Resonances in subatomic physics and principles of similitude and dimension

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Experimental data on hadronic resonances, dibaryonic resonances, the Darmstadt effect, and the ABC effect are systematized and discussed in detail from the point of view of the methods of similitude, dimension, and automodelity (scaling). A physical analogy between α decay and the decay of resonances in subatomic physics is advanced. On the basis of this analogy, the widths are reproduced for a large group of hadronic resonances. A diffraction approach to the analysis of resonance effects in the region of intermediate energies is presented. Approaches for further development of such investigations are discussed. © *1996 American Institute of Physics.* [S1063-7796(96)00301-8]

1. INTRODUCTION

The problem of constructing a theory of hadronic resonances is still, even after several decades, one of the most topical problems in modern elementary-particle physics. The fact is that, despite the impressive successes of quantum chromodynamics, calculation from first principles of the masses, widths, and magnetic moments of hadronic resonances is still impossible, since quantum-field perturbation theory is not at all suitable for these purposes in the range of intermediate energies in which we are interested. The currently most widely used practical methods of analyzing strong processes[1–9] (effective field approaches,[3–6] the Glauber approximation,[7] etc.) are model approaches in the sense that the calculated expressions which they yield contain a large number of phenomenological parameters. The model dependence of intermediate-energy physics is to a large degree due to the absence of small parameters like a dimensionless coupling constant.[5] In addition, the momentum transfers q characteristic of the physics of hadronic resonances are in general not small compared with the momentum P of the incident particles. In the intermediate-energy range, the condition of the long-wavelength approximation $kR \ll 1$, where k is the wave vector of the incident particle and R is the characteristic target radius, is not satisfied.

However, it is not only the absence of suitable small parameters that is responsible for the phenomenological nature of the theory of strong processes. It is well known that even for low-energy nuclear physics it is impossible to construct a theory from first principles. In the case of high energies, some hard processes can be successfully described in the framework of QCD.[9] However, on going to the intermediate-energy range, we encounter nonperturbative problems of QCD that are exceptionally complicated from both the physical and the computational point of view.

As a result, all currently known approaches in elementary-particle physics are, except for the purely kinematic approaches,[8] hybrid in nature. They are based both on fundamental principles of quantum theory and on phenomenological parametrizations of certain additional form factors that occur in calculations of the various observable quantities.

The problem is complicated by the fact that the degree of completeness of the system of first principles of quantum theory is unknown, while the meson theories traditionally employed in intermediate-energy physics (see, for example, Ref. 6) are nonrenormalizable.

During the last 30 years, a huge number of specific reactions have been investigated, and numerous models that reproduce the experiments fairly well have been developed. However, on account of the phenomenological nature of them noted above, these models and approaches themselves require additional investigation in order to clarify the physical picture of the phenomena. Thus, there arises the problem of analyzing the general physical concept of a resonance (quasistationary state), both in elementary-particle physics and in standard nuclear physics.

Historically, the first problem of finding the energy and lifetime of a quasistationary state in quantum mechanics was the theory of α decay. In 1928, Gamow[10] and Condon and Gurney[11] gave a theoretical explanation of the Geiger—Nuttall law[12] on the basis of tunneling-effect ideas. Moreover, from the very first study, the decay of any quasistationary state in quantum theory was treated as a two-step process. In the first stage of the decay, a cluster is formed in the interior region of the quantum system, and in the second stage this cluster tunnels as a single entity through a potential barrier. In accordance with this picture of a two-step decay of quasistationary states in quantum mechanics there arose several popular approaches to the description of such processes.[13] In one way or another, almost all of them involve factorization of the expression for the width of the quasistationary state. Moreover, they usually employ modifications of the R-matrix theory of nuclear reactions of Wigner and Eisenbud[14] with real boundary conditions at the "radius" of the reaction channel[15] or with complex boundary conditions (radiation conditions).[16]

Formally, R-matrix theory is rigorous. In principle, it can be used to describe all types of nuclear reactions. Indeed, the results of application of R-matrix theory to the description of various resonance reactions in nuclear physics are impressive (see, for example, the monographs of Refs. 17 and 18), and this theory has become a powerful tool for interpreting nuclear data.

It must be emphasized here that the R-matrix interpretation of experimental data on resonant nuclear reactions is unambiguous only for the description of narrow resonances. Moreover, it can be shown[19] that all approximate methods of calculating the positions and widths of narrow ($\Gamma \to 0$) resonance states are equivalent. For broad (overlapping) resonances, unique determination of a resonance (the resonance parameters) is impossible, since in this case there arises the unresolvable problem of separating the background from the resonance; the resonances overlap, and, what is more, the resonance parameters themselves become functions of the energy. Thus, the problem of the theoretical description of the positions and widths of broad resonances becomes a multiparameter and model-dependent problem.

