Long-range potentials in low-energy nuclear physics

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Recent theoretical investigations of long-range potential effects in elastic and inelastic collisions of light nuclei at low energies are reviewed. Attention is focused on the qualitative and mathematically correct description of the characteristic behavior of the elastic scattering phases and cross sections in the low-energy limit and analogs of the Ramsauer effect. In an effectively two-particle approximation, it is shown how to estimate the collision-energy range where the long-range potential cannot be neglected and how to determine correctly the range of such a potential. The problem of extrapolating the phases and cross sections of elastic collisions and the matrix elements of some nuclear reactions in the solar pp cycle in the experimentally inaccessible energy range is solved in the same approximation. The approach developed is used to investigate the effect of the polarization potential in pp scattering and in the reaction $pp \to de^+\nu$ as well as the role of the interaction of the neutron magnetic moments in triplet neutron–neutron scattering. © *1996 American Institute of Physics.*
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1. INTRODUCTION

In quantum mechanics,[1] nuclear theory,[2–5] and scattering theory[6–11] a potential $V^s(r)$ decaying with distance r in the manner of a Yukawa potential

$$V^s(r) \sim V_0^s r^{-1} \exp(-\mu r), \quad V_0^s = \text{const}, \quad \mu > 0, \quad r \to \infty, \tag{1}$$

or more rapidly is said to be short-range. A short-range potential V^s can be neglected in some region $r > r_s$. The lower limit r_s is ordinarily called the range of this potential. Potentials which decay more slowly, for example,

$$V^a(r) \sim V_0^a r^{-a}, \quad V_0^a = \text{const}, \quad a > 0, \quad r \to \infty, \tag{2}$$

are said to be long-range. The range of such potentials is considered to be infinite. For low-energy nuclear physics, the most important interactions are electromagnetic long-range interactions, such as the Coulomb interaction ($V^c \sim r^{-1}$) and the polarization interaction ($V^p \sim r^{-4}$), as well as the interaction of the magnetic moment of a particle with the Coulomb field of a nucleus ($V_{ls}^m \sim r^{-3}$) and the interaction of two magnetic moments ($V^m \sim r^{-3}$). No less interesting are the possible superpositions of these potentials and the short-range potential V^s describing the nuclear interaction.

The role of the Coulomb interaction in low-energy nuclear physics and diverse Coulomb–nuclear effects due to the superposition $V^{cs} = V^c + V^s$ has been investigated in detail both theoretically and experimentally. The results are reflected in many monographs[5–11] and reviews.[12–14] The effect of the interaction V_{ls}^m on neutron–nucleus scattering has also been well studied.[15–18] The results of recent theoretical investigations of the effects produced by the superpositions

$$V^{cps} = V^c + V^p + V^s, \tag{3}$$

$$V^{ms} = V^m + V^s \tag{4}$$

have not yet been summarized. This has motivated the present review, which is devoted to an analysis of the present status of the theory of potential scattering by the superpositions (3) and (4).

The main objectives of the present review are as follows: to show the importance of long-range potentials in low-energy nuclear physics, give a correct definition of short- and long-range potentials, describe methods for determining the energy range where the long-range potential cannot be neglected compared with the nuclear interaction, present qualitative estimates of the effect of long-range potentials on elastic and inelastic collisions, and present effective mathematical methods which make it possible to take account of this effect correctly.

The role of polarization potential in elastic nuclear and pion–nucleus collisions is discussed in Sec. 2. In Sec. 3 the effect of the same potential on nucleosynthesis reactions is analyzed. Section 4 is devoted to the concept of the range of the polarization potential. In Sec. 5 the effect of the interaction of the neutron magnetic moments on triplet neutron–neutron scattering is investigated.

2. POLARIZATION EFFECTS IN ELASTIC COLLISIONS

As is well known,[1,3,19] the most important details of the scattering of a slow particle by a complex of particles of like charge can be established and investigated on the basis of the two-particle model with an effective potential (3). In the system of units employed below, $\hbar, c, \mu = 1$ and the Coulomb potential has the form $V^c = 1/rR$, where r is the distance between the incident particles and the center of mass of the system and $R = \hbar^2/2\mu z_1 z_2 e^2$ is the Bohr radius of the system particle+complex. If the complex has a spherically symmetric charge distribution, which in the present section we assume to be the case, then the remainder of the Coulomb interactions between the slowly incident particle and the particles in the complex, i.e. the polarization potential, has in the adiabatic approximation[19] the asymptotic form[20]

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$$V^p(r)\sim -\alpha_e/2|R|r^4, \quad r\gg|R|, \tag{5}$$

where the electric polarizability of the complex is $\alpha_e>0$.

For example,[21] in the case of scattering of slow protons by a deuteron the pd polarization interaction has the asymptotic form (5). The experimental value[22] of the polarizability of the deuteron is $\alpha_e=0.7\pm0.015$ fm^3, the polarizability of ^{3}H, ^{3}He, and ^{4}He nuclei is an order of magnitude smaller,[23] $\alpha_e\approx0.07$ fm^3, and the polarizability of the proton[24] is even smaller, $\alpha_e=(1.07\pm0.11)\cdot10^{-3}$ fm^3.[25]

As noted in Ref. 26, a method for calculating accurately the effective interaction potential between a charged particle and a target consisting of two charged particles with arbitrary ratios of their charges and masses was developed in Refs. 27–29. No such method is known in the general case.

In the sum (3) the last term V^s, describing the nuclear (non-Coulomb) part of the effective interaction of a particle with the complex, decreases much more rapidly than the polarization interaction,

$$|V^s(r)/V^p(r)|\ll1, \quad r>r_s, \tag{6}$$

and as a result of the screening[7] by a repulsive centrifugal barrier $l(l+1)r^{-2}$, it has virtually no effect on slow collisions with nonzero orbital angular momentum l. Therefore, in the present section only S-wave ($l=0$) collisions of a particle and a nuclear cluster, which interact by means of the superposition (3), are discussed. According to Eq. (5), we assume that

$$V^p(r)\equiv-(\alpha_e/2|R|r^4)\Theta(r-r_p), \quad r_s<r_p\sim|R|, \tag{7}$$

where Θ is the theta function and r_p is an arbitrary but fixed parameter. In the region $r<r_p$ the polarization interaction is a complicated nonlocal potential[27] and can possess an r^{-2} dependence.[30] In this region the polarization interaction is ordinarily accounted for without prescribing its explicit form. We shall describe such a model-free method below. Since the nuclear interaction is known with a large uncertainty,[3,5] we impose on it the most general constraints[6] which are consistent with the theory of nuclear forces,[2,15] specifically,

$$\int_0^\infty|V^a(t)|tdt<\infty, \tag{8}$$

$$\lim_{r\to\infty} r^n\exp[4(r/R)^{1/2}]V^a(r)=0, \tag{9}$$

where $n=0,1,\ldots$ and $a=s$. In what follows, the superscripts a, ca or c, a denote a potential which is additional to the Coulomb potential and all functions characterizing the scattering by the superposition $V^{ca}=V^c+V^a$. We shall describe the notation for the phases in greater detail: δ^{ca} is the total scattering phase for the potential $V^{ca}=V^c+V^a$; the symbol $\delta^{c,a}$ will designate the scattering phase engendered by the potential V^a in a Coulomb field and measured from the Coulomb phase δ^c. For example, $\delta^{c,p}$ is the Coulomb–polarization phase (the case $a=p$) and $\delta^{c,s}$ is the Coulomb–nuclear phase (the case $a=s$). In the case $a=ps$, i.e., scattering by the superposition $V^{ca}=V^{cps}$ of three and not two potentials, two separations of the total phase are correct:

$$\delta^{cps}=\delta^c+\delta^{c,ps}=\delta^c+(\delta^{c,p}+\delta^{cp,s}), \tag{10}$$

$$\delta^{cps}=\delta^{cp}+\delta^{cp,s}=(\delta^c+\delta^{c,p})+\delta^{cp,s}. \tag{11}$$

These separations are physically nonequivalent in the following sense. In Ref. 10 the Coulomb–polarization-nuclear phase $\delta^{c,ps}$ is produced by the sum $V^{ps}=V^p+V^s$ in the Coulomb field and is measured from the Coulomb phase. In Eq. (11) the Coulomb-polarization–nuclear phase $\delta^{cp,s}$ is due to the potential V^s in the Coulomb–polarization field $V^{cp}=V^c+V^p$ and is measured from the Coulomb–polarization phase $\delta^{cp}\equiv\delta^c+\delta^{c,p}$. Analysis of the phase $\delta^{c,ps}$ makes it possible to judge the parameters of the sum V^{ps} but not the parameters of the terms in the sum. To reconstruct the form and parameters of the nuclear interaction, it is necessary to know the phase $\delta^{cp,s}$ and the coefficients in its low-energy expansion.

We shall now complete the description of the notation. The symbol $A(k)$ will denote the limit of the function $A(r;k)$ as $r\to\infty$, where $k=\sqrt{E}$ is the scattering momentum in the center-of-mass system; $\tilde{A}(r;k)$ and $\bar{A}(r;k)$ are the Born and WKB approximations of the same function; finally, $\hat{\delta}_{m,n}$ is the Kronecker δ function.

The basic formulas for our analysis will be the well-known low-energy (E, $k\to0$) or threshold asymptotic expansions of the Coulomb,[1] Columb–polarization,[31,32] and Coulomb–nuclear[33] scattering phases:

$$\delta^c(k)\sim\eta(\ln\eta-1)+\pi/4, \quad V^c>0, \tag{12}$$

$$\tan\delta^{c,p}(k)\sim\begin{cases}4\alpha_e k^5R^2/15, & V^c>0,\\ -a^{c,p}kC^2(\eta), & V^c<0,\end{cases} \tag{13}$$

$$\tan\delta^{c,s}(k)\sim-a^{c,s}kC^2(\eta)[1-a^{c,s}k^2(r^{c,s}/2+R/3)]. \tag{14}$$

Here $\eta\equiv1/2kR$ is the Sommerfeld parameter, $C^2(\eta)\equiv2\pi\eta/(\exp(2\pi\eta)-1)$ is the Coulomb barrier factor,[11] and the coefficients $a^{c,p}$, $a^{c,s}$, and $r^{c,s}$ are the scattering lengths and the effective range.[3,5,13] The coefficients in the low-energy expansions of the phases are the fundamental characteristics of low-energy nuclear collisions, making it possible to judge the different parameters in the nucleus–nucleus[3,34] and nucleon–nucleon[5,35] interactions. Therefore the correct determination of all of these coefficients is an important problem in the scattering theory.

According to this theory,[6,10] potential scattering at high (low) energies is determined mainly by the behavior of the potential at small (large) distances. Experimental data on the S phases of nuclear[3] and nucleon–nucleon[5] collisions are available for high energies ($E\geqslant10$ MeV), where the effect of the polarization tail (7) is negligibly small. Extrapolation of these data to low energies according to the formulas (14) for the short-range potential gives the Coulomb–nuclear parameters $a^{c,s}$ and $r^{c,s}$. Strictly speaking, the nuclear interaction V^s reconstructed from the high-energy scattering data contains all information about the polarization interaction cut off at some point $r\approx r_s$. In the region $r<r_s$ such a nuclear interaction is indeed a superposition V^{ps}, from which the individual terms cannot be separated. For $r>r_s$ the part of the effective interaction V^{cps} which is additional to the Coulomb

interaction is also a superposition $V^{ps}=V^{p}+V^{s}$, in which the explicit form (7) of the first term has been proved theoretically.

Thus, the assumptions (6) and (7) and the inequality $r_s<r_p$ are not only physically well-grounded but they also make it possible to avoid double-counting the polarization interaction in the region $r<r_s$. The method of including the polarization potential on the basis of these relations seems to be most accurate and the only possible method at the present time. Therefore we shall assume below that V^s is the nuclear interaction reconstructed from the high-energy experimental data and that $r_s<r_p$.

The threshold potential scattering picture is determined mainly by the behavior of the tail of the potential.[1,10] According to Eqs. (3) and (5)–(6), the tail of the potential V^{cps} is the sum V^c+V^p. If $V^c>0$ and $k\to 0$, then, on account of Eqs. (13) and (14), we have $|\delta^{c,p}/\delta^{c,s}|\geqslant 1$ and therefore[31]

$$\delta^{c,ps}(k)\sim\delta^{c,p}(k)\sim O(\alpha_e k^5 R^2). \tag{15}$$

In other words, if the Coulomb potential is repulsive, then the contribution of the polarization potential to the phase $\delta^{c,ps}$ dominates over the contribution of the nuclear potential at sufficiently low energies. Therefore, to analyze nuclear collisions correctly at such energies, the polarization potential must be taken into account.

Attention to this physically obvious fact was drawn once again in Ref. 36 and in the subsequent work of Ref. 37. In Ref. 37 the S phase $\delta^{c,ps}$ of pd scattering, approximated by the sum

$$\delta^{c,ps}(k)\approx\delta^{c,p}(k)+\delta^{c,s}(k), \tag{16}$$

was substituted into the formula

$$K^{c,ps}(k)\equiv kC^2(\eta)\cot\delta^{c,ps}(k)+h(\eta)/R \tag{17}$$

and then it was shown, with the aid of the relation (15), that the scattering-length function $A^{c,ps}(E;\alpha_e)$ at threshold ($E=0$) and the scattering length itself $a^{c,ps}$,

$$A^{c,ps}(E;\alpha_e)\equiv -1/K^{c,ps}(k),\quad a^{c,ps}\equiv\lim_{k\to 0}A^{c,ps}(E;\alpha_e), \tag{18}$$

are infinite:

$$A^{c,ps}(E;\alpha_e)\sim -(4\alpha_e k^5R^3/15\pi)\exp(\pi/kR),$$

$$k\to 0;\quad a^{c,ps}=-\infty. \tag{19}$$

Thus it was proved that if for the superposition (3) the effective-range function $K^{c,ps}$, the scattering-length function $A^{c,ps}(E;\alpha_e)$, and the scattering length itself $a^{c,ps}$ are determined by the formulas (17) and (18), i.e., by analogy with the Coulomb–nuclear effective-range function[1,33]

$$K^{c,s}(k)\equiv kC^2(\eta)\cot\delta^{c,s}(k)+h(\eta)/R$$

$$\sim -1/a^{c,s}+k^2r^{c,s}/2, \tag{20}$$

the scattering-length function $A^{c,s}(E)$, and the Coulomb–nuclear scattering length itself $a^{c,s}$,

$$A^{c,s}(E)\equiv -1/K^{c,s}(k),\quad a^{c,s}\equiv\lim_{k\to 0}A^{c,s}(E), \tag{21}$$

then a physically meaningless result is obtained: $a^{c,ps}=-\infty$. This means[14] that, strictly speaking, the scattering length $a^{c,ps}$ does not exist, and the concept of the scattering length of the potential V^s in the field V^{cp} must be defined differently.[38]

This fact, first noted in Ref. 31, stimulated in the 1980s intensive investigation[39–62] of effects produced by the electric polarizability of nuclei in elastic and inelastic nuclear reactions and the further elaboration[63–72] of the scattering theory for few-particle systems[8] in the low-energy limit. Before reviewing the results of these investigations, we note the following physically obvious fact. Since $|V^p/V^c|\ll 1$ everywhere, all effects due to the polarization interaction (polarization effects) are small corrections to the purely Coulomb picture of the collision being studied. These corrections should be investigated only by mathematically accurate methods. Only then will the subsequent physical results be reliable. We shall compare the effectiveness of three such methods. For this, we shall discuss the results obtained with them.

2.1. Effectively two-particle scattering problem

Polarization effects have been studied in a series of works[39–62] on the basis of the following effectively two-particle model. It was assumed that the S-wave function $u^{ca}(r;k)$ for the scattering of two nuclei is a regular ($u^{ca}(0;k)=0$) solution of the Schrödinger equation

$$[\partial_r^2+k^2-V^c(r)-V^a(r)]u^{ca}(r;k)=0, \tag{22}$$

normalized to unit flux density at infinity:

$$u^{ca}(r;k)\sim\sin[\rho-\eta\ln 2\rho+\delta^c(k)+\delta^{c,a}(k)],$$

$$\rho\equiv kr\to\infty. \tag{23}$$

The polarization potential was taken in the form (7), and it was assumed implicitly that all the relations (5)–(9) hold.

In Refs. 39 and 40 the role of the polarizability of the deuteron in S-wave pd scattering was investigated by numerical integration of Eq. (22) for the function u^{cps}. The phase $\delta^{c,ps}$ was extracted from the computed asymptotic expression (23). Next, the function $D(b)$ (see Eqs. (3)–(5) in Ref. 40), equal to the function

$$t^{c,a}(b;k)=\tan\delta^{c,a}(b;k), \tag{24}$$

which is[9] the tangent of the scattering phase $\delta^{c,a}(b;k)$ in the case of the potential V^a, $a=ps$, cut off at the point $r=b$, was studied. The scattering length $a^{c,ps}(b)$ of such a truncated potential was introduced by analogy with the Coulomb–nuclear scattering length $a^{c,s}$ from the expansion (20), i.e. as the limit

$$a^{c,ps}(b)\equiv -\lim_{k\to 0}t^{c,ps}(b;k)/kC^2(\eta). \tag{25}$$

Next, it was shown that the function (25) has the asymptotic form

$$a^{c,ps}(b)\sim -(\alpha_e/4R^2)\exp[\nu(b/R)^{1/2}],\quad 2<\nu<4,\quad b\gg R, \tag{26}$$

and the formulas (17), (18), and (24)–(26) were used to prove that the limit of the function (25) as $b\to\infty$ is the infinite "scattering length" $a^{c,ps}$ of the nontruncated potential V^{ps}.

All calculations performed in Ref. 40 are of low accuracy, since even the basic relation (26) found by numerical integration of Eq. (22) is at variance with the analytically derived asymptotic form[41,60]

$$a^{c,ps}(b)\sim(-\alpha_e R/16\pi b^3)\exp[4(b/R)^{1/2}],\quad b\gg R. \tag{27}$$

In connection with the critical remark and the further discussion, we shall explain why the wave function u^{cps} and the phase $\delta^{c,ps}$ cannot be calculated in the low-energy limit ($k\to 0$) by direct integration of Eq. (22). First, even in the case $V^a\equiv 0$ the regular F and irregular G solutions of Eq. (22), i.e. the Coulomb functions[73] with zero orbital angular momentum ($l=0$), can be calculated only by special methods[63,74,75] based on asymptotic expansions[79,80] which are well known in the theory of differential equations.[76–79] For example, for $r\ll r_c$, where $r_c\equiv 1/k^2R$ is the Coulomb turning point, it is convenient to use the Bessel–Clifford expansions[73] written in the form[63]

$$F(\rho,\eta)=kC(\eta)\sum_{n=0}^{\infty}k^{2n}f_n(r),$$

$$G(\rho,\eta)\sim C^{-1}(\eta)\sum_{n=0}^{\infty}k^{2n}g_n(r). \tag{28}$$

Second, if $k\to 0$, then the asymptotic function (23) oscillates rapidly, the phase δ^c is large, the phase $\delta^{c,ps}$ approaches zero, and the function u^{cps} decays rapidly[42,60] in the region $r<r_c$, where $u^{cps}=O(\rho C(\eta))$. Finally, to find the phase $\delta^{c,ps}$, solving Eq. (22), the function u^{cps} must be calculated[78] with relative accuracy $\varepsilon\leqslant|\delta^{c,ps}|$ for any $r\leqslant B$, where the upper limit B of the integration interval of Eq. (22) must be several units of r_c.[57,61,62]

Thus, if $k\to 0$, then increasingly better accuracy ($\varepsilon\to 0$) on an increasingly larger interval ($B\to\infty$) is required to calculate the wave function u^{cps} and the phase $\delta^{c,ps}$ accurately.

