_{\text{ph}}+\frac{d\,\text{Re}\, a(k)}{dk}(k-k_0).$$

In this expression we have taken into account the fact that $a(k)$ is a nearly real analytic function near nearly real values of k_0. Then

$$\text{Re}\, k_0=k+\frac{a_{\text{ph}}-\text{Re}\, a(k)}{d\,\text{Re}\, a(k)/dk},\quad \text{Im}\, k_0=-\frac{\text{Im}\, a(k)}{d\,\text{Re}\, a(k)/dk}.$$

A complex value of k_0 corresponds to a complex value of the energy $E=E_0-i\Gamma/2$, where

$$E_0=(\text{Re}\, k_0)^2-(\text{Im}\, k_0)^2,\quad \Gamma=\frac{4\,\text{Re}\, k_0\ \text{Im}\, a(k)}{d\,\text{Re}\, a(k)/dk}. \tag{16}$$

Therefore, for $k>0$ a pole whose real part is close to the physical value a_{ph} and whose imaginary part is small in magnitude will, under the condition $d\,\text{Re}\, a(k)/dk>0$ [if $\text{Im}\, a(k)>0$] or $d\,\text{Re}\, a(k)/dk<0$ [if $\text{Im}\, a(k)<0$], describe a resonance in a system of two interacting charged particles with CCC a_{ph} and orbital angular momentum l.

By imposing additional conditions on the potential it is possible to obtain more definite information about the location of the poles of the scattering matrix in the complex CCC plane.[8,11] For example, for a generalized Yukawa potential $V(z)$ admitting analytic continuation to the right-hand z half-plane and satisfying the conditions

$$\text{Re}\, V(i\rho)\leqslant 0,\quad |\text{Im}\, V(i\rho)|<N/\rho,\quad 0\leqslant\rho<\infty, \tag{17}$$

it has been proved that the "nuclear" poles of the S matrix are located in the upper a half-plane in the region $\text{Re}\, a<N$.

The region where $S(l,a,k)$ is meromorphic has a cut along the imaginary k axis from $k=-i\alpha/2$ to $k=i\alpha/2$. Part of this cut (from $k=0$ to $k=i\alpha/2$) is due to the function $f(l,a,k)$, and for real a it covers the S-matrix poles in the k plane corresponding to bound states. Since $f(l,a,-k)$ is an analytic function of the complex variables a and k in the region where the S matrix is meromorphic with a cut along the negative imaginary k axis from $k=-i\alpha/2$ to $k=0$, then according to the implicit-function theorem the solution $a=a_l(k)$ of the equation $f(l,a,-k)=0$ determines a as an analytic function of k everywhere in this region except for points at which $\partial f(l,a,-k)/\partial a=0$. The latter corresponds to the condition for the existence of multiple poles in the a plane. For $k^2<0$ the real a poles are simple. For $k^2>0$ they move from the real a axis to the complex a plane. It can be expected that here the order of the poles does not grow and they remain simple. In any case, we shall assume that this is so. Therefore, $a=a_l(k)$ for real k^2 is an analytic function.

The trajectory described in the a plane by the pole $a_l(k)$ for $-\alpha^2/4<k^2<\infty$ is of particular interest. This curve is the mapping into the a plane of the positive imaginary k axis from $k=i\alpha/2$ to $k=0$ and of the positive real k axis from $k=0$ to $k=\infty$. This trajectory can give useful information about the interaction of charged particles.

As noted earlier, the theory of complex CCC deals with the interaction between two groups of charged particles X and Y arising from the superposition of centrifugal, Coulomb, and short-range potentials, in which only the Coulomb component varies. This interaction between pairs of particles from group X and group Y leads to the appearance of bound and resonance states in the group of charged particles Q. Therefore, in the region $\text{Re}\, a>0$ this theory (according to the principle of the charge independence of nuclear forces) describes Coulomb effects in the system $X+Y\leftrightarrow Q$, where X, Y, and Q are isobaric multiplets. We note that with certain restrictions (about which more will be said later) the theory of complex CCC is also valid for nuclear isobars.

If $V(r)$ is a generalized Yukawa potential satisfying the conditions (17), the following characteristics of the behavior of a pole trajectory can be established on the basis of the preceding results. For a repulsive potential there are no poles in the region $\text{Re}\, a>0$ (there are no bound states or resonances). For an attractive potential, a pole lying on the real a axis for $k^2<0$ moves to the right with decreasing $|k^2|$, forming bound states with orbital angular momentum l in the system $X+Y$ for each passage through the physical value of the CCC. If $k^2>0$ the pole leaves the real a axis and moves into the first quadrant of the a plane. A pole whose real part is equal to the physical value a_{ph} and whose imaginary part is small in magnitude corresponds to a resonance with orbital angular momentum l of the member of the system $X+Y$ with CCC a_{ph}.

To determine the shift of the poles as a function of the strength of the potential written as $gV(r)$, it is necessary to find $\partial a/\partial g$. From Eq. (1) for $k^2<0$ it follows that

$$\frac{\partial a}{\partial g}=-\int_0^\infty V(r)|\varphi(l,a,i\varkappa,r)|^2 dr \times\left(\int_0^\infty \frac{|\varphi(l,a,i\varkappa,r)|^2}{r}\,dr\right)^{-1}. \qquad (18)$$

Then the following conclusions can be drawn on the basis of (18). For weak attractive potentials the pole leaves the real a axis without reaching even the first physical value a_{ph}, and there are no bound states. This, however, does not exclude the possibility of resonances for $k^2>0$. As the strength of the attractive potential increases the pole moves farther and farther to the right in the region $\operatorname{Re} a>0$, which leads to the appearance of bound and resonance states with ever larger physical values of the CCC.

Therefore, for a given interaction potential between the multiplets X and Y, the resulting bound and resonance states of the members of the multiplet Q are located on a single trajectory $a_l(k)$, that is, they can be described in a unified manner by the trajectory of a pole of the scattering matrix in the region $\operatorname{Re} a>0$. Thus, the construction of a certain type of trajectory from the experimental data in principle allows the masses or excitation energies of members of isobaric multiplets to be predicted by interpolation or extrapolation.

1.3. The number of bound states for a given superposition of centrifugal, repulsive Coulomb, and nuclear potentials

In the absence of a Coulomb potential in the Schrödinger equation (1), the number of bound states n_l with orbital angular momentum l satisfies the well known Bargmann inequality:

$$n_l<\frac{1}{2l+1}\int_0^\infty r|V_-(r)|dr, \qquad (19)$$

where $V_-(r)$ is the negative part of the nuclear potential $V(r)$ (Ref. 2).

In Ref. 12 the standard method of Ref. 13 was used to establish an upper limit like (19) on the number of bound states of a two-particle system for which the Schrödinger equation contains a repulsive Coulomb potential in addition to the centrifugal and nuclear potentials. In particular, it was shown that the number of bound states n_{la} of a system with orbital angular momentum l and CCC a satisfies the inequality

$$n_{la}<2\int_0^\infty rI_{2l+1}(2\sqrt{ar})K_{2l+1}(2\sqrt{ar})|V_-(r)|dr, \qquad (20)$$

where I_{2l+1} and K_{2l+1} are the modified Bessel functions, which for $l,a\geqslant 0$ are real and non-negative.

For $a=0$ the inequality (20) automatically becomes the inequality (19). Obviously, the inclusion of a repulsive Coulomb potential can lead only to a decrease in the number of bound states, so that the inequality (19) remains valid. From the estimate^9

$$I_{2l+1}(2\sqrt{ar})K_{2l+1}(2\sqrt{ar})\leqslant[2(2l+1)]^{-1}$$

it follows that if the Schrödinger equation contains a repulsive Coulomb potential, the inequality (20) gives a more accurate upper limit on the number of bound states of the system than does the Bargmann inequality (19). For $a\to\infty$ the inequality (20) takes the form

$$n_{l\infty}<\frac{1}{2\sqrt{a}}\int_0^\infty \sqrt{r}|V_-(r)|dr.$$

This means that for a very large repulsive Coulomb potential there are no bound states if the negative part of the nuclear potential satifies the condition $\int_0^\infty\sqrt{r}|V_-(r)|dr<\infty$. The following estimate for the number of bound states with orbital angular momentum l and CCC a having energies $E\leqslant -k^2$ was also obtained in Ref. 12:

$$n_{la}(E\leqslant -k^2) < \frac{\Gamma(l+1+a/2k)}{2k\Gamma(2l+2)}\int_0^\infty M_{-a/2k,l+1/2}(2kr) \times W_{-a/2k,l+1/2}(2kr)|V_-(r)|dr. \qquad (21)$$

Obviously, for $k\to 0$ the inequality (21) becomes the inequality (20).

Any potential for which the integrals on the right-hand sides of (20) and (21) are bounded leads to a finite number of bound states n_{la} and $n_{la}(E\leqslant -k^2)$. Substituting the values $n_{la}=1$ and $n_{la}(E\leqslant -k^2)=1$ into (20) and (21), we obtain the restrictions on the potential necessary (but not sufficient) for the existence of at least one bound state with orbital angular momentum l and CCC a possessing energy $E\leqslant 0$ or $E\leqslant -k^2$.

It has been shown that the bound and resonance states of the multiplet Q lie on a pole trajectory $a_l(k)$, which is a solution of the equation $f(l,a,-k)=0$ for real values of k^2. However, the equation $f(l,a,-k)=0$ can have several solutions for fixed l. Under certain conditions, a multiplet corresponds to each such solution (trajectory). Therefore, several multiplets Q with identical spins and parities J^π can exist. The members of these multiplets with identical isospin projections T_z have different masses or excitation energies E_x.

The inequalities given above can be used to obtain estimates for the number of trajectories of "nuclear" poles in the a plane due to the interaction potential of X and Y, and, accordingly, for the number of multiplets Q with $J^\pi=\text{const}$ which arise. Actually, the number of trajectories of fixed orbital angular momentum l located on the positive real a semiaxis and reached for increasing negative values $E\leqslant -k^2$ of the point a is equal to the number of bound states $n_{la}(E\leqslant -k^2)$ determined by the inequality (21). We also see that the total number of trajectories passing through the point $a=0$ into the right-hand a half-plane is equal to the total number of bound states $n_l=n_{l0}$. Therefore, the Bargmann estimate (19) is valid for the total number of trajectories $a_l(k)$ with $l=\text{const}$ in the region $\operatorname{Re} a>0$.

1.4. Effect of a charge-dependent potential on the behavior of the *S*-matrix poles in the *a* plane

For simplicity, here we restrict ourselves to the system $x+Y \leftrightarrow Q$, where x is a fixed charged cluster and Y and Q are multiplets. Analysis of nucleon elastic scattering on nuclei shows that this process can be explained by the inclusion of a charge-dependent interaction in the optical potential.[14] Therefore, the interaction between x and the members of Y with isospin projections T_z can be described by the superposition of a central potential and a symmetry potential,

$$V(r)+\frac{2T_z}{A}V_T(r)=V(r)+\left(1-\frac{2a}{za_0A}\right)V_T(r), \tag{22}$$

where z is the charge of the cluster x. The second term in (22), representing a small correction to the central potential $V(r)$, arises from the difference between the proton and neutron masses, and also the effect of the Coulomb forces.

The effect of a charge-dependent potential on the behavior of the poles of the scattering matrix in the a plane has been studied in detail in Ref. 15, where the following was shown. If the interaction between x and Y is described by the potential (22), then for $k^2<0$ the simple "nuclear" poles of the S matrix are located on the real positive a semiaxis, and at physical values of the CCC these correspond to bound states of the system $x+Y$. For increasing $k^2<0$ these poles, depending on the sign of the function $F_l(a,i\varkappa)=\int_0^\infty[1/r-2V_T(r)/za_0A]|\varphi(l,a,i\varkappa,r)|^2dr$, move to the left (if $F_l<0$) or to the right (if $F_l>0$) at a rate inversely proportional to the magnitude of F_l. The presence of the potential V_T causes the poles to shift, that is, the energy of the bound states of the system $x+Y$ is changed. For $k^2>0$ the poles move into the complex a plane, where for the conditions given above they correspond to resonances of the system $x+Y$.

Thus, the method of complex CCC can also be used when the principle of charge-independence of the nuclear forces is violated, which is taken into account by the inclusion of a charge-dependent term in the potential.