In elementary-particle physics, one is dealing with broad resonances in the majority of cases. As a standard example, let us consider the scattering process

$$\pi + N \to N^* \to \pi + N. \quad (1)$$

The generally accepted interpretation of this reaction is that in the process of the collision strongly excited intermediate states of a nucleon N^* with widths $\Gamma \approx 100-600$ MeV and lifetimes $\tau_R \approx (6-1)10^{-24}$ s are formed. Such states have become known as nucleon (baryonic) resonances. The lifetime τ_R of the nucleon resonances is approximately equal to or somewhat greater than the time required for the particles to pass through the interaction region (transit time): $\tau_{\rm tr}$ ($\tau_{\rm tr} \approx d/v$, where $d \approx 1F$ and v is the relative velocity of the colliding particles). Nevertheless, such a definition of resonances is helpful, since some of the properties of the resonances (quantum numbers) have a similarity to the properties of narrow resonances and stable particles, and they permit a semiquantitative description of processes of the type (1) in the language of models taken from nuclear physics.[22,23] Such an approach made it possible to systematize hadronic resonances on the basis of modern (quark) models.[24]

Several hundred hadronic resonances were classified on the basis of the well-known naive quark model [the model of $SU(3)$ supermultiplets].[24] In this model, mesons are regarded as bound states of valence quark–antiquark pairs ($M = q\bar{q}$), while the baryons consist of three valence quarks ($B = qqq$). However, the limitations of this naive quark model soon became clear (see Refs. 24 and 25). Therefore, more complicated states were then introduced into the model: multiquark states ($M = qq\bar{q}\bar{q}$, $B = qqqq\bar{q}$), glueballs ($M = gg, ggg$), and hybrids ($M = q\bar{q}g$, $B = qqqg$). All such new particles were called exotic hadrons. As a result, the model became a multiparameter model, unambiguous, and cumbersome. For example, even in a simple case ($M = qq\bar{q}\bar{q}$) modern calculations[26] are restricted to $(q^2\bar{q}^2)_{L=0}$ configurations.

It has now become obvious that modern models are incapable of describing many resonance effects in subatomic physics: dibaryonic resonances,[27–30] resonances in two-pion systems (the ABC effect),[31,32] resonances in e^+e^- systems observed in heavy-pion collisions,[33] etc.

Thus, the modern development of quark models proceeds in complete analogy with the historical development of the shell model in nuclear physics: First, simple quasiparticle states were considered, then multiquasiparticle states, and, finally, collective states—coherent superpositions of many simple quasiparticle configurations. It was found that the collective modes of motion in nuclei can be considered in the language of simple collective variables,[18,34,35] and this observation gave a powerful stimulus to the understanding of the exceptionally complicated spectrum of nuclear states.

Properly speaking, what we see here is the development of a certain culture of analysis and interpretation of experimental data and the search for methods of separating the contributions of different mechanisms to the cross sections of strong processes on the basis of model-independent criteria.

The emergence of many of the problems in this complex can be traced to the end of the twenties and beginning of the thirties, i.e., to the period of the formulation of the fundamental principles of quantum theory. Issues related directly to the physics of baryonic resonances began to appear after 1953, i.e., after the discovery by Fermi's group of the $\Delta(1232)$ isobar.

During the subsequent 40 years, an uncountable and unsurveyable number of studies on the physics of hadronic resonances were published. As a result of the analysis of the properties of these resonances, an entire zoo of particles ($N^*, \Delta, \Lambda, \Sigma, \Xi, \Omega, \ldots$) has been discovered together with corresponding quantum numbers such as strangeness, hypercharge, baryon number, etc. Moreover, it can be said that the entire quark problem, including QCD, has ultimately grown out of the physics of resonances.

Despite the huge number of impressive results obtained since the discovery of the Δ isobar, an understanding of the true role of this resonance in intermediate-energy physics began to appear only at the beginning of the eighties, when it became clear that many effects previously interpreted as manifestations of short-range nucleon–nucleon correlations are in fact associated with the contribution of baryonic resonances [in the first place, the $\Delta(1232)$ isobar]. By now there exist strong grounds for assuming that this resonance in intermediate-energy pion nuclear physics survives as a separate baryonic species even in a strongly interacting nuclear environment and can be regarded as a quasiparticle in exactly the same way as the nucleon.[6]

A semiphenomenological approach to the description of the mass spectrum of hadronic and dibaryonic resonances was proposed in Refs. 36–40. It was found that the simple mass formula (24) (see below) is similar, both in its physical nature and in its method of derivation, to the Balmer formula in the old quantum theory of Niels Bohr and describes not only the gross structure (i.e., the position of the "centroids" of multiplets) of the spectrum of hadronic masses but also the positions of individual levels with a precision not less than their widths. This result was to a high degree unexpected, since the Balmer-like mass formula did not contain any fitting parameters at all. Moreover, several new resonances were predicted.