For the reasons indicated above, as the energy decreases, it becomes increasingly more difficult to solve the Schrödinger problem (22) and (23) by direct numerical integration. It is obvious that this problem must be reformulated in a more convenient form for both numerical and analytical investigations. This was done in Refs. 32, 41, 42, and 55–64.

2.2. Nonlinear versions of the variable-phase method

It has been known[31] since 1965 that the Born approximation

$$\tan\delta^{c,a}(k)\approx\tan\tilde{\delta}^{c,a}(k)\equiv -k^{-1}\int_b^{\infty}V^a(t)F^2(kt,\eta)dt,$$

$$b\leqslant r_c, \tag{29}$$

correctly describes the threshold behavior of the scattering phase $\delta^{c,a}$ engendered by the correction (2) with $a>3$ to the repulsive Coulomb potential. In Ref. 32, Kvitsinskiĭ extended this assertion to the case $1<a<3$, having proved the asymptotic ($k\to 0$) relations

$$\delta^{c,a}(k)\sim\varphi^{c,a}(b;k)\equiv-\int_{r_c}^{b}[V^a(t)/2p^c(t;k)]dt, \tag{30}$$

$$\delta^{c,a}(k)\sim-(V_0^a/2R^{1-a})k^{2a-3}B(a-1,1/2), \tag{31}$$

where $b=\infty$, B is the beta function,[73] and $\varphi_0^{c,a}$ is the zeroth approximation of the solution $\varphi^{c,a}$ of Dashen's nonlinear phase equation[81]

$$\partial_r\varphi^{c,a}=(V^a/2p^c)[\cos 2(\chi^c+\varphi^{c,a})-1]$$
$$+(\partial_r p^c/p^c)\sin\varphi^{c,a}\cos(\varphi^{c,a}+2\chi^c) \tag{32}$$

with the single boundary condition $\varphi^{c,a}(r_c;k)\equiv 0$ and the functions

$$p^c(r;k)\equiv k(1-r_c/r)^{1/2}, \tag{33}$$

$$\chi^c(r;k)\equiv\arctan[p^c(r;k)F(\rho,\eta)/\partial_r F(\pi,\eta)]. \tag{34}$$

The inhomogeneous term in Eq. (32), containing the Coulomb quasimomentum (33), is singular at the boundary point $r=r_c$, and therefore Eq. (32) supplemented only by the boundary condition $\varphi^{c,a}(r_c;k)=0$ can possess many solutions.[76] For this boundary condition and any k the scattering phase $\delta^{c,a}(k)\equiv\varphi^{c,a}(\infty;k)$ does not depend on the behavior of the potential V^a for $r<r_c$, which seems absurd from the physical standpoint. As was pointed in Ref. 62, Eq. (32) must be regularized by Dashen's method[81] or supplemented by a second boundary condition, for example, the derivative $\partial_r\varphi^{c,a}$ must be prescribed at the point r_c and thereby include information about the behavior of the potential V^a for $r\leqslant r_c$. Since Eq. (32) contains the term $V^a/2p^c$ and the phase (34) which oscillates as $k\to 0$, it is inconvenient even for qualitative investigations. Indeed, the solution $\varphi^{c,a}(r;k)$ taken at some point $r=b$ is not the scattering phase of the potential V^a cut off at this point and converges to its limiting value $\varphi^{c,a}(\infty;k)$ much too slowly as $b\to\infty$, and moreover it oscillates in both cases $k\to 0$ and $b\to\infty$. It is more convenient to work with functions which are obviously physically meaningful for arbitrary r and which satisfy correct and simpler equations. Such functions are the phase function $\delta^{c,a}$,[41,57] the reduced amplitude functions cs^{ca} and sn^{ca},[42,58,59] and the amplitude functions c^{ca} and s^{ca}.[60–64]

Bencze and Chandler,[41] who used the relations (24) and (25) and the limiting forms of the series (28) in the limit $k\to 0$, reduced the well-known equation[10]

$$\partial_r\delta^{c,a}(r;k)=-k^{-1}V^a(r)[F(\rho,\eta)\cos\delta^{c,a}(r;k)$$
$$+G(\rho,\eta)\sin\delta^{c,a}(r;k)]^2 \tag{35}$$

with the boundary condition $\delta^{c,a}(0;k)=0$ to the boundary-value problem

$$\partial_r a^{c,a}(r)=-V^a(r)[f_0(r)-a^{c,a}(r)g_0(r)]^2,\quad a^{c,a}(0)=0 \tag{36}$$

for the function (25) and found analytically the asymptotic behavior of its derivative $\partial_r a^{c,a}$ in the limit $r\to\infty$ and $a=ps$. Integrating this asymptotic representation, it is easy to derive the formula (27). Although Bencze and Chandler[41]

TABLE I. The phase $\delta^{c,p}(k,r_p)$ (deg) of $\pi^+ d$ scatering as a function of the energy E (keV) and the parameter $r_p=\delta R$ of the polarization potential.

$E\backslash\gamma$	0.1	1	10
0.01	$2.0141\cdot10^{-15}$	$2.0141\cdot10^{-15}$	$2.0141\cdot10^{-15}$
0.1	$7.5129\cdot10^{-13}$	$7.5129\cdot10^{-13}$	$7.5129\cdot10^{-13}$
0.2	$4.3480\cdot10^{-12}$	$4.3479\cdot10^{-13}$	$4.3470\cdot10^{-13}$
0.4	$2.6058\cdot10^{-11}$	$2.5948\cdot10^{-11}$	$2.5774\cdot10^{-11}$
0.6	$7.9769\cdot10^{-11}$	$7.6569\cdot10^{-11}$	$7.1999\cdot10^{-11}$
0.8	$1.9417\cdot10^{-10}$	$1.7003\cdot10^{-10}$	$1.3951\cdot10^{-10}$
1.0	$4.1678\cdot10^{-10}$	$3.2091\cdot10^{-10}$	$2.1325\cdot10^{-10}$

noted that their main result ($|a^{c,ps}|=\infty$) was implied in Ref. 10, we recall the main result obtained by Babikov in Ref. 82, which will make it possible for us to formulate a helpful criterion. Representing the Coulomb functions by Bessel–Clifford series[73] and using for the phase function (24) the asymptotic expansion

$$t^{c,a}(r;k)\sim -kC^2(\eta)\sum_{n=0}^{\infty} k^{2n}A_n^{c,a}(r;h(\eta)), \tag{37}$$

Babikov[82] reduced the equation

$$\partial_r t^{c,a}(r;k)=-k^{-1}V^a(r)[F(\rho,\eta)+t^{c,a}(r;k)G(\rho,\eta)]^2 \tag{38}$$

with the boundary condition $t^{c,a}(0;k)=0$ to recurrence relations ($n=0,1,\ldots$) for the functions $A_n^{c,a}$. Applying to Eq. (38) the iteration method of Ref. 77, it can be proved that $|A_m^{c,a}(\infty;0)|<\infty$ if and only if the potential V^a satisfies the conditions (8) and (9) for all $n\leqslant m+1$. This criterion and the identity $a^{c,ps}(\infty)=A_0^{c,ps}(\infty;0)$, following from Eqs. (25) and (37), means that the equality $|a^{c,ps}|=\infty$ was already implied in Ref. 82.

An effective method[9,10] for studying the contribution from the tail of the potential V^a to the phase $\delta^{c,a}(k)$ is to analyze the ratio of the phase function $\delta^{c,a}(r;k)$ to its asymptotic value $\delta^{c,a}(\infty;k)$ equal to this phase. To clarify this method, we present the basic results of an investigation of the effect of the polarizability of the deuteron on S-wave $\pi^+ d$-scattering.[57]

In Ref. 57, Eq. (35) was rewritten in the dimensionless variable $\rho=kr$:

$$\partial_\rho\delta^{c,a}(\rho;k)=-k^{-2}V^a(\rho/k)[F(\rho,\eta)\cos\,\delta^{c,a}(\rho;k) +G(\rho,\eta)\sin\,\delta^{c,a}(\rho;k)]^2. \tag{39}$$

Next, the dependence of the phase function $\delta^{c,p}(\rho;k,r_p)$, i.e., the solutions of Eq. (39) with $a=p$ and the boundary condition $\delta^{c,p}(\rho_p;k,r_p)=0$, $\rho_p\equiv kr_p$, on the free parameter r_p of the polarization potential (7) was investigated. It was found that if $E\leqslant 1$ keV, then the fourth significant figure of the function $\delta^{c,p}(\rho;k,r_p)$ does not change as r increases in the region $\rho\geqslant 15\rho_c$, where $\rho_c=kr_c$. The relative accuracy $\varepsilon=10^{-4}$ was entirely sufficient, and therefore the values of the phases $\delta^{c,p}(k,r_p)$ were everywhere set equal to the values of the corresponding phase functions $\delta^{c,p}(15\rho_c;k,r_p)$.

As follows from Table I, at sufficiently low energies the

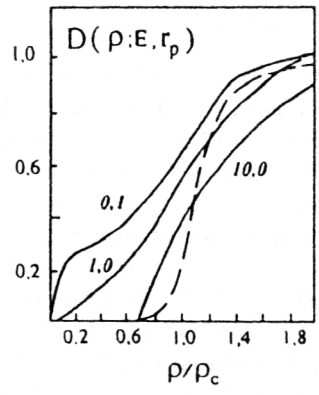

FIG. 1. $D(\rho;E,r_p)$ at energies $E=1$ keV (solid curves) and $E=0.1$ keV (dashed curve) and different values of the parameter $r_p=\gamma R$. The numbers on the curves are the values of γ.

Coulomb-polarization phase $\delta^{c,p}(k,r_p)$ is virtually independent of the parameter ρ_p. This result is also confirmed in Fig. 1, where plots of the functions

$$D(\rho;E,r_p)\equiv\delta^{c,p}(\rho;k,r_p)/\delta^{c,}(k,r_p) \tag{40}$$

are displayed. According to Fig. 1, the dependence of the phase functions and the functions (40) on the parameter r_p become increasingly weaker as the energy decreases. Indeed, for $E=0.1$ keV the functions (40), corresponding to three different values of the parameter $r_p=0.1R$, R, and $10R$, are equal to one another in the region $\rho>0.7\rho_c$ to within the relative accuracy $\varepsilon=10^{-4}$. The dashed curve is a general plot of these functions. It also follows from Fig. 1 that the region $\rho\leqslant 0.8\rho_c$ (i.e. $r\leqslant 0.8r_c$) plays only a small role in the formation of the phase $\delta^{c,p}(k,r_p)$. For $E\leqslant 1$ keV these phases are formed mainly in the interval $(0.8\rho_c,2\rho_c)$, i.e., at distances which are quite large on the nuclear scale ($R=104$ fm, whereas $r_c\approx 1.4\cdot10^4$ fm at $E=1$ keV), irrespective of the values of $\rho_p\geqslant 0.1R$. As the energy decreases, both limits of this interval increase as $O(E^{-1}R^{-1})$. This behavior of the phase functions $\delta^{c,p}(\rho;k,r_p)$ and their weak dependence on the parameter r_p are qualitatively explained by the screening[7,20] of the polarization potential by the repulsive Coulomb barrier in the quasiclassically inaccessible[80] range of distances ($r<r_c$). The mathematical proof[31] of the fact that the phase $\delta^{c,p}$ is virtually independent of the behavior of the polarization potential in this region is based on the following analysis of the Born integral (29), arising as a result of the first iteration of Eq. (39). As is well known,[75] $F^2=O(\rho^2C^2(\eta))$ for $r<r_c$ and $F^2=O(k/2p^c(r;k))$ for $r>r_c$. If $k\to 0$, then the contribution of the region $r<r_c$ to the integral (29) decreases as $O(\exp(-\pi/kR))$ and is negligibly small compared with the contribution (30) of the region $r>r_c$, decreasing according to (31) as $O(\alpha_e k^5R^2)$ to this same integral.

In Ref. 57 it was also established that the relative accuracy of the Born approximation $\delta^{c,p}\approx\widetilde{\delta}^{c,p}$ is no worse than 10^{-4}, if $E\leqslant 1$ keV. Later it was shown that the approximation $u^{cp}\approx u_{as}^{cp}$ of the wave function u^{cp} by its asymptotic form in the limit $\rho\to\infty$,

$$u_{as}^{ca}(r;k)\equiv F(\rho,\eta)\cos\,\delta^{c,a}(k,r_p)$$

$$+G(\rho,\eta)\sin\,\delta^{c,a}(k,r_p), \qquad (41)$$

where $a=p$, is valid to within the relative accuracy $\varepsilon=10^{-4}$, if $E\leqslant 1$ keV, $\rho_p\geqslant 0.1R$, and $r\geqslant 2r_c$, and the asymptotic representation (23) is valid with the same restrictions on E and r_p but with a lower relative accuracy $\varepsilon=0.1$ and at even larger distances, specifically, $r>10r_c$.

The assumption that the parameter r_p of the polarization potential (7) satisfies $r_s<r_p\sim R$ means that the distance ranges of the potentials V^s and V^p do not overlap. This made it possible to calculate in Ref. 57 the phase function $\delta^{c,ps}(\rho;k,r_p)$ in a model-independent manner, i.e. without prescribing a specific form of the π^+d nuclear potential V^s. For this, the standard[1] definition of the range r_s as the point on the r axis to the right of which it is possible to set $V^s=0$ was used. Therefore at the point $\rho_s=kr_s$ the function $\delta^{c,ps}(\rho;k,r_p)$ is equal to the Coulomb–nuclear phase $\delta^{c,s}$. Without knowing the potential V^s, this phase can be calculated at low energies according to the formula (14), using the experimental values[83] of the pion–deuteron scattering length $a^{c,s}=0.079$ fm. Therefore the assumption $r_s<r_p$ makes it possible to normalize the phase function $\delta^{c,ps}(\rho;k,r_p)$ at the point ρ_p by its experimental value equal to the Coulomb–nuclear π^+d phase $\delta^{c,s}(k)$.

Since $r_s<r_p$, it was assumed in Ref. 57 that $V^{ps}=V^p$ for $\rho\geqslant\rho_p$, and Eq. (39) was supplemented with the boundary condition $\delta^{c,ps}(\rho_p;k,r_p)=\delta^{c,s}(k)$. It was found that the phases $\delta^{c,ps}(k,r_p)\equiv\delta^{c,ps}(15\rho_c;k,r_p)$ calculated by solving such a boundary-value problem, like the phases $\delta^{c,p}(k;r_p)$, are virtually independent of the parameter r_p. The phase functions $\delta^{c,a}(k,r_p)$, $a=p,\,ps$, such as the scattering-length function $A^{c,ps}$ from Eq. (18) and the S-wave π^+d-scattering cross sections, exhibit the same property:

$$\sigma^{ca}(E)=4\pi[k^{-1}\sin(\delta^c(k)+\delta^{c,a}(k))]^2, \quad a=p,s,ps. \qquad (42)$$

It is pertinent to present here the scheme of the proof that the approximation (16), which is important for further analysis, is correct under the conditions (6) and (7). Let $a=ps$. For $r\leqslant r_s$ we have $V^{ps}=V^s$, and therefore the value of the phase function $\delta^{c,ps}(\rho;k)$ at the point ρ_s is equal to Coulomb–nuclear phase $\delta^{c,s}(k)$. We shall employ this value as the boundary condition $\delta^{c,ps}(\rho_s;k)=\delta^{c,s}(k)$ for Eq. (39) in the region $\rho\geqslant\rho_s$, where $V^{ps}\approx V^p$. We shall write this boundary-value problem in integral form and apply the iteration method of Ref. 77. After the first iteration the phase $\delta^{cp,s}$ will be a sum of the Coulomb–nuclear phase and the Born integral (29) with the lower limit $b=r_s$ and index $a=4$. The Born approximation is quite accurate, which makes it possible to replace in this sum $\widetilde{\delta}^{c,p}$ by $\sigma^{c,p}$ and to obtain the formula (16).

Using Eqs. (13), (14), and (16), we shall clarify the scattering-length function $A^{c,ps}$ and estimate the limits E^- and E^+ of the energy intervals $E\leqslant E^-$ and $E\geqslant E^+$ for which the contribution of the potential V^p and, correspondingly, V^s to the scattering is dominant. Estimates of this kind are arbitrary and depend on the assumed accuracy. Let us agree to the following definition: The contribution of $A^{c,p}$ from V^p (or the contribution of $A^{c,s}$ from V^s) to the function $A^{c,ps}$ under study dominates for $E\leqslant E^-$ (or for $E\geqslant E^+$) if $|A^{c,s}/A^{c,ps}|\leqslant 0.1$ for $E\leqslant E^-$ (or $|A^{c,s}/A^{c,ps}|\geqslant 0.9$ for $E\geqslant E^+$).

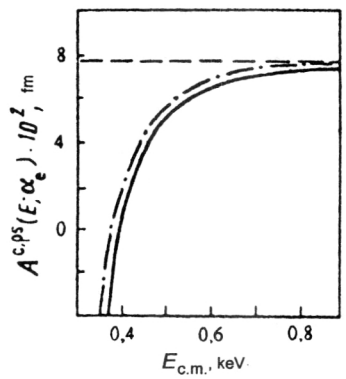

FIG. 2. π^+d scattering-length function $A^{c,ps}(E;\alpha_e)$. The calculations were performed by the variable-phase method (solid curve) and the WKB method (dot-dashed curve) for $\alpha_e=0.7$ fm^3. The dashed line represents the function $A^{c,ps}(E;\alpha_e)=A^{c,s}(E)$ at $\alpha_e=0$.

In Fig. 2 the solid curve represents the general plot of the three functions $A^{c,ps}(E;\alpha_e)$, calculated in Ref. 57 for $r_p=0.1R$, R, and $10R$ from Eqs. (17), (18), and (39). If the polarization interaction is included ($\alpha_e=0$), then $\delta^{c,ps}=\delta^{c,s}$ and the function $A^{c,ps}(E;0)$ is equal to the Coulomb–nuclear scattering-length function $A^{c,s}(E)$ displayed in Fig. 2 by the dashed line. On the basis (14), (20), and (21), the function $A^{c,s}$ is virtually independent of energy in the range studied, $E\leqslant 1$ keV, and has a finite limit at the point $E=0$, equal to the π^+d Coulomb–nuclear scattering length $a^{c,s}$. The polarization interaction (7) with $\alpha_e=0.7$ fm^3 radically changes the behavior of the function $A^{c,ps}(E;\alpha_e)$. As the energy decreases in the range $E\leqslant E^+=0.8$ keV, the phase $\delta^{c,s}$ in the sum (16) decreases more rapidly than the phase $\delta^{c,p}$, and therefore the function $A^{c,ps}$ deviates from its "horizontal asymptote" $A^{c,ps}(E;0)\approx a^{c,s}$. Since $a^{c,s}>0$, the signs of the phases $\delta^{c,p}$ and $\delta^{c,s}$ are opposite, and therefore their sum $\delta^{c,ps}$ and the function $A^{c,ps}$ vanish at $E=E_1\approx 0.4$ keV. As the energy decreases in the range $E\leqslant E^-=0.35$ keV, the phase $\delta^{c,p}$ dominates over the phase $\delta^{c,s}$, and as a result of the relation (13) the function $A^{c,ps}$ decreases without bound according to the law (19).