1.5. Threshold behavior of the *S*-matrix poles in the complex CCC plane

The threshold behavior of the "nuclear" poles of the scattering matrix in the region $\operatorname{Re} a>0$ (Refs. 8 and 10) is conveniently studied by introducing the function

$$\begin{aligned}
Z_l(a,E)&=B_l(a,E)\left[H(a,E)+\frac{2iC(a,E)}{S_n(l,a,E)-1}\right],\\
B_l(a,E)&=E^l\prod_{n=1}^{l}\left(1+\frac{a^2}{4En^2}\right),\\
H(a,E)&=a\left[\psi\left(\frac{ia}{2\sqrt{E}}\right)-\ln\left(\frac{ia}{2\sqrt{E}}\right)\right]-i\sqrt{E},\\
C(a,E)&=\pi a[\exp(\pi a/\sqrt{E})-1]^{-1},
\end{aligned} \tag{23}$$

where ψ is the psi function. If the potential satisfies the condition (2), then Z_l is single-valued and meromorphic in the direct product of the a plane and the strip $|\operatorname{Im}k|<\alpha/2$. For physical values of the CCC, it follows from (23), using (10), that

$$\begin{aligned}
Z_l(a,E)&=B_l(a,E)\{C(a,E)[\cot\delta_l(a,E)-i]+H(a,E)\}\\
&=-\frac{1}{A_l}+\frac{1}{2}R_lE+\cdots,
\end{aligned} \tag{24}$$

where A_l and R_l are the "scattering length" and "effective range." Equation (24) is a modified effective-range expansion,[16] which in contrast to the usual expansion is valid not only for $k>0$, but also for complex values of k from the region $|k|<\alpha/2$, including the origin. Thus, the function Z_l is the natural generalization of (24) for complex values of the CCC, since it is regular and can be expanded near the origin (except for points at which $S_n(l,a,E)=1$, where it has poles).

The approximation (24) is useful when only the first few terms can be kept in the series in powers of E. The rapid convergence of this series depends on the form and strength of the interaction potential, on the nature of the singularities of the partial amplitude, and on the value of the energy E. The conditions for the expansion (24) to be applicable are $|R_l/A_l|\leqslant 1$ and $kR_l<1$ (Ref. 17). Since large values of A_l correspond to strong potentials, (24) is satisfied well for short-range potentials V with large values of the coupling constant.

Using (23), we can write the "nuclear" scattering matrix as

$$S_n(l,a,E)=1+\frac{2iB_l(a,E)C(a,E)}{Z_l(a,E)-B_l(a,E)H(a,E)}.$$

From this, taking into account (23), we find

$$\begin{aligned}
Z_l[a_l(E),E]-E^l\prod_{n=1}^{l}\left[1+\frac{a_l^2(E)}{4En^2}\right]\Bigg\{a_l(E)\Bigg[\psi\left[\frac{ia_l(E)}{2\sqrt{E}}\right]\\
-\ln\left[\frac{ia_l(E)}{2\sqrt{E}}\right]\Bigg]-i\sqrt{E}\Bigg\}=0,
\end{aligned} \tag{25}$$

which determines the location of the S-matrix poles in the complex a plane.

To study the behavior of the pole $a_l(E)$ near the essential singularity $E=0$, we introduce the constants $X_l=\partial Z_l/\partial a_l[a_l(0),0]$ and $Y_l=\partial Z_l/\partial E[a_l(0),0]$, which are real because the function $Z_l(a,E)$ is real at threshold. We note that $Z_l[a_l(0),0]=0$, and the threshold value of the pole $a_l(0)>0$ is real. If $E<0$, the pole lies on the real positive a semiaxis, that is, the functions $\Psi[ia_l(E)/2\sqrt{E}]$ and $\ln[ia_l(E)/2\sqrt{E}]$ are real. Expanding $Z_l[a_l(E),E]$ near the point $[a_l(0),0]$ and keeping only the linear term, and also using the known formulas for the psi function,[18] from (25) we obtain

$$X_l[a_l(E)-a_l(0)]+Y_lE-\frac{a_l^{2l-1}(0)E}{3(2)^{2l}(l!)^2}\approx 0. \tag{26}$$

If $E>0$, the pole is located in the complex a plane, that is, the functions $\Psi[ia_l(E)/2\sqrt{E}]$ and $\ln[ia_l(E)/2\sqrt{E}]$ are complex. After expanding $Z_l[a_l(E),E]$ near the point $[a_l(0),0]$, using (25), we obtain

$$X_l[a_l(E)-a_l(0)]+Y_lE-\frac{a_l^{2l-1}(0)E}{3(2)^{2l}(l!)^2}-\frac{i\pi a_l^{2l+1}(0)}{(2)^{2l}(l!)^2}$$
$$\times\exp\left[-\frac{\pi a_l(0)}{\sqrt{E}}\right]\approx 0. \tag{27}$$

Then from (26) and (27) it follows that the trajectory of the pole of the scattering matrix has the form

$$a_l(E)\approx\begin{cases}c_1+c_2E, & E\leqslant 0,\\ c_1+c_2E+ic_3\exp(-c_4/\sqrt{E}), & E\geqslant 0,\end{cases} \tag{28}$$

where the c_i are real constants $(c_1,c_4>0)$ depending on l and $a_l(0)$. Equation (28) is very important for the applications studied in the following sections.

1.6. Asymptotic form of the Jost functions and the *S* matrix for large complex CCC

The determination of the asymptotic form of the solutions of the Schrödinger equation, the Jost functions, and the S matrix at large complex values of the parameters is interesting in connection with the solution of the inverse problem of quantum scattering theory. We shall consider the case $l=0$, and to simplify the notation we omit l in all expressions, so that instead of $\varphi(l,a,k,r)$, $f(l,a,\pm k,r)$, $f(l,a\pm k)$, $S(l,a,k)$, and $\delta_l(a,k)$ we use $\varphi(a,k,r)$, $f(a,\pm k,r)$, $f(a,\pm k)$, $S(a,k)$, and $\delta(a,k)$. We assume that the potential $V(z)$ is an analytic function in the z plane and that it satisfies the following auxiliary [relative to (2)] conditions:

$$\int_0^\infty|\sqrt{z}V(z)||dz|<\infty,\quad |zV(z)|<\infty. \tag{29}$$

Then, according to the results of Ref. 19 obtained by the WKB semiclassical method, the solutions of the Schrödinger equation, the Jost functions, and the S matrix at large $|a|$ behave like the corresponding functions of the free Schrödinger equation:

$$\varphi(a,k,r)\sim\varphi_0(a,k,r)\sim\frac{i^{-3/2}}{\sqrt{\pi}}a^{-3/4}r^{1/4}$$
$$\times\cos(2\sqrt{-ar}-3\pi/4),$$
$$f(a,-k,r)\sim f_0(a,-k,r)$$
$$\sim\frac{i\sqrt{k}(-ia/2k)^{-ia/2k}a^{-1/4}r^{1/4}\exp[i(2\sqrt{-ar}-3\pi/4)]}{\exp[a\pi/4k-ia/2k]},$$

$$f(a,\pm k)\sim f_0(a,\pm k)\sim\frac{\exp[a\pi/4k\mp ia/2k]}{\sqrt{2\pi}(\mp ia/2k)^{\mp ia/2k+1/2}},$$
$$S(a,k)\sim S_0(a,k)$$
$$\sim(-1)^{ia/2k+1/2}(ia/2k)^{ia/k}\exp(-ia/k). \tag{30}$$

1.7. Representation of the scattering matrix in the complex CCC plane

Since for $k>0$ and any finite complex values of the CCC the function $S_{\mathrm{n}}(a,k)$ is meromorphic, it cannot have pole accumulation points inside the region where it is meromorphic, i.e., at finite distances $|a|$. From Sec. 1.6 and (10) we find the equation

$$\lim_{|a|\to\infty}[S_{\mathrm{n}}(a,k)-1]=0. \tag{31}$$

Therefore, the poles of the function $S_{\mathrm{n}}(a,k)$ also cannot have an accumulation point at infinity. Consequently, the number n of "nuclear" poles $a_j(k)$ in the complex a plane is finite.

Let us consider the integral

$$I=\oint_C\frac{S_{\mathrm{n}}(a',k)-1}{a'-a}da',$$

where we take the integration contour C to be a circle of infinite radius centered at the point $a'=0$. The point a lies inside the circle C and does not coincide with the poles $a_j(k)$ of the function $S_{\mathrm{n}}(a',k)$. We shall calculate I in two ways. First, using residue theory, we obtain

$$I=2\pi i\left[S_{\mathrm{n}}(a,k)-1+\sum_{j=1}^{n}\frac{S_{\mathrm{n}j}(k)}{a_j(k)-a}\right],$$

where $S_{\mathrm{n}j}$ is the residue of the "nuclear" S matrix at the pole $a_j(k)$. Second, integrating along the contour C and using (31), we find $I=0$. Therefore,

$$S_{\mathrm{n}}(a,k)=\prod_{j=1}^{n}\frac{a-b_j(k)}{a-a_j(k)}, \tag{32}$$

where the $b_j(k)$ are complex numbers found from the equation

$$\prod_{j=1}^{n}[a-a_j(k)]-\prod_{j=1}^{n}[a-b_j(k)]$$
$$+\sum_{j=1}^{n}S_{\mathrm{n}j}(k)\prod_{\substack{i=1\\ i\neq j}}^{n}[a-a_i(k)]=0$$

by equating the coefficients of identical powers of a to zero. After substituting (32) into (10) and using (8), we obtain the representation of the S matrix for $l=0$ and $k>0$ in the complex a plane:

$$S(a,k)=\frac{\Gamma(1+ia/2k)}{\Gamma(1-ia/2k)}\prod_{j=1}^{n}\frac{a-b_j(k)}{a-a_j(k)}. \tag{33}$$

If the potential $V(r)$ is real for $r\geqslant 0$, then according to (10) and (11) for real a and k we have

$$S_n(a,k)=\frac{f_n(a,k)}{f_n(a,-k)}=\exp[2i\delta(a,k)]=\prod_{j=1}^{n}\frac{a-a_j^*(k)}{a-a_j(k)}. \tag{34}$$

Equation (34) can be used to determine the behavior of the poles $a_j(k)$ for $k\to\infty$. Setting $r=0$ and $\mathrm{Im}\,k=0$ in (9), we obtain $\lim_{k\to\infty}Q(0)=0$. Then from the equation $f(a,\pm k)=\lim_{r\to 0}f(a,\pm k,r)$ and the estimates of Sec. 1.1 for $l=0$ we obtain

$$\lim_{k\to\infty}f_n(a,\pm k)=\lim_{k\to\infty}\frac{f(a,\pm k)}{f_0(a,\pm k)}=1,$$

that is, $\lim_{k\to\infty}S_n(a,k)=1$. It can therefore be concluded from (34) that the "nuclear" poles of the S matrix $a_j(k)$ for $k\to\infty$ either lie on the real a axis or move to infinity with bounded imaginary part.

1.8. Modification of Levinson's theorem for the complex *a* plane

One of the fundamental results of quantum scattering theory is Levinson's theorem, which states a relation between the phase shift at zero energy and the number of bound states of a system. Levinson's theorem has been proved for various types of potential.[20] It also proved possible to obtain a Levinson-type relation for Regge poles.[21] An analogous result specifying the relation between the value of the s-wave phase shift at zero energy $\delta(a,0)$ and the poles of the S matrix in the a plane[22] is presented below.

Let us consider a class of analytic potentials satisfying the conditions (2), (17), and (29). Let n_a be the number of poles of the S matrix located for $k=+0$ on the real positive a semiaxis to the right of a specified fixed value of the CCC a. Then

$$\delta(a,0)=\begin{cases}n_a\pi, & \text{if } a>0 \text{ or } f_n(a,0)\neq 0 \quad \text{for } a=0,\\ (n_a+1/2)\pi, & \text{if } f_n(a,0)=0 \quad \text{for } a=0.\end{cases} \tag{35}$$

Equation (35) is valid for the "nuclear" phase shifts characterizing the difference between scattering by a nuclear–Coulomb potential and scattering by a purely Coulomb potential. It is valid for any real values of $a>0$, and not only for physical ones. It has a simple physical interpretation. Each pole of the set $n_a(k=+0)$ corresponds to its own trajectory, which for decreasing negative values of E must pass through a_{ph} lying to the left of the real positive a semiaxis. Therefore, the number of bound states is equal to the number of poles n_a.

