Construction of basis functions of the two-rotor nuclear model in the Fock–Bargmann space

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On the basis of a microscopic approach, a constructive method is proposed for constructing explicitly basis functions of the nuclear model of two axial rotors. The model is interpreted as a generalization of Elliott's SU(3) model, the basis of which is extended to a basis of the direct product SU(3)×SU(3). The solution to the problem is facilitated by the procedure of mapping the functions and operators to the Fock–Bargmann space. Explicit expressions are obtained in this space for the second-order Casimir operators of the group SU(3) and the Bargmann–Moshinsky operator. Ultimately, the wave functions of the model are expressed in terms of hypergeometric functions and spherical Wigner functions. A procedure for separating "Elliott states" from the complete set of constructed functions is also considered. A possibility of extending the region of applicability of the obtained basis is noted in the final part of the paper. © *1996 American Institute of Physics.* [S1063-7796(96)00202-X]

1. INTRODUCTION

In this paper, we consider a method for constructing wave functions in the two-rotor nuclear model on the basis of a microscopic approach, and we investigate the main properties of these functions. The main attention is devoted to the formal aspect of the problem, except in the Introduction and in part in Sec. 2, in which the physical formulation of the problem is also considered.

An important aspect of the approach presented below is the use of the Fock–Bargmann space, in which the basis functions of the model have a simple form. The transition to this space is made by means of wave packets that generate a complete basis of the model, the two vector parameters of the wave packets becoming independent variables in the Fock–Bargmann space.

In a phenomenological interpretation, the model of two axial rotors was proposed by Palumbo[1–3] and was later considered in various studies of other authors (see, for example, Refs. 4–7). We call the approach proposed in the present paper microscopic, since the original constructive elements of the nuclear wave functions are, as in Refs. 6 and 7, single-particle harmonic-oscillator functions that depend on the spatial coordinates and the spin–isospin variables of the individual nucleons. Depending on their classification, the nuclear states are described by definite linear combinations of products of such single-particle states.

It was shown in Ref. 8 that the dynamics of a single, in general nonaxial, rotor can be described on the basis of Elliott's microscopic SU(3) model.[9] This offers hope that in our case too it will be possible to make effective use of a classification of the states in accordance with their (λ,μ) symmetry. However, Elliott's model describes the dynamics of just three collective degrees of freedom of valence nucleons, whereas in the case of two rotors the number of degrees of freedom can vary from four (axial rotors) to six. It is therefore necessary to extend the basis of Elliott's model and introduce a new classification of nucleon configurations. In the classification that we propose, we use the SU(3) symmetry indices separately for the system of neutrons (λ_n,μ_n) and for the system of protons (λ_p,μ_p). The set of states constructed in this manner forms a basis of the direct product SU(3)×SU(3) of the two groups. The subsequent reduction of SU(3)×SU(3) with respect to the group SU(3) makes it possible to label the wave functions by the quantum numbers (λ,μ) that characterize the SU(3) symmetry of the neutron–proton system as a whole. Thus, in the considered model the states are identified by the following set of quantum numbers: (λ_n,μ_n), (λ_p,μ_p), (λ,μ), and K, L, M. The KLM quantum numbers are analogous to the quantum numbers of the states of a rigid rotor, for which L is the angular momentum, K is the projection of the angular momentum onto the axis of the rotor, and M is the projection of the angular momentum onto an external axis. In general, K is not an integral of the motion in the considered model, and therefore the wave function is a superposition of states with different K values. To ensure axial symmetry of the neutron and proton subsystems, the values of the quantum numbers that characterize the SU(3) symmetry of the subsystems are chosen in the form $(\lambda_n,\mu_n)=(n_1,0)$ and $(\lambda_p,\mu_p)=(n_2,0)$. The pair of numbers (λ,μ) characterizing the symmetry of the complete proton–neutron system can take the values $(n_1+n_2,0),(n_1+n_2-2,1),\ldots,(n_1-n_2,n_2)$. (For definiteness, we shall assume that $n_2<n_1$.)

As Elliott showed,[9] the irreducible representation (λ,μ) can be reduced to irreducible representations of the group R_3 of three-dimensional rotations with the following values of the angular momentum:

$$L=N,\ N+1,\ N+2,\ldots,\ N+B,$$

where $N=\min\{\lambda,\mu\}$, $\min\{\lambda,\mu\}-2$, $\min\{\lambda,\mu\}-4,\ldots,1$, or 0; $B=\max\{\lambda,\mu\}$, except for the case when $N=0$, in which $L=B,B-2,\ldots,1$, or 0.

The listed quantum numbers are sufficient for unique identification of the wave functions if $\lambda<2$ or $\mu<2$. If $\lambda\geqslant2$ and $\mu\geqslant2$, the need for an additional quantum number arises. It can, for example, be taken to be the Bargmann–Moshinsky integral ω (Ref. 10). In what follows, the required wave

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TABLE I. Order of filling by nucleons of nuclear shells in the considered model.

Nucleus	Ordering of filling of shells
$^{8}_{4}Be$	$\begin{cases} [000]_1^2\ [100]_1^2 \\ [000]_2^2\ [100]_2^2 \end{cases}$
$^{20}_{10}Ne$	$\begin{cases} [000]_1^2\ [100]_1^2\ [010]_1^2\ [001]_1^2\ [200]_1^2 \\ [000]_2^2\ [100]_2^2\ [010]_2^2\ [001]_2^2\ [200]_2^2 \end{cases}$
$^{44}_{22}Ti$	$\begin{cases} [000]_1^2\ [100]_1^2\ [010]_1^2\ [001]_1^2\ [200]_1^2\ \ldots [011]_1^2\ [300]_1^2 \\ [000]_2^2\ [100]_2^2\ [010]_2^2\ [001]_2^2\ [200]_2^2\ \ldots [011]_2^2\ [300]_2^2 \end{cases}$

functions will be denoted by $|(\lambda,\mu)LM\rangle$, the quantum numbers $\lambda_n=n_1$ and $\lambda_p=n_2$ being omitted for brevity.

2. THE METHOD OF GENERATING FUNCTIONS AND THE FOCK–BARGMANN REPRESENTATION

As we noted above, the wave functions $|(\lambda,\mu)LM\rangle$ are definite linear combinations of products of oscillator functions that depend on the spin coordinates and on the spin–isospin variables of the individual nucleons. It is well known that the explicit determination of many-particle wave functions with given quantum numbers is, in general, a nontrivial problem. The adequate description of nuclear states is considerably simplified by going over from functions that depend on the coordinates of the particles to the images of these functions in the Fock–Bargmann space.[11] Essentially, the mapping of the functions to their images has the effect of separating the dynamical variables of the model and, as it were, "freezing" the degrees of freedom that are not affected by the excitations considered in this model.

The procedure for mapping the functions and operators can be represented in general form as follows.

We denote by $\{\mathbf{u}_1,\mathbf{u}_2,\mathbf{u}_3\}$ a triplet of mutually orthogonal unit vectors, and by $\{\mathbf{v}_1,\mathbf{v}_2,\mathbf{v}_3\}$ another such triplet. Using these vectors, we write in invariant form the normalized three-dimensional single-particle harmonic-oscillator functions with allowance for their dependence on the nucleon spin–isospin variables:

$$\Psi_{q_1,q_2,q_3}(\mathbf{u}_1\cdot\mathbf{r},\mathbf{u}_2\cdot\mathbf{r},\mathbf{u}_3\cdot\mathbf{r}) = \frac{1}{\sqrt{2^{q_1+q_2+q_3}\cdot q_1!q_2!q_3!\cdot\pi^{3/2}}} \times H_{q_1}(\mathbf{u}_1\cdot\mathbf{r})\cdot H_{q_2}(\mathbf{u}_2\cdot\mathbf{r})\cdot H_{q_3}(\mathbf{u}_3\cdot\mathbf{r})\cdot e^{-r^2/2}\cdot\chi_M\cdot\tau_{1/2}, \tag{2.1}$$

$$\Psi_{q_1,q_2,q_3}(\mathbf{v}_1\cdot\mathbf{r},\mathbf{v}_2\cdot\mathbf{r},\mathbf{v}_3\cdot\mathbf{r}) = \frac{1}{\sqrt{2^{q_1+q_2+q_3}\cdot q_1!q_2!q_3!\cdot\pi^{3/2}}} \times H_{q_1}(\mathbf{v}_1\cdot\mathbf{r})\cdot H_{q_2}(\mathbf{v}_2\cdot\mathbf{r})\cdot H_{q_3}(\mathbf{v}_3\cdot\mathbf{r}) \times e^{-r^2/2}\cdot\chi_M\cdot\tau_{-1/2}, \tag{2.2}$$

where $H_{q_i}(z)$ are Chebyshev–Hermite polynomials, and $\mu=\pm1/2$ is the projection of the nucleon spin onto the quantization axis. The functions (2.1) and (2.2) describe the proton and neutron states, respectively, in accordance with the projection of the isotopic spin in them (τ_ν is the isospin function). Note that in the coordinate system in which the x, y, and z axes are directed along the vectors $\mathbf{u}_1$, $\mathbf{u}_2$, and $\mathbf{u}_3$ the coordinate part of the function (2.1) is a product of three one-dimensional oscillator functions, each of which depends on one of the Cartesian coordinates x, y, or z. If, however, the directions of the axes are specified by the vectors $\mathbf{v}_1$, $\mathbf{v}_2$, and $\mathbf{v}_3$, then it will be the coordinate part of the function (2.2) that has such a form.

As is standard in the nuclear shell model, we shall assume that all the single-particle states (2.1) and (2.2) with the same total number $q=q_1+q_2+q_3$ of oscillator quanta belong to the same shell. In this paper, we consider only the configurations in which only the outermost shell, i.e., the shell with the largest value of the quantum number q, is unfilled (open). As an illustration, Table I gives the order of filling of the shells for the nuclei $^{8}_{4}Be$, $^{20}_{10}Ne$, and $^{44}_{22}Ti$, for which there are two protons and two neutrons in the outermost shell. To each pair of square brackets in Table I there corresponds a definite form of the single-particle wave function with a definite set of oscillator quantum numbers q_1,q_2,q_3. The superscript outside the square brackets indicates the number of nucleons in the given state, and the subscript is equal to 1 or 2 according as the single-particle function has the form (2.1) or (2.2).

We list the main properties of the single-particle functions:

- Two functions with different spin or isospin quantum numbers are mutually orthogonal.
- Two functions belonging to different shells are orthogonal.
- The coordinate parts of two functions with different sets of numbers q_1,q_2,q_3 are mutually orthogonal if they both have the form (2.1), and the same is true if they both have the form (2.2).
- The coordinate parts of two functions for which one has the form (2.1) and the other has the form (2.2) are not in general mutually orthogonal if they belong to the same shell, i.e., if they have the same total number q of oscillator quanta.

It is obvious that closed shells with the single-particle functions (2.1) and (2.2) are spherically symmetric. Therefore, the shape of the nucleus is completely determined by the configuration of the nucleons in the outermost unfilled shell. It is usual to call the nucleons of the outermost shell valence nucleons, by analogy with valence electrons in atoms.

We now draw attention to the circumstance that for all three nuclei $^8_4$Be, $^{20}_{10}$Ne, and $^{44}_{22}$Ti the configurations of the outermost shell given in Table I have the form $[q_1 00]$. Therefore, the Chebyshev–Hermite polynomials in (2.1) do not depend on the vectors $\mathbf{u}_2$ and $\mathbf{u}_3$, and, since the exponential part of this function is spherically symmetric, the function (2.1) as a whole is invariant with respect to rotations around the axis directed along the vector $\mathbf{u}_1$. This means that the spatial distribution of the set of all the protons of the nucleus has axial symmetry with the orientation of the symmetry axis given by $\mathbf{u}_1$. Similar arguments enable us to conclude that for the considered configurations the distribution of the neutrons is also axisymmetric with symmetry axis along $\mathbf{v}_1$. The numbers n_1 and n_2, in terms of which the quantum number λ can be expressed, are equal to the product of the number q_1 of oscillator quanta in the corresponding (neutron or proton) outermost shell and the number of the corresponding nucleons in this shell.

We now consider many-particle functions in the form of Slater determinants whose elements are single-particle oscillator functions of the type (2.1) and (2.2). For each particular nucleus, the set of different single-particle functions in the determinant is determined by the corresponding shell configuration. An undoubted advantage of a many-particle function expressed as a Slater determinant is the automatic fulfillment in it of the fundamental Pauli principle and the possibility of separating the center-of-mass coordinate of the system of nucleons, so that the model can be treated as translationally invariant. For the concrete shell configurations considered here, the determinant functions are characterized by a definite SU(3) symmetry of the neutron and proton subsystems with quantum numbers of the form $(\lambda_p,0)$ and $(\lambda_n,0)$, but at the same time the neutron–proton system as a whole does not possess a definite SU(3) symmetry, and, therefore, the quantum numbers (λ,μ) are not integrals of the motion. In addition, in these states the total angular momentum L of the system and its projection M onto the external quantization axis do not have definite values.

However, it is known[12] that a Slater determinant formed from single-particle functions is a generating function that "generates" states $|(\lambda,\mu)LM\rangle$ with the quantum numbers that are necessary in the considered model. For shell configurations with axisymmetric neutron and proton subsystems, the generating function depends on two unit vectors $\mathbf{u}=\mathbf{u}_1$ and $\mathbf{v}=\mathbf{v}_1$, i.e., on four independent parameters.

As a new set of parameters, we choose the three Euler angles α,β,γ, which determine the orientation in the space of the "internal" coordinate system constructed on the basis of the vectors $\mathbf{u}$ and $\mathbf{v}$, and also the parameter $t=\cos\theta$, where θ is the angle between the directions of the vectors $\mathbf{u}$ and $\mathbf{v}$. We denote the set of variables $\{\mathbf{r}_1,\mathbf{r}_2,\ldots,\mathbf{r}_A\}$ on which the wave functions of the nucleus depend by r, and the set of parameters $\{\alpha,\beta,\gamma,\theta\}$ by Ω. Then the expansion of the proton–neutron generating function $\Psi(r,\Omega)$ with respect to states with definite (λ,μ) symmetry and a definite value of the angular momentum can be expressed in the form

$$\Psi(r,\Omega)=\sum_{(\lambda\mu)}\cdot\sum_{LM}\langle r|(\lambda,\mu)LM\rangle\cdot\langle(\lambda,\mu)LM|\Omega\rangle. \tag{2.3}$$

We shall call the function $\langle(\lambda,\mu)LM|\Omega\rangle$, which depends on the collective dynamical variables $\alpha,\beta,\gamma,\theta$, the "image" of the function $\langle r|(\lambda,\mu)LM\rangle$, which depends on the coordinates of the individual nucleons. The "image" contains that part of the information about the original that relates to the dynamics of the valence nucleons in the framework of the model of two axial rotors. The space in which the images are defined is called the Fock–Bargmann space.[11] The transformation of the functions brings with it a transformation of the operators, and it is therefore necessary to formulate the rule that enables us to associate with each operator a certain image that acts on functions depending on the variables Ω.