We shall now discuss the results of the comparative analysis ($a=p,s,ps$) in Ref. 57 of the π^+d cross sections (42). It is convenient to write the cross sections σ^{ca} as the sum

$$\sigma^{ca}=\sigma^c+8\pi k^{-2}\sin\,\delta^c\sin\,\delta^{c,a}\cos(\delta^c+\delta^{c,a})+\sigma^{c,a} \qquad (43)$$

of terms which oscillate as $k\to 0$: the Coulomb σ^c and interference cross sections[1] and the smooth term $\sigma^{c,a}$, which is usually investigated in experiments. On account of the relations (12)–(15), as $k\to 0$ we have

$$\sigma^c(E)\equiv 4\pi k^{-2}\sin^2\,\delta^c(k)=O(k^{-2}), \qquad (44)$$

$$\sigma^{c,a}(E)\equiv 4\pi k^{-2}\sin^2\,\delta^{c,a}(k)$$

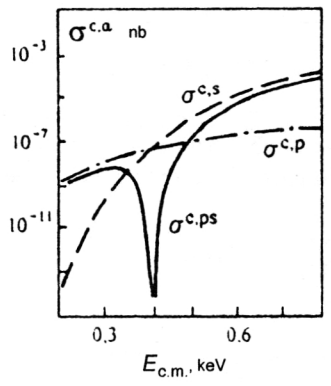

FIG. 3. Coulomb–polarization $\sigma^{c,p}$, Coulomb–nuclear $\sigma^{c,s}$, and Coulomb–polarization-nuclear $\sigma^{c,ps}$ S-wave $\pi^+ d$-scattering cross sections.

$$\sim \begin{cases} \pi\left(\dfrac{2\pi a^{c,s}}{kR}\right)^2 \exp(-2\pi/kR), & a=s, \\ 4\pi\left(\dfrac{4\alpha_e k^4 R^2}{15}\right)^2, & a=p,ps. \end{cases} \tag{45}$$

The Coulomb cross section (44) diverges in an oscillatory manner. If $a=p$ or $a=ps$, then the second term in the sum (43) also oscillates, but because the phase $\delta^{c,p}$ (13) decreases in a power-law manner as a function of k it exhibits a threshold behavior $O(k^3)$. For this reason, the Coulomb–polarization-nuclear cross section $\sigma^{c,ps}$, determined by the formula (45) for $a=ps$, decreases at threshold according to a power law. It is displayed in Fig. 3 by the solid curve. If $E \leqslant E^- = 0.35$ keV, then $\delta^{c,p} \gg |\delta^{c,s}|$, $\delta^{c,ps} \approx \delta^{c,p}$ and the cross section $\sigma^{c,ps}$ is determined mainly by the scattering on the potential V^p. The cross section $\sigma^{c,p}$ for such scattering is given by the formula (45) with $a=p$ and is displayed by the dot-dashed line in Fig. 3. In the region $E^- < E < E^+$, where $E^+ = 0.8$ keV, the phases $\delta^{c,p}$ and $\delta^{c,s}$ are comparable in order of magnitude but have opposite signs, and the two potentials V^p and V^s make approximately the same contribution to the cross section $\sigma^{c,ps}$. In this region, specifically, for $E = E_1 \approx 0.4$ keV, the phase $\delta^{c,ps}$ vanishes, and therefore the function $A^{c,ps}(E)$ and the cross section $\sigma^{c,ps}$ vanish at the same energy. For $E \geqslant E^+$ the cross section $\sigma^{c,ps}$ is close to the cross section $\sigma^{c,s}$, which at threshold decreases exponentially as in (45) (Fig. 3, the dashed curve). In this region $\sigma^{c,ps} \approx \sigma^{c,s}$ and the polarizability of the deuteron has virtually no effect on $\pi^+ d$ scattering. The minimum in the cross section $\sigma^{c,ps}(E)$ at $E \approx 0.4$ keV is due to the combined action of the repulsive pion–nucleus potential and the attractive polarization potential. This phenomenon is the S-wave $\pi^+ d$ analog of the Ramsauer effect,[84] well known in atomic physics.[19] We shall clarify this analogy. As observed in Ref. 84, the electron scattering cross section of the inert gases Ar, Kr, and Xe possesses a local minimum at an energy of about 0.5 eV. This phenomenon was termed the Ramsauer effect. It was later shown in Ref. 85 that this effect is due to the combined action of the short-range repulsive exchange potential and the long-range attractive polarization potential.

The role of the polarization potential (7) in S-wave $\pi^- d$ scattering was analyzed in Ref. 62 in analogy with the investigation of $\pi^+ d$ scattering.[57] The Coulomb $\pi^- d$ potential is attractive ($V^c < 0$) and does not screen the polarization potential. Therefore in both cases ($a=p,ps$) the phases $\sigma^{c,a}(k,r_p)$ are formed mainly in the finite interval $r_p < r < r_p + 2|R|$ and decrease rapidly as the parameter r_p increases. These conclusions demonstrate the results presented in Tables II and III. As one can see from Table III and Eq. (13), the Coulomb–polarization $\pi^- d$ scattering length $a^{c,p}$ is finite ($a^{c,p} \approx -4 \cdot 10^{-6}$ fm for $r_p = |R| = 104$ fm) and negligibly small compared with the Coulomb–nuclear scattering length $a^{c,s} = 0.079$ fm. Therefore it follows from Eq. (16) that $\delta^{c,ps} \approx \delta^{c,s}$ if $E \leqslant 1$ keV.

Thus, in the case $V^c < 0$ the potential V^p is only a very small correction to the nuclear interaction and all three ($a=p,s,ps$) cross sections $\sigma^{c,a}$ exhibit identical threshold behavior.

The results of Refs. 41, 57, and 62 mentioned above show that the nonlinear equations (35), (36), (38), and (39) make it possible to investigate in detail the role of the polarization potential (7) in elastic collisions. To investigate the effect of this interaction on a nuclear reaction it is necessary to know the function u^{ca} of the initial state. To construct such a function the problems (35) or (38) must be solved, and then the definite integrals[9,10] containing the functions $\delta^{c,a}(r;k)$ or $t^{c,a}(r;k)$ must be calculated. The linear version of the variable-phase method is more convenient, since in this method such integrals need not be calculated and the key equations are linear, so that they can be investigated by well-known asymptotic methods.[29]

TABLE II. The phase function $\delta^{c,p}(r;k,r_p)$ (deg) of $\pi^- d$ scattering as a function of the energy E (keV) and the distance $r=\gamma|R|$. All values are multiplied by 10^{11}.

$E \setminus \gamma$	0.1	1	10
1.04	1216	1182	908
1.08	2287	2216	1685
1.2	4813	4312	3402
1.4	7462	7183	4947
1.6	8971	8592	5621
1.8	9846	9388	5886
2.0	10356	9837	5976
4.0	11033	10343	6213
6.0	11075	10440	6272
10.0	11161	10503	6295
15.0	11165	10512	6300
30.0	11173	10518	6308

TABLE III. The phase $\delta^{c,p}(k,r_p)$ (deg) of $\pi^- d$ scattering as a function of the energy E (keV) and the parameter $r_p = \gamma|R|$. For $\gamma = 0.1$, 1, 10 the values of the phases are multiplied by 10^{10}, 10^{11}, 1014, respectively.

$\gamma \setminus E$	0.1	1	10
10^{-4}	69845	11249	12449
10^{-3}	69845	11249	12447
10^{-2}	69841	11242	12372
10^{-1}	69748	11173	11711
1.0	69572	10518	11469
10	69387	6308	6557

2.3. Linear version of the variable-phase method

The linear version of the variable-phase method, proposed by Calogero[86] for solving the problem (22) and (23), is based on the well-known[76] method of variation of "constant" coefficients c^{ca} and s^{ca}. Bencze *et al.*[42] used the same method, but they employed the functions $cs^{sa}=c^{ca}$ and $sn^{ca}=s^{ca}/kC^2$, and by making the substitution

$$u^{ca}(r;k)=cs^{ca}(r;k)[F(\rho,\eta)/kC(\eta)]+sn^{ca}(r;k)\times[C(\eta)G(\rho,\eta)] \tag{46}$$

they reduced Eq. (22) to the system of linear equations

$$\partial_r\begin{Bmatrix} cs^{ca}(r;k) \\ sn^{ca}(r;k)\end{Bmatrix}=V^a(r)u^{ca}(r;k)\begin{Bmatrix} C(\eta)G(\rho,\eta) \\ -F(\rho,\eta)/kC(\eta)\end{Bmatrix} \tag{47}$$

with the initial conditions $cs^{ca}=1$ and $sn^{ca}=0$ at the point $r=0$. Elaborating a well-known idea,[31,33,87] Bencze *et al.* employed instead of the decomposition (10) the representation (11), introduced a corresponding modified effective-range function

$$K^{cp,s}(k)\equiv kC_p^2(\eta)\cot\,\delta^{cp,s}(k)+h_p(\eta)/R \sim -1/a^{cp,s}+r^{cp,s}k^2/2, \tag{48}$$

and proved that in contrast to $a^{c,ps}$ the modified scattering length

$$a^{cp,s}\equiv-\lim_{k\to0}\tan\,\delta^{cp,s}(k)/kC_p^2(\eta) \tag{49}$$

is bounded, is physically meaningful, and can be calculated by extrapolating the phase $\sigma^{cp,s}$ to the point $E=0$ or by solving Eq. (22) at $k=0$.

The theory proposed in Ref. 42 for scattering on the superposition (3) of three potentials is incomplete. The functions $C_p(\eta)$ and $h_p(\eta)$ contained in Eqs. (48) and (49) have long been known as definite integrals,[87] but they have still not been found explicitly and, moreover, there is no effective method for constructing asymptotic expansions for these integrals in the limit $k\to0$. The next problem, which was not considered in Ref. 42, is to construct the low-energy expansions of the wave function u^{cps} and to investigate the effect of the polarization potential on its behavior. The solution of these problems is the subject of Refs. 60 and 62–64, where the linear version of the variable-phase method was developed in its original formulation.[86] In this version the desired solution u^{ca} of the problem (22) and (23) is written in the form

$$u^{ca}(r;k)=N^{c,a}(k)U^{ca}(r;k),$$
$$U^{ca}(r;k)\equiv c^{ca}(r;k)F(\rho,\eta)+s^{ca}(r;k)G(\rho,\eta), \tag{50}$$

the unknown amplitude functions c^{ca} and s^{ca} satisfy the equations

$$\partial_r\begin{Bmatrix} c^{ca}(r;k) \\ s^{ca}(r;k)\end{Bmatrix}=k^{-1}V^a(r)U^{ca}(r;k)\begin{Bmatrix} G(\rho,\eta) \\ -F(\rho,\eta)\end{Bmatrix} \tag{51}$$

with the boundary conditions $c^{ca}=1$ and $s^{ca}=0$ at the point $r=0$, and the phase $\delta^{c,a}(k)$ and the normalization factor $N^{c,a}(k)$ of the function (50) are equal to the limits of the corresponding functions as $r\to\infty$:

$$\delta^{c,a}(r;k)=\arctan[s^{ca}(r;k)/c^{ca}(r;k)], \tag{52}$$

$$N^{c,a}(r;k)=\cos\,\delta^{c,a}(r;k)/c^{ca}(r;k). \tag{53}$$

To construct low-energy asymptotic expansions, in Refs. 60 and 62 the Coulomb functions were replaced by the series (28) and the solutions of Eq. (51) were sought in the form

$$c^{ca}(r;k)\sim\sum_{n=0}^{\infty}k^{2n}c_n^{ca}(r),$$

$$s^{ca}(r;k)\sim kC^2(\eta)\sum_{n=0}^{\infty}k^{2n}s_n^{ca}(r). \tag{54}$$

The ultimate result was the asymptotic expansion

$$u^{ca}(r;k)\sim kC(\eta)N^{c,a}(k)\sum_{n=0}^{\infty}k^{2n}U_n^{ca}(r),\quad k\to0,\quad r\ll r_c, \tag{55}$$

$$U_n^{ca}(r)\equiv\sum_{m'+m=n}[c_{m'}^{ca}(r)f_m(r)+s_{m'}^{ca}(r)g_m(r)], \tag{56}$$

where the distance r is separated from the scattering momentum k, and the new unknown functions c_n^{ca} and s_n^{ca} do not depend on the energy and satisfy the recurrence relations $(n=0,1,\dots)$

$$\partial_r\begin{Bmatrix} c_n^{ca}(r) \\ s_n^{ca}(r)\end{Bmatrix}=V^a(r)\sum_{m'+m=n}U_{m'}^{ca}(r)\begin{Bmatrix} g_m(r) \\ -f_m(r)\end{Bmatrix} \tag{57}$$

with the initial conditions $c_n^{ca}(0)=\hat\delta_{n,0}$ and $s_n^{ca}(0)=0$.

For $r\geqslant r_c$ the series (28), (54), and (55) converge slowly. Therefore the functions $s^{ca(m)}$, $c^{ca(m)}$, and $u^{ca(m)}$ generated by the mth iteration of Eq. (51) and written in the equivalent integral form were used to approximate the functions c^{ca}, s^{ca}, and u^{ca} in the region $r\geqslant r_c$ and to construct the low-energy asymptotic expansions of the phase (52) and the normalization factor (53). The sufficient condition

$$(\alpha_e/3r_p^2R)\max\{3(\pi/2)^{1/2},\ |a^{c,s}/r_p|\}<1/2, \tag{58}$$

was proved by the method of contractions.[77] In this method these iterations converge uniformly on the entire half-line $r\geqslant r_p$ and the following asymptotic representations in the limit $k\to0$ hold: the more accurate approximation of the phase than the Born approximation (29),

$$\tan\,\delta^{c,p}(k)=\frac{B_2(\infty;k)}{1+B_3(\infty;k)}+O[B_2^2(\infty;k)], \tag{59}$$

$$B_n(r;k)\equiv-k^{-1}\int_{r_p}^{r}V^p(t)F(kt,\eta)[F(kt,\eta)\hat\delta_{n,2}-G(kt,\eta)\hat\delta_{n,3}]dt, \tag{60}$$

and the expansion of the normalization factor,

$$N^{c,a}(k)=\sum_{n=0}^{2}k^{2n}N_n^{c,a}+\Delta N^{c,a}(k),\quad a=p,s,ps, \tag{61}$$

$$\Delta N^{c,s}=O(k^6);\quad \Delta N^{c,p},\Delta N^{c,ps}=O(k^{16/3}). \tag{62}$$

Here $N_n^{c,a}$ are finite constants both for $a=s$, when the polarization potential is neglected, and for $a=p,ps$, when it is

taken into account. In these cases the remainder terms (62) exhibit different threshold behavior. This is how the effect of the polarization potential on the wave function is manifested in the region $kr \ll 1$, where according to Eqs. (50) and (51) it has the asymptotic form $u^{ca} \sim N^{c,a}(k)\rho C(\eta)$. The method proposed in Ref. 60 for constructing low-energy asymptotic expansions is semianalytic: The asymptotic expansions are series in explicitly known functions of the energy and functions of distance, which can be expressed quite simply in terms of the solution of the key equations (57). For example, by virtue of Eqs. (52) and (54), the coefficient $a^{c,a}$ from Eq. (19) or (20) is equal in the limit $r \to \infty$ to the corresponding function

$$a^{c,a}(r) = -s_0^{ca}(r)/c_0^{ca}(r). \tag{63}$$

A more general one-dimensional Schrödinger scattering problem than Eqs. (22) and (23) was investigated in Refs. 63 and 64: It was assumed that the angular momentum is $l \geqslant 0$, and the potential V^a satisfies only the condition (8). In Ref. 63 it was shown how to construct the regular and irregular wave functions in the form of asymptotic series, in which the momentum and distance are separated, in the limits $k \to 0$ and $r \ll r_c$. A new perturbation theory, which is an asymptotic method both in the limit of large angular momentum and in the limit of low energy, was proposed in Ref. 64. The zeroth approximation of this theory is calculated by integrating the equations for the amplitude functions on a definite finite segment $r \leqslant b < \infty$ and corresponds to the potential V^a cut off at the point $r = b$. All higher-order approximations which take account of the potential V^a in the region $r > b$ are constructed analytically by an iteration method.

In the case at hand, when $l = 0$, $a = 4$, and $V^c > 0$, the perturbation theory constructed in Ref. 64 is applicable on the entire semiaxis $r \geqslant 0$ if

$$\alpha_e / R r_p^2 < \sqrt{2/\pi} \ln 3. \tag{64}$$

Under this condition the relative estimate

$$|\delta^{(1)c,p}(k)/\delta^{c,p}(k) - 1| < O((kR)^{16/3}), \quad kR \to 0, \tag{65}$$

is valid. Here the phase $\delta^{(1)c,p}$, obtained in first-order perturbation theory, is expressed in terms of integrals (60) by the formula

$$\tan \delta^{(1)c,p}(k) = \exp(-2B_3(\infty;k)) \times \int_{r_p}^{\infty} \partial_t B_2(t;k) \exp(2B_3(t;k))\, dt. \tag{66}$$

The results of the works mentioned[32,41,42,57,60,62–64] prove convincingly that the variable-phase method is effective for investigating polarization effects and deriving different low-energy representations in an explicit form. However, a qualitative investigation of the distortions of the phase and wave function by the polarization interaction can be made by a simpler method, specifically, by the classical WKB method,[1] as was done in Ref. 55, or by a modification of this method, proposed in Ref. 56 and elaborated below.

2.4. WKB method in the small parameter $q = kR$

Introducing the dimensionless momentum $q \equiv kR$ and the variable $x = q^2 r/R$, we write Eq. (22) in the form

$$[q^2 \partial_x^2 + (p^{ca}(x;q))^2] u^{ca}(x;k) = 0 \tag{67}$$

with the small ($q \to 0$) parameter in front of the highest-order derivative and the function

$$(p^{ca}(x;q))^2 = (p^c(x))^2 + (R/q)^2 V^a(xR/q^2). \tag{68}$$

The momentum q in Eq. (67) plays the role of Planck's constant $\hbar$ in the standard quasiclassical approximation,[1] and the functions (68) and $p^c \equiv \sqrt{1 - x^{-1}}$ are analogs of the total and Coulomb (see Eq. (33)) quasimomenta. For this reason, we call the method of Ref. 79, which is well known in the theory of differential equations, for constructing the asymptotic expansions of the solutions of Eq. (67) in the limit $q \to 0$, the WKB method in the small parameter q. We shall now apply this method to Eq. (67).

First, we construct the WKB asymptotic expansions of the functions $u^{cp\pm}$, which satisfy Eq. (67) in the case $a = p$ and the normalized (at $\rho = xq^{-1} \to \infty$) conditions

$$u^{cp\pm}(x;q) \sim \sin[q^{-1}x - \eta \ln(2q^{-1}x) + \delta^c(q) + \delta^{c,p}(q) + (1 \mp 1)\pi/4]. \tag{69}$$

In this case the function (68) is given by the formulas

$$(p^{cp}(x;q))^2 = (p^c(x))^2 + \beta x^{-4} \Theta(x - x_p),$$
$$\beta \equiv \alpha_e q^6 / 2R^3, \tag{70}$$

it increases monotonically on the half-line $x \geqslant x_p \equiv q^2 r_p R^{-1}$, and it has one zero $x_{cp} = x_c + O(q^6)$, which approaches the root $x_c = 1$ of the equation $(p^c(x))^2 = 0$ as $q \to 0$. We shall not require the functions $u^{cp\pm}$ on the interval $x < x_p$, where the standard WKB method requires complicated modifications.[80] The well-known sufficient condition[79] for this method to be applicable on the half-line $x \geqslant x_p$ holds under the condition (58) or (64). Both inequalities hold for all collisions studied below.