1.9. Separation of the phase shift into purely nuclear and nuclear–Coulomb interference phase shifts

The separation of the nuclear–Coulomb interference (or purely nuclear) phase shift is an important problem, as this is needed for the correct interpretation of the experimental data. Actually, very often theoretical calculations must be compared with the experimentally obtained sum of the purely nuclear phase shift δ_n and the nuclear–Coulomb interference phase shift δ_i. In spite of the fact that the problem of separating δ_n and δ_i has been studied for a long time,[23] the situation is still murky. Here we use the method of complex CCC to separate the phase shift into the purely nuclear and the nuclear–Coulomb interference phase shifts (in the one-pole approximation, and also for small a).[24]

In the one-pole approximation ($n=1$), Eqs. (10) and (32) give for the s wave

$$\exp[2i\delta(a,k)]=\frac{a-b_1(k)}{a-a_1(k)}. \tag{36}$$

If the complex phase shifts $\delta(\pm a,k)$ corresponding to "particle$_1$+particle$_2$" and "particle$_1$+antiparticle$_2$" (or "antiparticle$_1$+particle$_2$") scattering are known for fixed values of k, then $a_1(k)$ and $b_1(k)$ can be determined from (36). Then, setting $a=0$ in (36), we can easily extract the purely nuclear phase shift $\delta_n(k)=\delta(0,k)$ and, accordingly, the interference phase shifts $\delta_i^{\pm}(a,k)=\delta(\pm a,k)-\delta_n(k)$:

$$\delta_n(k)=\frac{1}{2i}\ln\frac{b_1(k)}{a_1(k)},$$

$$\delta_i^{\pm}(a,k)=\frac{1}{2i}\ln\frac{a_1(k)[\pm a-b_1(k)]}{b_1(k)[\pm a-a_1(k)]}. \tag{37}$$

Equations (37) solve the problem of separating δ_n and $\delta_i^{\pm}$ in the one-pole approximation.

We shall show that in some cases δ_n and $\delta_{rmi}^{\pm}$ can be determined approximately without the one-pole approximation (36), but using only the experimental data. For this we expand the "nuclear" phase shifts $\delta(\pm a,k)$ in a series in the CCC:

$$\delta(\pm a,k)=\sum_{n=0}^{\infty}\frac{1}{n!}\left.\frac{d^{(n)}\delta(\pm a,k)}{da^n}\right|_{a=0}\cdot(\pm a)^n$$

$$\approx\delta_n(k)+\left.\frac{d\delta(\pm a,k)}{da}\right|_{a=0}\cdot(\pm a). \tag{38}$$

The expansion (38) can be used only for $|a|\ll\zeta$, where ζ is the modulus of the "nuclear" pole of the S matrix closest to the origin in the a plane. Here $\zeta>0$, because the "nuclear" S matrix has no pole at the origin.[11] In other words, $\delta(\pm a,k)$ is analytic in a circle of radius ζ (depending, in general, on the momentum $k>0$). In this linear approximation, from (38) it is easy to obtain expressions relating δ_n and $\delta_i^{\pm}$ to the experimentally known "nuclear" phase shifts $\delta(\pm a,k)$:

$$\delta_n(k)\approx\frac{1}{2}[\delta(a,k)+\delta(-a,k)],$$

$$\delta_i^{\pm}(a,k)\approx\pm\frac{1}{2}[\delta(a,k)-\delta(-a,k)].$$

In contrast to (37), these equations are valid for arbitrary orbital angular momentum.

Thus, the method of complex CCC including Coulomb effects in the scattering of particles of roughly identical mass is the natural basis for separating the purely nuclear and nuclear–Coulomb interference contributions to the "nuclear" phase shift.

1.10. Description of the isoelectronic series of hydrogen using trajectories of *S*-matrix poles in the *a* plane

The case of an attractive Coulomb field [$a<0$ and $V\equiv 0$ in Eq. (1)] is of particular interest. This corresponds to the problem of an electron moving in the field of a nucleus (the hydrogen atom, the He^+ ion, and other hydrogen-like ions) or the system e^-+Y, where Y are nuclei with charges $Z=1,2,\ldots$. The reduced mass of this system is equal to the electron mass, and the CCC is $a_{\mathrm{ph}}=-Za_0$. In this case there are no resonances, but the behavior of the S-matrix poles in the a plane can be clearly seen for the example of bound states.[25]

Equation (1) with a Coulomb potential has an exact solution corresponding to the scattering matrix (8). The S-matrix poles in the a plane are determined by the values of a for which

$$l+1+\frac{ia}{2k}=-n', \tag{39}$$

where $n'=0,1,2,\ldots$. The number n' coincides with the number of nodes of the radial wave function. It follows from (31) that $a_n(k)=2ikn$, where $n=n'+l+1$ is the principal quantum number. Therefore, the trajectory of a pole in the a plane is specified by the pair of numbers n',l and describes the state with quantum numbers nl for the isoelectronic series of hydrogen. For example, if $n'=l=1$, the pole trajectory corresponds to the $3p$ state of hydrogen-like ions. The number of trajectories described by poles in the a plane for varying real k^2 is infinite. The trajectories of poles with identical n (but different n' and l) coincide.

If $k^2<0$, $k=i\varkappa$ ($\varkappa>0$), then $a_n<0$ and $da_n/d\varkappa<0$, so that the pole lies on the negative real a semiaxis and for decreasing $\varkappa$ moves to the right along this semiaxis. When for some $\varkappa$ the pole passes through the physical value of the CCC a_{ph} ($Z=1,2,\ldots$), the hydrogen-like ion of the element with order number Z acquires a bound state with quantum numbers nl and energy $E_n=-a_{\mathrm{ph}}^2/4n^2$.

Since $a_n(0)\equiv 0$, the origin in the a plane is a pole accumulation point. For $k>0$ the pole lies on the positive imaginary a semiaxis, and as k increases it moves upward along this semiaxis.

States with fixed quantum numbers can be described in a similar way also for other isoelectronic series. For example, in Ref. 26 trajectories of poles of the scattering matrix in the a plane were constructed for states of the isoelectronic series of sodium, and the unknown energy levels of the ion Cl VII were constructed by interpolating them. Therefore, the method of complex CCC can be used successfully in atomic physics to determine the bound-state energies of isoelectronic series.

2. A NEW FORMULATION OF THE INVERSE SCATTERING PROBLEM

Solving the inverse problem of quantum scattering corresponds to reconstructing the interaction potential from the experimental data, different versions of which include the phase shifts, the S matrix, the energies and normalization constants of the bound states, and the scattering amplitude. Inverse problems have been under investigation for more than sixty years, and a huge number of studies have been published on this topic. The most important and fundamental results on the inverse scattering problem in general are contained in Refs. 20 and 27–30.

Until recently, two traditional formulations were known in which the inverse scattering problem has a solution: (1) if the phase shifts for a single value of the angular momentum at all energies, the bound-state energies, and the normalization constants are known; and (2) if the phase shifts for all values of the angular momenta at a single fixed energy are known, i.e., if the scattering amplitude is known.

As a rule, the inverse scattering problem is solved for various types of equations using modifications or analogs of the Gel'fand–Levitan or Marchenko methods. It has turned out that the Kreĭn method has not been widely used to solve the inverse scattering problem for more complicated types of equations.

In recent years, various aspects of the inverse scattering problem have been studied intensively. In the past fundamental results have been obtained for one-dimensional and one-channel problems, but in recent years more complicated systems—multichannel, multidimensional, and multiparticle systems and systems with a nonlocal interaction—have been studied.[20]

As noted above, in the traditional approaches the potential is reconstructed from the experimental data at fixed physical values of either the energy or the orbital angular momentum. Recently, a more general approach has been used in which the energy and the square of the orbital angular momentum satisfy a linear relation in the initial data.[31]

In the traditional formulations of the inverse scattering problem it is assumed that the CCC is fixed, so that the interaction potential for two fixed particles x and y is reconstructed. However, if instead the CCC is allowed to vary, it is possible to reformulate the inverse scattering problem in such a way that the interaction potential $V(r)$ is determined from the phase shifts at fixed values of the energy (momentum) and orbital angular momentum, but different physical values of the CCC. In this case *a single potential* $V(r)$ must describe the interaction between two groups of charged particles X and Y having roughly *identical reduced masses*. According to the principle of the charge independence of nuclear forces, such groups of particles X and Y are isobaric multiplets (or nuclear isobars). Therefore, in the new approach to the inverse scattering problem discussed here, it is possible in principle to reconstruct the interaction potential for two multiplets (or for a charged cluster and a multiplet) from the experimental data. The solution of the inverse scattering problem in this new formulation requires the results of a phase-shift analysis taking into account the Coulomb or charge splitting of the phase shifts. Examples of such a phase-shift analysis for the pN interaction in the ${}^1S_0$ state or for the $\pi^{\pm}p$ interaction in the S_{31} and P_{33} states are given in Refs. 32 and 33.

In this section we use the integral method (of the Gel'fand–Levitan type) to solve the newly formulated inverse scattering problem. As a result, the complex interaction potential belonging to the class of analytic functions is reconstructed from the spectral function given in the a plane.[24,34] We also give the expressions from the generalized version of Darboux–Crum–Kreĭn transformations for the case where the Schrödinger equation contains a Coulomb potential in addition to centrifugal and nuclear potentials.[35]

2.1. The completeness relation for the set of solutions of the Schrödinger equation

Let us consider the Schrödinger equation (1), where $l=0$, $k>0$, and the CCC takes complex values. We assume that the complex potential $V(r)$ is an analytic function and satisfies the conditions (2) and (29). Then the functions $\varphi(a,k,r)$, $f(a,\pm k,r)$, and $f(a,\pm k)$ are analytic, and the scattering matrix $S(a,k)$ is meromorphic in the complex a plane and can be written in the form (33). Consequently, the S matrix for $k>0$ has a finite number n of "nuclear" poles in the a plane. The number of these poles and their locations in the a plane depend on the specific form of the potential $V(r)$. In addition, there is an infinite set of simple Coulomb poles on the imaginary positive a axis at the points $2ik(n+1)$, where $n=0,1,\ldots$. Therefore, the spectrum in the a plane is discrete and consists of a denumerably infinite set of points a_j.

A basic element in solving the inverse scattering problem is the proof of the completeness relation. To derive this relation we consider the integral

$$I_1=\oint_C da\int_0^\infty \frac{h(r')}{\sqrt{rr'}}G(a,k;r,r')dr', \tag{40}$$

where $h(r)$ is an arbitrary square-integrable function satisfying the condition $h(0)=0$, and

$$G(a,k;r,r')=\begin{cases}\dfrac{\varphi(a,k,r)f(a,-k,r')}{f(a,-k)}, & r\leqslant r',\\[2ex] \dfrac{\varphi(a,k,r')f(a,-k,r)}{f(a,-k)}, & r\geqslant r'\end{cases}$$

is the Green function. For the integration contour C we take a circle of infinite radius R centered at the point $a=0$. We then calculate I_1 in two different ways: first, using residue theory, and second, by integrating along the contour C.

Since $\varphi(a,k,r)$ and $f(a,-k,r)$ are analytic in the a plane, the poles of the integrand in (40) at the points $a=a_j$ are the zeroes of the function $f(a,-k)$ or the poles of the S matrix. We recall that the Coulomb poles are simple. Using the arguments of Sec. 1.2, we assume that the "nuclear" poles are also simple. Since the Schrödinger equation leads to

$$\left.\frac{\partial f(a,-k)}{\partial a}\right|_{a=a_j}=\frac{2ik}{f(a_j,k)}\int_0^\infty\frac{\varphi^2(a_j,k,r)}{r}dr, \tag{41}$$

using (5) and (41), from residue theory we find

$$I_1=2\pi i\int_0^\infty h(r')\left[\sum_j C_j\frac{\varphi(a_j,k,r)}{\sqrt{r}}\frac{\varphi(a_j,k,r')}{\sqrt{r'}}\right]dr', \tag{42}$$

where the normalization constants are

$$C_j=\frac{2ik}{f(a_j,k)}\operatorname{Res}_{a=a_j}\frac{1}{f(a,-k)}=\frac{2ik}{f(a_j,k)\left.\dfrac{\partial f(a,-k)}{\partial a}\right|_{a=a_j}}=\left[\int_0^\infty\frac{\varphi^2(a_j,k,r)}{r}dr\right]^{-1}, \tag{43}$$

Now let us calculate I_1 by the second method. Taking into account the asymptotic expressions (30) and also the well known expressions for the δ function

$$\lim_{R\to\infty}\frac{\exp(iRx)}{x}=i\pi\delta(x),\quad \delta(f(x))=\sum_n\frac{\delta(x-x_n)}{|f'(x_n)|},$$

where the x_n are simple poles of the function $f(x)$, and integrating along the circle of infinite radius, we obtain

$$I_1=2\pi ih(r). \tag{44}$$

From (42) and (44) we find the completeness relation

$$h(r)=\int_0^\infty h(r')\left[\sum_j C_j\frac{\varphi(a_j,k,r)}{\sqrt{r}}\frac{\varphi(a_j,k,r')}{\sqrt{r'}}\right]dr',$$

which can be written as follows, using the Stieltjes integral:

$$\int_D\frac{\varphi(a,k,r)}{\sqrt{r}}\frac{\varphi(a,k,r')}{\sqrt{r'}}d\rho(a)=\delta(r-r'), \tag{45}$$

where D is a region including the complex a plane, and the spectral function $\rho(a)$ is given by

$$\frac{d\rho(a)}{da}=\sum_j C_j\delta(a-a_j). \tag{46}$$

Using the Schrödinger equation, it is easy to show that

$$C_j\int_0^\infty\frac{\varphi(a_j,k,r)\varphi(a_{j'},k,r)}{r}dr=\delta_{jj'},$$

from which, together with (45) and (46), it follows that the set of functions $F_j(r)=(C_j/r)^{1/2}\varphi(a_j,k,r)$ forms a complete orthonormal set on the semiaxis. We note that it is not the functions $F_j(r)$ and $F_{j'}^*(r)$ which are orthogonal (as in the usual case), but the functions $F_j(r)$ and $F_{j'}(r)$. This is the case because the eigenvalues a_j are complex.[36]

2.2. The solution of the inverse scattering problem using scattering data at fixed physical values of the energy and the orbital angular momentum

It is easily verified that the functions $\Phi_{1,2}(a,k,r)=\varphi_{1,2}(a,k,r)/\sqrt{r}$ satisfy the differential equation

$$\left\{r\frac{d^2}{dr^2}+\frac{d}{dr}+r[k^2-V_{1,2}(r)]-a-(1/4r)\right\}\Phi_{1,2}(a,k,r)=0 \tag{47}$$

and the boundary condition $\Phi_{1,2}(a,k,0)=0$. Here $\Phi_1(a,k,r)$ and $\Phi_2(a,k,r)$ are the solutions of (47) with known (or reference) potential $V_1(r)$ and unknown (reconstructed) potential $V_2(r)$.