In physics, we know a comparatively small number of examples of so-called coarse systems, the integrated properties of which can be excellently described in very simple approaches, whereas the accuracy in the description of the physical picture becomes worse as the picture is examined in more detail. Such behavior of systems in classical mechanics was first discovered by Henri Poincaré, who actually proposed the felicitous name for such systems. From the mathematical point of view, the lowering in the accuracy of description of a physical system with increasing number of model parameters has been very well studied in the theory of approximation by polynomials of tabulated functions. It is well known that with increasing degree of the approximating polynomial the approximation gets out of control in the intervals between the reference points, and the problem of correct extrapolation of experimental data becomes altogether impossible. In physics, a classical example of correct treatment of coarse systems is the theory of diffraction.

Finally, we mention one further fundamental approach in high-energy physics: the unification of the principle of self-similarity (automodelity) with the general principles of quantum theory. It was in this way that the famous quark counting rules were established by Matveev, Muradyan, and Tavkhelidze.[41] This approach was further developed in studies of Baldin[42] devoted to the relativistic theory of dynamical systems. In this review, we present the applications of the methods of the theory of similitude, dimension, and automodelity to the problem of calculating the mass spectrum of resonances in subatomic physics.

2. DEFINITION OF RESONANCES

A unique and generally accepted definition of quantization conditions can be given only for closed (isolated) quantum-mechanical systems. Further, a discrete spectrum exists only for systems that have finite motions, for which the corresponding wave functions at the origin ($r=0$) and at infinity ($r\to\infty$) are equal to zero. Examples of closed systems possessing a discrete spectrum are atoms, nuclei, etc. For the calculation of the discrete spectra of quantum systems, there have been proposed several semiclassical forms of the theory (including Bohr's old quantum theory) based on the correspondence principle. In the theory of waves and oscillations there is the well-known general physical principle of duality (reciprocity), which states that the result is invariant if the positions of the observer and source are interchanged. Mathematically, this means that the corresponding boundary-value problem for closed systems is self-adjoint. A widely adopted method for solving such boundary-value problems is Fourier expansion with respect to the natural modes of the system. In this way one obtains a simple model of a closed system as a set of independent oscillators.

Ehrenfest showed that for systems possessing periodic motions the quantities

$$\oint P_i dx_i = I_i \qquad (2)$$

are invariants (adiabatic invariants) of the motion under a slow (adiabatic) variation of the system parameters. Here P_i is a generalized momentum, and x_i is a generalized coordinate. Precisely this fact played a decisive role in the establishment of the quantization rules for hydrogen-like atoms in the "old" Bohr–Sommerfeld quantum theory (already before the work of Schrödinger and Heisenberg):

$$\frac{1}{2\pi}\oint P_i dx_i = n\hbar, \qquad (3)$$

where n is the so-called principal quantum number, which takes only integer values: 1,2,3,.... In other words, the Bohr–Sommerfeld quantization condition is a quantization condition for an adiabatic invariant.

Following de Broglie, the quantum conditions of the Bohr–Sommerfeld theory can be interpreted on the basis of wave ideas. According to this theory, the quantization condition for the angular momentum is

$$L = rP = n\hbar = n\frac{h}{2\pi}, \qquad (4)$$

where $P=mv$ is the electron momentum. The relation (4) can be rewritten as

$$2\pi r = n\lambda_D, \qquad (5)$$

where λ_D is the de Broglie wavelength:

$$\lambda_D = \frac{h}{P}. \qquad (6)$$

The left-hand side of (5) is the perimeter of a circle, so that the Bohr–Sommerfeld quantization condition means that an integer number of de Broglie wavelengths fits around this perimeter. Thus, the quantization condition (6) is identical to the condition for the existence of normal modes for closed microscopic and macroscopic systems that execute finite motions.

The quantization condition (5) admits one further elegant interpretation if we introduce the quantum wavelength

$$\lambda_C = \frac{\hbar}{mc}, \qquad (7)$$

and we rewrite (5) in the form

$$\lambda_D \alpha = n\lambda_C, \quad v = \alpha\frac{c}{n}, \qquad (8)$$

where $\alpha = e^2/\hbar c$ is the fine-structure constant (here and in what follows, $\lambda = \hbar/P$). The system can be quantized if the de Broglie wavelength is comparable with the Compton wavelength divided by the fine-structure constant α.