The function (70) is negative on the segment $[x_p, x_{cp} - \varepsilon]$, where ε is a small number, and therefore in this region the WKB asymptotic expansions of the functions $u^{cp\pm}$ have the form of decreasing and increasing exponentials:

$$u^{cp\pm}(x;q) \sim [2^{1\pm 1} |p^{cp}(x;q)|]^{-1/2} \exp(\pm q^{-1} \varphi^{cp}(x;q)), \tag{71}$$

$$\varphi^{cp}(x;q) \equiv \int_{x_{cp}}^{x} |p^{cp}(t;q)|\, dt. \tag{72}$$

We write the leading terms in the asymptotic expansions (71) in the form

$$u^{cp+}(x;q) \sim \bar{F}(x,q) P^+(x;q),$$
$$u^{cp-}(x;q) \sim \bar{G}(x,q) P^-(x;q), \tag{73}$$

separating the well-known WKB asymptotic expressions[73] of the Coulomb functions

$$\bar{F}(x,q)=\frac{\exp(q^{-1}\varphi^c(x))}{|2p^c(x)|^{-1/2}},\quad \bar{G}(x,q)=\frac{\sqrt{2}}{|p^c(x)|\bar{F}(x,q)}, \tag{74}$$

$$\varphi^c(x)\equiv\int_1^x|p^c(t)|dt=\pi/2-\sqrt{x(1-x)}-\arcsin\sqrt{x}, \tag{75}$$

and the functions describing the effect of the polarization potential

$$P^{\pm}(x;q)\equiv S(x;q)\exp(\mp Q(x;q)), \tag{76}$$

$$S(x;q)\equiv\left|\frac{p^c(x)}{p^{cp}(x;q)}\right|^{1/2},$$

$$Q(x;q)\equiv q^{-1}\int_x^1[|p^{cp}(t;q)|-|p^c(t)|]dt. \tag{77}$$

Under the condition (64) the derivative functions (76) are negligibly small and the functions (77) in the approximation linear in the parameter α_e/r_p^2R have the very simple form

$$S(r;k)\approx 1+\alpha_e/2r^3,\quad Q(r;k)\approx-\alpha_e/5r^2\sqrt{rR},$$

$$r\geq r_p,\quad k=0. \tag{78}$$

In the neighborhood $(x_{cp}-\varepsilon,\ x_{cp}+\varepsilon)$ of the point x_{cp} the WKB asymptotic expansions of the functions $u^{cp\pm}$ contain[79] the Airy functions.[73] The functions (70) and (72) are positive on the half-line $[x_{cp}+\varepsilon,\infty]$. When the functions (71) are continued analytically into this region, they pass into the corresponding oscillating functions

$$u^{cp\pm}(x;q)\sim(p^{cp}(x;q))^{-1/2}\sin[q^{-1}\varphi^{cp}(x;q)+(2\mp 1)\pi/4]. \tag{79}$$

To construct the leading terms of the asymptotic expansions (79), we set $x_{cp}=x_c$ and approximate the function (72) by a sum of two integrals

$$\varphi^{cp}(x;q)\sim\varphi^c(x)+\varphi^{c,p}(x;q)$$

$$\equiv\int_1^x|p^c(t)|dt+\beta\int_1^x\frac{t^8dt}{2|p^c(t)|^{1/2}}, \tag{80}$$

$$\varphi^c(x)=\sqrt{x(x-1)}-\ln(\sqrt{x}+\sqrt{x-1}), \tag{81}$$

$$\varphi^{c,p}(x;q)=(\alpha_e q^6/30R^3)\sqrt{1-x^{-1}}(8+4x^{-1}+3x^{-2}). \tag{82}$$

Using (70) and (80)–(82), we write the leading terms of the asymptotic expansions of the functions (79) in the limit $\rho\to\infty$ in the form

$$u^{cp\pm}(r;k)\sim\sin[\rho-\eta\ln 2\rho+\bar{\delta}^c+\bar{\delta}^{c,p}+(1\pm1)\pi/4]. \tag{83}$$

Comparing Eqs. (69) and (83), we conclude that the Coulomb and Coulomb–polarization WKB phases are determined by the limits

$$\bar{\delta}^c(k)\equiv\lim_{x\to\infty}q^{-1}\varphi^c(x),\quad \bar{\delta}^{c,p}(k)\equiv\lim_{x\to\infty}q^{-1}\varphi^{c,p}(x;q) \tag{84}$$

of the functions (81) and (82), and they are identical to the asymptotic expansions (12) and (13) of the corresponding phases δ^c and $\delta^{c,p}$.

We now construct the WKB approximation $\bar{u}^{cps}$ of the function u^{cps}. Let us switch to the variable r. For $r\leq r_p$ the polarization interaction (7) is absent, and therefore $u^{cps}\sim u^{cs}$. The relation $r_s<r_p$ makes it possible to approximate the function u^{cs} for $r>r_s$ by the function (41). Replacing in this approximation the Coulomb functions by their WKB asymptotic expansions (74) and (75), we have

$$\bar{u}^{cps}\approx\bar{N}^{cp,s}u_{as}^{cs}\sim\bar{N}^{cp,s}(\bar{F}\cos\delta^{c,s}+\bar{G}\sin\delta^{c,s}),$$

$$r_s<r\leq r_p, \tag{85}$$

where $\bar{N}^{cp,s}$ is a yet unknown normalization factor. The relation (6) makes it possible to neglect the nuclear potential for $r\geq r_p$ and to represent the solution $\bar{u}^{cps}$ by a linear combination of the functions (73):

$$\bar{u}^{cps}\approx u^{cp+}\cos\bar{\delta}^{cp,s}+u^{cp-}\sin\bar{\delta}^{cp,s},\quad r\geq r_p. \tag{86}$$

On account of Eq. (83), the function (86) has the required asymptotic form (23) with the phase

$$\delta^{cps}\approx\bar{\delta}^{cps}=\bar{\delta}^{cp}+\bar{\delta}^{cp,s}=(\bar{\delta}^c+\bar{\delta}^{c,p})+\bar{\delta}^{cp,s}, \tag{87}$$

corresponding to the decomposition (11). The terms $\bar{\delta}^c$ and $\bar{\delta}^{c,p}$ in the sum (87) are calculated according to the formulas (81), (82), and (84), and the unknown phase $\bar{\delta}^{cp,s}$, generated by the potential V^s and measured from the WKB phase $\bar{\delta}^{cp}$, is determined by "joining" the functions (85) and (86) at the point $r=r_p$. Equating the logarithmic derivatives of these functions at the indicated point and neglecting the derivatives of the functions (76), we obtain

$$\tan\bar{\delta}^{cp,s}(k)\approx\exp(-2Q(r_p;k))\tan\delta^{c,s}(k), \tag{88}$$

$$\bar{N}^{cp,s}(k)\approx S(r_p;k)\exp(-Q(r_p;k))\frac{\cos\bar{\delta}^{cp,s}(k)}{\cos\delta^{c,s}(k)}. \tag{89}$$

It follows from Eqs. (78) and (88) that the phases $\bar{\delta}^{cp,s}$ and $\delta^{c,s}$ have the same threshold behavior. It is described by the leading term of the threshold asymptotic expansion of the factor $kC^2(\eta)$. Let us divide the relation (88) by this factor and pass to the limit $k\to0$. Taking account of Eqs. (14) and (78), we obtain the WKB approximation of the modified scattering length (49):

$$\bar{a}^{cp,s}\equiv-\lim_{k\to0}\tan\bar{\delta}^{cp,s}(k)/kC^2(\eta)$$

$$\approx a^{c,s}(1+2\alpha_e/5r_p^2\sqrt{r_pR}). \tag{90}$$

Using Eqs. (78), (85), and (89), we derive the relation

$$\bar{u}^{cps}/u_{as}^{cs}\approx N^{cp,s}\approx 1+\alpha_e r_p^{-2}(1/2r_p+1/5\sqrt{r_pR}),\quad r_s\leq r\leq r_p, \tag{91}$$

indicating that the polarization interaction changes very little the probability density of finding the colliding nuclei at the limit $(r=r_s)$ of the range of the nuclear forces.

We shall now describe a simple method for finding estimates of $E^{\pm}$ that does not require construction of the func-

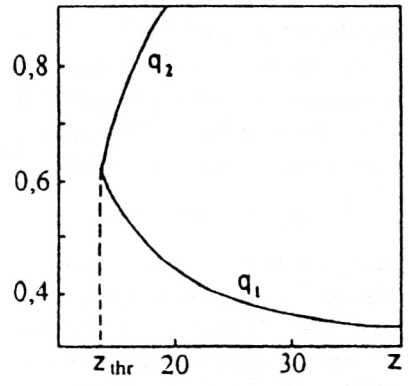

FIG. 4. Roots q_1 and q_2 of Eq. (93) as a function of the parameter z.

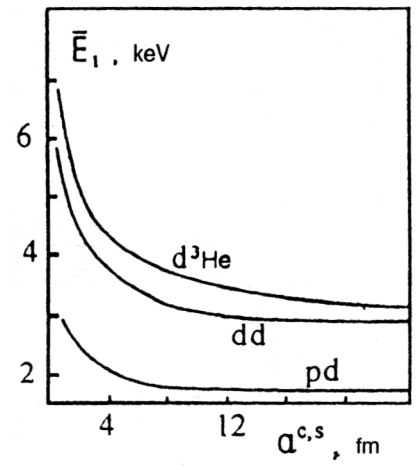

FIG. 5. Energy $\bar{E}_1$ corresponding to the root q_1 of Eq. (93) as a function of the Coulomb–nuclear scattering length $a^{c,s}$ for pd, dd, and d^3H collisions.

tion $A^{c,ps}$ and is based on the formulas (77), (84), (87), and (88). Let $a^{c,s}>0$, and let the root $k_1(\omega)$ of the equation

$$|\bar{\delta}^{c,p}(k)/\bar{\delta}^{cp,s}(k)|=w \tag{92}$$

with a fixed value of the parameter w satisfy the asymptotic condition $k_1(w)R\equiv q_1\ll 1$ of the applicability of the WKB method. It follows from the formulas $\bar{\delta}^{c,ps}=\bar{\delta}^{c,p}+\bar{\delta}^{cp,s}$ and (92) that the WKB estimates of $\bar{E}^{\pm}$ and $\bar{E}_1$ of the limits of the interval $E^-<E<E^+$, where $\delta^{c,p}=O(\delta^{c,s})$, and the positions of the zeros E^1 of the phase $\delta^{c,ps}$ (minimum of the cross section $\sigma^{c,ps}$) are given by the relations $\bar{E}^-=k_1^2(0{,}9)$, $\bar{E}^+=k_1^2(0{,}1)$, and $\bar{E}_1=k_1^2(1)$. We replace in Eq. (92) all phases by the corresponding leading terms of their asymptotic expansions (13), (14), and (88), and we switch to the dimensionless variable $q=kR$. Then we obtain the more convenient equation

$$\pi q^{-1}=\ln(1+zq^{-5}). \tag{93}$$

The positions of its roots $q_1(z)$ and $q_2(z)$ are determined not by the parameters α_e, $a^{c,s}$, and R, characterizing the collision under study, taken separately but rather by their dimensionless combination

$$z\equiv \pi wR^2|15a^{c,s}/4\alpha_e|\exp(-2Q(r_p;0)). \tag{94}$$

Equations and parameters of this type are said to be self-similar. Examples of such equations are the Navier–Stokes hydrodynamic equation[88] and the parameter in this equation—the Reynolds number.

The roots of Eq. (93), which were calculated in Ref. 56, are displayed in Fig. 4. The root q_1 always satisfies the condition $q_1<1$ and approaches zero as $z\to\infty$. If $z\geqslant z_{thr}=14.5$, then the roots are equal ($q_1=q_2$). Therefore if the values of the parameters α_e, $\alpha^{c,s}$, and R are such that $z\gg z_{thr}$, then Eq. (92) has a root $k_1\ll R^{-1}$. The self-similarity[88] of Eq. (93) makes it possible to obtain easily the WKB estimate $\bar{E}_1$ of the position of the S-wave Ramsauer minimum for any collision of a charged particle with a polarized cluster. For this, it is necessary to calculate the value of the self-similar parameter (94) for $w=1$ and then to use Fig. 4 to find the values of q_1 and the corresponding energy $\bar{E}_1=q_1R^{-2}$. As follows from the relation (94) and Fig. 4, for the same value of the polarizability α_e the energy $\bar{E}_1$ decreases as R and $a^{c,s}$ increase. These results are illustrated in Figs. 5–8, which are taken from Ref. 56. The energy $\bar{E}_1$ is displayed in Figs. 5 and 6 as a function of the Coulomb-nuclear scattering length $a^{c,s}$ for systems with different Bohr radii. The WKB approximations $\bar{A}^{c,ps}(E;\alpha_e)$ of the scattering-length functions are displayed in Figs. 7 and 8. This approximation was determined by the formulas (17) and (18), in which the exact phase $\delta^{c,ps}$ was replaced by its WKB asymptotic approximation $\bar{\delta}^{c,ps}=\bar{\delta}^{c,p}+\bar{\delta}^{cp,s}$. According to Fig. 7, as the Coulomb-nuclear pd scattering length $a^{c,s}$ decreases from 11.96 fm (Ref. 89) to 0.65 fm (Ref. 39), the zero of the pd function $\bar{A}^{c,ps}$ moves to higher energies. This same behavior is also confirmed in Fig. 8, according to which the zero of the $\pi^+{}^3$He function $\bar{A}^{c,ps}$ ($a^{c,s}=0.3$ fm (Ref. 90)) lies to the left of the zero of the $\pi^+{}^4$He function $A^{c,ps}$ ($a^{c,s}=0.143$ fm (Ref. 91)). The same π^+d function $\bar{A}^{c,ps}$ is displayed in Fig. 2 and in Fig. 8 by dashed and solid lines. As follows from Fig. 2, the relative accuracy ε of the approximation $A^{c,ps}\approx\bar{A}^{c,ps}$ is quite high ($\varepsilon\approx 0.06$) even at energies close to the position E_1 of the zero of the function $A^{c,ps}$ (the minimum of the cross section $\sigma^{c,ps}$). Hence the

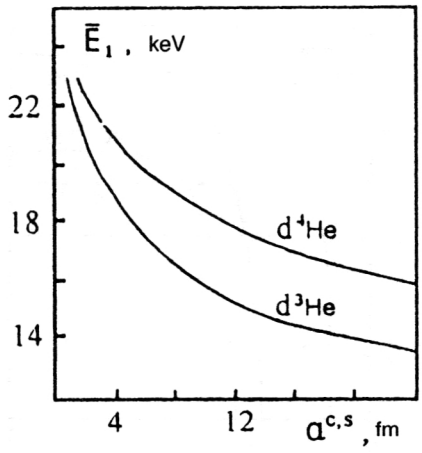

FIG. 6. Energy $\bar{E}_1$ corresponding to the root q_1 of Eq. (93) as a function of the Coulomb–nuclear scattering length $a^{c,s}$ for d^3He and d^4He collisions.

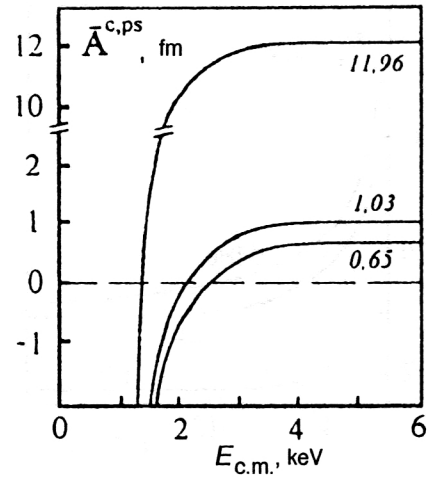

FIG. 7. pd scattering-length function $\bar{A}^{c,ps}(E;\alpha_e)$ for different values of the Coulomb–nuclear pd scattering length $a^{c,s}$. The numbers on the curves are the values of $a^{c,s}$ (fm).

estimates $\bar{E}_1$ of the positions of the S-wave Ramsauer minima in the cross section $\sigma^{c,ps}$ of the elastic pd, $\pi^+{}^3$He, and $\pi^+{}^4$He collisions can be found with approximately the same accuracy using Figs. 7 and 8. For example, for the quartet pd scattering $a^{c,s}=11.96$ fm and $E_1^{3/2}\approx\bar{E}_1\approx 1.2$ keV, and in the doublet case $a^{c,s}=1.03$ fm and $E_1^{1/2}\approx\bar{E}_1\approx 2.2$ keV. Therefore the total S-wave pd-scattering cross section should have two Ramsauer minima. Does this fact and the unitarity of the S matrix[7] mean that the pd-reaction $(d(p,\gamma)^3\text{He})$ cross section $\sigma_r(E)$ should have two maxima at approximately the same energies $E_1^{3/2}$ and $E_1^{1/2}$? How does the polarization interaction affect this and other nucleosynthesis reactions? References 43–56 and 58–62 are devoted to finding an answer to these questions.

3. POLARIZATION EFFECTS IN NUCLEOSYNTHESIS REACTIONS

The large difference (Davis' paradox) between the experimental value 2.0±0.2 SNU of the capture rate of solar neutrinos, measured in experiments with a ^{37}Cl detector,[92] and its theoretical estimate 8.0±3.0 SNU has still not been explained.[93–95] A large fraction of the solar neutrinos is one of the final products of the solar pp cycle[96] of sub-barrier fusion reactions of light nuclei. It is convenient to represent the cross section of such a reaction in the form[96,97]

$$\sigma_r^{ca}(E)=E^{-1}\exp(-2\pi\eta)S^{ca}(E), \tag{95}$$

thereby separating the most sharply energy-dependent Coulomb barrier factor. In modern astrophysics,[94,96] the factor S^{ca} in the cross section (95) of the nuclear reaction is proportional to the nuclear matrix element, which in the first-order perturbation theory in the Hamiltonian of the reaction under study it is convenient to express[98] as an integral containing its initial and final state functions.

For example, in the impulse approximation[96] in the β^+-capture Hamiltonian[4] the nuclear matrix element of the pp reaction $(pp\to de^+\nu)$,

$$\Lambda^{ca}(k)=[8\pi r_d^3k^2C^2(\eta)]^{-1/2}\int_0^\infty u^{ca}(r;k)u_d(r)dr, \tag{96}$$

is the overlap integral of the S wave function for pp scattering u^{ca} and the $^3S_1$ component of the wave function of the deuteron, whose radius is $r_d=4.316$ fm. The formula (96), proposed in the pioneering studies of Refs. 99 and 100, is still used.

The theoretical estimate of the solar-neutrino flux depends strongly on the S^{ca} factor of the initial and slowest (one event per 10^{29} yr) reaction $pp\to de^+\nu$ of the solar pp cycle.[96] This estimate is proportional to $(S^{cs})^{-5/2}$, so that for astrophysics it is important to investigate any corrections to the S^{cs} factors of nuclear reactions. In the 1980s a discussion of the infinite scattering length $a^{c,ps}$ initiated a series of investigations[43–56,58–62] of the role of the polarization potential in nucleosynthesis reactions. To show how the understanding of this role developed over time, we shall briefly review all these studies in the order in which they were published.

3.1. Estimates of the contribution of the polarization potential to nucleosynthesis reaction cross sections

The role of the polarization potential in nucleosynthesis reactions has been intensively studied since 1986, specifically, after Ref. 43, where it was predicted that for $E<2$ keV the cross sections of the reaction $(d,\gamma)^3$He, calculated taking account of (σ_r^{cps}) and neglecting (σ_r^{cs}) the polarization of the deuteron, "differ" strongly: $\sigma_r^{cps}/\sigma_r^{cs}\to\infty$ as $E\to 0$.