By definition, we shall say that the operator $\tilde{F}(\Omega)$ is the image of the operator $\hat{F}(r)$ if the following equation holds identically:

$$\langle\Psi(r,\tilde{\Omega})|\tilde{F}(\Omega)|\Psi(r,\Omega)\rangle=\langle\Psi(r,\tilde{\Omega})|\hat{F}(r)|\Psi(r,\Omega)\rangle, \tag{2.4}$$

where the brackets denote integration over the coordinates $\mathbf{r}_1,\mathbf{r}_2,\ldots,\mathbf{r}_A$ and summation over the spin–isospin variables of the nucleons, and the function $\Psi(r,\tilde{\Omega})$ is identical to the generating function $\Psi(r,\Omega)$ apart from the formal renotation $\Omega\rightarrow\tilde{\Omega}$. We simplify the form of the mathematical expressions by replacing the notation for the set of quantum numbers $\{(\lambda,\mu)LM\}$ by the single letter N. Then Eq. (2.3) takes the simple form

$$\Psi(r,\Omega)=\sum_N\langle r|N\rangle\langle N|\Omega\rangle. \tag{2.5}$$

Using this expression, and also taking into account the orthonormality of the functions $\langle r|(\lambda\mu)LM\rangle$ (written in abbreviated form as $\langle r|N\rangle$), we write down the value of the overlap integral of the generating invariants $\Psi(r,\tilde{\Omega})$ and $\Psi(r,\Omega)$:

$$J\equiv\langle\Psi(r,\tilde{\Omega})|\Psi(r,\Omega)\rangle=\sum_{N'}\sum_N\langle\tilde{\Omega}|N\rangle\cdot\delta_{N'N}\cdot\langle N|\Omega\rangle$$
$$=\sum_N\langle\tilde{\Omega}|N\rangle\cdot\langle N|\Omega\rangle. \tag{2.6}$$

Note that the overlap integral J is expressed as a binary sum of the images of the orthonormal functions $\langle r|(\lambda,\mu)LM\rangle$.

We now transform the left- and right-hand sides of the relation (2.4) with allowance for the expansion (2.5):

$$\langle\Psi(r,\tilde{\Omega})|\hat{F}(r)|\Psi(r,\Omega)\rangle=\sum_{N'}\sum_N F_{N'N}\langle\tilde{\Omega}|N'\rangle\cdot\langle N|\Omega\rangle, \tag{2.7}$$

where $F_{N'N}$ are the matrix elements of the operator $\hat{F}(r)$ on the states $\langle r|(\lambda,\mu)LM\rangle$;

$$\langle\Psi(r,\tilde{\Omega})|\tilde{F}(\Omega)|\Psi(r,\Omega)\rangle=\tilde{F}(\Omega)\cdot\langle\Psi(r,\tilde{\Omega})|\Psi(r,\Omega)\rangle$$
$$=\tilde{F}(\Omega)\cdot\sum_{N'}\langle\tilde{\Omega}|N'\rangle\cdot\langle N'|\Omega\rangle$$
$$=\sum_{N'}\langle\tilde{\Omega}|N'\rangle\cdot\langle N'|\tilde{F}(\Omega)|\Omega\rangle$$
$$=\sum_{N'}\sum_{N}\tilde{F}_{N'N}\langle\tilde{\Omega}|N'\rangle\cdot\langle N|\Omega\rangle. \tag{2.8}$$

In Eq. (2.8), $\tilde{F}_{N'N}$ are the coefficients of the expansion with respect to the functions $\langle N|\Omega\rangle$ of the result of applying the operator $\tilde{F}(\Omega)$ to $\langle N'|\Omega\rangle$:

$$\tilde{F}(\Omega)\cdot\langle N'|\Omega\rangle=\sum_{N}\tilde{F}_{N'N}\langle N|\Omega\rangle. \tag{2.9}$$

It follows from Eqs. (2.7) and (2.8) with allowance for (2.4) that

$$F_{N'N}=\tilde{F}_{N'N}. \tag{2.10}$$

On the basis of the obtained relations, we now list schematically the sequence of operations for constructing the required functions, and, where necessary, we make appropriate commentaries:

1. Regarding the coordinates of the vectors **u** and **v** as generator parameters, we construct the second-order Casimir operator of the SU(3) group of the proton–neutron system. Since the dynamical variables of the model are t,α,β,γ, the Casimir operator must also be expressed in terms of these variables.

2. Since the quantum numbers L and M are integrals of the motion, we shall seek the eigenfunctions of the Casimir operator in the form of products of functions that depend on the variable $t=\cos\theta$ and a Wigner D function: $\langle(\lambda,\mu)KLM|\Omega\rangle=F(t)\cdot D^{L}_{KM}(\alpha,\beta,\gamma)$. Since the eigenvalues of the Casimir operator of SU(3) are known, the determination of the explicit form of the functions $F(t)$ reduces to the solution of a quite definite second-order differential equation. The problem of finding the eigenfunctions of the Casimir operator and analyzing their properties will be considered below.

3. We obtain the required functions $\langle(\lambda,\mu)LM|\Omega\rangle$ in the form of linear combinations of the functions $\langle(\lambda,\mu)KLM|\Omega\rangle$ by summing over the quantum number K. The coefficients of the linear combination are found from the additional requirement that the required function be an eigenfunction of the Bargmann–Moshinsky operator. Therefore, this operator must also be constructed in terms of the variables t,α,β,γ.

Note, however, that the normalization of the function found in this manner remains undetermined. Moreover, the very concept of a normalized function needs to be reconsidered in this case, since the concept of a scalar product of functions that depend on the variables α,β,γ,t is not defined.

We shall assume that the functions $\langle(\lambda,\mu)LM|\Omega\rangle$ are correctly normalized (or simply "normalized") if the matrix elements of the transformed operators $\tilde{F}(\Omega)$ on these functions are equal to the matrix elements of the operators $\hat{F}(r)$ on the corresponding original functions $\langle r|(\lambda\mu)LM\rangle$ normalized to unity. By the matrix elements of the transformed operators in this definition, we must understand the expansion coefficients $\tilde{F}_{N'N}$ in the relation (2.9). Thus, the concept of normalization of the wave functions that depend on the collective variables α,β,γ,t is based on Eq. (2.10). It follows from this that in order to obtain normalized functions it is necessary to calculate the overlap integral of the generating functions and to represent it in the form of a binary linear combination (2.6). In this expansion, the functions $\langle(\lambda,\mu)LM|\Omega\rangle$ are then obtained automatically in normalized form. Note that if all the functions $\langle(\lambda,\mu)LM|\Omega\rangle$ are simultaneously multiplied by the same numerical factor, the values of the matrix elements $\tilde{F}_{N'N}$ are not changed [see Eq. (2.8)]. This means that the numerical factor with which the generating function is written down has no fundamental significance.

We now consider the procedure for calculating the overlap integral $\langle\Psi(r,\tilde{\Omega})|\Psi(r,\Omega)\rangle$. We formulate some propositions, which need no proofs, since they are obvious.

1. The overlap integral $\langle\Psi(r,\tilde{\Omega})|\Psi(r,\Omega)\rangle$ for a system of A nucleons can, apart from a numerical constant, be represented as the determinant of an $A\times A$ matrix. The elements of this matrix are overlap integrals (scalar products) of single-particle functions of the form (2.1) or (2.2).

2. The overlap integrals of two single-particle functions with different spin–isospin quantum numbers are equal to zero.

3. The overlap integrals of two single-particle functions belonging to different shells are equal to zero.

4. The overlap integrals of single-particle levels belonging to the same shell do not, in general, vanish.

5. The overlap integral $\langle\Psi(r,\tilde{\Omega})|\Psi(r,\Omega)\rangle$ can be represented as the product of analogous overlap integrals for the individual shells; moreover, the factors of this product corresponding to closed shells are constants.

6. For every separately taken shell, the overlap integral decomposes into factors, each of which is characterized by a definite value of the spin–isospin state of the nucleon.

Suppose that a certain configuration of a system of nucleons is specified. To calculate the overlap integral $\langle\Psi(r,\tilde{\Omega})|\Psi(r,\Omega\rangle$, it is necessary to perform the following sequence of operations:

a) Distinguish the single-particle states relating to individual unfilled shells (in the concrete examples given above, there is one such shell in each nucleus).

b) In each unfilled shell, distinguish the groups of single-particle functions with the same spin–isospin part.

c) For each group, form determinants from the overlap integrals of the corresponding single-particle functions.

d) Multiply the determinants obtained in this manner.

As an illustration, we calculate the overlap integral $\langle\Psi(r,\tilde{\Omega})|\Psi(r,\Omega)\rangle$ for the nucleon configurations in the nuclei ${}^{8}_{4}\mathrm{Be}$, ${}^{20}_{10}\mathrm{Ne}$, and ${}^{44}_{22}\mathrm{Ti}$. All four single-particle functions of the outermost shell in each of these nuclei differ from each other in the spin–isospin quantum numbers. The coordinate parts of these functions have the following form [see (2.1) and (2.2); in the following, the vectors $\mathbf{u}_2,\mathbf{u}_3,\mathbf{v}_2,\mathbf{v}_3$ are not encountered in the text, and for simplicity we shall write **u** and **v** instead of $\mathbf{u}_1$ and $\mathbf{v}_1$]:

$$\Psi^{(1)}_{q_1,0,0}(\mathbf{u}\cdot\mathbf{r})=\frac{1}{\sqrt{2^{q_1}\cdot q_1!\cdot\pi^{3/2}}}\cdot H_{q_1}(\mathbf{u}\cdot\mathbf{r})\cdot e^{-r^2/2}, \quad (2.1')$$

$$\Psi^{(2)}_{q_1,0,0}(\mathbf{v}\cdot\mathbf{r})=\frac{1}{\sqrt{2^{q_1}\cdot q_1!\cdot\pi^{3/2}}}\cdot H_{q_1}(\mathbf{v}\cdot\mathbf{r})\cdot e^{-r^2/2}, \quad (2.2')$$

where $q_1=1$ for the nucleus $^8_4$Be, $q_1=2$ for $^{20}_{10}$Ne, and $q_1=3$ for $^{44}_{22}$Ti. Taking into account the recommendations formulated above, we can write the required overlap integral in the form

$$\langle\Psi(r,\tilde{\Omega})|\Psi(r,\Omega)\rangle=[\langle\tilde{\Psi}^{(1)}_{q_1,0,0}|\Psi^{(1)}_{q_1,0,0}\rangle]^2 \times[\langle\tilde{\Psi}^{(2)}_{q_1,0,0}|\Psi^{(2)}_{q_1,0,0}\rangle]^2, \quad (2.11)$$

where $\tilde{\Psi}^{(1)}_{q,0,0}$ and $\tilde{\Psi}^{(2)}_{q,0,0}$ are obtained from the corresponding functions without a tilde by the replacement of u by $\tilde{u}$ and v by $\tilde{v}$. We find the value of the overlap integral of the single-particle functions:

$$\langle\tilde{\Psi}^{(1)}_{q_1,0,0}|\Psi^{(1)}_{q_1,0,0}\rangle=\frac{1}{2^{q_1}\cdot q_1!\,\pi^{3/2}}\cdot\int\int\int_{-\infty}^{+\infty} H_{q_1}(\tilde{\mathbf{u}}\cdot\mathbf{r})$$

$$\times H_{q_1}(\mathbf{u}\cdot\mathbf{r})\cdot e^{-(x^2+y^2+z^2)}dxdydz$$

$$=\frac{1}{2^{q_1}\cdot q_1!\,\pi^{3/2}}\int\int\int_{-\infty}^{+\infty} H_{q_1}((\mathbf{u}\cdot\tilde{\mathbf{u}})\cdot x+\ldots)$$

$$\times e^{-(x^2+y^2+z^2)}dxdydz$$

$$=\frac{1}{2^{q_1}\cdot q_1!\,\pi^{1/2}}\cdot(\mathbf{u}\cdot\tilde{\mathbf{u}})^{q_1}\int\int\int_{-\infty}^{+\infty} H_{q_1}(x)$$

$$\times H_{q_1}(x)\cdot e^{-x^2}dx=(\mathbf{u}\cdot\tilde{\mathbf{u}})^{q_1}.$$

We note that the calculations of the integral are conveniently made in a coordinate system with the x axis along the vector $\tilde{\mathbf{u}}_1$. By analogy, we write down the overlap integral of the functions $\tilde{\Psi}^{(2)}_{q_1,0,0}$ and $\Psi^{(2)}_{q_1,0,0}$:

$$\langle\tilde{\Psi}^{(2)}_{q_1,0,0}|\Psi^{(2)}_{q_1,0,0}\rangle=(\mathbf{v}\cdot\tilde{\mathbf{v}})^{q_1}.$$

Substituting these values of the overlap integrals in (2.11), we obtain

$$\langle\Psi(r,\tilde{\Omega})|\Psi(r,\Omega)\rangle=(\mathbf{u}\cdot\tilde{\mathbf{u}})^{2q_1}\cdot(\mathbf{v}\cdot\tilde{\mathbf{v}})^{2q_1},$$

or

$$\langle\Psi(r,\tilde{\Omega})|\Psi(r,\Omega)\rangle=\begin{cases}(\mathbf{u}\cdot\tilde{\mathbf{u}})^2\cdot(\mathbf{v}\cdot\tilde{\mathbf{v}})^2 & \text{for } ^8_4\text{Be}\\ (\mathbf{u}\cdot\tilde{\mathbf{u}})^4\cdot(\mathbf{v}\cdot\tilde{\mathbf{v}})^4 & \text{for } ^{20}_{10}\text{Ne}\\ (\mathbf{u}\cdot\tilde{\mathbf{u}})^6\cdot(\mathbf{v}\cdot\tilde{\mathbf{v}})^6 & \text{for } ^{44}_{22}\text{Ti}.\end{cases}$$

Note that in the model of two axial rotors the overlap integral in the most general case has the form

$$\langle\Psi(r,\tilde{\Omega})|\Psi(r,\Omega)\rangle=(\mathbf{u}\cdot\tilde{\mathbf{u}})^{n_1}\cdot(\mathbf{v}\cdot\tilde{\mathbf{v}})^{n_2}.$$

Thus, the overlap integral is a homogeneous polynomial of degree n_1 in the components of the vector **u** and of degree n_2 in the components of **v**.

3. GENERATORS OF THE SU(3)×SU(3) MODEL

As we noted above, we restrict ourselves in this paper to the case in which the indices of the irreducible representations of the group SU(3) of each of the two subsystems have the form $(\lambda_n,\mu_n)=(n_1,0)$ and $(\lambda_p,\mu_p)=(n_2,0)$. Each of these irreducible representations can be realized on tensor products of the vectors **u** and **v**:

$$\underbrace{\mathbf{u}\times\mathbf{u}\times\ldots\times\mathbf{u}}_{n_1\ \text{times}} \quad \text{for the first subsystem,}$$

$$\underbrace{\mathbf{v}\times\mathbf{v}\times\ldots\times\mathbf{v}}_{n_1\ \text{times}} \quad \text{for the second subsystem.}$$

The direct product $(n_1,0)\times(n_2,0)$ contains all irreducible representations of SU(3) having the symmetry indices

$$(\lambda,\mu)=(n_1+n_2-2m,m), \quad m=0,1,2,\ldots,n_2.$$

Suppose that we are given a Cartesian coordinate system, which in what follows we shall call the "laboratory" or "fixed" system. We take the six components $\{x_1,y_1,z_1;x_2,y_2,z_2\}$ of the vectors **u** and **v** in this system as the original generator parameters and associate with them in the usual manner generators of the group SU(3) for each of the two subsystems. In what follows, we shall also use, besides the Cartesian coordinates of **u** and **v**, their spherical components: $\{u,\theta_1,\varphi_1;v,\theta_2,\varphi_2\}$.

