Hamiltonian approach in the theory of condensed media with spontaneously broken symmetry

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The dynamics of condensed media with spontaneously broken symmetry is considered on the basis of the Hamiltonian formalism. A method is formulated for obtaining the Poisson brackets of dynamical variables based on specification of the kinematic part of the Lagrangian and interpretation of the integrated terms in the variation of the action as the generators of canonical transformations. Differential conservation laws associated with the symmetries of the Hamiltonian of the system are studied. Examples of canonical transformations that play an important physical role are considered, and their generators are obtained. The connection between the Hamiltonian and Lagrangian approaches is established. Specific physical systems studied in the review are liquid and quantum crystals, many-sublattice magnets, superfluids (^{4}He, ^{3}He-B), and nematic elastomers.
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1. INTRODUCTION

The Hamiltonian formalism is an effective method for constructing nonlinear dynamical equations for many physical systems. The Hamiltonian approach played a most important role in the creation of quantum mechanics[1,2] and quantum field theory.[3–5] It has also been widely used in the theory of exactly integrable systems[6–8] and in the construction of perturbation theory for weakly perturbed nonlinear equations when an exactly integrable problem is taken as the zeroth approximation.[9] In this last case, the method of averaging proposed in Refs. 10 and 11 can be used effectively, and also the idea of a hierarchy of space–time scales, which are the basis of modern kinetic theory.[12] The Hamiltonian approach makes it possible to investigate the dynamics of both classical condensed media and macroscopic quantum objects. For classical condensed media, the Hamiltonian formalism has been developed for normal liquids,[13,14] solids,[15–17] and liquid crystals.[18–20] For quantum condensed media, the approach has been applied to superfluids,[21–24] magnets,[25–27] and quantum crystals.[28]

In the Hamiltonian approach, a decisive role is played by the definition of the Poisson brackets (PB) of the dynamical variables. In the case of condensed media, in contrast to ordinary mechanical systems, the PB of the dynamical variables have a nontrivial structure. For normal physical systems, a reduced description in the hydrodynamic stage of evolution can be constructed by using the densities of the additive integrals of the motion, the PB of which are well known. To describe systems with spontaneously broken symmetry, one introduces additional hydrodynamic parameters, which are not associated with conservation laws but are due to the existence of the broken symmetry. It is the construction of the Poisson brackets for these variables (both with the densities of the additive integrals of the motion and with one another) that is the main difficulty.

In Ref. 16, the Poisson brackets were derived by considering the variations of the variables under the group transformations corresponding to the broken symmetry of the system. In a number of cases, the PB were obtained by replacing the commutators, calculated in quantum mechanics, by Poisson brackets. In the approach proposed here, the decisive role in obtaining the PB is played by the structure of the kinematic part of the action (or the Lagrangian); the kinematic part is defined as the part that contains the time derivatives of the dynamical variables linearly and homogeneously. Namely, we shall show how it is possible to obtain the PB for the variables of several physical systems by proceeding from transformations that leave the kinematic part of the action invariant. The integrated terms in the variation of the action are interpreted here as the generators of these transformations. An important feature of the approach is the introduction of additional hydrodynamic variables in terms of the quantities conjugate to the densities of the additive integrals of the motion. In the clarification of the structure of the kinematic part of the Lagrangian, an important role is played by the construction of the translation operators and their densities for the various physical fields that appear in the action.

On the basis of this approach, we consider classical continuous media (elasticity theory, hydrodynamics, some phases of liquid crystals, nematic elastomers) and also magnetically ordered and superfluid systems.[29]

In the case of the classical continuous media, having established for them the PB algebra of the dynamical variables, we separate out in each case from this general algebra closed subalgebras of variables of the system in order to describe the considered physical state. A closed dynamics is then obtained under the assumption that the variables that do not belong to the subalgebra are cyclic (the Hamiltonian H does not depend on such variables). This enables us to trace in a unified manner the way in which it is possible to obtain the well-known equations of elasticity theory, hydrodynamics, and the dynamics of liquid-crystal phases from the general dynamical equations of continuous media.

In the study of magnetically ordered systems, we find the general PB algebra that corresponds to a magnet with complete symmetry breaking with respect to spin rotations, and as subalgebras we separate the PB of the variables that describe a uniaxial helical magnet and an antiferromagnet. We also show how it is possible, proceeding from the expression for the kinematic part of the action in terms of the generalized coordinates and momenta, to obtain the PB for the variables of a quadrupole magnet. We also consider the dynamics of magnetoelastic media for which the dynamical variables include both variables corresponding to continuous media and variables corresponding to magnetism.

In the study of systems with broken phase invariance, we consider quantum crystals and the ^{4}He, ^{3}He-B superfluid phases, and we also take into account the spin degrees of freedom in the dynamical equations of quantum crystals (quantum spin crystals).

We emphasize that the inclusion of the additional thermodynamic parameters not associated with conservation laws (order parameters) in the complete system of nonlinear hydrodynamic equations is not a trivial problem. To consider these problems, we use the Hamiltonian approach, which makes it possible, without loss of physical transparency, to obtain consistently nondissipative hydrodynamic equations that take into account automatically the required symmetry properties of the Hamiltonian and also to find the flux densities of the additive integrals of the motion in terms of the densities of the additive integrals of the motion. We now turn to the fundamentals of the formalism.

1. FUNDAMENTALS OF THE FORMALISM

We represent the Lagrangian of the system in the form

$$L=L_k(\varphi,\dot{\varphi})-H(\varphi)\equiv\int d^3xF_\alpha(x;\varphi)\dot{\varphi}_\alpha(x)-H(\varphi), \tag{1.1}$$

where $L_k(\varphi,\dot{\varphi})$ is the kinematic part of the Lagrangian, $H(\varphi)$ is the Hamiltonian, and $F_\alpha(x;\varphi(x'))$ is some functional of the dynamical variables $\varphi_\alpha(x)$. We consider infinitesimal transformations of the field variables $\varphi_\alpha(x)$:

$$\varphi_\alpha(x,t)\rightarrow\varphi'_\alpha(x,t)=\varphi_\alpha(x,t)+\delta\varphi_\alpha(x,t) \tag{1.2}$$

(in what follows, we shall not explicitly show the argument t in the dynamical variables and their variations). The variation of the action $W=\int_{t_1}^{t_2}L\,dt$ due to the transformations (1.2) has the form

$$\delta W=G(t_2,\varphi)-G(t_1,\varphi)+\int_{t_1}^{t_2}dt\int d^3x'\,\delta\varphi_\beta(x')\times\left(\int d^3xJ_{\beta\alpha}(x',x;\varphi)\dot{\varphi}_\alpha(x)-\frac{\delta H}{\delta\varphi_\beta(x')}\right), \tag{1.3}$$

where

$$G(t,\varphi)=\int d^3xF_\alpha(x,\varphi)\delta\varphi_\alpha(x);$$

$$J_{\alpha\beta}(x,x';\varphi)=\frac{\delta F_\beta(x';\varphi)}{\delta\varphi_\alpha(x)}-\frac{\delta F_\alpha(x;\varphi)}{\delta\varphi_\beta(x')}.$$

It follows from the principle of stationary action that the equations for the field components $\varphi_\alpha(x)$ have the form

$$\dot{\varphi}_\alpha(x)=\int d^3x'J^{-1}_{\alpha\beta}(x,x';\varphi)\,\frac{\delta H}{\delta\varphi_\beta(x')}. \tag{1.4}$$

Here the inverse matrix $J^{-1}_{\alpha\beta}(x,x')$ is defined by

$$\int d^3x''J_{\alpha\nu}(x,x'')J^{-1}_{\nu\beta}(x'',x')=\delta_{\alpha\beta}\delta(x-x').$$

By virtue of the antisymmetry of $J_{\alpha\beta}(x,x')$ with respect to the substitutions $\alpha\leftrightarrow\beta$, $x\leftrightarrow x'$, the matrix $J_{\alpha\beta}(x,x')$ is nondegenerate only in the case of an even number of the variables $\varphi_\alpha(x)$.

We define the Poisson bracket of arbitrary functionals A and B of the dynamical variables $\varphi_\alpha(x)$ by

$$\{A,B\}=\int d^3xd^3x'\,\frac{\delta A}{\delta\varphi_\alpha(x)}\,J^{-1}_{\alpha\gamma}(x,x';\varphi)\,\frac{\delta B}{\delta\varphi_\beta(x')}. \tag{1.5}$$

Then the equations of motion (1.4) take the Hamiltonian form

$$\dot{\varphi}_\alpha(x)=\{\varphi_\alpha(x),H\}. \tag{1.6}$$

It follows from the definition (1.5), with allowance for (1.3), that the PB of the arbitrary functionals A and B satisfies the relations

$$\{A,B\}=-\{B,A\},\quad \{AB,C\}=A\{B,C\}+B\{A,C\}$$

and the Jacobi identity

$$\{A,\{B,C\}\}+\{B,\{C,A\}\}+\{C,\{A,B\}\}=0.$$

This last identity holds by virtue of the equation

$$\frac{\delta J_{\alpha\beta}(\varphi)}{\delta\varphi_\gamma(x)}+\frac{\delta J_{\beta\gamma}(\varphi)}{\delta\varphi_\alpha(x)}+\frac{\delta J_{\gamma\alpha}(\varphi)}{\delta\varphi_\beta(x)}=0.$$

Note that Hamiltonian mechanics in arbitrary variables for systems with finite degrees of freedom were studied by Pauli[30] (see also Ref. 31), who, in particular, obtained PB analogous to (1.5).

We consider finite transformations

$$\varphi_\alpha(x)\rightarrow\varphi'_\alpha(x)=\varphi'_\alpha(x;\varphi(x')). \tag{1.7}$$

The transformations (1.7) are called canonical if the following condition holds:

$$\int d^3xF_\alpha(x;\varphi)\delta\varphi_\alpha(x)-\int d^3xF_\alpha(x;\varphi')\delta\varphi'_\alpha(x)=\delta Q(\varphi). \tag{1.8}$$

Here $Q(\varphi)$ is some functional of the dynamical variables φ_α that depends on the structure of the canonical transformations. Since the relation (1.8) is equivalent to

$$\frac{\delta Q}{\delta\varphi_\alpha(x)}=F_\alpha(x,\varphi)-\int d^3x_1F_\lambda(x_1,\varphi')\,\frac{\delta\varphi'_\lambda(x_1)}{\delta\varphi_\alpha(x)}, \tag{1.9}$$

we can, by taking into account the formula

$$\frac{\delta^2Q}{\delta\varphi_\alpha(x)\delta\varphi_\beta(x')}=\frac{\delta^2Q}{\delta\varphi_\beta(x')\delta\varphi_\alpha(x)},$$

represent the condition of canonicity in the form

$$J_{\alpha\beta}(x,x';\varphi)$$

$$=\int d^3x_1 d^3x_2 \frac{\delta\varphi'_\lambda(x_1)}{\delta\varphi_\alpha(x)}\frac{\delta\varphi'_\nu(x_2)}{\delta\varphi_\beta(x')}J_{\lambda\nu}(x_1,x_2;\varphi'). \tag{1.10}$$

In the case of infinitesimal transformations

$$\varphi_\alpha(x)\to\varphi'_\alpha(x)=\varphi_\alpha(x)+\delta\varphi_\alpha(x;\varphi(x')),$$

the relation (1.9) becomes

$$\int d^3x' J_{\alpha\beta}(x,x';\varphi)\delta\varphi_\beta(x')=\frac{\delta G}{\delta\varphi_\alpha(x)},$$

$$G=\delta Q+\int d^3x F_\alpha(x,\varphi)\delta\varphi_\alpha(x),$$

or, with allowance for (1.5),

$$\delta\varphi_\alpha(x)=\{\varphi_\alpha(x),G\}, \tag{1.11}$$

where G is the generator of infinitesimal canonical transformations. It is readily verified that the PB (1.5) is invariant with respect to the canonical transformations (1.7) and (1.10).

We now consider transformations (1.7) that leave the kinematic part of the Lagrangian invariant. Such transformations satisfy the relation

$$F_\alpha(x;\varphi)=\int d^3x' F_\beta(x';\varphi')\frac{\delta\varphi'_\beta(x';\varphi)}{\delta\varphi_\alpha(x)},$$

and, in accordance with (1.9), are canonical with $Q(\varphi)$ $=$const. For infinitesimal transformations, this last equation can be written in the form (1.11) with the generator $G=\int d^3x F_\alpha(x;\varphi)\delta\varphi_\alpha(x)$, determined by (1.3). The interpretation of the integrated terms in the variation of the action as generators of infinitesimal transformations was first implemented for the quantum case by Schwinger.[3]

The considered class of variations can be extended by adding to the Lagrangian a total derivative with respect to the time of an arbitrary functional $\chi(\varphi)$. The corresponding equation for the extended class of variations has the form

$$\int d^3x' J_{\alpha\beta}(x,x';\varphi)\delta\varphi_\beta(x')-\frac{\delta}{\delta\varphi_\alpha(x)}$$

$$\times\left(G+\int d^3x' \frac{\delta\chi(\varphi)}{\delta\varphi_\beta(x')}\delta\varphi_\beta(x')\right).$$

For finite transformations (1.7), the condition of invariance of the kinematic part of the Lagrangian can, with allowance for the indeterminacy in the choice of L,

$$L\to L'=L+\int d^3x \frac{\delta\chi(\varphi)}{\delta\varphi_\alpha(x)}\dot\varphi_\alpha(x),$$

be written in the form

$$F_\alpha(x;\varphi)=\int d^3x' F_\beta(x';\varphi')\frac{\delta\varphi'_\beta(x';\varphi)}{\delta\varphi_\alpha(x)}+\frac{\delta\chi(\varphi')}{\delta\varphi_\alpha(x)}$$

$$-\frac{\delta\chi(\varphi)}{\delta\varphi_\alpha(x)}. \tag{1.12}$$

The transformations (1.7) and (1.12) are canonical with $Q=\chi(\varphi')-\chi(\varphi)$.

In the framework of the Hamiltonian approach, one can readily formulate the differential conservation laws associated with the various symmetry properties of the Hamiltonian. The equation of motion for the density of an arbitrary physical variable $A=\int d^3x a(x)$ can be represented in the form

$$\dot a(x)=\{a(x),H\}\equiv\{A,\varepsilon(x)\}-\nabla_k a_k(x), \tag{1.13}$$

where

$$a_k(x)=\int d^3x' x'_k\int_0^1 d\lambda\{a(x+\lambda x'),\varepsilon(x-(1-\lambda)x')\}.$$

If A is the generator G of a symmetry group of the Hamiltonian, then Eq. (1.13) has the form of a differential conservation law (see below). At the same time, the generator G itself does not depend on the time. Indeed,

$$\dot G=\{G,H\}=-\{H,G\}=-\delta H, \tag{1.14}$$

where δH is the variation of the Hamiltonian due to the transformations (1.11). Hence, if $\delta H=0$, then also $\dot G=0$.

Setting in (1.13) $a(x)=\rho(x)$, where $\rho(x)$ is the matter density, and assuming that

$$\{M,\varepsilon(x)\}=0,\quad M\equiv\int d^3x\rho(x),$$

we obtain the differential conservation law for the mass density:

$$\dot\rho(x)=-\nabla_k j_k(x),$$

$$j_k(x)=\int d^3x' x'_k\int_0^1 d\lambda\{\rho(x+\lambda x'),\varepsilon(x-(1-\lambda)x')\}. \tag{1.15}$$

Here j_k is the mass flux density.

If $a(x)=\pi_i(x)$, where $\pi_i(x)$ is the momentum density, and the energy density satisfies the condition of translational invariance

$$\{P_i(x),\varepsilon(x)\}=\nabla_i\varepsilon(x),\quad P_i\equiv\int d^3x\pi_i(x),$$

then from (1.13) we obtain the differential conservation law for the momentum:

$$\dot\pi_i(x)=-\nabla_k t_{ik}(x),$$

$$t_{ik}(x)=-\varepsilon(x)\delta_{ik}+\int d^3x' x'_k\int_0^1 d\lambda\{\pi_i(x$$

$$+\lambda x'),\varepsilon(x-(1-\lambda)x')\}. \tag{1.16}$$

Here t_{ik} is the stress tensor.

In (1.13), we set $a(x)=s_\alpha(x)$, where $s_\alpha(x)$ is the spin density. If the condition of rotational invariance of the energy density with respect to spin rotations is satisfied,

$$\{S_\alpha,\varepsilon(x)\}=0,\quad S_\alpha=\int d^3x s_\alpha(x),$$

then (1.13) leads to the conservation law

$$\dot{s}_\alpha(x) = -\nabla_k j_{\alpha k}(x),$$

$$j_{\alpha k}(x) = \int d^3x' x'_k \int_0^1 d\lambda \{s_\alpha(x+\lambda x'), \varepsilon(x-(1-\lambda)x')\}. \quad (1.17)$$

Here $j_{\alpha k}$ is the spin flux density.

For $a(x)=\varepsilon(x)$, we obtain from (1.13) the differential conservation law for the energy:

$$\dot{\varepsilon}(x) = -\nabla_k q_k(x),$$

$$q_k(x) = \frac{1}{2}\int d^3x' x'_k \int_0^1 d\lambda \{\varepsilon(x+\lambda x'), \varepsilon(x-(1-\lambda)x')\}, \quad (1.18)$$

where q_k is the energy flux density.

2. DYNAMICS OF CLASSICAL CONTINUOUS MEDIA

We now consider the dynamics of continuous media.[32] In terms of the Lagrangian variables ξ_i, the position of a particle of the medium is characterized by functions $x_k(\xi,t)$ (the Lagrangian coordinates ξ_i can be regarded as the coordinates of the particle in the initial position corresponding to the undeformed state). We write the Lagrangian of the system in the form

$$L = L_k - \int d^3\xi \underline{\varepsilon}(\xi), \quad (2.1)$$

where $\underline{\varepsilon}(\xi) = \underline{\varepsilon}(\underline{\pi}(\xi); \partial x/\partial\xi)$ is the energy density, a functional of the derivatives $\partial x_i/\partial \xi_j$, L_k is the kinematic part of the Lagrangian,

$$L_k = \int d^3\xi \underline{\mathscr{L}}_k(\xi), \quad \underline{\mathscr{L}}_k(\xi) = \underline{\pi}_i(\xi)\dot{x}_i(\xi), \quad (2.2)$$

and $\underline{\pi}_i(\xi)$ is the momentum density in the Lagrangian variables.

In the expression (2.2) for the density of the kinematic part $\underline{L}_k(\xi)$, we go over to the Eulerian variables x_k. For this, we introduce the displacement vector of the particles of the continuous medium in accordance with

$$x_i(t) = \xi_i + u_i(x,t). \quad (2.3)$$

Noting that

$$b_{ij}(x)\dot{x}_j = \dot{u}_i(x), \quad b_{ij}(x) = \delta_{ij} - \nabla_j u_i(x), \quad (2.4)$$

we represent the density of the kinematic part of the Lagrangian in the form

$$\mathscr{L}_k(x) = \pi_i(x) b_{ij}^{-1}(x)\dot{u}_j(x), \quad (2.5)$$

where

$$\pi_i(x) = \left|\frac{\partial\xi}{\partial x}\right| \underline{\pi}_i(\xi)$$

is the momentum density in the Eulerian variables. Using this kinematic part, we can obtain the PB for the variables $u_i(x), \pi_i(x)$. In addition, to formulate the equation of adiabaticity, we need the PB that contain the entropy density $\sigma(x)$. To find these PB, we rewrite the kinematic part of the Lagrangian in the form

$$\mathscr{L}_k(x) = p_j(x)\dot{u}_j(x) - \sigma(x)\dot{\psi}(x), \quad (2.6)$$

where

$$p_j(x) = (\pi_i(x) - \sigma(x)\nabla_i\psi(x)) b_{ij}^{-1}(x).$$

The variable $\psi(x)$, which is conjugate to the variable $\sigma(x)$, is introduced into the kinematic part of the Lagrangian formally, and when we write down the equations of motion will be regarded as cyclic (the Hamiltonian H does not depend on ψ). The origin of the second term in (2.6) is rather transparent, but the structure of p_j requires clarification. For this, we note that the momentum density $\pi_i(x)$ in (2.5) is associated with translations in the space of the variables $u_i(x), \pi_i(x)$. On the introduction of the new variables $\sigma(x), \psi(x)$, the momentum density $\pi_i(x)$ then becomes associated with translations in the complete space of the variables $u_i(x), \pi_i(x), \sigma(x), \psi(x)$ and can be represented in the form

$$\pi_i(x) = \pi_i^*(x) + \pi_i^\sigma(x),$$

where $\pi_i^*(x)$ is the momentum density associated with translations in the space of only the variables $u_i(x), \pi_i(x)$, while $\pi_i^\sigma(x)$ is associated with translations in the space $\sigma(x), \psi(x)$. As will be seen later, $\pi_i^\sigma(x)$ is given by

$$\pi_i^\sigma(x) = \sigma(x)\nabla_i\psi(x)$$

[this formula can be readily anticipated by noting that $\psi(x)$ and $-\sigma(x)$ are generalized coordinates and momenta]. Therefore, the momentum density $\pi_i^*(x)$ associated with the translations in the space of the variables $u_i(x), \pi_i(x)$ has the form

$$\pi_i^*(x) = \pi_i(x) - \sigma(x)\nabla_i\psi(x),$$

and it is from this that the structure of p_j in (2.6) follows. These arguments are merely heuristic and not rigorous. Note that in accordance with the general formalism it would be necessary to find the inverse matrix $J_{\alpha\beta}^{-1}$ in order to obtain the PB of the basic dynamical variables. However, we shall proceed differently and adopt the following scheme. First, we find certain infinitesimal canonical transformations $\delta\varphi_\alpha(x;\varphi)$ that leave the kinematic part of the Lagrangian invariant. Knowing, on the one hand, their explicit form and, on the other, representing $\delta\varphi_\alpha$ in the form

$$\delta\varphi_\alpha(x) = \{\varphi_\alpha(x), G\}$$

with generator

$$G = \int d^3x F_\alpha(x,\varphi)\delta\varphi_\alpha(x)$$

(we retain here the general notation and for convenience give once more the expressions of Sec. 1), it is easy, by comparing the left- and right-hand sides of (1.11), to find the PB of the various dynamical variables.