The unexpected results obtained in Ref. 43, specifically, the proof that the polarizability of the deuteron causes the $S^{cps}(E)$ factor of the pd reaction to diverge as $E\to 0$, i.e. it destroys unitarity,[2,3] stimulated the first investigation[56] of the WKB estimate $\bar{D}^p(E)$ for the contribution

$$D^p(E)\equiv S^{cps}(E)/S^{cp}(E)-1 \tag{97}$$

from the pd-polarization potential (7) to the S^{cps} factor. In Ref. 56 it was shown from the relation (91) that the estimate $\bar{D}^p(E)$ is a smooth function of the energy of pd collisions and does not exceed $5\cdot 10^{-3}$ for $E\leq 6$ keV. Therefore the

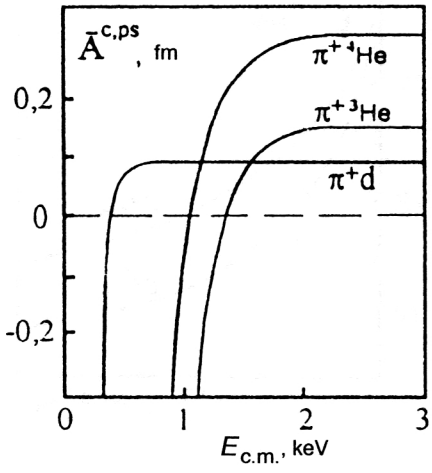

FIG. 8. π^+d, $\pi^+{}^3$He, and $\pi^+{}^4$He scattering-length functions.

$S^{cps}(E)$ factor of the pd reaction is also a continuous and finite function, which is at variance with the above-mentioned result $S^{cps}(0)=\infty$ of Ref. 43.

An attempt to restore unitarity of the method employed in Ref. 43 was undertaken in Refs. 44 and 45. It was unsuccessful: An unphysical result was once again obtained, indicating that because of the polarization interaction between the protons the pp-reaction cross section σ_r^{cps} should have a very large ($\sigma_r^{cps}/\sigma_r^{cs}\approx 10^{30}$) and narrow minimum.

The next work was L'vov's preprint.[55] Using the elements of the WKB method[1] and perturbation theory in the polarization interaction (7), L'vov constructed on the basis of Eq. (22) an approximate pd-scattering function u^{cps}, obtained the WKB estimate $\bar{D}^p\approx 10^{-3}$ for the contribution (97), and thereby confirmed the estimate derived previously in Ref. 56.

Later it was predicted in Ref. 46 that the cross section σ_r^{cps} of the reaction $^3\mathrm{H}(d,n)^4\mathrm{He}$. should have a narrow maximum ($\sigma_r^{cps}/\sigma_r^{cs}\approx 100$), produced by the polarization interaction (7) of a deuteron with tritium.

The results obtained in Refs. 43–46 were discussed in the report of Ref. 47 with no mention of the critical remarks made by L'vov in Ref. 55 and Levashev in Ref. 48. In this connection it should be especially noted that L'vov was the first to explain the fact that the main error in Refs. 43–46 was the unjustified replacement of the function u^{cp} by its asymptotic form u_{as}^{cp} (see Eq. (41)) for arbitrary $r\leqslant r_s$. Later this error, which is common to all the studies of Refs. 43–46, was pointed out by Levashev in a series of abstracts[48] and then in 49–52. In these studies Levashev assumed that the S^{ca} factor of the pp reaction is proportional to the squared nuclear matrix element (96). Replacing the pp function u^{cps} in Eq. (96) by the function $\tilde{u}^{cps}$ which he constructed in the Born approximation in the polarization potential (7), he obtained the Born approximations $\tilde{\Lambda}^{cps}$ and $\tilde{D}^p$ for the integral (96) and the contribution (97). Levashev's main result means that the contribution $\tilde{D}^p$ from the polarization potential in the Born approximation $\tilde{S}^{cps}$ for the S^{cps} factor of the pp reaction is determined by the value of the polarization potential at the range limit of nuclear forces ($\tilde{D}^p(E)\approx\alpha_e/2r_p^2R$), and therefore this contribution is negligibly small.

Prior to the appearance of the journal versions[50,51] of the preprint of Ref. 49 Bencze suggested in Ref. 53 a novel and physically clear method for estimating the contribution (97). This method is based on R-matrix theory[3,7] and is therefore more general than the other methods developed in Refs. 48–52 and 58–62. Unfortunately, Bencze's last formula (Eq. (14) in Ref. 53), indicating that the contribution of D^p is equal to $\tan^2\delta^{c,p}$, is wrong. As Levashev noted in Ref. 52, a term was lost in the derivation of this formula. Nonetheless, this sad fact does not change Bencze's main conclusion[53] that the electrical polarizability of the nuclei makes a negligibly small contribution to the cross section for low-energy nuclear reactions. This conclusion was confirmed by precise calculations performed in Ref. 58.

In Ref. 58 the following model was employed to describe pp collisions. A soft-core Reid potential[5] was taken as the nuclear potential, and the representation (7) with the parameter $r_p=4$ fm and the doubled experimental value[25] ($\alpha_e=2\cdot 10^{-3}$ fm^3) of the polarizability of the proton was used for the polarization interaction. As a result of the calculations performed on the basis of Eq. (47) and the representations (46), (96), and (97), it was shown that for $E\leqslant 6$ keV the contribution (97) is negligibly small but is an increasing function of energy: $D^p(0)=2.11\cdot 10^{-6}$ and $D^p(E)/D^p(0)\approx 1.02$ at $E=6$ keV. This result stimulated subsequent investigations[59,60] of the energy dependence of the functions (96) and (97). They were performed on the basis of the same model of pp collisions.

In the announcement[59] of Ref. 60 it was proved that

$$\Lambda^{ca}(k)=\sum_{n=0}^{2}k^{2n}\Lambda_n^{ca}+\Delta\Lambda^{ca}(k),\quad k\to 0, \tag{98}$$

where Λ_n^{ca} are constants, irrespective of whether ($a=ps$) or not ($a=s$) the polarization interaction is taken into account, but as $k\to 0$ the quantity $\Delta\Lambda^{cps}$ decreases more slowly than $\Delta\Lambda^{cs}$. In other words, the polarization interaction does not change the energy dependence of the first three terms in the low-energy expansion of the nuclear matrix element, the contribution D^p, and hence also the cross section for the pp reaction.

A quite complete theory of pp collisions, which takes account of the polarizability of the proton, was constructed in Ref. 60. As a result of calculations performed using Eqs. (50)–(61) and (95)–(98), it was proved that the contribution D^p has the value $2.1108\cdot 10^{-6}$ at $E=0$, reaches its maximum value of $3.3773\cdot 10^{-6}$ at the point $E\approx 400$ keV, and then decreases slowly as the energy increases further.

The reaction $^7\mathrm{Be}(p,\gamma)^8\mathrm{B}$ was investigated in Ref. 54. To construct the function $\tilde{u}^{cp}$, approximating the p^7Be scattering function u^{cps}, the well-known[9] Born approximation $\tilde{t}^{cp}$ was used for the function t^{cp} satisfying Eq. (38). The final result was physically plausible, indicating that the estimate $\tilde{D}^p$ of the contribution (97) from the p^7Be polarization interaction (7) to the factor S^{cps} of the reaction $^7\mathrm{Be}(p,\gamma)^8\mathrm{B}$ is approximately $2.6\cdot 10^{-3}$.

The main result of Ref. 62 was announced in the abstract of Ref. 61. It consisted of the development of the concept of the range of the polarization potential, which we shall discuss in Sec. 4, and the proof that this potential has a short range in nuclear reactions.

The general result of Refs. 48–62, obtained on the basis of an effectively two-particle model, consists of the following: The polarization interaction does not change the threshold behavior of the cross sections of nucleosynthesis reactions, and its contribution to these cross sections is negligibly small. This conclusion is also confirmed by many results obtained on the basis of the few-body problem. We shall mention the most interesting ones.

An algebraic version of the resonating-groupg method[101] was employed in Ref. 102 to investigate the S^{cps} factors of the reactions $^3\mathrm{H}(\alpha,\gamma)^7\mathrm{Li}$ and $^3\mathrm{He}(\alpha,\gamma)^7\mathrm{Be}$ at zero energy. No polarization effects were observed.

Faddeev's three-particle integral equations[8] as modified in Ref. 103 were employed in Ref. 104 to calculate the

S^{cps} factor of the reaction $d(p,\gamma)^3$He at zero energy. It was proved for the first time on the basis of the three-body problem that this factor is finite in the low-energy limit. However, the authors were unable to calculate it in the energy range $E \leqslant 20$ keV of interest for astrophysics and take account of the long-range part (5) of the pd polarization interaction for a very simple reason: There is no effective method for solving the problem of three-body scattering at ultralow energies. A possible method for constructing such a method for the $3\to3$ scattering problem in a system of three pairwise noninteracting particles is based on a synthesis[65–70] of the variable-phase method and the theory of hyperspherical harmonics.[105,106]

In Ref. 65 a low-energy asymptotic expansion of the phase for the problem of $3\to3$ scattering in a system of three uncharged particles was obtained by the WKB method. The low-energy expansions for the same problem in the case of the truncated hyperradius were derived in Ref. 66. In the same case but for a system of three charged particles, the three-particle analog of the $(3\to3)$ effective-range function was proposed in Ref. 67. A semianalytic method for constructing the N-body analog of the Jost function for a system of N uncharged particles was discussed in Ref. 68. An accurate determination of the Jost function in the problem of the scattering of three neutrons was proposed in Ref. 69 and elaborated in Ref. 70.

As is well known,[106] the terms in the expansions of the pair Coulomb potentials in the basis of three-particle hyperspherical harmonics[105] can be calculated explicitly, but such series converge extremely slowly. This is why the approach proposed in Refs. 65–70 is unpromising for developing an effective method for investigating the problem of three charged particles in the low-energy limit. The first step in the construction of such a method is to investigate the low-energy asymptotic expansions in the two-particle case. In the next section the method developed in Refs. 60–64 for constructing such two-particle asymptotic expansions is used to analyze polarization effects in pp reactions. The model of Ref. 58, mentioned above, is used to describe the pp collisions.

3.2. Polarization effects in pp reactions

As is well known,[95,96] the average energy of a pp collision in the deep interior of the sun is approximately 6 keV. In the energy range $E<20$ keV, the squared Coulomb-nuclear ($a=s$) matrix element (96) is customarily[96] approximated by a linear function of the energy[107]

$$(\Lambda^{ca}(E))^2=(\Lambda^{ca}(0))^2(1+\lambda^{ca}E)+O(E^2) \qquad (99)$$

with the coefficients $(\Lambda^{cs}(0))^2=7.0\pm0.18$ and $\lambda^{cs}=2.2$ MeV^{-1}. The errors in determining them[97] are mainly due[98] to the uncertainties (≈0.025) in the nucleon–nucleon potential[5] and the contribution (≈0.02) of the meson exchange currents.[108]

Reid's soft-core nuclear potential[5] V^s possesses a Yukawa tail (1) and satisfies Eqs. (8) and (9) for all n. Therefore[60] the Coulomb–nuclear functions $c_0^{cs}(r)$, $s_0^{cs}(r)$, and $a^{c,s}(r)$, calculated from (57) and (63), are everywhere finite. The fourth significant figures of these functions remain

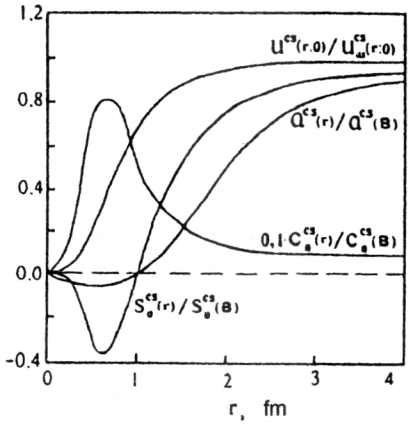

FIG. 9. Ratios of the Coulomb–nuclear functions $c_0^{cs}(r)$, $s_0^{sr}(r)$, $a^{c,s}(r)$, and $u^{cs}(r;E)$ to their asymptotic forms for $r\geqslant B=80$ fm and $E=0$ keV.

unchanged as r increases in the range $r\geqslant B=80$ fm. The ratios of these functions to their values at $r=B$ and the ratio of the wave function u^{cs}, calculated from the formulas (53)–(57), to its asymptotic form (41) are displayed in Fig. 9. According to this figure, the function $s_0^{cs}(r)$ reaches its limiting value in the limit $r\to\infty$ more slowly than the function $c_0^{cs}(r)$; the Coulomb–nuclear scattering length $a^{c,s}$ is formed mainly in the inner distance range ($r<4$ fm); and the approximation $u^{cs}(r;0)/u_{as}^{cs}(r;0)\approx1$ is valid only in the outer region ($r>4$ fm). The polarization potential (7) radically changes the behavior of the functions $s_n^{cps}(r)$ and $a^{c,ps}(r)$ in the region $r\gg r_p$. According to Eq. (57), as $x\equiv2\sqrt{r/R}\to\infty$, for $n=0,1,2$ the functions $s_n^{cps}(r)$ increase without bound and the functions $c_n^{cps}(r)$ remain bounded:[60]

$$s_n^{cps}(r)/c_0^{cps}(\infty)\sim(8\alpha_e/\pi R^2)(R/2)^{2n}x^{3n-6}\exp(2x),$$

$$c_n^{cps}(r)/c_0^{cps}(\infty)=1+32\alpha_e/5R^3x^5+O(x^{-10}). \qquad (100)$$

On the basis of Eq. (100), the scattering length $a^{c,ps}(b)$ determined by the formula (63) diverges according to the law (27) as $b\to\infty$. Since in any case ($a=p,s,ps$) the functions $c_n^{ca}(r)$ with index $n=0,1,2$ are bounded, for $E\to0$ and $a=p,s,ps$ the normalization factor (53) can be approximated by a linear function (61) of the energy:

$$N^{c,a}(k)\approx N_0^{c,a}+EN_1^{c,a}, \quad N_0^{c,a}=\frac{1}{c_0^{ca}(\infty)},$$

$$N_1^{c,a}=-\frac{c_1^{ca}(\infty)}{(c_0^{ca}(\infty))^2}. \qquad (101)$$

Likewise, in the three cases $a=p,s,ps$ the low-energy representation of the wave function follows from the formulas (55), (56), and (101):

$$u^{ca}(r;k)\sim kC(\eta)[u_0^{ca}(r)+Eu_1^{ca}(r)], \quad r\leqslant r_c, \qquad (102)$$

where the distance is separated from the energy, and the functions

$$u_0^{ca}(r)\equiv N_0^{c,a}U_0^{ca}(r), \quad u_1^{ca}\equiv N_0^{c,a}U_1^{ca}(r)+N_1^{c,a}U_0^{ca}(r), \qquad (103)$$

on the basis of Eqs. (55), (56), and (100), possess at $x \equiv 2\sqrt{r/R} \to \infty$ the asymptotic expansions

$$u_n^{ca}(r) = N_n^{c,a}[R^{2n+1}x^{3n}/(1+23n)]\sqrt{x/8\pi}\,\exp(x)(1 + O(x^{-1})). \quad (104)$$

Replacing in the integral (96) the pp-scattering wave function by its asymptotic approximation (102), we obtain the low-energy representation (99), whose coefficients are the integrals

$$\Lambda^{ca}(0) = (8\pi r_d^3)^{-1/2}\int_0^\infty u_0^{ca}(r)u_d(r)dr, \quad (105)$$

$$\lambda^{ca} = (\Lambda^{ca}(0))^{-1}(2\pi r_d^3)^{-1/2}\int_0^\infty u_1^{ca}(r)u_d(r)dr, \quad (106)$$

which converge at the upper limit, on account of the relations (104) and the rapid decay of the deuteron wave function: $u_d = O(\exp(-r/r_d))$ as $r \to \infty$. From the representations (99) it follows for the functions Λ^{cs} and Λ^{cps} that the contribution (97) has a linear low-energy asymptotic behavior:

$$D^p(E) \approx (\Lambda^{cps}(0)/\Lambda^{cs}(0))^2[1 + (\lambda^{cps} - \lambda^{cs})E] - 1. \quad (107)$$

Thus, in the proposed method the problem of extrapolating the nuclear matrix element Λ^{cps} and the contribution D^p into the region of astrophysically low energies reduces to integrating the linear equations (57) and then calculating the extrapolation coefficients according to the formulas (101)–(107). Since the key equations (57) do not depend on energy, such calculations are difficult to perform accurately. We shall discuss their results.

The calculated values of the coefficients (105) and (106) in the Coulomb–nuclear function (99),

$$(\Lambda^{cs}(0))^2 = 6.96072905, \quad \lambda^{cs} = 2.42552113\ \text{MeV}^{-1}, \quad (108)$$

are close to the coefficients $(\Lambda^{cs}(0))^2 = 6.934$ and $\lambda^{cs} = 2.5$ MeV^{-1}, obtained in Ref. 109 by the standard method based on calculating the functions u^{cs} by integrating the Schrödinger equation (22) with the soft-core Reid potential at low energies. The fundamental deficiencies of this approach, which were discussed in Sec. 2.1, make it impossible to use it for investigating polarization effects because their contribution is very small. In the method proposed, such investigations are entirely realistic.

It is obvious that the weak attractive polarization interaction increases very little the coefficients (105) and (106) and hence also the cross section for the pp reaction. Indeed, taking account of this interaction,

$$(\Lambda^{cps}(0))^2 = 6.96074375, \quad \lambda^{cps} = 2.42552961\ \text{MeV}^{-1}. \quad (109)$$

If this is necessary to do, then for $E < 20$ keV the contribution (97) from the polarization interaction can be described to within $\varepsilon \approx 0.02$ by the linear function (107) of the energy with the coefficients $D^p(0) = 2.1108 \cdot 10^{-6}$ and $\lambda^{cps} - \lambda^{cs} = 8.48 \cdot 10^{-6}$ MeV^{-1} found from the formulas (108) and (109).

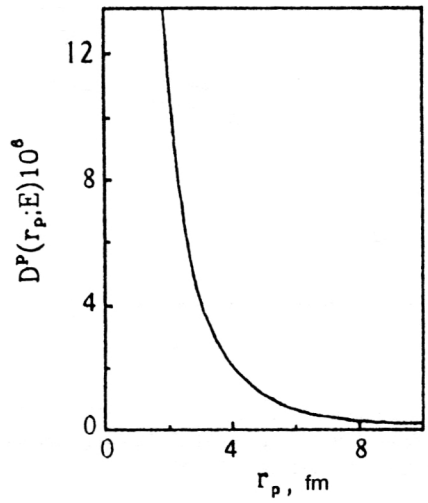

FIG. 10. Contribution $D^p(r_p;E)$ of the polarization potential to the S^{cps} factor for the pp reaction as a function of the parameter r_p for $E \leq 20$ keV.

The calculated dependence $D^p(r_p;E)$ of the contribution D^p on the parameter r_p of the polarization potential (7) at fixed energy ($E \leq 20$ keV) is demonstrated in Fig. 10. Even if this parameter reaches its lower, and physically reasonable limit, equal to the distance between the centers of mass of the protons (≈ 2 fm), the contribution remains negligibly small ($D^p \sim 10^{-5}$) compared with the existing uncertainties (≈ 0.025) of the nuclear interaction.

The ratio of the function $(\Lambda^{cps}(E))^2$ calculated without using the low-energy expansions to its linear approximation (99) with the coefficients (109) is displayed in Fig. 11. The relative accuracy ε of this approximation becomes worse as the energy increases: $\varepsilon \leq 0.003$ for $E \leq 10$ keV, $\varepsilon = 0.014$ at $E = 20$ keV, and $\varepsilon = 0.4$ at $E = 50$ keV. It is obvious that for $E > 50$ keV the linear approximation (99) cannot be used. In this energy range the pp reaction can be studied effectively on the basis of the formulas (50)–(53). We present here the most interesting results obtained by this method.

