The method of approximate functional integration in mathematical physics

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The latest results obtained by various authors using the method of approximate functional integration for solving a wide range of problems in physics and mathematics are reported. This method has been used in quantum mechanics, in quantum field theory, in the study of differential operators, and for the solution of various types of differential equations in mathematical physics. The rigorous definition of the functional integral in a complete separable metric space is given. Studies of the continuum limit of lattice models are reviewed, along with the results of using functional integrals to study nonperturbative phenomena in quantum field theory, the topological structure of the vacuum in quantum gauge theories, in tunneling problems, and for describing multiparticle quantum systems, string theory, and quantum gravity. Results obtained by the present authors using functional integration with a new approach to path integrals based on functional analysis and measure theory are presented. These results are relevant to the theoretical study of functional integrals, the development of a new method for their approximate solution, and their use in studying various quantum systems. © *1996 American Institute of Physics.* [S1063-7796(96)00401-7]

INTRODUCTION

The method of functional integration is a powerful tool for studying a wide variety of problems in various areas of physics and mathematics.[1] First used in quantum mechanics by Feynman,[2] functional integration now forms the foundation of modern constructive quantum field theory,[3] and is the principal method of numerically studying nonperturbative phenomena in quantum gauge theory.[4–6] Functional integrals have extensive applications in quantum mechanics, field theory,[7–13] quantum electrodynamics,[14–16] statistical physics,[9,17–19] nuclear physics,[20] solid-state physics,[19] quantum statistics,[21] superconductivity theory,[22] ferromagnetism,[23,24] gravitational theory,[25] quantum optics, statistical radio engineering, the radiation physics of high-energy particles, and many other areas.[26] Functional integration is regularly the subject of large international conferences such as Path Integration: Achievements and Goals (Trieste, Italy, 1987); Colloquium on Functional Integration Theory and Applications (Louvain-la-Neuve, Belgium, 1979); Adriatico Research Conference on Path Integration and Its Applications (Trieste, Italy, 1991); International Conference on Path Integrals in Physics (Bangkok, Thailand, 1993); and the series of regular conferences Path Integrals from meV to MeV (Albany, New York, 1983; Bielefeld, 1985; Bangkok, 1989; Tutzing, Munich, 1992).

The extensive use of functional integrals has stimulated the development of the theory and methods of calculating them approximately. Because the "Feynman measure" does not satisfy the condition of countable additivity, i.e., it is not a measure in the rigorous mathematical sense, a variety of other approaches have been developed for the construction and approximate calculation of Feynman integrals. Among the fundamental approaches are those of Nelson, Itô, DeWitt-Morette, Albeverio and Hoegh-Krohn, and Truman. More details about these and other approaches can be obtained from Refs. 27–30 and references therein. The history of the development of the theory, the methodology, and the use of path integrals are discussed in Ref. 31. Most of the results pertain to Lagrangian path integrals in configuration space, but a great deal of attention has also been given to Hamiltonian integrals in phase space (see, for example, Refs. 32–34). In Ref. 35 and in Ref. 36, which develops the results of Ref. 35 further for the special case of path integrals on phase space using the WKB approximation, a generalization was obtained of the Duistermaat–Heckman theorem stating the conditions for exactness of the WKB method. As noted in Ref. 36, the fact that the path integral can be WKB-exact under certain conditions is interesting for two reasons: because it becomes possible to extend the number of known examples of quantum systems where the Feynman integral can be evaluated exactly, and because it becomes possible to understand the geometrical structure of quantum systems more deeply. However, as noted in Refs. 35 and 36, the results of these studies have a limited range of applicability and require that a number of conditions be satisfied, which does not always occur in practice. As noted in Ref. 36, the analysis performed in that study is valid only in the case of phase spaces which are two-dimensional symplectic sets, and generalization to higher numbers of dimensions is not possible.

Path integrals in curved configuration space have been studied (see Refs. 7 and 37). The authors of Ref. 37 studied the dynamics of a quantum system on a sphere and obtained a representation for the Green function (the matrix element of the evolution operator) in the form of a path integral, for which the semiclassical approximation was used.

The authors of Refs. 35–37 discuss integrals with a measure specified not on a function space, but in the form of the $N\to\infty$ limit of N-fold Riemann integrals. This measure does not satisfy the condition of countable additivity, which, as mentioned above, complicates the mathematical justification of such integrals. Studies have appeared in recent years which interpret such constructions as quasimeasures or

pseudomeasures (see Ref. 38). No numerical calculations were made in Refs. 35–37.

The mathematically rigorous study of functional integrals with countably additive measures was begun in the studies by Wiener,[39] who introduced the functional-integration measure bearing his name in the space of functions continuous on a segment. It should be noted that the term "functional integral" should be used for integrals with a definite measure in a given space, although in Russian this term is often used synonymously with "path integral," which under certain conditions is the physical interpretation of a functional integral in the special case of quantum physics. In recent years researchers around the world have obtained important results in the theory and use of functional integrals. For a long time the best studied examples were Wiener integrals (see Ref. 40, for example) related to Feynman integrals by rotation to imaginary time, i.e., to the Euclidean formulation of the theory. However, special attention has recently been paid to the generalization of the Wiener integral,[41] and to the study of functional integrals over other measures in the corresponding spaces (for example, Refs. 26 and 42–48). Great advances have also been made in the development of methods for the approximate calculation of functional integrals without preliminary discretization of space and time and with no need for simplifying assumptions like the mean-field approximation, the semiclassical approximation, and so on (see, for example, Refs. 38 and 49–52). Various authors have also worked on compiling tables of exactly computable functional integrals with countably additive measures, and also Feynman path integrals.[53,54] The current status of the mathematical theory of functional integration is described in Refs. 38, 43, and 55.

In recent years a huge number of studies have been devoted to the use of functional integrals for the numerical solution of various problems. Let us list a few of these as examples: the calculation of the propagator of a charged particle in a magnetic field;[56] the calculation of special functions;[57] the numerical study of helium gas absorbed by graphite;[58] the calculation of the spectrum of the effective Hamiltonian in $SU(3)$;[59] the solution of the inverse problem for the three-dimensional Schrödinger equation;[60] the numerical study of the interaction of a charged particle with a Kaluza–Klein monopole;[61] the study of phase transitions[62–64] and turbulent fluid flow;[65] the calculation of the second virial coefficient of the anyon gas.[66] Some problems involving the use of functional integrals will be discussed in more detail below. We shall pay special attention to the functional integrals of Euclidean quantum physics. We shall mainly discuss the calculation of functional integrals with countably additive measure in separable metric spaces; the calculation of path integrals not satisfying this condition can form the subject of a separate discussion. In the present review we also present results of approximate functional integration that we have obtained during the last few years.

1. DEFINITION OF THE FUNCTIONAL INTEGRAL

Let us define the functional integral on the basis of recent advances in measure theory and the theory of integration in function spaces (see, for example, Refs. 38 and 43). Let there be given a space X on which is defined a σ algebra G of subsets of this space together with a measure μ given on G.

A functional $F[x]$ defined on $x \in X$ and taking real values is called measurable on X if for every Borel set B of the real line the corresponding set $\{x \in X: F[x] \in B\}$ belongs to the σ algebra G.

A simple functional is a measurable functional taking no more than a countable number of different values. We write such a functional as

$$F[x]=\sum_j a_j \chi_{A_j}[x],$$

where a_j are various numbers; A_j are nonintersecting sets from G, and $\chi_{A_j}[x]$ are the characteristic functions of the sets A_j:

$$\chi_{A_j}[x]=\begin{cases}1 & \text{for } x \in A_j,\\ 0 & \text{for } x \notin A_j.\end{cases}$$

The integral of a simple functional $F[x]$ over the space X, denoted by $\int_X F[x]d\mu$, is given by the equation

$$\int_X F[x]d\mu=\sum_{k=1}^{\infty} a_k \mu(A_k),$$

where $A_k=\{x \in X: F[x]=a_k\} \in G$,

if the series on the right-hand side converges absolutely. In this case the simple functional $F[x]$ is termed integrable with measure μ on the space X.

Let $F[x]$ be an arbitrary functional measurable on X. We use the notation $F^+[x]=\max\{F[x],0\}$, $F^-[x]=-\min\{F[x],0\}$. Obviously, $F[x]=F^+[x]-F^-[x]$, where $F^+[x]$ and $F^-[x]$ are functionals which are non-negative and measureable on X.

The quantity

$$\int_X F[x]d\mu=\lim_{n\to\infty}\int_X F_n[x]d\mu$$

is called the integral of a measurable non-negative functional $F[x]$ over the space X. Here $\{F_n[x]\}_{n=1}^{\infty}$ is an arbitrary, monotonically nondecreasing sequence of non-negative simple functionals converging to $F[x]$, and the limit on the right-hand side is independent of the choice of sequence.

Let $F[x]$ be an arbitrary measurable functional. If one of the integrals

$$\int_X F^+[x]d\mu, \quad \int_X F^-[x]d\mu \tag{1}$$

is finite, the integral of the functional $F[x]$ is given by the equation

$$\int_X F[x]d\mu=\int_X F^+[x]d\mu-\int_X F^-[x]d\mu. \tag{2}$$

If both integrals in (1) are finite, the integral (2) is also finite, and the functional $F[x]$ is termed integrable with measure μ on the space X.

We shall study spaces X which are complete separable metric spaces, as these are most commonly encountered in

applications. In particular, an example of such a space is $C[a,b]$, the space of functions continuous on a segment.

If we have $\mu(X)=1$, the measure μ is termed a probability measure on the space X.

Let X' be the space conjugate to X of linear continuous functionals on X. For every $\varphi\in X'$ and any $x\in X$ we shall denote the value of the functional φ on the element x by $\langle\varphi,x\rangle$.

A set of the form

$$Q_{\varphi_1\ldots\varphi_n}(A_n)=\{x\in X:[\langle\varphi_1,x\rangle,\langle\varphi_2,x\rangle,\ldots,\langle\varphi_n,x\rangle]\in A_n\} \tag{3}$$

is called a cylinder set in the space X. Here $\varphi_1\ldots\varphi_n$ are any linearly independent elements of the space X', and A_n is an arbitrary Borel set in n-dimensional Euclidean space R^n $(n=1,2,\ldots)$. The set A_n is called the base of the cylinder set $Q_{\varphi_1\ldots\varphi_n}$.

Let $K(\varphi,\psi)$ be a bilinear functional and $M(\varphi)$ be a linear functional given on the conjugate space X', $\varphi,\psi\in X'$. The probability measure μ given on the σ algebra G generated by all cylinder sets of the space X is termed Gaussian if its value on any cylinder set $Q_{\varphi_1\ldots\varphi_n}$ is given by the expression

$$\mu(Q_{\varphi_1\ldots\varphi_n}(A_n))=(2\pi)^{-n/2}(\det K)^{-1/2}\times\int_{A_n}\exp\{-\tfrac{1}{2}(K^{-1}[u-M(\varphi)],[u-M(\varphi)])\}du, \tag{4}$$

where K^{-1} is the inverse of the matrix $K=\|K(\varphi_i,\varphi_j)\|$,

$M(\varphi)=\{M(\varphi_1),M(\varphi_2),\ldots,M(\varphi_n)\}$;

$u=(u_1,u_2,\ldots,u_n)$;

$$(K^{-1}[u-M(\varphi)],[u-M(\varphi)])=\sum_{i,j=1}^{n}K_{i,j}^{-1}[u_i-M(\varphi_i)]\times[u_j-M(\varphi_j)],$$

$K_{i,j}^{-1}$ are the elements of the matrix K^{-1}, and $du=du_1du_2\ldots du_n$.

The Gaussian measure defined as in (4) is a countably additive probability measure on the Borel σ algebra of the space X. Here the functional integral is a Lebesgue integral in X with measure μ. The functional $K(\varphi,\psi)$ is called the correlation functional, and $M(\varphi)$ is the average value of the measure μ.

This definition of the measure differs from the well known measure in n-dimensional Euclidean space R^n, because it is not possible to introduce a measure in the usual manner in the infinite-dimensional case $(n\to\infty)$, since the volume of the n-dimensional parallelepiped will tend to infinity. As will be shown below, the solutions of various differential and integral equations, and also the observable values of operators in quantum physics, can be expressed in terms of functional integrals with Gaussian measure.

In many aspects of measure theory an important role is played by the functional defined on X':

$$\chi(\varphi)=\int_X\exp\{i\langle\varphi,x\rangle\}d\mu(x),$$

which is termed the characteristic functional. As shown in Ref. 50, in the case of a Gaussian measure

$$\int_X\exp\{i\langle\varphi,x\rangle\}d\mu(x)=\exp\left\{iM(\varphi)-\frac{1}{2}K(\varphi,\varphi)\right\},\quad\varphi\in X',$$

where

$$K(\varphi,\psi)=\int_X[\langle\varphi,x\rangle-M(\varphi)]\times[\langle\psi,x\rangle-M(\psi)]d\mu(x),\quad\psi\in X',$$

$$M(\varphi)=\int_X\langle\varphi,x\rangle d\mu(x).$$

A Gaussian measure is uniquely defined by its characteristic functional $\chi(\varphi)$ or by the functionals $K(\varphi,\psi)$ and $M(\varphi)$ (Ref. 50).

The concept of Gaussian measure includes a large class of measures most commonly used in applications, including the conditional Wiener measure on the space of continuous functions $x(t)\in C[0,1]$ satisfying $x(0)=x(1)=0$. In this case $M(\varphi)=0$. Since in the space $C_0\equiv\{C[0,1],\ x(0)=x(1)=0\}$ we have

$$\langle\varphi,x\rangle=\int_0^1x(t)d\varphi(t)$$

for functions $\varphi(t)$ with bounded variation, in this case

$$K(\varphi,\psi)=\int_{C_0}\langle\varphi,x\rangle\langle\psi,x\rangle d\mu(x)=\int_0^1\int_0^1\mathscr{K}(t,s)d\varphi(t)d\psi(s),$$

where the integrated kernel

$$\mathscr{K}(t,s)=\int_{C_0}x(t)x(s)d\mu(x)$$

is called the correlation function of the measure. The conditional Wiener measure d_Wx is characterized by the following correlation function:

$$\mathscr{K}(t,s)=\min\{t,s\}-ts,\quad t,s\in[0,1].$$

A clearer definition of the Wiener measure based on the theory of Brownian motion is given in, for example, the paper by Gel'fand and Yaglom.[67]

2. RELATION BETWEEN FUNCTIONAL INTEGRALS AND SOLUTIONS OF DIFFERENTIAL AND INTEGRAL EQUATIONS

The relation between partial differential equations and functional integrals was revealed by Feynman[2] in developing a new approach to quantum mechanics. A mathematically rigorous proof of the representation of the solution of the

heat-conduction equation (the Schrödinger equation in imaginary time) is given by Kac.[68] According to the Feynman–Kac formula (see also Ref. 67), the Green function Z of the equation

$$\frac{\partial}{\partial t} Z(x,x_0,t)=\frac{1}{2}\frac{\partial^2}{\partial x^2} Z(x,x_0,t)-V(x)Z(x,x_0,t),$$

$$Z(x,x_0,0)=\delta(x-x_0), \quad x\in(-\infty,\infty), \quad t>0,$$

$$Z(x,x_0,t)\to 0 \quad \text{for } |x|\to\infty, \tag{5}$$

where $V(x)$ is a given function, is represented as an integral with conditional Wiener measure

$$Z(x,x_0,t)=\int_C \exp\left\{-\int_0^t V[\omega(\tau)]d\tau\right\} d_W\omega \tag{6}$$

over the space $C[0,t]$ of continuous functions $\omega(\tau)$ satisfying the conditions $\omega(0)=x_0, \omega(t)=x$.

The Green function $Z(x,x_0,t)$ can be used to find the solutions $Z(x,t)$ of the problem (5) for various initial conditions $Z(x,0)=Z_0(x)$ in the form

$$Z(x,t)=\int_{-\infty}^{\infty} Z_0(x_0)Z(x,x_0,t)dx_0.$$

In Ref. 67 the relation between functional integrals and the solutions of differential equations was studied from the viewpoint of applications to problems in quantum physics. There the problem was posed of generalizing this relation to a larger class of differential equations, including parabolic ones of higher order. Daletskiĭ has obtained such a generalization for abstract evolution equations, including equations and systems of equations of the parabolic, hyperbolic, and Schrödinger types (Refs. 69–71).