From the Cartesian or spherical generator parameters, we can form different combinations of them that have different meanings. For example, the angle θ between the vectors **u** and **v** is related to $\theta_1,\varphi_1,\theta_2,\varphi_2$ by

$$\cos\theta=\sin\theta_1\sin\theta_2\cos(\varphi_2-\varphi_1)+\cos\theta_1\cos\theta_2 \quad (3.1)$$

Following Refs. 1–3, we construct by means of the vectors **u** and **v** the "intrinsic," or "moving," rectangular coordinate system with unit vectors

$$\mathbf{e}_1=\frac{\mathbf{n}_1+\mathbf{n}_2}{|\mathbf{n}_1+\mathbf{n}_2|},\quad \mathbf{e}_2=\frac{\mathbf{n}_2-\mathbf{n}_1}{|\mathbf{n}_2+\mathbf{n}_1|},\quad \mathbf{e}_3=\frac{\mathbf{n}_1\times\mathbf{n}_2}{|\mathbf{n}_1\times\mathbf{n}_2|},$$

where

$$n_1=\frac{\mathbf{u}}{|\mathbf{u}|},\quad n_2=\frac{\mathbf{v}}{|\mathbf{v}|}.$$

In the intrinsic system, the components of the vectors **u** and **v** can be written in the form

$$u=\{\xi_1,\xi_2,0\},\qquad v=\{\eta_1,\eta_2,0\},$$

$$\xi_1=u\cdot\cos\frac{\theta}{2},\qquad \eta_1=v\cdot\cos\frac{\theta}{2},\tag{3.2}$$

$$\xi_2=-u\cdot\sin\frac{\theta}{2},\qquad \eta_2=v\cdot\sin\frac{\theta}{2}.$$

Of course, ξ_1,ξ_2,n_1,n_2 can be expressed in terms of the spherical components $u,\theta_1,\varphi_1,v,\theta_2,\varphi_2$ if we use the relation (3.1). The orientation of the moving (intrinsic) coordinate system with respect to the laboratory system can be specified by means of a rotation matrix, the elements of which are direction cosines:

$$d_{11}=(\mathbf{i}\cdot\mathbf{e}_1)=\frac{1}{2\cos\frac{\theta}{2}}\cdot(\sin\theta_1\cos\varphi_1+\sin\theta_2\cos\varphi_2),$$

$$d_{21}=(\mathbf{j}\cdot\mathbf{e}_1)=\frac{1}{2\cos\frac{\theta}{2}}\cdot(\sin\theta_1\sin\varphi_1+\sin\theta_2\sin\varphi_2),$$

$$d_{31}=(\mathbf{k}\cdot\mathbf{e}_1)=\frac{1}{2\cos\frac{\theta}{2}}\cdot(\cos\theta_1+\cos\theta_2),$$

$$d_{12}=(\mathbf{i}\cdot\mathbf{e}_2)=\frac{1}{2\sin\frac{\theta}{2}}\cdot(-\sin\theta_1\cdot\cos\varphi_1+\sin\theta_2\cdot\cos\varphi_2),$$

$$d_{22}=(\mathbf{j}\cdot\mathbf{e}_2)=\frac{1}{2\sin\frac{\theta}{2}}\cdot(-\sin\theta_1\cdot\sin\varphi_1+\sin\theta_2\cdot\sin\varphi_2),$$

$$d_{32}=(\mathbf{k}\cdot\mathbf{e}_2)=\frac{1}{2\sin\frac{\theta}{2}}\cdot(-\cos\theta_1+\cos\theta_2),$$

$$d_{13}=(\mathbf{i}\cdot\mathbf{e}_3)=\frac{1}{\sin\theta}\cdot(\sin\theta_1\cdot\sin\varphi_1\cdot\cos\theta_2-\cos\theta_1\cdot\sin\theta_2\times\sin\varphi_2),$$

$$d_{23}=(\mathbf{j}\cdot\mathbf{e}_3)=\frac{1}{\sin\theta}\cdot(\cos\theta_1\cdot\sin\theta_2\cdot\cos\varphi_2-\sin\theta_1\times\cos\varphi_1\cdot\cos\theta_2),$$

$$d_{33}=(\mathbf{k}\cdot\mathbf{e}_3)=\frac{1}{\sin\theta}\cdot\sin\theta_1\cdot\sin\theta_2\cdot\sin(\varphi_2-\varphi_1).$$

Equating the elements of the rotation matrix expressed in terms of $\theta_1,\varphi_1,\theta_2,\varphi_2$ to the corresponding elements expressed in terms of the Euler angles α,β,γ (Ref. 13) and taking into account the relation (3.1), we can establish the relationship between the variables $\{\theta_1,\varphi_1,\theta_2,\varphi_2\}$, on the one hand, and $\{\theta,\alpha,\beta,\gamma\}$, on the other. Thus, in what follows we shall use the following equivalent sets of independent variables:

$$\begin{Bmatrix} x_1\\ y_1\\ z_1\\ x_2\\ y_2\\ z_2\end{Bmatrix}\leftrightarrow\begin{Bmatrix} u\\ \theta_1\\ \varphi_1\\ v\\ \theta_2\\ \varphi_2\end{Bmatrix}\leftrightarrow\begin{Bmatrix} u\\ v\\ \theta\\ \alpha\\ \beta\\ \gamma\end{Bmatrix}.\tag{3.3}$$

In the final column in (3.3), we have explicitly separated the dynamical variables $\theta,\alpha,\beta,\gamma$ of the collective motions in the nuclei in the framework of the model that we are considering.

One of our tasks in this paper is to construct the Bargmann–Moshinsky operator in terms of the variables $\theta,\alpha,\beta,\gamma$. For this, we first find the generators of SU(3) in the intrinsic coordinate system, doing this both for each of the subsystems and for the complete system. In (3.2) above, we have already given the Cartesian coordinates ξ_i and η_i of the vectors $\mathbf{u}$ and $\mathbf{v}$ in the intrinsic system, and, therefore, to construct the group generators it is now necessary to write down the expressions for the derivatives $\partial/\partial\xi_i$ and $\partial/\partial\eta_i$. Since $\partial/\partial\xi_i$ are the components of the nabla operator in the moving coordinate system, they can be expressed in terms of the corresponding components $\partial/\partial x_i^{(1)}=\{\partial/\partial x_1\partial/\partial y_1\partial/\partial z_1\}$ of this vector in the laboratory system by means of a rotation matrix:

$$\frac{\partial}{\partial\xi_1}=\sum_i d_{1i}\frac{\partial}{\partial x_i^{(1)}};\quad \frac{\partial}{\partial\xi_2}=\sum_i d_{2i}\frac{\partial}{\partial x_i^{(1)}};$$

$$\frac{\partial}{\partial\xi_3}=\sum_i d_{3i}\frac{\partial}{\partial x_i^{(1)}}.\tag{3.4}$$

Similarly,

$$\frac{\partial}{\partial\eta_1}=\sum_i d_{1i}\frac{\partial}{\partial x_i^{(2)}};\quad \frac{\partial}{\partial\eta_2}=\sum_i d_{2i}\frac{\partial}{\partial x_i^{(2)}};$$

$$\frac{\partial}{\partial\eta_3}=\sum_i d_{3i}\frac{\partial}{\partial x_i^{(2)}}.\tag{3.4'}$$

Equations (3.4) and (3.4′) in conjunction with (3.2) enable us to express the derivatives $\partial/\partial\xi_i$ and $\partial/\partial\eta_i$ in terms of the variables $u,v,\theta,\alpha,\beta,\gamma$. Making the necessary calculations, we obtain

$$\frac{\partial}{\partial\xi_1}=\cos\frac{\theta}{2}\cdot\frac{\partial}{\partial u}-\frac{1}{u}\sin\frac{\theta}{2}\cdot\left(\frac{\partial}{\partial\theta}+\frac{1}{2}M_3\right),$$

$$\frac{\partial}{\partial\xi_2}=-\sin\frac{\theta}{2}\cdot\frac{\partial}{\partial u}-\frac{1}{u}\cos\frac{\theta}{2}\cdot\left(\frac{\partial}{\partial\theta}+\frac{1}{2}M_3\right),$$

$$\frac{\partial}{\partial\xi_3}=\frac{1}{2u}\cdot\left(\frac{1}{\sin\frac{\theta}{2}}\cdot M_1+\frac{1}{\cos\frac{\theta}{2}}\cdot M_2\right),$$

$$\frac{\partial}{\partial\eta_1}=\cos\frac{\theta}{2}\cdot\frac{\partial}{\partial v}+\frac{1}{v}\sin\frac{\theta}{2}\cdot\left(-\frac{\partial}{\partial\theta}+\frac{1}{2}M_3\right),$$

$$\frac{\partial}{\partial\eta_2}=\sin\frac{\theta}{2}\cdot\frac{\partial}{\partial v}-\frac{1}{v}\cos\frac{\theta}{2}\cdot\left(-\frac{\partial}{\partial\theta}+\frac{1}{2}M_3\right),$$

$$\frac{\partial}{\partial\eta_3}=\frac{1}{2v}\left(-\frac{1}{\sin\frac{\theta}{2}}\cdot M_1+\frac{1}{\cos\frac{\theta}{2}}\cdot M_2\right), \tag{3.4''}$$

where M_1, M_2, M_3 are, apart from a factor i (the imaginary unit), the operators of the projections of the angular momentum of the complete system onto the moving coordinate axes, and they are determined by the equations

$$M_1=-\left(\xi_2\frac{\partial}{\partial\xi_3}-\xi_3\frac{\partial}{\partial\xi_2}+\eta_2\frac{\partial}{\partial\eta_3}-\eta_3\frac{\partial}{\partial\eta_2}\right)$$

$$=-\sin\gamma\cdot\frac{\partial}{\partial\beta},$$

$$M_2=-\left(\xi_3\frac{\partial}{\partial\xi_1}-\xi_1\frac{\partial}{\partial\xi_3}+\eta_3\frac{\partial}{\partial\eta_1}-\eta_1\frac{\partial}{\partial\eta_3}\right)$$

$$=-\cos\gamma\cdot\frac{\partial}{\partial\beta},$$

$$M_3=-\left(\xi_1\frac{\partial}{\partial\xi_2}-\xi_2\frac{\partial}{\partial\xi_1}+\eta_1\frac{\partial}{\partial\eta_2}-\eta_2\frac{\partial}{\partial\eta_1}\right)=-\frac{\partial}{\partial\gamma}.$$

We now write down all the group generators associated with the vector $\mathbf{u}$ in the moving coordinate system:

$$\xi_1\frac{\partial}{\partial\xi_1}=\cos^2\frac{\theta}{2}\cdot u\frac{\partial}{\partial u}-\sin\frac{\theta}{2}\cdot\cos\frac{\theta}{2}\cdot\left(\frac{\partial}{\partial\theta}+\frac{1}{2}M_3\right),$$

$$\xi_1\frac{\partial}{\partial\xi_2}=-\sin\frac{\theta}{2}\cdot\cos\frac{\theta}{2}\cdot u\frac{\partial}{\partial u}-\cos^2\frac{\theta}{2}$$

$$\cdot\left(\frac{\partial}{\partial\theta}+\frac{1}{2}M_3\right),$$

$$\xi_1\frac{\partial}{\partial\xi_3}=\frac{1}{2}M_2+\frac{1}{2}\cot\frac{\theta}{2}\cdot M_1,$$

$$\xi_2\frac{\partial}{\partial\xi_1}=-\sin\frac{\theta}{2}\cdot\cos\frac{\theta}{2}\cdot u\frac{\partial}{\partial u}+\sin^2\frac{\theta}{2}\cdot\left(\frac{\partial}{\partial\theta}+\frac{1}{2}M_3\right),$$

$$\xi_2\frac{\partial}{\partial\xi_2}=\sin^2\frac{\theta}{2}\cdot u\frac{\partial}{\partial u}+\sin\frac{\theta}{2}\cdot\cos\frac{\theta}{2}\cdot\left(\frac{\partial}{\partial\theta}+\frac{1}{2}M_3\right),$$

$$\xi_2\frac{\partial}{\partial\xi_3}=\frac{1}{2}M_1+\frac{1}{2}\tan\frac{\theta}{2}\cdot M_2,$$

$$\xi_3\frac{\partial}{\partial\xi_1}=\xi_3\frac{\partial}{\partial\xi_3}=\xi_3\frac{\partial}{\partial\xi_3}=0.$$

We can write down similarly the generators associated with the vector $\mathbf{v}$. It is obvious that the expressions for the generators $\eta_i\partial/\partial\eta_j$ can be obtained from $\xi_i\partial/\partial\xi_j$ by the formal substitutions $u\to v, \theta\to-\theta$.