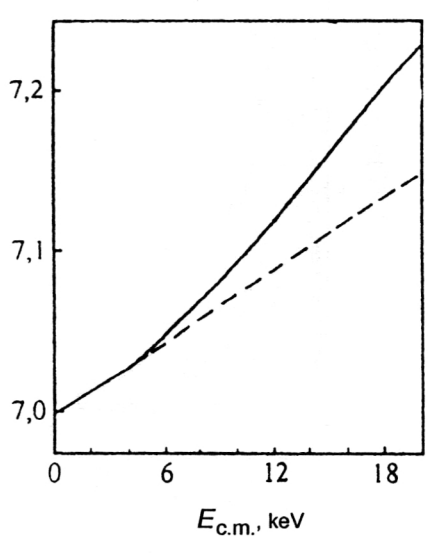

FIG. 11. Solid curve—$(\Lambda^{cps}(E))^2$; dashed curve—its linear approximation $(\Lambda^{csp}(0))^2(1 + \lambda^{cps}E)$.

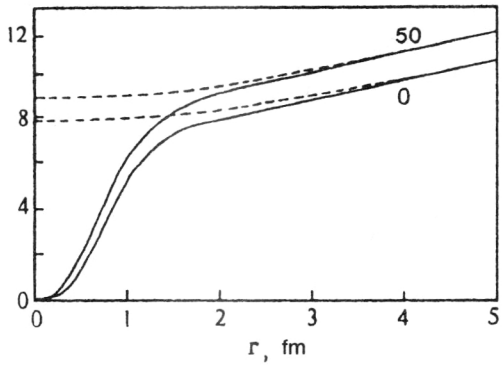

FIG. 12. pp-scattering wave function $u^{cps}(r;E)/kC(\eta)$ (solid curves) and its asymptotic form $u_{as}^{cps}(r;E)/kC(\eta)$ (dashed curves). The numbers on the curves are the pp-collision energies E (keV).

The functions $u_{as}^{cps}/kC(\eta)$ and $u^{cps}/kC(\eta)$ calculated according to the formulas (41) and (50) are displayed in Fig. 12. As one can see from Figs. 9 and 12, the polarization potential hardly changes the lower limit of the outer region ($r>4$ fm), where $u^{cps}\approx u_{as}^{cps}$ to within $\varepsilon\leqslant 0.01$.

Since the deuteron function decreases exponentially, the product $u^{cps}u_d/kC(\eta)$ displayed in Fig. 13 converges rapidly to zero as $r\to\infty$ and the region $r_-<r<r_+$, $r_-\approx 0.5$ fm, $r_+\approx 20$ fm, where the pp reaction mainly occurs, makes the main contribution to the integral (96). The upper limit r_+ of this region is approximately fives times greater than the size of the deuteron, because the function $u_{cps}/kC(\eta)$ increases as $r\to r_c$ (see Fig. 12). Both limits r_- and r_+ can be estimated more accurately. For this, it is convenient to introduce the function $\Lambda^{ca}(r;E)$, defined by the integral (96) with upper limit r, and then to analyze the ratio of this function to its limiting value as $r\to\infty$. The squares of these ratios are displayed in Fig. 14. We shall clarify their energy dependence. At very low energies only a small fraction of the colliding protons overcomes the repulsive Coulomb barrier, which is slightly lowered by the polarization attraction. As the energy increases, this barrier becomes increasingly more transparent, the function $u^{cps}u_d/kC(\eta)$ increases uniformly, and therefore the integral (96) also increases. As the energy increases further, the pp functions acquire zeros in the reaction region $r_-<r<r_+$. As a result, the function $u^{cps}u_d$ in the integrand in Eq. (96) starts to oscillate and the integral $(\Lambda^{cps})^2$ starts to decrease in magnitude. In the limit $E\to\infty$ the pp-collision time and hence the pp-reaction probability become infinitesimally small and the function u^{cps} converges uniformly to the function for free motion, everywhere equal to $\sin kr$ and orthogonal to the deuteron function. Therefore $\Lambda^{cps}\to 0$ as $E\to\infty$.

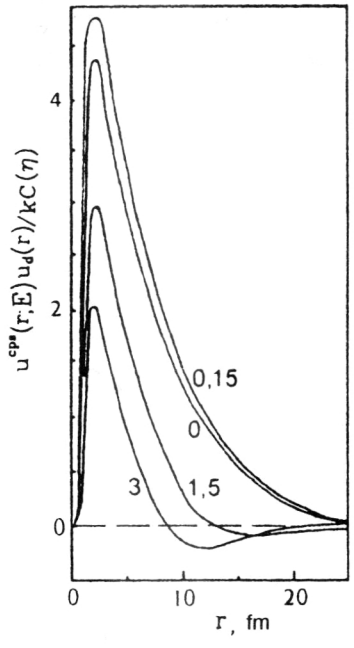

FIG. 13. Product $u^{cps}(r;E)u_d(r)/kC(\eta)$ of the pp-scattering and deuteron wave functions. The numbers on the curves are the pp-collision energies E (MeV).

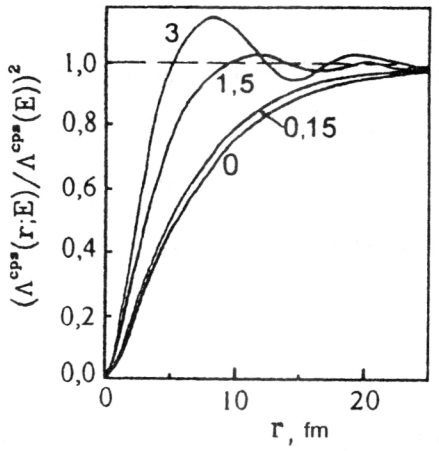

FIG. 14. Squared functions $\Lambda^{cps}(r;E)/\Lambda^{cps}(E)$. The numbers on the curves are the pp-collision energies E (MeV).

It follows from the qualitative analysis presented above that the function $(\Lambda^{cps})^2$ is bounded at any energy and should have a local maximum. These results are confirmed in Fig. 15. The figure displays the calculated squares of the matrix element Λ^{cps} and its standard[96] approximation Λ_{as}^{cs} obtained by making the substitution $u^{cps}\to u_{as}^{cs}$ in the integral (96). Both functions shown possess a maximum at $E\approx 130$ keV.

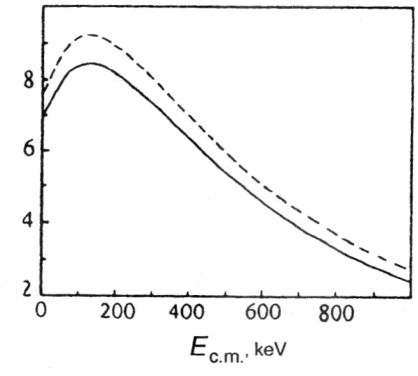

FIG. 15. Squared nuclear matrix elements of the pp reaction. Solid curve—$(\Lambda^{cps}(E))^2$; dashed curve—$(\Lambda_{as}^{cs}(E))^2$.

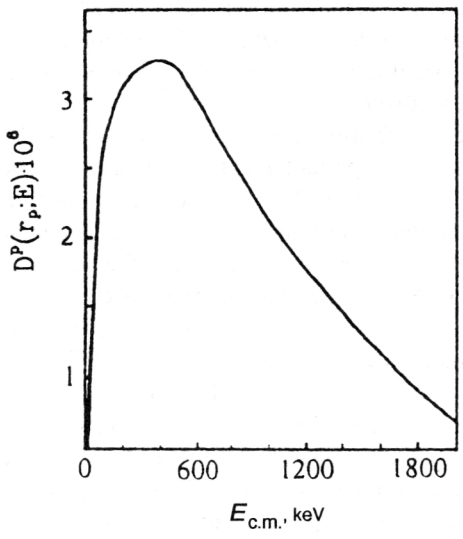

FIG. 16. Contribution $D^p(E)$ of the polarization potential to the pp-reaction cross section.

It is well known from the quasiclassical theory[80] that in the case $V^c>0$ only the part of the potential V^p in the region $r>r_c=1/ER$ affects elastic scattering. As the pp-collision energy increases from zero, the Coulomb turning point r_c, i.e. the lower limit of the range of distances where the attractive polarization potential influences scattering, moves leftward and therefore the contribution $D^p(E)$ increases. At the energy $E\approx E^p_{\max}=1/r_pR$, where $r_c=r_p$ and the polarization interaction (7) is at full strength from the standpoint of the quasiclassical theory, the contribution D^p reaches its maximum value. As the energy increases further, the relation $E>|V_p|$ comes into force and therefore the contribution D^p decreases monotonically. The qualitative results presented above are confirmed in Fig. 16, where the function $D^p(E)$ calculated for $r_p=4$ fm is displayed. The function has a maximum at the point $E\approx 400$ keV, which is in complete agreement with the qualitative estimate $E^p_{\max}\approx 372$ keV.

Thus, the main contribution from the interaction V^p to the normalization factor $N^{c,ps}$ of the function u^{cps} and the S^{cps} factor of the pp reaction comes from its interior part $V^p\Theta(r_+-r)$, where $r_+\approx 20$ fm. As was shown in Sec. 2.2, the main contribution from this interaction to the phase $\delta^{c,ps}$ and the corresponding cross section $\sigma^{c,ps}$ is due to its tail $V^p\Theta(r-r_c)$ if $V^c>0$, or to the inner part $V_p\Theta(2|R|+r_p-r)$ if $V^c<0$. Therefore the distance range where, strictly speaking, the potential V^p cannot be neglected depends strongly on the type of physical process being studied. This remark already shows that the range of the polarization potential must be taken into account correctly.

4. RANGE OF THE POLARIZATION POTENTIAL

In the physics literature[1,2] the range of a potential V is taken to be the point B on the r axis to the right of which it is possible to set $V\equiv 0$. In the computational mathematics[78] the range B of a potential V for the desired function A is determined as the upper limit of the segment $0\leq r\leq B$ in which the problem is being solved numerically. This limit is chosen so that the calculated function would be close, in a certain sense, to the exact solution of the problem. In asymptotic methods of the theory of differential equations[79] the point B is close in meaning to the one mentioned earlier and is determined as the lower limit of the interval $B\leq r\leq\infty$, where the desired function can be replaced, once again in a certain sense, by its asymptotic expression for $r\to\infty$.

Following Ref. 62, we shall determine the range B of the potential V^p for the desired function A as the solution $B=B(A;\varepsilon,\{\mathscr{P}\})$ of the inequality

$$d(A,B)\equiv|A(B;\{\mathscr{P}\})/A(\infty;\{\mathscr{P}\})-1|\leq\varepsilon,\quad 0\leq\varepsilon<1, \tag{110}$$

where $\{\mathscr{P}\}$ is the set of all parameters of Eq. (22). For A we shall study the phase function $\delta^{c,p}(r;k)$, the regular solution $u^{ca}(r;k,B)$ $(a=p,ps)$ of Eq. (22) with the potential V^p truncated at the point $r=B$, the normalization factor $N^{c,ps}(B;k)$ of the function $u^{cps}(r;k,B)$, and the matrix element $\Lambda^{cps}(B;k)$ defined by the integral (96), where the function $u^{ca}(r;k)$ is replaced by the function $c^{cps}(r;k,B)$. In each case the desired function A, the quantities d and ε, and the relation (110) have a clear meaning. By definition $A(B;k)$ is the desired function $A(k)=A(\infty;k)$ in the case when the potential V^p is truncated at the point $r=B$. If B satisfies the inequality (110), then the function $A(B;k)$ approximates the function $A(k)$, corresponding to the real polarization potential which has not been truncated, with relative accuracy d less than or equal to the required accuracy ε.

In practice, the function $A(B;k)$ is always calculated instead of the function $A(\infty;k)$, and the quantity B is determined so that the numbers $A(B;k)$ and $A(B';k)$ are identical to M significant figures for any $B'>B$. To construct B as a function of M, the sequence of functions $\{A(B_i;k\}_{i=1}^N$, corresponding to the sequence of potentials $V^p\Theta(B_i-r)$, truncated at the points $B_1<B_2<\ldots<B_N$, is calculated. As a result of a numerical experiment, the point B_N is chosen to be quite far away and is taken as a point at infinity. Next, Eq. (110) is reduced to the inequality

$$d(A,B_M)=|A(B_M;k)/A(B_N;k)-1|<10^{-M}, \tag{111}$$

which holds for all $M=1,\ldots,N-1$. This gives the solution $B(A;\varepsilon,\{\mathscr{P}\})=B_M$ of the inequality (110) with $\varepsilon=10^{-M}$, $M=1,\ldots,N-1$, and fixed values of the set $\{\mathscr{P}\}$ of all parameters.

It was shown by the method described in Ref. 62 that the range of the polarization potential for the $\pi^{\pm}d$ phases $\delta^{c,p}$ is given by the formulas

$$B(\delta^{c,p}(k);\varepsilon,\{k,R,r_p\})\approx\begin{cases}(0.3+0.6\varepsilon^{-1/3})r_c, & R>0,\\ 0.6r_p-(0.2-0.5\varepsilon^{-1/3})|R|, & R<0.\end{cases} \tag{112}$$

These formulas are valid if $\varepsilon=10^{-M}$ with $M=1,\ldots,5$, $0.1\leq r_p/|R|\leq 10$, and $E\leq 10$ keV, and they show that to calculate the $\pi^{\pm}d$ phases with relative error $d(\delta^{c,p},B)$ less than ε, the polarization potential (7) can be neglected in the region $r>B$, whose lower limit depends strongly both on the set $\{\mathscr{P}\}=\{k,R,r_p\}$ of all parameters and on the prescribed accuracy ε. It is obvious from the formulas (112) that in the

case $V^c>0$ the polarization potential has a longer range than in the case $V^c<0$. Its range for the $\pi^+ d$ phase $\delta^{c,p}$ increases with decreasing energy and is virtually energy-independent in the case of $\pi^- d$-scattering. Since the phase $\delta^{c,p}$ occurs in the asymptotic approximation (23) of the function u^{cp}, for this function the range of the polarization potential cannot be less than its range (112) for the phase. As was noted in Ref. 62, the relations

$$B(u^{cp}(r;k);\varepsilon,\{k,R,r_p\})=B(\delta^{c,p}(k);\varepsilon,\{k,R,r_p\}) \quad (113)$$

hold for $\pi^{\pm} d$-scattering if ε, r_p, and E have the same values as in the estimates (112). The ranges of the polarization potential for the normalization factor of the pp-scattering function and the nuclear matrix element are also equal, and for $\varepsilon<10^{-M}$ with $M=1,\ldots,5$, $E<10$ keV, and $r_p=4$ fm they are described by the relations[62]

$$B(N^{c,ps};\varepsilon,\{k,R,r_p\})=B(\Lambda^{cps};\varepsilon,\{k,r_p\})$$
$$\approx r_p+1.3\cdot 10^{-3}R\varepsilon^{-2/5}. \quad (114)$$

As follows from Eqs. (112) and (114), the range of the potential V^p for the integral Λ^{cps} and hence also for the pp-reaction cross section σ_r^{cps} does not depend on the energy and is much smaller than the range of the same potential for the phase $\delta^{c,p}$ and therefore for the elastic pp-scattering cross section $\sigma^{c,ps}$.

Another method for determining the range B of the polarization potential for the desired function A is an analytic method. We shall use this method to estimate the range of the polarization potential for the WKB phase $\bar{\delta}^{c,p}$. We substitute into Eq. (110) the phase function (82) for A. Solving for B the obtained inequality, we have

$$B(\bar{\delta}^{c,p}(k);\varepsilon,\{k,R\})\approx(16\varepsilon/5)^{-1/3}r_c\approx 0.68 r_c\varepsilon^{-1/3}. \quad (115)$$

We shall now find the range of the polarization potential for the normalization factor $N^{c,ps}(k)$ for $k=0$ and $V^c>0$. From Eqs. (53), (54), and (101) we have $N^{c,ps}(r;0)=1/c_0^{cps}(r)$. Using the asymptotic approximation (110) for the function $c_0^{cps}(r)$, we solve the inequality (110) with $A=N^{c,ps}$. The result is

$$B(N^{c,ps}(0);\varepsilon,\{\alpha_e,R\})\approx R(\alpha_e/10R^3\varepsilon)^{2/5}. \quad (116)$$

The formulas (112)–(116) show that the lower limit B of the interval $B\leqslant r\leqslant\infty$, where the polarization potential can be replaced identically by zero, is in the general case a functional of the desired function A, the accuracy ε required for calculating A, the collision energy, and the parameters of the Coulomb and polarization potentials. This limit must always be determined with respect to the desired function A, as the solution of the inequalities (110) or (111), by an analytic or numerical method.

Thus, the concept of a long-range potential is relative. We shall give an additional argument confirming this conclusion. Besides a slow decrease, a characteristic indication of a long-range potential (2) is slow convergence[6,13] of the series of partial amplitudes f_l^a to the total scattering amplitude $f^a(\theta)$ and, in consequence of this convergence, a peak-like behavior of the differential cross section $d\sigma^a(\theta)$ at small scattering angles ($\theta\to 0$). For example, if $a=3$, $k\to 0$, and $l\geqslant 1$, then the scattering phases δ_l^a of the potential (2) are linear in the momentum,[110] and for this reason the series of partial amplitudes which determines the cross section $d\sigma^a(\theta)$ diverges as $O(\ln(\theta/2))^2$ in the limit $\theta\to 0$. As $k\to 0$, the polarization interaction (the case $a=4$) likewise engenders two important features[37] in the behavior of the scattering amplitude $f^{c,ps}$ for the superposition (3). The first one is that the contribution $f^{c,p}$ from this interaction decreases not exponentially,[11] as does the contribution $f^{c,s}$ from the nuclear interaction, but rather according to a power law: $f^{c,p}\sim k^3$. The second one is the nonuniform character of the asymptotic behavior of the amplitude $f^{c,ps}$: As $k\to 0$, the contribution to this amplitude from the sum of the partial amplitudes $f_l^{c,ps}$ with finite l is small, and its leading term is produced by the terms $f_l^{c,ps}\sim f_l^{c,p}$ with large $l\sim\eta\cot(\theta/2)$.

However, a long-range potential can engender a rapidly converging series of partial amplitudes and a nonsingular scattering cross section. An example of such a potential is the interaction of magnetic moments. We shall prove this assertion by means of a phase-shift analysis of nn scattering.

5. EFFECT OF THE INTERACTION OF THE NEUTRON MAGNETIC MOMENTS ON NEUTRON–NEUTRON SCATTERING

After the studies of Mott[111] and Schwinger[112] the effect of the interaction[15]

$$V_{ls}^m=\frac{\hbar}{M}\gamma(\mathbf{l}\cdot\boldsymbol{\mu})r^{-3};\quad \gamma\equiv\frac{\mu Ze^2}{2Mc^2}, \quad (117)$$

of the magnetic moment $\boldsymbol{\mu}$ of the neutron with the electric field of a nucleus with charge Z was studied many times both theoretically and experimentally.[15–18] As a result, it was established that the differential scattering cross section for an unpolarized neutron beam scattered by a spinless nucleus is described by the formulas

$$d\tilde{\sigma}(\theta)=d\sigma^s(\theta)+d\tilde{\sigma}_{ls}^m(\theta),$$
$$d\tilde{\sigma}_{ls}^m(\theta)=(\gamma\cot(\theta/2))^2, \quad (118)$$

obtained by Schwinger[112] in the Born approximation.[7] The differential cross section $d\sigma^s(\theta)$ for nuclear scattering due to the nuclear interaction of a neutron with a nucleus is a smooth function of the scattering angle θ. The differential scattering cross section $d\tilde{\sigma}_{ls}^m(\theta)$ of the potential (117), first named in Ref. 113 Schwinger scattering, grows without bound as $\theta\to 0$. The sum (118) of these cross sections at small scattering angles, where $d\tilde{\sigma}(\theta)\approx d\tilde{\sigma}_{ls}^m(\theta)$, also exhibits peak-like behavior. It is known from scattering theory[6] that this effect is due to the slow decrease of the potential (117).