Resonances are among the most remarkable effects in microscopic and macroscopic physics. In their simplest manifestation, they lead to sharp peaks in the total scattering cross section as a function of the energy of the colliding particles. Many different models have by now been proposed for the explanation of resonance effects (see the reviews of Refs. 19–21). A feature common to these models is that well-defined maxima of the cross sections at $E=E_R$ are associated with the existence of long-lived almost stable states consisting of the target and the incident particle. If the energy of the incident particle is E_R, then it is "captured" in such a long-lived intermediate state. This state then decays, radiating the corresponding particles to infinity. Such an explanation of resonance scattering is standard. Another example of resonance effects is provided by the decay of radioactive nuclei or unstable particles. In a rigorous sense, a decaying system does not possess a discrete spectrum of energies, since the particles from it escape to infinity, i.e., the motion of the system is infinite.

Certain decaying systems have a low decay probability. For such systems, the concept of "quasistationary states" is usually introduced. In such states, the particles move for a long time interval τ in a bounded region of space and then leave this region and escape to infinity. For such states, one can introduce the concept of the density of states in the continuum[43,44] in the form

$$\rho(E)=\frac{1}{\pi}\frac{d\delta(E)}{dE}, \tag{9}$$

where $\delta(E)$ is the scattering phase shift. The energy spectrum of such states is "quasidiscrete," and the width Γ determines the lifetime τ of the system in the bounded region of space in accordance with the relation

$$\tau=\frac{\hbar}{\Gamma}. \tag{10}$$

Such a definition of "quasistationary" states as almost bound states is consistent with the correspondence principle. Indeed, in the limit $\Gamma\rightarrow 0$ the density of quasistationary states (9) goes over[19] into the density of states of the discrete spectrum:

$$\lim_{\Gamma\rightarrow 0}\rho(E)=\delta(E-E_\nu), \tag{11}$$

where E_ν are the energies of the discrete spectrum of the system.

Bound states for closed systems are considered by solving the Schrödinger equation with zero-value boundary conditions at infinity. Unbound systems decay, radiating particles to infinity. It therefore appears natural in this case to seek solutions of the Schrödinger equation that are outgoing spherical waves at infinity, thus adequately reflecting the physics of the problem. Since in this case the boundary conditions are complex, the energy eigenvalues are also complex. In fact, a method of complex eigenvalues was first employed by J. J. Thomson[45] in 1884, who used this method to study the problem of the electromagnetic oscillations of a charge on a perfectly conducting sphere. A natural generalization of the solutions of the Schrödinger equation corresponding to discrete bound states for positive (complex $E=E_R-i\Gamma/2$, with $E_R>0$ and $\Gamma>0$) energy was introduced by Gamow[10] in the description of α decay. Like bound states, states with complex eigenvalues have energies that correspond to poles of the S matrix. A representation of the S matrix by means of its poles was given by Peierls.[46] A theory of nuclear reactions in which these poles are used to expand the reaction amplitude was developed by Humblet and Rosenfeld.[47] Later, the method of complex eigenvalues found wide application in many branches of physics,[48] and in the theory of diffraction and propagation of electromagnetic waves[49] and in the theory of resonators[50] it has become standard.

Since the Gamow functions increase exponentially with increasing r, the ordinary definitions of normalization, orthogonality, and completeness are inapplicable for them. Several methods were later proposed for normalizing Gamow functions, but they are all equivalent to Zel'dovich's method[51] and are inapplicable for the functions of antibound states and for functions with $|\mathrm{Re}\ k_n|<|\mathrm{Im}\ k_n|$. Much more recently,[19] we proved completeness of the Gamow wave functions in a bounded region of space $r\leqslant R$, where R is the distance over which the nuclear forces can be set equal to zero. In the proof of the completeness, the Mittag-Leffler theorem in the Cauchy formulation[48] played an important role. Thus, the continuum wave functions, Green's functions, and S matrix can be expanded in a Mittag-Leffler series, and the resulting series converge uniformly (for $r\leqslant R$).

The resonance effects associated with unstable quantum systems can be described by means of the complete system of Gamow functions. Although we expand the wave function of an unstable system with respect to Gamow functions in a bounded region of space ($0\leqslant r\leqslant R$), it is actually known in the complete space, since the S matrix is expanded with respect to the same resonance poles. The basis Gamow functions have asymptotic properties (radiation conditions) adequate for the physics of the investigated problem, and therefore in this approach it is possible to investigate many important questions in the theory of unstable states for the example of exactly solvable models: the analytic properties of the wave functions of such states, determination of the mean lifetime of an unstable particle from the scattering of its decay products by one another, etc. Therefore, in the framework of this approach it is possible to investigate the accuracy of other approximate methods and to establish the limits of their applicability. We have already shown[19] that all approximate methods of calculating the positions and widths of narrow ($\Gamma\rightarrow 0$) quasistationary states are equivalent. In this case, the corresponding cross sections for mutual scattering of the decay products of an unstable particle are described by a standard formula of Breit–Wigner type, the parameters of which can be uniquely determined from the experimental data. The physical foundations of this derivation are obvious and are discussed in detail in Refs. 17, 19, and 52. We mention here that although the wave function of a quasistationary state with small width ($\Gamma\rightarrow 0$ and $E_R\gg\Gamma$) does formally belong to the continuous spectrum and in principle is extended to the whole of space, in actual fact it is localized in the interior region of the classical motion (i.e., in the region in which the short-range attractive potential acts), being separated from the exterior region by a broad and high potential barrier with low penetrability.[17]