Representation of the solution of a differential equation as a functional integral allows the original problem to be reduced to quadratures in function space, which in many cases is convenient for both theoretical study and numerical solution. The functional-integration approach offers additional possibilities for studying a solution, for solving the problem in an unrestricted region without replacing the boundary conditions at infinity by conditions on the boundary of a fixed region, and so on.[72] The methods used in the theory of the Lebesgue integral (in particular, substitution of variables in the integral, construction of quadrature formulas in function spaces, and so on) can be applied to the solutions of differential equations in this case.[73] In Ref. 67 the functional-integration method was used, in particular, to study the asymptotic behavior of the eigenfunctions and eigenvalues of an elliptic differential operator, to study the classical limit and the semiclassical approximation of quantum mechanics, to calculate the partition function of quantum statistics, to estimate the lowest eigenvalue of the Schrödinger equation, and so on.

The representation of the solutions of several differential equations in the form of functional integrals in phase space has been studied in Refs. 74 and 75.

The solution of the Cauchy and Gursat problems for certain equations of the hyperbolic type can be expressed as an integral with generalized Wiener measure in the space C_2 of continuous functions of two variables $\omega=\omega(t,s)$ given on the square $Q=[0,1]\times[0,1]$ and satisfying the conditions $\omega(t,0)=\omega(0,s)$; the same can be done for systems of differential equations of the hyperbolic type if the space is $C_2^p=C_2\times C_2\ldots\times C_2$ (Ref. 41). For example, the solution of the problem

$$\frac{\partial^2 u_k(t,s)}{\partial t\partial s}=\sum_{l=1}^{p} a_{kl}(t,s)u_l(t,s)+b_k(t,s),$$

$$u_k(t,0)=u_k(0,s)=0, \quad k=1,\ldots,p; \quad t,s\in Q$$

in the case where $a_{kl}(t,s)$ and $b_k(t,s)$ are continuous functions on the square Q can be represented as the following functional integral:[41]

$$\begin{aligned} u_k(t,s)=\exp\Bigg\{&-\frac{1}{4}\sum_{k=1}^{p}\int_Q a_{kk}(\tau,\sigma)d\tau d\sigma \\ &-\sum_{k=1}^{p}\int_Q b_k^2(\tau,\sigma)d\tau d\sigma\Bigg\}\int_{C_2^p} x_k(t,s) \\ &\times\exp\Bigg\{2\sum_{k=1}^{p}\int_Q b_k(\tau,\sigma)d_{\tau,\sigma}\Bigg[x_k(\tau,\sigma) \\ &-\sum_{l=1}^{p}\int_0^\tau\int_0^\sigma a_{kl}(\tau_1,\sigma_1)x_l(\tau_1,\sigma_1)d\tau_1 d\sigma_1\Bigg] \\ &+2\sum_{k,l=1}^{p}\int_Q a_{kl}(\tau,\sigma)x_l(\tau,\sigma)dx_k(\tau,\sigma) \\ &-\sum_{k=1}^{p}\int_Q\Bigg[\sum_{l=1}^{p} a_{kl}(\tau,\sigma)x_l(\tau,\sigma)\Bigg]^2 d\tau d\sigma\Bigg\} d_W x. \end{aligned}$$

The solution of the Cauchy problem for the integro-differential equation

$$\begin{aligned} \frac{\partial u(t,x)}{\partial t}=&a\frac{\partial u(t,x)}{\partial x}+\frac{b}{2}\frac{\partial^2 u(t,x)}{\partial x^2}+\int_R\Bigg[u(t,x+y) \\ &-u(t,x)-\frac{y^2}{1+y^2}\frac{\partial u(t,x)}{\partial x}\Bigg]\pi(dy) \\ &+c(x)u(t,x), \end{aligned}$$

$$u(0,x)=f(x), \quad t>0, \quad x\in(-\infty,\infty),$$

where $c(x)$ and $f(x)$ are given functions and the measure π satisfies the conditions

$$\pi\{0\}=0, \quad \int_R \frac{y^2}{1+y^2}\pi(dy)<\infty,$$

can be written as follows for certain restrictions on c and f (Refs. 76 and 77):

$$u(t,x)=\int_D \exp\left\{\int_0^t c(\xi_t+x)dt\right\} f(\xi_t+x)d\mu(\xi), \tag{7}$$

where μ is the measure given on the σ algebra of the Borel sets of the space D of functions on $[0,t]$ without discontinuities of the second kind and corresponding to a random uniform process with independent increments ξ_τ,

$0 \leqslant \tau\, 0 \leqslant \tau \leqslant t < \infty$, satisfying the condition $\xi_0 = 0$ and taking values in R. The characteristic functional $\chi(l)$ of the measure μ has the form[50]

$$\chi(l) = \exp\left\{ il(m) + \sum_{k=2}^{\infty} \frac{i^k}{k!}\, \sigma_k \int_0^t l^k(\rho(u))du \right\},$$

where

$$m(t) = t\left(a + \int_R \frac{y^2}{1+y^2}\, \pi(dy) \right),$$

$$\sigma_2 = b + \int_R y^2 \pi(dy),$$

$$\sigma_k = \int_R y^k \pi(dy), \quad k = 3,4,\ldots$$

$$\rho_t(u) = 1_{[u,t]}(t).$$

According to (7), by using random processes of various types (the process of Brownian motion or the Wiener process, the Poisson process, the Laplace process, the telegraph process, and so on) it is possible to obtain the solutions of various differential equations in the form of functional integrals in the corresponding spaces X (Ref. 50). In particular, if ξ_t is the Wiener process $[m(t) = ta;\ \sigma_2 = b;\ \sigma_k = 0,\ k > 2;\ \pi(dy) = 0]$ and $a = 0$, $b = 1$, then the integral (7) gives the solution of the Cauchy problem for Eq. (5).

Functional integrals can be used to represent the solutions of various integral equations. For example, the solution of the Fredholm integral equation of the second kind

$$y(t,s) = x(t,s) + \int_Q K(t,s,t_1,s_1)x(t_1,s_1)dt_1 ds_1, \qquad (8)$$

where

$$y(t,s) \in C_2; \quad \frac{\partial^2 y_k(t,s)}{\partial t \partial s} \in L_2(Q),$$

and the kernel $K(t,s,t_1,s_1)$ satisfies certain conditions, can be written as[41]

$$x(t,s) = \exp\left\{ -\int_Q \left[\frac{\partial^2 y(t,s)}{\partial t \partial s} \right]^2 dtds \right\} \int_{C_2} z(t,s)$$
$$\times \exp\left\{ 2\int_Q \frac{\partial^2 y(t,s)}{\partial t \partial s}\, d_{t,s} \left[z(t,s) + \int_Q K(t,s,t_1,s_1)z(t_1,s_1)dt_1 ds_1 \right] \right.$$
$$\left. - \int_Q \left[\frac{\partial^2}{\partial t \partial s} \int_Q K(t,s,t_1,s_1)z(t_1,s_1)dt_1 ds_1 \right]^2 \right.$$
$$\times dtds - 2\int_Q \left[\frac{\partial^2}{\partial t \partial s} \int_Q K(t,s,t_1,s_1)z(t_1,s_1) \right.$$
$$\left. \left. \times dt_1 ds_1 \right] dz(t,s) \right\} d_W z.$$

The solutions of Volterra integral equations of the second kind and also systems of Fredholm and Volterra integral equations of the second kind can be written in a similar manner.[41] Finding the solutions of systems of partial differential equations and of systems of Fredholm and Volterra integral equations of the second kind in the form of functional integrals has also been studied in Refs. 78 and 79.

Functional integration is a convenient method of studying differential equations in Banach spaces. In particular, the functional integral generated by the process $x_t = (-1)^{\xi_t}$, where ξ_t is the Poisson process, can be used to express the solution of the Cauchy problem for the abstract telegraph equation:

$$u_{TT} + \alpha u_T = A^2 u \quad (\alpha > 0),$$
$$u(0) = f, \quad u_T(0) = g, \qquad (9)$$

where u_T and u_{TT} are strong derivatives of the function $u(T)$ taking values in the Banach space Y, and f and g are elements of the region in which the linear operator A^2 acting in Y is defined. If A generates a strongly continuous group of bounded linear operators in Y, the solution of the problem (9) can be represented as (Ref. 80; see also Ref. 50)

$$u(T) = \frac{1}{2} \sum_{j=0}^{1} \int_X \left[w_1\left((-1)^j \int_0^T x_t dt \right) \right.$$
$$\left. + (-1)^j w_2\left((-1)^j \int_0^T x_t dt \right) \right] d\mu(x),$$

where X is the space of step functions continuous on the right and taking values ± 1 beginning at unity and having any finite number of points of discontinuity; w_1 and w_2 are unique solutions of the Cauchy problems for the abstract wave equation

$$w_{i_{TT}} = A^2 w_i, \quad i = 1,2;$$
$$w_1(0) = f; \quad w_{2_T}(0) = g; \quad w_{1_T}(0) = w_2(0) = 0.$$

When A^2 is a Laplace operator, the solutions w_1 and w_2 are well known.

Functional integrals over a space of branching trajectories can be used to represent the solutions of certain nonlinear differential equations.[43,81]

There are many studies devoted to the use of functional integrals for solving various differential equations in mathematical physics. These include studies of the solutions of parabolic equations,[82] the Schrödinger equation,[30] third- and fourth-order differential equations,[83,84] the wave equation,[85] the Fokker–Planck equation,[86] the Dirac equation,[87,88] the nonlinear Langevin equation,[89] the telegraph equation,[90] and so on.

3. FUNCTIONAL INTEGRALS IN QUANTUM PHYSICS

One of the main areas in which functional integrals are used is quantum physics.[67,91] The approach to quantum physics based on functional integrals with Gaussian measure and the Wiener measure as a special case of a Gaussian measure are discussed in Ref. 92 (the recent edition). In recent years important results have been obtained in the construction of functional-integration measures in quantum physics (see, for example, Refs. 93–100). These include the Euclidean mea-

sure for the electromagnetic field,[97] the measure in Lagrangian gauge theories,[98] and the Gaussian measure on an extended Grassmann algebra for functional integrals over fermionic fields.[100] The functional integrals of Euclidean quantum physics and the foundations of the method for their approximate calculation developed by the present authors are discussed in Ref. 101.

The method of integration in function spaces has reached its highest level of development in quantum mechanics (Refs. 10, 11, 67, and 102). The use of approximate functional integration for various calculations in quantum mechanics is studied in, for example, Refs. 103 and 104. The various physical characteristics of a system can be determined on the basis of calculating the Green function for the Schrödinger equation in the form of a functional integral.[105] The author of Ref. 106 proposed a method of calculating the Green function based on representing it as a Feynman integral (in imaginary time), which offers significant advantages in numerical calculations. We have developed this method for the approximate calculation of integrals with conditional Wiener measure.[73] The use of functional integrals for studying energy spectra, particle scattering, and other problems is discussed in Refs. 107–116. The method of approximate calculation of functional integrals developed by the present authors for quantum-mechanical problems is described in Refs. 117 and 118.

The use of functional integrals in calculations in quantum electrodynamics is discussed in, for example, Refs. 119–121, and their use in relativistic theory is described in Refs. 122–127. Functional integrals are used in quantum optics[26,128] and in the quantum theory of magnetism,[129] and they play an important role in the study of conformal anomalies in gauge theory.[130–137] The method of functional integration is widely used for analytical and numerical studies in theories of strings, membranes,[138–149] and the polaron.[150–152] In recent years many studies have been devoted to the functional-integration measure and the use of functional integrals in quantum gravity[153–166] and in quantum cosmology.[167–172] Many authors are occupied with the theoretical study of functional integrals in various areas of quantum physics,[173–190] and also with the development of methods for computing them approximately.[189–201]

The introduction of a space-time lattice is one of the methods of defining functional integrals in quantum theory by reducing them to ordinary (Riemann) integrals of high multiplicity.[202] A number of important numerical results have been obtained in this way (see Refs. 203–207). In lattice calculations it is necessary to deal with the problems of studying the existence and uniqueness of the continuum limit,[208–213] the dependence of the results on the lattice spacing, the appearance of finite-size effects and lattice artifacts,[214–216] and problems with topology on the lattice (ambiguities in determining the topological charge, and so on).[217–220] The continuum limit is being studied by many authors.[221–228] Other topics under study are the method of working in the continuum limit directly (see Refs. 95 and 229–231), the development of a continuous quantum gauge theory,[6] methods of nonperturbative regularization at the continuum level,[231–235] and methods of calculating functional integrals without the introduction of a lattice.[198,236] The methods that we have developed for approximate integration in functional spaces also do not require a preliminary lattice-type discretization of space-time.[237]

In Euclidean quantum mechanics, according to the Feynman–Kac formula the matrix element of the evolution operator $\exp\{-\beta H\}$,

$$Z(x_i,x_f,\beta)=\langle x_f|e^{-\beta H}|x_i\rangle, \tag{10}$$

where H is the Hamiltonian of the system, $H=\frac{1}{2}\hat{p}^2+V$, is written as a functional integral with conditional Wiener measure

$$Z(x_i,x_f,\beta)=\int_C \exp\left\{-\int_0^\beta V(x(t))dt\right\} d_W x, \tag{11}$$

where the integration runs over all functions $x(t)\in C[0,\beta]$ satisfying the condition $x(0)=x_i$, $x(\beta)=x_f$. The quantity (10) represents the probability density of finding the particle at the point with coordinate x_f at time $t=\beta$ with the condition that it had coordinate x_i at time $t=0$. In contrast to the notation with Z in the form of a path integral, the argument of the exponential in (11) contains not the action functional S, but only the part of it corresponding to the potential energy. The kinetic term $(dx/dt)^2$ is absent inside the integral, because it is included in the integration measure (see Ref. 67, for example). The Green function Z with periodic boundary conditions $x_i=x_f=x$ provides the basis for finding the various physical characteristics of the quantum system. After the corresponding variable substitution,[238] the functional integral $Z(x,x,\beta)$ can be written as a standard integral with conditional Wiener measure in the space $C_0\equiv\{C[0,1];x(0)=x(1)=0\}$:

$$Z(x,x,\beta)=\frac{1}{\sqrt{2\pi\beta}}\int_{C_0}\exp\left\{-\beta\int_0^1 V(\sqrt{\beta}x(t)+x)dt\right\} d_W x. \tag{12}$$

The expressions for the free energy $f(\beta)$, the ground-state energy E_0, the propagator $G(\tau)$, the wave function $\psi_0(x)$, and the difference of the energies of the ground and first excited states ΔE are written in the form of functional integrals with conditional Wiener measure as follows:[238]

$$f(\beta)=-\frac{1}{\beta}\ln Z(\beta),$$

$$E_0=\langle 0|H|0\rangle=\lim_{\beta\to\infty}\frac{1}{Z(\beta)}(2\pi\beta)^{-1/2}\int_{-\infty}^{\infty}\int_{C_0}\exp\left\{-\beta\int_0^1 V(\sqrt{\beta}x(t)+x)dt\right\}\left[\frac{1}{2}xV'(x)+V(x)\right]d_W x\,dx; \tag{13}$$

$$G(\tau)=\langle 0|x(0)x(\tau)|0\rangle$$

$$=\lim_{\beta\to\infty}\frac{1}{Z(\beta)}(2\pi\beta)^{-1/2}\int_{-\infty}^{\infty}\int_{C_0}\exp$$

$$\left\{-\beta\int_0^1 V(\sqrt{\beta}x(t)+x)dt\right\}[\sqrt{\beta}x(\tau/t)$$

$$+x]d_W x\ x\ dx,$$

$$\Delta E=E_1-E_0=-\lim_{\tau\to\infty}\frac{d}{d\tau}\ln G(\tau), \tag{14}$$

$$|\psi_0(x)|^2=\lim_{\beta\to\infty}(\exp\{E_0\beta\}Z(x,x,\beta)).$$

Here

$$Z(\beta)=\mathrm{Tr}\exp\{-\beta H\}=\int_{-\infty}^{\infty}Z(x,x,\beta)dx,$$

$\beta=1/kT$, k is the Boltzmann constant, and T is the absolute temperature.

Analogous expressions for observables in the form of functional integrals hold in the case of Euclidean quantum field theory (see Ref. 92). For example, in two-dimensional Euclidean field theory with polynomial interactions of bosonic fields $P(\varphi)_2$ (Ref. 239), the Gaussian measure μ is defined on the Schwartz space of tempered generalized functions $S'(R^2)$. The renormalizations in the $P(\varphi)_2$ model are limited to a subtraction related to Wick ordering. The functional integrals of the $P(\varphi)_2$ model and their approximate calculation will be studied in more detail below.