4. THE CASIMIR OPERATOR G_2

The Casimir operator G_2 is the quadratic scalar contraction of the generators $A_{ij}=u_i\partial/\partial u_j+v_i\partial/\partial v_j$ of SU(3) [or the group U(3)]: $G_2=A_{ij}A_{ji}$. Our next task is to transform the Casimir operator from the Cartesian components $\{u_i, v_i\}$ of the vectors $\mathbf{u}$ and $\mathbf{v}$ to the dynamical variables $\{\theta,\alpha,\beta,\gamma\}$. For this, we make some preliminary transformations:

$$G_2=\left(u_i\cdot\frac{\partial}{\partial u_j}+v_i\cdot\frac{\partial}{\partial v_j}\right)\left(u_j\cdot\frac{\partial}{\partial u_i}+v_j\cdot\frac{\partial}{\partial v_i}\right)$$

$$=u_i\frac{\partial}{\partial u_j}u_j\frac{\partial}{\partial u_i}+u_i\frac{\partial}{\partial u_j}v_j\frac{\partial}{\partial v_i}+v_i\frac{\partial}{\partial v_j}u_j\frac{\partial}{\partial u_i}$$

$$+v_i\frac{\partial}{\partial v_j}v_j\frac{\partial}{\partial v_i}. \tag{4.1}$$

By a simple commutation of the operators, the first term on the right-hand side of Eq. (4.1) can be transformed as follows:

$$u_i\frac{\partial}{\partial u_j}u_j\frac{\partial}{\partial u_i}=\left(u_i\frac{\partial}{\partial u_i}\right)^2+2u_i\frac{\partial}{\partial u_i}$$

$$=\left(u\frac{\partial}{\partial u}\right)^2+2u\frac{\partial}{\partial u}. \tag{4.2}$$

We recall that in the considered problem all the operators (including G_2) are defined in the space of homogeneous polynomials of degree n_1+n_2 with respect to the components of the vectors $\mathbf{u}$ and $\mathbf{v}$, where n_1 is the sum of the degrees of the components of the vector $\mathbf{u}$, and n_2 is the same for $\mathbf{v}$. Such polynomials are eigenfunctions of the operators

$$u_i\frac{\partial}{\partial u_i}\equiv u\frac{\partial}{\partial u}\quad\text{and}\quad v_i\frac{\partial}{\partial v_i}\equiv v\frac{\partial}{\partial v}$$

with eigenvalues n_1 and n_2,

respectively. Therefore, in (4.1) the operator (4.2) can be replaced by its eigenvalue:

$$u_i\frac{\partial}{\partial u_j}u_j\frac{\partial}{\partial u_i}=n_1(n_1+2),$$

and similarly

$$v_i\frac{\partial}{\partial v_j}v_j\frac{\partial}{\partial v_i}=n_2(n_2+2).$$

Further,

$$u_i\frac{\partial}{\partial u_j}v_j\frac{\partial}{\partial v_i}=u_i\frac{\partial}{\partial u_j}\left(\frac{\partial}{\partial v_i}v_j-\delta_{ij}\right)$$

$$=u_i\frac{\partial}{\partial v_i}v_j\frac{\partial}{\partial u_j}-u_i\frac{\partial}{\partial u_i}$$

$$=(\mathbf{u}\cdot\nabla^v)(\mathbf{v}\cdot\nabla^u)-n_1.$$

Having made analogous transformations in the following term of the expression (4.1), we obtain

$$v_i\frac{\partial}{\partial v_j}u_j\frac{\partial}{\partial u_i}=(\mathbf{v}\cdot\nabla^u)(\mathbf{u}\cdot\nabla^v)-n_2.$$

The right-hand side of this last equation can be written differently as

$$v_i \frac{\partial}{\partial v_j} u_j \frac{\partial}{\partial u_i} = u_j v_i \frac{\partial}{\partial v_j} \frac{\partial}{\partial u_j}$$

$$= u_j \left(\frac{\partial}{\partial v_j} v_i - \delta_{ij} \right) \frac{\partial}{\partial u_i}$$

$$= (\mathbf{u}\cdot\boldsymbol{\nabla}^v)(\mathbf{v}\cdot\boldsymbol{\nabla}^u) - n_1 .$$

Therefore, in the considered function space the commutator of the operators $(\mathbf{u}\cdot\boldsymbol{\nabla}^v)$ and $(\mathbf{v}\cdot\boldsymbol{\nabla}^u)$ is simply equal to the difference of the numbers n_1 and n_2:

$$[(\mathbf{u}\cdot\boldsymbol{\nabla}^v),(\mathbf{v}\cdot\boldsymbol{\nabla}^u)] = n_1 - n_2 . \tag{4.3}$$

Thus, the Casimir operator can be represented by one of the following expressions:

$$G_2 = 2(\mathbf{u}\cdot\boldsymbol{\nabla}^v)(\mathbf{v}\cdot\boldsymbol{\nabla}^U) + n_1^2 + n_2(n_2+1),$$

$$G_2 = 2(\mathbf{v}\cdot\boldsymbol{\nabla}^u)(\mathbf{u}\cdot\boldsymbol{\nabla}^v) + n_2^2 + n_1(n_1+1),$$

$$G_2 = (\mathbf{u}\cdot\boldsymbol{\nabla}^v)(\mathbf{v}\cdot\boldsymbol{\nabla}^u) + (\mathbf{v}\cdot\boldsymbol{\nabla}^u)(\mathbf{u}\cdot\boldsymbol{\nabla}^v) + n_1(n_1+1) + n_2(n_2+1). \tag{4.4}$$

In what follows, we shall use the third form of the operator G_2 given here; it is symmetric with respect to the interchange of $\mathbf{u}$ and $\mathbf{v}$.

Taking into account the relations[13]

$$\boldsymbol{\nabla}_u = \mathbf{n}_u \frac{\partial}{\partial u} - \frac{i}{u}[\mathbf{n}_u \times \mathbf{l}_u]; \quad \boldsymbol{\nabla} v = \mathbf{n}_v \frac{\partial}{\partial v} - \frac{i}{v}[\mathbf{n}_v \times \mathbf{l}_v],$$

where $\mathbf{n}_u = \mathbf{u}/u$, $\mathbf{n}_v = \mathbf{v}/v$, and $\mathbf{l}_u$ and $\mathbf{l}_v$ are the operators of the orbital angular momentum acting on the variables u_i and v_i, we transform the scalar products $(\mathbf{u}\cdot\boldsymbol{\nabla}_u)$ and $(\mathbf{v}\cdot\boldsymbol{\nabla}_u)$:

$$(\mathbf{u}\cdot\boldsymbol{\nabla}_v) = u(\mathbf{n}_u\cdot\mathbf{n}_v) \frac{\partial}{\partial v} - i \frac{u}{v} (n_u \cdot [\mathbf{n}_v \times \mathbf{l}_v])$$

$$= t\cdot u \frac{\partial}{\partial v} - i \frac{u}{v} ([\mathbf{n}_u \times \mathbf{n}_v]\cdot \mathbf{l}_v)$$

$$= tu \frac{\partial}{\partial v} - i \frac{u}{v} \sqrt{1-t^2}\, l_v^{(3)},$$

where $t = \cos\theta$, and $l_v^{(3)}$ is the projection of the angular-momentum operator $\mathbf{l}_v$ onto the third axis of the moving (intrinsic) coordinate system.

Proceeding similarly, we obtain

$$(\mathbf{v}\cdot\boldsymbol{\nabla}_u) = tv \frac{\partial}{\partial u} + i \frac{v}{u} \sqrt{1-t^2}\, l_u^{(3)} .$$

Therefore

$$(\mathbf{u}\cdot\boldsymbol{\nabla}_v)(\mathbf{v}\cdot\boldsymbol{\nabla}_u)$$

$$= \left[tu \frac{\partial}{\partial v} - i \frac{u}{v} \sqrt{1-t^2}\, l_v^{(3)} \right] \cdot \left[tv \frac{\partial}{\partial u} + i \frac{v}{u} \sqrt{1-t^2}\, l_u^{(3)} \right]$$

$$= n_1(n_2+1)t^2 - i n_1 \sqrt{1-t^2}\, l_v^{(3)} t + i(n_2+1) t \sqrt{1-t^2}\, l_u^{(3)} + \sqrt{1-t^2}\, l_v^{(3)} \sqrt{1-t^2}\, l_u^{(3)} ;$$

$$(\mathbf{v}\cdot\boldsymbol{\nabla}_u)(\mathbf{u}\cdot\boldsymbol{\nabla}_v)$$

$$= n_2(n_1+1)t^2 - i(n_1+1) t \sqrt{1-t^2}\, l_v^{(3)} + i n_2 \sqrt{1-t^2}\, l_u^{(3)} t + \sqrt{1-t^2}\, l_u^{(3)} \sqrt{1-t^2}\, l_v^{(3)} . \tag{4.5}$$

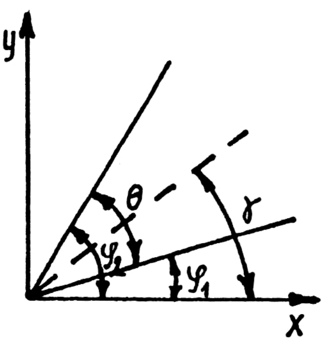

FIG. 1. Relation between the degrees of freedom φ_1 and φ_2 of the two subsystems with variables θ, φ characterizing the position of the system as a whole.

We now obtain the explicit form of the operators $l_u^{(3)}$ and $l_v^{(3)}$. It is obvious that they can be represented in the form

$$l_u^{(3)} = \frac{1}{2} L_3 + \tilde{l}_u^{(3)}; \quad l_v^{(3)} = \frac{1}{2} L_3 + \tilde{l}_v^{(3)},$$

where L_3 is the projection of the total angular momentum onto the third axis of the intrinsic coordinate system, and $\tilde{l}_u^{(3)}$ and $\tilde{l}_v^{(3)}$ are the projections onto the same axis of the angular momenta of the "intrinsic" motion of the subsystems.

Since in the considered model there is only one degree of freedom, θ of the intrinsic motion, it is obvious that

$$l_u^{(3)} = -i\left(\frac{1}{2} \frac{\partial}{\partial \gamma} + c \frac{\partial}{\partial \theta} \right); \quad l_v^{(3)} = -i\left(\frac{1}{2} \frac{\partial}{\partial \gamma} - c \frac{\partial}{\partial \theta} \right),$$

where c is a constant. The value of this constant can be found by considering the relationship between the degrees of freedom φ_1 and φ_2 of the subsystems and the variables θ and γ in the simple case of motion relative to a fixed axis (see Fig. 1):

$$\left.\begin{aligned} \varphi_1 &= \gamma - \frac{\theta}{2} \\ \varphi_2 &= \gamma + \frac{\theta}{2} \end{aligned}\right\} \Rightarrow \begin{aligned} \gamma &= \frac{1}{2}(\varphi_1 + \varphi_2) \\ \theta &= \varphi_2 - \varphi_1 \end{aligned}$$

$$l_u^{(3)} = -i \frac{\partial}{\partial \varphi_1} = -i\left(\frac{\partial}{\partial \gamma} \frac{\partial \gamma}{\partial \varphi_1} + \frac{\partial}{\partial \theta} \frac{\partial \theta}{\partial \varphi_1} \right)$$

$$= -i\left(\frac{1}{2} \frac{\partial}{\partial \gamma} - \frac{\partial}{\partial \theta} \right),$$

$$l_v^{(3)} = -i \frac{\partial}{\partial \varphi_2} = -i\left(\frac{1}{2} \frac{\partial}{\partial \gamma} + \frac{\partial}{\partial \theta} \right).$$

Thus, it follows from the last equation that the constant c is equal to zero.

Bearing in mind that $\partial/\partial\theta = -\sin\theta\, \partial/\partial(\cos\theta) = -\sqrt{1-t^2}\,\partial/\partial t$, we finally obtain

$$l_u^{(3)} = -i\left(\frac{1}{2} \frac{\partial}{\partial \gamma} + \sqrt{1-t^2} \frac{\partial}{\partial t} \right);$$

$$l_v^{(3)} = -i\left(\frac{1}{2}\frac{\partial}{\partial\gamma} - \sqrt{1-t^2}\,\frac{\partial}{\partial t}\right). \tag{4.6}$$

Substituting (4.6) in (4.5), we write down the required expression for the Casimir operator G_2:

$$G_2 = 2(1-t^2)^2\frac{\partial^2}{\partial t^2} + 2(n_1+n_2-1)t(1-t^2)\frac{\partial}{\partial t} + (n_2-n_1)t\sqrt{1-t^2}\,\frac{\partial}{\partial\gamma} - \frac{1}{2}(1-t^2)\cdot\frac{\partial^2}{\partial\gamma^2} - 2n_1n_2(1-t^2) + (n_1+n_2)^2 + 2(n_1+n_2). \tag{4.7}$$

5. EIGENFUNCTIONS OF THE CASIMIR OPERATOR G_2

We consider the problem of finding the function $\varphi(t,\alpha,\beta,\gamma)$, which depends on all four dynamical variables of the model and is an eigenfunction of not only the operator of the square of the angular momentum and its projections onto the external and internal axes but also the Casimir operator G_2. We shall seek the solution to this problem in the form

$$\varphi(t,\alpha,\beta,\gamma) = y(t)\cdot D^L_{KM}(\alpha,\beta,\gamma).$$

Independently of the choice of $y(t)$, the function $\varphi(t,\alpha,\beta,\gamma)$ is characterized by the quantum numbers K,L,M. If these functions are to transform in accordance with an irreducible representation of SU(3), they must be eigenfunctions of the Casimir operator G_2. The eigenvalues g_2 of G_2 are known and can be expressed simply in terms of the indices (λ,μ) of the irreducible representations of SU(3) [or the group U(3)]:[15]

$$g_2 = \lambda^2 + 2\lambda\mu + 2\mu^2 + 2\lambda + 2\mu.$$

In the model that we consider, $\lambda = n_1+n_2-2m$ and $\mu = m$, and therefore

$$g_2 = (n_1+n_2)^2 - 2(m-1)(n_1+n_2-m).$$

We note that the Wigner D functions are eigenfunctions of the operators $\partial/\partial\gamma$ and $\partial^2/\partial\gamma^2$, which occur in G_2, since the dependence of the D functions on the variable γ is determined by the factor $e^{iK\gamma}$. Thus, the problem of finding the eigenfunctions of the Casimir operator reduces to the solution of the second-order ordinary differential equation

$$(1-t^2)^2\cdot\frac{d^2y}{dt^2} + (n_1+n_2-1)t(1-t^2)\frac{dy}{dt} + \left[\left(\frac{1}{4}K^2 - n_1n_2\right)(1-t^2) - i\frac{1}{2}K(n_1-n_2)t\sqrt{1-t^2} + m(n_1+n_2-m+1)\right]\cdot y(t) = 0. \tag{5.1}$$

We consider the symmetry properties of Eq. (5.1). We note, first, that for $K=0$ it is unchanged by transposition of the parameters n_1 and n_2. In the general case $K\neq 0$, this equation is complex, and it is invariant with respect to simultaneous application of any pair of the following three operations: the transposition $n_1\leftrightarrow n_2$, the substitution $K\to -K$, and the operation of complex conjugation.

Making in (5.1) the substitution

$$y(t) = (1-t^2)^{m/2}\cdot u(t), \tag{5.2}$$

we write down the equation for the new unknown function $u(t)$:

$$(1-t^2)\frac{d^2u}{dt^2} + (N_1+N_2-1)t\frac{du}{dt} + \left[\frac{1}{4}K^2 - N_1N_2 - i\frac{1}{2}K(N_1-N_2)\frac{t}{\sqrt{1-t^2}}\right]\cdot u(t) = 0, \tag{5.3}$$

where $N_1 = n_1 - m$, $N_2 = n_2 - m$.

All the arguments given above concerning the symmetry of Eq. (5.1) are also true for Eq. (5.3) if for it we use in place of the transposition $n_1\leftrightarrow n_2$ the transposition $N_1\leftrightarrow N_2$.

We rewrite the last equation, replacing N_1 and N_2 in it by the integer parameters $\lambda = N_1+N_2 = n_1+n_2-2m$ and $q = N_1-N_2 = n_1-n_2$:

$$(1-t^2)\cdot\frac{d^2u}{dt^2} + (\lambda-1)t\cdot\frac{du}{dt} + \frac{1}{4}\left[K^2+q^2-\lambda^2 - i2Kq\cdot\frac{t}{\sqrt{1-t^2}}\right]\cdot u(t) = 0. \tag{5.3'}$$

It follows directly from this form of the equation that it is invariant with respect to transposition of the parameters K and q.

We first consider the solution of Eq. (5.3) in some special cases.