Let us consider the expression

$$B(r,r')=\int_D \Phi_2(a,k,r)\Phi_1(a,k,r')d[\rho_2(a)-\rho_1(a)],$$

where $\rho_1(a)$ and $\rho_2(a)$ are the spectral functions corresponding to the potentials $V_1(r)$ and $V_2(r)$. On the basis of (47), $B(r,r')$ satisfies the partial differential equation

$$\left\{r\,\frac{\partial^2}{\partial r^2}+\frac{\partial}{\partial r}+r[k^2-V_2(r)]-(1/4r)\right\}B(r,r')$$
$$=\left\{r'\,\frac{\partial^2}{\partial r'^2}+\frac{\partial}{\partial r'}+r'[k^2-V_1(r')]\right.$$
$$\left.-(1/4r')\right\}B(r,r') \tag{48}$$

with the boundary conditions $B(0,r')=B(r,0)=0$.

Let $\Phi_2(a,k,r)$ be related to $\Phi_1(a,k,r)$ as

$$\Phi_2(a,k,r)=\Phi_1(a,k,r)-\int_0^r B(r,r')\Phi_1(a,k,r')dr'. \tag{49}$$

After substituting (49) into (47) with subscript 2 and using (48), we find

$$\left\{r[V_2(r)-V_1(r)]+B(r,r)+r\,\frac{dB(r,r)}{dr}\right.$$
$$\left.+r\,\frac{\partial B(r,r')}{\partial r}\bigg|_{r'=r}\right\}\Phi_1(a,k,r)$$
$$+rB(r,r)\,\frac{d\Phi_1(a,k,r)}{dr}+\int_0^r \Phi_1(a,k,r')\left\{r'\,\frac{\partial^2}{\partial r'^2}\right.$$
$$\left.+\frac{\partial}{\partial r'}+r'[k^2-V_1(r')]-a-(1/4r')\right\}B(r,r')dr'=0. \tag{50}$$

It follows from the boundary condition for $\varphi(a,k,r)$ that

$$\Phi_1(a,k,r)\underset{r\to 0}{\sim}\sqrt{r},\quad \frac{d\Phi_1(a,k,r)}{dr}\underset{r\to 0}{\sim}\frac{1}{2\sqrt{r}},$$

$$B(r,r')\underset{r'\to 0}{\sim}\text{const}\cdot\sqrt{r'},\quad \frac{\partial B(r,r')}{\partial r'}\underset{r'\to 0}{\sim}\frac{\text{const}}{2\sqrt{r'}}. \tag{51}$$

Integrating by parts and using (51), we easily find

$$\int_0^r \Phi_1(a,k,r')\left(r'\,\frac{\partial^2}{\partial r'^2}+\frac{\partial}{\partial r'}\right)B(r,r')dr'$$
$$=\int_0^r B(r,r')\left(r'\,\frac{d^2}{dr'^2}+\frac{d}{dr'}\right)\Phi_1(a,k,r')dr'$$
$$+r\,\frac{\partial B(r,r')}{\partial r'}\bigg|_{r'=r}\cdot\Phi_1(a,k,r)$$
$$-rB(r,r)\,\frac{d\Phi_1(a,k,r)}{dr}. \tag{52}$$

Then, after substituting (52) into (50) and using (47) with subscript 1, we find

$$V_2(r)=V_1(r)-\frac{2}{\sqrt{r}}\frac{d}{dr}[\sqrt{r}B(r,r)]. \tag{53}$$

Now we multiply (49) by $\Phi_1(a,k,r')$ and integrate the difference of the functions $\rho_2(a)-\rho_1(a)$ in the region D. As a result, we obtain an integral equation of the Gel'fand–Levitan type:

$$B(r,r')=A(r,r')-\int_0^r A(r',r'')B(r,r'')dr'', \tag{54}$$

the symmetric kernel of which has the form

$$A(r,r')=\int_D \Phi_1(a,k,r)\Phi_1(a,k,r')d[\rho_2(a)-\rho_1(a)]. \tag{55}$$

We have thus constructed the following procedure for solving the inverse scattering problem. We find the spectral function from the discrete spectrum in the a plane. It follows from the definition of the spectral function (46) that the Coulomb poles in the difference $\rho_2(a)-\rho_1(a)$ cancel and

$$d[\rho_2(a)-\rho_1(a)]$$
$$=\left[\sum_{j_2} C_{j_2}\delta(a-a_{j_2})-\sum_{j_1} C_{j_1}\delta(a-a_{j_1})\right]da,$$

where a_{j_1} and a_{j_2} are the "nuclear" poles due to the potentials V_1 and V_2. Therefore, the kernel (55) depends only on the "nuclear" poles, is degenerate, and contains a finite number of terms. Consequently, the determination of $B(r,r')$ from (54) reduces to solving a system of linear equations for certain functions. Then, using $B(r,r')$, from (49) and (53) we can calculate the regular solution $\varphi_2(a,k,r)$ and the potential $V_2(r)$.

In Ref. 24, Eqs. (43), (46), (49), and (53)–(55) were used to obtain an explicit expression for the potential in the one-pole approximation, and it was shown that it is finite at the origin, and that at large distances it falls off as $(r^2\ln r)^{-1}$. In addition, to illustrate the use of the method for a specific calculation, the quasipotential of the $\pi^{\pm}p$ interaction in the S_{31} state was reconstructed in the one-pole approximation of the scattering matrix for two fixed energies.

Thus, the new formulation of the inverse scattering problem has a clear physical interpretation, and the method proposed for solving it, given a sufficient number of experimental phase shifts including their Coulomb splitting, can be used to reconstruct the interaction potential for a charged cluster and a multiplet or for two multiplets. We note that this new formulation of the inverse scattering problem has also been studied in Ref. 37, where a different (algebraic) method of solution was used.

2.3. Generalized Darboux–Crum–Kreĭn transformations

The problem of finding exact solutions of the Schrödinger equation in closed analytic form is interesting in general, and also in particular, because these solutions and their corresponding potentials can be used as the basis for solving the inverse scattering problem. It is well known that the Schrödinger equation has exact solutions in terms of elementary or special functions for a number of very simple potentials: potentials linear or quadratic in r, the centrifugal potential, the Coulomb potential, and so on. The use of the inverse scattering formalism has greatly broadened the possibilities of finding exact solutions of the Schrödinger equation for the class of Bargmann potentials, which in various formulations of the inverse scattering problem correspond to specification of the S matrix as a rational function of momentum, orbital angular momentum, or the CCC.[38]

Another effective method of constructing exact solutions of the Schrödinger equation is that of Darboux–Crum–Kreĭn transformations.[20] Using these transformations, it is possible to determine an exact solution φ_2 and its corresponding potential V_2 from a known solution φ_1 of the Schrödinger equation with potential V_1. Below, we give a generalized version of the Darboux–Crum–Kreĭn transformations for the superposition of centrifugal, Coulomb, and nuclear potentials.

Let us consider the Schrödinger equation (1) for real values of the parameters $E=k^2$, $\lambda=l+\frac{1}{2}$, and a. Let the functions $\varphi_1(E_0,\lambda_0,a_0,r)$ and $\varphi_1(E,\lambda,a,r)$ satisfy the Schrödinger equation with known potential $V_1(r)$ and with fixed parameters (E_0,λ_0,a_0) and arbitrary (E,λ,a), respectively, while $\varphi_2(E_0,\lambda_0,a_0,r)$ and $\varphi_2(E,\lambda,a,r)$ satisfy the Schrödinger equation with parameters (E_0,λ_0,a_0) and (E,λ,a), respectively, for unknown potential $V_2(r)$. We assume that $V_{1,2}(r)$ are continuous functions on the interval $\alpha<r<\beta$, where α and β are some numbers. Then, according to Ref. 35, the following statement is valid.

If on the interval (α,β) $\varphi_1(E_0,\lambda_0,a_0,r)\neq 0$, $Ar^2+Br+C\neq 0$ (where A, B, and C are real numbers), and the spectral parameters E, λ, a satisfy an arbitrary linear dependence of the form

$$\frac{E-E_0}{-A}=\frac{\lambda^2-\lambda_0^2}{C}=\frac{a-a_0}{B}, \tag{56}$$

then the expressions

$$\varphi_2(E_0,\lambda_0,a_0,r)=\frac{q(r)}{\varphi_1(E_0,\lambda_0,a_0,r)},$$

$$\varphi_2(E,\lambda,a,r)=\frac{q(r)W[\varphi_1(E_0,\lambda,a_0,r),\varphi_1(E,\lambda,a,r)]}{\varphi_1(E_0,\lambda_0,a_0,r)}, \tag{57}$$

where

$$q(r)=\frac{C_1 r}{\sqrt{|Ar^2+Br+C|}}, \tag{58}$$

W is the Wronskian, and C_1 is an arbitrary constant, determine solutions of the Schrödinger equation on this interval with potential independent of E, λ, and a:

$$V_2(r)=V_1(r)-2\frac{d^2}{dr^2}\ln\varphi_1(E_0,\lambda_0,a_0,r)$$
$$-2\frac{q'(r)}{q(r)}\frac{d}{dr}\ln\varphi_1(E_0,\lambda_0,a_0,r)+\frac{q''(r)}{q(r)}. \tag{59}$$

If $\varphi_1(E_0,\lambda_0,a_0,r)=0$ or $Ar^2+Br+C=0$, singularities arise in the transformed solutions (57) and potentials (59). The family of potentials $V_2(r)$ depends on the six parameters E_0, λ_0, a_0, A, B, and C. Equation (56) defines a line in (E,λ^2,a) space passing through the point with coordinates (E_0,λ_0^2,a_0) and having directional coefficients A, B, and C. Therefore, by varying the parameters E_0, λ_0, a_0, A, B, and C we obtain different $V_2(r)$ and the corresponding solutions $\varphi_2(E,\lambda,a,r)$ with values E, λ, and a on lines in (E,λ^2,a) space passing through the point (E_0,λ_0^2,a_0). By repeated application of the transformation (56)–(59) it is possible to obtain new potentials $V_2(r)$ and the exact solutions corresponding to them.

Let us consider some special cases.

1. $\lambda=$const, $a=$const, $E\neq$const, that is, $C=B=0$, $A\neq 0$.
2. $E=$const, $a=$const, $\lambda\neq$const, that is, $A=B=0$, $C\neq 0$.
3. $E=$const, $\lambda=$const, $a\neq$const, that is, $A=C=0$, $B\neq 0$.
4. $a=$const, $E\neq$const, $\lambda\neq$const, that is, $B=0$, $A\neq 0$, $C\neq 0$.
5. $\lambda=$const, $E\neq$const, $a\neq$const, that is, $C=0$, $A\neq 0$, $B\neq 0$.
6. $E=$const, $\lambda\neq$const, $a\neq$const, that is, $A=0$, $C\neq 0$, $B\neq 0$.

According to (56), in cases 1, 2, and 3 the lines on which we obtain exact solutions are, respectively, parallel to the E, λ^2, and a axes in (E,λ^2,a) space, while in cases 4, 5, and 6 they lie in the planes respectively perpendicular to the a, λ^2, and E axes.

The transformation formulas (56)–(59) in case 1 reduce to ordinary Darboux–Crum–Kreĭn transformations,[20] and in cases 2 and 4 they reduce to the expressions obtained in Refs. 38 and 39. We note that the transformation formulas for the special cases 3, 5, and 6 were first obtained in Ref. 35.