This conclusion is not true for broad resonances. Even if such a broad resonance is isolated from others, there is still the problem of separating the resonance contribution from the background (which can be settled only in the framework of model assumptions), and the dependence of the resonance width Γ on the energy is unknown.

Thus, we must recognize that at the present time it is impossible to give a universal and exact definition of a resonance, since the characteristic properties of this phenomenon gradually disappear with increasing width of the quasistationary state, and the overlapping resonance and inelastic threshold effects cannot be separated from each other.

In our view, the quantum-mechanical definition of resonances presented above is the most consistent one. Such a definition of resonances is now widely accepted in nuclear physics, and for 70 years it has also been used to interpret resonances in elementary-particle physics. Such a definition of resonances in elementary-particle physics was revived only a year or two ago.[23]

In practice, the definition of a resonance as a pole of the S matrix and the extraction of the parameters of this pole from experimental data require some further assumptions. It is usually assumed that the interaction has a finite range. It is then possible to introduce a time interval (this concept was first introduced by Wigner[53]) equal to the difference between the time of arrival and time of departure of the wave packet from the interaction region. In some cases, the time interval has a characteristic energy dependence with a sharp peak in a narrow energy interval. This delaying in time of the wave packet in the interaction region is interpreted as the formation of a short-lived unstable particle, which then decays into the corresponding open channels.[52,54,55]

The delay time can be calculated by means of the S matrix. Since the parameters of the resonance are usually determined in the framework of a phase-shift analysis (using an Argand diagram), one considers the quantities[23]

$$T(W)=\frac{1}{2i}\left(\eta(W)\exp[2i\delta(W)]-1\right), \tag{12}$$

$$SP(W)=\left|\frac{dT(W)}{dW}\right|, \tag{13}$$

where $\delta(W)$ is the real part of the phase shift, $\eta(W)$ is the absorption parameter, and W is the total energy in the center-of-mass system. The delay time is proportional to $SP(W)$. It is not always related to the lifetime of the unstable particle. The delay can be due to the opening of a new decay channel of the system (threshold effect) or to the formation of a virtual state (for details, see Ref. 17). A classical example of a virtual state is provided by low-energy proton–neutron scattering in the ^1S state.

We emphasize that a delay time is a necessary but not a sufficient condition for the occurrence of a resonance. The positions of resonances are then associated with poles of the function $T(W)$ situated not far from the real axis. The function $T(W)$ is usually parametrized by a Breit–Wigner formula with mass M and constant Γ; then the maximum value of the delay time will be at $W=M$. A last subtle detail is that the resonance must also be observed in other decay channels.

The results of investigations of the excited states of the nucleon on the basis of the concepts presented above are well known.[23] A different, so-called standard method is adopted by the Particle Data Group.[24] A comparative analysis shows that there is a fundamental difference between the results of analysis of the two groups; this is important for modern quark models of resonances. It must be emphasized that there are large differences between the results obtained by different groups (see the compilation in Ref. 24). In our view, this indicates a model dependence of the results of analysis of the experimental data on resonances and imperfection of the modern approaches to the description of resonance effects in elementary-particle physics. We must mention here the general imperfection of the theory of resonances with large width $E_R \sim \Gamma$, not only in elementary-particle physics but also in nuclear[19,56] and atomic[57] physics.

3. ASYMPTOTIC QUANTIZATION

The observation of bright sharp lines is one of the most interesting effects in geometrical optics. These lines, which are due to concentration of the light rays (the light rays are tangent to them) were called caustics. A classical example of a caustic is the rainbow. An explanation of this effect was apparently first given already in 1857 by the Reverend Hamnet Holdich (see Ref. 58). In the case of the rainbow, the reflected light rays have an envelope, which is the caustic. The standard explanation of the brightness of a caustic is very simple: The light rays that form the caustic are tangent to it and are almost coincident, i.e., infinitely close rays accumulate on the caustic. As a consequence of this, many more rays are concentrated in a restricted region of space than in the other regions, and this explains the brightness of the caustic. It should be recalled that although the classical description of a rainbow in the language of geometrical optics correctly gives the position and shape of the caustic it does predict an infinitely high intensity of the rainbow, and this is in principle incorrect.