4. APPROXIMATE CALCULATION OF FUNCTIONAL INTEGRALS WITH GAUSSIAN MEASURE

The first results on the approximate calculation of functional integrals with Gaussian measure were obtained by Cameron,[240] who found the analog of Simpson's rule for Wiener integrals over a space of continuous functions, and by Gel'fand and Chentsov.[241] Vladimirov[242] constructed quadrature formulas of third-order accuracy for integrals with Wiener measure, and also formulas which are exact for a certain type of functional of fourth degree. Ideas about the construction of approximation formulas of a given order of accuracy for functional integrals with Wiener measure and with conditional Wiener measure are developed in Refs. 243–246. The approximate calculation of functional integrals with arbitrary Gaussian measures was studied by Yanovich (see Ref. 49 and references therein). He, in particular, constructed approximation formulas of the interpolation type for functional integrals, quadrature formulas for integrals of special types of functionals (for example, ones depending on the norm of the argument, and so on), approximation formulas of a given order of accuracy, formulas based on approximation of the argument in the functional, and a number of others. Approximation formulas have also been obtained for integrals with measures corresponding to various uniform random processes with independent increments (Wiener, Poisson, Laplace, telegraph, and so on), approximation formulas for integrals with Gaussian measure based on interpolation of the characteristic functional of the measure, and some others.[50] One of the most widespread approaches to the approximate calculation of functional integrals is the construction of approximation formulas which are exact on a class of functional multinomials of given degree, the idea of which originated with Cameron. We have used this approach to construct approximation formulas of an arbitrary given order of accuracy for functional integrals with arbitrary Gaussian measure.[247]

Let $\mathscr{H}$ be a Hilbert space which is dense almost everywhere in a complete separable metric space X and generated by a Gaussian measure μ, and let $\{e_k\}_{k=1}^{\infty}$ be an orthonormal basis in $\mathscr{H}$. We have obtained the following formula, in the form of a theorem, for the approximate calculation of the functional integral with measure μ of an arbitrary real functional $F[x]$ integrable on X (Ref. 247):

Theorem 1. *Let* $S_n(x)=\sum_{k=1}^{n}(e_k,x)e_k$,

$$U_n(u)=\sum_{k=1}^{n}u_k e_k,$$

$$\theta_m(v)=\sum_{k=1}^{m}c_{m,k}\rho(v_k),\quad v\in R^m,\quad u\in R^n,$$

$[c_{m,k}]^2$, $k=1,\ldots,m$ *be the roots of the multinomial* $Q_m(r)=\sum_{k=0}^{m}(-1)^k r^{m-k}/k!$; $r\in R$ *and the function* $\rho(r):R\to X$ *satisfy the conditions*

$$\rho(r)=-\rho(-r),$$

$$\int_R\langle\xi,\rho(r)\rangle\langle\eta,\rho(r)\rangle d\nu(r)=K(\xi,\eta),$$

$$\prod_{i=1}^{j}\langle\xi_i,\rho(r)\rangle\in L(R,\nu),$$

$$1\leqslant j\leqslant 2m+1,\quad \eta,\xi,\xi_i\in X',$$

where $K(\xi,\eta)$ *is the correlation functional of the measure* μ.

Then the approximate formula

$$\int_X F[x]d\mu(x)=(2\pi)^{-n/2}\int_{R^n}\exp\left\{-\frac{1}{2}(u,u)\right\}$$

$$\times\int_{R^m}F[\theta_m(v)-\theta_{m,n}(v)+U_n(u)]$$

$$\times d\nu^{(m)}(v)du+\mathscr{R}_{m,n}(F) \tag{15}$$

is exact for any functional multinomial of degree $\leqslant 2m+1$.

Here $\mathscr{R}_{m,n}(F)$ is the remainder term, $\theta_{m,n}(v)=S_n(\theta_m(v))$, and the measure $\nu^{(m)}(v)$ in R^m is the Cartesian product of symmetric probability measures ν on R. In the case $m=1$, Eq. (15) is the same as the result obtained in Ref. 49.

A functional of the form

$$P_m[x]=\sum_{k=0}^{m}p_k[x],$$

where $p_k[x]$ is a homogeneous form of kth order continuous on X, is called a functional multinomial of degree m.

The approximation formula (15) replaces the method of finding a functional integral by calculating an $(n+m)$-fold Riemann integral. Practical calculations (see below) show that in some cases good accuracy (equal to or better than 0.1%) can be obtained by choosing small values of n and m, even values of unity.

Let us study the convergence of the approximations obtained using (15) to the exact value of the integral for $n\to\infty$. We assume that for almost all $v\in R^m$ pertaining to the measure ν for $n\to\infty$ we have convergence:

$$\theta_{m,n}(v)\to\theta_m(v). \tag{16}$$

In the case of the space $C[a,b]$ it is obvious that (16) is satisfied.

Theorem 2. *(Ref. 247) Let $F[x]$ be a functional continuous on X satisfying the condition*

$$|F[x]|\leq g(A(x,x)),$$

where $A(x,x)$ is a non-negative quadratic functional of the form

$$A(x,x)=\sum_{k=1}^{\infty}\gamma_k(x,e_k)^2 \tag{17}$$

with

$$\sum_{k=1}^{\infty}\gamma_k<\infty,\quad \gamma_k\geq 0,$$

$g(u)$ is a nondecreasing function, and

$$\int_{R^m}\int_X g[A(\theta_m(v),\theta_m(v))+A(x,x)]d\mu(x)d\nu(v)<\infty.$$

Then $\mathscr{R}_{m,n}(F)\to 0$ for $n\to\infty$.

The estimate of the remainder of the approximation formula (15) is given by the following theorem:[247]

Theorem 3. *Let the functional $F[x]$ integrated with measure μ admit the representation*

$$F[x+x_0]=P_{2m+1}[x]+r_{2m+1}(x,x_0), \tag{18}$$

where P_{2m+1} is a functional multinomial of degree $\leq 2m+1$, and the remainder r_{2m+1} is estimated by the expression

$$|r_{2m+1}(x,x_0)|\leq(A(x,x))^{m+1}[L_1\exp\{L_2A(x+x_0,x+x_0)\}+L_3\exp\{L_2A(x_0,x_0)\}]. \tag{19}$$

Here $A(x,x)$ is defined as in (17), x_0 is a fixed element from X, $L_i>0$ $(i=1,2,3)$;

$$1-2L_2\gamma_k\geq\alpha>0,\quad k=1,2,\ldots,$$

$$\sum_{k=1}^{\infty}\gamma_k a_k<\infty,$$

where a_k is given by the relations

$$(e_k,\theta_m(v))^2\leq a_k \quad \text{for all } v\in R^m.$$

Then for the remainder of the approximation formula (15) we have the estimate

$$|\mathscr{R}_{m,n}(F)|\leq G_m\left(\sum_{k=n+1}^{\infty}\gamma_k\right)^{m+1}+H_m\left(\sum_{k=n+1}^{\infty}\gamma_k a_k\right)^{m+1},$$

where G_m and H_m are positive constants depending on m (Ref. 247).

In the case $m=1$, Theorem 3 coincides with the result obtained in Ref. 49.

As an example, let us consider the rate of convergence to the exact result of approximations obtained using (15) in the case of an integral with Wiener measure in the space $C=\{C[0,1],x(0)=0\}$. The Wiener measure is characterized by zero average value and correlation function

$$\mathscr{K}(t,s)=\min\{t,s\},\quad t,s\in[0,1].$$

In this case $e_k(t)=\sqrt{\lambda_k}\varphi_k(t)$, $k=1,2,\ldots$, where λ_k and φ_k are the eigenvalues and eigenfunctions of the kernel $\mathscr{K}(t,s)$:

$$\varphi_k(t)=\sqrt{2}\sin\left(k-\frac{1}{2}\right)\pi t,\quad k=1,2,\ldots,$$

$$\lambda_k=\frac{4}{(2k-1)^2\pi^2}.$$

In the case of the Wiener measure the approximation formula (15) takes the form

$$\int_C F[x]d_W x=(2\pi)^{-n/2}\times\int_{R^n}\exp\left\{-\frac{1}{2}(u,u)\right\}\times\frac{1}{2^m}\int_{-1}^{1}\cdots\int_{-1}^{1}F[\theta_m(v,t)-\theta_{m,n}(v,t)+U_n(u,t)]dvdu+\mathscr{R}_{m,n}(F), \tag{20}$$

where

$$\theta_m(v,t)=\sum_{i=1}^{m}c_{m,i}\rho(v,t),$$

$$\rho(v,t)=\begin{cases}\operatorname{sign}(v), & 0\leq|v|<t\leq 1,\\ 0, & 0\leq t<|v|\leq 1,\end{cases}$$

$$\theta_{m,n}(v,t)=4\sum_{k=1}^{n}\sin\left(k-\frac{1}{2}\right)\pi t\,\frac{1}{(2k-1)\pi}\times\sum_{i=1}^{m}c_{m,i}\operatorname{sign}(v_i)\cos\left(k-\frac{1}{2}\right)\pi v_i,$$

$$U_n(u,t)=2\sqrt{2}\sum_{k=1}^{n}\sin\left(k-\frac{1}{2}\right)\pi t\,\frac{u_k}{(2k-1)\pi}.$$

It is possible to take $\int_0^1 x^2(t)dt$ as $A(x,x)$. It is easy to see that in this case the conditions (17) are satisfied. On the basis of Theorem 3 we find that the order of convergence of the

TABLE I. Values of the Wiener integral calculated using the compound approximation formula.

n	$I_n^{(1)}$	$I_n^{(2)}$
1	1.35980945	1.360432541
2	1.36034801	1.360446160
3	1.36041602	1.360446774
4	1.36043360	1.360446855
5	1.36044001	1.360446873
10	1.36044601	1.360446880

approximation formula (20) for $n\to\infty$ in the case where the functional $F[x]$ satisfies the conditions (18) and (19) with the selected $A(x,x)$ is $O(n^{-(m+1)})$.

As an example, let us consider the approximate calculation of the integral with Weiner measure of the functional

$$F[x]=\exp\left\{\frac{1}{2}\int_0^1 x^2(t)dt\right\}.$$

The exact value of the functional integral in this case is known:[67]

$$I=\int_C \exp\left\{\frac{1}{2}\int_0^1 x^2(t)dt\right\}d_W x=\frac{1}{\sqrt{\cos 1}}$$
$$\approx 1.3604468816\ldots .$$

We use $I_n^{(m)}$ to denote the approximate values obtained using (20), where

$$I_n^{(1)}=\prod_{k=1}^{n}\frac{1}{\sqrt{1-\lambda_k}}\int_0^1 \exp\left\{\frac{1}{2}w_n(v)\right\}dv,$$

$$I_n^{(2)}=\prod_{k=1}^{n}\frac{1}{\sqrt{1-\lambda_k}}2\int_0^1 \exp\left\{\frac{1}{4}w_n(v_1)\right\}$$
$$\times\int_0^{v_1}\exp\left\{\frac{1}{4}w_n(v_2)\right\}\cos\left[\frac{1}{4}(w_n(v_1)\right.$$
$$\left.-w_n(v_2))\right]\cosh\left\{\frac{\sqrt{2}}{2}\left(1-v_1-2\sum_{j=1}^{n}\lambda_j\right.\right.$$
$$\left.\left.\times\cos\left(j-\frac{1}{2}\right)\pi v_1\cos\left(j-\frac{1}{2}\right)\pi v_2\right)\right\}dv_1dv_2.$$

Here

$$w_n(v)=1-v-2\sum_{j=1}^{n}\lambda_j\cos^2\left(j-\frac{1}{2}\right)\pi v.$$

The approximate values $I_n^{(1)}$ and $I_n^{(2)}$ obtained by computer are given in Table I. We note that when using Eqs. (15) and (20) to find a functional integral it is necessary to calculate an $(n+m)$-fold integral. We see from Table I that for a given multiplicity of the integrals, the approximation formula (20) for $m=2$ gives a result more accurate than for $m=1$.

5. THE CONDITIONAL WIENER MEASURE

In the case of the conditional Wiener measure $d_W x$ in the space $C_0=\{C[0,1];x(0)=x(1)=0\}$ the approximation formula (15) has the form

$$\int_{C_0}F[x]d_W x=(2\pi)^{-n/2}$$
$$\times\int_{R^n}\exp\left\{-\frac{1}{2}(u,u)\right\}2^{-m}$$
$$\times\int_{-1}^{1}\cdots\int_{-1}^{1}F[\tilde\theta_m(v,\cdot)-\tilde\theta_{m,n}(v,\cdot)$$
$$+\tilde U_n(u,\cdot)]dvdu+\mathscr{R}_{m,n}(F), \qquad (21)$$

where

$$\tilde\theta_m(v,t)=\sum_{k=1}^{m}c_{m,k}\tilde\rho(v_k,t),$$

$$\tilde\rho(v,t)=\begin{cases}-t\ \mathrm{sign}(v), & t\leqslant|v|,\\ (1-t)\mathrm{sign}(v), & t>|v|,\end{cases}$$

$$\tilde\theta_{m,n}(v,t)=\sum_{k=1}^{n}\frac{2}{k\pi}\sin(k\pi t)\sum_{j=1}^{m}c_{m,j}\ \mathrm{sign}(v_j)\cos(k\pi v_j),$$

$$\tilde U_n(u,t)=\sum_{k=1}^{n}\frac{\sqrt{2}}{k\pi}u_k\sin(k\pi t).$$

As in (20), the rate of convergence of the approximations obtained using (21) is of order $\mathscr{R}_{m,n}(F)=O(n^{-(m+1)})$.

In many cases it is convenient to isolate part of the integrated functional in the form of a weight and to use weighted approximation formulas. In Ref. 248 we constructed a family of approximation formulas for functional integrals with conditional Wiener measure

$$\int_{C_0}P[x]F[x]d_W x$$

with weight

$$P[x]=\exp\left\{\int_0^1[p(t)x^2(t)+q(t)x(t)]dt\right\},$$

$$p(t),q(t)\in C[0,1]. \qquad (22)$$

Theorem 4. *(Ref. 248) Let $B(s)$ be the solution of the differential equation*

$$(1-s)B'(s)-(1-s)^2B^2(s)-3B(s)-2p(s)=0,$$

$$s\in[0,1] \qquad (23)$$

with the initial condition

$$B(1)=-\frac{2}{3}p(1).$$

Let

$$W(t)=\exp\left\{\int_0^1(1-s)B(s)ds\right\},$$

$$a(t)=\int_0^t L(s)ds-\frac{1-t}{W(t)}\int_0^t B(s)W(s)\left[\int_0^s L(u)du\right]ds,$$

$$L(t)=\int_0^t[B(s)W(s)H(s)-q(s)]ds+c,$$

$$H(t)=\int_t^1 q(s)\frac{1-s}{W(s)}ds, \tag{24}$$

where the constant c is determined by the condition $\int_0^1 L(s)ds=0$.

Then the approximation formula

$$\int_{C_0}P[x]F[x]d_W x=[W(1)]^{-1/2}\exp\left\{\frac{1}{2}\int_0^1 L^2(t)dt\right\}2^{-m}\times\int_{-1}^1\cdots\int_{-1}^1 F[\Psi_m(v,\cdot)+a(\cdot)]\times dv_1\cdots dv_m+\mathscr{R}_m(F), \tag{25}$$

where

$$\Psi_m(v,\cdot)=\sum_{k=1}^m c_{m,k}\Psi(v_k,\cdot),$$

$$\Psi(r,\cdot)=f(r,\cdot)-\sigma(r,\cdot);\quad \sigma(r,t)=\begin{cases}\operatorname{sign}(r), & t\leqslant|r|,\\ 0, & t>|r|,\end{cases}$$

$$f(r,t)=\operatorname{sign}(r)\frac{1-t}{W(t)}\left[1+\int_0^{\min\{|r|,t\}}B(s)W(s)ds\right],$$

is exact for any functional multinomial of degree $\leqslant 2m+1$.

The proof of Theorem 4 is based on the properties of a linear transformation $x(t)\to y(t)$ that we have found and studied:[249] $y=x+Ax$, where

$$Ax(t)=(1-t)\int_0^t B(s)x(s)ds,\quad B(s)\in C[0,1].$$

This transformation gives a one-to-one mapping of the space C_0 onto itself. The inverse transformation has the form[249]

$$x(t)=\hat{A}y(t)=y(t)-\frac{1-t}{W(t)}\int_0^1 B(s)W(s)x(s)ds,$$

where $W(t)$ satisfies (24).