As noted above, exact solutions of the Schrödinger equation and the corresponding Bargmann potentials can also be determined by the inverse scattering method. Accordingly, cases 1–6 correspond to different formulations of the inverse scattering problem, i.e., special cases of the generalized approach to the inverse scattering problem in which the initial scattering data are specified for physical values of the parameters E, λ, and a lying on a line in (E,λ^2,a) space.[40] The special cases 1 and 2 of the inverse scattering formalism are well known and well studied,[20,29] but case 3 has only been studied fairly recently in Refs. 24, 34, and 37. The formalism of case 4 has been studied in Ref. 31. The formalism of cases 5 and 6 has not yet been considered in the literature.

3. RULES FOR THE SEPARATION OR DECAY OF ISOBARS INTO FRAGMENTS

The theory of complex CCC describes Coulomb effects in the systems $x+Y\leftrightarrow Q$ or $X+Y\leftrightarrow Q$ (where x is a charged cluster, and X, Y, and Q are isobaric multiplets), because the reduced mass of the system $x+Y$ or $X+Y$ is practically constant, and the interaction potential for x and Y or for X and Y is a superposition of centrifugal, repulsive Coulomb, and nuclear potentials, in which only the Coulomb component varies. These conditions are also satisfied in the case where x is a charged cluster and Y and Q are nuclear isobars in states with identical spins and parities $[(J^\pi)_Y=\text{const}$ and $(J^\pi)_Q=\text{const}]$.

We shall assume that the nuclear interaction, which is actually a multiparticle problem, can be reduced to a two-body problem. Accordingly, the isobars Q will henceforth be treated as bound or resonance states arising in the interaction of the cluster x with the isobars Y. Here the energy of separation ($E_x<0$) or decay ($E_x>0$) of the isobar Q into fragments x and Y is

$$E_x=M_Q-M_x-M_Y, \tag{60}$$

where M_Q, M_Y, and M_x are the mass excesses of the isobars Q, Y, and the cluster x.

In Sec. 1.2 we showed that bound states and resonances of the isobar Q are located on a trajectory $a_l(E_x)$ of a pole of the scattering matrix in the a plane. For example, a pole lying on the real positive a semiaxis for $E_x<0$ moves to the right with increasing E_x, leading to bound states with orbital angular momentum l and energy E_x of the isobar Q in each passage through a_{ph}. For $E_x>0$ the pole $a_l(E_x)$ moves into the complex a plane. If Re $a_l(E_x)=a_{\text{ph}}$ and Im $a_l(E_x)$ is small, the pole $a_l(E_x)$ corresponds to a resonance of the isobar Q with orbital angular momentum l, CCC a_{ph}, energy E_x, and width Γ_x given by (16). In the notation used here this expression takes the form

$$\Gamma_x=\frac{2\,\text{Im}\,a_l(E_x)}{\left(\dfrac{d\,\text{Re}\,a_l(E_x)}{dE_x}\right)}. \tag{61}$$

To determine the parameters of the bound and resonance states of the system $x+Y\leftrightarrow Q$ in the theory of complex CCC it is necessary to (1) specify the interaction potential of x and Y; (2) solve the Schrödinger equation with this potential and find the S matrix; (3) construct the trajectory $a_l(E_x)$ of the pole of the scattering matrix in the a plane; (4) use the pole trajectory to find the energies E_x of the bound and resonance states, and also the resonance widths Γ_x.

However, it is difficult to do this mainly because the interaction potential for x and Y either is not known at all, or is not known exactly. Therefore, in practice (for example, in Regge theory) one uses so-called semiempirical pole trajectories. Such trajectories can be constructed by using the dependence (28), with E replaced by E_x. It is impossible to calculate the constants c_i in (28) in the theory of complex CCC. However, they are easily determined from the experimental values of the parameters of the bound and resonance states belonging to the trajectory $a_l(E_x)$.

Equations (28) and (61) can be used to easily find regularities in the separation or decay of the isobar Q. Since from (28) we see that the physical values of the CCC depend linearly on $E_x\gtrless 0$, the energies of separation (decay) of the isobar Q into fragments x and Y are equidistant. In addition, from (28) and (61) for $E_x>0$ we find the relation

$$\log\Gamma_x=B-(D/\sqrt{E_x}),\quad B=\log(2c_3/c_2),\quad D=c_4\log e, \tag{62}$$

between the decay energies E_x and the partial widths Γ_x of resonances of the isobar Q located on the pole trajectory $a_l(E_x)$.

The theory of complex CCC is based on two assumptions: constancy of the reduced mass of the system $x+Y$ and the existence of exactly one interaction potential for the cluster x and isobars Y. Calculations show that the reduced mass changes insignificantly. However, the nuclear interaction potential for x and Y can vary dramatically, owing to shell effects. Closed shells of the isobar Q can also be very important. Therefore, the dependences (28) and (62) must be satisfied for isobars Y and Q which do not contain nuclei with the numbers of neutrons or protons near those of filled shells. Otherwise, they can be violated, which would cause such nuclear isobars to "leave" the pole trajectory (28). On the basis of these arguments we can state the following:

1. If the isobars Q and Y with spins and parities $(J^\pi)_Q=\text{const}$ and $(J^\pi)_Y=\text{const}$ do not contain nuclei with the numbers of neutrons or protons near those of filled shells, the energies E_x of separation (decay) of the parent isobars Q into a charged cluster x and the corresponding daughter isobars Y are equidistant (the equidistance rule).
2. If the isobars Q and Y with spins and parities $(J^\pi)_Q=\text{const}$ and $(J^\pi)_Y=\text{const}$ do not contain nuclei with the numbers of neutrons or protons near those of filled shells, the energies E_x and the corresponding partial widths Γ_x for the decay of the parent isobars Q into a charged cluster x and the corresponding daughter isobars Y satisfy the relation $\log\Gamma_x=B-(D/\sqrt{E_x})$, which is similar to the Geiger–Nuttall law.

In this section we shall check these rules and describe ways of estimating unknown values of some nuclear characteristics.

3.1. Equidistance of the separation (decay) energies of even–even isobars

To check the equidistance of the E_x, in Refs. 8 and 41 the isobars Q were taken to be even–even nuclei in their ground states ($J^\pi=0^+$), the fragments x were taken to be the proton, diproton (a single particle with twice the charge and mass of the proton), α particle, or ^{14}C nucleus, and Y were taken to be the corresponding isobars in their ground states. The graphs of the dependences of E_p, E_{2p}, E_α, and $E_{^{14}\text{C}}$ on the number of neutrons confirm the approximate equidistance of the E_x (Ref. 8), since the points (nuclei) fit onto a set of straight lines if the isobars Q and Y do not contain magic numbers of neutrons or protons. In these cases a regularity is observed for nuclei of the same isobar Q: as the number of neutrons in the nucleus decreases, the value of E_x increases linearly, because increase of the relative proton content in the nucleus enhances the role of the Coulomb energy. We note that equidistance of the E_x also occurs for fragments heavier than the ^{14}C nucleus.[8]

Obviously, by interpolation or near extrapolation of this regularity it is possible to estimate reliably the value of E_x for known or unknown nuclei. The unknown values of E_p, E_{2p}, E_α, and $E_{^{14}\text{C}}$ for a large number of nuclei were calculated in this way in Refs. 8 and 41, and on the whole they are in good agreement with the results of other studies.

The study of ultra-neutron-deficient nuclei is of great interest, because in regions close to the limits of nucleon stability decay modes such as proton and two-proton radioactivity can be expected to appear. For even–even isobars with $A=70-84$, 128–136, and 224–248 it has proved possible to extrapolate the values of E_p and E_{2p} all the way to the region of ultra-neutron-deficient nuclei, since the nuclei of these isobars Q and the corresponding isobars Y do not contain magic numbers of neutrons or protons. Segments of the p- and $2p$-stability boundaries identified in this manner are given in Ref. 41. In the same study the values of $E_{2p}>0$ were analyzed for a group of unknown even–even nuclei in order to estimate the possibility of $2p$ decay.

If the mass excess of the daughter (parent) nucleus is known, then (60) can be used to calculate the mass excess of the parent (daughter) nucleus. In Ref. 42 the mass excesses of 52 as yet unknown nuclei were found by interpolation or near extrapolation of the values of E_p, E_{2p}, E_α, and $E_{^{14}C}$ using (60), and the accuracy of the calculations was evaluated. This method of determining nuclear masses is inapplicable when shell effects are important. Its advantages are good accuracy, simplicity of the calculations, and the possibility of cross-checking the masses of a given nucleus estimated by using different fragments x.

3.2. The Geiger–Nuttall law for even–even isobars

At present, equation (62) can be checked only for the example of α decay, although in the foreseeable future it may be possible to check it for heavier fragments x, owing to the discovery of new types of radioactivity with cluster emission. Let us apply the result obtained above to the α decay of isobar nuclei Q located in ground states with $J^\pi=\text{const}$. Each α-radioactive parent nucleus Q will be treated as a resonance state of a compound nucleus—a "daughter-nucleus+α-particle" system, which after some time decays into a daughter nucleus and an α particle. Since the isobars Q have identical J^π and the effective interaction potentials of the α particles and the daughter nuclei Y are identical, the S-matrix poles corresponding to them in the a plane are located on a single pole trajectory (28). Using the fact that the half-life is $T=\ln 2/\Gamma_x$, the dependence (62) can be written as

$$\log T=C+(D/\sqrt{E}),\quad C=\log\left(\frac{c_2\ln 2}{2c_3}\right), \tag{63}$$

where E is the α-decay energy. Equations (62) and (63) are the Geiger–Nuttall law for isobars with identical J^π. In the formalism of the complex a plane the dependence (62) can be interpreted as the mapping of the trajectory of the S-matrix pole (28) onto the $(E_x^{-1/2},\log\Gamma_x)$ plane according to Eq. (61) (Ref. 43).

In Refs. 44 and 45 the α-decay data[46] were used to compare the Geiger–Nuttall law for isobars with experiment. It was shown that if even–even isobars do not contain nuclei with magic numbers of neutrons or protons, then $\log T$ depends linearly on $E^{-1/2}$. Here the observed deviations from the dependence (63) are due to the nuclear shell structure. The unknown half-lives of a group of heavy neutron-deficient even–even nuclei were estimated in Ref. 44 by extrapolating the Geiger–Nuttall law (63).

3.3. Semiempirical formulas for describing the half-lives of α-active heavy even–even nuclei

One of the first successful applications of quantum mechanics to nuclear physics was the classical theory of α decay proposed in 1928 by Gamow and independently by Condon and Gurney, in which the idea of tunneling provides the theoretical basis of the Geiger–Nuttall law. Since 1928 a great many authors have worked on this problem, either to improve the mathematical approximations, or to take into account various physical effects which had initially been neglected. For well known reasons, the theoretical expressions for the half-lives of α-active nuclei are not accurate enough. The difference between the values of T calculated using these expressions and the experimental values often reach several orders of magnitude. Therefore, much more accurate semiempirical formulas are usually used to estimate half-lives. A thorough account of the advantages and disadvantages of these formulas and a comparison of their accuracy are given in Ref. 47. The development of semiempirical methods for calculating half-lives has continued in recent years.[48] Here we derive new semiempirical formulas for the half-lives of heavy even–even nuclei.

If for some sets of nuclei the values of C and D in (63) are constant, then these nuclei lie on a set of smooth straight lines (Geiger–Nuttall lines) in the $(E^{-1/2},\log T)$ plane. The Geiger–Nuttall law is usually checked by selecting and identifying groups of nuclei which would lie fairly well on Geiger–Nuttall lines. At first (owing to lack of sufficient data), the validity of the Geiger–Nuttall law was checked only for radioactive series. Later, as new experimental data appeared, additional possibilities for checking the Geiger–Nuttall law arose. For example, in Refs. 46 and 49 the validity of the Geiger–Nuttall law for the ground states of even–even isotopes and isotones was proved using experimental data. Deviations from the Geiger–Nuttall law are observed in these cases only for nuclei with the number of nucleons close to the magic numbers. Therefore, it can be concluded that the Geiger–Nuttall law (63) is valid for the ground states of even–even isobars, isotopes, and isotones if they do not contain magic numbers of nucleons.