It is important for us that a caustic divides space into two parts: one part filled with rays (the illuminated part) and another part, into which the rays do not penetrate (caustic shadow). In the illuminated part, two rays pass through each point—one of them touches the caustic, and the other does not. As the caustic is approached from the side of the illuminated region, there is a growth in the field amplitude, i.e., a local maximum is observed. The field decreases on the transition through the caustic and with increasing distance from it into the shadow region. In the direction of the normal to the caustic, the field in the illuminated part of space has the nature of a standing wave, which is due to the interference of the two ray fields. Along the caustic, the field has the nature of a traveling wave.

In the theory of diffraction of electromagnetic and acoustic waves, there exist so-called glancing waves,[59] which "glide" along the caustic (are concentrated on the caustic) and continuously give up their energy to the surrounding space. The existence of glancing waves has not been proved in the general case, but if the variables separate, they can be constructed explicitly.[59] In the theory of open resonators (in particular, in the theory of antenna emission), such waves play a fundamental role.[50] An analog of glancing waves for closed systems can be found in so-called whispering-gallery waves, which were first discovered by Rayleigh (1910)[60] when considering the phenomenon of whispering galleries in acoustics for a circular region. These waves are localized near the surface of the gallery as a result of their reflection from the boundary at a glancing angle.

Glancing waves and whispering-gallery waves have been studied in detail,[17,59,61] and the condition for their existence in the case of spherical symmetry is fulfillment of the relation

$$L \approx kr_0, \tag{14}$$

where L is the index of the Bessel–Hankel function, kr_0 is its argument, and the circle $r=r_0$ is the caustic for the glancing waves. For the whispering-gallery waves, the circle $r=r_0$ is the wall of the gallery. It is obvious that the condition (14) is none other than the condition of conservation of the corresponding adiabatic invariant in classical mechanics, in the given case conservation of the angular momentum L. In quantum mechanics, this means quantization of the angular momentum $L\hbar$.

This well-known fact can be most transparently illustrated in the semiclassical approximation, in which the wave function can be written in the form[17]

$$\psi(q_i,t)=A(q_i)e^{iS(q_i,t)/\hbar}, \tag{15}$$

where $S(q_i,t)$ is the total action, and $A(q_i)$ is a single-valued function of the coordinates q_i. It follows from the requirement that the wave function be single-valued that for any closed curve in the q_i space the following equation must hold:

$$\Delta S = \oint P_i dq_i = 2\pi\hbar n_i. \tag{16}$$

This is a quantization condition with integers, but it is not always true. In the case of an oscillator, n_i must be replaced by $n_i+1/2$. In the general case, the quantization condition has the form

$$\Delta S = \oint P_i dq_i = 2\pi\hbar(n_i+\gamma_i), \tag{16a}$$

where γ_i is a number of order unity that depends on the nature of the boundary conditions for the considered degree of freedom.

Thus, strongly localized waves arise at the interface of two media. A classical example of this phenomenon is provided by refraction waves[62] in hydrodynamics that arise when a spherical wave is reflected at an interface of two media with different densities. The nature of the inhomogeneities of the medium governs the manner in which the scattering of waves by such singularities of the medium occurs. Neither in classical nor in quantum theory is there a general result. Very promising observations are accumulating in catastrophe theory[63] and in the theory of chaotic systems and their semiclassical quantization,[64] but these issues go beyond the ambit of our review. Here we merely mention that a paper of Einstein,[67] which was forgotten and discovered 40 years later by Keller[65] (see also Ref. 66, where many interesting examples are given), has a direct bearing on the understanding of the Bohr–Sommerfeld quantization condition. Einstein introduced a quantization condition on a torus, namely, he required continuity of the wave function for any closed curve on the torus. This quantization is now called EBK quantization (from the first letters of the authors: Einstein, Brillouin, Keller) and takes the form

$$\Delta S = \oint P_i dq_i = 2\pi\hbar\left(n_i+\frac{\beta_i}{4}\right), \tag{16b}$$

where $n_i \geqslant 0$ and $\beta_i \geqslant 0$ must be integers. The integer β_i is called the Maslov number.[68,69] The Einstein–Brillouin–Keller approach is very general, and the conditions of EBK quantization are also valid in cases when the variables do not separate. Practical calculations of Maslov numbers have been made comparatively recently.[70]

In nuclear physics, it is now a widely held view[18,71,72] that semiclassical quantization of motion in multidimensional periodic orbits plays a significant role in the appearance of the appreciable decreases and increases in the densities of nuclear levels.