In the special case $p(t)\equiv p=\text{const}$, $q(t)\equiv q=\text{const}$ often encountered in applications, the Riccati equation (23) is solved explicitly,

$$B(s)=\frac{1}{1-s}\left[\sqrt{2p}\cot(\sqrt{2p}(1-s))-\frac{1}{1-s}\right],$$

and the approximation formula (25) takes the form

$$\int_{C_0}\exp\left\{\int_0^1[px^2(t)+qx(t)]dt\right\}F[x]d_W x=\left(\frac{\sqrt{2p}}{\sin\sqrt{2p}}\right)^{1/2}\exp\left\{\frac{q^2}{(2p)^{3/2}}\left[\tan\sqrt{\frac{p}{2}}-\sqrt{\frac{p}{2}}\right]\right\}\times 2^{-m}\int_{-1}^1\cdots\int_{-1}^1 F[\Psi_m(v,\cdot)+a(\cdot)]dv_1\cdots dv_m+\mathscr{R}_m(F), \tag{26}$$

where

$$a(t)=q\left(p\cos\sqrt{\frac{p}{2}}\right)^{-1}\sin\left(\sqrt{\frac{p}{2}}t\right)\sin\left(\sqrt{\frac{p}{2}}(1-t)\right).$$

The estimate of the remainder of the approximation formula (26) as a function of m is given by the following theorem:[248]

Theorem 5. *Let the functional $F[x]$ integrated with weight $P[x]$ and conditional Wiener measure admit the representation*

$$F[x]=P_{2m+1}[x]+r_{2m+1}[x],\quad m=1,2,3,\ldots,$$

where $P_{2m+1}[x]$ is a functional multinomial of degree $\leqslant 2m+1$, and the remainder $r_{2m+1}[x]$ is estimated as

$$|r_{2m+1}[x]|\leqslant c_1(m)\exp\left\{c_2(m)\int_0^1 x^2(t)dt\right\},$$

$$c_1(m),c_2(m)\geqslant 0;$$

$$0\leqslant c_3(m)<\frac{\pi^2}{2},\quad c_3(m)=p+c_2(m).$$

Then

$$|\mathscr{R}_m(F)|\leqslant c_1(m)\left[\left(\frac{\sqrt{2c_3(m)}}{\sin\sqrt{2c_3(m)}}\right)^{1/2}\exp\left\{\frac{q^2}{(2c_3(m))^{3/2}}\times\left[\tan\sqrt{\frac{1}{2}c_3(m)}-\sqrt{\frac{1}{2}c_3(m)}\right]\right\}+\left(\frac{\sqrt{2\pi}}{\sin\sqrt{2\pi}}\right)^{1/2}\exp\left\{\frac{q^2}{(2p)^{3/2}}\left[\tan\sqrt{\frac{p}{2}}-\sqrt{\frac{p}{2}}\right]+\frac{1}{6}c_2(m)b(m)\right\}\right],$$

where

$$b(m)=(d_1\sqrt{m}+d_2)^2,$$

and d_1 and d_2 are constants independent of m (Ref. 248).

Corollary. *A sufficient condition for the remainder in (26) $\mathscr{R}_m(F)$ to tend to zero for $m\to\infty$ is that the remainder $r_{2m+1}[x]$ also tend to zero, where*

$$c_2(m)\equiv c_2=\text{const},$$

$$c_1(m)\exp\left\{\frac{1}{6}c_2(d_1\sqrt{m}+d_2)^2\right\}\to 0.$$

As an illustration, let us consider the approximate calculation of the integral

$$I=\int_{C_0} P[x]\exp\left\{-\int_0^1 [px^2(t)+qx(t)]dt\right\} d_W x, \qquad (27)$$

where $P[x]$ is determined according to (22) for $p(t)\equiv p$, $q(t)\equiv q$. The exact answer is obviously $I=1$. The results $I^{(1)}$ of the approximate calculation of the integral (27) using (26) with $m=1$ for several values of the parameters p and q are given in Tables II and III. In Table III we give the values of the actual error $\varepsilon^{(1)}=I-I^{(1)}$ and the theoretical estimate of the error $R^{(1)}=\mathscr{R}_1(F)$, obtained using Theorem 5, for the case $p=0$. We see from this table that good accuracy of the approximations obtained using the "elementary" formula (26) for $m=1$ is obtained only for small p and q, i.e., when the integrated functional does not deviate too much from a functional multinomial of degree $\leqslant 2m+1=3$. More accurate results for a wide range of functionals can be obtained using "compound" (i.e., containing an additional integration over R^n) approximation formulas of the type (21). Combining the methods of constructing approximation formulas that we developed in Refs. 247 and 248, we obtain new formulas involving a weight, which possess the advantages of compound approximation formulas.[250]

Theorem 6. *When the conditions of Theorems 1 and 4 are satisfied, the approximation formula*

$$\begin{aligned}\int_{C_0} P[x]F[x]d_W x=(2\pi)^{-n/2}[W(1)]^{-1/2}&\\ \times\exp\left\{\frac{1}{2}\int_0^1 L^2(t)dt\right\}2^{-m}&\\ \times\int_{R^n}\exp\left\{-\frac{1}{2}(u,u)\right\}&\\ \times\int_{-1}^1\cdots\int_{-1}^1\Phi[\tilde{\theta}_m(v,\cdot)-\tilde{\theta}_{m,n}(v,\cdot)&\\ +\tilde{U}_n(u,\cdot)]dvdu+\mathscr{R}_{m,n}(F),&\qquad (28)\end{aligned}$$

where $P[x]$ satisfies (22), $\tilde{\theta}_m$, $\tilde{\theta}_{m,n}$, and $\tilde{U}_n$ correspond to (21), and $\Phi[x]=F[\hat{A}x(\cdot)+a(\cdot)]$, is exact for any functional multinomial of degree $\leqslant 2m+1$.

We have proved, under certain conditions, the convergence of approximations obtained using (28) to the exact value of the integral for $n\to\infty$ (Ref. 250) and have found an estimate of the remainder as a function of m and n, from which it follows, in particular, that $\mathscr{R}_{m,n}=O(n^{-(m+1)})$ for $n\to\infty$ (see Ref. 250).

Let us consider the use of the compound and elementary formulas for the example of the functional integral (27) with the exact value $I=1$. The results[250] of computing this integral using the elementary formula (26) with $m=1$ and $m=2$ and also using the compound formula (28) with $m=n=1$ are shown in Figs. 1 and 2. We see from these graphs that the compound formula with a weight gives more accurate approximations than does the elementary formula for the same multiplicity of the Riemann integrals contained in them. We also see that when the compound formula is used, good results can be obtained even if the integrated functional deviates significantly from a functional multinomial of degree $\leqslant 2m+1$.

6. QUANTUM-MECHANICAL MODELS

Let us study the use of the approximation formulas for functional integrals and their effectiveness for some quantum-mechanical models. First we consider the simple but very instructive example of the harmonic oscillator with potential $V(x)=\frac{1}{2}x^2$, $x\in(-\infty,\infty)$. It is well known that a quantum-mechanical system with such a potential possesses a discrete energy spectrum $E_n=n+\frac{1}{2}$. When it is studied using the functional-integration method, the Green function Z is written as an integral with conditional Wiener measure:

$$\begin{aligned}Z(x,x,\beta)=\frac{1}{\sqrt{2\pi\beta}}\exp\left\{-\frac{\beta}{2}x^2\right\}&\\ \times\int_{C_0}\exp\left\{-\frac{\beta}{2}\int_0^1[\beta x^2(t)&\\ +2x\sqrt{\beta}x(t)]dt\right\}d_W x.&\qquad (29)\end{aligned}$$

It is easy to see that the exact result can be obtained by using the weighted formula (26) for the integral (29). Substituting $p=\frac{1}{2}\beta^2$, $q=x\beta^{3/2}$, and $F[x]\equiv 1$ into (26), we have

$$Z(x,x,\beta)=\frac{1}{\sqrt{2\pi\sinh\beta}}\exp\left\{-\tanh\left(\frac{\beta}{2}\right)x^2\right\}. \qquad (30)$$

In contrast to the traditional formulation of quantum mechanics, in this case the functional-integral approach allows the system to be studied analytically at any finite temperature $T=1/k\beta$ and the passage of its characteristics to the thermodynamical limit $T\to 0$ ($\beta\to\infty$) can be clearly demonstrated. For example, according to (13) for the ground-state energy E_0 we obtain

$$E_0^{(\beta)}=\frac{\int_{-\infty}^{\infty}x^2Z(x,x,\beta)dx}{\int_{-\infty}^{\infty}Z(x,x,\beta)dx}.$$

Substituting (30) into this, we find

$$E_0^{(\beta)}=\frac{1}{2}\coth\left(\frac{\beta}{2}\right). \qquad (31)$$

Obviously, for $\beta\to\infty$ the quantity $E_0^{(\beta)}$ tends to the value known from quantum mechanics:

$$E_0^{(\beta)}\to E_0=\tfrac{1}{2}.$$

Similarly, for the energy difference of adjacent quantum levels ΔE and the ground-state wave function $\Psi_0(x)$ at finite temperature we have[238]

TABLE II. Values of the functional integral obtained using the weighted approximation formula.

$p;q$	0	1	2	3	4
0	1	0.9997	0.9948	0.9756	0.9311
1	0.9900	0.9985	1.0131	1.0071	0.9531
2	0.9550	0.9842	1.0416	1.0568	0.9687
−1	0.9931	0.9896	0.9777	0.9539	0.9138
−2	0.9768	0.9724	0.9589	0.9350	0.8990

$$\Delta E^{(\beta)}=\tanh\left(\frac{\beta}{2}\right),$$

$$|\Psi_0^{(\beta)}(x)|^2=\frac{1}{\sqrt{2\pi\sinh\beta}}\exp\left\{\frac{\beta}{2}\coth\left(\frac{\beta}{2}\right)\right\}\times\exp\left\{-\tanh\left(\frac{\beta}{2}\right)x^2\right\}. \tag{32}$$

Obviously, for $\beta\to 0$,

$$\Delta E^{(\beta)}\to\Delta E=1,$$

$$|\Psi_0^{(\beta)}(x)|^2\to|\Psi_0(x)|^2=\frac{1}{\sqrt{\pi}}e^{-x^2}.$$

The nature of the passage to the thermodynamical limit can be seen from Tables IV and V, in which the values of $E_0^{(\beta)}$, $\Delta E^{(\beta)}$, and $|\Psi_0^{(\beta)}(x)|^2$ are given for various β.

In Table IV we also give the results of the approximate calculation of $E_0^{(\beta)}$ using the compound approximation formula (21) for $m=1$ and various n, i.e., by calculating an $(n+2)$-fold integral. There we also give the time needed to find $E_0^{(\beta,n)}$ using a CDC-6500 computer to calculate the Riemann integrals by Gaussian quadratures with a relative error of down to 10^{-4}.

In Fig. 3 the crosses on a logarithmic scale show the results of calculating the propagator $G(\tau)=\langle 0|x(0)x(\tau)|0\rangle$ using the compound approximation formula (21) for $m=1$, $n=2$, and $\beta=6$. The computing time was about 10 sec per point τ. We note that the exact value of $G^{(\beta)}(\tau)$ for arbitrary β obtained from (13) and (30) is

$$G^{(\beta)}(\tau)=\frac{1}{2}\coth\left(\frac{\beta}{2}\right)\left(\cosh\tau-\tanh\left(\frac{\beta}{2}\right)\sinh\tau\right),$$

from which

$$\lim_{\beta\to\infty}G^{(\beta)}(\tau)=G(\tau)=\frac{1}{2}e^{-\tau}. \tag{33}$$

TABLE III. Error in calculating the functional integral.

q	$I^{(1)}$	$\varepsilon^{(1)}$	$R^{(1)}$
0	1	0	0
0.1	0.9999938	0.62×10^{-5}	2.39×10^{-5}
0.2	0.9999972	0.28×10^{-5}	4.30×10^{-4}
0.3	0.9999962	0.38×10^{-5}	2.48×10^{-3}
0.4	0.9999891	0.11×10^{-4}	9.01×10^{-3}
0.5	0.9999777	0.22×10^{-4}	2.58×10^{-2}

The quantity $G(\tau)$ in the thermodynamical limit (33) is shown by the solid line in Fig. 3. From the calculated values of $G^{(\beta,n)}(\tau)$ shown by the crosses in Fig. 3, we used the least-squares method to find[238]

$$\Delta E^{(\beta,n)}=-\frac{d}{d\tau}\ln G^{(\beta,n)}(\tau)=1.0198.$$

For comparison, we give the results of similar calculations performed by other authors. In Ref. 103 the value of E_0 was calculated by a lattice approximation of the functional integral with $N=51$ sites and lattice spacing $a=0.5$. When the N-fold integral was found exactly, the result was

$$E_0^{(N)}=0.447;\quad \Delta E^{(N)}=0.9875,$$

and when it was calculated by modeling $N_E=100$ lattice configurations[103] the result was

$$E_0^{(N,N_E)}=0.45;\quad \Delta E^{(N,N_E)}=0.99.$$

In Ref. 236 the trajectories in the functional integral were approximated and the $(N=4)$-fold integral was found using $N_R=100$ runs of 10^4 passes each (without introducing a lattice). The result was

$$E_0^{(N,N_R)}=0.4932\pm0.145;\quad \Delta E_0^{(N,N_R)}=0.8801\pm0.202,$$

which required 19×100 sec of computing time on a Vax 780. For $N=10$ and $N_R=100$ the result of Ref. 236 was

$$E_0^{(N,N_R)}=0.4979\pm0.051;\quad \Delta E_0^{(N,N_R)}=0.9331\pm0.129,$$

and the computing time was 67×100 sec.

In Fig. 4 the solid line shows the theoretical value of the square of the ground-state wave function in the thermodynamical limit, and the dashed line shows the best result of

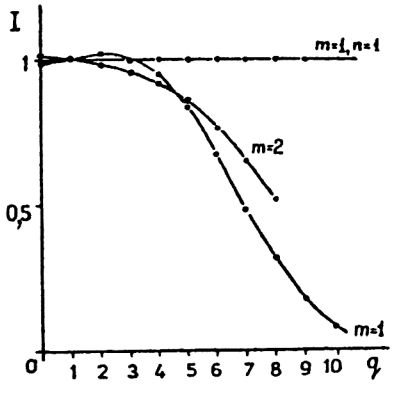

FIG. 1. Results of calculating the functional integral for $p=1$.

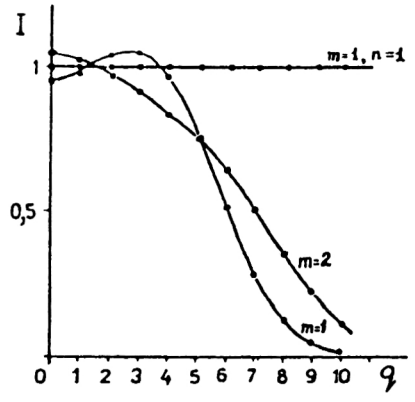

FIG. 2. Results of calculating the functional integral for $p=2$.

Ref. 103, obtained using a lattice with $N=100$ sites and lattice spacing $a=1$ with the N-fold integral found exactly, i.e.,

$$|\Psi_0^{(N)}(x)|^2=0.59\exp(-1.1x^2)$$

$$=1.05\frac{1}{\sqrt{\pi}}\exp(-1.1x^2).$$

In the same figure the crosses show the values calculated using the compound approximation formula (21) for $m=1$, $n=2$, and $\beta=6$. The numerical integration was performed by means of Gaussian quadratures with a relative error of at best 10^{-5}. The computing time for a CDC-6500 was around 0.2 sec per point x.

Let us also give the results of the calculations for the anharmonic oscillator with potential $V(x)=\frac{1}{2}x^2+gx^4$. In this case the functional integral $Z(x,x,\beta)$ cannot be evaluated exactly. We calculated it using the weighted approximation formula (26) and the compound approximation formula (21). The results of the calculations[118] are given in Tables VI and VII. The total time to calculate $E_0^{(\beta,1)}$ and $G^{(\beta,1)}(0)$ was about 0.5 min per point g for formula (21) and about 10 min per point g for formula (26). The "exact" results[236,251] are denoted by E_0^*, $G^*(0)$, and ΔE^*. The results obtained in Ref. 236 using 10 runs of 3000 passes each are denoted by $G^{(N)}(0)$ and $E^{(N)}$. The computing time quoted in Ref. 236 is 10×25 sec per point g for $N=4$ and 10×17 min per point g for $N=20$. We use $\Delta E^{(\beta,1)}$ to denote the logarithmic derivative of the quantity $G^{(\beta,1)}(\tau)$ that we obtained with the values of $G^{(\beta,1)}(\tau)$ calculated using (21) for $n=m=1$ (Ref. 238).

These results show that the calculation of functional integrals using approximation formulas with the minimum values of the parameters m and n can give results which are as (and in some cases even more) accurate as those in Refs. 103 and 236, while the multiplicity of Riemann integrals calculated and, consequently, the required computer time are significantly smaller. This makes it possible to calculate integrals using deterministic methods (as opposed to the Monte Carlo method commonly used in lattice calculations, which gives only a probabilistic estimate of the results), and leads to significant savings of computing time and memory.

TABLE IV. Ground-state energy of the harmonic oscillator.