We shall restrict ourselves to heavy nuclei with $Z>82$ and $N>126$, where there are fairly many α-active even–even nuclei. Then any α-active nucleus $^A_ZX_N$ simultaneously belongs to isobaric ($A=\text{const}$), isotopic ($Z=\text{const}$), and isoneutronic ($N=\text{const}$) Geiger–Nuttall lines, that is, it is located at the intersection of these lines. Therefore, Eq. (63) can be written as follows for isobars, isotopes, and isotones, respectively:

$$\log T=(\alpha_1 A+\alpha_2)+(\beta_1 A+\beta_2)E^{-1/2}, \tag{64}$$

$$\log T=(\alpha_1 Z+\alpha_2)+(\beta_1 Z+\beta_2)E^{-1/2}, \tag{65}$$

$$\log T=(\alpha_1 N+\alpha_2)+(\beta_1 N+\beta_2)E^{-1/2}. \tag{66}$$

TABLE I. Parameters α_i, β_i $(i=1,2,3)$ and rms deviations σ of the logarithms of the half-lives calculated using Eqs. (67)–(69) from the experimental values.

Formula	α_1	α_2	α_3	β_1	β_2	β_3	σ
(67)	0.5824	-1.2504	-9.1375	-0.7212	2.6712	-71.6069	0.37
(68)	0.4895	-0.1827	-70.6508	-0.2997	0.5421	92.6265	0.25
(69)	$-2.793\cdot 10^{-2}$	$1.446\cdot 10^{-2}$	-47.7094	$-1.309\cdot 10^{-2}$	1.1966	32.1976	0.19

We note that the semiempirical formulas of Ref. 50 are based on (65). Now, taking pairwise linear combinations of (64)–(66), we obtain the following expressions for the logarithms of the half-lives:

$$\log T=(\alpha_1 A+\alpha_2 N+\alpha_3)+(\beta_1 A+\beta_2 N+\beta_3)E^{-1/2}, \tag{67}$$

$$\log T=(\alpha_1 Z+\alpha_2 N+\alpha_3)+(\beta_1 Z+\beta_2 N+\beta_3)E^{-1/2}, \tag{68}$$

$$\log T=(\alpha_1 A+\alpha_2 Z+\alpha_3)+(\beta_1 A+\beta_2 Z+\beta_3)E^{-1/2}. \tag{69}$$

Here the parameters α_i and β_i $(i=1,2,3)$ in (67)–(69) take different values.

In Table I we give the values of α_i and β_i $(i=1,2,3)$ found from the data of Ref. 46 by the least-squares method. Equations (67)–(69), where the decay energies are expressed in MeV and the half-lives in seconds, are the desired semiempirical formulas for the half-lives. To estimate the accuracy with which (67)–(69) hold for these parameters using the expression $\sigma=[(\Sigma_{i=1}^{n}\delta_i^2)/n]^{1/2}$, where $\delta_i=\log(T_i/T_i^{\exp})$ and n is the number of cases $(n=60)$, we determined the rms deviations of the values of $\log T$ calculated using (67)–(69) from their experimental values $\log T^{\exp}$, which are given in Table I. The values of σ show that Eqs. (67)–(69) are reliably and accurately satisfied. Equation (69) is the most accurate $(\sigma=0.19)$. The values of $|\delta_i|$ for it are distributed as follows: for 28 nuclei they lie in the range 0–0.1, for 19 nuclei they lie in the range 0.1–0.2, for seven nuclei they lie in the range 0.2–0.3, and for six nuclei they exceed 0.3. These results indicate that Eq. (69) is very accurate.

Since $A=Z+N$, Eqs. (67) and (68) can be reduced to (69), where the parameters α_i and β_i $(i=1,2,3)$ will take different values. This is the case because Eqs. (67), (68), and (69) represent fundamentally different approximations to the logarithms of the half-life, since they are based on linear combinations of Eqs. (64) and (66), (65) and (66), and (64) and (65), expressing the Geiger–Nuttall law for isobars and isotones, isotopes and isotones, and isobars and isotopes. The degrees of accuracy with which (64), (65), and (66) are satisfied differ. Therefore, the degrees of accuracy of (67), (68), and (69) will also differ.

In Table II we give the half-lives calculated using (69). The α-decay energies used in the calculations are taken from the systematics of Ref. 51. The results of Table II are close to the data of Ref. 51, obtained by a different method. The semiempirical relations (67)–(69) therefore depend on a relatively small number of parameters and can be used for fairly reliable estimates of the half-lives of heavy even–even α-active nuclei.

4. REGULARITIES IN THE SEPARATION OR DECAY OF MEMBERS OF ISOBARIC MULTIPLETS INTO FRAGMENTS

Let the energies of separation (decay) of Q into X and Y or Q into x and Y be

$$E=M(Q)-M(X)-M(Y),\quad E=M(Q)-M(x)-M(Y), \tag{70}$$

where $M(Q)$, $M(X)$, $M(Y)$, and $M(x)$ are the mass excesses of the members of the multiplets Q, X, Y, and the cluster x. Then (28) and (61) lead to the following statements.[52,53]

1. For any two possible modes of separation (decay) of the members of the multiplet Q into fragments belonging to multiplets X and Y, the change of the separation (decay) energy is directly proportional to the change of the physical value of the CCC.

TABLE II. Half-lives of a group of heavy even–even α-active nuclei calculated from Eq. (69).

Nucleus	T	Nucleus	T	Nucleus	T
^{234}Th	$6\cdot 10^{12}$ yr	^{254}Cm	$2\cdot 10^{9}$ yr	^{260}No	30 min
^{236}Th	$5\cdot 10^{15}$ yr	^{256}Cm	$7\cdot 10^{11}$ yr	^{262}No	7 h
^{238}Th	$3\cdot 10^{18}$ yr	^{238}Cf	2 sec	^{264}No	5 d
^{240}U	$2\cdot 10^{12}$ yr	^{256}Cf	$9\cdot 10^{3}$ yr	^{266}No	100 d
^{242}U	$6\cdot 10^{16}$ yr	^{258}Cf	$8\cdot 10^{5}$ yr	^{268}No	30 yr
^{228}Pu	30 msec	^{260}Cf	10^{8} yr	^{258}Ku	70 msec
^{230}Pu	10 sec	^{242}Fm	0.1 sec	^{260}Ku	0.8 sec
^{246}Pu	$2\cdot 10^{11}$ yr	^{244}Fm	0.5 sec	^{262}Ku	7 sec
^{248}Pu	$2\cdot 10^{10}$ yr	^{258}Fm	90 d	^{264}Ku	1 min
^{250}Pu	$3\cdot 10^{13}$ yr	^{260}Fm	7 yr	^{266}Ku	10 min
^{234}Cm	1 min	^{262}Fm	300 yr	^{268}Ku	3 h
^{236}Cm	1 h	^{264}Fm	10^{4} yr	^{270}Ku	5 d
^{252}Cm	10^{7} yr	^{258}No	2 min	^{272}Ku	3 yr

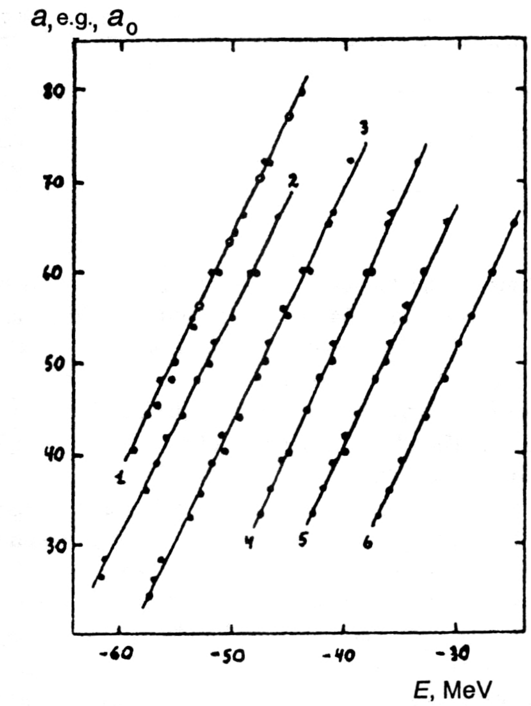

FIG. 1. Dependence of the CCC on the separation energy. The points are the experimental data, and the lines were obtained using the least-squares method (see explanation in the text).

2. The energies of separation (decay) of the members of the multiplet Q into a charged cluster x and members of the corresponding daughter multiplet Y are equidistant (the equidistance rule).

3. The energies E and corresponding partial widths Γ for decay of the members of the multiplet Q into fragments X and Y or into fragments x and Y satisfy the relation

$$\log \Gamma = B - (D/\sqrt{E}), \tag{71}$$

where B and D are constants.

In this section we shall check the reliability of the above statements and establish relations between the basic characteristics of members of an isobaric multiplet.

4.1. Comparison with experiment

Recently, a great deal of work has been done on the identification and systematics of members of isobaric multiplets. These results are given in a compact and convenient form in Ref. 54, where the masses of members of multiplets with $A \leqslant 60$ are listed. The data of Ref. 54 can be used to compare several theoretical regularities in the separation (decay) of multiplets into fragments with the experimental data. Here we use the notation (A,T,J^{π}) to refer to a multiplet with mass number A, isospin T, and spin and parity J^{π}.

In Fig. 1 we show the energy dependence of the CCC for the following processes of separation of Q into X and Y:

1) $(32,2,0^+) \rightarrow (12,2,0^+) + (20,2,0^+)$,

2) $(32,1,1^+) \rightarrow (8,2,0^+) + (24,2,0^+)$,

3) $(32,2,0^+) \rightarrow (8,2,0^+) + (24,2,0^+)$,

4) $(32,2,0^+) \rightarrow (9,3/2,3/2^-) + (23,3/2,5/2^+)$,

5) $(32,2,0^+) \rightarrow (8,1,2^+) + (24,2,0^+)$,

6) $(32,2,0^+) \rightarrow (8,1,2^+) + (24,1,4^+)$.

The numbers 1–6 in Fig. 1 refer to these cases. The lines in Fig. 1 confirm the linear dependence of a on E, i.e., statement 1 above.

This regularity can be used to estimate the masses of the members of multiplets. Let us explain this for the example of finding the unknown mass of the nucleus ^{12}N ($T_z = -1$) belonging to the quintet $(12,2,0^+)$. Using the pole trajectory $a(E) = 198.0946 + 2.6835E$ found by the least-squares method for case 1, we can calculate the values of E in terms of a for four possible versions of separation into fragments, one of which is ^{12}N (circles on curve 1 in Fig. 1). Then, from the known masses of the ^{32}P, ^{32}S, ^{32}Cl, ^{32}Ar and ^{20}O, ^{20}F, ^{20}Ne, ^{20}Na members of the quintets $(32,2,0^+)$ and $(20,2,0^+)$, using (70) for the four variants of separation, we find the ^{12}N mass excesses to be 29.923, 29.87, 29.755, and 29.535 MeV. The ^{12}N mass excess calculated from the isobaric mass formula is 29.585(12) MeV (Ref. 54). In Fig. 2 the data of Ref. 54 are used to construct graphs of the dependence of the separation energy on the isospin projection for the following variants of separation of Q into fragments x and Y:

$$(50,1,0^+) \rightarrow \begin{cases} ^{14}\mathrm{N} + (36,1,2^+) \\ ^{28}\mathrm{Si} + (22,1,0^+), \\ ^{42}\mathrm{Sc} + (8,1,2^+) \end{cases}$$

$$(41,3/2,3/2^+) \rightarrow \begin{cases} ^{12}\mathrm{C} + (29,3/2,5/2^+) \\ ^{22}\mathrm{Na} + (19,3/2,5/2^+) \\ ^{32}\mathrm{S} + (9,3/2,3/2^-) \end{cases}$$

$$(32,2,0^+) \rightarrow \begin{cases} ^{2}\mathrm{Be} + (24,2,0^+) \\ ^{12}\mathrm{C} + (20,2,0^+) \\ ^{24}\mathrm{Mg} + (8,2,0^+) \end{cases}.$$

The lines in Fig. 2 confirm the equidistance rule 2. For brevity, in Figs. 1 and 2 we give only a limited number of graphs confirming statements 1 and 2. However, analysis of the experimental data of Ref. 54 showed that these regularities are general and are well satisfied (particularly the equidistance rule) for multiplets with $A \leqslant 60$.