In particular, it is readily seen that the variations

$$\delta p_i(x) = \delta\sigma(x) = 0, \quad \delta u_i(x) = f_i(x), \quad \delta\psi(x) = \chi(x) \quad (2.7)$$

[where the functions $f_i(x)$ and $\chi(x)$ do not depend on $u_i(x), p_i(x), \sigma(x), \psi(x)$] leave the kinematic part of the Lagrangian (2.6) invariant. Representing them in the form

$$\delta p_i(x)=\{p_i(x),G\},\quad \delta u_i(x)=\{u_i(x),G\},$$

$$\delta\sigma(x)=\{\sigma(x),G\},\quad \delta\psi(x)=\{\psi(x),G\},\tag{2.8}$$

where G, the generator of the transformations (2.7), is

$$G=\int d^3x(p_i(x)f_i(x)-\sigma(x)\chi(x)),\tag{2.9}$$

we obtain from (2.8) and (2.9) the PB

$$\{u_i(x),p_j(x')\}=\delta_{ij}\delta(x-x'),\quad \{\sigma(x),\psi(x')\}=\delta(x-x')\tag{2.10}$$

(we have given only the nontrivial PB). We can now find the PB of the physical variables $u_i(x),\pi_i(x),\sigma(x),\psi(x)$. It is readily seen that from

$$\{\pi_i^*(x),\psi(x')\}=\{\pi_i^*(x),\sigma(x')\}=0,$$

where

$$\pi_i^*(x)\equiv\pi_i(x)-\sigma(x)\nabla_i\psi(x)=p_j(x)b_{ji}(x)$$

we obtain

$$\{\pi_i(x),\sigma(x')\}=-\sigma(x)\nabla_i\delta(x-x'),$$

$$\{\pi_i(x),\psi(x')\}=\nabla_i\psi(x)\delta(x-x').\tag{2.11}$$

With allowance for (2.10), for $\pi_i^\sigma(x)\equiv\sigma(x)\nabla_i\psi(x)$ we obtain

$$\{\pi_i^\sigma(x),\pi_k^\sigma(x')\}=\pi_k^\sigma(x)\nabla_i'\delta(x-x')-\pi_i^\sigma(x')\nabla_k\delta(x-x').$$

Hence and from

$$\{p_i(x),p_k(x')\}=0$$

we obtain

$$\{\pi_i(x),\pi_k(x')\}=\pi_k(x)\nabla_i'\delta(x-x')-\pi_i(x')\nabla_k\delta(x-x').\tag{2.12}$$

The first of the relations (2.10) leads to the equation

$$\{u_i(x),\pi_k(x')\}=b_{ik}(x)\delta(x-x').\tag{2.13}$$

Equations (2.10)–(2.13) determine the system of nontrivial PB of the continuum:

$$\{\sigma(x),\psi(x')\}=\delta(x-x'),$$

$$\{\pi_i(x),\sigma(x')\}=-\sigma(x)\nabla_i\delta(x-x'),$$

$$\{\pi_i(x),\psi(x')\}=\nabla_i\psi(x)\delta(x-x'),$$

$$\{u_i(x),\pi_k(x')\}=b_{ik}(x)\delta(x-x'),$$

$$\{\pi_i(x),\pi_k(x')\}=\pi_k(x)\nabla_i'\delta(x-x')-\pi_i(x')\nabla_k\delta(x-x').\tag{2.14}$$

We now note that if $\tilde{\rho}$ is the matter density per unit undeformed volume, then the true density ρ is determined by

$$\rho=\tilde{\rho}\det(b_{ij}).\tag{2.15}$$

Taking into account (2.13) and (2.15), we find the PB of the variables $\rho(x),\pi_i(x)$:

$$\{\pi_i(x),\rho(x')\}=\rho(x)\nabla_i'\delta(x-x').\tag{2.16}$$

We now turn to the dynamical equations of the continuous medium. In the general case, the Hamiltonian of the system has the form

$$H=\int d^3x\varepsilon(x),\quad \varepsilon(x)=\varepsilon(x;\sigma(x'),\pi_i(x'),b_{ij}(x')).$$

Here $\varepsilon(x)$ is the energy density in the Eulerian variables; it is a functional of $\sigma(x),\pi_i(x),b_{ij}(x)$ (the variable ψ is cyclic) and is related to the energy density in the Lagrangian variables by

$$\varepsilon(x)=\left|\frac{\partial\xi}{\partial x}\right|\underline{\varepsilon}(\xi).$$

Note that by virtue of the invariance of $\varepsilon(x)$ with respect to translations of the Lagrangian coordinates [which will be assumed in what follows; see (3.12) and (3.14)] the energy density $\varepsilon(x)$ depends, not on the $u_i(x)$ themselves, but only on their derivatives $\partial u_i/\partial x_j$ or, which is the same thing, on $b_{ij}(x)$. Therefore, as dynamical variables it is convenient to choose, besides the remaining variables, the quantities $b_{ij}(x)$ directly. Their nonvanishing PB with the momentum density $\pi_i(x)$ is

$$\{b_{ij}(x),\pi_k(x')\}=-b_{ik}(x')\nabla_j\delta(x-x').\tag{2.17}$$

Using the PB (2.14) and (2.17), we obtain the dynamical equations of the continuous medium in the form

$$\dot{\sigma}(x)=-\nabla_i\left(\sigma(x)\frac{\delta H}{\delta\pi_i(x)}\right),\quad \dot{b}_{ik}(x)=-\nabla_k\left(b_{ij}(x)\frac{\delta H}{\delta\pi_j(x)}\right),$$

$$\dot{\pi}_i(x)=-\pi_j(x)\nabla_i\frac{\delta H}{\delta\pi_j(x)}-\nabla_j\left(\pi_i(x)\frac{\delta H}{\delta\pi_j(x)}\right)-b_{ki}(x)\nabla_j\frac{\delta H}{\delta b_{kj}(x)}-\sigma(x)\nabla_i\frac{\delta H}{\delta\sigma(x)}.\tag{2.18}$$

In the theory of continuous media, it is customary, in place of the second equation in (2.18), to write down an equation directly for the displacement vector:

$$\dot{u}_i(x)=-b_{ij}(x)\frac{\delta H}{\delta\pi_j(x)}.$$

If we assume that the Hamiltonian of the system possesses the property of Galilean invariance,

$$H=H_0+V(b(x')),\quad H_0=\int d^3x\frac{\pi^2(x)}{2\rho(x)},\tag{2.19}$$

then from (2.18) we obtain

$$\dot{\sigma}(x)=-\nabla_i\left(\sigma(x)\frac{\pi_i(x)}{\rho(x)}\right),$$

$$\dot{b}_{ik}(x)=-\nabla_k\left(b_{ij}(x)\frac{\pi_j(x)}{\rho(x)}\right),$$

$$\dot{\pi}_i(x) = -\nabla_j \frac{\pi_i(x)\pi_j(x)}{\rho(x)} - b_{ki}(x)\nabla_j \frac{\delta V}{\delta b_{kj}(x)} - \sigma(x)\nabla_i \frac{\delta V}{\delta\sigma(x)}. \tag{2.20}$$

If the density of the interaction energy $v(x)$ $[V=\int d^3xv(x)]$ depends on $b(x)$, $\sigma(x)$ only locally, $v(x)=v(b(x),\sigma(x))$, then for the last of Eqs. (2.20) we have

$$\dot{\pi}_i = -\nabla_j t_{ij}, \quad t_{ij} = \frac{\pi_i\pi_j}{\rho} + b_{ki}\frac{\partial v}{\partial b_{kj}} + \left(-v+\sigma\frac{\partial v}{\partial\sigma}\right)\delta_{ij}. \tag{2.21}$$

In what follows, we require the property of invariance of the Hamiltonian of the continuous medium with respect to rotations, which we write in the form

$$\{\mathscr{L}_i, \varepsilon(x)\} = \varepsilon_{ikl}x_k\frac{\partial\varepsilon(x)}{\partial x_l}, \quad \mathscr{L}_i = \int d^3x\varepsilon_{ikl}x_k\pi_l(x). \tag{2.22}$$

3. DIFFERENTIAL CONSERVATION LAWS

We consider the physical interpretation of some transformations that leave the kinematic part of the Lagrangian invariant. Let $x_i \to x_i' = x_i'(x)$ be an arbitrary finite transformation under which the displacement vector transforms in accordance with the law

$$u_i(x,t) \to u_i'(x',t) = u_i(x,t) + x_i' - x_i \tag{3.1}$$

[the law (3.1) corresponds to displacement of the particle with the Lagrangian coordinate ξ_i from the point x_i to the point x_i']. Since $\dot{u}_i'(x',t)=\dot{u}_i(x,t)$ [the functions $x_i'(x)$ do not depend on t] and

$$b_{ij}'(x') = b_{is}(x)\frac{\partial x_s}{\partial x_j'}$$

[see (2.4) and (3.1)], the density of the kinematic part of the Lagrangian can be rewritten in the form

$$\mathscr{L}_k = (\pi_i'(x') - \sigma'(x')\nabla_i'\psi'(x'))\left|\frac{\partial x'}{\partial x}\right| b_{ij}'^{-1}(x')\dot{u}_j'(x') - \sigma'(x')\left|\frac{\partial x'}{\partial x}\right|\dot{\psi}'(x'),$$

where we have defined the transformation law of the momentum density

$$\pi_i(x) \to \pi_i'(x') = \left|\frac{\partial x}{\partial x'}\right| \frac{\partial x_s}{\partial x_i'}\pi_s(x) \tag{3.2}$$

and of the quantities $\psi(x)$ and $\sigma(x)$,

$$\psi(x) \to \psi'(x') = \psi(x), \quad \sigma(x) \to \sigma'(x') = \left|\frac{\partial x}{\partial x'}\right|\sigma(x) \tag{3.3}$$

in such a way that the kinematic part L_k of the Lagrangian is invariant with respect to the finite transformations $x_i \to x_i' = x_i'(x)$. In the new variables, for L_k we have

$$L_k = \int d^3x'\{p_j'(x')\dot{u}_j'(x') - \sigma'(x')\dot{\psi}'(x')\} = \int d^3x\{p_j'(x)\dot{u}_j'(x) - \sigma'(x)\dot{\psi}'(x)\}, \tag{3.4}$$

where

$$p_j'(x) = (\pi_i'(x) - \sigma'(x)\nabla_i\psi'(x))b_{ij}'^{-1}(x).$$

It follows from (3.4) that the variations of the dynamical variables $u_i(x), \pi_i(x), \psi(x), \sigma(x)$ under infinitesimal transformations $x_i \to x_i' = x_i + \chi_i(x), |\chi_i(x)| \ll 1$, must be defined by

$$\delta f(x) = f'(x) - f(x), \quad f = \{u_i, \pi_i, \psi, \sigma\}. \tag{3.5}$$

Taking into account (3.1)–(3.3), we obtain for the variations (3.5)

$$\delta u_i(x) = b_{ij}(x)\chi_j(x), \quad \delta\psi(x) = -\chi_i(x)\nabla_i\psi(x),$$
$$\delta\sigma(x) = -\nabla_i(\chi_i(x)\sigma(x)),$$
$$\delta\pi_i(x) = -\nabla_j(\chi_j(x)\pi_i(x)) - \pi_j(x)\nabla_i\chi_j(x). \tag{3.6}$$

In accordance with the general theory, these variations leaving the kinematic part of the Lagrangian invariant are infinitesimal canonical transformations with the generator

$$G = \int d^3x\pi_i(x)b_{ij}^{-1}(x)\delta u_j(x) = \int d^3x\pi_i(x)\chi_i(x). \tag{3.7}$$

We consider the transformations (3.6) for χ_i=const:

$$\delta u_i(x) = \chi_i - \chi_k\nabla_k u_i(x), \quad \delta\psi(x) = -\chi_i(x)\nabla_i\psi(x),$$
$$\delta\sigma(x) = -\chi_i(x)\nabla_i\sigma(x), \quad \delta\pi_i(x) = -\chi_j(x)\nabla_j\pi_i(x). \tag{3.8}$$

The variations (3.8) correspond to infinitesimal translations of the Eulerian coordinates $\delta x_k = \chi_k$, $\delta\xi_k = 0$ [with allowance for the constraint $\xi_i = x_i - u_i(x)$] with the generator

$$G = \chi_k P_k, \quad P_k = \int d^3x\pi_k(x). \tag{3.9}$$

From (3.8) and (3.9), we find

$$\delta\varepsilon(x) = \{\varepsilon(x), G\} = -\chi_k\nabla_k\varepsilon(x) \tag{3.10}$$

and, therefore, $\delta H = \delta(\int dx\varepsilon(x)) = 0$. Therefore, in accordance with (1.14), P_k is an integral of the motion and can be interpreted as the momentum of the medium. With allowance for (1.14) and (3.10), the differential conservation law for the momentum density of the medium has the form (1.16). If the Hamiltonian of the system possesses Galilean invariance [see (2.19)], then for the stress density t_{ik} we obtain

$$t_{ik} = t_{ik}^0 + t_{ik}', \quad t_{ik}^0 = \frac{\pi_i\pi_k}{\rho},$$

$$t_{ik} = -v(x)\delta_{ik} + \int d^3x'x_k'\int_0^1 d\lambda\{\pi_i(x+\lambda x'), v(x-(1-\lambda)x')\}. \tag{3.11}$$

We now consider infinitesimal transformations (2.7) with f_i=const, $\chi=0$:

$$\delta u_i(x)=f_i,\quad \delta\psi(x)=0,\quad \delta\pi_i(x)=0,\quad \delta\sigma(x)=0. \tag{3.12}$$

Since $x_i=\xi_i+u_i(x)$, these transformations correspond to translations of the Lagrangian coordinates $\delta\xi_i=-f_i$ ($\delta x_i=0$). The generator of the transformations (3.12) has the form

$$G=f_iR_i,\quad R_i=\int d^3xp_i(x),\quad p_i(x)=\pi_j^*(x)b_{ji}^{-1}(x). \tag{3.13}$$

Since for the transformations (3.12)

$$\delta\varepsilon(x)=\{\varepsilon(x),G\}=0$$

and, therefore, $\delta H=0$, it follows that R_i in the expression (3.13) for the generator of the translations of the Lagrangian coordinates is an integral of the motion. The corresponding differential conservation law has the form

$$\dot{p}_i(x)=-\nabla_k\tilde{\sigma}_{ik}(x),\quad \tilde{\sigma}_{ik}(x)=\int d^3x'x_k'\int_0^1 d\lambda\{p_i(x+\lambda x'),\varepsilon(x-(1-\lambda)x')\}. \tag{3.14}$$

This conservation law can be called the conservation law for the generalized momentum [see (2.6)]. We shall show that the conservation law associated with translations of the Lagrangian variables leads to the concept of the momentum of quasiparticles. Suppose that the medium contains a scattering center, whose momentum we denote by q_i. In this case, $P_i=\int dx\pi_i(x)$ will not be conserved, but P_i+q_i will be (since the Hamiltonian is invariant with respect to simultaneous shifts of the Eulerian coordinates and the coordinates of the particle). In contrast, the invariance of the Hamiltonian with respect to translations of the Lagrangian variables will be formulated in the previous form. Therefore, R_i will be an integral of the motion. We represent it in the form

$$R_i=P_i-\mathscr{P}_i,\quad \mathscr{P}_i=\int d^3xk_i(x),$$

$$k_i(x)=-\pi_j^*(x)\frac{\partial u_j(x)}{\partial\xi_i}.$$

Because R_i is conserved, we arrive at the relation $\dot{P}_i=\dot{\mathscr{P}}_i$, and since $\dot{P}_i+\dot{q}_i=0$, it follows that $\dot{q}_i+\dot{\mathscr{P}}_i=0$. Therefore, $\mathscr{P}_i$, which is quadratic in the variables $u_i(x)$, $\pi_i^*(x)$, can be interpreted as the momentum of quasiparticles. [One can show that on quantization of the mechanical system $\mathscr{P}_i$ is transformed into the phonon momentum operator: $\mathscr{P}_i=\Sigma_{kj}k_ic_j^+(k)c_j(k)$.] Thus, the change in the momentum of the medium is equal to the change in the total momentum of the quasiparticles. Noting that $p_i(x)=\pi_i^*(x)-k_i(x)$ and taking into account the expressions (3.11) and (3.14), we find

$$\dot{k}_i(x)=-\nabla_k\sigma_{ik}(x),$$

where the momentum flux density σ_{ik} of the quasiparticles is given by

$$\sigma_{ik}(x)=-\varepsilon(x)\delta_{ik}+\int d^3x'x_k'\int_0^1 d\lambda\{k_i(x+\lambda x'),\varepsilon(x-(1-\lambda)x')\}.$$

It follows from (3.8) and (3.12) that $K=\chi_i\int d^3xk_i(x)$ is a generator of translation transformations

$$\delta u_i(x)=-\chi_k\nabla_ku_i(x),\quad \delta\pi_i(x)=-\chi_k\nabla_k\pi_i(x),$$
$$\delta\psi(x)=-\chi_k\nabla_k\psi(x),\quad \delta\sigma(x)=-\chi_k\nabla_k\sigma_i(x)$$

of the dynamical variables u_i, π_i, ψ, σ regarded as field variables. Therefore, the momentum of the quasiparticles is equal to the field momentum.

4. ELASTICITY. HYDRODYNAMICS

We now consider how the equations of elasticity theory can be obtained from the general dynamical equations (2.20)–(2.21) for continuous media. The assumption of invariance of the Hamiltonian of the system with respect to arbitrary rotations of the lattice [see (2.22)] has the consequence that the interaction energy density will depend on quite definite combinations of the matrix b_{ij}. Since under rotations of the body the Euler coordinates x_k corresponding to the positions of the particles of the medium in the deformed state are transformed, while the Lagrangian coordinates ξ_k remain unchanged, we can choose as invariants

$$K_{ij}=\frac{\partial\xi_i}{\partial x_k}\frac{\partial\xi_j}{\partial x_k}=b_{ik}b_{jk}.$$

Thus, $v(x)=v(K_{ij}(x))$, and, as a consequence of the general equations (2.20)–(2.21), we obtain the equations of motion in the case of an elastic medium:

$$\dot{\sigma}=-\nabla_i\left(\sigma\frac{\pi_i}{\rho}\right),\quad \dot{b}_{ik}=-\nabla_k\left(b_{ij}\frac{\pi_j}{\rho}\right),$$

$$\dot{\pi}_i=-\nabla_jt_{ij},\quad t_{ij}=\frac{\pi_i\pi_j}{\rho}+2b_{mi}b_{lj}\frac{\partial v}{\partial K_{ml}}+\left(-v+\sigma\frac{\partial v}{\partial\sigma}\right)\delta_{ij}. \tag{4.1}$$

If the interaction energy density depends only on the density $\rho(x)$, which is related to the invariants K_{ij} by

$$\rho=\tilde{\rho}\det(b_{ij})=\tilde{\rho}\sqrt{\det(K_{ij})}$$

($\tilde{\rho}$ is the density of the undeformed medium), then from the equations of motion of the elastic medium we obtain the ordinary equations of ideal hydrodynamics:

$$\dot{\rho}+\nabla_k\pi_k=0,\quad \dot{\sigma}=-\nabla_i\left(\sigma\frac{\pi_i}{\rho}\right),$$

$$\dot{\pi}_i=-\nabla_kt_{ik},\quad t_{ik}=\frac{\pi_i\pi_k}{\rho}+p\delta_{ik}, \tag{4.2}$$

where

$$p=\rho\frac{\partial v}{\partial\rho}+\sigma\frac{\partial v}{\partial\sigma}-v. \tag{4.3}$$

Equations (4.2) can also be derived directly on the basis of the PB algebra of the variables ρ, π_i, σ [see (2.12) and

(2.16) and the first equation in (2.11)], which is a subalgebra of the algebra (2.14) of the variables of the continuous medium. To obtain (4.3), we note that, as is well known from thermodynamics, $p=-\omega$ (ω is the density of the Gibbs thermodynamic potential). In a frame of reference moving together with the fluid, the thermodynamic equation

$$\sigma=\frac{1}{T}(-\omega+\varepsilon-\mu\rho), \qquad (4.4)$$

where μ is the chemical potential, holds. Bearing in mind that

$$d\sigma=\frac{1}{T}(d\varepsilon-\mu d\rho),$$

we obtain

$$\mu=\left(\frac{\partial\varepsilon}{\partial\rho}\right)_{\sigma},$$

from which, using (4.4), we arrive at (4.3). If a solution of (4.2) for the variables ρ, π_k, and σ has been found, then the dependence of the variable u_i on the time can be recovered from the known time dependence of ρ and π_k:

$$\dot{u}_i=b_{ij}\frac{\pi_j}{\rho}.$$

5. LIQUID CRYSTALS

In this section it will be shown how one can obtain the PB for the dynamical variables that describe the state of the various phases of liquid crystals.[32] In doing this, we shall proceed from the PB algebra (2.14), (2.17) found in the study of continuous media.

5.1. Nematic liquid crystals

In the nematic phase of liquid crystals, there is spontaneous breaking of the rotational invariance, and therefore the dynamical variables that describe the state of the isotropic liquid—the mass density ρ, the momentum density π_k, and the entropy density σ—must be augmented by a further parameter—the unit vector $\mathbf{n}$ (director) associated with the breaking of the rotational symmetry. We shall consider two possibilities for introducing the unit vector, one of which corresponds to a nematic with rod-shaped molecules and the other to a nematic with disk-shaped molecules.