β	$E_0^{(\beta)}$	n	$E_0^{(\beta,n)}$	computing time, sec
5	0.50678	1	0.5077	1.8
6	0.50248	2	0.5073	3.1
7	0.50091	3	0.5010	5.5
8	0.50034	5	0.5002	9.7

7. TUNNELING IN THE DOUBLE-WELL POTENTIAL

Let us consider the ground state of the quantum-mechanical system described by the Hamiltonian $H=-\frac{1}{2}\Delta+V$ with the potential

$$V(x)=\lambda(x^2-f^2)^2,\quad x\in(-\infty,\infty),\tag{34}$$

having minima at $\pm f$. The double-well system is interesting because it is a convenient model for studying tunneling and instanton effects. There are several studies devoted to such systems (see, for example, Refs. 252–257). The properties of this system are used in studying ferroelectrics, semiconductors, and so on.[258,259] Owing to tunneling, the ground-state wave function is an even superposition of the wave functions of each well. The main effect due to instantons is the splitting of the energy levels (which are doubly degenerate if tunneling is neglected). In the dilute instanton-gas approximation the ground-state energy of the quantum system with potential (34) is[260]

$$E_0=\frac{\omega}{2}-\sqrt{\frac{2\omega^3}{\pi\lambda}}\frac{\omega}{2}\exp\left\{-\frac{\omega^3}{12\lambda}\right\},$$

where $\omega=8\lambda f^2$. In this approximation the difference of the energies of the ground and first excited states is equal to the instanton density

$$\Delta E=E_1-E_0=\frac{dn}{d\tau_0},$$

where

$$\frac{dn}{d\tau_0}=\omega\sqrt{\frac{2\omega^3}{\pi\lambda}}\exp\left\{-\frac{\omega^3}{12\lambda}\right\}.$$

The method of functional integrals in the Euclidean metric (imaginary time) is a convenient technique for studying tunneling effects.[260–262] The results of our calculations of E_0

TABLE V. Difference of the energies of adjacent quantum levels and the ground-state wave function of the harmonic oscillator.

β	$\Delta E^{(\beta)}$	$\lvert\Psi_0^{(\beta)}(x)\rvert^2$
5	0.9866	$1.0345\frac{1}{\sqrt{\pi}}\exp(-0.9866x^2)$
6	0.9951	$1.0150\frac{1}{\sqrt{\pi}}\exp(-0.9951x^2)$
7	0.9982	$1.0064\frac{1}{\sqrt{\pi}}\exp(-0.9982x^2)$
8	0.9993	$1.0027\frac{1}{\sqrt{\pi}}\exp(-0.9993x^2)$

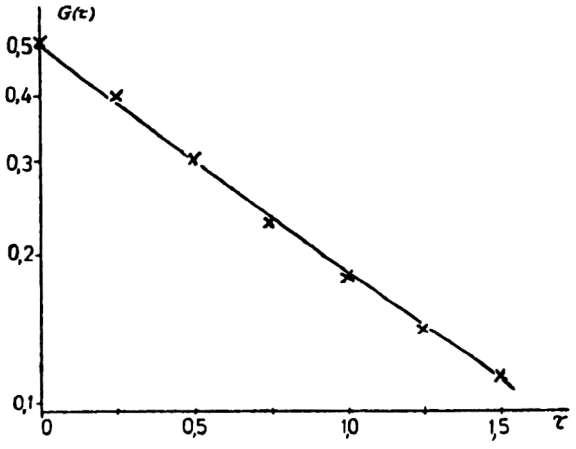

FIG. 3. Propagator of the harmonic oscillator.

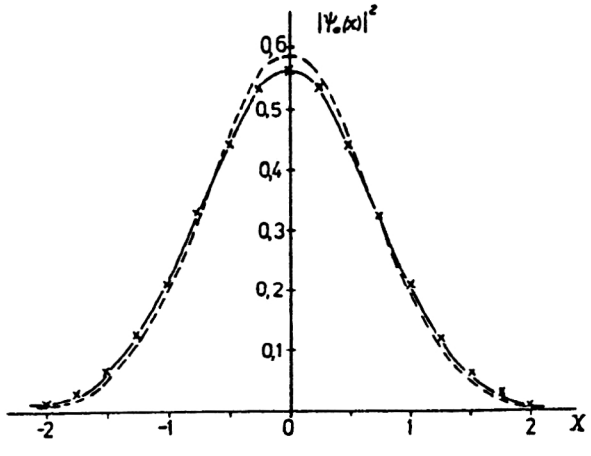

FIG. 4. Ground-state wave function of the harmonic oscillator.

and ΔE for the system (34) with $\lambda=\frac{1}{2}$using the compound approximation formula (21) with $n=m=1$ (Ref. 263) are shown by the solid circles in Figs. 5 and 6. The computing time for each value of f^2 was about 10 sec on a CDC-6500. The open circles show the results of Ref. 104 using the lattice Monte Carlo method. The squares show the results obtained in Ref. 103 on a lattice with $N=303$ sites and lattice spacing $a=0.25$ with calculation of the N-fold integral by averaging over 10 Monte Carlo iterations. The solid line shows the "exact" result,[264] and the dashed line shows the dilute instanton-gas approximation.

The crosses in Fig. 7 show the squared modulus of the ground-state wave function that we calculated using (21) with $n=m=1$, $\beta=4.5$, and $f^2=2$. The points show the results of Ref. 103 obtained on a lattice with $N=200$ and $a=0.25$ by averaging over 100 Monte Carlo iterations. The stars are the values of $|\Psi_0(\pm f)|^2=\frac{1}{2}\sqrt{2f/\pi}$ obtained in the dilute-gas approximation.[260] The solid and dashed lines connecting the points are just to aid visualization.

These results demonstrate the efficiency of our method for the approximate calculation of functional integrals.

8. THE TOPOLOGICAL SUSCEPTIBILITY. THE QUANTUM PENDULUM

One of the most important areas of application of functional integrals is the study of the topological structure of the vacuum in quantum gauge theory. The values of the topological susceptibility calculated by various authors using the lattice Monte Carlo method (see Refs. 265–275) differ significantly from each other and from phenomenological estimates (Refs. 276–278). This difference might be due[265] both to a difference in the determinations of the topological charge on the lattice, and to the presence of a particular systematic error related to the finiteness of the lattice discretization length.[104] Some authors are studying the topological charge directly in the continuum limit.[279,280]

Let us see how to find the topological charge and the topological susceptibility without preliminary space-time discretization for the example of the quantum pendulum, i.e., the system described by the Hamiltonian

$$H=\frac{1}{2}\lambda\hat{p}^2+V, \tag{35}$$

where

$$V(x)=\frac{\omega^2}{\lambda}(1-\cos x), \tag{36}$$

ω is the frequency of small oscillations, and λ is the coupling constant. In the case where $V(x)$ is a bounded periodic function (with period 2π) on $x\in(-\infty,\infty)$, the analysis of the Hamiltonian (35) reduces to study of the operator

$$H(\theta)=-\frac{\lambda}{2}\left(\frac{d^2}{dx^2}\right)_\theta+V,$$

where $(d^2/dx^2)_\theta$ is the operator d^2/dx^2 on $L^2[0,2\pi]$ with boundary conditions of the form

$$\psi(2\pi)=e^{i\theta}\psi(0); \quad \psi'(2\pi)=e^{i\theta}\psi'(0),$$

and θ is a fixed parameter, $|\theta|\leq\pi$. According to the Bloch theorem (see Ref. 282, for example), if $\psi(x)$ is an eigenfunction of the Hamiltonian (35) with periodic $V(x)$, then

$$\psi(x+2\pi)=e^{i\theta}\psi(x).$$

TABLE VI. Ground-state energy and propagator of the anharmonic oscillator.

		$G^{(N)}(0)$ (Ref. 236)			Formula with weight (26)	
g	E_0^* (Ref. 251)	$N=4$	$N=20$	β	$E_0^{(\beta,1)}$	$G^{(\beta,1)}(0)$
0.1	0.559146	0.443±0.16	0.409±0.06	3	0.552	0.406
0.2	0.602405	-	-	2.5	0.592	0.364
0.5	0.696176	0.296±0.07	0.293±0.04	2	0.685	0.293
1.0	0.803771	0.269±0.08	0.267±0.08	1.5	0.774	0.257

TABLE VII. Energies of the ground and first excited states of the anharmonic oscillator.

			$\Delta E^{(N)}$ (Ref. 236)		Composite formula (21)		
g	ΔE^* (Ref. 251)	$G^*(0)$ (Ref. 236)	$N=20$	β	$E_0^{(\beta,1)}$	$\Delta E^{(\beta,1)}$	$G^{(\beta,1)}(0)$
0.1	1.2104	0.4125	1.03±0.29	4.5	0.570	1.14	0.419
0.2	1.3481	-	-	4	0.616	1.32	0.377
0.5	1.6282	0.3058	1.50±0.67	3	0.707	1.55	0.313
1.0	1.9341	0.2571	1.56±0.81	2.5	0.832	1.88	0.263

In this case for the functional integral Z we will have[263]

$$Z(x,x+2\pi N,\beta)=e^{i\theta N}Z(x,x,\beta). \tag{37}$$

Performing a transformation to remove the phase factor[283] $\psi(x)\to e^{-i(x/2\pi)\theta}\psi(x)$, we find that the form of the Feynman–Kac formula is preserved if the action functional is replaced by the new functional (the "θ action")

$$S_\theta=S-i\theta Q,$$

where Q is the topological charge. In this case the value of the topological charge is[260]

$$Q(x)=-\frac{1}{2\pi}\int_0^\beta \dot{x}\,dt=\left[\frac{x}{2\pi}\right]$$

$$(x(0)=x,\quad x(\beta)=x \bmod 2\pi),$$

where [] denotes the integer part.

Writing $Z(\beta)$ as

$$Z(\beta)=\int_{-\infty}^{\infty}Z(x,x,\beta)dx=\sum_{n=0}^{\infty}\int_{2\pi n}^{2\pi(n+1)}Z(x,x,\beta)dx$$

$$+\sum_{n=-\infty}^{0}\int_{2\pi(n-1)}^{2\pi n}Z(x,x,\beta)dx,$$

after the change of variable $x=\bar{x}+2\pi n$ using (37) we obtain

$$Z(\beta)=\sum_{n=0}^{\infty}e^{i\theta n}\int_0^{2\pi}Z(\bar{x}+2\pi n,\bar{x},\beta)d\bar{x}$$

$$+\sum_{n=-\infty}^{0}e^{i\theta n}\int_{-2\pi}^{0}Z(\bar{x}+2\pi n,\bar{x},\beta)d\bar{x},$$

where

$$Z(\bar{x}+2\pi n,\bar{x},\beta)=\frac{e^{-\omega^2\beta/\lambda}}{\sqrt{2\pi}}\sqrt{\frac{\lambda}{\beta}}\,e^{-2\pi^2\lambda/\beta n^2}$$

$$\times\int_{C_0}\exp\left\{\frac{\omega^2\beta}{\lambda}\int_0^1\cos\left[\sqrt{\frac{\beta}{\lambda}}x(t)\right.\right.$$

$$\left.\left.+2\pi nt+\bar{x}\right]dt\right\}d_W x.$$

Therefore,

$$Z(\beta)=\frac{e^{-\omega^2\beta/\lambda}}{\sqrt{2\pi}}\sqrt{\frac{\lambda}{\beta}}\sum_{n=0}^{\infty}e^{-2\pi^2\lambda/\beta n^2}[e^{i\theta n}Z_n(\beta)$$

$$+e^{-i\theta n}Z_{-n}(\beta)],$$

where

$$Z_n(\beta)=\int_0^\pi[I_n(x,\beta)+I_n(-x,\beta)]dx;$$

$$I_n(x,\beta)=\int_{C_0}\exp\left\{\frac{\omega^2\beta}{\lambda}\int_0^1\cos\left[\sqrt{\frac{\beta}{\lambda}}x(t)+2\pi nt\right.\right.$$

$$\left.\left.+x\right]dt\right\}d_W x.$$

Using the parity of the cosine and the property of the functional integral

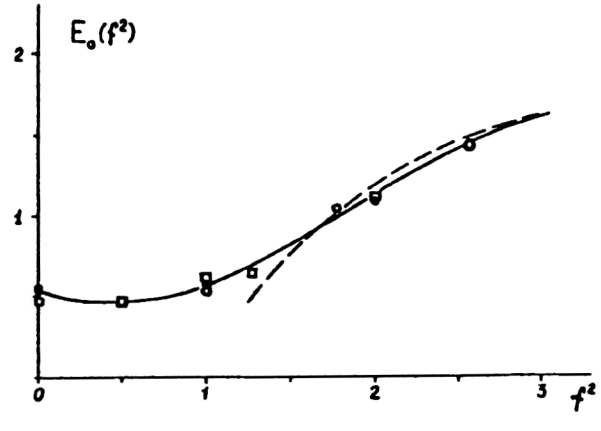

FIG. 5. Ground-state energy of the double well.

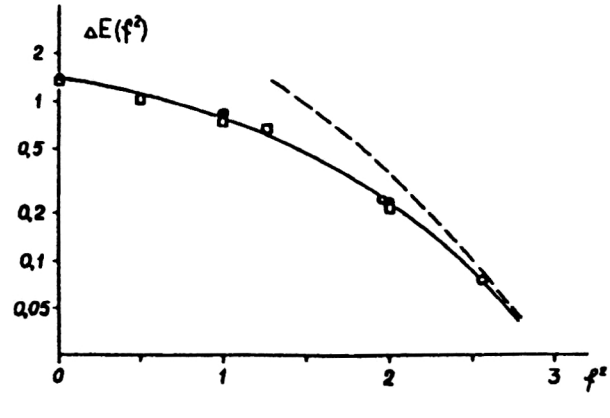

FIG. 6. Difference of the energies of the ground and first excited states of the double well.

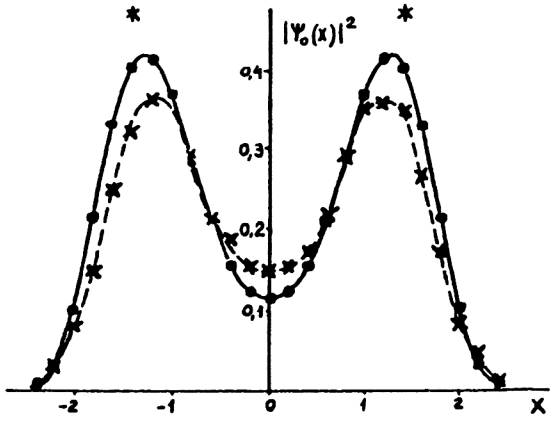

FIG. 7. Ground-state wave function of the double well.

$$\int_{C_0} F[x(\cdot)]d_W x = \int_{C_0} F[-x(\cdot)]d_W x,$$

we find that $I_n(x,\beta)=I_{-n}(x,\beta)$, and thus $Z_n(\beta)=Z_{-n}(\beta)$ and

$$Z(\beta)=\frac{2e^{-\omega^2\beta/\lambda}}{\sqrt{2\pi}}\sqrt{\frac{\lambda}{\beta}}\sum_{n=0}^{\infty}\cos(\theta n) \times\exp\left\{-2\pi^2\frac{\lambda}{\beta}n^2\right\}Z_n(\beta), \tag{38}$$

where $Z_n\to 2\pi$ for $n\to\infty$ (Ref. 263). The points on a logarithmic scale in Fig. 8 show the topological susceptibility $(1/\beta_0)\langle Q^2\rangle$, where the average topological charge is

$$\langle Q^2\rangle=\frac{1}{Z(\beta)}\frac{2e^{-\omega^2\beta/\lambda}}{\sqrt{2\pi}}\sqrt{\frac{\lambda}{\beta}}\sum_{n=0}^{\infty}\cos(\theta n)n^2 \times\exp\left\{-2\pi^2\frac{\lambda}{\beta}n^2\right\}Z_n(\beta), \tag{39}$$

as a function of the parameter $\rho=\omega/\lambda$, $\beta_0=\omega\beta$, calculated using the elementary (not compound) approximation formula for the functional integral $Z_n(\beta)$ for $m=1$, $\beta_0=7$, $\theta=0$, and $\lambda=1$. The calculations showed that four terms of the series in (38) and (39) are sufficient to obtain three correct decimal places. The computing time on a CDC-6500 was about 10 sec per point ρ. The crosses in Fig. 8 show the ρ dependence of the quantity

$$\frac{1}{\varepsilon N}\langle Q^2\rangle$$

obtained in Ref. 284 by the lattice Monte Carlo method with $N=100$, $\varepsilon=\omega a=1$ (N is the number of sites, and a is the lattice spacing). The solid line for $\rho>0.5$ shows the results obtained in the dilute instanton-gas approximation (see Ref. 284):

$$\frac{1}{\beta_0}\langle Q^2\rangle=\frac{8}{\sqrt{\pi}}\sqrt{\rho}\,e^{-8\rho}\left[1-\frac{7}{64}\rho^{-1}-\ldots\right]. \tag{40}$$

The curve for $\rho<0.5$ is the high-temperature expansion in the continuum limit ($\varepsilon\to 0$, $N\to\infty$ with εN fixed):

$$\frac{1}{\beta_0}\langle Q^2\rangle=\frac{1}{4\pi^2\rho}.$$

For a more accurate comparison of $\langle Q^2\rangle$ with the theoretical predictions let us consider the quantity

$$D=\frac{1}{\beta_0}\langle Q^2\rangle\rho^{-1/2}e^{S\rho}.$$

Obviously, in the continuum limit ($S=8$) and for sufficiently large ρ (the semiclassical region) we must have

$$D=\frac{8}{\sqrt{\pi}}\simeq 4.51 \tag{41}$$

The results of our calculations are shown by the points in Fig. 9; they are in good agreement with (41). The crosses show the results of Ref. 284 obtained for $\varepsilon=1$, $N=100$, and $S=7.87$ (the instanton action on this lattice). The difference between the results of Ref. 284 and the theoretical prediction is not explained by the two-loop correction in (40). Therefore, strictly speaking, from these results it is impossible to arrive at any conclusion about the validity of the dilute instanton-gas approximation. To clarify the situation, the authors of Ref. 284 also performed calculations for $\varepsilon=0.6$ (at the point $\rho=0.7$). The value $D=3.3$ was obtained, which is larger than for $\varepsilon=1$, but still smaller than the theoretical estimate. It was not possible to decrease ε further in Ref. 284, owing to the difficulties which arise from the increase of the instanton size and the slowing of the convergence of the iterations. Such problems obviously do not arise in our calculations, because the calculations are always performed without space discretization: $\varepsilon=0$.