Solution of Eq. (5.3) for $n_1 = n_2$

Let $N_1 = N_2 = N$ (or, which is the same thing, $n_1 = n_2 = n$). Then after the introduction of the new independent variable $x = t^2$ and the substitution $K = 2l$, we obtain

$$x(1-x)\cdot\frac{d^2u}{dx^2} + \left[\frac{1}{2} - (1-N)x\right]\cdot\frac{du}{dx} - \frac{N^2-l^2}{4}\cdot u(x) = 0. \tag{5.4}$$

It is obvious that (5.4) is the hypergeometric equation, the canonical form of which is usually written as

$$x(1-x)\cdot\frac{d^2u}{dx^2} + [\gamma-(\alpha+\beta+1)x]\cdot\frac{du}{dx} - \alpha\beta\cdot u = 0. \tag{5.5}$$

If γ is not an integer, then as the two linearly independent solutions of Eq. (5.5) we can choose

$$u_1(x) = F(\alpha,\beta,\gamma;x),$$

$$u_2(x) = x^{1-\gamma}F(\alpha-\gamma+1,\beta-\gamma+1,2-\gamma;z),$$

where $F(\alpha,\beta,\gamma;z)$ is the hypergeometric function.

Comparing the coefficients of Eqs. (5.4) and (5.5), we determine the values of the parameters α,β,γ in (5.4):

$$\alpha = -\frac{N-l}{2} = -\frac{n-m-l}{2},$$

$$\beta = -\frac{N+l}{2} = -\frac{n-m+l}{2}, \quad \gamma = \tfrac{1}{2}.$$

Therefore, the linearly independent solutions of Eq. (5.4) are

$$u_1(t)=F\left(-\frac{n-m-l}{2},-\frac{n-m+l}{2},\frac{1}{2},t^2\right),$$

$$u_2(t)=t\cdot F\left(-\frac{n-m-l-1}{2},-\frac{n-m+l-1}{2},\frac{3}{2};t^2\right).$$

In the considered model, the only solutions of interest are those that are polynomials of finite degree in t, and l must vary in the following ranges:

$$-(n-m)\leqslant l\leqslant n-m$$

if l is an integer of the same parity as $n-m$, and

$$-(n-m-1)\leqslant l\leqslant n-m-1$$

if the parity of l differs from that of $n-m$.

Thus, with allowance for the factor $(1-t^2)^{m/2}$ introduced in (5.2), the required solutions of Eq. (5.1) are the functions

$$y(t)=\psi_l^{n,m}(t)$$

$$=\begin{cases}(1-t^2)^{m/2}F\left(-\frac{n-m-l}{2},-\frac{n-m+l}{2},\frac{1}{2};t^2\right), \\ \quad \text{if } n-m-l \text{ is even,} \\ (1-t^2)^{m/2}t\cdot F\left(-\frac{n-m-l-1}{2},-\frac{n-m+l-1}{2},\frac{3}{2};t^2\right) \\ \quad \text{otherwise.}\end{cases} \tag{5.6}$$

Note that by virtue of the symmetry of the hypergeometric function $F(\alpha,\beta,\gamma;z)$ with respect to the transposition $\alpha\leftrightarrow\beta$ the function $\psi_l^{n,m}(t)$ is unchanged under the substitution $l\rightarrow -l$.

We now rewrite (5.6), going over from n,m,l to the standard indices (λ,μ) that characterize SU(3) symmetry and to the parameter $K=2l$:

$$\psi_K^{\lambda,\mu}(t)=\begin{cases}(1-t^2)^{\mu/2}F\left(-\frac{\lambda-K}{4},-\frac{\lambda+K}{4},\frac{1}{2};t^2\right), \\ \quad \text{if } \frac{\lambda-K}{2} \text{ is even,} \\ (1-t^2)^{\mu/2}F\left(-\frac{\lambda-K-2}{4},-\frac{\lambda-K+2}{4},\frac{3}{2};t^2\right) \\ \quad \text{otherwise.}\end{cases}$$

For fixed values of the indices (λ,μ), the set of functions $\psi_K^{\lambda,\mu}(t)$ with different admissible values of K can be regarded as a multiplet with given SU(3) symmetry. The number K, which takes only even values $(K=2l)$, varies in the range $-\lambda\leqslant K\leqslant\lambda$. Thus, for given λ there are $\lambda+1$ functions. Note, however, that in the considered case $n_1=n_2=n$ the functions $\psi_K^{\lambda,\mu}$ are identical to $\psi_{-K}^{\lambda,\mu}$. Therefore, there are just $\lambda/2+1$ different functions. Invariance of the functions $\psi_K^{\lambda,\mu}(t)$ with respect to the substitution $K\rightarrow -K$ follows directly from the invariance of Eq. (5.4) under the substitution $l\rightarrow -l$ and from the fact that this equation for given N and l (or, which is the same thing, for given λ and K) has a unique polynomial solution. In the general case $n_1\neq n_2$, the functions $\psi_K^{\lambda,\mu}$ and $\psi_{-K}^{\lambda,\mu}$ are not equal, as we shall show below.

Solution of Eq. (5.3) for $n_1\neq n_2$ and $K=0$, $K=\pm 1$

For $n_1\neq n_2$, the parameter $\lambda=n_1+n_2-2m=N_1+N_2$ takes even values if the numbers n_1 and n_2 (or, which is the same thing, N_1 and N_2) have the same parity. In this case, K takes even values in the range $-\lambda\leqslant K\leqslant\lambda$. For $K=0$, Eq. (5.3) can be reduced by the substitution $t^2=x$ to the hypergeometric equation (5.5) with parameters $\alpha=-N_1/2$, $\beta=-N_2/2$, $\gamma=1/2$. Therefore, the following functions are two linearly independent solutions of this equation:

$$u_1(t)=F\left(-\frac{N_1}{2},-\frac{N_2}{2},\frac{1}{2};t^2\right);$$

$$u_2(t)=t\cdot F\left(-\frac{N_1-1}{2},-\frac{N_2-1}{2},\frac{3}{2};t^2\right),$$

where, as above, $F(\alpha,\beta,\gamma;z)$ is the hypergeometric function. Choosing from the two solutions only functions that are finite polynomials in the variable t, we write down the form of the functions $y(t)$:

$$y(t)=\begin{cases}(1-t^2)^{m/2}F\left(-\frac{n_1-m}{2},-\frac{n_2-m}{2},\frac{1}{2};t^2\right), \\ \quad \text{if } n_1-m \text{ and } n_2-m \text{ are even numbers,} \\ (1-t^2)^{m/2}t\cdot F\left(-\frac{n-m-1}{2},-\frac{n-m-1}{2},\frac{3}{2};t^2\right), \\ \quad \text{if } n_1-m \text{ and } n_2-m \text{ are odd numbers.}\end{cases}$$

We rewrite this function, using in place of n_1, n_2, and m the parameters λ, μ, and q:

$$y(t)=\psi_q^{\lambda,\mu}(t)$$

$$=\begin{cases}(1-t^2)^{\mu/2}F\left(-\frac{\lambda+q}{4},-\frac{\lambda-q}{4},\frac{1}{2};t^2\right), \\ \quad \text{if } \frac{\lambda+q}{2} \text{ are even numbers,} \\ (1-t^2)^{\mu/2}t\cdot F\left(-\frac{\lambda+q-2}{2},-\frac{\lambda-q-2}{2},\frac{3}{2};t^2\right), \\ \quad \text{if } \frac{\lambda+q}{2} \text{ are odd numbers.}\end{cases}$$

For fixed values of n_1 and n_2, the parameter q also remains unchanged (in particular, $q=0$ for $n_1=n_2=n$), whereas λ can take the values

$$\lambda=n_1+n_2,\quad n_1+n_2-2,\dots,1,\text{ or } 0.$$

The set of functions $\psi_q^{\lambda,\mu}$ for fixed values of the parameters (λ,μ) but with different values of q can be regarded as a multiplet with definite SU(3) symmetry. The parameter q takes the values $q=\lambda,\lambda-2,\dots,-\lambda+2,-\lambda$ ($\lambda+1$ values in all); at the same time, the functions $\psi_q^{\lambda,\mu}$ are equal to $\psi_{-q}^{\lambda,\mu}$. Recall that we are now considering the case $K=0$.

Note also that $\psi_q^{\lambda,\mu}$ can be obtained from the analogous functions $\psi_K^{\lambda,\mu}$ by the formal renotation $K\rightarrow q$. This is due to the fact that Eq. (5.3′) in the two considered special cases has the same form, provided that K is replaced by q.

Finally, we give the solution of Eq. (5.3) for one further special case: when N_1 and N_2 are numbers of opposite parity and at the same time $K=1$. It can be directly verified that if $N_1=n_1-m$ is even and $N_2=n_2-m$ is odd, Eq. (5.3) is satisfied by the function

$$y(t)=(1-t^2)^{m/2}\cdot\left\{F\left(-\frac{N_1}{2},-\frac{N_2-1}{2},\frac{1}{2},t^2\right)(\sqrt{1+t}+i\sqrt{1-t})+N_1t\cdot F\left(-\frac{N_1-2}{2},-\frac{N_2-1}{2},\frac{3}{2},t^2\right)\times(\sqrt{1+t}-i\sqrt{1-t})\right\}.$$

6. SOLUTION OF EQ. (5.3) FOR ARBITRARY VALUES OF K

To find the solution of Eq. (5.3) for arbitrary admissible values of K, we make the change $t=\cos\theta$ of the independent variable in it:

$$\frac{d^2u}{d\theta^2}-(N_1+N_2)\cot\theta\,\frac{du}{d\theta}+\left[\frac{1}{4}K^2-N_1N_2+i\frac{1}{2}K(N_1-N_2)\cot\theta\right]u=0. \tag{6.1}$$

We transform Eq. (6.1), replacing $u(\theta)$ by the new function $w(\theta)$ defined by

$$u(\theta)=w(\theta)e^{i\theta(N_1-K/2)}. \tag{6.2}$$

After the substitution of (6.2) in (6.1), we obtain

$$\frac{d^2w}{d\theta^2}+[i(2N_1-K)-(N_1+N_2)\cot\theta]\times\frac{dw}{d\theta}-\{N_1(N_1+N_2)-KN_1+i\cot\theta\times[N_1(N_1+N_2)-KN_2]\}\cdot w=0. \tag{6.3}$$

After the introduction of the new independent variable $z=e^{-i2\theta}$, Eq. (6.3) takes the form

$$z(1-z)\frac{d^2w}{dz^2}+\frac{1}{2}[(2+K-N_1+N_2)+(3N_1+N_2-K-2)z]\times\frac{dw}{dz}+\frac{1}{2}N_1(K-N_1-N_2)w=0. \tag{6.4}$$

It is readily seen that (6.4) is the hypergeometric equation, a solution of which in the form of a polynomial of finite degree in z can be written as

$$w(z)=F\left(-N_1,-\frac{N_1+N_2-K}{2},-\frac{N_1-N_2-K}{2}+1;z\right). \tag{6.5}$$

The case when the parameter γ of the hypergeometric function $F(\alpha,\beta,\gamma;z)$ is equal to zero or some negative integer is singular. Let $\gamma=1-s$, where s is a natural number. Then the hypergeometric equation will be solved by the function[14]

$$w(z)=z^s\cdot F(\alpha+m,\beta+m,1+m;z).$$

Therefore, in our case

$$w(z)=z^{N_1-N_2-K/2}\cdot F\left(-N_2,-\frac{N_1+N_2+K}{2},\frac{N_1-N_2-K}{2}+1;z\right),\quad \text{if } N_1-N_2-K>0.$$

Substituting these expressions in (6.2) and returning to the variable θ, we obtain

$$u(\theta)=\begin{cases}F\left(-N_1,-\frac{N_1+N_2-K}{2},-\frac{N_1-N_2-K}{2}+1;e^{-i2\theta}\right)\times e^{i\theta(N_1-K/2)}, & \text{if } N_1-N_2-K\leqslant 0,\\ F\left(-N_2,-\frac{N_1+N_2+K}{2},-\frac{N_2-N_1+K}{2}+1;e^{-i2\theta}\right)\times e^{i\theta(N_2+K/2)}, & \text{if } N_1-N_2-K>0.\end{cases} \tag{6.6}$$

The expressions on the right-hand side of (6.6) can be transformed by means of the well-known relations[14] in such a way that at $\theta=0$ the hypergeometric functions are equal to unity:

$$u(\theta)=\begin{cases}F\left(-N_1,-\frac{N_1+N_2-K}{2},-N_1-N_2;i-e^{-i2\theta}\right)\times e^{i\theta(N_1-K/2)}, & (6.6a)\\ F\left(-N_2,-\frac{N_1+N_2+K}{2},-N_1-N_2;1-e^{-i2\theta}\right)\times e^{i\theta(N_2+K/2)}. & (6.6b)\end{cases}$$

Despite the superficial difference, the expressions (6.6a) and (6.6b) represent the same function, as is readily seen by taking into account the well-known identity

$$F(\alpha,\beta,\gamma)=(1-z)^{\gamma-\alpha-\beta}F(\gamma-\alpha,\gamma-\beta,\gamma;z).$$

Note that if N_1 and N_2 have the same (respectively, opposite) parities, then the parameter K in (6.6a) and (6.6b) can take only even (respectively, only odd) values in the range $-(N_1+N_2)\leqslant K\leqslant N_1+N_2$.

Returning to the expression (5.2), we now write down the eigenfunction of the Casimir operator with a fixed value of K:

$$\psi_K=\sin^m\theta\times\begin{cases}F\left(-N_1,-\frac{N_1+N_2-K}{2},-N_1-N_2;1-e^{-i2\theta}\right)\times e^{i\theta(N_1-K/2)},\\ \text{or}\\ F\left(-N_2,-\frac{N_1+N_2+K}{2},-N_1-N_2;1-e^{-i2\theta}\right)\times e^{i\theta(N_2+K/2)}.\end{cases} \tag{6.7}$$

7. THE BARGMANN–MOSHINSKY OPERATOR

By the Bargmann–Moshinsky (BM) operator, we shall understand the scalar contraction of generators defined by

$$\Omega=L_iA_{ij}L_j, \tag{7.1}$$

where $\mathbf{L}$ is the operator of the total angular momentum, i.e., the sum of the angular momenta of the two subsystems:

$$\mathbf{L}=\mathbf{l}^u+\mathbf{l}^v, \quad \text{and } A_{ij}=u_i\nabla_j^u+v_i\nabla_j^v, \tag{7.2}$$

where ∇ is the nabla operator. In what follows, we shall make some refinements in the definition of the BM operator in order to bring it into correspondence with the analogous definition used in the book of Ref. 15.