Suppose that the particles of the medium consist of rod-shaped molecules. Then in the equilibrium state it is possible to find a family of curves to which the tangents at each point coincide with the direction of the rods. Let $\xi_i=\xi_i(\alpha)$ be the parametric equations of one of the curves of this family. Then the direction of the rods at each point is characterized by the vector with coordinates $d\xi_i/d\alpha\equiv a_i$. If the medium is deformed, the directions of the molecules change, and therefore the curves of the family are deformed. Let $x_i=x_i(\alpha)$ be the new parametric equations of the already considered curve of the family after the deformation. Then the direction of the rods at each point after the deformation is characterized by the vector $dx_i/d\alpha\equiv b_i$. Bearing in mind that $x_i=x_i(\xi)$, we readily see that the vectors a_i and b_i are related by

$$b_i=b_{ij}^{-1}a_j,$$

where $b_{ij}\equiv\partial\xi_i/\partial x_j$ [see (2.3) and (2.4)].

It follows from this that the unit vector associated with the direction of the rods can be introduced in accordance with

$$n_i=\frac{b_i}{b},\quad b_i=b_{ij}^{-1}a_j. \qquad (5.1)$$

Here a_i is some constant vector. Using the definition of b_{ij} [see (2.4)] and the PB (2.17), we can readily find the PB

$$\{\pi_i(x),b_{sl}^{-1}(x')\}=\delta(x-x')\nabla_i b_{sl}^{-1}(x)$$
$$-\delta_{is}b_{kl}^{-1}(x')\nabla_k'\delta(x-x'). \qquad (5.2)$$

From (5.2) we obtain the PB for the variables $\pi_i(x)$ and $n_i(x)$:

$$\{\pi_i(x),n_j(x')\}=\delta(x-x')\nabla_i n_j(x)$$
$$-\delta_{ij}^{\perp}(x')n_k(x')\nabla_k'\delta(x-x'), \qquad (5.3)$$

$$\delta_{ij}^{\perp}(x)\equiv\delta_{ij}-n_i(x)n_j(x).$$

The brackets (5.3), together with the brackets

$$\{\pi_i(x),\sigma(x')\}=-\sigma(x)\nabla_i\delta(x-x'),$$

$$\{\pi_i(x),\rho(x')\}=\rho(x)\nabla_i'\delta(x-x'),$$

$$\{\pi_i(x),\pi_j(x')\}=\pi_j(x)\nabla_i'\delta(x-x')-\pi_i(x')\nabla_j\delta(x-x') \qquad (5.4)$$

form the algebra of variables of the nematic with rod-shaped molecules (we have given only the nontrivial PB). We emphasize that, on the basis of the method of introduction of $\rho(x)$, $n_i(x)$ [see (2.15) and (5.1)] it follows that this algebra is a subalgebra of the algebra (2.14), (2.17) of dynamical variables of a continuous medium. Using the PB (5.3), (5.4) and assuming that the energy density of the system has the form $\varepsilon(x)=\varepsilon(\sigma(x),\rho(x),\pi(x),n(x),\nabla n(x))$, we obtain the dynamical equations of a nematic with ideal rod-shaped molecules (see also Ref. 20):

$$\dot{\sigma}=-\nabla_i\left(\sigma\frac{\pi_i}{\rho}\right),\quad \dot{\rho}=-\nabla_k\pi_k;$$

$$\dot{n}_i=-\left(\frac{\pi_k}{\rho}\nabla_k\right)n_i+n_k\delta_{ij}^{\perp}\nabla_k\frac{\pi_j}{\rho};$$

$$\dot{\pi}_i=-\nabla_k t_{ik},$$

$$t_{ik}=p\,\delta_{ik}+\frac{\pi_i\pi_k}{\rho}+\nabla_i n_j\frac{\partial\varepsilon}{\partial\nabla_k n_j}$$
$$-n_k\delta_{ij}^{\perp}\left(\frac{\partial\varepsilon}{\partial n_j}-\nabla_l\frac{\partial\varepsilon}{\partial\nabla_l n_j}\right). \qquad (5.5)$$

If we use the condition (2.22) of rotational invariance of the energy density $\varepsilon(x)$, which in our case is given by

$$\varepsilon_{ikj}\left(\frac{\partial\varepsilon}{\partial n_k}n_j+\frac{\partial\varepsilon}{\partial\nabla_s n_k}\nabla_s n_j+\frac{\partial\varepsilon}{\partial\nabla_k n_s}\nabla_j n_s\right)=0, \qquad (5.6)$$

then we can readily see that the divergence $\nabla_k t_{ik}$ can be represented as the divergence of the following manifestly symmetric tensor $\tilde{t}_{ik}$:

$$\tilde{t}_{ik}=p\,\delta_{ik}+\frac{\pi_i\pi_k}{\rho}+\frac{1}{2}(g_{km}\nabla_i n_m+g_{im}\partial_k n_m)-\frac{1}{2}(n_i h_k+n_k h_i)+\tfrac{1}{2}\nabla_m(g_{ik}n_m+g_{ki}n_m-g_{im}n_k-g_{km}n_i).$$

Here

$$g_{ik}=\frac{\partial\varepsilon}{\partial\nabla_i n_k},\quad h_i=\left(\frac{\partial\varepsilon}{\partial n_j}-\nabla_k\frac{\partial\varepsilon}{\partial\nabla_k n_j}\right)\delta^{\perp}_{ij}.$$

We now consider a nematic consisting of disk-shaped molecules. The direction of their orientation is specified by the unit vector of their normal. If we introduce a family of surfaces for which their tangent planes at each point coincide with the planes of the disks, then, as follows from the previous consideration, the two noncollinear vectors d_i and f_i that determine the positions of the planes can be represented in the form $d_i=b^{-1}_{ij}m_j$, $f_i=b^{-1}_{ij}n_j$, where m_i and n_i are certain constant noncollinear vectors that determine the positions of the planes in the undeformed state. Then the vector of the normal to the plane spanned by the vectors d_i and f_i is $c_i=l_k\partial\xi_k/\partial x_i\equiv l_k b_{ki}$, where $l_k=(\mathbf{m}\times\mathbf{n})_k$, and, therefore, the unit vector of the normal to the plane of the disk-shaped molecule is determined by

$$n_i=\frac{c_i}{c},\quad c_i=l_k b_{ki}. \tag{5.7}$$

Here l_k is an arbitrary constant vector that determines the direction of the director in the undeformed state. Using the definition of n_i and the brackets (2.17), we obtain for the variables $\pi_i(x)$, $n_i(x)$ the PB

$$\{\pi_i(x),n_j(x')\}=\delta(x-x')\nabla_i n_j(x)+\delta^{\perp}_{jk}(x')n_i(x')\nabla'_k\delta(x-x'). \tag{5.8}$$

The PB (5.4), (5.8) form the algebra of dynamical variables of the nematic with disk-shaped molecules, and this algebra is a subalgebra of the algebra (2.14), (2.17) of the dynamical variables of a continuous medium. Note that both brackets (5.3) and (5.8) were found in Ref. 18 by the use of the transformation laws of contravariant and covariant vectors under translations. Assuming again that $\varepsilon(x)=\varepsilon(\sigma(x),\rho(x),\pi(x),n(x),\nabla n(x))$, we obtain the dynamical equations of the nematic with disk-shaped molecules in the form

$$\dot\sigma=-\nabla_i\left(\sigma\frac{\pi_i}{\rho}\right),\quad \dot\rho=-\nabla_k\pi_k,$$

$$\dot n_i=-\left(\frac{\pi_k}{\rho}\nabla_k\right)n_i-n_k\delta^{\perp}_{ij}\nabla_j\frac{\pi_k}{\rho};$$

$$\dot\pi_i=-\nabla_k t_{ik},$$

$$t_{ik}=p\,\delta_{ik}+\frac{\pi_i\pi_k}{\rho}+\nabla_i n_j\frac{\partial\varepsilon}{\partial\nabla_k n_j}+n_i\delta^{\perp}_{jk}\left(\frac{\partial\varepsilon}{\partial n_j}-\nabla_l\frac{\partial\varepsilon}{\partial\nabla_l n_j}\right). \tag{5.9}$$

Taking into account the condition of rotational invariance for $\varepsilon(x)$, which can be written in the previous form (5.6), we transform the divergence $\nabla_k t_{ik}$ to the divergence of the following manifestly symmetric tensor $\tilde{t}_{ik}$:

$$\tilde{t}_{ik}=p\,\delta_{ik}+\frac{\pi_i\pi_k}{\rho}+\frac{1}{2}(g_{km}\nabla_i n_m+g_{im}\nabla_k n_m)+\frac{1}{2}(n_i h_k+n_k h_i)+\frac{1}{2}\nabla_m(g_{ik}n_m+g_{ki}n_m-g_{im}n_k-g_{km}n_i).$$

Note that for the nematic the part of the energy density that is associated with the dependence on the unit vector $\mathbf{n}$ and has the property of rotational invariance can be written in the form[33]

$$\varepsilon(n)=\tfrac{1}{2}K_1(\nabla_i n_i)^2+\tfrac{1}{2}K_2(\mathbf{n}\ \mathrm{curl}\ \mathbf{n})^2+\tfrac{1}{2}K_3(n_i\nabla_i n_k)^2. \tag{5.10}$$

For nematics consisting of disk-shaped molecules, it follows from the definition (5.7) that the invariant $\mathbf{n}$ curl $\mathbf{n}$ vanishes, and therefore this term must be omitted for the considered method of introducing $\mathbf{n}$. The relation $\mathbf{n}$ curl $\mathbf{n}=0$ is compatible with the equations of motion (5.9) for the vector $\mathbf{n}$, since it follows from them that

$$\dot w=-\nabla_j\left(\frac{\pi_j}{\rho}w\right)+2wn_in_j\nabla_i\frac{\pi_j}{\rho};\quad w\equiv\mathbf{n}\ \mathrm{curl}\ \mathbf{n}.$$

5.2. Other types of liquid-crystal ordering

We briefly consider the description of the dynamics of other possible liquid-crystal states.

In the cholesteric phase of liquid crystals, the macroscopic symmetry group of the phase does not contain an inversion. The spontaneous breaking of the translational and rotational symmetry of the equilibrium state leads to helical ordering of the director in the ground state.[33] In accordance with Ref. 20, the cholesteric dynamics is similar to the dynamics of nematics; a difference appears in the structure of the expansion of the energy density with respect to the gradients of the director. By virtue of what we have said above, the energy density of the cholesteric can contain terms odd with respect to ∇.

In smectic liquid crystals, the translational invariance along one of the directions is broken.[33] According to Ref. 19, the hydrodynamic variables for a smectic include the smectic variable $W(x)$ instead of a director. We define the smectic variable $W(x)$ by $W(x)\equiv m_j\xi_j(x)$, where m_j is an arbitrary constant vector, and $\xi_j(x)$ is the Lagrangian coordinate of the particles of the medium. Taking into account (2.3) and the fact that $b_{ij}=\nabla_j\xi_i$, we find from (2.13) the bracket

$$\{\pi_k(x),\xi_i(x')\}=\delta(x-x')\nabla_k\xi_i. \tag{5.11}$$

From this, contracting both sides of (5.11) with the vector m_i, we obtain

$$\{\pi_k(x),W(x')\}=\delta(x-x')\nabla_k W(x). \tag{5.12}$$

The algebra (5.4), (5.12) is the basis for constructing the smectic dynamics. This construction is almost identical to the nematic case, and it is merely necessary to take into account the fact that the energy density is a function of $\nabla_i W$: $\varepsilon(x)=\varepsilon(\sigma(x),\rho(x),\pi(x),\nabla W(x))$.

Discotic liquid crystals are characterized by spontaneous symmetry breaking in two directions. There are two additional variables W_α ($\alpha=1,2$) associated with translational invariance, which we introduce in accordance with $W_\alpha(x)=m_i^\alpha \xi_i(x)$ (cf. the smectic case). The PB of the momentum density $\pi_i(x)$ with $W_\alpha(x)$ has the form

$$\{\pi_k(x),W_\alpha(x')\}=\delta(x-x')\nabla_i W_\alpha(x). \tag{5.13}$$

The further construction of the discotic hydrodynamic equations is done on the basis of the algebra (5.4), (5.13), following the scheme for nematics and with allowance for the fact that $\varepsilon(x)=\varepsilon(\sigma(x),\rho(x),\pi(x),\nabla W_\alpha(x))$. Note that for both smectics and discotics the algebras [(5.4), (5.12) and (5.4), (5.13)] are subalgebras of the continuum dynamical variables. The closed dynamics for the smectic or discotic variables is obtained, as in all the previously considered cases, from the general case of the continuum dynamics by assuming that the variables in the subalgebra are cyclic.

5.3. Nematic elastomers

Here we consider some features of the dynamics of nematic liquid-crystal elastomers, which were synthesized at the beginning of the eighties.[34] There has recently been a strong growth of interest in them[35,36] on account of the new possibilities for experimental study of them.[37,38] Nematic elastomers are characterized by spontaneous breaking of rotational symmetry and of the symmetry with respect to spatial translations. Further hydrodynamic variables are associated with these broken symmetries: a director n_i (as in nematic liquid crystals) and a displacement vector u_i (as in crystalline solids). Physically, the fact that the displacement vector is a dynamical variable corresponds to the circumstance that in nematic elastomers there exists a certain lattice of polymer chains possessing a finite shear modulus in the hydrodynamic limit (as $\omega\to 0$). For the director to be a well-defined hydrodynamic variable, the mean separation between the sites of neighboring chains must be sufficiently great and, therefore, the coupling between the chains must be weak.

Assuming that the Hamiltonian of the system is invariant with respect to homogeneous rotations (and, therefore, that the energy density depends on the displacement vector only through the quantities K_{ij}; see Sec. 4), we represent the Hamiltonian in the form

$$H=\int d^3x\,\varepsilon(x),\quad \varepsilon=\frac{\pi^2}{2\rho}+v(\sigma,n_i,\nabla_j n_i,K_{ij}). \tag{5.14}$$

The PB algebra of the dynamical variables of the system can be written down on the basis of the brackets (2.14), (2,16), (5.3), and (5.8) already obtained and has the form

$$\begin{gathered}\{\pi_i(x),\sigma(x')\}=-\sigma(x)\nabla_i\delta(x-x'),\\ \{b_{ij}(x),\pi_k(x')\}=-b_{ik}(x')\nabla_j\delta(x-x'),\\ \{\pi_i(x),\rho(x')\}=\rho(x)\nabla_i'\delta(x-x'),\\ \{\pi_i(x),\pi_k(x')\}=\pi_k(x)\nabla_i'\delta(x-x')-\pi_i(x')\nabla_k\delta(x-x'),\end{gathered} \tag{5.15}$$

$$\{\pi_i(x),n_j(x')\}=\delta(x-x')\nabla_i n_j(x)-\delta_{ij}^{\perp}(x'1)n_k(x')\nabla_k'\delta(x-x'), \tag{5.15a}$$

$$\{\pi_i(x),n_j(x')\}=\delta(x-x')\nabla_i n_j(x)+\delta_{jk}^{\perp}(x')n_i(x')\nabla_k'\delta(x-x'). \tag{5.15b}$$

The brackets (5.15a) and (5.15b) correspond to rod- and disk-shaped nematic elastomers, respectively. The equations of the low-frequency dynamics corresponding to the energy functional (5.14) and the PB (5.15) have the form

$$\dot\sigma=-\nabla_i\left(\sigma\frac{\pi_i}{\rho}\right),\quad \dot b_{ik}=-\nabla_k\left(b_{ij}\frac{\pi_j}{\rho}\right),\quad \dot\rho=-\nabla_k\pi_k,$$

$$\dot\pi_i=-\nabla_j t_{ij},\quad t_{ij}=p\,\delta_{ij}+\frac{\pi_i\pi_j}{\rho}+2b_{mi}b_{lj}\frac{\partial v}{\partial K_{ml}}+\widetilde{t}_{ij}(n), \tag{5.16}$$

$$\dot n_i=-\left(\frac{\pi_k}{\rho}\nabla_k\right)n_i+n_k\delta_{ij}^{\perp}\nabla_k\frac{\pi_j}{\rho}, \tag{5.16a}$$

$$\dot n_i=-\left(\frac{\pi_k}{\rho}\nabla_k\right)n_i-n_k\delta_{ij}^{\perp}\nabla_j\frac{\pi k}{\rho}. \tag{5.16b}$$

Here

$$\widetilde{t}_{ik}(n)=\tfrac12(g_{km}\nabla_i n_m+g_{im}\nabla_k n_m)\mp\tfrac12(n_ih_k+n_kh_i)+\tfrac12\nabla_m(g_{ik}n_m+g_{ki}n_m-g_{im}n_k-g_{km}n_i),$$

and the upper and lower signs correspond to the rod- and disk-shaped molecules, respectively. The actual structure of the energy functional $v=v(\sigma,n_i,\nabla_j n_i,K_{ij})$, which is rather cumbersome, is given in Ref. 36.

6. MAGNET WITH COMPLETE SYMMETRY BREAKING WITH RESPECT TO SPIN ROTATIONS

In this and the following sections, we shall study magnetically ordered systems with broken symmetry with respect to spin rotations. The case of complete symmetry breaking with respect to spin rotations in magnetic media was first considered in Ref. 39. Linear dynamical equations for such systems were obtained in Refs. 40 and 41, and a nonlinear generalization of them in the framework of the method of phenomenological Lagrangians is given in Refs. 25 and 42, and in the Hamiltonian approach in Refs. 16 and 27. The possible spectra of spin waves are analyzed in Refs. 25, 41, and 42 without allowance for relaxation processes and under the assumption that the equilibrium state of the magnetic systems is translationally invariant.

For an adequate description of the thermodynamics and kinetics of systems with spontaneously broken symmetry, it is necessary to introduce into the theory additional thermodynamic parameters associated, not with conservation laws, but the physical nature of the phase of the investigated state. It is well known[43] that in the case of systems with complete spontaneous symmetry breaking with respect to spin rotations these dynamical variables are the rotation angles φ_α that parametrize the group of three-dimensional rotations of the spin space or the real rotation matrix $a(\varphi)$ associated with them ($a\tilde{a}=1$, where here and in what follows the tilde denotes the transposition operation). In the exponential parametrization, this matrix has the form

$$a(\varphi)=\exp(-\varepsilon\varphi), \quad (\varepsilon\varphi)_{\alpha\beta}\equiv\varepsilon_{\alpha\beta\gamma}\varphi_\gamma. \tag{6.1}$$

With the rotation matrix $a(\varphi)$ there are associated the right $\underline{\omega}_{\alpha k}$ and left $\omega_{\alpha k}$ differential Cartan forms defined by

$$\underline{\omega}_{\alpha k}=\tfrac{1}{2}\varepsilon_{\alpha\beta\gamma}a_{\beta\lambda}\nabla_k a_{\gamma\lambda}, \quad \omega_{\alpha k}=\tfrac{1}{2}\varepsilon_{\alpha\beta\gamma}a_{\lambda\gamma}\nabla_k a_{\lambda\beta}. \tag{6.2}$$

To obtain the PB algebra of the dynamical variables of the magnet, we write the kinematic part of the Lagrangian in the form

$$L_k=\int d^3x\mathscr{L}_k(x), \quad \mathscr{L}_k(x)=-s_\alpha(x)\omega_\alpha(x), \tag{6.3}$$

where ω_α is the left Cartan form associated with the time derivative:

$$\omega_\alpha=\tfrac{1}{2}\varepsilon_{\alpha\beta\gamma}(\tilde{a}\dot{a})_{\gamma\beta}.$$

We shall determine which transformations leave the kinematic part invariant. To this end, we consider variations $\delta a_{\alpha\beta}$ of the rotation matrix for which the orthogonality condition $\tilde{a}a=1$ is preserved:

$$\delta\tilde{a}a+\tilde{a}\delta a=0.$$

Since the variations $\delta a_{\alpha\beta}$ are determined by three infinitesimal parameters [for example, by the variations of the angles φ_α in (6.1)], these parameters can be taken to be

$$R_{\alpha\beta}=\tilde{a}_{\alpha\mu}\delta a_{\mu\beta}, \quad \tilde{R}=-R.$$

The variation of the left Cartan form for the transformations

$$\delta a=aR \tag{6.4}$$

is

$$\delta\omega_\alpha=\dot{R}_\alpha+(\boldsymbol{\omega}\times\mathbf{R})_\alpha, \quad R_\alpha\equiv\tfrac{1}{2}\varepsilon_{\alpha\beta\gamma}R_{\gamma\beta}. \tag{6.5}$$

We consider time-independent ($\dot{\mathbf{R}}=0$) variations (6.5) and, besides them, the transformations of the spin density:

$$\delta\omega_\alpha=(\boldsymbol{\omega}\times\mathbf{R})_\alpha, \quad \delta s_\alpha=(\mathbf{s}\times\mathbf{R})_\alpha. \tag{6.6}$$

The transformation law for the spin density is defined in such a way that the kinematic part of the Lagrangian is invariant with respect to the given class of variations, $\delta L_k=0$.