As a general conclusion of our discussions, we advance as a working hypothesis the following physical definition of a resonance: It is a periodic motion and reflection of waves in a bounded region of space and their interference leading to strong localization of waves near inhomogeneities of the medium. In the framework of the R-matrix theory of nuclear reactions,[15] we use a radiation boundary condition for the physical particles into which the considered system can decay. At the same time, the complete space is divided into two parts: an interaction region, where the system "lives" for a certain time, and an asymptotic region, where the decay products of the unstable particle do not interact with one another. In accordance with the concept of this theory, it is possible to introduce a complete system of wave functions in the interaction region $r \leqslant r_0$, and therefore it is possible to define logarithmic radial derivatives for the interior wave functions:

$$\left.\left(\frac{r}{u_{\rm in}(r)}\frac{du_{\rm in}(r)}{dr}\right)\right|_{r=r_0-0} = f = \frac{1}{R}. \tag{17}$$

These interior wave functions can be calculated in the framework of modern quark models; they must then be projected onto the physical observable states (this procedure is developed in detail in the framework of the P-matrix approach[73–76]) and fitted to the exterior wave functions.

For simplicity, we consider unstable systems that decay only through one dominant open channel. As we have already said, the decay of hadronic resonances can be treated in complete analogy with the radiation of open classical electromagnetic resonators.[50] Therefore, the corresponding mathematical formalism can be used with slight modifications (see the Appendix). Consequently, the boundary conditions for the decay products of the unstable particle must correspond to the radiation condition

$$\left.\left(\frac{r}{h_L^{(1)}(Pr)}\frac{dh_L^{(1)}(Pr)}{dr}\right)\right|_{r=r_0+0} = f, \tag{18}$$

where $h_L^{(1)}(Pr)$ are Ricatti–Bessel spherical functions. In the framework of our approach, it appears to us entirely natural to require that the interior wave function have a maximum at the point of fitting $r=r_0$ to the exterior wave function, i.e.,

$$\left.\frac{du_{\rm in}(r)}{dr}\right|_{r=r_0-0} = 0. \tag{19}$$

Physically, this means that we require maximum localization of the wave function of the unstable system on the surface $r=r_0$. In the approximation of the P-matrix approach, the boundary condition was the opposite one, namely, the interior wave function at the point of fitting $r=r_0$ to the exterior wave function had to vanish,

$$u_{\text{in}}(r)|_{r=r_0-0}=0, \tag{20}$$

in order to satisfy the confinement requirement.

Here we wish to make an important remark. No one has ever pointed out (except in our studies of Refs. 36–39) the fact that there is a deep physical analogy between the widely adopted models of open resonators in classical electrodynamics and in wave mechanics and the R-matrix approach for the description of resonance reactions in nuclear physics. Open resonators have a real physical surface that divides space into two parts: an interior part of the resonator, where the corresponding eigenstates of the resonator are generated, and the exterior part of the resonator to which the waves escape from the resonator. In the R-matrix approach, it is stipulated that the radius $r=r_0$ at which the interior and exterior wave functions are fitted must be chosen in such a way that the influence of the nuclear forces can be ignored. This is correct, but the requirement must be made more precise. The point is that when Einstein's invariant torus is constructed it is necessary to determine the caustic surfaces, and it can then be shown that the wave eigenfunctions are concentrated within these caustic surfaces. Thus, the caustics in problems involving the determination of normal modes of all systems play the role of "physical surfaces" that confine for a definite time the system within such a torus. We arrive at the conclusion that the radiation boundary condition must be satisfied near the outer caustic surface. Moreover, it can be shown rigorously[61] that the specification of definite boundary conditions on the channel radius $r=r_0$ uniquely determines the potential of the interaction between the decay products of the unstable system at this point. Finally, the condition for determining the eigenfrequencies of open resonators[50] is completely analogous to the quantization condition (16b) in accordance with the EBK method.

Thus, resonances in wave systems arise when the ratio of the radius r_0 of the resonating system ("cavity") to the corresponding wavelength λ is determined by an expression of the type

$$Pr_0=(n+\gamma); \quad n=1,2,3,\ldots, \tag{21}$$

$$2\pi r_0 \equiv r_{\text{eff}}=(n+\gamma)\lambda, \tag{21a}$$

where $P=2\pi/\lambda$, and γ is a number of order 1 ($0\leq\gamma\leq 1$) that depends on the boundary conditions for the considered degree of freedom and on the type of the wave equation for the resonating system. In the Appendix, we show that the quantization condition (21) for the asymptotic momentum for hadronic resonances can be obtained in the framework of the R-matrix approach by using a radiation boundary condition on the surface of a resonance that decays into two hadrons. We reach the conclusion that $\gamma=0$ or $1/2$ (one can advance arguments[67] indicating that in principle the cases $\gamma=1/4$ and $3/4$ can also occur). The case $Pr_0=n+1/2$ is interpreted as radial quantization, while the case $Pr_0=L$ can be regarded as the well-known Bohr–Sommerfeld orbital quantization condition. We shall attempt to analyze the spectrum of the Δ isobars from this point of view.