The interaction of the magnetic moments $\boldsymbol{\mu}=2\mu_i\mathbf{s}_i$ of two ($i=1,2$) nucleons is described by the tensor potential[2,15]

$$V^m=-[3(\boldsymbol{\mu}_1\cdot\mathbf{r})(\boldsymbol{\mu}_2\cdot\mathbf{r})r^{-2}-(\boldsymbol{\mu}_1\cdot\boldsymbol{\mu}_2)]r^{-3}, \quad (119)$$

with the same r dependence as that of the potential (117). Although the potential (119) is long-range, so far no effects due to its influence on pp and np scattering have been observed.[35] It is known[114] that the contribution of $f^{c,m}$ from V^m to the pp scattering amplitude, in addition to the Cou-

lomb contribution, is nonsingular in the limit $\theta\to 0$ and is small compared with the contribution of $f_{ls}^{c,m}$ from V_{ls}^{m}. Therefore the interaction (119) is usually neglected in the analysis of the experimental data on elastic pp and np scattering.[35] Until the recent studies of Refs. 71 and 72, the effect of this interaction on nn scattering apparently had not been investigated, either theoretically or experimentally. Such investigations are not mentioned in the monographs of Refs. 15–18, which are devoted to the fundamental properties of the neutron.

As is well known,[15,115,116] the problem of measuring directly the characteristics of elastic neutron–neutron scattering is still open. The possibility of measuring directly the nn scattering length in a neutron beam, discussed in Ref. 117, stimulated our investigations[71,72] of the effect of the potential (119) on the energy dependence of the nn scattering phases and cross sections. In the present section the qualitative analysis[72] of the role of the interaction of the magnetic moments of neutrons in triplet nn scattering is supplemented by the results of recent calculations performed in three cases ($a=m,s,ms$) corresponding to different assumptions about the total interaction V^a of two neutrons. In the first case ($a=m$) scattering in the potential (119) is investigated. The characteristics of such model scattering are said to be magnetic. In the second case ($a=s$) the characteristics of nuclear scattering are investigated. In the third case ($a=ms$) account of both potentials V^m and V^s is taken in order to study the magnetic-nuclear scattering, and the total interaction is taken in the form of the superposition (4). In each of these three cases all functions are labeled by a corresponding superscript a.

Let us recall the main notation and formulas from the theory of two-nucleon scattering.[2] Let s, l, and j be the total spin ($\mathbf{s}=\mathbf{s}_1\oplus\mathbf{s}_2$), the angular momentum, and the total angular momentum ($\mathbf{j}=\mathbf{l}\oplus\mathbf{s}$) of a system of two neutrons with total energy $E=k^2$, where k is the scattering momentum in the center-of-mass system of the neutrons. According to the theory of the nucleon–nucleon interaction,[5] s and j are conserved quantum numbers; if $s=1$, then l is conserved in the pure states with $j=0$, $l=1$ or $l=j=1,3,5,\ldots$ and it is not conserved in the mixed states, for which $j=2,3,4,\ldots$, and $l=j\pm 1$. The mixing of the states is due to the tensor term in the nuclear potential. The interaction (119) differs from this term only by the dependence on the distance r between two neutrons and vanishes in their spin-singlet ($s=0$) states. For this reason, we shall investigate only triplet ($s=1$) nn scattering.

In the variable-phase method[10] the scattering parameters (the phases $\delta_{l,j}^a(k)$ and the mixing parameter $\varepsilon_j^a(k)$), introduced by Stapp *et al.*[118] in the case $a=s$, are determined as the limits of the corresponding phase functions $\delta_{l,j}^a(r;k)$ and $\varepsilon_j^a(r;k)$ as $r\to\infty$. These functions are equal to zero at $r=0$, and for $r>0$ they satisfy the following equations:[119]

$$\partial_r\delta_{l,j}^a=-k^{-1}\sec 2\varepsilon_j\{V_{l,l}^a(P_l^2\cos^4\varepsilon_j^a-Q_l^2\sin^4\varepsilon_j^a)-V_{l,l'}^a\sin 2\varepsilon_j^a[2^{-2}(P_{l'}^2-Q_{l'}^2)\sin 2\varepsilon_j^a+P_lQ_{l'}\cos^2\varepsilon_j^a-P_{l'}Q_l\sin^2\varepsilon_j^a]\},$$

$$\partial_r\varepsilon_j^a=-k^{-1}\left\{V_{l,l'}^a(P_lP_{l'}\cos^2\varepsilon_j^a+Q_lQ_{l'}\sin^2\varepsilon_j^a)-2^{-1}\sin 2\varepsilon_j^a\sum_{l''=j\pm1}V_{l'',l''}^aP_{l''}Q_{l''}\right\}. \tag{120}$$

In these equations l, $l'=j\pm1$, $l\neq l'$;

$$P_l\equiv j_l\cos\delta_{l,j}^a-n_l\sin\delta_{l,j}^a,\quad Q_l\equiv j_l\sin\delta_{l,j}^a+n_l\cos\delta_{l,j}^a;$$

j_l and n_l are the regular and irregular Riccati–Bessel functions;[73] the symbols $V_{l,l'}^a$ denote the matrix elements of the nn interaction V^a in the basis of vectorial spherical functions. For the pure states $\varepsilon_j^a\equiv 0$ and the system (120) decomposes into two independent equations. In this basis the matrix elements of the tensor potential (119) have the form

$$V_{l,l}^m=2\tau r^{-3}\{\hat\delta_{l,j}-[l\hat\delta_{l,j-1}+(l+1)\hat\delta_{l,j+1}]/(2j+1)\},$$
$$V_{l,l'}^m=\tau r^{-3}\sqrt{6j(j+1)}/(2j+1), \tag{121}$$

they are bounded for all l, and they contain the constant $\tau\equiv-\mu_n^2m_n=-0.0014$ fm, where μ_n and m_n are the magnetic moment and mass of the neutron.

The triplet scattering of two nucleons is described[119] by the matrix $\mathbf{M}$ defined in the spin space by the equalities

$$f_n^a(\Omega;k)=\sum_{n'=-1}^{1}M_{n,n'}^a(\Omega;k)a_{n'}(k).$$

They relate the three ($n=-1,0,1$) scattering amplitudes $f_n^a(\Omega;k)$ in states with projection $s_z=n$ of the total spin $s=1$ with amplitudes $a_{n'}$ of the states with projections $s_z=n'$ in the incident wave. The matrix elements $M_{n,n'}^a$ depend on the scattering angles $\Omega=(\theta,\varphi)$ and are given in Table III of Ref. 118 in the form of infinite series. For example,

$$M_{0,0}^a(\Omega;k)=(ik)^{-1}\sum_{l=1,3,\ldots}^{\infty}P_l^0(x)\{(l+1)\alpha_{l,l+1}^a+l\alpha_{l,l-1}^a+[(l+1)(l+2)]^{1/2}\alpha_{l+1}^a+[l(l-1)]^{1/2}\alpha_{l-1}^a\}. \tag{122}$$

Here $P_l^n(x)$ are the associated Legendre polynomials[73] in the variable $x=\cos\theta$, and the matrix α^a is defined as follows: $\alpha_{l,j}^a=0$ if $j<0$, and $\alpha_j^a=0$ if $j\leqslant 2$; in all other cases

$$\alpha_{l,j}^a=\cos 2\varepsilon_j^a[\exp(2i\delta_{l,j}^a)-1],$$
$$\alpha_j^a=i\sin 2\varepsilon_j^a\exp[i(\delta_{j+1,j}^a+\delta_{j-1,j}^a)]. \tag{123}$$

According to Table II of the same work,[118] the differential cross section for triplet nn scattering is determined by the formula

$$d\sigma^a(\theta;E)=\pi\sum_{n=0}^{1}\sum_{n'=-\hat\delta_{n,1}}^{1}(1-\hat\delta_{n,0}\hat\delta_{n',0}/2)|M_{n,n'}^a|^2. \tag{124}$$

We shall now proceed to the investigation of the characteristics of triplet nn scattering.

5.1. Characteristics of triplet *nn* scattering

We shall present the main results of a qualitative analysis[72] of the behavior of the nn scattering parameters. The first iteration of Eq. (120) in the case $a=m$ gives the magnetic phases $\delta^m_{l,j}$ and the mixing parameters ε^m_j in the Born approximation,

$$\tilde{\delta}^m_{l,j}(k)=\frac{(-1)^{l-j+1}\tau k}{\max\{l,j\}(2j+1-j\hat{\delta}_{l,j})},$$

$$\tilde{\varepsilon}^m_j(k)=\frac{-\tau k}{\sqrt{j(j+1)}(2j+1)}, \tag{125}$$

whose accuracy is such that

$$\delta^m_{l,j}(k)=\tilde{\delta}^m_{l,j}(k)+o(\tau k),\quad \varepsilon^m_j(k)=\tilde{\varepsilon}^m_j(k)+o(\tau k),\quad k\to 0. \tag{126}$$

It follows from Eqs. (125) and (126) that the leading terms of the low-energy asymptotic expansions of the magnetic-scattering parameters are linear in the momentum k for arbitrary l and j. The parameters of nuclear scattering (the case $a=s$) decrease more rapidly:[7,10]

$$\delta^s_{l,j}=A^s_{l,j}k^{2l+1}(1+O(k^2)),\quad \varepsilon^s_j=A^s_j k^{2j+1}(1+O(k^2)),$$

$$k\to 0. \tag{127}$$

The coefficients $A^s_{l,j}$ and A^s_j can be found either by extrapolating the corresponding experimental data of Refs. 5 and 35, according to the formulas (127), for the phases and the mixing parameters of NN scattering at the point $E=0$ or by calculation using any nuclear interaction describing these data. We shall clarify this important assertion, which implies that all of the subsequent conclusions are independent of the model used for the nuclear interaction. All known parametrizations of the nuclear interaction (5) have the same Yukawa tail (1), where μ is the mass of the π meson. This tail determines the behavior of the parameters of nuclear scattering at low energies. It is virtually independent of the form of V^s at finite distances, and therefore in what follows only one such potential, specifically, the soft-core Reid potential,[5] is used as the nuclear potential V^s. Integrating Eq. (120) with this potential gives

$$A^s_{1,0}=2.74\ \text{fm}^3,\quad A^s_{1,1}=-1.53\ \text{fm}^3;$$

$$A^s_{1,2}=0.25\ \text{fm}^3,\quad A^s_{3,2}=1.36\ \text{Fm}^7,\quad A^s_2=-2.00\ \text{fm}^5. \tag{128}$$

The parameters of magnetic-nuclear scattering (the case $a=ms$) are approximated[72] by the following sums:

$$\delta^{ms}_{l,j}(k)\approx\delta^s_{l,j}(k)+\delta^m_{l,j}(k),\quad \varepsilon^{ms}_j(k)\approx\varepsilon^s_j(k)+\tilde{\varepsilon}^m_j(k),$$

$$k\to 0. \tag{129}$$

On the strength of Eqs. (125)–(129), three energy ranges can be conditionally distinguished. In the first region, i.e. at quite low energies ($E\leqslant E^-_{l,j}$), the magnetic terms (125) in the sum (129) are much larger than the nuclear terms (127) and therefore the magnetic-nuclear scattering parameters (129) are close to the magnetic-scattering parameters (126) and are therefore linear in k for arbitrary l,j. In the third region, i.e. at quite high energies ($E\geqslant E^+_{l,j}$), the inverse relations hold and the magnetic-nuclear parameters are close to the nuclear parameters. In the second, intermediate, region ($E^-_{l,j}<E<E^+_{l,j}$) the magnetic and nuclear terms in the sums (129) are comparable in order of magnitude and the scattering is due to the interplay of the potentials V^m and V^s. As a result of this interplay, the magnetic-nuclear parameters (129) can have zeros if the nuclear and magnetic parameters have opposite signs. This condition is satisfied for the ${}^3P_0$, ${}^3P_1$, and ${}^3P_2-{}^3F_2$ states of the nn system. The approximate formulas

$$E^{ms}_{l,j}\approx\left(\frac{(-1)^{l-j}\tau}{A^s_{l,j}\max\{l,j\}(2j+1-j\hat{\delta}_{l,j})}\right)^{1/l},$$

$$E^{ms}_j\approx\left(\frac{\tau}{A^s_j\sqrt{j(j+1)}(2j+1)}\right)^{1/j}, \tag{130}$$

which relate the zeros of the phases $\delta^{ms}_{l,j}$ and of the mixing parameters ε^{ms}_j with the constant τ of the potential (119) and the coefficients of the nuclear asymptotic expansions (127), follow from Eqs. (125), (127), and (129). Substituting the values (128) of these coefficients into Eq. (130), we obtain the following estimates:

$$E^{ms}_{1,0}\approx 21\ \text{keV},\quad E^{ms}_{1,1}\approx 19\ \text{keV},\quad E^{ms}_{1,2}\approx 23\ \text{keV},$$

$$E^{ms}_{3,2}\approx 1.7\ \text{MeV},\quad E^{ms}_2\approx 313\ \text{keV}. \tag{131}$$

According to these estimates, all three ($j=0,1,2$) phases $\delta^{ms}_{1,j}$ have a zero at about the same energy, equal to approximately 20 keV. Therefore the total magnetic-nuclear scattering cross section σ^{ms} has a local minimum at the same energy $E\approx 20$ keV if the nn states with $j\leqslant 2$ make the main contribution to this cross section. Although the potential V^{ms} is long-range, this condition is realized at quite low energies. We shall describe the proof of this assertion.

Each series $M^a_{n,n'}$ is represented as a sum of two terms

$$M^a_{n,n'}=M^a_{n,n'}(j\leqslant j_{\max})+M^a_{n,n'}(j>j_{\max}). \tag{132}$$

The first term is a finite sum of this series, in all terms of which the index j does not exceed $j_{\max}$, and it describes the contribution from nn scattering states with $j\leqslant j_{\max}$ to the series being studied. To construct the low-energy approximations of the series (132), their subsums, and the cross sections (124), the elements (123) of the matrix α^a are approximated by the first terms of their expansions in a Taylor series: $\alpha^a_{l,j}\approx 2i\sigma^a_{l,j}$ and $\alpha^a_j\approx 2i\varepsilon^a_j$, and the scattering parameters are replaced by the corresponding leading terms in their asymptotic expansions (125), (127), and (129).

In the Born approximation all series $\tilde{M}^m_{n,n'}$ with the exception of $\tilde{M}^m_{1,-1}$, are equal to finite sums of their terms with the index $l=1$ and corresponding to nn states with $j\leqslant 2$. All other terms in the series vanish, since the Born magnetic scattering phases in the pure and mixed states have opposite signs. For example, the series (122) and the series $\tilde{M}^m_{1,0}$ are equal to the contributions from the ${}^3P_0$ state, and the series $\tilde{M}^m_{0,1}$ and $\tilde{M}^m_{1,1}$ are equal to the contributions from the states with $j=1,2$. The infinite series $\tilde{M}^m_{1,-1}$ contains contributions from all states with $j\geqslant 2$, but it converges rapidly and is

tabulated in Ref. 73. This structure of the magnetic Born series makes it possible to sum them explicitly:

$$\widetilde{M}^m_{0,1}=(\widetilde{M}^m_{1,0})^*=-\sqrt{2}\tau\exp(i\varphi)P^1_1(x),\quad \widetilde{M}^m_{1,1}=-\tau P_1(x),$$

$$\widetilde{M}^m_{0,0}=2\tau P_1(x),\quad \widetilde{M}^m_{1,-1}=\tau\exp(-2i\varphi)P_1(x). \tag{133}$$

Substituting them for the corresponding series $M^m_{n,n'}$ in Eq. (124), we obtain the Born approximations $d\widetilde{\sigma}^m=4\pi\tau^2$ and $\widetilde{\sigma}^m=8\pi\tau^2$ of the differential and total magnetic cross sections. On account of Eq. (126), the accuracy of these approximations improves as $o(\tau k)$ if $k\to 0$. In this limit the magnetic differential cross section $d\sigma^m$ becomes isotropic, in contrast to the Schwinger differential cross section $d\widetilde{\sigma}^m_{ls}$ (see Eq. (118)). The large difference in the angular dependence of these cross sections is due to the operator structure of the potentials (117) and (119). As the neutron-nuclear angular momentum l increases, the matrix elements of the ls potential V^m_{ls} in the spin–angular neutron–nucleus basis grow as $O(l)$. Therefore all amplitudes with large numbers $l\sim 1/\theta$ make the main contribution to the series $d\widetilde{\sigma}^m_{ls}(\theta)$ of partial Schwinger scattering amplitudes, as a result of which this series diverges as $\theta\to 0$. The matrix elements (121) of the potential V^m are bounded in any nn states. In addition, this interaction is attractive (repulsive) in the pure (mixed) nn scattering states. At low energies this attraction is compensated by a repulsion in each state with $j\geqslant 3$. As a result, a finite number of nn states with $j\leqslant 2$ makes the main contribution to the series $M^m_{n,n'}$, and therefore the cross section $d\widetilde{\sigma}^m(\theta)$ is nonsingular.

Since the nuclear-scattering parameters decrease rapidly according to (127), the series $M^s_{n,n'}$ in an approximation linear in the energy

$$M^s_{1,0}\approx\sqrt{2}\exp(-i\varphi)E(A^s_{1,2}-A^s_{1,0})P^1_1(x),\quad M^s_{1,-1}=0,$$

$$M^s_{0,1}\approx(3/\sqrt{2})\exp(i\varphi)E(A^s_{1,1}-A^s_{1,2})P^1_1(x),$$

$$M^s_{1,1}\approx 3E(A^s_{1,1}+A^s_{1,2})P_1(x),$$

$$M^s_{0,0}\approx 2E(A^s_{1,0}+2A^s_{1,2})P_1(x), \tag{134}$$

are equal to the contributions from the states with $j\leqslant 2$, and the contributions from all other states to the sum (132) decrease as $O(k^4)$. Substituting the expressions (134) into Eq. (124), we obtain the total cross section of nuclear scattering:

$$\sigma^s(E)\approx 4\pi E^2\sum_{j=0}^{2}(2j+1)(A^s_{1,j})^2, \tag{135}$$

in the standard approximation,[35] in which the contribution from states with $j\geqslant 3$ is neglected. This contribution decreases as $O(k^6)$ in the limit $k\to 0$.

The approximations (126), (127), and (129) engender the approximate (with accuracy $O(\tau k)$) representations in the form of the sums

$$M^{ms}_{n,n'}\approx\widetilde{M}^m_{n,n'}+M^s_{n,n'} \tag{136}$$

with the terms (133) and (134). Substituting the expression (136) into Eq. (124), we obtain the corresponding approximate formula

$$\sigma^{ms}(E)\approx\widetilde{\sigma}^m(E)+(4\pi/3)\tau E(2A^s_{1,0}-3A^s_{1,1}+A^s_{1,2})+\sigma^s(E) \tag{137}$$

for the total magnetic-nuclear cross section. The accuracy of this approximation is $o(\tau k)$ in the limit $k\to 0$. The contribution of $\widetilde{\sigma}^m$ from V^m to the sum (137) is constant and equal to $8\pi\tau^2$. For the values (128) of the coefficients $A^s_{1,j}$ its second term is negative and linear in the energy. It describes the relative contribution from V^m and V^s. The last term, i.e. the contribution (135) from V^s, is positive and quadratic in the energy. This structure of the relation (137) makes it possible to separate conditionally three energy ranges. In the first range of quite low energies ($E\leqslant E^-$) the cross section σ^{ms} is approximately constant and equal to the cross section $\widetilde{\sigma}^m$. At these energies the nuclear potential can be neglected in the sum (4). In the third range the energy is so high ($E\geqslant E^+$) that $\sigma^{ms}\approx\sigma^s$ and the potential V^m can be neglected. In this range the cross section σ^{ms} increases with the energy. It is obvious that in the second, intermediate, energy range ($E^-<E<E^+$) the magnetic-nuclear cross section (137) should possess a local minimum. This phenomenon is the neutron–neutron analog of the Ramsauer effect, because, like the well-known Ramsauer effect itself,[84] it is engendered by the combined action of the short- and long-range potentials with opposite signs. The zero of the first energy derivative of the function (137) determines the position of the minimum of the total magnetic–nuclear nn scattering cross section,

$$E^{ms}_{\min}\approx(-\tau/2)(2A^s_{1,0}-3A^s_{1,1}+A^s_{1,2})\Big/\sum_{j=0}^{2}(2j+1)(A^s_{1,j})^2, \tag{138}$$

in terms of the constant τ in the ptoential V^m and the coefficients $A^s_{1,j}$ of the asymptotic expansions (127) of the three nuclear scattering phases. Substituting the values (128) of these coefficients in Eq. (138), we obtain $E^{ms}_{\min}\approx 20$ keV. The formula (138) is the analog of the relations first obtained in Ref. 120 between the position of the Ramsauer minimum with polarization constant α_e and the scattering length a^s of the short-range exchange potential.