In accordance with the general theory, the generator of the transformations (6.4) and (6.6) is

$$G=-\frac{1}{2}\varepsilon_{\alpha\beta\gamma}\int d^3x s_\alpha(x)\tilde{a}_{\gamma\mu}\delta a_{\mu\beta}(x)=$$
$$-\int d^3x s_\alpha(x)R_\alpha(x).$$

We represent the canonical transformations (6.4) and (6.6) in the form

$$\delta a_{\alpha\beta}(x)=\{a_{\alpha\beta}(x),G\}, \quad \delta s_\alpha(x)=\{s_\alpha(x),G\}. \tag{6.7}$$

Expressing the variations δa and δs in terms of R,

$$\delta a_{\alpha\beta}=-a_{\alpha\rho}\varepsilon_{\rho\beta\gamma}R_\gamma, \quad \delta s_\alpha=\varepsilon_{\alpha\beta\gamma}s_\beta R_\gamma$$

and comparing the left- and right-hand sides of (6.7), we find, R_α being arbitrary, the PB

$$\{a_{\alpha\beta}(x),s_\gamma(x')\}=a_{\alpha\rho}(x)\varepsilon_{\rho\beta\gamma}\delta(x-x'),$$
$$\{s_\alpha(x),s_\beta(x')\}=\varepsilon_{\alpha\beta\gamma}s_\gamma(x)\delta(x-x'). \tag{6.8}$$

To obtain the PB $\{a_{\alpha\beta}(x),a_{\mu\nu}(x')\}$, we use the exponential parametrization (6.1) of the rotation matrix and, by calculating the derivative $\dot{a}$ in ω_α, represent the density of the kinematic part of the Lagrangian in the form

$$\mathscr{L}_k(x)=g_\lambda(x)\dot{\varphi}_\lambda(x), \quad g_\lambda\equiv-\frac{1}{2}\varepsilon_{\alpha\beta\gamma}s_\alpha\left(\tilde{a}\frac{\partial a}{\partial\varphi_\lambda}\right)_{\gamma\beta}. \tag{6.9}$$

Besides (6.9), we also consider the density of the kinematic part:

$$\mathscr{L}_k'(x)=-\varphi_\lambda(x)\dot{g}_\lambda(x) \tag{6.10}$$

[the expression (6.10) differs from (6.9) by the total time derivative $(\varphi_\lambda\cdot\dot{g}_\lambda)$]. We consider the variations

$$\delta\varphi_\lambda(x)=0, \quad \delta g_\lambda(x)=\chi_\lambda(x) \tag{6.11}$$

(the functions χ_λ do not depend on $\varphi_\lambda, g_\lambda$) that leave the kinematic part invariant. To the variations (6.11) there corresponds the generator

$$G=-\int d^3x\varphi_\alpha(x)\chi_\alpha(x).$$

Since

$$\delta\varphi_\alpha(x)=\{\varphi_\alpha(x),G\}=0,$$

we obtain from this

$$\{\varphi_\lambda(x),\varphi_\mu(x')\}=0.$$

It follows from this relation, with allowance for (6.1), that

$$\{a_{\alpha\beta}(x),a_{\mu\nu}(x')\}=0. \tag{6.12}$$

The relation $\chi_\lambda(x)=\{g_\lambda(x),G\}$ is automatically satisfied by virtue of (6.8) and the definition (6.9) of g_λ.

The system of Poisson brackets (6.8) and (6.12) forms the algebra for the variables of a magnet with complete symmetry breaking with respect to spin rotations—the spin density $s_\alpha(x)$ and the rotation matrix $a_{\alpha\beta}(x)$.

We now consider the construction of the generator of spatial translations in the space of the variables $s_\alpha(x)$, $a_{\alpha\beta}(x)$. For finite transformations

$$x\to x'=x'(x)$$

we define the law of transformation of the dynamical variables $s_\alpha(x)$, $a_{\alpha\beta}(x)$ in such a way that the kinematic part of the Lagrangian remains invariant, $L_k(s(x),a(x))=L_k(s(x'),a(x'))$:

$$s_\alpha(x)\to s_\alpha'(x')=\left|\frac{\partial x}{\partial x'}\right|s_\alpha(x), \quad a_{\alpha\beta}(x)\to a_{\alpha\beta}'(x')=a_{\alpha\beta}(x). \tag{6.13}$$

Indeed, it follows from (6.13) that

$$\omega_{\alpha\beta}(x)\to\omega_{\alpha\beta}'(x')=\omega_{\alpha\beta}(x)$$

and therefore

$$L_k=-\int d^3x' s'_\alpha(x')\omega'_\alpha(x')=-\int d^3xs'_\alpha(x)\omega'_\alpha(x). \tag{6.14}$$

In accordance with (6.14), the variations of the dynamical variables s, a under infinitesimal transformations $x_i\to x'_i=x_i+\delta x_i(x)$ must be defined by

$$\delta s_\alpha(x)=s'_\alpha(x)-s_\alpha(x),\quad \delta a_{\alpha\beta}=a'_{\alpha\beta}(x)-a_{\alpha\beta}(x). \tag{6.15}$$

With allowance for (6.13), for the variations (6.15) we obtain

$$\delta s_\alpha(x)=-\nabla_k(s_\alpha(x)\delta x_k(x)),$$

$$\delta a_{\alpha\beta}(x)=-\delta x_k(x)\nabla_k a_{\alpha\beta}(x). \tag{6.16}$$

The generator of the transformations (6.16) has the form

$$G=-\frac{1}{2}\varepsilon_{\alpha\beta\gamma}\int d^3xs_\alpha(x)(\tilde{a}\delta a)_{\gamma\beta}$$

$$=\int d^3xs_\alpha(x)\omega_{\alpha k}(x)\delta x_k(x)$$

or

$$G=\int d^3x\pi^s_k(x)\delta x_k(x),\quad \pi^s_k\equiv s_\alpha\omega_{\alpha k}. \tag{6.17}$$

Here $\pi^s_k(x)$, which determines the generator of the spatial translations in the space of the variables s and a, is the momentum density of the magnons. Since for canonical transformations we have the representation

$$\delta s_\alpha(x)=\{s_\alpha(x),G\},\quad \delta a_{\alpha\beta}(x)=\{a_{\alpha\beta}(x),G\},$$

we obtain, substituting here the expressions for the variations (6.16) and remembering that the functions $\delta x_i(x)$ are arbitrary, the PB

$$\{s_\alpha(x),\pi^s_k(x')\}=s_\alpha(x')\nabla'_k\delta(x-x'),$$

$$\{a_{\alpha\beta}(x),\pi^s_k(x')\}=-\delta(x-x')\nabla_k a_{\alpha\beta}(x). \tag{6.18}$$

To find the PB $\{\pi^s_k(x),\pi^s_l(x)\}$, we write down the variation of the left Cartan form under the transformations (6.16). It follows from the definition (6.2) that

$$\delta\omega_{\alpha k}(x)=-\delta x_l(x)\nabla_l\omega_{\alpha k}(x)-\omega_{\alpha l}(x)\nabla_k\delta x_l(x). \tag{6.19}$$

Taking into account the expression for π^s_k, we obtain from this

$$\delta\pi^s_k(x)=-\pi^s_l(x)\nabla_k\delta x_l(x)-\nabla_l(\pi^s_k(x)\delta x_l(x)).$$

On the other hand, representing the variation $\delta\pi^s_k$ in the form

$$\delta\pi^s_k(x)=\{\pi^s_k(x),G\},\quad G=\int d^3x\pi^s_l(x)\delta x_l(x) \tag{6.20}$$

and comparing the left- and right-hand sides of (6.20), and using the arbitrariness of the functions $\delta x_i(x)$, we find the PB

$$\{\pi^s_k(x),\pi^s_l(x')\}=\pi^s_l(x)\nabla'_k\delta(x-x')-\pi^s_k(x')\nabla_l\delta(x-x'). \tag{6.21}$$

As follows from (6.21), the PB for the magnon momentum density have the standard form [see (2.12)].

In the general case, the energy density of the considered magnetic system is a functional of the spin densities $s_\alpha(x)$ and of the orthogonal rotation matrix $a(x)$:

$$\varepsilon(x)=\varepsilon(x,s(x'),a(x')). \tag{6.22}$$

The equations of motion corresponding to the PB (6.8), (6.12) and to the functional expression (6.22) can be written in the form

$$\dot{s}_\alpha=\varepsilon_{\alpha\beta\gamma}\left(\frac{\delta H}{\delta s_\beta}s_\gamma+\frac{\delta H}{\delta a_{\rho\beta}}a_{\rho\gamma}\right),$$

$$\dot{a}_{\alpha\beta}=a_{\alpha\rho}\varepsilon_{\rho\beta\gamma}\frac{\delta H}{\delta s_\gamma} \tag{6.23}$$

and are a generalization of the Landau–Lifshitz equation to magnetic systems with complete spontaneous symmetry breaking. As follows from (6.23), in the general case s^2 is not conserved. Note that from the algebra (6.8), (6.12) we can separate the subalgebra of the PB of only the spin variables $s_\alpha(x)$. This corresponds to consideration of a ferromagnet. A closed dynamics for the variables s_α is obtained under the assumption that the Hamiltonian of the system does not depend on the rotation matrix $a_{\alpha\beta}$. Formulation of a Hamiltonian approach solely for the variables s_α, without inclusion of the rotation matrix $a_{\alpha\beta}(\varphi)$, is not possible, since in this case the number of independent dynamical variables s_α is odd.

We now write the dynamical equations in local form [in contrast to Eqs. (6.23) in nonlocal form]. We shall assume that the energy density is a function of the variables s, a, ∇a or, which is the same thing, a function of the variables s, a, ω_k:

$$\varepsilon(x;s(x'),a(x'))=\varepsilon(s(x),a(x),\omega_k(x)). \tag{6.24}$$

In addition, since the main interactions in the system have an exchange nature, we shall assume that the energy density is invariant with respect to homogeneous rotations in the spin space that are described by the matrix b:

$$\varepsilon(x;bs(x'),a(x')\tilde{b})=\varepsilon(x;s(x'),a(x')). \tag{6.25}$$

It follows from (6.25) that

$$\varepsilon(s,\omega_k,a)=\varepsilon(as,a\omega_k,1)\equiv\varepsilon(\underline{s},\underline{\omega}_k), \tag{6.26}$$

where $\underline{s}=as$, $\underline{\omega}=a\omega_k$. Taking into account (6.2), (6.8), and (6.12), we obtain the relations

$$\{s_\alpha(x),\underline{\omega}_{\beta k}(x')\}=a_{\beta\alpha}(x')\nabla'_k\delta(x-x'),$$

$$\{s_\alpha(x),\underline{s}_\beta(x')\}=0,$$

which lead to the property of invariance of $\underline{s}$ and $\underline{\omega}_k$ with respect to global rotations:

$$\{S_\alpha,\underline{\omega}_{\beta k}(x)\}=\{S_\alpha,\underline{s}_\beta(x)\}=0,$$

and this corresponds to invariance of the energy density (6.26):

$$\{S_\alpha,\varepsilon(x)\}=0,\quad S_\alpha=\int d^3xs_\alpha(x). \tag{6.27}$$

Using (6.27), we can write down a differential conservation law for the spin density $s_\alpha(x)$ in the form (1.17). Calculating the PB in (1.17) and using (6.24), (6.8), and (6.12), we obtain for the spin flux density $j_{\alpha k}$ the expression

$$j_{\alpha k}=\frac{\partial\varepsilon}{\partial\omega_{\alpha k}}.$$

The differential energy conservation law of the system has the form (1.18). Calculation of the PB in (1.18) leads to the following expression for the energy flux density:

$$q_k=\frac{\partial\varepsilon}{\partial s_\alpha}\frac{\partial\varepsilon}{\partial\omega_{\alpha k}}.$$

Earlier, we obtained an expression for the generator of the infinitesimal spatial translations (6.16). From (6.16) and (6.19) we obtain the condition of translational invariance of the energy density:

$$\{P_k,\varepsilon(x)\}=\nabla_k\varepsilon(x),\quad P_k\equiv\int d^3x s_\alpha\omega_{\alpha k}.$$

From this, we obtain for the magnon momentum density π_k^s the differential conservation law (1.16). Calculation of the PB leads to the expression

$$t_{ik}=-\left(\varepsilon-s_\alpha\frac{\partial\varepsilon}{\partial s_\alpha}\right)\delta_{ik}+\omega_{\alpha i}\frac{\partial\varepsilon}{\partial\omega_{\alpha k}}.$$

Here t_{ik} is the magnon momentum flux density (stress tensor).

Thus, the dynamical equations for magnets with spontaneously broken symmetry have in the long-wavelength case the form

$$\dot{s}_\alpha=-\nabla_k\frac{\partial\varepsilon}{\partial\omega_{\alpha k}},\quad \dot{a}_{\alpha\beta}=a_{\alpha\rho}\varepsilon_{\rho\beta\gamma}\frac{\partial\varepsilon}{\partial s_\gamma}.$$

From this and (6.24) we obtain

$$\dot{\varepsilon}=-\nabla_k\frac{\partial\varepsilon}{\partial s_\alpha}\frac{\partial\varepsilon}{\partial\omega_{\alpha k}},$$

in agreement with the expression for the energy flux density q_k.

By virtue of the definition of the energy density (6.26), it is expedient to choose as independent variables $\underline{s}$ and $\underline{\omega}_k$, in terms of which

$$\dot{\varepsilon}=-\nabla_k\frac{\partial\varepsilon}{\partial\underline{s}_\alpha}\frac{\partial\varepsilon}{\partial\underline{\omega}_{\alpha k}},\quad \dot{a}_{\alpha\beta}=\varepsilon_{\alpha\rho\gamma}a_{\rho\beta}\frac{\partial\varepsilon}{\partial\underline{s}_\gamma},$$

$$\dot{\underline{s}}_\alpha=-\nabla_k\frac{\partial\varepsilon}{\partial\underline{\omega}_{\alpha k}}+\varepsilon_{\alpha\beta\gamma}\left(\underline{s}_\beta\frac{\partial\varepsilon}{\partial\underline{s}_\gamma}+\underline{\omega}_{\beta k}\frac{\partial\varepsilon}{\partial\underline{\omega}_{\gamma k}}\right).\qquad(6.28)$$

From the equation of motion for the matrix a in (6.28) there follows the equation of motion for $\underline{\omega}_{\alpha k}$:

$$\dot{\underline{\omega}}_{\alpha k}=-\nabla_k\frac{\partial\varepsilon}{\partial\underline{s}_\alpha}+\varepsilon_{\alpha\beta\gamma}\underline{\omega}_{\beta k}\frac{\partial\varepsilon}{\partial\underline{s}_\gamma}.\qquad(6.29)$$

Equations (6.28) determine the dynamical properties of the system with neglect of the dissipative processes and describe the low-frequency dynamics of a many-sublattice magnet with exchange interaction when at sufficiently long times the strong exchange results in the formation of rigid spin complexes that are practically subject to no deformation and have orientation given by the rotation matrix a.

7. UNIAXIAL HELICAL MAGNET

We consider the case of a uniaxial helical magnet when the energy density as a function of the spin vector $\underline{s}_\alpha$ and $\underline{\omega}_{\alpha k}$ has the form

$$\varepsilon=\varepsilon(l_\alpha\underline{s}_\alpha,l_\alpha\underline{\omega}_{\alpha k})\equiv\varepsilon(s,p_k),\qquad(7.1)$$

where l_α is the unit vector of spontaneous "anisotropy" and is independent of the coordinates and the time [we emphasize that, as before, the energy density (7.1) is invariant with respect to spin rotations]. The algebra of the dynamical variables—the component of the spin s along the anisotropy axis and the helical vector p_k—can be obtained with allowance for the definition (7.1) from the more general algebra of the variables $\underline{s}_\alpha$ and $\underline{\omega}_{\alpha k}$,

$$\{\underline{s}_\alpha(x),\underline{s}_\beta(x')\}=-\varepsilon_{\alpha\beta\gamma}\underline{s}_\gamma\delta(x-x'),$$

$$\{\underline{\omega}_{\alpha k}(x),\underline{\omega}_{\beta l}(x')\}=0,$$

$$\{\underline{s}_\alpha(x),\underline{\omega}_{\beta k}(x')\}=\varepsilon_{\alpha\nu\beta}\underline{\omega}_{\nu k}(x)\delta(x-x')+\delta_{\alpha\beta}\nabla'_k\delta(x-x'),$$

and has the form

$$\{s(x),p_k(x')\}=\nabla'_k\delta(x-x'),$$

$$\{s(x),s(x')\}=\{p_k(x),p_l(x')\}=0.$$

In accordance with (6.28) and (6.29), s and p_k satisfy the equations of motion

$$\dot{s}=-\nabla_k\frac{\partial\varepsilon}{\partial p_k},\quad \dot{p}_k=-\nabla_k\frac{\partial\varepsilon}{\partial s}.\qquad(7.2)$$

The transverse values of the spin, $\underline{s}_\alpha^\perp$, and of the Cartan form, $\underline{\omega}_{\alpha k}^\perp$, defined by

$$\underline{s}_\alpha=sl_\alpha+\underline{s}_\alpha^\perp,\quad \underline{\omega}_{\alpha k}=l_\alpha p_k+\underline{\omega}_{\alpha k}^\perp$$

satisfy equations of the form

$$\dot{\underline{s}}_\alpha^\perp=\varepsilon_{\alpha\beta\gamma}l_\gamma\left(\underline{s}_\beta^\perp\frac{\partial\varepsilon}{\partial s}+\underline{\omega}_{\beta k}^\perp\frac{\partial\varepsilon}{\partial p_k}\right),\quad \dot{\underline{\omega}}_{\alpha k}^\perp=\varepsilon_{\alpha\beta\gamma}l_\gamma\frac{\partial\varepsilon}{\partial s}\underline{\omega}_{\beta k}^\perp.$$

Using the restrictions imposed on the Cartan forms $\underline{\omega}_{\alpha k}$ by the Maurer–Cartan relations

$$\nabla_k\underline{\omega}_{\alpha i}-\nabla_i\underline{\omega}_{\alpha k}=\varepsilon_{\alpha\beta\gamma}\underline{\omega}_{\beta k}\underline{\omega}_{\gamma i},$$

we readily obtain the equations

$$\nabla_k p_i-\nabla_i p_k=\varepsilon_{\alpha\beta\gamma}l_\gamma\underline{\omega}_{\alpha k}^\perp\underline{\omega}_{\beta i}^\perp,$$

$$\nabla_k\underline{\omega}_{\gamma i}^\perp-\nabla_i\underline{\omega}_{\gamma k}^\perp=(\delta_{\gamma\lambda}-l_\gamma l_\lambda)\varepsilon_{\lambda\alpha\beta}\underline{\omega}_{\alpha k}^\perp\underline{\omega}_{\beta i}^\perp$$
$$+\varepsilon_{\alpha\beta\gamma}(\underline{\omega}_{\alpha k}^\perp l_\beta p_i+\underline{\omega}_{\beta i}^\perp l_\alpha p_k).\qquad(7.3)$$

It is easily seen that the final relation contains the trivial solution $\underline{\omega}_{\gamma i}^\perp=0$, as a consequence of which the condition (7.3) for the helical vector p_k takes the form

$$\operatorname{curl}\mathbf{p}=0.$$

For the transverse component $\underline{s}_\alpha^\perp$ of the spin, we have the equation of motion

$$\dot{s}^{\perp}_{\alpha}=\varepsilon_{\alpha\beta\gamma}l_{\gamma}\underline{s}^{\perp}_{\beta}\frac{\partial\varepsilon}{\partial s}. \tag{7.4}$$

Linearization of Eqs. (7.2) and (7.4) with respect to the equilibrium values $s=s^0$, $p_k=p_k^0$ leads to the two Goldstone modes[44]

$$\omega_{\pm}(k)=k_i\frac{\partial^2\varepsilon}{\partial p_i\partial_s}\pm\sqrt{\frac{\partial^2\varepsilon}{\partial s^2}\frac{\partial^2\varepsilon}{\partial p_i\partial p_l}k_ik_l}$$

and the one activation mode

$$\omega^2=\left(\frac{\partial\varepsilon}{\partial s}\right)^2.$$

8. ANTIFERROMAGNETS

We now consider the case[45] when a dependence on the rotation matrix $a_{\alpha\beta}$ in the Hamiltonian of the system enters only through the combination $\underline{l}_{\beta}a_{\beta\alpha}\equiv l_{\alpha}$ ($\underline{l}_{\beta}$ is a certain constant vector)

$$\varepsilon=\varepsilon(s_{\alpha},\underline{l}_{\beta}a_{\beta\alpha})\equiv\varepsilon(s_{\alpha},l_{\alpha}). \tag{8.1}$$

The antiferromagnetism vector l_{α} and the spin density s_{α} are the main dynamical variables in terms of which the hydrodynamic (low-frequency) theory of antiferromagnets is constructed. The algebra of the variables s_{α} and l_{α} can be obtained with allowance for the definition (8.1) from the more general algebra (6.8), (6.12) of the variables $a_{\alpha\beta}$ and s_{α} and has the form

$$\{s_{\alpha}(x),s_{\beta}(x')\}=\varepsilon_{\alpha\beta\gamma}s_{\gamma}(x)\delta(x-x'),$$
$$\{s_{\alpha}(x),l_{\beta}(x')\}=\varepsilon_{\alpha\beta\gamma}l_{\gamma}(x)\delta(x-x'),$$
$$\{l_{\alpha}(x),l_{\beta}(x')\}=0. \tag{8.2}$$

Setting successively in (1.6) $\varphi_{\alpha}(x)=s_{\alpha}(x)$ and $\varphi_{\alpha}(x)=l_{\alpha}(x)$, we obtain the equations of motion for the spin s_{α} and for the antiferromagnetism vector l_{α} in the form

$$\dot{s}_{\alpha}=\varepsilon_{\alpha\beta\gamma}\left(\frac{\delta H}{\delta s_{\beta}}s_{\gamma}+\frac{\delta H}{\delta l_{\beta}}l_{\gamma}\right),$$
$$\dot{l}_{\alpha}=\varepsilon_{\alpha\beta\gamma}\frac{\delta H}{\delta s_{\beta}}l_{\gamma}. \tag{8.3}$$