Substituting (7.2) in (7.1) and bearing in mind that

$$(\mathbf{l}^u\cdot\mathbf{u})\equiv 0, \quad (\mathbf{l}^v\cdot\mathbf{v})\equiv 0, \quad (\nabla^u\cdot\mathbf{l}^u)\equiv 0, \quad \text{and } (\nabla^v\cdot\mathbf{l}^v)\equiv 0, \tag{7.3}$$

we obtain

$$\Omega=(\mathbf{u}\cdot\mathbf{l}^v)(\mathbf{l}^v\cdot\nabla^u)+(\mathbf{v}\cdot\mathbf{l}^u)(\mathbf{l}^u\cdot\nabla^v).$$

Taking into account the identity (7.3), we can replace the operators $\mathbf{l}^u$ and $\mathbf{l}^v$ on the right-hand side of this equation by the operator of the total angular momentum $\mathbf{L}$:

$$\Omega=(\mathbf{u}\cdot\mathbf{L})(\mathbf{L}\cdot\nabla^u)+(\mathbf{v}\cdot\mathbf{L})(\mathbf{L}\cdot\nabla^v). \tag{7.4}$$

We rewrite (7.4), making the substitution $\mathbf{L}=i\mathbf{M}$:

$$-\Omega=(\mathbf{u}\cdot\mathbf{M})(\mathbf{M}\cdot\nabla^u)+(\mathbf{v}\cdot\mathbf{M})(\mathbf{M}\cdot\nabla^v). \tag{7.5}$$

We express the scalar products in this last equation in terms of the components of the vectors in the intrinsic coordinate system:

$$(\mathbf{u}\cdot\mathbf{M})=u\cdot\cos\frac{\theta}{2}\cdot M_1-u\cdot\sin\frac{\theta}{2}\cdot M_2,$$

$$(\mathbf{v}\cdot\mathbf{M})=v\cdot\cos\frac{\theta}{2}\cdot M_1+v\cdot\sin\frac{\theta}{2}\cdot M_2,$$

$$(\mathbf{M}\cdot\nabla^u)=M_1\cdot\frac{\partial}{\partial\xi_1}+M_2\cdot\frac{\partial}{\partial\xi_2}+M_3\cdot\frac{\partial}{\partial\xi_3}+\delta,$$

$$(\mathbf{M}\cdot\nabla^v)=M_1\cdot\frac{\partial}{\partial\eta_1}+M_2\cdot\frac{\partial}{\partial\eta_2}+M_3\cdot\frac{\partial}{\partial\eta_3}+\delta. \tag{7.5$'$}$$

The presence of the additional terms δ in the last two equations is due to the fact that the operators $\mathbf{M}$ do not commute with the matrices of the transformation to the intrinsic coordinate system. In the intermediate calculations, we omit these terms; their contribution will be taken into account below when we write down the final expression for the BM operator.

We substitute in (7.5′) the values of the derivatives $\partial/\partial\xi_i$ and $\partial/\partial\eta_i$ in the variables $\{\alpha,\beta,\gamma,\theta,u,v\}$ (3.4″) and rewrite (7.5) in the form

$$\Omega=-\Bigg\{(n_1+n_2)\left(\cos^2\frac{\theta}{2}\cdot M_1^2+\sin^2\frac{\theta}{2}\cdot M_2^2\right)+\frac{1}{2}(n_2-n_1)$$
$$\times\sin\theta\cdot(M_1\cdot M_2+M_2\cdot M_1)+\sin\theta\frac{\partial}{\partial\theta}\cdot(M_2^2-M_1^2)$$
$$+\left(\sin^2\frac{\theta}{2}\cdot M_2M_1M_3-\cos^2\frac{\theta}{2}\cdot M_1M_2M_3\right)$$
$$+M_1M_3M_2-M_2M_3M_1\Bigg\}. \tag{7.6}$$

We again make the substitution $\mathbf{M}=-i\mathbf{L}$, $\cos\theta=t$ and introduce in place of the operators L_1,L_2,L_3 the operators $L_+=L_1+iL_2$, $L_-=L_1-iL_2$, $L_0=L_3$. Taking into account the well-known commutation relations for the projections of the angular momentum onto the intrinsic axes, $[L_i,L_j]=-i\varepsilon_{ijk}L_k$, we obtain

$$\Omega=(n_1+n_2)\left[\frac{1}{4}t(L_+^2+L_-^2)+\frac{1}{2}(\mathbf{L}^2-L_0^2)\right]+i\frac{1}{4}(n_1-n_2)$$
$$\times\sqrt{1-t^2}(L_+^2-L_-^2)+\frac{1}{2}(1-t^2)\frac{\partial}{\partial t}(L_+^2-L_-^2)$$
$$+\frac{1}{4}[t(L_+^2-L_-^2)+2L_0]L_0+\mathbf{L}^2-2L_0^2+\gamma. \tag{7.7}$$

On the right-hand side of the last equation, γ is the correction that brings (7.7) into correspondence with the BM operator in the book of Ref. 15. In the direct determination of this correction, it must be borne in mind that in Ref. 15 the trace of the generators A_{ij} is equal to zero and that in the BM operator the factors are written in a manner somewhat different from (2.1), namely, $\Omega=A_{ij}L_iL_j$. In addition, it is necessary to take into account the correction δ mentioned above.

However, γ can be found in a different way (since there is every reason for assuming that it has a simple form), namely, by the choice of a γ for which the eigenvalues of the operator (7.7) for the lowest values of L are equal to the corresponding eigenvalues of the BM operator in Ref. 15.

We represent the final expression for the Bargmann–Moshinsky operator as a sum of three operators, $\Omega=R_++R_-+R_0$, where

$$R_+=\frac{1}{4}\left[(n_1+n_2)t+i(n_1-n_2)\sqrt{1-t^2}+2(1-t^2)\frac{\partial}{\partial t}\right]$$
$$\times L_+^2+\frac{1}{4}t\cdot L_+^2\cdot L_0,$$

$$R_-=\frac{1}{4}\left[(n_1+n_2)t-i(n_1-n_2)\sqrt{1-t^2}+2(1-t^2)\frac{\partial}{\partial t}\right]$$
$$\times L_-^2-\frac{1}{4}t\cdot L_-^2\cdot L_0,$$

$$R_0=\left[\frac{1}{6}(n_1+n_2)+\frac{1}{2}\right]\cdot\mathbf{L}^2-\frac{1}{2}(n_1+n_2+3)L_0^2. \tag{7.8}$$

The basis functions between which we shall in what follows calculate the matrix elements of the operators are eigenfunctions of the Casimir operator and are products of Wigner D functions and functions $\varphi_K(t)$ that depend only on the variable t [see (6.7)]. The functions $\varphi_K(t)\cdot D^L_{MK}$ are eigenfunctions of the operator R_0:

$$R_0\varphi_K(t)\cdot D^L_{MK}=\left\{\left[\left[\frac{1}{6}(n_1+n_2)+\frac{1}{2}\right]\cdot L(L+1)-\frac{1}{2}\cdot(n_1+n_2+3)K^2\right]\varphi_K(t)\cdot D^L_{MK}.\right.$$

The operators R_+ and R_-, when applied to the functions $\varphi_K(t)\cdot D^L_{MK}$, change the quantum number K by two units:

$$R_+\varphi_K(t)\cdot D^L_{MK}=\text{const}\cdot\varphi_{K-2}(t)\cdot D^L_{MK-2},$$

$$R_-\varphi_K(t)\cdot D^L_{MK}=\text{const}\cdot\varphi_{K+2}(t)\cdot D^L_{MK+2}. \quad (7.9)$$

8. MATRIX ELEMENTS OF THE BARGMANN–MOSHINSKY OPERATOR

We calculate the matrix elements of the Bargmann–Moshinsky operator on the functions $\varphi_K(t)\cdot D^L_{MK}$. Bearing in mind that the D functions are eigenfunctions of the operator L_0, we can write the operators $R_\pm$ in (7.9) in factorized form:

$$R_+=\tfrac{1}{4}\cdot L_+^2\cdot\tilde{R}_+,\quad R_-=\tfrac{1}{4}\cdot L_-^2\cdot\tilde{R}_-,$$

where

$$\tilde{R}_+=(n_1+n_2+K)t+i(n_2-n_1)\sqrt{1-t^2}+2(1-t^2)\frac{\partial}{\partial t},$$

$$\tilde{R}_-=(n_1+n_2-K)t-i(n_2-n_1)\sqrt{1-t^2}+2(1-t^2)\frac{\partial}{\partial t},$$

or

$$\tilde{R}_+=(n_1+n_2+K)\cos\theta-\sin\theta\cdot\left[i(n_1-n_2)+2\frac{\partial}{\partial\theta}\right],$$

$$\tilde{R}_-=(n_1+n_2-K)\cos\theta+\sin\theta\cdot\left[i(n_1-n_2)-2\frac{\partial}{\partial\theta}\right].$$

In what follows, we shall, when it is expedient, denote functions that depend only on the variable t and correspond to a definite value of K simply by $|K\rangle$, and the D functions by $|LK\rangle$.

Since the operators $\tilde{R}_+$ and $\tilde{R}_-$ when applied to the functions $|K\rangle$ change the value of the parameter K by two units,

$$\tilde{R}_+|K\rangle=c_1|K-2\rangle,$$

$$\tilde{R}_-|K\rangle=c_2|K+2\rangle, \quad (8.1)$$

it follows that for each given value of the quantum number L the set of matrix elements of the BM operator forms (for $L\leqslant\max\{\lambda,\mu\}$) a tridiagonal $(L+1)\times(L+1)$ matrix having on the principal diagonal the matrix elements of the operator R_0 and on the two side diagonals the matrix elements of the operators R_+ and R_-. At the same time, it is obvious that the nonvanishing matrix elements of the last two operators can be represented in the factorized form

$$\langle R_+\rangle=\tfrac{1}{4}\langle LK-2|L_+^2|LK\rangle\cdot\langle K-2|\tilde{R}_+|K\rangle,$$

$$\langle R_-\rangle=\tfrac{1}{4}\langle LK+2|L_-^2|LK\rangle\cdot\langle K+2|\tilde{R}_-|K\rangle. \quad (8.2)$$

The calculation of the matrix elements of the operators R_+ and $\tilde{R}_-$ reduces to determination of the coefficients c_1 and c_2 in the relations (8.1). To find the latter, it is sufficient to compare the first terms of the expansion in powers of θ on the left- and right-hand sides of (8.1). Bearing in mind that for small values of θ we have

$$\tilde{R}_+=(n_1+n_2+K)-2\theta\cdot\frac{\partial}{\partial\theta}+\ldots,$$

$$\tilde{R}_-=(n_1+n_2-K)-2\theta\cdot\frac{\partial}{\partial\theta}+\ldots,$$

we obtain from (8.1)

$$\left[(n_1+n_2+K)-2\theta\frac{\partial}{\partial\theta}\right]\cdot\theta^m=c_1\cdot\theta^m,$$

$$\left[(n_1+n_2-K)-2\theta\frac{\partial}{\partial\theta}\right]\cdot\theta^m=c_2\cdot\theta^m.$$

It follows from this that

$$\langle K-2|\tilde{R}_+|K\rangle=n_1+n_2+K-2m=N_1+N_2+K=\lambda+K,$$

$$\langle K+2|\tilde{R}_-|K\rangle=n_1+n_2-K-2m=N_1+N_2-K=\lambda-K.$$

Taking into account the known values of the matrix elements of the ladder operators $L_\pm$, we write

$$\langle LK-2|L_+^2|LK\rangle=[(L+K)(L+K-1)(L-K+1)\times(L-K+2)]^{1/2},$$

$$\langle LK+2|L_-^2|LK\rangle=[(L-K)(L-K-1)(L+K+1)\times(L+K+2)]^{1/2}.$$

The diagonal matrix elements of the BM operator are readily found, since the functions $\varphi_K(t)\cdot D^L_{MK}$ are eigenfunctions of the operator R_0. The final expressions for the matrix elements of the Bargmann–Moshinsky operator Ω are

$$\langle K'|\Omega|K\rangle=\begin{cases}\langle K+2|R_-|K\rangle=\tfrac{1}{4}(\lambda-K)\cdot[(L-K)(L-K-1)\times(L+K+1)(L+K+2)]^{1/2}\\ \langle K|R_0|K\rangle=\left[\frac{1}{6}(\lambda+2\mu)+\frac{1}{2}\right]\cdot L(L+1)-\tfrac{1}{2}(\lambda+2\mu+3)K^2\\ \langle K-2|R_+|K\rangle=\tfrac{1}{4}(\lambda+K)\cdot[(L+K)(L+K-1)\times(L-K+1)(L-K+2)]^{1/2}.\end{cases}$$

As an example, we give the table of matrix elements of the operator Ω for odd values of λ and angular momentum $L=1$ (Table II). In this case, K can take the values 1 and -1.

We also write down the table of matrix elements of the operator Ω for even values of λ and angular momentum $L=2$ (Table III). In this case, K can take the values 2, 0, and -2.

9. EIGENFUNCTIONS AND EIGENVALUES OF THE BARGMANN–MOSHINSKY OPERATOR

We consider the procedure for finding the eigenvalues and eigenfunctions of the BM operator in detail for the special example for which $L=2$ with $L \leqslant \max\{\lambda,\mu\}$. For even λ, the eigenfunction of the BM operator is to be sought in the form of a linear superposition of functions with all admissible values of K:

$$|\psi\rangle = C_0|K=0\rangle\cdot D^2_{M0}+C_2|K=2\rangle\cdot D^2_{M2} + C_{-2}|K=-2\rangle\cdot D^2_{M-2}, \tag{9.1}$$

and the problem then reduces to finding the coefficients C_0, C_2, C_{-2} of the superposition. Since the matrix elements of the BM operator between functions with given values of K have been found (see Table III), the eigenvalues and eigenfunctions can be found by the usual matrix diagonalization method.

We denote by ω the eigenvalues of the BM operator and write down the secular equation for finding them:

$$\begin{vmatrix} -(\lambda+\mu+3)-\omega & \sqrt{\frac{3}{2}}\cdot\lambda & 0 \\ \sqrt{\frac{3}{2}}\cdot(\lambda+2) & (\lambda+2\mu+3)-\omega & \sqrt{\frac{3}{2}}\cdot(\lambda+2) \\ 0 & \sqrt{\frac{3}{2}}\cdot\lambda & -(\lambda+\mu+3)-\omega \end{vmatrix} = 0.$$

This algebraic equation of third degree represented as a determinant factorizes, and as a result its roots are readily found:

$$\omega_1 = -(\lambda+\mu+3);$$

$$\omega_{2,3} = \pm\sqrt{(2\lambda+\mu+3)^2+3\mu(\mu+2)}$$

Substituting, as usual, ω_i in the system of homogeneous equations, we find the required coefficients C_0, C_2, C_{-2}:

$$C_2 = C_{-2} = \sqrt{\frac{3}{2}}\cdot\frac{\lambda}{(\lambda+2\mu)+\omega_i}\cdot C_0; \quad \omega_i=\omega_{2,3}.$$

If $\omega_i=\omega_1=-(\lambda+2\mu+3)$, then $C_0=0$ and $C_{-2}=-C_2$. In what follows, if C_0 is not equal to zero, we shall normalize the functions (9.1) in such a way that $C=1$. Thus, the eigenfunctions of the BM operator have the form

$$|\psi\rangle_1 = |K=2\rangle\cdot D^2_{M2} - |K=-2\rangle\cdot D^2_{M-2},$$

$$|\psi\rangle_{2,3} = |K=0\rangle\cdot D^2_{M0} + \sqrt{\frac{3}{2}}\cdot\frac{\lambda}{(\lambda+2\mu)+\omega_{2,3}} \times[|K=2\rangle\cdot D^2_{M2}+|K=-2\rangle\cdot D^2_{M-2}].$$

We rewrite the expression on the right-hand side of the equation in terms of the functions

$$D^2_{M2+} = \frac{1}{\sqrt{2}}\cdot(D^2_{M2}+D^2_{M-2})$$

and $D^2_{M2-} = \frac{1}{\sqrt{2}}\cdot(D^2_{M2}-D^2_{M-2})$:

$$|\psi\rangle_1 = |K_{2-}\rangle\cdot D^2_{M2+} + |K_{2+}\rangle\cdot D^2_{M2-},$$

$$|\psi\rangle_{2,3} = |K=0\rangle\cdot D^2_{M0} + \sqrt{3}\cdot\frac{\lambda}{(\lambda+2\mu+3)+\omega_{2,3}} \times[|K_{2+}\rangle\cdot D^2_{M2+}+|K_{2-}\rangle\cdot D^2_{M2-}].$$

In the last equation, we have introduced the notation

$$|K_{2+}\rangle = \frac{1}{2}\cdot(|K=2\rangle+|K=-2\rangle)$$

and $|K_{2-}\rangle = \frac{1}{2}\cdot(|K=2\rangle-|K=-2\rangle)$.