We shall now list the basic results of the numerical analysis of nn scattering. The scattering parameters in states with $j\leqslant 2$, calculated by integrating Eq. (120), are displayed in Figs. 17–21. These data show graphically the effect of the potential (119) on nn scattering and make it possible to estimate the limits $E^-_{l,j}$ and $E^+_{l,j}$ If they are determined as the maximum and minimum energies at which the corresponding inequalities $|\delta^s_{l,j}/\delta^{ms}_{l,j}|\leqslant 0.1$ and $|\delta^s_{l,j}/\delta^{ms}_{l,j}|\geqslant 0.9$ are satisfied, then the following estimates obtain:

$$E^-_{1,j}\approx 2\ \text{keV},\quad E^+_{1,j}\approx 0.3\ \text{MeV},\quad j=0,1,2;$$

$$E^-_{3,2}\approx 0.7\ \text{MeV},\quad E^+_{3,2}\approx 7\ \text{MeV}.$$

As it turned out, the relative accuracy of the Born approximation is quite high: The Born expressions (125) approximate the magnetic parameters of nn scattering with relative accuracy $\varepsilon\approx 10^{-4}$ for all $j\leqslant 10$ and $E\leqslant 10$ MeV. The relative accuracy of the approximation of the nuclear scattering parameters by the leading terms in their asymptotic expan-

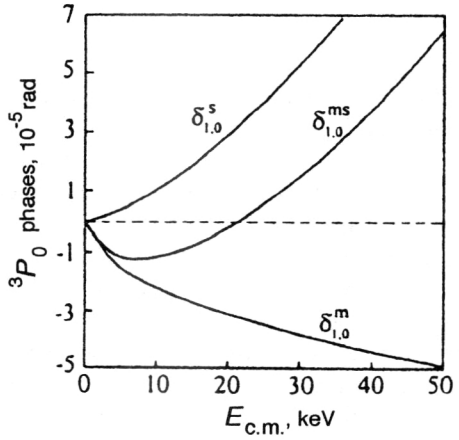

FIG. 17. Magnetic $\delta^m_{1,0}$, nuclear $\delta^s_{1,0}$, and magnetic-nuclear $\delta^{ms}_{1,0}$ nn-scattering phases in the $^3P_0$ state.

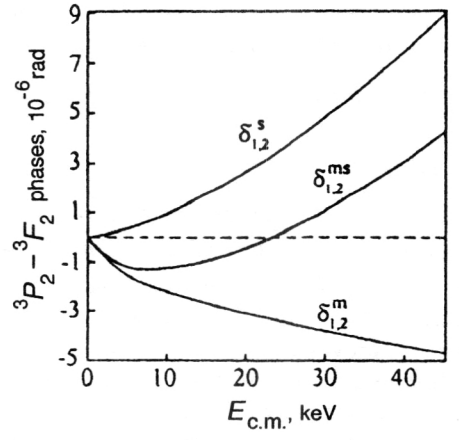

FIG. 19. Magnetic $\delta^m_{1,2}$, nuclear $\delta^s_{1,2}$, and magnetic-nuclear $\delta^{ms}_{1,2}$ nn-scattering phases in the $^3P_2-{}^3F_2$ state.

sions (127) is not worse than 10^{-2} if $l=0,1$, $j=0,1,2$, and $E<7$ MeV. The approximations (129), (130), and (131) hold with the same accuracy and under the same conditions. Since the accuracy of all key approximations mentioned above is quite high, the approximations

$$M^m_{n,n'}\approx\widetilde{M}^m_{n,n'},\quad M^s_{n,n'}\approx M^s_{n,n'}(j\leqslant 2),$$
$$M^{ms}_{n,n'}\approx M^{ms}_{n,n'}(j\leqslant 2)+\widetilde{M}^m_{n,n'}(j>2) \qquad (139)$$

should hold with relative accuracy $\varepsilon\approx 10^{-2}$ at energies $E<7$ MeV. This result was checked numerically and made it possible to use the approximations (139) to calculate the cross sections (124).

Figure 22 illustrates the variation of the angular dependence of the differential cross section $d\sigma^{ms}(\theta)$ with increasing energy. At $E=5$ keV the scattering is due mainly to the potential V^m, and the cross section $d\sigma^{ms}$ is close to the magnetic cross section $d\sigma^m$ and is therefore isotropic. In the intermediate range $d\sigma^{ms}$ decreases uniformly with increasing energy up to the value $E\approx 20$ keV. Then, $d\sigma^{ms}$ increases and becomes nonisotropic as a result of the increasing influence of the nuclear interaction.

The effect of the potential V^m on the total nn-scattering cross section and the neutron–neutron analog of the Ramsauer effect are demonstrated in Fig. 23. This figure shows simultaneously the cross section σ^m, σ^s, and σ^{ms}. As one can see, σ^s decreases rapidly with decreasing energy, σ^m is virtually independent of energy, and $\sigma^{ms}=\sigma^m$ at $E=0$, has a local minimum at $E\approx 20$ keV, and converges to σ^s as the energy increases further.

For completeness, the contributions of $d\sigma^{ms}_{j_{\max}}$ to the cross section $d\sigma^{ms}$ from states with $j\leqslant j_{\max}$ were investigated. By definition $d\sigma^{ms}_{j_{\max}}$ is the sum (124), whose terms are replaced by the contributions $M^{ms}_{n,n'}$ $(j\leqslant j_{\max})$, calculated according to the exact formulas. In this definition the case $j_{\max}=\infty$ corresponds to the real situation, where all scattering states are taken into account, so that $d\sigma^{ms}_{\infty}\equiv d\sigma^{ms}$. The contributions of $\sigma^{ms}_{j_{\max}}$ to the total cross section σ^{ms}_{∞} and the cross section itself are displayed in Fig. 24. According to this figure, the $^3P_0$ and $^3P_1$ states make the main contribution to σ^{ms}_1. Both contributions σ^{ms}_0 and σ^{ms}_1 vanish approximately at the same energy. On account of the $^3P_2-{}^3F_2$ states, the minimum value of the contribution σ^{ms}_2 is different from zero, and the position of the minimum is displaced somewhat

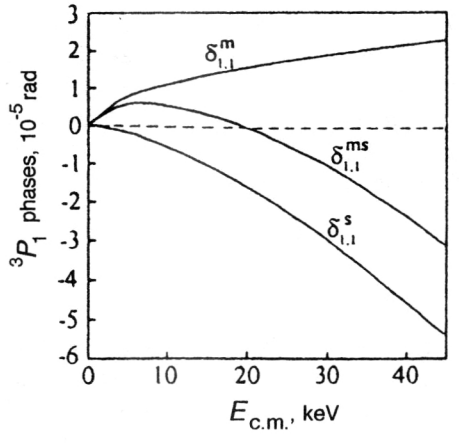

FIG. 18. Magnetic $\delta^m_{1,1}$, nuclear $\delta^s_{1,1}$, and magnetic-nuclear $\delta^{ms}_{1,1}$ nn-scattering phases in the $^3P_1$ state.

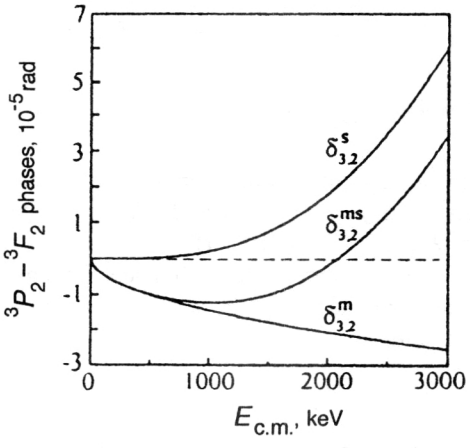

FIG. 20. Magnetic $\delta^m_{3,2}$, nuclear $\delta^s_{3,2}$, and magnetic-nuclear $\delta^{ms}_{3,2}$ nn-scattering phases in the $^3P_2-{}^3F_2$ state.

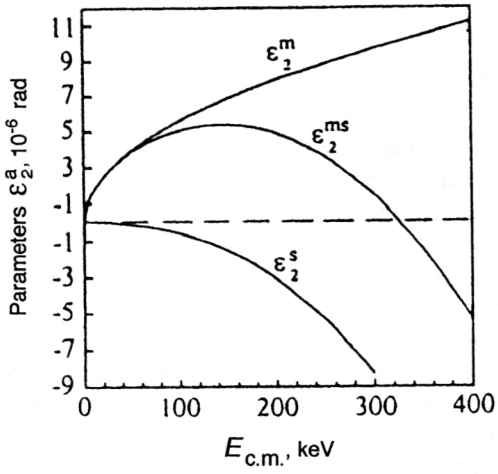

FIG. 21. Magnetic ε_2^m, nuclear ε_2^s, and magnetic-nuclear ε_2^{ms} mixing parameters of nn-scattering phases in the ${}^3P_2-{}^3F_2$ state.

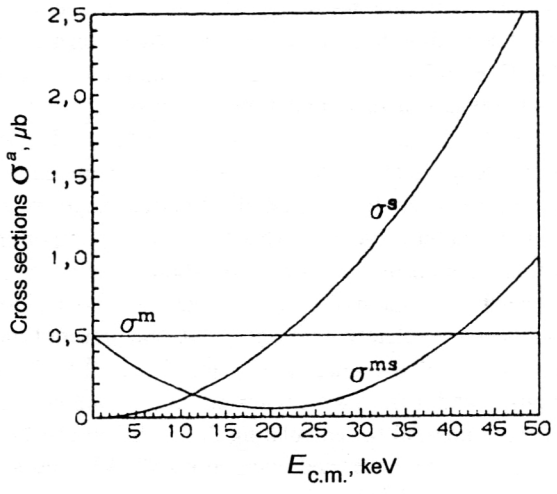

FIG. 23. Magnetic σ^m, nuclear σ^s, and magnetic–nuclear σ^{ms} total cross sections for triplet nn scattering.

to the right. As a result of all other states with $j \geqslant 3$, the cross section σ^{ms} is uniform in energy but is somewhat higher than the contribution of σ_2^{ms}.

Thus, the numerical calculations confirm all the results of the qualitative analysis[72] of the role of the potential (119) in triplet nn scattering. This makes it possible to draw the following conclusions.

The interaction of the neutron magnetic moments must be taken into account in order to describe correctly triplet nn scattering at low energies ($E \leqslant 300$ keV) for the following reasons. On account of this interaction, in the low-energy limit ($E \leqslant 2$ keV) the scattering phases must be proportional to the scattering momentum, the differential cross section should be isotropic, and the total cross section should not vanish. As a result of the interference of the nuclear and magnetic scatterings at the energy $E \approx 20$ keV, the scattering phases in the states with $l=1$ and $j=0,1,2$ should vanish and the total cross section should possess a local minimum. The formulas (129) and (137) can be used to extrapolate the parameters and the total cross section for triplet nn scattering, taking account of the interaction of the magnetic moments.

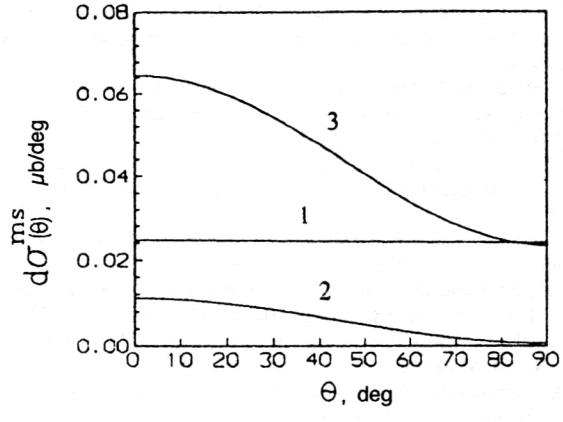

FIG. 22. Magnetic-nuclear differential cross section $d\sigma^{ms}(\theta)$ for triplet nn scattering at 5, 20, and 40 keV (curves 1, 2, and 3, respectively).

For $E \leqslant 300$ keV, these formulas describe quite accurately not only the phases and the total nn-scattering cross section but also their experimentally interesting structural features: the positions of the zeros and the local minimum.

6. CONCLUSIONS

This review was concerned mainly with the theoretical analysis of the effects produced by the polarization interaction and the interaction of magnetic moments. Each section was concluded by a summary of this analysis, so that only some of them will be listed here. This is necessary in order to formulate the most important problems, which were never solved in the cited studies of the theory of scattering and the methods for analyzing the experimental data.

Thus, the polarization interaction radically changes the threshold ($E \to 0$) behavior of the S phases $\delta^{c,ps}$ and cross sections $\sigma^{c,ps}$ of the elastic nuclear and pion–nucleus collisions, but its contribution to the cross section for nucleosynthesis reactions is very small. On account of this interaction,

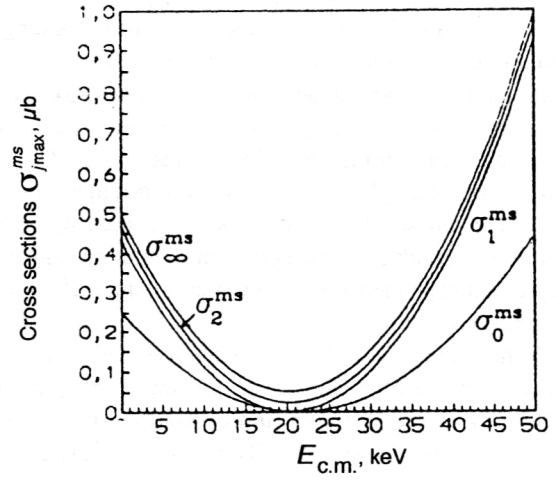

FIG. 24. Contributions $\sigma^{ms}_{j_{max}}$, $j_{max}=0,1,2$, to the magnetic–nuclear total cross section σ^{ms}_∞ for triplet nn scattering.

the phases $\delta^{c,ps}$ and the cross sections $\sigma^{c,ps}$ decrease at threshold according to a power law and not exponentially, as do the Coulomb–nuclear phases $\delta^{c,s}$ and cross sections $\sigma^{c,s}$. As a result of this comparatively slow decrease of the phases $\delta^{c,ps}$, the quantity $a^{c,ps}$ is unbounded and is not a scattering length. The modification $a^{cp,s}$, contained in the expansion (48) of the modified effective-range function $K^{cp,s}$, is a scattering length. This is especially important in order to extract correctly information about the nuclear interaction itself from the experimental data. The expansion has still not been completely investigated. At the present time the explicit forms of the analogs C_p and h_p of the Coulomb functions C and h from the expansion (20) are unknown, and an effective method for calculating the modified coefficients $a^{cp,s}$ and $r^{cp,s}$ for prescribed interactions V^c, V^p, and V^s has not been proposed. Most importantly, there is no reliable practical method for determining these quantities from the data on the elastic scattering cross section σ^{cps} of the superposition (3) of such potentials at quite high energies ($E>10$ MeV) and therefore energies accessible for modern experimental investigations of nuclear collisions. To solve this important problem, it is necessary to subtract from the measured cross section σ^{cps} the contribution σ^{cp} from the superposition of the Coulomb and polarization interactions. However, even this cannot be done sufficiently accurately in a wide energy range for the simple reason that, in contrast to the Coulomb cross section σ^c, the explicit forms of the cross sections σ^{cp} and $\sigma^{c,p}$ are unknown.

As follows from this remark and the entire review, at the present time the only possible method for taking account of the polarization interaction correctly is to use the low-energy expansion of the desired function $A^{cp,s}$, corresponding to the decomposition (10) of the phase δ^{cps}. In this method the approximations (12)–(14), (16) and (55), (61), and (99) are the main ingredients. They make it possible to extrapolate correctly the phases $\delta^{c,ps}$, the cross sections $\sigma^{c,ps}$, the normalization factors $N^{c,ps}$, and the nuclear matrix elements, such as the integral (96), into the energy range $E\leqslant E^+$, where direct experimental measurements are impossible to perform. For such an extrapolation, it is necessary to know, first, how to separate the three energy ranges $E\leqslant E^-$, $E^-<E<E^+$, and $E\geqslant E^+$ in which the corresponding approximations $\delta^{c,ps}\approx\delta^{c,p}$, $\delta^{c,ps}\approx\sigma^{c,p}+\delta^{c,s}$, and $\delta^{c,ps}\approx\delta^{c,s}$ are valid, and, second, the accuracy of these approximations. This is why in the present review we discussed in detail how to estimate the limits $E^{\pm}$ by investigating the ratio $\delta^{c,s}/\delta^{c,ps}$ of the phases, the scattering-length function $A^{c,ps}$, or the self-similar equation (93). For the same reason, the sufficient conditions for each approximation and its accuracy and independence from the nuclear-interaction model were analyzed.

We note that the main approximation (16) can be substantially improved by using instead of the Born approximation (29) the more accurate approximations (59), (60) or (65), (66) for the phases $\delta^{c,p}$. The formulas (31) and (66) make it possible to extend all the results discussed above on the threshold behavior of the phases and cross sections to the case of scattering by the more general, than (3), superposition $V^c+V^a+V^s$, containing the long-range term (2) with arbitrary $a>2$.

One of the most interesting results discussed in this review is the indication of the analog of the Ramsauer effect in pd scattering. It should be underscored once again that this indication pertains only to the S-wave ($l=0$) cross section $\sigma^{c,ps}$. The question of the minimum in the total cross section, which accounts for the pd scattering in all partial waves ($l\leqslant 0$), remained open for both theoretical and experimental investigations. Experimental measurements of this cross section in the region $E\leqslant E^+\approx 10$ keV, where its S-wave quartet and doublet parts possess local minima at close energies (see Fig. 7), would make it possible to answer this question. At the present time such measurements are impossible, and therefore it is especially interesting to investigate theoretically pd scattering at energies $E\leqslant 10$ keV.

Although the question of the pd analog of the Ramsauer effect remains unsolved, the theoretical indication of the existence of such an analog in triplet nn scattering is quite convincing. Indeed, the theoretical analysis was performed in Sec. 5 for the total cross section σ^{ms} in a manner independent of the model of the nuclear interaction and was confirmed by the results of careful numerical investigations. All this suggests that the conclusions presented in Sec. 5 that the interaction of magnetic moments influences triplet nn scattering at energies below 300 keV could be confirmed in future experiments. A theoretically possible scheme for such an experiment is to measure the total cross section for triplet nn scattering in a spin-polarized neutron beam. Such measurements would be of fundamental value and would make it possible to refine current ideas of the nucleon–nucleon interaction at low energies. However, the preparation of a polarized neutron beam and the measurement of such a small magnetic-nuclear cross section (see Fig. 23) are still unsolved problems in experimental neutron physics.

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