The integration of (8.3) is significantly simplified in the long-wavelength limit, in which the spatial inhomogeneities of the dynamical variables are weak. Note that in obtaining the algebra (8.2) we assumed that the antiferromagnetism vector has arbitrary length. Such a description is equivalent to the description of antiferromagnets in terms of a unit vector l_{α} if it is assumed that the energy functional of the system depends on l_{α} only through the ratio l_{α}/l. In what follows, we shall everywhere make the explicit assumption that l_{α} is a unit vector: $l_{\alpha}^2=1$. At the same time, since l_{α}^2 is an integral of the motion, the PB (8.2) also retains the same form for the case $l_{\alpha}^2=1$. Assuming that the energy density is a function of $s_{\alpha},l_{\alpha},\nabla_k l_{\alpha}$ or, which is the same thing, a function of $s_{\alpha},l_{\alpha},v_{\alpha k}\equiv-\varepsilon_{\alpha\beta\gamma}l_{\beta}\nabla_k l_{\gamma}$,

$$\varepsilon(x;s(x'),l(x'))=\varepsilon(s(x),l(x),v_k(x)),$$

we obtain dynamical equations in local form. Since the variations of l_{α} and $v_{\alpha k}$ are not independent,

$$l_{\alpha}\delta l_{\alpha}=0,\quad l_{\alpha}\delta v_{\alpha k}+v_{\alpha k}\delta l_{\alpha}=0,$$

we fix the derivatives $\partial a/\partial l_{\alpha},\partial a/\partial v_{\alpha k}$ (a is an arbitrary function of s,l,v_k) by the additional requirements

$$l_{\alpha}\frac{\partial a}{\partial l_{\alpha}}=0,\quad l_{\alpha}\frac{\partial a}{\partial v_{\alpha k}}=0.$$

With allowance for this and the expressions (1.17) and (1.18), we represent the dynamical equations of antiferromagnets in the long-wavelength limit in the form

$$\dot{s}_{\alpha}=-\nabla_k\frac{\partial\varepsilon}{\partial v_{\alpha k}},\quad \dot{\varepsilon}=-\nabla_k\frac{\partial\varepsilon}{\partial s_{\alpha}}\frac{\partial\varepsilon}{\partial v_{\alpha k}},\quad \dot{l}_{\alpha}=\varepsilon_{\alpha\beta\gamma}\frac{\partial\varepsilon}{\partial s_{\beta}}l_{\gamma}. \tag{8.4}$$

It follows from this that

$$\dot{v}_{\alpha k}=-(\delta_{\alpha\beta}-l_{\alpha}l_{\beta})\nabla_k\frac{\partial\varepsilon}{\partial s_{\beta}}+\varepsilon_{\alpha\beta\gamma}\frac{\partial\varepsilon}{\partial s_{\beta}}v_{\gamma k}.$$

The complete system of equations (8.4) determines the dynamical properties of antiferromagnets with neglect of dissipative processes. Equations (8.4) admit helical solutions for which the dependence on the coordinates and time of the quantities $s_{\alpha}(x)$ and $l_{\alpha}(x)$ are determined by

$$\mathbf{s}(\mathbf{x},t)=\underline{\mathbf{s}}a(\mathbf{x},t),\quad \mathbf{l}(\mathbf{x},t)=\underline{\mathbf{l}}a(\mathbf{x},t), \tag{8.5}$$

where $\underline{\mathbf{s}}$ and $\underline{\mathbf{l}}$ are certain constant vectors, and the matrix $a(\mathbf{x},t)$ has the form

$$a(\mathbf{x},t)=\exp\varepsilon(\mathbf{px}-\underline{h}t)b. \tag{8.6}$$

Here

$$\varepsilon_{\mu\nu}\equiv\varepsilon_{\mu\lambda\nu}n_{\lambda},\quad \underline{h}n_{\lambda}\equiv\frac{\partial\varepsilon}{\partial\underline{s}_{\lambda}},$$

in which n_{λ} is a unit vector, and b is an arbitrary rotation matrix that does not depend on the coordinates or the time. In (8.6), p_k is the helical vector. From (8.4), we obtain restrictions on the independent values of $p_k,\underline{s}_{\alpha},\underline{l}_{\alpha},\underline{v}_{\alpha k}$:

$$\varepsilon_{\alpha\rho\nu}n_{\nu}\left\{\underline{s}_{\rho}\left(n_{\lambda}\frac{\partial\varepsilon}{\partial\underline{s}_{\lambda}}\right)-p_k\frac{\partial\varepsilon}{\partial\underline{v}_{\rho k}}\right\}=0.$$

Note that for the helical solutions (8.5) and (8.6) we have

$$\underline{v}_{\alpha k}\equiv a_{\alpha\beta}v_{\beta k}=(\delta_{\alpha\beta}-\underline{l}_{\alpha}\underline{l}_{\beta})n_{\beta}p_k.$$

Linearization of (8.4) near the equilibrium state $l_{\alpha}=l_{\alpha}^0$, $p_k=0$, $s_{\alpha}=0$ leads to the spin-wave spectrum[45]

$$\omega^2=c^2k^2,\quad c^2=\frac{1}{12}\frac{\partial^2\varepsilon}{\partial v_k^2}(\delta_{\alpha\beta}-l_{\alpha}l_{\beta})\frac{\partial^2\varepsilon}{\partial s_{\alpha}\partial s_{\beta}},$$

and this agrees with the result of Ref. 46.

9. MAGNETS WITH A QUADRUPOLE ORDER PARAMETER

In this section, we show how, proceeding from the standard expression for the action

$$W=\int_{t_1}^{t_2}dt\int d^3x\{c_{\alpha\beta}(x)\dot{a}_{\beta\alpha}(x)-H(c,a)\} \qquad (9.1)$$

($c_{\alpha\beta}$ and $a_{\alpha\beta}$ are dynamical variables, and H is the Hamiltonian), we can construct the dynamics of magnetically ordered systems with a quadrupole order parameter and relating the formal dynamical variables $c_{\alpha\beta}(x)$, $a_{\alpha\beta}(x)$ with the physical dynamical variables that are used in the theory of magnetism.[47] In contrast to the previous treatment, in (9.1) the matrix $a_{\alpha\beta}$ is understood as the matrix of an arbitrary affine transformation. Our reason for using the same notation will become clear from the following exposition. Using the known expressions for the PB of the dynamical variables a and c,

$$\{a_{\alpha\beta}(x),a_{\mu\nu}(x')\}=\{c_{\alpha\beta}(x),c_{\mu\nu}(x')\}=0,$$
$$\{a_{\alpha\beta}(x),c_{\mu\nu}(x')\}=\delta_{\alpha\nu}\delta_{\beta\mu}\delta(x-x'), \qquad (9.2)$$

we can find the PB for the physical dynamical variables of the quadrupole magnet.

We first obtain relations between the dynamical variables a and c and the dynamical variables that have direct physical meaning. We introduce the spin of the system as the generator of homogeneous infinitesimal rotations characterized by the angle $\delta\varphi_\alpha$. Since under the rotation described by the matrix b the matrices a and c transform in accordance with

$$a\to a'=a\tilde{b};\quad c\to c'=bc$$

[under these transformations, the kinematic part of the action $W_k=\int_{t_1}^{t_2}dt\int d^3x c_{\alpha\beta}(x)\dot{a}_{\beta\alpha}(x)$ remains invariant], it follows that, setting $b=1+\varepsilon$, $\varepsilon_{\mu\nu}\equiv\varepsilon_{\mu\lambda\nu}\delta\varphi_\lambda$, we obtain

$$\delta a_{\alpha\beta}=\varepsilon_{\gamma\beta\nu}a_{\alpha\gamma}\delta\varphi_\nu,\quad \delta c_{\alpha\beta}=\varepsilon_{\gamma\alpha\nu}c_{\gamma\beta}\delta\varphi_\nu. \qquad (9.3)$$

It is readily seen that the variations (9.3) can be represented in the form

$$\delta a_{\alpha\beta}(x)=\{a_{\alpha\beta}(x),G\},\quad \delta c_{\alpha\beta}(x)=\{c_{\alpha\beta}(x),G\},$$

where

$$G=\delta\varphi_\alpha\int d^3x\varepsilon_{\alpha\beta\gamma}c_{\gamma\nu}(x)a_{\nu\beta}(x)$$

is the generator of the transformations (9.3). Writing it in the form

$$G=\delta\varphi_\alpha\int d^3x s_\alpha(x),$$

we obtain for the spin density $s_\alpha(x)$ the expression

$$s_\alpha(x)=\varepsilon_{\alpha\beta\gamma}c_{\gamma\nu}(x)a_{\nu\beta}(x). \qquad (9.4)$$

In what follows, we shall find it convenient to introduce the tensor $g_{\alpha\beta}$:

$$g_{\alpha\beta}=c_{\alpha\nu}a_{\nu\beta}. \qquad (9.5)$$

Then the spin density can be expressed in terms of the antisymmetric part of $g_{\alpha\beta}$:

$$s_\alpha(x)=\varepsilon_{\alpha\beta\gamma}g^a_{\gamma\beta}(x),\quad g^a_{\mu\nu}\equiv\tfrac{1}{2}(g_{\mu\nu}-g_{\nu\mu}). \qquad (9.6)$$

In its turn, as follows from (9.6), the antisymmetric part of the tensor $g_{\alpha\beta}$ is uniquely determined by the spin density:

$$g^a_{\alpha\beta}=-\tfrac{1}{2}\varepsilon_{\alpha\beta\gamma}s_\gamma.$$

Denoting

$$g^s_{\alpha\beta}\equiv f_{\alpha\beta},\quad g^s_{\alpha\beta}=\tfrac{1}{2}(g_{\alpha\beta}+g_{\beta\alpha}), \qquad (9.7)$$

we write

$$g_{\alpha\beta}=f_{\alpha\beta}-\tfrac{1}{2}\varepsilon_{\alpha\beta\gamma}s_\gamma.$$

Thus, as dynamical variables of the system it is necessary to choose the spin density $s_\alpha(x)$, the matrix $a_{\alpha\beta}(x)$ of arbitrary affine transformations, and the symmetric matrix $f_{\alpha\beta}(x)$ (in what follows, we shall show that the matrix $f_{\alpha\beta}$ is the matrix of the quadrupole moment). Taking into account the representation (9.5) of the matrix $g_{\alpha\beta}$ and the expression (9.2), we readily obtain the following algebra for the variables $a_{\alpha\beta}(x)$, $g_{\alpha\beta}(x)$:

$$\{a_{\alpha\beta}(x),g_{\mu\nu}(x')\}=\delta_{\beta\mu}a_{\alpha\nu}(x)\delta(x-x'),$$
$$\{g_{\alpha\beta}(x),g_{\mu\nu}(x')\}=(g_{\alpha\nu}(x)\delta_{\beta\mu}-g_{\mu\beta}(x)\delta_{\alpha\nu})\delta(x-x'). \qquad (9.8)$$

The variables $s_\alpha(x)$, $f_{\alpha\beta}(x)$ are related to the variables $g_{\alpha\beta}(x)$ by (9.6) and (9.7). Hence and from (9.8) we find the PB for the dynamical variables $s_\alpha(x)$, $\alpha_{\alpha\beta}(x)$, $f_{\alpha\beta}(x)$:

$$\{f_{\alpha\beta}(x),f_{\mu\nu}(x')\}=\tfrac{1}{4}(\varepsilon_{\alpha\gamma\nu}\delta_{\beta\mu}+\varepsilon_{\beta\gamma\mu}\delta_{\alpha\nu}+\varepsilon_{\beta\gamma\nu}\delta_{\alpha\mu}+\varepsilon_{\alpha\gamma\mu}\delta_{\beta\nu})s_\gamma(x)\delta(x-x'),$$
$$\{s_\alpha(x),f_{\beta\gamma}(x')\}=(\varepsilon_{\alpha\beta\rho}f_{\gamma\rho}(x)+\varepsilon_{\alpha\gamma\rho}f_{\beta\rho}(x))\delta(x-x'),$$
$$\{s_\alpha(x),s_\beta(x')\}=\varepsilon_{\alpha\beta\gamma}s_\gamma(x)\delta(x-x'), \qquad (9.9)$$
$$\{a_{\alpha\beta}(x),s_\mu(x')\}=\varepsilon_{\beta\mu\rho}a_{\alpha\rho}(x)\delta(x-x'),$$
$$\{\alpha_{\alpha\beta}(x),\alpha_{\mu\nu}(x')\}=0,$$
$$\{a_{\alpha\beta}(x),f_{\mu\nu}(x')\}=\tfrac{1}{2}(\delta_{\beta\mu}a_{\alpha\nu}(x)+\delta_{\beta\nu}a_{\alpha\mu}(x))\delta(x-x'). \qquad (9.10)$$

The first three expressions (9.9) determine the subalgebra of the dynamical variables s and f. We give the physical interpretation of this subalgebra. To this end, we replace, in accordance with the general rules of quantum mechanics, the dynamical variables $s_\alpha, f_{\alpha\beta}$ by the operators $\hat{s}_\alpha, \hat{f}_{\alpha\beta}$ and the Poisson brackets {...,...} by the commutators $(1/i)$[...,...]. Then the relations (9.9) take the form

$$[\hat{f}_{\alpha\beta}(x),\hat{f}_{\mu\nu}(x')]=\frac{i}{4}(\varepsilon_{\alpha\gamma\nu}\delta_{\beta\mu}+\varepsilon_{\beta\gamma\mu}\delta_{\alpha\nu}+\varepsilon_{\beta\gamma\nu}\delta_{\alpha\mu}+\varepsilon_{\alpha\gamma\mu}\delta_{\beta\nu})\hat{s}_\gamma(x)\delta(x-x'),$$
$$[\hat{s}_\alpha(x),\hat{f}_{\beta\gamma}(x')]=i(\varepsilon_{\alpha\beta\rho}\hat{f}_{\gamma\rho}(x)+\varepsilon_{\alpha\gamma\rho}\hat{f}_{\beta\rho}(x))\delta(x-x'),$$
$$[\hat{s}_\alpha(x),\hat{s}_\beta(x')]=i\varepsilon_{\alpha\beta\gamma}\hat{s}_\gamma(x)\delta(x-x').$$

It is easy to realize this algebra in the language of spin matrices corresponding to spin $s=1$ $[(s_\alpha)_{\mu\nu}=i\varepsilon_{\alpha\mu\nu}]$, namely

$$\hat{f}_{\alpha\beta}=\frac{1}{2}\left(\hat{s}_\alpha\hat{s}_\beta+\hat{s}_\beta\hat{s}_\alpha-\frac{2}{3}\delta_{\alpha\beta}\hat{s}^2\right). \qquad (9.11)$$

Since the operator $\hat{f}_{\alpha\beta}$ is the operator of the quadrupole moment (see Ref. 48), we shall, on the basis of this, assume that $f_{\alpha\beta}$ in (9.9) is the matrix of the quadrupole moment of the spin $s=1$, and at the same time $\mathrm{Tr}\, f_{\alpha\beta}=f_{\alpha\alpha}=0$ [the relation $f_{\alpha\alpha}=0$ is compatible with the subalgebra (9.9)]. Using the PB (9.9), we obtain the dynamical equations for s_α and $f_{\alpha\beta}$ (quadrupole magnet):

$$\dot{s}_\alpha(x)=\varepsilon_{\alpha\beta\gamma}\frac{\delta H}{\delta s_\beta(x)}s_\gamma(x)+2\varepsilon_{\alpha\beta\rho}f_{\mu\rho}(x)\frac{\delta H}{\delta f_{\mu\beta}(x)},$$

$$\dot{f}_{\alpha\beta}(x)=\frac{1}{2}\frac{\delta H}{\delta f_{\gamma\nu}(x)}(\delta_{\gamma\beta}\varepsilon_{\nu\mu\alpha}+\delta_{\alpha\nu}\varepsilon_{\gamma\beta\mu})s_\mu(x)$$

$$+(\varepsilon_{\beta\gamma\rho}f_{\alpha\rho}(x)+\varepsilon_{\alpha\gamma\rho}f_{\beta\rho}(x))\frac{\delta H}{\delta s_\gamma(x)}.$$

From the general algebra (9.9)–(9.10), we can separate the subalgebra of the dynamical variables $a_{\alpha\beta}$ and s_α (this subalgebra does not contain $f_{\alpha\beta}$), which is compatible with the subsidiary condition $a\tilde{a}=1$. This subalgebra determines the low-frequency dynamics of a many-sublattice magnet (see Sec. 6).

The quantum-mechanical dynamics of a quadrupole magnet with spin $s=1$ was studied in Refs. 49 and 50. However, the authors considered only the dynamics of pure states in terms of four independent variables. In contrast, to describe the dynamics of a quadrupole magnet, we use, taking into account the representation (9.11), the complete set of expectation values of the eight operators $\hat{s}_\alpha$ and $\hat{f}_{\alpha\beta}$ that form the Lie algebra of SU(3). This agrees with the number of variables used in Ref. 51 to study the ground state and spectrum of a magnet with single-ion anisotropy for spin $s=1$.

10. CONNECTION BETWEEN THE HAMILTONIAN AND LAGRANGIAN FORMALISMS

As we have already noted, the dynamical equations for magnetic systems with complete spontaneous symmetry breaking with respect to spin rotations were formulated using model phenomenological Lagrangians in Refs. 25 and 42. In this section, we shall obtain the dynamical equations of such systems in the Lagrangian approach in general form without the use of any model assumptions.

The Lagrangian approach is extremely convenient for formulating the symmetry properties of a system. Introducing any particular symmetry, we obtain in explicit form integrals of the motion for the considered magnetic systems.

The Lagrangian of magnetic systems with complete symmetry breaking with respect to spin rotations is a functional of a rotation matrix a and also Cartan forms ω_α and $\omega_{\alpha k}$:

$$L=\int dx\mathscr{L}(a(x),\omega(x),\omega_k(x)). \tag{10.1}$$

In the exchange approximation, the Lagrangian density $\mathscr{L}$ is invariant with respect to right homogeneous rotations in the spin space:

$$\mathscr{L}(\omega,\omega_k,a)=\mathscr{L}(\omega b,\omega_k b,ab), \tag{10.2}$$

where b is an arbitrary rotation matrix. Setting $b=\tilde{a}$, we obtain

$$\mathscr{L}=\mathscr{L}(\underset{\sim}{\omega},\underset{\sim}{\omega}_k,1)\equiv\mathscr{L}(\underset{\sim}{\omega},\underset{\sim}{\omega}_k).$$

We introduce $\underset{\sim}{R}_\alpha$, which is related to the variation δa of the rotation matrix by

$$R_\alpha=\tfrac{1}{2}\varepsilon_{\alpha\beta\gamma}(\delta a\tilde{a})_{\gamma\beta}.$$

In terms of the introduced quantity, the variations $\delta\underset{\sim}{\omega}_\alpha$ and $\delta\underset{\sim}{\omega}_{\alpha k}$ of the Cartan forms are

$$\delta\underset{\sim}{\omega}_\alpha=\dot{\mathbf{R}}_\alpha+(\underset{\sim}{\mathbf{R}}\times\underset{\sim}{\boldsymbol{\omega}})_\alpha,\quad \delta\underset{\sim}{\omega}_{\alpha k}=\nabla_k\underset{\sim}{\mathbf{R}}_\alpha+(\underset{\sim}{\mathbf{R}}\times\underset{\sim}{\boldsymbol{\omega}}_k)_\alpha. \tag{10.3}$$

Using (10.3), we write down the total variation of the action $W=\int_{t_1}^{t_2}Ldt$:

$$\delta W=\int_{t_1}^{t_2}dt\int d^3x\left\{\frac{\partial}{\partial t}\left(\underset{\sim}{R}_\alpha\frac{\partial\mathscr{L}}{\partial\underset{\sim}{\omega}_\alpha}\right)+\frac{\partial}{\partial t}(\mathscr{L}\delta t)\right\}$$

$$+\int_{t_1}^{t_2}dt\int d^3x\underset{\sim}{R}_\alpha\left\{\left(\underset{\sim}{\boldsymbol{\omega}}\times\frac{\partial\mathscr{L}}{\partial\underset{\sim}{\boldsymbol{\omega}}}\right)_\alpha+\left(\underset{\sim}{\boldsymbol{\omega}}_k\times\frac{\partial\mathscr{L}}{\partial\underset{\sim}{\boldsymbol{\omega}}_k}\right)_\alpha\right.$$

$$\left.-\frac{\partial}{\partial t}\frac{\partial\mathscr{L}}{\partial\underset{\sim}{\omega}_\alpha}-\nabla_k\frac{\partial\mathscr{L}}{\partial\underset{\sim}{\omega}_{\alpha k}}\right\}.$$

Setting $\delta t=0$ and $\delta a(t)|_{t_1}=\delta a(t)|_{t_2}=0$, we obtain from the principle of stationary action the Lagrange equations

$$\frac{\partial}{\partial t}\frac{\partial\mathscr{L}}{\partial\underset{\sim}{\omega}_\alpha}+\nabla_k\frac{\partial\mathscr{L}}{\partial\underset{\sim}{\omega}_{\alpha k}}=\left(\underset{\sim}{\boldsymbol{\omega}}\times\frac{\partial\mathscr{L}}{\partial\underset{\sim}{\boldsymbol{\omega}}}\right)_\alpha+\left(\underset{\sim}{\boldsymbol{\omega}}_k\times\frac{\partial\mathscr{L}}{\partial\underset{\sim}{\boldsymbol{\omega}}_k}\right)_\alpha. \tag{10.4}$$

Near the actual motion of the system, the variation of the action has the form

$$\delta W=G(t_2)-G(t_1),\quad G(t)\equiv\int d^3x\left(\underset{\sim}{R}_\alpha\frac{\partial\mathscr{L}}{\partial\underset{\sim}{\omega}_\alpha}+\mathscr{L}\delta t\right). \tag{10.5}$$

We consider infinitesimal transformations associated with time translations. In this case $\delta W=0$ (since L does not depend explicitly on the time t) and $\underset{\sim}{R}_\alpha=-\underset{\sim}{\omega}_\alpha\delta t$. Therefore, from (10.5) we deduce the energy conservation law

$$H=\int d^3x\left(-\mathscr{L}+\underset{\sim}{\omega}_\alpha\frac{\partial\mathscr{L}}{\partial\underset{\sim}{\omega}_\alpha}\right)\equiv\int d^3x\varepsilon(x). \tag{10.6}$$

Under infinitesimal spatial translations, $\delta W=0$ and $\underset{\sim}{R}_\alpha=-\underset{\sim}{\omega}_{\alpha k}\delta x_k$. The corresponding integral of the motion—the momentum P_k—has the form

$$P_k=-\int d^3x\underset{\sim}{\omega}_{\alpha k}\frac{\partial\mathscr{L}}{\partial\underset{\sim}{\omega}_\alpha}.$$

We now consider infinitesimal spin rotations. Then $\delta a_{\gamma\lambda}=a_{\gamma\rho}\varepsilon_{\rho\lambda\sigma}\delta\varphi_\sigma$ and $\underset{\sim}{R}_\alpha=-a_{\alpha\beta}\delta\varphi_\beta$. In terms of the Lagrangian, the corresponding conserved quantity—the spin S_α—has the form

$$S_\alpha=-\int d^3x\frac{\partial\mathscr{L}}{\partial\omega_\alpha}.$$

By virtue of the relation (10.6), the energy density ε is related to the Lagrangian by

$$\varepsilon(s,\omega_k,a)=-\mathscr{L}(\omega(s,\omega_k),\omega_k,a)-s_\alpha\omega_\alpha(s,\omega_k),$$

from which we obtain the expressions

$$\left(\frac{\partial\varepsilon}{\partial s_\alpha}\right)_{\omega_k}=-\omega_\alpha,\quad \left(\frac{\partial\varepsilon}{\partial\omega_{\alpha k}}\right)_s=-\left(\frac{\partial\mathscr{L}}{\partial\omega_{\alpha k}}\right)_\omega=j_{\alpha k}.$$

Therefore, Eqs. (10.4) are equivalent to the system of equations

$$\dot a_{\alpha\beta}=a_{\alpha\rho}\varepsilon_{\rho\beta\lambda}\frac{\partial\varepsilon}{\partial s_\lambda},\quad \dot s_\alpha=-\nabla_k\frac{\partial\varepsilon}{\partial\omega_{\alpha k}},$$

which were obtained earlier in the framework of the Hamiltonian approach.