For $\mu=0$ and even λ, the wave functions have the form

$$|\psi\rangle_2 = |K=0\rangle\cdot D^2_{M0} + \frac{1}{\sqrt{3}}\cdot\frac{\lambda}{(\lambda+2)}\cdot[|K_{2+}\rangle\cdot D^2_{M2+} + |K_{2-}\rangle\cdot D^2_{M2-}],$$

$$|\psi\rangle_3 = |K=0\rangle\cdot D^2_{M0} - \sqrt{3}\cdot[|K_{2+}\rangle\cdot D^2_{M2+}+|K_{2-}\rangle \times D^2_{M2-}].$$

If λ is odd (i.e., if n_1 and n_2 are numbers with opposite parity), then the set of matrix elements of the BM operator for a given even value of the angular momentum L (this can be only in the case when $\mu\neq 0$) forms an $L\times L$ matrix. For $L=2$, K can take only two values, $K=\pm 1$, and the matrix elements are given in Table IV.

Solving the secular equation, we find the eigenvalues of the Bargmann–Moshinsky operator:

$$\omega_1 = 2\lambda+\mu+3, \quad \omega_2 = -\lambda+\mu.$$

In the considered case, the corresponding eigenfunctions of the BM operator have the form

$$\begin{aligned} |\psi\rangle_1 &= |K=1\rangle\cdot D^2_{M1} + |K=-1\rangle\cdot D^2_{M-1} \\ &= \sqrt{2}(|K_{1+}\rangle\cdot D^2_{M1+}+|K_{1-}\rangle)\cdot D^2_{M1-}, \\ |\psi\rangle_2 &= |K=1\rangle\cdot D^2_{M1} - |K=-1\rangle\cdot D^2_{M-1} \\ &= \sqrt{2}(|K_{1-}\rangle\cdot D^2_{M1+}+|K_{1-}\rangle)\cdot D^2_{M1+}. \end{aligned} \tag{9.2}$$

It can be seen from (9.2) that the coefficients of the superposition in the considered special case do not depend on λ and μ.

The functions $|K=0\rangle$, $|K_{2+}\rangle$, $|K_{2-}\rangle$, $|K_{1+}\rangle$, and $|K_{1-}\rangle$ for each set of concrete values N_1, N_2, m, K can be readily written down in explicit form by using the expressions (7.9). We give the form of these functions for the lowest values of N_1 and N_2 (Table V). Since all the functions $|K\rangle$ with $m\neq 0$ contain the same factor $(1-t^2)^{m/2}$, for brevity we shall omit this factor in the expressions for the functions. We take into account the fact that for even λ the numbers N_1 and N_2 have the same parity, while for odd λ they have opposite parities, since these numbers are related by $\lambda=N_1+N_2$.

The eigenvalues and eigenfunctions of the BM operator for angular momenta $L=0$, 1, 2, and 3 are given in Table VI.

As an example, we give the explicit form of the eigenfunctions of the BM operator in the cases $L=2$ and $L=3$ for some particular values of λ and μ. Note that for $\mu=0$ the parity of λ must be equal to the parity of L.

The eigenfunctions of the BM operator are given explicitly in the cases $L=2$ and $L=3$ in Tables VII and VIII.

10. TRANSFORMATION OF THE WAVE FUNCTIONS UNDER ROTATION OF THE COORDINATE SYSTEM AND SELECTION OF "ELLIOTT STATES"

The coordinate system that we have chosen (for convenience) in the present paper differs from the system chosen earlier by Elliott.[9] Therefore, to compare our eigenfunctions of the BM operator with Elliott's functions, it is first necessary to transform these functions from one coordinate system to another.

The transition from Elliott's coordinate system to our system is made by transforming the unit vectors that define the directions of the axes in accordance with the scheme $e_x\to e_z, e_y\to -e_y, e_z\to e_x$ with subsequent rotation around the new z axis through the angle $\theta/2$. If the matrix that transforms the Cartesian components of a vector on the transition from the "tilded" Elliott coordinate system to the untilded system is denoted by

$$\hat{D}=\begin{vmatrix}\alpha_{11} & \alpha_{12} & \alpha_{13}\\ \alpha_{21} & \alpha_{22} & \alpha_{23}\\ \alpha_{31} & \alpha_{32} & \alpha_{33}\end{vmatrix},$$

then the transition from the tilded Elliott system to our coordinate system is made by means of the matrix

$$\hat{D}'=\begin{vmatrix}\alpha_{13}\cos\frac{\theta}{2}-\alpha_{12}\sin\frac{\theta}{2} & -\alpha_{13}\sin\frac{\theta}{2}-\alpha_{12}\cos\frac{\theta}{2} & \alpha_{13}\\ \alpha_{23}\cos\frac{\theta}{2}-\alpha_{22}\sin\frac{\theta}{2} & -\alpha_{23}\sin\frac{\theta}{2}-\alpha_{22}\cos\frac{\theta}{2} & \alpha_{23}\\ \alpha_{33}\cos\frac{\theta}{2}-\alpha_{32}\sin\frac{\theta}{2} & -\alpha_{33}\sin\frac{\theta}{2}-\alpha_{32}\cos\frac{\theta}{2} & \alpha_{33}\end{vmatrix}.$$

The transition from the untilded Elliott system to our coordinate system can also be specified by the three Euler angles $\alpha=\pi$, $\beta=-\pi/2$, and $\gamma=\theta/2$, and therefore the transformation of states with definite angular momentum L will then be made by means of the Wigner functions $D^L(\pi,-\pi/2,\theta/2)$.

We consider some concrete examples of transformation of the Elliott functions on the transition from the original (Elliott) coordinate system to our system.

Example 1. Let $(\lambda,\mu)=(\lambda,0)$ and $\lambda=N_1$, $N_2=0$. Then the overlap integral $(\mathbf{u}\cdot\tilde{\mathbf{u}})^{n_1}\cdot(\mathbf{v}\cdot\tilde{\mathbf{v}})^{n_2}$, represented as a superposition of expressions with definite values of the quantum numbers (λ,μ), will contain the unique term $(\mathbf{u}\cdot\tilde{\mathbf{u}})^{N_1}$. Since $(\mathbf{u}\cdot\tilde{\mathbf{u}})^{N_1}=(\cos\beta)^{N_1}$ does not depend on the angles α and γ, it can be represented as a sum of Wigner D functions with zero values of the projections of the angular momentum:

$$(\mathbf{u}\cdot\tilde{\mathbf{u}})^{N_1}=\sum C_L\cdot D^L_{00}.$$

Representing D^L_{00} as a sum of products of the tilded and untilded states, $D^L_{00}=\Sigma_M D^{*L}_{M0}(\tilde{\Omega})\cdot D^L_{M0}(\Omega)$, we arrive at the conclusion that the required Elliott function is equal to $D^L_{M0}(\Omega)$ except for a normalization factor.

The transformation of this function to our coordinate system is made in accordance with the scheme

TABLE II. Matrix elements of the operator Ω for angular momentum $L=1$.

	$K=1$	$K=-1$
$K'=1$	$-\frac{1}{2}\left[1+\frac{1}{3}(\lambda+2\mu)\right]$	$\frac{1}{2}\cdot(\lambda+1)$
$K'=-1$	$\frac{1}{2}\cdot(\lambda+1)$	$-\frac{1}{2}\left[1+\frac{1}{3}(\lambda+2\mu)\right]$

TABLE III. Matrix elements of the operator Ω for angular momentum $L=2$ (λ even).

	$K=2$	$K=0$	$K=-2$
$K'=2$	$-(\lambda+2\mu+3)$	$\sqrt{\frac{3}{2}}\cdot\lambda$	0
$K'=0$	$\sqrt{\frac{3}{2}}\cdot(\lambda+2)$	$\lambda+2\mu+3$	$\sqrt{\frac{3}{2}}\cdot(\lambda+2)$
$K'=-2$	0	$\sqrt{\frac{3}{2}}\cdot\lambda$	$-(\lambda+2\mu+3)$

TABLE IV. Matrix elements of the operator Ω for $L=2$ (λ odd, $\mu\neq 0$).

	$K=1$	$K=-1$
$K'=1$	$\frac{1}{2}\cdot(\lambda+2\mu+3)$	$\frac{3}{2}\cdot(\lambda+1)$
$K'=-1$	$\frac{3}{2}\cdot(\lambda+1)$	$\frac{1}{2}\cdot(\lambda+2\mu+3)$

TABLE V. The functions $|K=0\rangle$, $|K_{2+}\rangle$, $|K_{2-}\rangle$, $|K_{1+}\rangle$, and $|K_{1-}\rangle$ for some values of N_1 and N_2.

		λ even	
N_1 even, $N_2=0$	$\lvert K=0\rangle=1;$	$\lvert K_{2+}\rangle=t;$	$\lvert K_{2-}\rangle=i\cdot\sqrt{1-t^2}$
$N_1=1$, $N_2=1$	$\lvert K=0\rangle=t$	$\lvert L_{2+}\rangle=1;$	$\lvert K_{2-}\rangle=0$
$N_1=2$, $N_2=2$	$\lvert K=0\rangle=\frac{1}{3}\cdot(1+2t^2);$	$\lvert K_{2+}\rangle=t;$	$\lvert K_{2-}\rangle=0$
$N_1=4$, $N_2=2$	$\lvert K=0\rangle=\frac{1}{5}\cdot(1+4t^2);$	$\lvert K_{2+}\rangle=\frac{11}{15}\cdot t\left(1+\frac{4}{11}\cdot t^2\right);$	$\lvert K_{2-}\rangle=-i\,\frac{1}{15}(1+4t^2)\cdot\sqrt{1-t^2}$
$N_1=3$, $N_2=1$	$\lvert K=0\rangle=t$	$\lvert K_{2+}\rangle=\frac{1}{2}\cdot(1+t^2)$	$\lvert K_{2-}\rangle=i\cdot\frac{1}{2}\,t\cdot\sqrt{1-t^2}$
$N_1=3$, $N_2=3$	$\lvert K=0\rangle=\frac{3}{5}\,t\cdot\left(1+\frac{2}{3}\,t^2\right)$	$\lvert K_{2+}\rangle=\frac{1}{5}(1+4t^2)$	$\lvert K_{2-}\rangle=0$
		λ odd	
N_1 odd, $N_2=0$	$\lvert K_{1+}\rangle=\frac{1}{\sqrt{2}}\cdot\sqrt{1+t}$		$\lvert K_{1-}\rangle=i\cdot\frac{1}{\sqrt{2}}\cdot\sqrt{1-t}$
$N_1=2$, $N_2=1$	$\lvert K_{1+}\rangle=\frac{1}{3\sqrt{2}}\cdot(1+2t)\cdot\sqrt{1+t};$		$\lvert K_{1-}\rangle=-i\cdot\frac{1}{3\sqrt{2}}\cdot(1-2t)\cdot\sqrt{1-t}$

TABLE VI. Eigenvalues and eigenfunctions of the Bargmann–Moshinsky operator for angular momenta $L=0$, 1, 2, and 3.

Momentum	Eigenvalues of BM operator	Eigenfunctions of BM operator
$L=0$	$\omega=0$	$\lvert\Psi\rangle=\lvert K=0\rangle$
		$\lvert\Psi\rangle=\lvert K=0\rangle\cdot D^1_{M0}$
	$\omega=1+\frac{1}{3}(\lambda+2\mu)$	$\lvert\Psi\rangle=\lvert K_{1-}\rangle\cdot D^1_{M1+}+\lvert K_{1+}\rangle\cdot D^1_{M1-}$
$L=1$	$\omega=-[1+\frac{1}{3}(2\lambda+\mu)]$	$\lvert\Psi\rangle=\lvert K_{1+}\rangle\cdot D^1_{M1+}+\lvert K_{1-}\rangle\cdot D^1_{M1-}$
	$\omega=\frac{1}{3}(\lambda-\mu)$	
	$\omega=2\lambda+\mu+3$	$\lvert\Psi\rangle=\lvert K_{1+}\rangle\cdot D^2_{M1+}+\lvert K_{1-}\rangle\cdot D^2_{M1-}$
$L=2$	$\omega=-\lambda+\mu$	$\lvert\Psi\rangle=\lvert K_{1-}\rangle\cdot D^2_{M1+}+\lvert K_{1+}\rangle\cdot D^2_{M1-}$
	$\omega_1=-(\lambda+2\mu+3)$	$\lvert\Psi\rangle=\lvert K_{2-}\rangle\cdot D^2_{M2}+\lvert K_{2+}\rangle\cdot D^2_{M2-}$
	$\omega_{2,3}=\pm\sqrt{(2\lambda+\mu+3)^2+3\mu(\mu+2)}$	$\lvert\Psi\rangle=\lvert K=0\rangle\cdot D^2_{M0}+\sqrt{\frac{3}{2}}\cdot\frac{\lambda}{(\lambda+2\mu+3)+\omega_{2,3}}(\lvert K_{2+}\rangle\cdot D^2_{M2+}+\lvert K_{2-}\rangle\cdot D^2_{M2-})$
	$\omega_1=0$	$\lvert\Psi\rangle=\lvert K_{2-}\rangle\cdot D^3_{M2+}+\lvert K_{2+}\rangle\cdot D^3_{M2-}$
	$\omega_{2,3}=\lambda+2\mu+3\pm\sqrt{(\lambda+2\mu+3)^2+15\lambda(\lambda+2)}$	$\lvert\Psi\rangle=\lvert K=0\rangle\cdot D^3_{M0}+\frac{\lambda\sqrt{15}}{\omega_{2,3}}[\lvert K_{2+}\rangle\cdot D^3_{M2+}+\lvert K_{2-}\rangle\cdot D^3_{M2-}]$
$L=3$		
	$\omega_{4,5}=-(2\lambda+\mu+3)$ $\pm\frac{1}{2}\sqrt{(\lambda+8\mu+9)^2+15(\lambda-1)(\lambda+3)}$	$\lvert\Psi\rangle=\lvert K_{1-}\rangle\cdot D^3_{M1+}+\lvert K_{1+}P)\cdot D^3_{M1-}+\frac{\sqrt{15}(\lambda-1)}{5(\lambda+2\mu+3)+2\omega_{4,5}}$ $\times[\lvert K_{3-}\rangle\cdot D^3_{M3+}+\lvert K_{3+}\rangle\cdot D^3_{M3-}]$
	$\omega_{6,7}=\lambda-\mu$ $\pm\frac{1}{2}\sqrt{(7\lambda+8\mu+15)^2+15(\lambda-1)(\lambda+3)}$	$\lvert\Psi\rangle=\lvert K_{1+}\rangle\cdot D^3_{M1}+\lvert K_{1-}\rangle\cdot D^3_{M1-}+\frac{\sqrt{15}(\lambda-1)}{5(\lambda+2\mu+3)+2\omega_{6,7}}\cdot[\lvert K_{3+}\rangle\cdot D^3_{M3+}+\lvert K_{3-}\rangle$ $\times D^3_{M3-}]$

TABLE VII. Eigenfunctions of the Bargmann–Moshinsky operator in explicit form for angular momentum $L=2$.