The equidistance rule can be used to predict the masses (energy levels) of the members of a multiplet. In Table III we give the results of calculating the unknown mass of the ^{12}N nucleus ($T_z = -1$) belonging to the quintet $(12,2,0^+)$. The following five variants of separation of the quintet Q into fragments x and Y were studied:

$$(12,2,0^+) \rightarrow {}^4\mathrm{He} + (8,2,0^+), \quad (20,2,0^+) \rightarrow {}^8\mathrm{Be} + (12,2,0^+),$$

$$(24,2,0^+) \rightarrow {}^{12}\mathrm{C} + (12,2,0^+), \quad (32,2,0^+) \rightarrow {}^{20}\mathrm{Ne} + (12,2,0^+),$$

$$(36,2,0^+) \rightarrow {}^{24}\mathrm{Mg} + (12,2,0^+).$$

For each of these variants the separation energy E was determined by interpolating the linear dependence of E on T_z for the parent quintet Q to the point $T_z = -1$. Then Eq. (70) was used to calculate the ^{12}N mass, which is close to the mass obtained from the isobaric mass formula.[54]

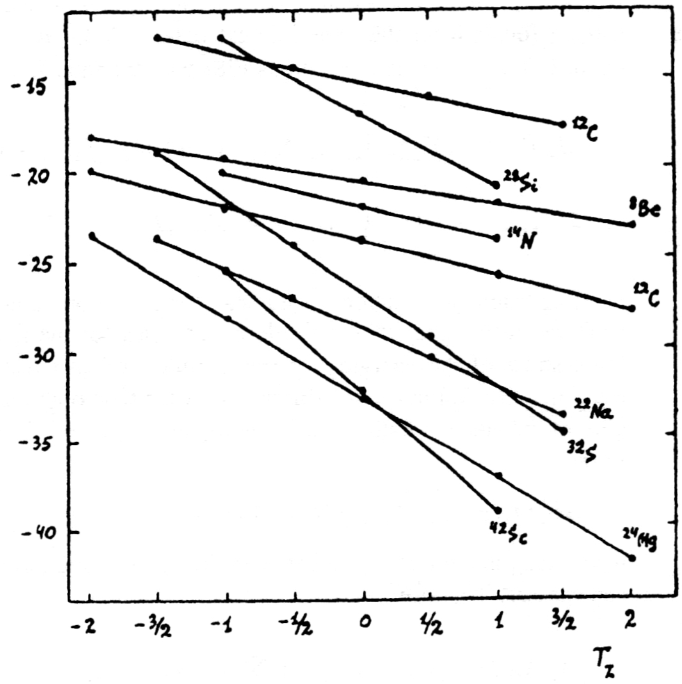

FIG. 2. Dependence of the separation energy of members of the triplet $(50,1,0^+)$, the quartet $(41,3/2,3/2^+)$, and the quintet $(32,2,0^+)$ on the isospin projection (see explanation in the text).

Among the advantages of this method of calculating the mass are good accuracy, simplicity of the calculations, and the possibility of cross-checking the estimates of an unknown mass of a member of a multiplet by using (as in the above example) different versions of the separation (decay) of the parent multiplet into a cluster and the corresponding daughter multiplet. This method was used with the equidistance rule in a series of studies[52,55–58] to calculate a very large number of unknown excitation energies and masses of light, intermediate, and heavy nuclei.

In going to intermediate and heavy nuclei the energies E become positive, which is a necessary condition for decay. To verify statement 3 we need: (1) identification of at least three members of a multiplet in the region of intermediate or heavy nuclei; (2) the measured decay energies E for these members of the multiplet and the corresponding partial widths Γ. Unfortunately, owing to the absence of this type of experimental data, it is not possible at present to check (71) or statement 3.

TABLE III. Calculation of the mass of the ^{12}N nucleus belonging to the quintet $(12,2,0^+)$ using the equidistance rule for the separation (decay) energies.

Nucleus	Mode of separation into fragments	$M-A$, MeV	
		Using (70)	Ref. 54
	$^{12}N \to {}^4He + {}^8B$	29.606	
	$^{20}Na \to {}^8Be + {}^{12}N$	29.61	
^{12}N	$^{24}Al \to {}^{12}C + {}^{12}N$	29.594	29.585(12)
	$^{32}Cl \to {}^{20}Ne + {}^{12}N$	29.607	
	$^{36}K \to {}^{24}Mg + {}^{12}N$	29.569	

4.2. Recursion relations between the masses and differences of the Coulomb energies of members of an isobaric multiplet

The isotopic invariance of the strong interaction in nuclear isobars is manifested in the existence of analog states. Such nuclear states form an isobaric multiplet (A,T,J^π). In going from one member of the multiplet to another, the isospin T remains fixed and its projection T_z changes. Therefore, the nuclei belonging to the multiplet must have energy levels of the same structure, so that the known masses or excitation energies of certain members of the multiplet can be used to estimate the analogous characteristics of the other members.

For example, this can be done using the isobaric mass formula[54]

$$M(n)=\sum_{m=0}^{k} C_{m+1}T_z^m(n), \quad T_z(n)=T+1-n, \tag{72}$$

where $M(n)$ is the mass excess of the nth member of the multiplet (the parent member of the multiplet with $T_z=T$ is taken to be the first member). The isobaric mass formula relates the masses of the $2T+1$ members of a multiplet with isospin T. In Eq. (72) the coefficients C_{m+1} for $m=3,4,\ldots,k$ are usually set equal to zero. Then the known masses of the first three members of the multiplet can be used to find the coefficients C_1, C_2, and C_3 of the isobaric mass formula, after which this formula can be used to calculate the masses of the other members.

Here we use the equidistance rule to obtain new relations between the masses and the differences of the Coulomb energies of the members of an isobaric multiplet.[59] Let the interaction between a charged cluster x and the members of a multiplet Y give rise to bound and resonance states which are members of a multiplet Q. Then the energies of the separation (decay) of members of the multiplet Q into fragments x and Y are

$$E(n)=M_Q(n)-M_x-M_Y(n),$$

$$n=1,2,\dots,\min(2T_Q+1,2T_Y+1), \tag{73}$$

where $M_Q(n)$ and $M_Y(n)$ are the mass excesses of the nth members of the multiplets Q and Y, M_x is the mass excess of the cluster x, and T_Q, $T_Y\geqslant 1$ are the isospins of the multiplets Q and Y. Therefore, the equidistance rule for the separation (decay) energies of members of the multiplet can be written as

$$\frac{E(n)-E(k)}{n-k}=\frac{E(n)-E(k-1)}{n-k+1}=\alpha,$$

$$k=2,3,\dots,n-1;\quad k<n;$$

$$n=3,4,\dots,\min(2T_Q+1,2T_Y+1), \tag{74}$$

where α is a constant. Combining (74), we easily obtain the following relations for the separation (decay) energies:

$$E(n)-(-1)^k\sum_{j=1}^{k} a_{kj}(n)E(j)=0,$$

$$a_{kj}(n)=(-1)^j\prod_{\substack{i=1\\ i\neq j}}^{k}\frac{n-i}{|j-i|}. \tag{75}$$

Here the coefficients $a_{kj}(n)$ satisfy the equations

$$\sum_{j=1}^{k} j^m a_{kj}(n)=(-1)^k[n^m-\delta_{km}f_k(n)],$$

$$f_k(n)=\prod_{i=1}^{k}(n-i),$$

$$\delta_{km}=\begin{cases}1, & k=m,\\ 0, & k\neq m,\end{cases}\quad m=0,1,\dots,k. \tag{76}$$

Substitution of (73) into (75), using (76) for $m=0$, leads to formulas relating the mass excesses of the members of any two multiplets Q and Y:

$$[M_Q(n)-M_Y(n)]-(-1)^k\sum_{j=1}^{k} a_{kj}(n)[M_Q(j)-M_Y(j)]=0. \tag{77}$$

Then from (77) we find relations between the mass excesses of members of the multiplet:

$$M(n)-(-1)^k\sum_{j=1}^{k} a_{kj}(n)M(j)-\gamma_{k+1}=0,$$

$$k=2,3,\dots,n-1;$$

$$k<n;\quad n=3,4,\dots,2T+1;\quad T\geqslant 1, \tag{78}$$

where γ_{k+1} are constants. To find the γ_{k+1} we substitute (72) into (78) and use Eq. (76) for $m=0,1,\dots,k$. As a result, we obtain

$$\gamma_{k+1}=(-1)^k C_{k+1}f_k(n),\quad k=2,3,\dots,n-1, \tag{79}$$

where c_{k+1} are k-dependent constants, the values of which can be found from the experimental data. Taking into account (79), the recursion relations (78) take the final form

$$M(n)-(-1)^k\sum_{j=1}^{k} a_{kj}(n)M(j)-(-1)^k C_{k+1}f_k(n)=0,$$

$$k=2,3,\dots,n-1;\quad k<n;\quad n=3,4,\dots,2T+1;\quad T\geqslant 1. \tag{80}$$

Important information on nuclear configurations comes from the differences of the Coulomb energies (or the Coulomb shifts of the energies) of the members of an isobaric multiplet. By definition, the difference of the Coulomb energies of adjacent members of a multiplet numbered j and $j-1$ is

$$\Delta E_c(j,j-1)=M(j)-M(j-1)+\Delta_{n\mathrm{H}},$$

where $\Delta_{n\mathrm{H}}$ is the difference between the neutron and hydrogen masses. Therefore,

$$M(j)=M(1)-(j-1)\Delta_{n\mathrm{H}}+\sum_{i=2}^{j}\Delta E_c(i,i-1). \tag{81}$$

To derive relations between the differences of the Coulomb energies of the members of a multiplet we write (80) for $M(n-1)$:

$$M(n-1)-(-1)^k\sum_{j=1}^{k} a_{kj}(n-1)M(j)$$

$$-(-1)^k C_{k+1}f_k(n-1)=0. \tag{82}$$

Then, taking the difference between (80) and (82), from Eqs. (81) and (76) for $m=0,1$ and the equation

$$f_k(n)-f_k(n-1)=\frac{kf_k(n)}{n-1}$$

we find recursion relations between the differences of the Coulomb energies of the members of a multiplet:

$$\Delta E_c(n,n-1)-(-1)^k\sum_{j=2}^{k}[a_{kj}(n)-a_{kj}(n-1)]$$

$$\times\left[\sum_{i=2}^{j}\Delta E_c(i,i-1)\right]-(-1)^k C_{k+1}\frac{kf_k(n)}{n-1}=0,$$

$$k=2,3,\dots,n-1;\quad k<n;\quad n=3,4,\dots,2T+1;\quad T\geqslant 1. \tag{83}$$

Since at present the masses of no more than five members of multiplets have been identified,[54,60] for clarity we write out Eqs. (80) and (83) explicitly for $k=2,3,4$:

$$M(n)-(n-1)M(2)+(n-2)M(1)-(n-1)(n-2)C_3=0,$$

$$\Delta E_c(n,n-1)-\Delta E_c(2,1)-2(n-2)C_3=0,\quad k=2;$$

$$n=3,4,\dots,2T+1; \tag{84}$$

$$M(n)-\tfrac{1}{2}(n-1)(n-2)M(3)+(n-1)(n-3)M(2)$$

$$-\tfrac{1}{2}(n-2)(n-3)M(1)+(n-1)(n-2)(n-3)C_4=0,$$

TABLE IV. Values of C_{k+1} and rms deviations σ from zero of the left-hand sides of (80) and (83).

k	n	T	N	C_{k+1}, MeV	σ, keV
2	3	1	61	0.2274	111
		3/2	24		49
		2	16		105
		3,4,22	4		162
	4	3/2	29		139
		2	5		117
	5	2	11		214
	As a whole		150		122
3	4	3/2	23	$1.019\cdot10^{-3}$	37
		2	5		38
	5	2	11		133
	As a whole		39		77
4	5	2	5	$3.846\cdot10^{-4}$	120
	For all multiplets		194		114

$$\Delta E_c(n,n-1)-(n-2)\Delta E_c(3,2)+(n-3)\Delta E_c(2,1) + 3(n-2)(n-3)C_4=0,$$

$$k=3;\quad n=4,5,\ldots,2T+1; \tag{85}$$

$$M(n)-\tfrac{1}{6}(n-1)(n-2)(n-3)M(4)+\tfrac{1}{2}(n-1)(n-2)(n-4)M(3)-\tfrac{1}{2}(n-1)(n-3)(n-4)M(2) + \tfrac{1}{6}(n-2)(n-3)(n-4)M(1)-(n-1)(n-2)(n-3)(n-4)C_5=0,$$

$$\Delta E_c(n,n-1)-\tfrac{1}{2}(n-2)(n-3)\Delta E_c(4,3)+(n-2)(n-4)\Delta E_c(3,2)-\tfrac{1}{2}(n-3)(n-4)\Delta E_c(2,1)-4(n-2)(n-3)(n-4)C_5=0,\quad k=4;\quad n=5,6,\ldots,2T+1. \tag{86}$$

The constant C_{k+1} is the coefficient of $T_z^k(n)$ in Eq. (72). According to the calculations of Ref. 54, the coefficients C_{k+1} for $2\leqslant k\leqslant 4$ take very similar values for different multiplets. Therefore, C_{k+1} can be found by the least-squares method. C_{k+1} was calculated using the data[54,60] on the masses of members of multiplets with $T\geqslant 1$. The values of C_{k+1} obtained using (80) are given in Table IV and represent averages over all multiplets of the value of the coefficient of the isobaric mass formula for $T_z^k(n)$. Thus, Eqs. (80) and (83) [or (84)–(86)], where the mass excesses and differences of the Coulomb energies are expressed in MeV, together with the numerical values of the C_{k+1} from Table IV are the desired linear recursion relations between the masses and differences of the Coulomb energies of the members of multiplets with isospins $T\geqslant 1$.