By determining the parameters by the least-squares method from the expression

$$\ln\left[\frac{1}{\beta_0}\langle Q^2\rangle\right]=\ln D-S\rho+p\ln\rho,$$

the authors of Ref. 284 obtained $D=2.98$ and $p=0.46$. According to our values (points in Fig. 9), we have $D=4.25$ and $p=0.493$, in good agreement with the theoretical predictions (in the continuum limit the theoretical result is $p=0.5$).

Let us now calculate the vacuum energy. Owing to instanton effects, the ground-state energy E is smeared to form a band and is characterized by the angle θ which violates CP invariance. The values of $E(\theta)$ that we obtained from

$$E(\theta)=-\frac{1}{\beta}\ln Z(\beta),\quad \omega=1,\quad \rho=\frac{1}{\lambda}$$

using the elementary approximation formula for $m=1$ are shown in Fig. 10 for $\rho=1$ and $\rho=1.2$ by the solid circles and the squares, respectively. We see from this figure that the results are in good agreement with the theoretical estimates obtained in the dilute-gas approximation:

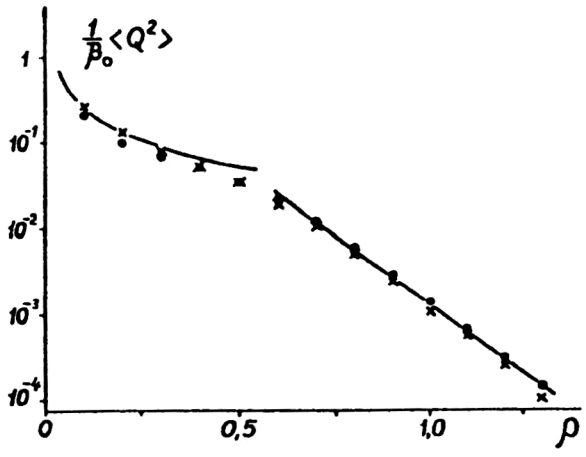

FIG. 8. Topological susceptibility in the model of the quantum pendulum.

$$E(\theta)=\left(\frac{1}{2}-\frac{1}{8\cdot 4}\rho^{-1}-\frac{1}{32\cdot 16}\rho^{-2}-\ldots\right)$$
$$-\cos\theta\,\frac{8}{\sqrt{\pi}}\rho^{-1/2}e^{-8\rho}\left[1-\frac{7}{64}\rho^{-1}-\ldots\right],$$

which are shown by the solid lines in the figure.

9. APPROXIMATE CALCULATION OF MULTIPLE FUNCTIONAL INTEGRALS

When studying real physical processes it is often necessary to consider systems with more than one degree of freedom. Here in the functional-integral approach it becomes necessary to calculate multiple functional integrals. This can be done, in particular, by successive application of approximation formulas for single functional integrals (for example, the formulas that we have already studied, which are exact on a class of functional multinomials). However, it is more economical to use formulas possessing a given total order of accuracy.[285] Let us consider the construction of approximation formulas of third-order total accuracy for m-fold functional integrals with Gaussian measures:

$$\int_X\cdots\int_X F[x_1,\ldots,x_m]d\mu(x_1)\ldots d\mu(x_m)$$
$$\equiv\int_{X^m}F[\mathbf{x}]d\mu^{(m)}(\mathbf{x}),\tag{42}$$

viewed as a Lebesgue integral, constructed on the Cartesian product of complete separable metric spaces $X^m=X\times X\times\ldots\times X$, ($m$ times), with the Cartesian product of Gaussian measures μ specified by the correlation functional $K(\xi,\eta)$ and the average value $M(\xi)$; $\xi,\eta\in X'$.

Let $\mathscr{H}$ be the Hilbert space with scalar product $(\cdot,\cdot)$ dense almost everywhere in X and generated by the Gaussian measure μ, and $\{e_k\}_{k=1}^{\infty}$ be an orthonormal basis in $\mathscr{H}$. As before, let $\rho(r)$ be a function odd in r specified on R with value in X, for which

$$\int_R\langle\xi,\rho(r)\rangle\langle\eta,\rho(r)\rangle d\nu(r)=K(\xi,\eta).$$

Then we have the following theorem.

Theorem 7. *(Ref. 285) Let $F[x]$ be an arbitrary real integrable functional. Let*

$$S_{n_i}(x_i)=\sum_{j=1}^{n_i}(e_j,x_i)e_j;\quad U_{n_i}(\mathbf{u}^{(i)})=\sum_{j=1}^{n_i}u_j^{(i)}e_j;$$
$$N=\sum_{i=1}^{m}n_i.$$

Then the (compound) approximation formula

$$\int_{X^m}F[\mathbf{x}]d\mu^{(m)}(\mathbf{x})=(2\pi)^{-N/2}$$
$$\times\int_{R^N}\exp\left\{-\frac{1}{2}\sum_{i=1}^{m}(\mathbf{u}^{(i)},\mathbf{u}^{(i)})\right\}$$
$$\times\frac{1}{m}\sum_{i=1}^{m}\int_R F(U_{n_1}(\mathbf{u}^{(1)}),\ldots,$$
$$\Sigma_i(\sqrt{m}\rho(v,\cdot),\mathbf{u}^{(i)}),$$
$$\ldots,U_{n_m}(\mathbf{u}^{(m)}))d\mathbf{u}d\nu(v)+\mathscr{R}_N(F),\tag{43}$$

where $\nu(v)$ is the symmetric probability measure on R and

$$\Sigma_i(x_i,\mathbf{u}^{(i)})=x_i-S_{n_i}(x_i)+U_{n_i}(\mathbf{u}^{(i)}),$$

is exact for functional multinomials of third total degree on X^m.

A functional of the form

$$F[x_1,\ldots,x_m]=\prod_{i=1}^{m}F_{k_i}(x_i),$$

where $k_1+k_2+\ldots+k_m\leq k$ and $F_{k_i}(x_i)$ is a homogeneous functional multinomial of degree k in the variable x_i, is called a functional multinomial of kth total degree.

In the special case where X is a space of continuous functions $C_0\equiv\{C[0,1],x(0)=x(1)=0\}$ with conditional Wiener measure characterized by zero average value and correlation function $B(t,s)=\min(t,s)-ts$, the measure $d\nu=\frac{1}{2}dv$, $v\in[0,1]$, we obtain the compound approximation formula for an m-fold functional integral with conditional Wiener measure:

$$\int_{C^m}F[\mathbf{x}]d_W\mathbf{x}=(2\pi)^{-N/2}$$
$$\times\int_{R^N}\exp\left\{-\frac{1}{2}\sum_{i=1}^{m}(\mathbf{u}^{(i)},\mathbf{u}^{(i)})\right\}$$
$$\times\frac{1}{m}\sum_{i=1}^{m}\int_R F(\tilde{U}_{n_1}(\mathbf{u}^{(1)}),\ldots,$$
$$\sqrt{m}\tilde{\Sigma}_i(\rho(v,\cdot),\mathbf{u}^{(i)}),$$
$$\ldots,\tilde{U}_{n_m}(\mathbf{u}^{(m)}))d\mathbf{u}dv+R_N(F),\tag{44}$$

where

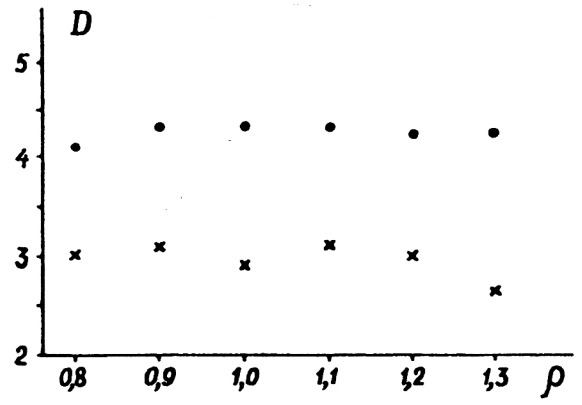

FIG. 9. Comparison of the results of calculating the topological susceptibility.

$$dv=\frac{1}{2}\,dv, \quad \rho(v,t)=\begin{cases} -t\ \mathrm{sign}(v) & t\leqslant|v| \\ (1-t)\mathrm{sign}(v) & t>|v| \end{cases},$$

$$\tilde{\Sigma}_i(\rho(v,t),\mathbf{u}^{(i)})=\rho(v,t)-S_{n_i}(\rho(v,t))+\tilde{U}_{n_i}(\mathbf{u}^{(i)}),$$

$$S_{n_i}(\rho,(v,t))=2\sum_{j=1}^{n_i}\frac{1}{j\pi}\sin(j\pi t)\mathrm{sign}(v)\cos(j\pi t),$$

$$\tilde{U}_{n_i}(\mathbf{u}^{(i)})=\sqrt{2}\sum_{j=1}^{n_i}u_j^{(i)}\frac{1}{j\pi}\sin(j\pi t)$$

for all $i=1,2,\ldots,m$.

The sufficient condition for convergence of the approximations obtained using (43) and, accordingly, (44) to the exact value of the integral as n_i $(i=1,2,\ldots,m)$ tends to infinity is given by the following theorem:[285]

Theorem 8. *Suppose that for almost all $v\in R$ pertaining to the measure $\nu(v)$ we have convergence*

$$S_{n_i}(\rho(v))\to\rho(v) \quad \text{for } n_i\to\infty,\ i=1,2,\ldots,m.$$

Let $F[x]$ be a functional continuous on X^m satisfying almost everywhere on X^m the condition

$$|F[x]|\leqslant g(A^1(x_1,x_1),\ldots,A^m(x_m,x_m)),$$

where $A^k(x_k,x_k)$ is a non-negative quadratic functional of the form

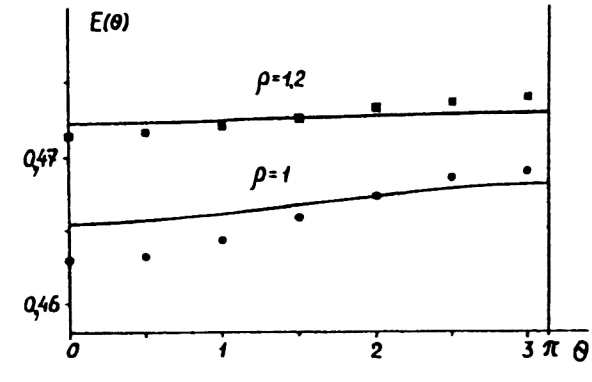

FIG. 10. Energy of the θ vacuum in the model of the quantum pendulum.

$$A^k(x_k,x_k)=\sum_{i=1}^{\infty}\gamma_i^k(x_k,e_i)^2, \quad k=1,2,\ldots,m, \tag{45}$$

$$\sum_{i=1}^{\infty}\gamma_i^k<\infty, \quad \gamma_i\geqslant 0, \quad i=1,2,\ldots, \tag{46}$$

and $g(x)$ is a positive function which is nondecreasing in all variables, such that

$$\int_{X^m}\int_R g(A^1(x_1,x_1),\ldots,$$

$$A^k(\sqrt{m}\rho(v),\sqrt{m}\rho(v))+A^k(x_k,x_k),$$

$$\ldots,A^m(x_m,x_m))\,d\nu(v)\,d\mu^{(m)}(x)<\infty.$$

Then

$$\mathscr{R}_N(F)\to 0 \quad \text{for } n_i\to\infty,\ i=1,2,\ldots,m.$$

The rate of convergence of the approximations obtained using (43) to the exact value of the integral is estimated by the following theorem.[286]

Theorem 9. *Let a functional integrated with measure $\mu^{(m)}(x)$ be represented as*

$$F[x+x_0]=P_3(x)+r(x,x_0),$$

where $P_3(x)$ is a functional multinomial of third total degree and

$$|r(x,x_0)|\leqslant\prod_{i=1}^{m}(A^i(x_i,x_i))^2(c_1\exp\{c_2A^i(x_i+x_i^0,x_i+x_i^0)\}+c_3\exp\{c_2A^i(x_i,x_i^0)\}),$$

in which the $A^i(x_i,x_i)$ satisfy (45) and (46), x_0 is a fixed point from X^m, and the c_i are positive constants such that

$$\tfrac{1}{2}-c_2\gamma_k^{(i)}\geqslant 0, \quad k=1,2,\ldots,m,$$

$$\sum_{k=1}^{\infty}\gamma_k^{(i)}a_k<\infty; \quad (e_k,\sqrt{m}\rho(v))^2\leqslant a_k; \quad a_k,v\in R.$$

Then for the remainder of the approximation formula (43) we have the estimate

$$\mathscr{R}_N(F)=O\left(\prod_{i=1}^{m}\left(\sum_{k=n_i+1}^{\infty}\gamma_k^{(i)}\right)^2\right)+\sum_{i=1}^{m}O\left(\left(\sum_{k=n_i+1}^{\infty}\gamma_k^{(i)}a_k\right)^2\right).$$

As an example, let us consider the approximate calculation of the two-fold functional integral with conditional Wiener measure

$$I=\int_{C_0^2}\exp\left\{\int_0^1(px^2(t)+qx(t)+py^2(t)+qy(t))dt\right\}d_Wx\,d_Wy$$

using (44). The results of the calculation[287] for $q=5$ and various p, n_1, and n_2 are given in Tables VIII and IX. We use I^* to denote the exact value found by us in Ref. 248. The

dependence of I on p and q for $n_1=n_2=20$ is given in Tables X and XI, where $\hat{I}$ denotes the result obtained using the "elementary" approximation formula for multiple functional integrals, taken from Ref. 49. The times to compute I and $\hat{I}$ at each point (p,q) were about 1 sec on a CDC-6500. We see from these tables that the use of compound approximation formulas leads to more accurate approximations.

As for the single functional integral, we have constructed and studied[285] an approximation formula of third-order total accuracy for multiple functional integrals with conditional Wiener measure and a weight:

$$\int_{C_0^m} P[x]F[x]d_W^{(m)}x,$$

$$x=(x_1,\ldots,x_m);\quad d_W^{(m)}x=d_Wx_1\ldots d_Wx_m;$$

$$P[x]=\exp\left\{\sum_{i=1}^{m}\int_0^1 (p_i(t)x_i^2(t)+q_i(t)x_i(t))dt\right\},$$

$$p_i(t),q_i(t)\in C[0,1] \text{ for all } i=1,2,\ldots,m.$$

In Ref. 286 we used the approximation formulas for multiple functional integrals to calculate the ground-state energy and wave function of the harmonic oscillator for 2- and 3-dimensional space. Comparison of the results with the theoretical values shows that our approximation formulas are highly efficient.

10. THE CALOGERO MODEL. THE TRITIUM NUCLEUS

Let us consider a quantum system of n particles in one dimension characterized by the Hamiltonian[288]

$$H=-\sum_{k=1}^{n}\frac{\partial^2}{\partial x_k^2}+\frac{1}{2}\omega^2\sum_{i<j}^{n}(x_i-x_j)^2+g\sum_{i<j}^{n}(x_i-x_j)^{-2}, \tag{47}$$

corresponding to the pairwise interaction of particles via the repulsive force of a centrifugal potential and a linear attraction. This model has been studied by many authors (see Refs. 289–293). An exact analytic solution exists for it,[288] so that it can be used to study the efficiency of various numerical methods. As noted in Refs. 291 and 294, the method of functional integration is the most suitable way of numerically analyzing quantum systems with a large number of degrees of freedom, i.e., of solving multidimensional problems of high dimensionality, where other methods become inefficient.