Eigenvalues of BM operator	Values of μ, λ, N_1, N_2	Eigenfunctions of BM operator for $L=2$
$\omega=2\lambda+3$	$\mu=0\ N_2=0$ $\lambda=N_1$ even	$\lvert\Psi\rangle=D^2_{M0}+\frac{1}{\sqrt{3}}\cdot\frac{N_1}{N_1+2}\cdot(t\cdot D^2_{M2+}+i\sqrt{1-t^2}\cdot D^2_{M2-})$
$\omega=-(2\lambda+3)$	$\mu=0\ N_2=0$ $\lambda=N_1$ even	$\lvert\Psi\rangle=D^2_{M0}-\sqrt{3}\cdot(t\cdot D^2_{M2+}+i\sqrt{1-t^2}\cdot D^2_{M2-})$
$\omega=2\lambda+3=7$	$\mu=0\quad N_1=1$ $\lambda=2\quad N_2=1$	$\lvert\Psi\rangle=t\cdot D^2_{M0}+\frac{1}{2\sqrt{3}}\cdot D^2_{M2+}$
$\omega=-(2\lambda+3)=-7$		$\lvert\Psi\rangle=t\cdot D^2_{M0}-\sqrt{3}\cdot D^2_{M2+}$
$\omega=2\lambda+3=11$	$\mu=0\quad N_1=2$ $\lambda=4\quad N_2=2$	$\lvert\Psi\rangle=\frac{1}{3}\cdot(1+2t^2)\cdot D^2_{M0}+\frac{2}{3\sqrt{3}}\cdot t\cdot D^2_{M2+}$
$\omega=-(2\lambda+3)=-11$		$\lvert\Psi\rangle=\frac{1}{3}\cdot(1+2t^2)\cdot D^2_{M0}-\sqrt{3}\cdot t\cdot D^2_{M2+}$
$\omega=2\lambda+3=15$	$\mu=0\quad N_1=3$ $\lambda=6\quad N_2=3$	$\lvert\Psi\rangle=\frac{3}{5}\cdot t\left(1+\frac{2}{3}t^2\right)\cdot D^2_{M0}+\frac{\sqrt{3}}{5\cdot 4}\cdot(1+4t^2)\cdot D^2_{M2+}$
$\omega=-(2\lambda+3)=-15$		$\lvert\Psi\rangle=\frac{3}{5}\cdot t\left(1+\frac{2}{3}t^2\right)\cdot D^2_{M0}-\frac{\sqrt{3}}{5}\cdot(1+4t^2)\cdot D^2_{M2+}$
$\omega=-(\lambda+2\mu+3)=-7$	$\mu=1\quad N_1=1$ $\lambda=2\quad N_2=1$	$\lvert\Psi\rangle=\sqrt{2}\cdot\sqrt{1-t^2}\cdot D^2_{M2-}$
	$\mu=1\quad N_1=2$ $\lambda=2\quad N_2=0$	$\lvert\Psi\rangle=\sqrt{2}[i(1-t^2)\cdot D^2_{M2+}+t\cdot\sqrt{1-t^2}\cdot D^2_{M2-}$
$\omega=\sqrt{(2\lambda+\mu+3)^2+3\mu(\mu+2)}=\sqrt{105}$	$\mu=2\quad N_1=1$ $\lambda=2\quad N_2=1$	$\lvert\Psi\rangle=(1-t^2)\cdot\left[t\cdot D^2_{M0}+\frac{2\sqrt{3}}{9+\sqrt{105}}\cdot D^2_{M2+}\right]$
$\omega=-105$		$\lvert\Psi\rangle=(1-t^2)\cdot\left[t\cdot D^2_{M0}+\frac{2\sqrt{3}}{9-\sqrt{105}}\cdot D^2_{M2+}\right]$
$\omega=105$	$\mu=2\quad N_1=2$ $\lambda=2\quad N_2=0$	$\lvert\Psi\rangle=(1-t^2)\cdot\left[D^2_{M0}+\frac{2\sqrt{3}}{9+\sqrt{105}}(tD^2_{M2+}+i\sqrt{1-t^2}\cdot D^2_{M2-}\right]$
$\omega=-105$		$\lvert\Psi\rangle=(1-t^2)\cdot\left[t\cdot D^2_{M0}+\frac{2\sqrt{3}}{9-\sqrt{105}}(tD^2_{M2+}+i\sqrt{1-t^2}\cdot D^2_{M2-}\right]$

$$D^L_{M0}(\Omega)=\langle LM|\hat{D}(\alpha,\beta,\gamma)|L0\rangle \rightarrow\left\langle LM\left|\hat{D}(\alpha,\beta,\gamma)\cdot D^L\left(\pi,-\frac{\pi}{2},\frac{\theta}{2}\right)\right|L0\right\rangle =\sum_K D^L_{MK}(\alpha,\beta,\gamma)\cdot D^L_{K0}\left(\pi,-\frac{\pi}{2},\frac{\theta}{2}\right). \quad (10.1)$$

For $L=1$, this transformation has the form

$$D^1_{M0}(\Omega)\rightarrow\frac{1}{\sqrt{2}}\cdot e^{i\theta/2}\cdot D^1_{M1}(\Omega)-\frac{1}{\sqrt{2}}\cdot e^{-i\theta/2}\cdot D^1_{M-1}(\Omega) =\frac{1}{\sqrt{2}}\cdot[\sqrt{1+t}\cdot D^1_{M1-}(\Omega)+i\sqrt{1-t} \times D^1_{M1+}(\Omega)].$$

The right-hand side of the last equation is equal to one of the functions given above in the table.

Example 2. Let $L=2$, $N_2=0$, $\mu=0$, and λ be an even number. Then the corresponding transformation of the Elliott function is made in accordance with the scheme $D^2_{00}=\Sigma_M D^{*2}_{M0}(\tilde{\Omega})\cdot D^2_{M0}(\Omega)$:

$$D^2_{M0}(\Omega)\rightarrow\sum_K D^2_{MK}(\alpha,\beta,\gamma)\cdot D^2_{K0}\left(\pi,-\frac{\pi}{2},\frac{\theta}{2}\right)-\frac{1}{2} \times\{D^2_{M0}-\sqrt{3}\cdot[t\cdot D^2_{M2+}+i\sqrt{1-t^2}\cdot D^2_{M2-}]\}. \quad (10.2)$$

This function is also equal to one of the functions in Table VIII.

Example 3. We write down the pair of functions with $L=2$, $\lambda=\mu$ given in the Elliott coordinate system:

$$\Psi_1=\frac{\sqrt{3}}{2}\cdot D^2_{M0}-\frac{1}{2}\cdot D^2_{M2+},$$

$$\Psi_2=\frac{1}{2}\cdot D^2_{M0}+\frac{\sqrt{3}}{2}\cdot D^2_{M2+}.$$

On the transition to our coordinate system, D^2_{M0} transforms in accordance with (10.2), and the transformation of D^2_{M2+} can be found by using the general rule (10.1):

DM2+2=DM22+DM−22

TABLE VIII. Eigenvalues of the Bargmann–Moshinsky operator in explicit form for $L=3$.

Eigenvalues of BM operator	Values of μ, λ, N_1, N_2	Eigenfunctions of the BM operator for $L=3$.
$\omega_2=20$	$\mu=1$ $N_2=1$ $\lambda=2$ $N_1=1$	$\vert\Psi\rangle=\sqrt{1-t^2}\cdot\left[t\cdot D^3_{M0}+\frac{\sqrt{3}}{2\sqrt{5}}\cdot D^3_{M2+}\right]$
	$\mu=1$ $N_2=0$ $\lambda=2$ $N_1=1$	$\vert\Psi\rangle=\sqrt{1-t^2}\cdot\left\{D^3_{M0}+\frac{\sqrt{3}}{2\sqrt{5}}\cdot[t\cdot D^3_{M2}+i\sqrt{i-t^2}\cdot D^3_{M2-}]\right\}$
$\omega_3=-6$	$\mu=1$ $N_2=1$ $\lambda=2$ $N_1=1$	$\vert\omega\rangle=\sqrt{1-t^2}\cdot\left[t\cdot D^3_{M0}-\frac{\sqrt{5}}{\sqrt{3}}\cdot D^3_{M2+}\right]$
	$\mu=1$ $N_2=0$ $\lambda=2$ $N_1=2$	$\vert\Psi\rangle=\sqrt{1-t^2}\cdot\left\{D^3_{M0}-\frac{\sqrt{5}}{\sqrt{3}}\cdot[tD^3_{M2+}+i\sqrt{1-t^2}\cdot D^3_{M2-}]\right\}$
$\omega=0$	$\mu=1$ $N_2=0$ $\lambda=2$ $N_1=2$	$\vert\Psi\rangle=\sqrt{1-t^2}\cdot[t\cdot D^3_{M2-}+i\sqrt{1-t^2}\cdot D^3_{M2+}]$
	$\mu=0$ $N_2=1$ $\lambda=2$ $N_1=1$	$\vert\Psi\rangle=\sqrt{1-t^2}\cdot D^3_{M2-}$
$\omega_4=0$	$\mu=0$ $N_2=0$ $\lambda=3$ $N_1=3$	$\vert\Psi\rangle=\{\sqrt{1+t}D^3_{M1-}+i\sqrt{1-t}\cdot D^3_{M1+}\}+\frac{1}{\sqrt{15}}\cdot\{(2t-1)\sqrt{1+t}\cdot D^3_{M3-}+i(2t+1)\sqrt{1-t}\cdot D^3_{M3+}\}$
	$\mu=0$ $N_2=1$ $\lambda=3$ $N_1=2$	$\Psi=\frac{1}{3}\{(2t+1)\sqrt{1+t}\cdot D^3_{M1-}+i(2t-1)\sqrt{1-t}\cdot D^3_{M1+}\}+\frac{1}{\sqrt{15}}\cdot\{\sqrt{1+t}\cdot D^3_{M3-}+i\sqrt{1-t}\cdot D^3_{M3+}\}$
$\omega_5=-18$	$\mu=0$ $N_2=0$ $\lambda=3$ $N_1=3$	$\vert\Psi\rangle=\{\sqrt{1+t}\cdot D^3_{M1-}+i\sqrt{1-t}\cdot D^3_{M1+}\}+\frac{\sqrt{15}}{3}\cdot\{(2t-1)\sqrt{1+t}\cdot D^3_{M3-}+i(2t+1)\sqrt{1-t}\cdot D^3_{M3+}\}$
	$\mu=0$ $N_2=1$ $\lambda=3$ $N_1=2$	$\Psi=\{(2t+1)\sqrt{1+t}\cdot D^3_{M1-}+i(2t-1)\sqrt{1-t}\cdot D^3_{M1+}\}-15\cdot\{\sqrt{1+t}\cdot D^3_{M3-}+i\sqrt{1-t}\cdot D^3_{M3+}\}$
$\omega_{6,7}=3(1\pm\sqrt{41})$	$\mu=0$ $N_2=0$ $\lambda=3$ $N_1=3$	$\vert\Psi\rangle=\{\sqrt{1+t}\cdot D^3_{M1+}+i\sqrt{1-t}\cdot D^3_{M1-}\}+\frac{\sqrt{15}}{3(6\pm\sqrt{41})}\cdot\{(2t-1)\sqrt{1+t}\cdot D^3_{M3+}+i(2t+1)\sqrt{1-t}\cdot D^3_{M3-}\}$
	$\mu=0$ $N_2=1$ $\lambda=3$ $N_1=2$	$\Psi=\{(2t+1)\sqrt{1+t}\cdot D^3_{M1+}+i(2T-1)\sqrt{1-t}\cdot D^3_{M1-}\}+\frac{\sqrt{15}}{(6\pm\sqrt{41})}\cdot\{\sqrt{1+t}\cdot D^3_{M3+}+i\sqrt{1-t}\cdot D^3_{M3-}\}$

$$\to\sqrt{\frac{3}{2}}\cdot D^2_{M0}+\frac{1}{\sqrt{2}}\cdot[t\cdot D^2_{M2+}+i\sqrt{1-t^2}\cdot D^2_{M2-}].$$

Therefore, the functions Ψ_1 and Ψ_2 are transformed to

$$\Psi_1\to\Psi_1'=-\frac{\sqrt{3}}{2}\cdot D^2_{M0}+\frac{1}{2}[t\cdot D^2_{M2+}+i\sqrt{1-t^2}\cdot D^2_{M2-}],$$

$$\Psi_2\to\Psi_2'=\frac{1}{2}\cdot D^2_{M0}+\frac{\sqrt{3}}{2}\cdot[t\cdot D^2_{M2+}+i\sqrt{1-t^2}\cdot D^2_{M2-}].$$

The transformation of the coordinate system without rotation through the angle $\theta/2$ is tantamount to the following transformation of the unit vectors:

$$\mathbf{e}_1\to\mathbf{e}_3,\quad \mathbf{e}_2\to-\mathbf{e}_2,\quad \mathbf{e}_3\to\mathbf{e}_1. \tag{10.3}$$

In the new coordinate system, the functions Ψ_1 and Ψ_2 then take the form

$$\Psi_1\to\Psi_1''=-\left[\frac{\sqrt{3}}{2}\cdot D^2_{M0}-\frac{1}{2}\cdot D^2_{M2+}\right]=-\Psi_1,$$

$$\Psi_2\to\Psi_2''=\frac{1}{2}\cdot D^2_{M0}+\frac{\sqrt{3}}{2}\cdot D^2_{M2+}=\Psi_2.$$

We elucidate our result. For $\lambda=\mu$, the expression $\alpha^{\lambda}_{33}\cdot\alpha^{\mu}_{11}$ is invariant with respect to the transformation (10.3). Since $\alpha^{\lambda}_{33}\cdot\alpha^{\mu}_{11}$ can be expanded in a series in which each term is a product of functions of the type Ψ_1 or Ψ_2 with the same functions (in complex-conjugate form) of the tilded variables, invariance of the left-hand side of the equation entails invariance of its right-hand side. Therefore, under the coordinate transformation (10.3) the functions Ψ_1 and Ψ_2 are unchanged. However, there may be a change in the sign of the function due to the fact that the expansion is a binary expansion.

11. CONCLUSIONS

Thus, the problem of constructing basis functions of the model of two rotors from products of single-particle harmonic-oscillator functions can be carried through to the end, and in the Fock–Bargmann space these functions can be represented explicitly by expressing them in terms of hypergeometric functions, spherical Wigner functions, and the eigenfunctions of the matrix of the Bargmann–Moshinsky operator. The calculation of the matrix elements of the operators in the Fock–Bargmann space reduces to a simple recursive procedure for hypergeometric functions and Wigner functions.

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