Under infinitesimal left spin rotations, $\delta a_{\alpha\beta}=\varepsilon_{\alpha\mu\rho}\delta\varphi_\rho a_{\mu\beta}$ and $\underset{\sim}{R}_\alpha=-\delta\varphi_\alpha$. If the Lagrangian is simultaneously invariant with respect to right and left spin rotations, then in accordance with (10.2) and

$$\mathscr{L}(\omega,\omega_k,a)=\mathscr{L}(b\omega,b\omega_k,ba)$$

we have

$$\left(\underset{\sim}{\omega}\times\frac{\partial\mathscr{L}}{\partial\underset{\sim}{\omega}}\right)_\alpha+\left(\underset{\sim}{\omega}_k\times\frac{\partial\mathscr{L}}{\partial\underset{\sim}{\omega}_k}\right)_\alpha=0,$$

and in the considered magnetic system not only the spin S_α but also

$$\underset{\sim}{S}_\alpha=-\int d^3x\,\frac{\partial\mathscr{L}}{\partial\underset{\sim}{\omega}_\alpha}$$

is conserved. As a concrete example of a phenomenological Lagrangian of magnetic systems with complete spontaneous symmetry breaking, we give the Lagrangian proposed in Ref. 25:

$$\mathscr{L}=\frac{1}{2}(a_{\alpha\beta}\underset{\sim}{\omega}_\alpha\underset{\sim}{\omega}_\beta+2b_\alpha\underset{\sim}{\omega}_\alpha-c_{\alpha i,\beta k}\omega_{\alpha i}\omega_{\beta k}-2d_{\alpha k}\omega_{\alpha k}),$$

where $a_{\alpha\beta},b_\alpha,c_{\alpha i,\beta k},d_{\alpha k}$ are certain phenomenological constants. The actual structure of these constants, corresponding to different magnetic media, is given in Ref. 42.

11. MAGNETOELASTIC SYSTEMS

In Secs. 1–5, we considered the PB algebra of the variables of a classical continuous medium and, as special cases, subalgebras of this general algebra and also the dynamics corresponding to these brackets. A similar treatment was given in Secs. 6–9 for a number of magnetically ordered systems. In the present section, we shall study magnetoelastic systems, the dynamical variables of which include both variables corresponding to continuous media $[u_i(x),\pi_i(x),\sigma(x),\psi(x)]$ and variables corresponding to magnetism $[s_{\alpha(x)},a_{\alpha\beta}(x)]$.

We represent the density of the kinematic part of the Lagrangian in the form

$$\mathscr{L}_k(x)=\pi_i^*(x)b_{ij}^{-1}\dot u_j(x)-\sigma(x)\dot\psi(x)-s_\alpha(x)\omega_\alpha(x),\tag{11.1}$$

where

$$\pi_i^*=\pi_i-\sigma\nabla_i\psi-s_\alpha\omega_{\alpha i}.$$

For the expression (11.1) we can make the same clarifying comments as for the expression (2.6). Namely, the final term in (11.1) is the density of the kinematic part of the Lagrangian for the magnetic systems. Accordingly, the momentum density π_i is the generator of spatial translations in the space of the variables $u_i,\pi_i,\sigma,\psi,s_\alpha,a_{\alpha\beta}$, the momentum density $\pi_i^\sigma=\sigma\nabla_i\psi$ is the same in the space of the variables σ, ψ, and the momentum density $\pi_i^s=s_\alpha\omega_{\alpha i}$ is the same in the space of the variables $s_\alpha,a_{\alpha\beta}$ [see (6.17)]. Therefore, the momentum density associated with translations in the space of the variables u_i and π_i has the form

$$\pi_i^*=\pi_i-\sigma\nabla_i\psi-s_\alpha\omega_{\alpha i},$$

and from this the structure of the density of the kinematic part (11.1) follows. To obtain the PB of the variables of the magnetoelastic medium, we rewrite the density of the kinematic part of the Lagrangian in the form

$$\mathscr{L}_k(x)=p_j(x)\dot u_j(x)-\sigma(x)\dot\psi(x)-s_\alpha(x)\omega_\alpha(x),$$

where

$$p_j=(\pi_i-\sigma\nabla_i\psi-s_\alpha\omega_{\alpha i})b_{ij}^{-1}\equiv\pi_i^*b_{ij}^{-1}.\tag{11.2}$$

Then, considering variations (2.7), (6.4), and (6.6) that leave the kinematic part of the Lagrangian invariant, we find the PB (2.10), (6.8), and (6.12), in which the generalized momentum p_j is now defined by (11.2). From the relations

$$\{\pi_i^*(x),\psi(x')\}=\{\pi_i^*(x),\sigma(x')\}=\{\pi_i^*(x),s(x')\}$$
$$=\{\pi_i^*(x),a(x')\}=0$$

and the PB (6.18) we obtain the relations (2.11) and the PB

$$\{\pi_i(x),a_{\alpha\beta}(x')\}=\delta(x-x')\nabla_i a_{\alpha\beta}(x),$$
$$\{\pi_i(x),s_\alpha(x')\}=-s_\alpha(x)\nabla_i\delta(x-x').$$

In turn, from the relations

$$\{p_i(x),p_j(x')\}=0,$$
$$\{\pi_i^b(x),\pi_k^b(x')\}=\pi_k^b(x)\nabla_i'\delta(x-x')-\pi_i^b(x')\nabla_k\delta(x-x'),$$
$$b=(\sigma,s)$$

we find the PB (2.12). Combining the results, we write the PB algebra of the dynamical variables of the magnetoelastic medium in the form

$$\{\pi_i(x),\sigma(x')\}=-\sigma(x)\nabla_i\delta(x-x'),$$
$$\{\pi_i(x),\psi(x')\}=\delta(x-x')\nabla_i\psi(x),$$
$$\{\sigma(x),\psi(x')\}=\delta(x-x'),$$
$$\{u_i(x),\pi_k(x')\}=b_{ik}(x)\delta(x-x'),$$
$$\{\pi_i(x),\pi_k(x')\}=\pi_k(x)\nabla_i'\delta(x-x')-\pi_i(x')\nabla_k\delta(x-x'),$$
$$\{\pi_i(x),a_{\alpha\beta}(x')\}=\delta(x-x')\nabla_i a_{\alpha\beta}(x),$$
$$\{\pi_i(x),s_\alpha(x')\}=-s_\alpha(x)\nabla_i\delta(x-x'),$$
$$\{s_\alpha(x),s_\beta(x')\}=\varepsilon_{\alpha\beta\gamma}s_\gamma(x)\delta(x-x'),$$
$$\{s_\alpha(x),a_{\beta\gamma}(x')\}=\varepsilon_{\alpha\gamma\rho}a_{\beta\rho}(x)\delta(x-x')\tag{11.3}$$

(we have given here only the nontrivial PB). As in the case of continuous media, we shall assume that the variable ψ is cyclic. Assuming that the Hamiltonian of the system is invariant with respect to spatial translations, we represent it in the form

$$H=\int d^3x\varepsilon(x),$$
$$\varepsilon(x)=\varepsilon(x;\sigma(x'),\pi_i(x'),b_{ij}(x'),s_\alpha(x'),a_{\alpha\beta}(x')). \tag{11.4}$$

The equations of motion for the variables of the magnetoelastic medium that correspond to (11.3) and (11.4) have the form

$$\dot{\sigma}(x)=-\nabla_i\left(\sigma(x)\frac{\delta H}{\delta\pi_i(x)}\right),\quad \dot{u}_i(x)=-b_{ij}(x)\frac{\delta H}{\delta\pi_j(x)},$$
$$\dot{\pi}_i(x)=-\pi_j(x)\nabla_i\frac{\delta H}{\delta\pi_j(x)}-\nabla_j\left(\pi_i(x)\frac{\delta H}{\delta\pi_j(x)}\right)-b_{ki}(x)\nabla_j\frac{\delta H}{\delta b_{kj}(x)}-\sigma(x)\nabla_i\frac{\delta H}{\delta\sigma(x)}-s_\alpha(x)\nabla_i\frac{\delta H}{\delta s_\alpha(x)}+\frac{\delta H}{\delta a_{\alpha\beta}(x)}\nabla_i a_{\alpha\beta}(x),$$
$$\dot{s}_\alpha(x)=-\nabla_i\left(s_\alpha(x)\frac{\delta H}{\delta\pi_i(x)}\right)+\varepsilon_{\alpha\beta\gamma}\left(\frac{\delta H}{\delta s_\beta(x)}s_\gamma(x)+\frac{\delta H}{\delta a_{\rho\beta}(x)}a_{\rho\gamma}(x)\right),$$
$$\dot{a}_{\alpha\beta}(x)=-\frac{\delta H}{\delta\pi_i(x)}\nabla_i a_{\alpha\beta}(x)+a_{\alpha\rho}(x)\varepsilon_{\rho\beta\gamma}\frac{\delta H}{\delta s_\gamma(x)}. \tag{11.5}$$

The structure of (11.5) simplifies appreciably in the long-wavelength limit, in which the spatial inhomogeneities of the dynamical variables are weak. Assuming that the energy density is a function of the variables $\sigma,\pi_i,b_{ik},s_\alpha,a_{\alpha\beta},\nabla_k a_{\alpha\beta}$ (or, which is the same thing, not $\nabla a_{\alpha\beta}$ but the left Cartan form $\omega_{\alpha k}$), we obtain dynamical equations in local form. If we assume that the energy density is invariant with respect to spin rotations, $\{S_\alpha,\varepsilon(x)\}=0$, then we obtain a differential conservation law for the spin density $s_\alpha(x)$ with flux $j_{\alpha k}$ given by (1.17). Calculation of the PB in (1.17) gives

$$\dot{s}_\alpha=-\nabla_k j_{\alpha k},\quad j_{\alpha k}=s_\alpha\frac{\partial\varepsilon}{\partial\pi_k}+\frac{\partial\varepsilon}{\partial\omega_{\alpha k}}. \tag{11.6}$$

The differential conservation law for the energy density has the form (1.18). On the basis of the algebra (11.3), we find

$$\dot{\varepsilon}=-\nabla_k q_k,\quad q_k=\frac{\partial\varepsilon}{\partial\pi_k}\left(\sigma\frac{\partial\varepsilon}{\partial\sigma}+\pi_l\frac{\partial\varepsilon}{\partial\pi_l}+s_\alpha\frac{\partial\varepsilon}{\partial s_\alpha}\right)+\frac{\partial\varepsilon}{\partial\omega_{\alpha k}}\left(\frac{\partial\varepsilon}{\partial s_\alpha}+\omega_{\alpha i}\frac{\partial\varepsilon}{\partial\pi_i}\right)+b_{ij}\frac{\partial\varepsilon}{\partial b_{ik}}\frac{\partial\varepsilon}{\partial\pi_j}. \tag{11.7}$$

Finally, if the energy density $\varepsilon(x)$ has the property of translational invariance, $\{P_i,\varepsilon(x)\}=\nabla_i\varepsilon(x)$, we obtain from this the differential conservation law for the momentum density (1.16). Calculation of the PB leads to the result

$$\dot{\pi}_i=-\nabla_k t_{ik},\quad t_{ik}=p\delta_{ik}+\pi_i\frac{\partial\varepsilon}{\partial\pi_k}+\omega_{\alpha i}\frac{\partial\varepsilon}{\partial\omega_{\alpha k}}+b_{ji}\frac{\partial\varepsilon}{\partial b_{jk}},$$
$$p\equiv-\varepsilon+\sigma\frac{\partial\varepsilon}{\partial\sigma}+\pi_l\frac{\partial\varepsilon}{\partial\pi_l}+s_\alpha\frac{\partial\varepsilon}{\partial s_\alpha}. \tag{11.8}$$

Here t_{ik} is the stress tensor.

If the energy density $\varepsilon(x)$ does not depend on the rotation matrix $a_{\alpha\beta}$, then the equations of motion (11.6)–(11.8) are identical to the corresponding equations of motion in Ref. 52 if in the latter we ignore the influence of electromagnetic fields.

For an energy density invariant with respect to spin rotations, we have [see (6.26)]

$$\varepsilon(\ldots,s,\omega_k,a)=\varepsilon(\ldots,\underline{s},\underline{\omega}_k)$$

(we indicate the dependence on only the variables that change under the considered transformations), and therefore it is expedient to choose as independent variables the right variables $\underline{s}$ and $\underline{\omega}_k$. Then, using the PB (11.3), we obtain the dynamical equations of the magnetoelastic medium in the long-wavelength limit in the form

$$\dot{\sigma}=-\nabla_k\left(\sigma\frac{\partial\varepsilon}{\partial\pi_k}\right),\quad \dot{\pi}_i=-\nabla_k\left(p\delta_{ik}+\pi_i\frac{\partial\varepsilon}{\partial\pi_k}+\underline{\omega}_{\alpha i}\frac{\partial\varepsilon}{\partial\underline{\omega}_{\alpha k}}+b_{ji}\frac{\partial\varepsilon}{\partial b_{jk}}\right),$$
$$\dot{b}_{ik}=-\nabla_k\left(b_{ij}\frac{\partial\varepsilon}{\partial\pi_j}\right),\quad \dot{\underline{s}}_\alpha=-\nabla_k\left(\underline{s}_\alpha\frac{\partial\varepsilon}{\partial\pi_k}+\frac{\partial\varepsilon}{\partial\underline{\omega}_{\alpha k}}\right)+\varepsilon_{\alpha\beta\gamma}\left(\underline{s}_\beta\frac{\partial\varepsilon}{\partial\underline{s}_\gamma}+\underline{\omega}_{\beta k}\frac{\partial\varepsilon}{\partial\underline{\omega}_{\gamma k}}\right),$$
$$\dot{\underline{\omega}}_{\alpha k}=-\nabla_k\left(\frac{\partial\varepsilon}{\partial\underline{s}_\alpha}+\underline{\omega}_{\alpha l}\frac{\partial\varepsilon}{\partial\pi_l}\right)+\varepsilon_{\alpha\beta\gamma}\underline{\omega}_{\beta k}\left(\frac{\partial\varepsilon}{\partial\underline{s}_\gamma}+\underline{\omega}_{\gamma l}\frac{\partial\varepsilon}{\partial\pi_l}\right), \tag{11.9}$$

where

$$p\equiv-\varepsilon+\sigma\frac{\partial\varepsilon}{\partial\sigma}+\pi_l\frac{\partial\varepsilon}{\partial\pi_l}+\underline{s}_\alpha\frac{\partial\varepsilon}{\partial\underline{s}_\alpha}$$

is the pressure. The complete system of equations (11.9) determines the dynamical properties of the magnetoelastic medium when dissipative processes are ignored.

12. QUANTUM CRYSTALS

Hitherto, we have considered normal systems. We now turn to the study of superfluid systems for which the equilibrium state is a state with spontaneously broken symmetry with respect to phase transformations. We first consider the derivation of the PB algebra of the variables of a quantum crystal, and from this algebra we then separate the subalgebra corresponding to superfluid ^{4}He. We shall then take into account the effect of spin degrees of freedom and describe the dynamics of a quantum spin crystal in the ground state of which invariance with respect to spin rotations and also the translational and phase invariance are broken. As a subalgebra, we shall here separate the PB algebra of the variables of superfluid ^{3}He−B.

Let us consider a quantum crystal. The phenomenon of superfluidity of a quantum crystal[53] arises from the possibility of two forms of motion. The first of them is the motion of the sites of a lattice characteristic of a solid, and the second type is the transport of mass by quasiparticles with fixed lattice sites that is characteristic of a superfluid liquid. Linear dynamical equations of a quantum crystal were obtained in a phenomenological approach in Ref. 53, and the studies of Refs. 54 and 55 were devoted to a nonlinear generalization of them. We shall obtain nonlinear dynamical equations of quantum crystals on the basis of the Hamiltonian approach.

The variables that describe the breaking of the phase and translational invariance of the equilibrium state of a quantum crystal are the superfluid phase $\phi(x)$ and the displacement vector $u_i(x)$ of the lattice sites in the configuration space. Therefore, the energy density $\varepsilon(x)$ in the general case is a functional of the following variables: the superfluid phase $\phi(x)$, the displacement vector $u_i(x)$, and the densities of the entropy $\sigma(x)$, mass $\rho(x)$, and momentum $\pi_i(x)$. Thus,

$$\varepsilon(x)=\varepsilon(x;\sigma(x'),\rho(x'),\pi_i(x'),\phi(x'),u_i(x')). \tag{12.1}$$

We now obtain the PB of the dynamical variables of the system. We first make the following remark. It is well known[56] that in continuum mechanics the variables $\rho(x)$ and $u_i(x)$ are not independent but are related by

$$\rho=\tilde{\rho}\det\|\delta_{ij}-\nabla_j u_i\|, \tag{12.2}$$

where $\tilde{\rho}$ is the density of the undeformed matter. However, since we consider the quantum-crystal phase, for which the number of atoms and the number of lattice sites are not equal,[53] the variables $\rho(x)$ and $u_i(x)$ must be regarded as independent, so that the relation (12.2) no longer holds for them. Therefore, the PB of these variables must also be specified independently.