We have calculated[295] the Green function Z for the system (47) in the form of a functional integral with conditional Wiener measure using a weighted approximation formula. The values of the ground-state energy E_0 obtained in the three-particle case ($n=3$) for $g=1.5$ and various ω are given in Table XII. The time needed to calculate E_0 at each point ω was 11 sec on a CDC-6500.

The values of E_0 calculated for $g=1.5$ and $\omega=0.25$ for various numbers of particles n are given in Table XIII. For comparison, we give the results obtained in Ref. 291 using the Monte Carlo method (1000 discretization points, 100 iterations), denoted by E_{MC}. The exact result is denoted by E^*. The time needed to find E_0 for 11 particles using the weighted approximation formula was 3 min on a CDC-6500, and the calculation of E_{MC} required 15 min on the same computer.[291] Comparison of the numerical values indicates that our deterministic approach for functional integrals gives more accurate results for shorter computing time than the Monte Carlo method, where there was not even any guarantee that the results would coincide with the exact values[291] within the error.

The numerical study of a system of three interacting particles is one of the fundamental problems of mathematical physics. Let us consider the ground state of the triton (the nucleus of the tritium atom), i.e., the system described by the Hamiltonian

$$H=\sum_{i=1}^{3}\frac{\hbar^2}{2m_i}\frac{\partial^2}{\partial x_i^2}+\sum_{i<j}V(|r_{ij}|). \tag{48}$$

Here we use $x_i=(x_i^{(1)},x_i^{(2)},x_i^{(3)})$, $i=1,2,3$, to denote the coordinates of the three nucleons with masses m_i and

$$r_{ij}=x_i-x_j.$$

A rather common method of studying such a system numerically is the variational method; since the early 1970s there have been many studies (see, for example, Refs. 296–300) on the development of this method for this nine-dimensional problem. The results of these studies are upper and lower bounds on the binding energy, for which the accuracy is fairly high when sufficient computing time (something like several hours) is used. In some studies the binding energy was calculated by the Monte Carlo method (Refs. 291, 301, and 302). The ground-state energy was obtained in Ref. 291 with an accuracy of order 0.1 MeV for a computing time of about one hour on a CDC-6500.

Let us consider the following model of the triton, used in Ref. 296 and subsequent studies: three identical particles of mass $m=m_p=938.279$ MeV interact pairwise via a spherically symmetric spin-independent potential:

$$V(r)=-51.5\exp\left\{-\frac{r^2}{b^2}\right\}\ \text{MeV},\quad b=1.6\ \text{F}. \tag{49}$$

In the studies cited above, the following values were obtained for the ground-state energy of the system (48) with the potential (49):

$E=-9.77\pm0.06$ MeV (Ref. 291),

$E=-9.42$ MeV (Ref. 296),

$E=-9.47\pm0.4$ MeV (Ref. 301),

-9.99 ± 0.05 MeV$<E<-9.75\pm0.04$ MeV

(Ref. 297),

$E=-9.78$ MeV (Refs. 297, 299).

(These differ from each other by more than the quoted error.) We have calculated[286] the ground-state energy of the triton using the approximation formula of third-order total accuracy for multiple functional integrals. After changing to dimensionless variables, the Hamiltonian of the system is rewritten as[286]

TABLE VIII. Values of the two-fold functional integral for $p=0.5$.

	$I^*=12.03390688$				
$n_1;n_2$	1	2	3	5	10
1	12.0332950				
2	12.0342440	12.0332116			
3	12.0395110	12.0356295	12.0339518		
5	12.0423562	12.0374153	12.0342150	12.0339122	
10	12.0447756	12.0390693	12.0347689	12.0340572	12.0339069

$$H_0 \equiv \frac{1}{R} H = -\frac{1}{2} \sum_{i=1}^{3} \frac{\partial^2}{\partial x_i^2} + \sum_{i<j} \frac{1}{R} V(|r_{ij}|),$$

where

$$R = \frac{(197.93)^2}{938.279} \text{ MeV} \cdot \text{F}^2.$$

The matrix element $Z(x_1,\ldots,x_9,\beta)$ is written as

$$Z(x_1,\ldots,x_9,\beta) = (2\pi\beta)^{-9/2} \times \int_{C_0^9} \exp\left\{ -\frac{\beta}{R} \int_0^1 \sum_{i<j}^{9} V(|(\sqrt{\beta}u_i(t)+x_i) - (\sqrt{\beta}u_j(t)+x_j)|)dt \right\} d_W^{(9)}u,$$

$$x_1=(x_1,x_2,x_3), \quad x_2=(x_3,x_4,x_5), \quad x_3=(x_7,x_8,x_9).$$

The values of the ground-state energy E_0 in MeV calculated for various β are given in Table XIV. The multiple Riemann integrals were calculated by the Korobov method with relative error $\epsilon=1\%$. The time for computing the energy E_0 for each β was about 15 min on a CDC-6500. We see from this table that the improved value of the ground-state energy that we obtained, $E_0=-9.7$ MeV, is consistent with the results of other authors, while for us the computing time was smaller than that quoted in the other studies.

11. FUNCTIONAL INTEGRALS IN $P(\varphi)_2$ QUANTUM FIELD THEORY

One of the areas of quantum field theory in which the question of the functional-integration measure has been studied most deeply is that of two-dimensional Euclidean field theory with polynomial interactions of bosonic fields.[239] In the $P(\varphi)_2$ model it is possible to study, in particular, such processes as phase transitions, critical phenomena, particle interactions, scattering, and bound states. This model has been studied by many authors (see Ref. 92). In particular, in Ref. 303 the method of functional integration was applied to this model to study the vacuum energy density in infinite volume.

The Lagrangian of the $P(\varphi)_2$ model is written as[92]

$$L(\varphi(x)) =: \tfrac{1}{2}(\nabla\varphi)^2 + \tfrac{1}{2}m^2\varphi^2(x) + \lambda P(\varphi(x)): \quad . \qquad (50)$$

Here $x \in R^2$; $\varphi(x) \in S'(R^2)$, the space of tempered generalized functions; and P is a polynomial bounded from below. The basis-function space is the Schwartz space of rapidly decreasing functions $\mathcal{S}(R^2)$. The value of φ on a basis function $f \in \mathcal{S}(R^2)$ is defined as

$$\varphi(f) \equiv \langle \varphi, f \rangle = \int_{R^2} \varphi(x) f(x) dx.$$

The Wick ordering is given by the expression

$$:\varphi(x)^n: = \lim_{\varkappa\to\infty} \sum_{j=0}^{[n/2]} \frac{(-1)^j n!}{(n-2j)!\,j!\,2^j} k_\varkappa^j(x)(\varphi_\varkappa(x))^{n-2j},$$

where

$\varphi_\varkappa(x) = \int_{R^2} \varphi(y)\delta_{\varkappa,x}(y)dy$ is the momentum cutoff of the field φ;

$\delta_{\varkappa,x}(y) = \varkappa^2 h(\varkappa(x-y))$ is the "smeared" δ function,

$$h \in C_0^\infty(R^2), \quad h(y) \geqslant 0, \quad \int_{R^2} h(y)dy = 1,$$

$$k_\varkappa(x) = \langle \delta_{\varkappa,x}, K\delta_{\varkappa,x} \rangle.$$

Here K is the correlation functional of the measure, and $K(f,g) = \langle f, Kg \rangle$ [a continuous, nondegenerate bilinear form on the product space $\mathcal{S}(R^2) \times \mathcal{S}(R^2)$]:

TABLE IX. Values of the two-fold functional integral for $p=-0.5$.

	$I^*=5.655919950$				
$n_1;n_2$	1	2	3	5	10
1	5.65338831				
2	5.65472254	5.65515020			
3	5.65465495	5.65557558	5.65573296		
5	5.65487646	5.65554252	5.65583832	5.65587219	
10	5.65513682	5.65553213	5.65597514	5.65593299	5.65591293

TABLE X. The q *dependence of the values of the two-fold functional integral for* $p=0.5$.

q	I^*	I	$\hat{I}$
0	1.188395106×10^{0}	1.188394988×10^{0}	1.18469738×10^{0}
2	1.721203344×10^{0}	1.721293187×10^{0}	1.65205903×10^{0}
4	5.229331352×10^{0}	5.229330998×10^{0}	3.71647298×10^{0}
6	3.332750443×10^{1}	3.332750366×10^{1}	1.04813216×10^{1}
8	4.455557798×10^{2}	4.455558015×10^{2}	0.32888278×10^{2}
10	1.249522331×10^{4}	1.249522528×10^{4}	0.01094908×10^{4}

$$K(f,g)=\int [\langle\varphi,f\rangle-M(f)][\langle\varphi,g\rangle-M(g)]d\mu(\varphi),$$

where $M(f)=\int\langle\varphi,f\rangle d\mu(\varphi)$ is the average value of the measure $d\mu(\varphi)$.

The functional measure in the space $\mathscr{S}'(R^2)$ is defined as follows.[92] In the restricted region $\Lambda\subset R^2$ one introduces the measure

$$\partial\mu_\Lambda=Z^{-1}e^{-V(\Lambda)}d\varphi_{K_{\partial\Lambda}},$$

where

$$V(\Lambda)=\int_\Lambda :P(\varphi(x)):_{K_\varnothing}dx;$$

$$Z=Z(\Lambda)=\int e^{-V(\Lambda)}d\varphi_{K_{\partial\Lambda}}.$$

The Wick ordering is done with respect to the free correlation functional $K_\varnothing$ (Ref. 92). Here $d\varphi_{K_{\partial\Lambda}}$ is a Gaussian measure with correlation functional $K_{\partial\Lambda}$ satisfying Dirichlet conditions on $\partial\Lambda$—the boundary of the region Λ—and having zero average value $M(f)=0$. As already noted, the correlation functional and the average value determine the Gaussian measure uniquely.

An important result obtained by Glimm and Jaffe is the proof that the measure exists in an unrestricted region. They showed that for certain conditions on P and for $f\in C_0^\infty$ the sequence of characteristic functionals of measures in finite volumes

$$S_\Lambda\{f\}=\int e^{i\varphi(f)}d\mu_\Lambda$$

has the limit

$$S\{f\}=\lim_{\Lambda\uparrow R^2}S_\Lambda\{f\},$$

TABLE XI. The q dependence of the values of the two-fold functional integral for $p=-0.5$.

q	I^*	I	$\hat{I}$
0	8.509181282×10^{-1}	8.509180438×10^{-1}	8.48798027×10^{-1}
2	1.152142647×10^{0}	1.152141525×10^{0}	1.14025833×10^{0}
4	2.859971347×10^{0}	2.859970984×10^{0}	2.40410599×10^{0}
6	1.301529645×10^{1}	1.301529441×10^{1}	0.64445955×10^{1}
8	1.085883915×10^{2}	1.085883710×10^{2}	0.19531843×10^{2}
10	1.660923723×10^{3}	1.660923370×10^{3}	0.06347512×10^{3}

TABLE XII. Ground-state energy in the Calogero model for three particles.

ω	E_0	E_{MC} (Ref. 291)	E^* (Ref. 288)
0.10	1.346	-	1.3472
0.20	2.700	-	2.6944
0.25	3.366	3.35 ± 0.004	3.3680
0.50	6.738	-	6.7361

where the limiting functional $S\{f\}$ satisfies the Euclidean axioms of analyticity, regularity, and invariance under translations, rotations, and reflections, as required in constructing the quantum field φ (Ref. 92).

Therefore, the observable values of operators defined as vacuum averages can be found by calculating the functional integral

$$\langle 0|F(\varphi)|0\rangle$$

$$=\lim_{\Lambda\uparrow R^2}\frac{\int_{\mathscr{S}'(\Lambda)}\exp\{-\lambda\int_\Lambda :P(\varphi(x)):_{K_\varnothing}dx\}F(\varphi)d\varphi_{K_{\partial\Lambda}}}{\int_{\mathscr{S}'(\Lambda)}\exp\{-\lambda\int_\Lambda :P(\varphi(x)):_{K_\varnothing}dx\}d\varphi_{K_{\partial\Lambda}}}. \tag{51}$$

It is important that for $d=2$ the renormalizations in the $P(\varphi)_d$ model are limited to a subtraction related to the Wick ordering, i.e., there are no divergences in (51).

In the construction of approximation formulas for functional integrals it is necessary to have an explicit expression for the correlation functional of the measure. Writing $K(f,g)$ as

$$K(f,g)=\int_{R^2\times R^2}\mathscr{K}(x,y)f(x)g(y)dxdy,$$

let us consider the integrated kernel $\mathscr{K}(x,y)$, $x,y\in R^2$:

$$\mathscr{K}(x,y)=\int_{\mathscr{S}'(R^2)}\varphi(x)\varphi(y)d\varphi_K.$$

For the integrated kernel of the correlation functional $K_{\partial\Lambda}$ with Dirichlet boundary conditions on the boundary of an arbitrary region $\Lambda\subset R^2$ there exists a representation in terms of a functional integral with conditional Wiener measure:[92]

$$\mathscr{K}_{\partial\Lambda}(x,y)=\int_0^\infty dt\ e^{-m^2t}\int_{C_{x,y}[0,t]}\chi_{\partial\Lambda}(\omega)d_W\omega,$$

where the integration runs over the set $C_{x,y}[0,t]$ of continuous functions $\omega(\tau)$, $\tau\in[0,t]$, satisfying the conditions $\omega(0)=x$, $\omega(t)=y$. Here $\chi_{\partial\Lambda}(\omega)$ is the characteristic function of the set of trajectories having no points of intersection with $\partial\Lambda$:

TABLE XIII. Ground-state energy in the Calogero model for various numbers of particles.

n	E_0	E_{MC} (Ref. 291)	E^* (Ref. 288)
5	13.447	13.37 ± 0.04	13.4397
7	32.249	32.34 ± 0.09	32.2718
9	61.473	61.31 ± 0.10	61.5183
11	102.865	102.31 ± 0.14	102.6028

TABLE XIV. Ground-state energy for the tritium nucleus.

β	2.0	3.0	4.0	4.5	4.7	4.8	5.0
E_0 (MeV)	−33.4	−15.2	−10.5	−9.9	−9.8	−9.8	−9.7

$$\chi_{\partial\Lambda}(\omega)=\begin{cases}0, & \text{if } \exists\tau_0\in[0,t]:\omega(\tau_0)\in\partial\Lambda,\\ 1, & \text{otherwise.}\end{cases}$$

Since we shall be interested in the limit $\Lambda\uparrow R^2$, without loss of generality we can assume that $x,y\in\Lambda\backslash\partial\Lambda$; in the case $x\in\partial\Lambda$ and/or $y\in\partial\Lambda$ it is obvious that $\mathscr{K}(x,y)=0$. If we write

$$C^{\Lambda}_{x,y}[0,t]=\{\omega(0,\tau)\in C[0,t]:\omega(0)=x,\ \omega(t)=y,$$

$$\omega(\tau)\in\Lambda\backslash\partial\Lambda\,\forall\,\tau\in[0,t]\},$$

then

$$\mathscr{K}_{\partial\Lambda}(x,y)=\int_0^\infty dt\ e^{-m^2t}\int_{C^{\Lambda}_{x,y}} d_W\omega$$

$$=\int_0^\infty M(x,y,t)e^{-m^2t}dt,$$

where $M(x,y,t)=\text{mes}\,C^{\Lambda}_{x,y}[0,t]$ is the Wiener volume of the set $C^{\Lambda}_{x,y}$.

Therefore, to find the kernel $\mathscr{K}_{\partial\Lambda}(x,y)$ it is sufficient to determine the volume of the set of two-dimensional functions continuous on a segment with fixed values at the ends of the segment and taking values inside the given region Λ. We have found this quantity in Ref. 304. A consequence of this is an expression for $\mathscr{K}_{\partial\Lambda}(x,y)$ (Refs. 304 and 305).