To estimate the accuracy with which (80) and (83) [or (84)–(86)] hold for these C_{k+1} using the expression $\sigma=[(\Sigma_{i=1}^N\delta_i^2)/N]^{1/2}$, where the δ_i are the values of the left-hand side of (80) and N is the number of cases, we calculated the rms deviations of the left-hand side from zero. These are given in Table IV. Since Eqs. (83) follow directly from (80), they are satisfied with the same accuracy as the mass formulas (80). The values of σ show that the equidistance rule (74), the expressions for the separation (decay) energies (75), and the recursion relations (80) and (83) are reliable and accurate for the values of k and n given in Table IV. We note that as new experimental data on the masses of members of multiplets appear, the values of the C_{k+1} in Table IV will need to be improved. In addition, by determining the mass of the sixth and higher members of multiplets it will be possible to extend the range of applicability of Eqs. (80) and (83), because the values of C_{k+1} for $k>4$ can then be found.

If the masses of the first k members of a multiplet are known, then (80) can be used to easily find the masses of the other members. Equations (83) allow the differences of the Coulomb energies between the terms of a multiplet to be obtained from the known differences of the Coulomb energies for the first k terms. Here the same mass excesses and Coulomb-energy differences can be calculated in several ways by using different allowed combinations of k and n in (80) and (83). The unknown values of the mass excesses and differences of the Coulomb energies of the members of light multiplets found from the data of Ref. 54 and from Eqs. (80) and (83) are given in Ref. 59.

The advantages of Eqs. (80) over the isobaric mass formula (72) are the following. First, when the isobaric mass formula is used to find the masses of the members of a multiplet, it is first necessary to find its coefficients. Equation (80) establishes a direct relation between the masses of the members of a multiplet. Second, the isobaric mass formula with its coefficients describes the masses of the members of only a given multiplet. Equations (80) are more general and universal in the sense that they are valid for any multiplet.

4.3. Relations between the masses of the ground states of mirror nuclei

The development of the theory of nuclear structure and various nuclear models has led to a rather large number of semiempirical mass formulas of varying quality. Work on finding new methods of determining the masses of nuclei far from the stability line has been carried out intensively in recent years.[61] The approaches used to solve this problem differ greatly from each other in the accuracy of the resulting expressions and in the number of parameters used. In some

TABLE V. Values of the parameters a_k, b_k, c_k, and d_k and the rms deviations σ from zero of the left-hand side of Eq. (89) for various k.

k	a_k, MeV	b_k, MeV	c_k, MeV	d_k, MeV	σ, keV
2	2.7379	−3.0336	0	0.239	257
3	2.7419	−2.9462	0.1909	−0.0135	252
4	2.4484	−1.2271	0.1709	0.0064	554
5	2.4653	−1.3163	0.1707	−0.0104	576
$6,7,...,2T-1$	2.7379	−3.0336	0.239	0	257
$2T$	2.7357	−2.6736	0.1751	−0.0045	294

studies the number of parameters reaches several hundred. A smaller number of parameters is used in other studies. In the most accurate methods the rms deviations of the calculated masses from the experimental values are usually 190–310 keV (Ref. 62).

Special mention should be made of methods of determining the nuclear mass based on the principle of the isotopic invariance of nuclear forces. It follows from isotopic invariance that the wave functions of mirror nuclei coincide exactly, apart from replacement of the neutrons by protons and vice versa. Therefore, the states (including the ground state) of such nuclei belong to the same isobaric multiplet and possess very similar properties. In principle, the known mass of a neutron-rich nucleus can therefore be used to calculate the unknown mass of the mirror neutron-deficient nucleus.

Relations between the masses of mirror nuclei can easily be obtained from the results of Sec. 4.2 (Ref. 63). After substitution of (81) and (84) for $\Delta E_c(n,n-1)$ into (80), and using (76) for $m=0,1,2$, we find

$$M(n)-M(1)+(n-1)[\Delta_{nH}-\Delta E_c(2,1)]-C_3[(n-1)\times(n-2)-\delta_{k2}f_k(n)]-(-1)^k C_{k+1}f_k(n)=0. \qquad (87)$$

Now we set $n=2T+1$ in (87) and introduce new, more convenient notation for the masses of mirror nuclei and differences of the Coulomb energies of the second and first members of the multiplet:

$$M(1)=M(A,T), \quad M(2T+1)=M(A,-T),$$
$$\Delta E_c(2,1)=\Delta E_c(A;T-1,T).$$

In the end we obtain the following relations between the masses of mirror nuclei:

$$M(A,-T)-M(A,T)+2T[\Delta_{nH}-\Delta E_c(A;T-1,T)-C_3[2T(2T-1)-\delta_{k2}f_k(2T+1)]-(-1)^k C_{k+1}f_k(2T+1)=0, \quad k=2,3,...,2T. \qquad (88)$$

If we also include the fact that the difference of the Coulomb energies $\Delta E_c(A;T-1,T)$ depends linearly on $(Z+0.5)A^{-1/3}$ (Ref. 54), then (88) can be rewritten as

$$M(A,-T)-M(A,T)-[a_k(Z+0{,}5)A^{-1/3}+b_k]T-c_k[2T(2T-1)-\delta_{k2}f_k(2T+1)]-d_k\times(-1)^k f_k(2T+1)=0, \quad k=2,3,...,2T, \qquad (89)$$

where Z is the charge of the first member of the multiplet [the nucleus (A,T)], and a_k, b_k, c_k, and d_k are k-dependent parameters. Equations (89) establish a general relation between the masses of isobar-analog (both ground and excited) states of mirror nuclei. However, now we shall restrict ourselves to only ground states, so that $M(A,\pm T)$ will correspond to the mass excesses of the ground states of the mirror nuclei (A,T) and $(A,-T)$.

The experimental data[64] on the masses of the ground states of mirror nuclei were used to find the values of the parameters. At present, the masses of 95 pairs of mirror nuclei with isospins $T=\frac{1}{2}-\frac{5}{2}$(the control group) are known for $A\geqslant 17$. In Table V we give the values of the parameters in Eq. (89) for different k determined from the masses of the nuclei of the control group by the least-squares method. The mathematical structure of Eqs. (89) causes them to coincide for the values $k=2$ and $k=6,7,\ldots,2T-1$ for nuclei from the control group (with $T_{\max}=5/2$). To estimate the accuracy with which (89) holds, we used the calculated values of the parameters of the nuclei of the control group to find the rms deviations of the left-hand side of (89) from zero (Table V). We see that the most accurate expression is (89) for $k=3$ ($\sigma=252$ keV). This formula was used in Ref. 63 to find the values of the mass excesses for 135 unknown neutron-deficient nuclei with $T\leqslant 5/2$. Those results for the mass excesses are in good agreement with the results of Ref. 61.

We conclude that, Eqs. (89) are reliable, and for $k=2,3,6,7,\ldots,2T$ they are satisfied with an accuracy comparable to that of the best current methods for determining nuclear masses. These relations are quite simple and depend on a small number (three or four) of parameters. In addition, Eqs. (89) can be used to cross-check the estimates of unknown nuclear mass excesses by using different values of k in the calculations.

4.4. Rms charge radii of members of an isobaric multiplet

An important characteristic of nuclei is the rms charge radius, which describes the spatial charge distribution. At present the accuracy with which charge distributions are measured is so high that differences of the rms charge radii in nuclear isobars can be determined. The rms charge radii can be measured directly if the lifetimes of the ground and excited states are fairly large (at least on the order of several hours). Therefore, there exist many phenomenological and theoretical models for calculating the rms charge radii. Here we describe yet another method for finding the rms charge

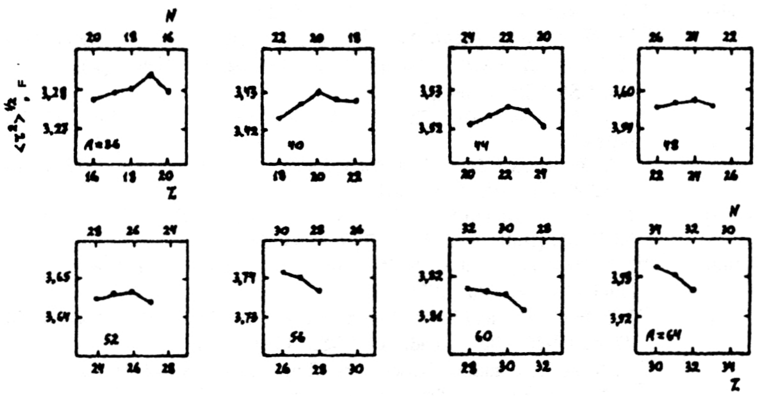

FIG. 3. Dependence of the rms charge radii of members of $4p$ quintets with $36 \leqslant A \leqslant 64$ on the number of protons and neutrons.

radii of both ground and excited states of nuclei, based on the use of the differences of the Coulomb energies between the members of an isobaric multiplet.[65]

In Ref. 66 the following expression was obtained for the Coulomb energies of nuclei with mass numbers $A \geqslant 20$:

$$E_c = cZ(Z-1)/\langle r^2 \rangle^{1/2}, \tag{90}$$

where $\langle r^2 \rangle^{1/2}$ is the rms charge radius and $c = 0.64145$ F·MeV. If Z is the charge of the first member of the multiplet and $\langle r^2(n) \rangle^{1/2}$ is the rms charge radius of the nth member of the multiplet, then according to (90) the Coulomb energies of members of the multiplet numbered 1 and n are

$$E_c(1) = cZ(Z-1)\langle r^2(1) \rangle^{-1/2},$$

$$E_c(n) = c(Z+n-1)(Z+n-2)\langle r^2(n) \rangle^{-1/2}.$$

Therefore,

$$\Delta E_c(n,1) = E_c(n) - E_c(1) = c[(Z+n-1)(Z+n-2) \times \langle r^2(n) \rangle^{-1/2} - Z(Z-1)\langle r^2(1) \rangle^{-1/2}]. \tag{91}$$

Since

$$\Delta E_c(n,1) = M(n) - M(1) + (n-1)\Delta_{n\mathrm{H}},$$

Eq. (91) gives relations between the rms charge radii of the members of a multiplet numbered n and 1:

$$\langle r^2(n) \rangle^{1/2} = (Z+n-1)(Z+n-2)\{Z(Z-1)\langle r^2(1) \rangle^{-1/2} + c^{-1}[M(n) - M(1) + (n-1)\Delta_{n\mathrm{H}}]\}^{-1},$$

$$n = 2,3,\dots,2T+1. \tag{92}$$

Since the values of the rms charge radii are known from experiment only for the first members of multiplets, it is impossible to check (92) at present. However, Eqs. (92) can be used to estimate the rms charge radii of members of multiplets with $n > 1$. For example, in Refs. 65 and 67 the known values of $\langle r^2(1) \rangle^{1/2}$ (Ref. 68) were used to calculate the rms charge radii of the members of isobaric $4p$ quintets with $36 \leqslant A \leqslant 64$, and also the members of multiplets with isospins 3 and 4. In Fig. 3 we show the dependences of the rms charge radii of the members of quintets on the numbers of protons and neutrons; these clearly show the effect of the nuclear shell structure. In fact, as a rule the rms charge radii of the members of a quintet are decreased if these nuclei contain magic (submagic) numbers of not only protons, but also neutrons. This is consistent with the data[69] on nuclear ground states.

CONCLUSION

In this review we have described a new approach to the study of the bound and resonance states of two-particle systems based on the assumption that the Coulomb coupling constant can take not only physical but also complex values. In atomic physics the theory of complex CCC describes isoelectronic atomic and ionic systems, and in nuclear physics it describes isobars and isobaric multiplets. The method of complex CCC has made it possible to obtain regularities in the separation (decay) of isobars and isobaric multiplets into fragments, and also to formulate and solve the inverse scattering problem in a fundamentally new way which has a clear physical interpretation.

This method is useful not only for analyzing the general properties of two-particle systems, but also for practical calculations. Here among the theoretical results we have described new semiempirical methods of calculating the following nuclear characteristics: the half-lives of α-active nuclei, the excitation energies, the differences of the Coulomb energies, and the rms charge radii of the members of isobaric multiplets. In addition, we have suggested efficient ways of separating the phase shift into its purely nuclear and nuclear–Coulomb interference parts, of finding new exact solutions of the Schrödinger equation, and of constructing two-particle interaction potentials using the phase shifts. Therefore, as regards the number of applications and the importance of the results obtained in nuclear physics, the method of complex CCC has turned out to be just as useful and productive as the Regge method of complex orbital angular momentum proposed earlier.

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