We write the density of the kinematic part of the Lagrangian in the form

$$\mathscr{L}_k(x)=\pi_i^*(x)b_{ij}^{-1}\dot{u}_j(x)-\sigma(x)\dot{\psi}(x)-\rho(x)\dot{\phi}(x), \tag{12.3}$$

where

$$\pi_i^*=\pi_i-\sigma\nabla_i\psi(x)-\rho\nabla_i\phi.$$

The expression (12.3) is constructed in the same way as was done in the case of classical continuous media (see the explanation on the inclusion of the term $\sigma\dot{\psi}$). Namely, the final term in (12.3) takes into account the effect of the superfluidity; accordingly, in the momentum density π_i^* associated with spatial translations in the space of the variables u_i and π_i it is necessary to take into account the additional term $\rho\nabla_i\phi$. To obtain the PB, it is also necessary, as for classical continuous media, to consider variations (2.7) with generalized momentum $p_j=\pi_i^* b_{ij}^{-1}$ and, besides them, variations

$$\delta\rho(x)=0,\quad \delta\phi(x)=g(x).$$

To these variations there corresponds the generator

$$G=\int d^3x(p_i f_i-\sigma\chi-\rho g).$$

Making the further calculations like those in Sec. 2, we arrive at the following PB algebra of the variables of quantum crystals:

$$\begin{aligned}
&\{\pi_i(x),u_k(x')\}=-(\delta_{ik}-\nabla_i u_k(x))\delta(x-x'),\\
&\{\pi_i(x),\sigma(x')\}=-\sigma(x)\nabla_i\delta(x-x'),\\
&\{\pi_i(x),\rho(x')\}=-\rho(x)\nabla_i\delta(x-x'),\\
&\{\pi_i(x),\phi(x')\}=\delta(x-x')\nabla_i\phi(x),\\
&\{\pi_i(x),\psi(x')\}=\delta(x-x')\nabla_i\psi(x),\\
&\{\sigma(x),\psi(x')\}=\{\rho(x),\phi(x')\}=\delta(x-x'),\\
&\{\pi_i(x),\pi_k(x')\}=\pi_k(x)\nabla_i'\delta(x-x')-\pi_i(x')\nabla_k\delta(x-x').
\end{aligned} \tag{12.4}$$

In all that follows, the variable ψ will be assumed to be cyclic. Note that by virtue of the invariance of $\varepsilon(x)$ with respect to global phase transformations and spatial translations, the energy density $\varepsilon(x)$ depends not on the variables $\phi(x)$ and $u_i(x)$ themselves but only on their derivatives $\nabla_i\phi\equiv p_i,\nabla_k u_i$ (or, in the last case, on $b_{ik}=\delta_{ik}-\nabla_k u_i$):

$$\varepsilon(x)=\varepsilon(x;\sigma(x'),\rho(x'),\pi_i(x'),p_i(x'),b_{ik}(x')). \tag{12.5}$$

The vector p is the superfluid momentum. It is therefore convenient to choose as the dynamical variables, in addition to the remaining variables, p_i and b_{ik} directly. Using the PB (12.4), we obtain nonlocal dynamical equations of the quantum crystals in the form

$$\begin{aligned}
&\dot{\sigma}(x)=-\nabla_k\left(\sigma\frac{\delta H}{\delta\pi_k}\right),\quad \dot{\rho}=-\nabla_k\left(\rho\frac{\delta H}{\delta\pi_k}+\frac{\delta H}{\delta p_k}\right),\\
&\dot{\pi}_i=-\sigma\nabla_i\frac{\delta H}{\delta\sigma}-\pi_j\nabla_i\frac{\delta H}{\delta\pi_j}-\nabla_j\left(\pi_i\frac{\delta H}{\delta\pi_j}\right)-\rho\nabla_i\frac{\delta H}{\delta\rho}\\
&\qquad -p_i\nabla_j\frac{\delta H}{\delta p_j}-b_{ki}\nabla_j\frac{\delta H}{\delta b_{kj}},\\
&\dot{p}_i=-\nabla_i\left(p_j\frac{\delta H}{\delta\pi_j}+\frac{\delta H}{\delta\rho}\right);\quad \dot{b}_{ik}=-\nabla_k\left(b_{ij}\frac{\delta H}{\delta\pi_j}\right).
\end{aligned} \tag{12.6}$$

The structure of (12.6) simplifies appreciably in the long-wavelength limit, in which the spatial inhomogeneities of the dynamical variables are weak. Assuming that the energy density is a function of the variables σ, ρ, π_i, p_i, and b_{ik}, we write the dynamical equations in local form. To do this, we use the results of Sec. 1. For an energy density that satisfies the condition $\{M,\varepsilon(x)\}=0$, $M=\int d^3x\rho(x)$, we obtain the differential conservation law (1.15) for the mass density. Using (12.4), we obtain for the mass flux density j_k the expression

$$j_k=\rho\frac{\partial\varepsilon}{\partial\pi_k}+\frac{\partial\varepsilon}{\partial p_k}. \tag{12.7}$$

The calculation of the PB in (1.16) and (1.18) leads to the following expressions for the energy flux density q_k and the momentum flux density t_{ik}:

$$q_k=\frac{\partial\varepsilon}{\partial\pi_k}\left(\sigma\frac{\partial\varepsilon}{\partial\sigma}+\rho\frac{\partial\varepsilon}{\partial\rho}+\pi_l\frac{\partial\varepsilon}{\partial\pi_l}\right)+\frac{\partial\varepsilon}{\partial p_k}\left(\frac{\partial\varepsilon}{\partial\rho}+p_i\frac{\partial\varepsilon}{\partial\pi_i}\right)+b_{ij}\frac{\partial\varepsilon}{\partial b_{ik}}\frac{\partial\varepsilon}{\partial\pi_j},$$

$$t_{ik}=p\delta_{ik}+\pi_i\frac{\partial\varepsilon}{\partial\pi_k}+p_i\frac{\partial\varepsilon}{\partial p_k}+b_{ji}\frac{\partial\varepsilon}{\partial b_{jk}},$$

$$p\equiv-\varepsilon+\sigma\frac{\partial\varepsilon}{\partial\sigma}+\pi_l\frac{\partial\varepsilon}{\partial\pi_l}+\rho\frac{\partial\varepsilon}{\partial\rho}. \tag{12.8}$$

Thus, the dynamical equations of the quantum crystals in the long-wavelength limit have the form

$$\dot\sigma=-\nabla_k\left(\sigma\frac{\partial\varepsilon}{\partial\pi_k}\right),\quad \dot\rho=-\nabla_k\left(\rho\frac{\partial\varepsilon}{\partial\pi_k}+\frac{\partial\varepsilon}{\partial p_k}\right),$$

$$\dot\pi_i=-\nabla_k\left(p\delta_{ik}+\pi_i\frac{\partial\varepsilon}{\partial\pi_k}+p_i\frac{\partial\varepsilon}{\partial p_k}+b_{ji}\frac{\partial\varepsilon}{\partial b_{jk}}\right),$$

$$\dot p_i=-\nabla_i\left(p_j\frac{\partial\varepsilon}{\partial\pi_j}+\frac{\partial\varepsilon}{\partial\rho}\right),\quad \dot b_{ik}=-\nabla_k\left(b_{ij}\frac{\partial\varepsilon}{\partial\pi_j}\right). \tag{12.9}$$

These equations are identical to the ones obtained earlier by the phenomenological method in Ref. 54 and on the basis of a microscopic approach in Ref. 55.

Note that the expressions (12.8) for the flux densities can be expressed in compact form by introducing the density of the thermodynamic potential:

$$\omega=Y_a\zeta_a-\sigma,\quad a=(0,i,4);\quad \zeta_a=(\varepsilon,\pi_i,\rho), \tag{12.10}$$

where Y_a are the thermodynamic forces defined in accordance with

$$\frac{\partial\varepsilon}{\partial\sigma}\equiv\frac{1}{Y_0},\quad \frac{\partial\varepsilon}{\partial\pi_i}\equiv-\frac{Y_i}{Y_0},\quad \frac{\partial\varepsilon}{\partial\rho}\equiv-\frac{Y_4}{Y_0}. \tag{12.11}$$

It follows from (12.10) and (12.11) that

$$\frac{\partial\omega}{\partial Y_a}=\zeta_a$$

and, since $\omega=\omega(Y_a,p_i,b_{ik})$, the second law of thermodynamics has the form

$$d\omega=\zeta_a dY_a+\frac{\partial\omega}{\partial p_i}dp_i+\frac{\partial\omega}{\partial b_{ik}}db_{ik}. \tag{12.12}$$

With allowance for (12.8) and (12.12), we represent the flux densities in the form

$$\zeta_{ak}=-\frac{\partial}{\partial Y_a}\frac{\omega Y_k}{Y_0}+\frac{\partial\omega}{\partial p_k}\frac{\partial}{\partial Y_a}\frac{Y_4+Y_ip_i}{Y_0}+\frac{\partial\omega}{\partial b_{jk}}\frac{\partial}{\partial Y_a}\frac{b_{jl}Y_l}{Y_0}. \tag{12.13}$$

Here $\zeta_{0k}\equiv q_k,\zeta_{ik}\equiv t_{ik},\zeta_{4k}\equiv j_k$. The dynamics of the densities of the additive integrals of the motion and of the parameters that describe the broken symmetry is determined in accordance with (12.9) and (12.13) by the equations

$$\dot\zeta_a=-\nabla_k\zeta_{ak},\quad \dot p_i=\nabla_i\left(\frac{Y_4+Y_jp_j}{Y_0}\right),$$

$$\dot b_{ik}=\nabla_k\left(b_{ij}\frac{Y_j}{Y_0}\right).$$

In order to interpret the thermodynamic forces Y_a, we write down the fundamental thermodynamic identity for the energy density $\varepsilon=\varepsilon(\sigma,\rho,\pi_k,p_i,b_{ik})$:

$$d\varepsilon=Td\sigma+\mu d\rho+v_kd\pi_k+\frac{\partial\varepsilon}{\partial p_i}dp_i+\frac{\partial\varepsilon}{\partial b_{ik}}db_{ik}, \tag{12.14}$$

where T is the temperature, μ is the chemical potential, and v_k is the normal velocity. It follows from (12.11) and (12.14) that

$$T=\frac{1}{Y_0},\quad v_i=-\frac{Y_i}{Y_0},\quad \mu=-\frac{Y_4}{Y_0}.$$

Let us assume that the considered system has the property of Galilean invariance. Then the thermodynamic potential ω depends only on the following combinations of the thermodynamic forces Y_a (Ref. 55):

$$\omega=\omega(Y_0',Y_k',Y_4',b_{ik}),$$

$$Y_0'=Y_0,\quad Y_k'=Y_k+Y_0\frac{p_k}{m},\quad Y_4'=Y_4+Y_kp_k+Y_0\frac{p^2}{2m}. \tag{12.15}$$

With allowance for (12.15), for the flux densities we obtain the expressions

$$j_k=\frac{1}{m}\frac{\partial\omega}{\partial Y_k'}+\frac{p_k}{m}\frac{\partial\omega}{\partial Y_4'},$$

$$t_{ik}=-\frac{\omega}{Y_0}\delta_{ik}+\frac{p_ip_k}{m}\frac{\partial\omega}{\partial Y_4'}+\frac{p_i}{m}\frac{\partial\omega}{\partial Y_k'}-\frac{Y_k}{Y_0}\frac{\partial\omega}{\partial Y_i'}+\frac{b_{ji}}{Y_0}\frac{\partial\omega}{\partial b_{jk}},$$

$$q_k=-\frac{Y_4+Y_lp_l}{mY_0}\pi_k-\frac{Y_k}{Y_0^2}\left(\sigma-Y_l\frac{\partial\omega}{\partial Y_l'}\right)-\frac{b_{jl}Y_l}{Y_0^2}\frac{\partial\omega}{\partial b_{jk}}. \tag{12.16}$$

The expressions (12.16) are identical to the analogous expressions in Ref. 55.

13. SUPERFLUID ^{4}He

In the equilibrium state of superfluid ^{4}He, phase invariance is broken. The energy density $\varepsilon(x)$ is a functional of the densities of the entropy, $\sigma(x)$, mass, $\rho(x)$, momentum, $\pi_i(x)$, and the superfluid phase $\phi(x)$:

$$\varepsilon(x)=\varepsilon(x;\sigma(x'),\rho(x'),\pi_i(x'),\phi(x')). \tag{13.1}$$

The algebra of the dynamical variables of superfluid ^{4}He forms a subalgebra of the algebra (12.4) of the variables of quantum crystals and is determined by the brackets

$$\{\pi_i(x),\sigma(x')\}=-\sigma(x)\nabla_i\delta(x-x'),$$

$$\{\pi_i(x),\rho(x')\}=-\rho(x)\nabla_i\delta(x-x'),$$

$$\{\pi_i(x),\phi(x')\}=\delta(x-x')\nabla_i\phi(x),$$

$$\{\pi_i(x),\psi(x')\}=\delta(x-x')\nabla_i\psi(x),$$

$$\{\sigma(x),\psi(x')\}=\{\rho(x),\phi(x')\}=\delta(x-x'),$$

$$\{\pi_i(x),\pi_k(x')\}=\pi_k(x)\nabla_i'\delta(x-x')-\pi_i(x')\nabla_k\delta(x-x'). \tag{13.2}$$

We give here the interpretation of the variables x_i and u_i introduced in Sec. 2. Since we were there considering a normal system, x_i and u_i were, respectively, Eulerian coordinates and the displacement vector of the particles of the medium. In the case of superfluid ^{4}He, it is necessary to clarify to which component these variables correspond. First of all, the normal velocity is determined by

$$v_i(x)=\frac{\delta H}{\delta\pi_i(x)}. \tag{13.3}$$

We obtained the equation of motion for the displacement vector earlier [see (2.18)] and, with allowance for (13.3), it has the form

$$\dot{u}_i=b_{ij}v_j. \tag{13.4}$$

We define the function $x=x(\xi,t)$ implicitly by means of the equation

$$x_i(\xi,t)=\xi_i+u_i(x(\xi,t),t). \tag{13.5}$$

Differentiation of both sides of (13.5) with respect to the time leads to the relation

$$\dot{u}_i=b_{ij}\dot{x}_j. \tag{13.6}$$

Hence, comparing Eqs. (13.4) and (13.6), we obtain

$$b_{ij}\dot{x}_j=b_{ij}v_j$$

and, therefore,

$$\dot{x}_j=v_j.$$

Thus, we conclude that the function $x_i(\xi,t)$ defined by the relation (13.5) and u_i are, respectively, the Eulerian coordinate and the displacement vector of the particles of the normal component.

If the variable ψ is cyclic, then the local dynamical equations corresponding to the PB (13.2) have the form

$$\dot{\sigma}=-\nabla_k\left(\sigma\frac{\partial\varepsilon}{\partial\pi_k}\right),\quad \dot{\rho}=-\nabla_k\left(\rho\frac{\partial\varepsilon}{\partial\pi_k}+\frac{\partial\varepsilon}{\partial p_k}\right),$$

$$\dot{\pi}_i=-\nabla_k\left(p\delta_{ik}+\pi_i\frac{\partial\varepsilon}{\partial\pi_k}+p_i\frac{\partial\varepsilon}{\partial p_k}\right),$$

$$\dot{p}_i=-\nabla_i\left(p_j\frac{\partial\varepsilon}{\partial\pi_j}+\frac{\partial\varepsilon}{\partial\rho}\right), \tag{13.7}$$

where

$$p=-\varepsilon+\sigma\frac{\partial\varepsilon}{\partial\sigma}+\pi_l\frac{\partial\varepsilon}{\partial\pi_l}+\rho\frac{\partial\varepsilon}{\partial\rho}$$

is the pressure and $p_i=\nabla_i\phi$ is the superfluid momentum.

We note the following interesting circumstance. If the energy density $\varepsilon(x)$ contains a dependence on the variable ψ, then on the right-hand side of the equation of motion for the entropy density an additional term appears. Indeed, under the assumption that

$$\varepsilon(x;\ldots,\psi(x'))=\varepsilon(\ldots,\nabla\psi(x))$$

and, therefore,

$$\frac{\delta H}{\delta\psi}=-\nabla_k\frac{\partial\varepsilon}{\partial\nabla_k\psi}$$

we obtain

$$\dot{\sigma}=-\nabla_k\left(\sigma\frac{\partial\varepsilon}{\partial\pi_k}+\frac{\partial\varepsilon}{\partial\nabla_k\psi}\right).$$

The second term in the brackets need not be associated with motion of the normal component, and in the expression for the entropy flux density there arises as a consequence a term due to the superfluid motion.

14. QUANTUM SPIN CRYSTALS

When we considered the dynamics of quantum spin crystals, we did not take into account the effect of the spin degrees of freedom, which can be important for quantum solid ^{3}He. Bearing in mind that in the superfluid liquid phases of ^{3}He there is spontaneous breaking of the symmetry with respect to homogeneous spin rotations, we consider the case of complete breaking of the spin invariance (the parameter that describes such breaking is a real rotation matrix); this corresponds to the B phase of ^{3}He. In other words, we consider a quantum spin crystal in the equilibrium state of which the phase, translational, and spin invariances are simultaneously broken.[28]

If in the ground state there is not complete breaking of the symmetry with respect to spin rotations, then the spin dynamics of the quantum crystal can be described in terms of just the spin density in the framework of the well-known Landau–Lifshitz equation with a convection term.[52]

The dynamical variables of a quantum spin crystal with broken symmetry with respect to spin rotations are the densities of the entropy, $\sigma(x)$, the mass, $\rho(x)$, the momentum, $\pi_i(x)$, and the spin, $s_\alpha(x)$, and also the orthogonal rotation matrix $a_{\alpha\beta}(x)$, the superfluid phase $\phi(x)$, and the displacement vector $u_i(x)$. Therefore, in the general case the energy density $\varepsilon(x)$ is a functional of these variables:

$$\varepsilon(x)=\varepsilon(x;\sigma(x'),\rho(x'),\pi,(x'),s_\alpha(x'),\\ a_{\alpha\beta}(x'),\phi(x'),u_i(x')). \tag{14.1}$$

The density of the kinematic part of the Lagrangian of quantum spin crystals can be written in the form [see (6.3) and (12.3)]

$$\mathscr{L}_k(x)=\pi_i^*(x)b_{ij}^{-1}\dot{u}_j(x)-\sigma(x)\dot{\psi}(x)-\rho(x)\dot{\phi}(x)\\ -s_\alpha(x)\omega_\alpha(x), \tag{14.2}$$

where

$$\pi_i^*=\pi_i-\sigma\nabla_i\psi-\rho\nabla_i\phi-s_\alpha\omega_{\alpha i}.$$

The density of the kinematic part of the Lagrangian (14.2) corresponds to the following PB algebra of the dynamical variables:

$$\{\pi_i(x),u_k(x')\}=-(\delta_{ik}-\nabla_i u_k(x))\delta(x-x'),$$
$$\{\pi_i(x),\psi(x')\}=\nabla_i\psi(x)\delta(x-x'),$$
$$\{\pi_i(x),\sigma(x')\}=-\sigma(x)\nabla_i\delta(x-x'),$$
$$\{\pi_i(x),\rho(x')\}=-\rho(x)\nabla_i\delta(x-x'),$$
$$\{\pi_i(x),s_\alpha(x')\}=-s_\alpha(x)\nabla_i\delta(x-x'),$$
$$\{\pi_i(x),a_{\alpha\beta}(x')\}=\delta(x-x')\nabla_i a_{\alpha\beta}(x),$$
$$\{\pi_i(x),\pi_k(x')\}=\pi_k(x)\nabla_i'\delta(x-x')-\pi_i(x')\nabla_k\delta(x-x'),$$
$$\{\pi_i(x),\phi(x')\}=\delta(x-x')\nabla_i\phi(x),$$
$$\{\rho(x),\phi(x')\}=\{\sigma(x),\psi(x')\}=\delta(x-x'),$$
$$\{s_\alpha(x),s_\beta(x')\}=\varepsilon_{\alpha\beta\gamma}s_\gamma(x)\delta(x-x'),$$
$$\{s_\alpha(x),a_{\beta\gamma}(x')\}=\varepsilon_{\alpha\gamma\rho}a_{\beta\rho}(x)\delta(x-x'). \tag{14.3}$$

Note that by virtue of the invariance of $\varepsilon(x)$ with respect to global phase transformations and spatial translations the energy density $\varepsilon(x)$ depends not on the variables $\phi(x)$ and $u_i(x)$ themselves but only on their derivatives $\nabla_i\phi\equiv p_i, \nabla_k u_i$ (or, in the last case, on the variables $b_{ik}\equiv\delta_{ik}-\nabla_k u_i$):

$$\begin{aligned}\varepsilon(x)=\varepsilon(x;\sigma(x'),\rho(x'),\pi_i(x'),s_\alpha(x'),\\ a_{\alpha\beta}(x'),p_i(x'),b_{ik}(x')).\end{aligned} \tag{14.4}$$

The vector p has the meaning of the superfluid momentum. As dynamical variables, it is therefore convenient to choose, in addition to the remaining variables, the variables p_i and b_{ik} directly. Using the PB (14.3) and the general functional expression (14.4), we obtain nonlocal dynamical equations of the quantum spin crystal in the form

$$\dot\sigma(x)=-\nabla_k\left(\sigma\frac{\delta H}{\delta\pi_k}\right),\quad \dot\rho=-\nabla_k\left(\rho\frac{\delta H}{\delta\pi_k}+\frac{\delta H}{\delta p_k}\right),$$
$$\dot\pi_i=-\sigma\nabla_i\frac{\delta H}{\delta\sigma}-\pi_j\nabla_i\frac{\delta H}{\delta\pi_j}-\nabla_j\left(\pi_i\frac{\delta H}{\delta\pi_j}\right)-s_\alpha\nabla_i\frac{\delta H}{\delta s_\alpha}$$
$$+\frac{\delta H}{\delta a_{\alpha\beta}}\nabla_i a_{\alpha\beta}-\rho\nabla_i\frac{\delta H}{\delta\rho}-p_i\nabla_j\frac{\delta H}{\delta p_j}-b_{ki}\nabla_j\frac{\delta H}{\delta b_{kj}},$$
$$\dot s_\alpha=-\nabla_i\left(s_\alpha\frac{\delta H}{\delta\pi_i}\right)+\varepsilon_{\alpha\beta\gamma}\left(\frac{\delta H}{\delta s_\beta}s_\gamma+\frac{\delta H}{\delta a_{\rho\beta}}a_{\rho\gamma}\right) \tag{14.5}$$

and also equations for the parameters that describe the broken symmetry:

$$\dot a_{\alpha\beta}=-\frac{\delta H}{\delta\pi_i}\nabla_i a_{\alpha\beta}+a_{\alpha\rho}\varepsilon_{\rho\beta\gamma}\frac{\delta H}{\delta s_\gamma};$$
$$\dot p_i=-\nabla_i\left(p_j\frac{\delta H}{\delta\pi_j}+\frac{\delta H}{\delta\rho}\right),\quad b_{ik}=-\nabla_k\left(b_{ij}\frac{\delta H}{\delta\pi_j}\right). \tag{14.6}$$