Theorem 10. *For an arbitrary connected, restricted region $\Lambda\subset R^2$ with piecewise-smooth boundary $\partial\Lambda$, the integrated kernel of the correlation functional of the $P(\varphi)_2$ Gaussian measure in the space of tempered generalized functions $\mathscr{S}'(R^2)$ can be represented as*

$$\mathscr{K}_{\partial\Lambda}(x,y)=\sum_n \frac{1}{E_n+m^2}\,\vartheta_n(x)\vartheta_n(y),\quad x,y\in\Lambda\backslash\partial\Lambda,$$

where E_n and ϑ_n are the eigenvalues and eigenfunctions of the problem

$$\begin{cases}-\dfrac{1}{2}\,\Delta\vartheta(x)=E\vartheta(x), & x\in\Lambda\backslash\partial\Lambda,\\ \vartheta(x)=0, & x\in\partial\Lambda.\end{cases}$$

The explicit expression for $\mathscr{K}_{\partial\Lambda}(x,y)$ depends on the shape of the region Λ. In the special case where Λ is a rectangle, $\Lambda=\{[0,2a]\times[0,2b]\}$, $\mathscr{K}_{\partial\Lambda}(x,y)$ can be written as[304]

$$\mathscr{K}(x,y)=\frac{1}{ab}\sum_{k_1,k_2=1}^{\infty}\frac{1}{m^2+\left(\dfrac{\pi k_1}{2a}\right)^2+\left(\dfrac{\pi k_2}{2b}\right)^2}$$

$$\times\sin\left(\frac{\pi k_1}{2a}x_1\right)\sin\left(\frac{\pi k_2}{2b}x_2\right)\sin\left(\frac{\pi k_1}{2a}y_1\right)$$

$$\times\sin\left(\frac{\pi k_2}{2b}y_2\right),\qquad(52)$$

where $x=(x_1,x_2)$ and $y=(y_1,y_2)$.

The eigenfunctions of the kernel $\mathscr{K}(x,y)$ are

$$\beta_{k_1,k_2}(x)=\frac{1}{\sqrt{ab}}\sin\left(\frac{\pi k_1}{2a}x_1\right)\sin\left(\frac{\pi k_2}{2b}x_2\right),$$

corresponding to the eigenvalues

$$\lambda_{k_1,k_2}=\frac{1}{m^2+\left(\dfrac{\pi k_1}{2a}\right)^2+\left(\dfrac{\pi k_2}{2b}\right)^2}.$$

In what follows we shall consider a square region Λ: $b=a$, and we take $m=1$.

Since for any $\varphi\in\mathscr{S}'(\Lambda)$ we have the expansion

$$\varphi=\sum_{k=1}^{\infty}(e_k,\varphi)e_k,$$

where $\{e_k\}_{k=1}^{\infty}$ is an orthonormal basis in the above-mentioned Hilbert space $\mathscr{H}$ generated by the Gaussian measure, which converges in the topology of the space $\mathscr{S}'(\Lambda)$ for nearly all $\varphi\in\mathscr{S}'(\Lambda)$, we have the following theorem, which we proved in Ref. 306:

Theorem 11. *Let $F[\varphi]$ be a functional continuous nearly everywhere on $\mathscr{S}'(\Lambda)$ and satisfying*

$$|F[\varphi]|<\Phi[\varphi],$$

where $\Phi[\varphi]$ is a non-negative integrated functional such that $\Phi[\sum_{k=1}^{n}(e_k,\varphi)e_k]$ does not fall off for $n\to\infty$. Then

$$\int_{\mathscr{S}'(\Lambda)}F[\varphi]d\mu(\varphi)=\lim_{n\to\infty}(2\pi)^{-n/2}$$

$$\times\int_{R^n}\exp\left\{-\sum_{k=1}^{\infty}\frac{u_k^2}{2}\right\}$$

$$\times F\left[\sum_{k=1}^{\infty}u_ke_k\right]du.\qquad(53)$$

This theorem allows the functional integral to be calculated in the form of an n-fold Riemann integral for a large class of functionals. However, the rate of convergence of the approximations obtained using it for $n\to\infty$ is not high [of

order $O(1/n)$]. Let us consider the construction of approximation formulas which are exact on a class of functional multinomials of a given degree, and which give good approximations when the norm of the integrated functional F is close to that of a functional multinomial of degree $\leqslant 2m+1$.

Theorem 12. *(Ref. 49) Let ν be a symmetric probability measure on R, and K be the given correlation functional of the measure $d\mu(x)$ in a complete separable space X, and let the function $\rho(v):R\to X$ satisfy*

$$\rho(v)=-\rho(-v)$$
$$\times\int_R\langle\xi,\rho(v)\rangle\langle\eta,\rho(v)\rangle d\nu(v)=K(\xi,\eta),$$
$$\times\prod_{i=1}^{l}\langle\xi_i,\rho(v)\rangle\in L(R,\nu)$$
$$\text{for } 1\leqslant l\leqslant 2m+1,\quad \xi,\eta,\xi_i\in X'. \tag{54}$$

Then the approximation formula

$$\int_X F[x]d\mu(x)\approx\int_{R^m}F[\theta_m(v)]d\nu^{(m)}(v) \tag{55}$$

is exact for any functional multinomial of degree $\leqslant 2m+1$. Here

$$\theta_m(v)=\sum_{k=1}^{m}c_k^{(m)}\rho(v_k),$$

$[c_k^{(m)}]^2$ are the roots of the multinomial $\sum_{k=0}^{m}(-1)^k(t^{m-k}/k!)$, and the measure $\nu^{(m)}$ on R^m is the Cartesian product of the measures ν on R.

Let us consider an integer lattice (n_1,n_2) on R^2 and define the measure ν on the set $Q\in R^2$ as

$$\nu(Q)=\sum_{(n_1,n_2)\in Q}\nu(n_1,n_2), \tag{56}$$

where

$$\nu(n_1,n_2)=\frac{1}{\left[1+\left(\frac{\pi n_1}{2a}\right)^2+\left(\frac{\pi n_2}{2a}\right)^2\right]^2\cdot Z},$$

$$Z=\sum_{(n_1,n_2)\in R^2}\frac{1}{\left[1+\left(\frac{\pi n_1}{2a}\right)^2+\left(\frac{\pi n_2}{2a}\right)^2\right]^2}.$$

Since $\nu(R^2)=1$, the measure $\nu(x_1,x_2)$ is a probability measure in R^2 symmetric in both arguments. The integral with this measure is written as

$$\int_{R^2}f(x_1,x_2)d\nu(x_1,x_2)$$
$$=\frac{1}{Z}\sum_{n_1,n_2=-\infty}^{\infty}\frac{1}{\left[1+\left(\frac{\pi n_1}{2a}\right)^2+\left(\frac{\pi n_2}{2a}\right)^2\right]^2}f(n_1,n_2).$$

We choose the function $\rho(v)$ to have the form

$$\rho(v)\equiv\rho_v(x)=\frac{1}{2a}\sqrt{Z}\left[1+\left(\frac{\pi v_1}{2a}\right)^2+\left(\frac{\pi v_2}{2a}\right)^2\right]^2$$
$$\times\sin\left(v_1\frac{\pi x_1}{2a}\right)\sin\left(v_2\frac{\pi x_2}{2a}\right), \tag{57}$$

where $v=(v_1,v_2)\in R^2$ and $x=(x_1,x_2)\in\Lambda$. The function ρ is an odd function of the arguments v_1 and v_2. In Ref. 306 we showed that it also satisfies the other conditions in (54), i.e., we have the following theorem.

Theorem 13. *The approximation formula*

$$\int_{\mathscr{S}'(\Lambda)}F[\varphi]d\mu(\varphi)\approx\int_{R^{2m}}F[\theta_m(v)]d\nu^{(m)}(v), \tag{58}$$

where the notation is the same as for the preceding theorem, the measure ν is defined in (56), and the function $\rho(v)$ is given in (57), is exact for any functional multinomial of degree $\leqslant 2m+1$.

We shall term (58) "elementary," in contrast to the "compound" formulas constructed for integrals with measure $d\mu_K(\varphi)$ in the space $\mathscr{S}'(\Lambda)$, as for the formulas for integrals with abstract Gaussian measure μ in complete separable spaces X:

Theorem 14. *(Ref. 306) For any real functional $F[\varphi]$ integrated with measure $d\mu(\varphi)$ the (compound) approximation formula*

$$\int_{\mathscr{S}'(\Lambda)}F[\varphi]d\mu(\varphi)=(2\pi)^{-n/2}\int_{R^n}\exp\left\{-\frac{1}{2}(u,u)\right\}$$
$$\times\int_{R^{2m}}F[\rho_m(v)-\rho_m^n(v)+\Psi_n(u)]$$
$$\times d\nu(v)du+\mathscr{R}_m^n(F) \tag{59}$$

is exact for an arbitrary functional multinomial of degree $\leqslant 2m+1$. Here

$$\rho_m(v)=\sum_{k=1}^{m}c_k^{(m)}\rho(v_k);\quad \rho_m^n(v)=S_n(\rho_m(v)),$$
$$v\in R^{2m},\quad u\in R^n,$$
$$S_n(x)=\sum_{k=1}^{n}(e_k,x)e_k,\quad \Psi_n(u)=\sum_{k=1}^{n}u_ke_k,$$

e_k is an orthonormal basis in $\mathscr{H}$, and $(u,u)=\sum_{k=1}^{n}u_k^2$.

In Ref. 306 we showed that, like integrals with abstract Gaussian measure, for integrals with measure $d\mu_K(\varphi)$ in the space $\mathscr{S}'(\Lambda)$ there are theorems about the convergence of the approximations in (59) and about the estimate of the remainder, from which it follows, in particular, that $\mathscr{R}_m^n(F)=O(n^{-(m+1)})$ for $n\to\infty$.

Let us compare the efficiency of the various approximation formulas for some numerical examples. First we calculate the integral

$$I_1=\int_{\mathscr{S}'(\Lambda)}\langle l,\varphi\rangle^4 d\mu_K(\varphi),$$

$$\text{where } l(x,y)=\begin{cases}1, & 0\leqslant x,y\leqslant 1,\\ 0, & \text{otherwise.}\end{cases} \tag{60}$$

TABLE XV. Values of the functional integral I_1 for various sizes of the region Λ.

a	$I_1^* \times 10^2$	$I_1^{\text{basis}(5)} \times 10^2$	$I_1^{\text{simple}(1)} \times 10^2$	$I_1^{\text{composite}(1,1)} \times 10^2$
1	4.07	3.90	1.98	4.13
2	4.13	3.73	3.83	4.21
3	4.13	1.58	4.76	5.66
4	4.13	0.37	5.28	6.61
5	4.13	0.08	5.61	6.50
10	4.13	0.01	6.30	6.85

This integral is the central moment of fourth order of the Gaussian measure $d\mu_K$. Its exact value is known:[50]

$$I_1^* = 3K^2(l,l).$$

In Table XV we give the results of the approximate calculation of the integral (60) using (53) with $n=5$ $[I_1^{\text{basis}(5)}]$, using the elementary formula of third-order accuracy (58) with $m=1$ $[I_1^{\text{simple}(1)}]$, and using the compound approximation formula (59) with $n=m=1$ $[I_1^{\text{compound}(1,1)}]$ for various sizes a of the region Λ. We see from this table that the results become less accurate as the size of the region Λ increases. However, for each fixed a the results obtained using (53), $I_1^{\text{basis}(n)}$, and using the compound formula (59), $I_1^{\text{compound}(1,1)}$, converge to the exact value of the integral for $n \to \infty$, as illustrated in Tables XVI and XVII. We see from these tables that the compound approximation formula ensures a higher rate of convergence of the approximations. The elementary approximation formula (58) with $m=1$ does not give good approximations in this case, because the functional (60) is not close to a functional multinomial of third degree. However, we note that for $m=2$ (fifth-order accuracy), this formula already gives the exact value of this integral for arbitrary size of the region Λ.

Let us now consider the calculation of the integral

$$I_2 = \int_{\mathcal{S}'(\Lambda)} \exp\{-\langle l,\varphi\rangle\} d\mu_K(\varphi)$$

using (53) $[I_2^{\text{basis}(n)}]$, the elementary formulas (58) of third-order accuracy $[I_2^{\text{simple}(1)}]$ and fifth-order accuracy $[I_2^{\text{simple}(2)}]$, and the compound formulas (59) of third- and fifth-order accuracy $[I_2^{\text{composite}(1,n)}$ and $I_2^{\text{composite}(2,n)}]$. The results are given in Table XVIII. We see that the approximations converge for $n \to \infty$ to the same number 1.060, which according to the convergence theorem should be considered the exact value of the integral.

TABLE XVI. Convergence to the exact value of functional-integral approximations I_1 for size of the region $a=1$.

	$I_1^* = 4.068\times10^{-2}$; $I_1^{\text{simple}(1)} = 1.98\times10^{-2}$	
n	$I_1^{\text{basis}(n)}$	$I_1^{\text{composite}(1,n)}$
1	0.673×10^{-2}	4.133×10^{-2}
2	3.520×10^{-2}	4.105×10^{-2}
3	3.849×10^{-2}	4.098×10^{-2}
4	3.850×10^{-2}	4.097×10^{-2}
5	3.903×10^{-2}	4.096×10^{-2}
10	4.053×10^{-2}	4.070×10^{-2}
15	4.063×10^{-2}	4.069×10^{-2}
20	4.065×10^{-2}	4.068×10^{-2}

TABLE XVII. Convergence to the exact value of functional-integral approximations I_1 for size of the region $a=10$.

	$I_1^* = 4.129\times10^{-2}$; $I_1^{\text{simple}(1)} = 6.30\times10^{-2}$	
n	$I_1^{\text{basis}(n)}$	$I_1^{\text{composite}(1,n)}$
1	4.07×10^{-13}	7.29×10^{-2}
2	2.19×10^{-10}	6.30×10^{-2}
10	5.56×10^{-4}	6.04×10^{-2}
20	2.27×10^{-2}	5.52×10^{-2}
30	3.86×10^{-2}	4.22×10^{-2}
40	3.91×10^{-2}	4.21×10^{-2}
50	3.93×10^{-2}	4.20×10^{-2}
60	4.03×10^{-2}	4.16×10^{-2}
70	4.09×10^{-2}	4.13×10^{-2}
80	4.10×10^{-2}	4.13×10^{-2}

CONCLUSION

Analysis of the literature of the last few years and the experience of participating in international conferences devoted to functional integrals show that the method of functional integration is finding ever broader application in various areas of science. The following two important trends can be seen in the use of this method in quantum physics. On the one hand, methods are being developed which use a lattice regularization scheme. These include searching for new modifications of the action functional with improved continuum properties, and improvement of lattice computational algorithms, including the use of parallel processors. On the other hand, there is steady development of methods based on nonperturbative functional regularization and studies performed directly in the continuum. As has widely been noted, this approach is attractive because it allows the study of general nonperturbative phenomena in quantum field theory. This approach is promising computationally because it does not encounter the numerous serious (and still not fully resolved) difficulties which arise in the lattice approach, in particular, the problem of the existence and uniqueness of the continuum limit and the problem of obtaining different results when different discretizations are used. There are also no problems associated with the appearance of metastability and slowing of convergence of the iterations as the lattice

TABLE XVIII. Approximate values of the functional-integral I_2.

	$I_2^{\text{simple}(1)} = 0.74$; $I_2^{\text{simple}(2)} = 0.622$		
n	$I_2^{\text{basis}(n)}$	$I_2^{\text{composite}(1,n)}$	$I_2^{\text{composite}(2,n)}$
1	1.023	0.888	1.357
2	1.055	1.029	1.124
3	1.058	1.046	1.088
4	1.058	1.052	1.082
5	1.059	1.059	1.073
6	1.060	1.060	1.066
15	1.060	1.060	1.061
50	1.060	1.060	1.060

spacing tends to zero, and there are no lattice artifacts, which in some cases lead to incorrect results (for example, the appearance of false "lattice" instantons). Some authors think that in lattice quantum chromodynamics the instanton sizes roughly correspond to the spacing of the lattices used for calculations at present (even when supercomputers and parallel computers are used), and that this, in particular, may be the reason for the strong disagreement between the values of the topological susceptibility obtained by different methods. Recently, special attention has been paid to the rigorous definition of the functional integrals of quantum physics on the basis of the concept of cylinder sets, the Gaussian measure, and so on (see Ref. 92). The modern treatment of the foundations of quantum field theory is also based on the method of functional integration.[307] The method of approximate calculation of functional integrals that we have developed within this approach makes it possible to obtain mathematically rigorous physical results with computational error which is specified *a priori* (by theorems). It is most useful in cases where the problem is highly sensitive to the discretization length, for example, in the study of singularities like first- and second-order phase transitions; where the introduction of a lattice can spoil the original topology of the space; in the study of systems of high dimensionality (including systems with a multiparticle interaction, since in this approach no problems arise with the inversion of dense matrices of high order), and so on. Comparison of the numerical results shows that the multiplicity of the ordinary (Riemann) integrals arising when our approximation formulas are used is 1–2 orders of magnitude lower than in the lattice Monte Carlo method (in the quantum-mechanical problems studied, the numbers are 2–3 versus 100–200), which ensures significant savings in required computer memory and makes it possible to use the more advantageous deterministic methods, which give a definite rather than a probabilistic error. Here the computing time turns out to be an order of magnitude smaller than in the Monte Carlo method. At the present time we are working on the further development of the method and its generalization to systems with nonzero spin and models of quantum field theory of dimensionality $d>2$.

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