The dynamical equations (14.5) and (14.6) and the general functional expression (14.4) for the energy density describe the nonequilibrium properties of a quantum spin crystal with arbitrary nature of the spatial inhomogeneities. Assuming that the energy density is a function of the form

$$\varepsilon(x)=\varepsilon(\sigma(x),\rho(x),\pi_i(x),p_i(x),b_{ik}(x),s_\alpha(x),\omega_{\alpha k}(x)), \tag{14.7}$$

we obtain dynamical equations in local form. Using the expressions (1.15)–(1.18), we find for the mass, spin, energy, and momentum flux densities

$$j_k=\rho\frac{\partial\varepsilon}{\partial\pi_k}+\frac{\partial\varepsilon}{\partial p_k},\quad j_{\alpha k}=s_\alpha\frac{\partial\varepsilon}{\partial\pi_k}+\frac{\partial\varepsilon}{\partial\omega_{\alpha k}},$$
$$q_k=\frac{\partial\varepsilon}{\partial\pi_k}\left(\sigma\frac{\partial\varepsilon}{\partial\sigma}+\rho\frac{\partial\varepsilon}{\partial\rho}+\pi_l\frac{\partial\varepsilon}{\partial\pi_l}+s_\alpha\frac{\partial\varepsilon}{\partial s_\alpha}\right)+\frac{\partial\varepsilon}{\partial p_k}\left(\frac{\partial\varepsilon}{\partial\rho}\right.$$
$$\left.+p_i\frac{\partial\varepsilon}{\partial\pi_i}\right)+\frac{\partial\varepsilon}{\partial\omega_{\alpha k}}\left(\frac{\partial\varepsilon}{\partial s_\alpha}+\omega_{\alpha i}\frac{\partial\varepsilon}{\partial\pi_i}\right)$$
$$+b_{ij}\frac{\partial\varepsilon}{\partial b_{ik}}\frac{\partial\varepsilon}{\partial\pi_j},$$
$$t_{ik}=p\delta_{ik}+\pi_i\frac{\partial\varepsilon}{\partial\pi_k}+p_i\frac{\partial\varepsilon}{\partial p_k}+\omega_{\alpha i}\frac{\partial\varepsilon}{\partial\omega_{\alpha k}}+b_{ji}\frac{\partial\varepsilon}{\partial b_{jk}},$$
$$p\equiv-\varepsilon+\sigma\frac{\partial\varepsilon}{\partial\sigma}+\pi_l\frac{\partial\varepsilon}{\partial\pi_l}+\rho\frac{\partial\varepsilon}{\partial\rho}+s_\alpha\frac{\partial\varepsilon}{\partial s_\alpha}. \tag{14.8}$$

Here p is the pressure. Assuming that, as before, the energy density has the property (6.25) of rotational invariance in the spin space, we obtain

$$\varepsilon(\ldots,s,\omega_k,a)=\varepsilon(\ldots,as,a\omega_k,1)\equiv\varepsilon(\ldots,\underline{s},\underline{\omega}_k).$$

As independent variables, it is therefore convenient to choose, besides the remaining variables, $\underline{s}$ and $\underline{\omega}_k$. Then, calculating the flux densities in the new variables using the PB algebra (14.3), we obtain the dynamical equations of quantum spin crystals in the long-wavelength limit in the form

$$\dot\sigma=-\nabla_k\left(\sigma\frac{\partial\varepsilon}{\partial\pi_k}\right),\quad\dot\rho=-\nabla_k\left(\rho\frac{\partial\varepsilon}{\partial\pi_k}+\frac{\partial\varepsilon}{\partial p_k}\right),$$
$$\dot\pi_i=-\nabla_k\left(p\delta_{ik}+\pi_i\frac{\partial\varepsilon}{\partial\pi_k}+p_i\frac{\partial\varepsilon}{\partial p_k}+\underline{\omega}_{\alpha i}\frac{\partial\varepsilon}{\partial\underline{\omega}_{\alpha k}}+b_{ji}\frac{\partial\varepsilon}{\partial b_{jk}}\right),$$
$$\dot{\underline{s}}=-\nabla_k\left(\underline{s}_\alpha\frac{\partial\varepsilon}{\partial\pi_k}+\frac{\partial\varepsilon}{\partial\underline{\omega}_{\alpha k}}\right)+\varepsilon_{\alpha\beta\gamma}\left(\underline{s}_\beta\frac{\partial\varepsilon}{\partial\underline{s}_\gamma}\right.$$
$$\left.+\underline{\omega}_{\beta k}\frac{\partial\varepsilon}{\partial\underline{\omega}_{\gamma k}}\right), \tag{14.9}$$

where

$$p=-\varepsilon+\sigma\frac{\partial\varepsilon}{\partial\sigma}+\pi_l\frac{\partial\varepsilon}{\partial\pi_l}+\rho\frac{\partial\varepsilon}{\partial\rho}+\underline{s}_\alpha\frac{\partial\varepsilon}{\partial\underline{s}_\alpha},$$

and equations for the parameters that describe the broken symmetry:

$$\dot a_{\alpha\beta}=\varepsilon_{\alpha\rho\gamma}\left(\frac{\partial\varepsilon}{\partial\underline{s}_\gamma}+\frac{\partial\varepsilon}{\partial\pi_i}\underline{\omega}_{\gamma i}\right)a_{\rho\beta},$$

$$\dot{p}_i=-\nabla_i\left(p_j\frac{\partial\varepsilon}{\partial\pi_j}+\frac{\partial\varepsilon}{\partial\rho}\right),\quad \dot{b}_{ik}=-\nabla_k\left(b_{ij}\frac{\partial\varepsilon}{\partial\pi_j}\right). \tag{14.10}$$

A consequence of Eqs. (14.9) and (14.10) is the following differential conservation law for the energy density:

$$\dot{\varepsilon}=-\nabla_k\left[\frac{\partial\varepsilon}{\partial\pi_k}\left(\sigma\frac{\partial\varepsilon}{\partial\sigma}+\rho\frac{\partial\varepsilon}{\partial\rho}+\pi_l\frac{\partial\varepsilon}{\partial\pi_l}+s_\alpha\frac{\partial\varepsilon}{\partial s_\alpha}\right)\right.$$
$$+\frac{\partial\varepsilon}{\partial p_k}\left(\frac{\partial\varepsilon}{\partial\rho}+p_i\frac{\partial\varepsilon}{\partial\pi_i}\right)+\frac{\partial\varepsilon}{\partial\omega_{\alpha k}}\left(\frac{\partial\varepsilon}{\partial s_\alpha}+\omega_{\alpha i}\frac{\partial\varepsilon}{\partial\pi_i}\right)$$
$$\left.+b_{ij}\frac{\partial\varepsilon}{\partial b_{ik}}\frac{\partial\varepsilon}{\partial\pi_j}\right]. \tag{14.11}$$

To obtain compact expressions for the flux densities (14.8), we introduce the thermodynamic potential ω:

$$\omega=Y_a\zeta_a-\sigma,\quad a=(0,i,\alpha,4);\quad \zeta_a=(\varepsilon,\pi_i,s_\alpha,\rho), \tag{14.12}$$

where Y_a are the thermodynamic forces determined by the expressions

$$\frac{\partial\varepsilon}{\partial\sigma}\equiv\frac{1}{Y_0},\quad \frac{\partial\varepsilon}{\partial\pi_i}\equiv-\frac{Y_i}{Y_0},\quad \frac{\partial\varepsilon}{\partial s_\alpha}\equiv-\frac{Y_\alpha}{Y_0},\quad \frac{\partial\varepsilon}{\partial\rho}\equiv-\frac{Y_4}{Y_0} \tag{14.13}$$

($\omega'=-\omega/Y_0$ is the Gibbs potential). With allowance for (14.12) and (14.13),

$$\frac{\partial\omega}{\partial Y_a}=\zeta_a$$

and the second law of thermodynamics for the potential ω can be written in the form

$$d\omega=\zeta_a dY_a+\frac{\partial\omega}{\partial p_i}dp_i+\frac{\partial\omega}{\partial b_{ik}}db_{ik}+\frac{\partial\omega}{\partial\omega_{ak}}d\omega_{ak}. \tag{14.14}$$

It follows from (14.8) and (14.14) that

$$\zeta_{ak}=-\frac{\partial}{\partial Y_a}\frac{\omega Y_k}{Y_0}+\frac{\partial\omega}{\partial p_k}\frac{\partial}{\partial Y_a}\frac{Y_4+Y_ip_i}{Y_0}$$
$$+\frac{\partial\omega}{\partial b_{jk}}\frac{\partial}{\partial Y_a}\frac{b_{jl}Y_l}{Y_0}+\frac{\partial\omega}{\partial\omega_{\alpha k}}\frac{\partial}{\partial Y_a}\frac{Y_\alpha+Y_l\omega_{\alpha l}}{Y_0}. \tag{14.15}$$

The dynamics of the densities ζ_a of the additive integrals of the motion and of the parameters that describe the broken symmetry is given by the equations

$$\dot{\zeta}_a=-\nabla_k\zeta_{ak},\quad \dot{p}_i=\nabla_i\left(\frac{Y_4+Y_jp_j}{Y_0}\right),\quad \dot{b}_{ik}=\nabla_k\left(b_{ij}\frac{Y_j}{Y_0}\right),$$

$$\dot{a}_{\alpha\beta}=\frac{1}{Y_0}(Y_i\nabla_i a_{\alpha\beta}-\varepsilon_{\alpha\rho\gamma}Y_\gamma a_{\rho\beta}).$$

In order to interpret the thermodynamic forces Y_a, we write down the fundamental thermodynamic identity for the energy density:

$$d\varepsilon=Td\sigma+\mu d\rho+v_k d\pi_k+h_\alpha ds_\alpha+\frac{\partial\varepsilon}{\partial p_i}dp_i$$
$$+\frac{\partial\varepsilon}{\partial b_{ik}}db_{ik}+\frac{\partial\varepsilon}{\partial\omega_{\alpha k}}d\omega_{\alpha k}. \tag{14.16}$$

Here h_α is the magnetizing field. From (14.13) and (14.16), we obtain

$$T=\frac{1}{Y_0},\quad v_i=-\frac{Y_i}{Y_0},\quad \mu=-\frac{Y_4}{Y_0},\quad h_\alpha=-\frac{Y_\alpha}{Y_0}.$$

Let us assume that the considered system has the property of Galilean invariance. Then the thermodynamic potential ω depends only on the following combinations of the thermodynamic forces Y_a (Ref. 55):

$$\omega=\omega(Y_0',Y_k',Y_\alpha',Y_4',b_{ik},\omega_{\alpha k}),$$

$$Y_0'=Y_0,\quad Y_k'=Y_k+Y_0\frac{p_k}{m},\quad Y_\alpha'=Y_\alpha,$$

$$Y_4'=Y_4+Y_kp_k+Y_0\frac{p^2}{2m}.$$

Noting that

$$\frac{\partial\omega}{\partial p_l}=\frac{Y_0}{m}\frac{\partial\omega}{\partial Y_l'}+\left(Y_l+Y_0\frac{p_l}{m}\right)\frac{\partial\omega}{\partial Y_4'},$$

$$\frac{\partial\omega}{\partial Y_l}=\frac{\partial\omega}{\partial Y_l'}+p_l\frac{\partial\omega}{\partial Y_4'},$$

we obtain for the flux densities the expressions

$$j_k=\frac{1}{m}\frac{\partial\omega}{\partial Y_k'}+\frac{p_k}{m}\frac{\partial\omega}{\partial Y_4'},\quad j_{\alpha k}=-\frac{Y_k}{Y_0}\frac{\partial\omega}{\partial Y_\alpha'}+\frac{1}{Y_0}\frac{\partial\omega}{\partial\omega_{\alpha k}},$$

$$t_{ik}=-\frac{\omega}{Y_0}\delta_{ik}+\frac{p_ip_k}{m}\frac{\partial\omega}{\partial Y_4'}+\frac{p_i}{m}\frac{\partial\omega}{\partial Y_k'}-\frac{Y_k}{Y_0}\frac{\partial\omega}{\partial Y_i'}$$
$$+\frac{\omega_{\alpha i}}{Y_0}\frac{\partial\omega}{\partial\omega_{\alpha k}}+\frac{b_{ji}}{Y_0}\frac{\partial\omega}{\partial b_{jk}},$$

$$q_k-\frac{Y_4+Y_lp_l}{mY_0}\pi_k-\frac{Y_k}{Y_0^2}\left(\sigma-Y_l\frac{\partial\omega}{\partial Y_l'}\right)-\frac{b_{jl}Y_l}{Y_0^2}\frac{\partial\omega}{\partial b_{jk}}$$
$$-\frac{Y_\alpha+Y_l\omega_{\alpha l}}{Y_0^2}\frac{\partial\omega}{\partial\omega_{\alpha k}}.$$

At the same time, the second law of thermodynamics can be written in the form

$$Td\sigma=d\varepsilon+\frac{Y_k'}{Y_0}d\pi_k+\frac{Y_\alpha'}{Y_0}ds_\alpha+\frac{Y_4'}{Y_0}d\rho$$
$$-\frac{1}{Y_0}\frac{\partial\omega}{\partial\omega_{\alpha k}}d\omega_{\alpha k}-\frac{1}{Y_0}\frac{\partial\omega}{\partial b_{ik}}db_{ik}.$$

15. SUPERFLUID ^{3}He-B

In the case of the superfluid B phase of ^{3}He, there is spontaneous breaking of the symmetry with respect to phase transformations and spin rotations. The parameters that describe this breaking are the superfluid phase $\phi(x)$ and the orthogonal rotation matrix $a_{\alpha\beta}(x)$. Accordingly, the energy density is a functional of the form

$$\varepsilon(x)=\varepsilon(x;\sigma(x'),\rho(x'),\pi_i(x'),\phi(x'),s_\alpha(x'),a_{\alpha\beta}(x')). \tag{15.1}$$

The PB of the dynamical variables of the superfluid ^{3}He B phase form a subalgebra of the PB algebra (14.3) of quantum spin crystals. For them, we have

$$\begin{aligned}
&\{\pi_i(x),\psi(x')\}=\delta(x-x')\nabla_i\psi(x),\\
&\{\pi_i(x),\sigma(x')\}=-\sigma(x)\nabla_i\delta(x-x'),\\
&\{\pi_i(x),\rho(x')\}=-\rho(x)\nabla_i\delta(x-x'),\\
&\{\pi_i(x),s_\alpha(x')\}=-s_\alpha(x)\nabla_i\delta(x-x'),\\
&\{\pi_i(x),a_{\alpha\beta}(x')\}=\delta(x-x')\nabla_i a_{\alpha\beta}(x),\\
&\{\pi_i(x),\phi(x')\}=\delta(x-x')\nabla_i\phi(x),\\
&\{\rho(x),\phi(x')\}=\{\sigma(x),\psi(x')\}=\delta(x-x'),\\
&\{s_\alpha(x),s_\beta(x')\}=\varepsilon_{\alpha\beta\gamma}s_\gamma(x)\,\delta(x-x'),\\
&\{s_\alpha(x),a_{\beta\gamma}(x')\}=\varepsilon_{\alpha\gamma\rho}a_{\beta\rho}(x)\,\delta(x-x'),\\
&\{\pi_i(x),\pi_k(x')\}=\pi_k(x)\nabla_i'\delta(x-x')-\pi_i(x')\nabla_k\delta(x-x').
\end{aligned} \tag{15.2}$$

By virtue of the global phase invariance, the dependence of the energy density (15.1) on the phase $\phi(x)$ occurs only through $p_i\equiv\nabla_i\phi$. The nonlocal dynamical equations for the densities of the additive integrals of the motion of superfluid ^{3}He-B have the form

$$\begin{aligned}
&\dot\sigma(x)=-\nabla_k\left(\sigma\frac{\delta H}{\delta\pi_k}\right),\quad \dot\rho=-\nabla_k\left(\rho\frac{\delta H}{\delta\pi_k}+\frac{\delta H}{\delta p_k}\right),\\
&\dot\pi_i=-\pi_j\nabla_i\frac{\delta H}{\delta\pi_j}-\nabla_j\left(\pi_i\frac{\delta H}{\delta\pi_j}\right)-\sigma\nabla_i\frac{\delta H}{\delta\sigma}-s_\alpha\nabla_i\frac{\delta H}{\delta s_\alpha}\\
&\qquad+\frac{\delta H}{\delta a_{\alpha\beta}}\nabla_i a_{\alpha\beta}-\rho\nabla_i\frac{\delta H}{\delta\rho}-p_i\nabla_j\frac{\delta H}{\delta p_j},\\
&\dot s_\alpha=-\nabla_i\left(s_\alpha\frac{\delta H}{\delta\pi_i}\right)+\varepsilon_{\alpha\beta\gamma}\left(\frac{\delta H}{\delta s_\beta}s_\gamma+\frac{\delta H}{\delta a_{\rho\beta}}a_{\rho\gamma}\right),
\end{aligned} \tag{15.3}$$

while the equations for the parameters that describe the broken symmetry have the form

$$\begin{aligned}
&\dot a_{\alpha\beta}=-\frac{\delta H}{\delta\pi_i}\nabla_i a_{\alpha\beta}+a_{\alpha\rho}\varepsilon_{\rho\beta\gamma}\frac{\delta H}{\delta s_\gamma};\\
&\dot p_i=-\nabla_i\left(p_j\frac{\delta H}{\delta\pi_j}+\frac{\delta H}{\delta\rho}\right).
\end{aligned} \tag{15.4}$$

In the long-wavelength limit

$$\varepsilon(x)=\varepsilon(\sigma(x),\rho(x),\pi_i(x),p_i(x),s_\alpha(x),a_{\alpha\beta}(x),\omega_{\alpha k}(x)),$$

and the flux densities of the integrals of the motion are, in accordance with (1.15)–(1.18),

$$\begin{aligned}
&j_k=\rho\frac{\partial\varepsilon}{\partial\pi_k}+\frac{\partial\varepsilon}{\partial p_k},\quad j_{\alpha k}=s_\alpha\frac{\partial\varepsilon}{\partial\pi_k}+\frac{\partial\varepsilon}{\partial\omega_{\alpha k}},\\
&q_k=\frac{\partial\varepsilon}{\partial\pi_k}\left(\sigma\frac{\partial\varepsilon}{\partial\sigma}+\rho\frac{\partial\varepsilon}{\partial\rho}+\pi_l\frac{\partial\varepsilon}{\partial\pi_l}+s_\alpha\frac{\partial\varepsilon}{\partial s_\alpha}\right)+\frac{\partial\varepsilon}{\partial p_k}\left(\frac{\partial\varepsilon}{\partial\rho}+p_i\frac{\partial\varepsilon}{\partial\pi_i}\right)+\frac{\partial\varepsilon}{\partial\omega_{\alpha k}}\left(\frac{\partial\varepsilon}{\partial s_\alpha}+\omega_{\alpha i}\frac{\partial\varepsilon}{\partial\pi_i}\right),\\
&t_{ik}=p\,\delta_{ik}+\pi_i\frac{\partial\varepsilon}{\partial\pi_k}+p_i\frac{\partial\varepsilon}{\partial p_k}+\omega_{\alpha i}\frac{\partial\varepsilon}{\partial\omega_{\alpha k}},\\
&p\equiv-\varepsilon+\sigma\frac{\partial\varepsilon}{\partial\sigma}+\pi_l\frac{\partial\varepsilon}{\partial\pi_l}+\rho\frac{\partial\varepsilon}{\partial\rho}+s_\alpha\frac{\partial\varepsilon}{\partial s_\alpha}.
\end{aligned} \tag{15.5}$$

If the energy density is invariant with respect to spin rotations, then

$$\varepsilon(\ldots,s,\omega_k,a)=\varepsilon(\ldots,\underline{s},\underline{\omega}_k). \tag{15.6}$$

The local equations of motion for the superfluid ^{3}He B phase that correspond to the energy density (15.6) have the form

$$\begin{aligned}
&\dot\sigma=-\nabla_k\left(\sigma\frac{\partial\varepsilon}{\partial\pi_k}\right),\quad \dot\rho=-\nabla_k\left(\rho\frac{\partial\varepsilon}{\partial\pi_k}+\frac{\partial\varepsilon}{\partial p_k}\right),\\
&\dot\pi_i=-\nabla_k\left(p\,\delta_{ik}+\pi_i\frac{\partial\varepsilon}{\partial\pi_k}+p_i\frac{\partial\varepsilon}{\partial p_k}+\underline{\omega}_{\alpha i}\frac{\partial\varepsilon}{\partial\underline{\omega}_{\alpha k}}\right),\\
&\dot{\underline{s}}_\alpha=-\nabla_k\left(\underline{s}_\alpha\frac{\partial\varepsilon}{\partial\pi_k}+\frac{\partial\varepsilon}{\partial\underline{\alpha}_{\alpha k}}\right)+\varepsilon_{\alpha\beta\gamma}\left(\underline{s}_\beta\frac{\partial\varepsilon}{\partial\underline{s}_\gamma}+\underline{\omega}_{\beta k}\frac{\partial\varepsilon}{\partial\underline{\omega}_{\gamma k}}\right),
\end{aligned} \tag{15.7}$$

while the equations for the parameters that describe the broken symmetry have the form

$$\begin{aligned}
&\dot a_{\alpha\beta}=\varepsilon_{\alpha\rho\gamma}\left(\frac{\partial\varepsilon}{\partial\underline{s}_\gamma}+\frac{\partial\varepsilon}{\partial\pi}\underline{\omega}_{\gamma i}\right)a_{\rho\beta},\\
&\dot p_i=-\nabla_i\left(p_j\frac{\partial\varepsilon}{\partial\pi_j}+\frac{\partial\varepsilon}{\partial\rho}\right).
\end{aligned} \tag{15.8}$$

Equations (15.7) and (15.8) are identical to the ones obtained earlier in Ref. 43.

CONCLUSIONS

Thus, we have constructed a systematic method for obtaining the Poisson brackets of dynamical variables based on specifying the kinematic part of the Lagrangian and interpreting the integrated terms in the variation of the action as the generators of canonical transformations. In the construction of the kinematic part of the Lagrangian, the order parameters are defined in terms of quantities conjugate to the densities of the additive integrals of the motion. In the approach that we have considered, the entropy density is a dynamical variable, and this requires the introduction of an additional conjugate variable (the variable ψ), which, however, does not occur in the equation of motion, owing to the assumption that it is cyclic. An important role in finding the kinematic part of the Lagrangian is played by the definition of the operators of spatial shift associated with the various physical fields that occur in the Lagrangian such as the entropy field, the field of the phases of the order parameters, etc. On the basis of the approach developed here, we have been able to treat from a unified point of view very varied physical systems, beginning with classical continuous media and ending with macroscopic quantum objects (various magnetically ordered systems, superfluid liquids, quantum crystals). The generality of the treatment is achieved both by the universality of the formalism and by the device of using separation of the PB subalgebras of the dynamical variables from a more general algebra with the subsequent assumption that the variables that do not occur in the subalgebra are cyclic.

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