TABLE I.

Resonance	P_π (c.m.s.) (calc.) GeV/c	$1/\tilde{P}_\pi$ (c.m.s.) F	$\frac{\langle r_0\rangle}{n+\gamma}$ F	$n+\gamma$
$\Delta(1232)$	0.227	0.86	0.86	1
$\Delta(1620)$	0.526	0.38	0.43	2
$\Delta(1950)$	0.741	0.27	0.29	3
$\Delta(2420)$	1.023	0.19	0.21	4

If we ignore the effects of the L dependence and the spin–orbit interaction and consider only the gross structure of this spectrum, it may be noted that we are dealing with four multiplets separated from each other by about 400 MeV. The "quasioscillator" nature of the gross structure of the Δ spectrum forces us to suggest that the Δ-resonance state of the πN system can be approximately described by means of an oscillator potential with parameter $\langle r_0\rangle=\sqrt{\hbar/m\omega}\approx 0.86$ F, which is close to the nucleon electromagnetic radius.

It can be seen from Table I that for the pions produced by the decay of the Δ isobars the values of the c.m.s. momenta are consistent with such an estimate made under the assumption that $\gamma=0$ and are practically identical with the analogous values of the momenta of the pion beams that excite the Δ resonances. In addition, the momenta of the emitted pions are to a good degree of accuracy integer multiples of the characteristic momentum $\tilde{P}_\pi^{(0)}\approx 0.23$ GeV/c determined by the nucleon radius $\langle r_0\rangle=1/\tilde{P}_\pi^0(\text{c.m.s.})=0.86$ F.

Similar calculations were made for the case of N^* resonances. Table II gives the systematics of N^* resonances based on the use of the "principal quantum number" n and "constant of the boundary conditions" $\gamma=1/2$. It is obvious that the absence of "gaps" in the systematics of the Δ and N^* resonances for $1\leq n\leq 4$ can be regarded as an indication of the possible existence of the hypothetical $N^*(1125)$ resonance with mass $m_{N^*}\approx 1115$–1130 MeV and width $\Gamma<30$ MeV, in agreement with the results of statistical analysis of the $\pi^- p$ cross sections (Fig. 1).

A distinctive feature of the nucleon–nucleon interaction with production of a Δ isobar is that: 1) it can be excited by a virtual and not a real pion; 2) the colliding nuclei are identical particles and, therefore, a minimum of two coherent processes contribute to the $NN\to N\Delta$ reaction.

TABLE II.

Resonance	P_π (c.m.s.) (calc.) GeV/c	$1/\tilde{P}_\pi$ (c.m.s.) F	$\frac{\langle r_0\rangle}{n+\gamma}$ F	$n+\gamma$
$N^*(1125)$	0.114	1.73	1.73	1/2
$N^*(1440)$	0.397	0.50	0.57	3/2
$N^*(1710)$	0.587	0.34	0.34	5/2
$N^*(2200)$	0.897	0.22	0.25	7/2
$N^*(2600)$	1.126	0.18	0.19	9/2

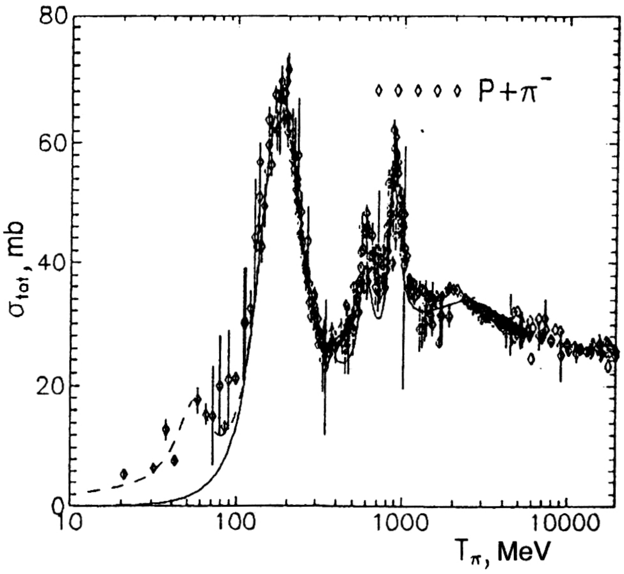

FIG. 1. Total cross sections of $\pi^- p$ scattering. The dashed curved is plotted with allowance for the contribution of the hypothetical $N^*(1125)$ resonance.

For example, the $p(p,n)_\Delta$ charge-exchange reaction can be described by a superposition of direct (D) and exchange (E) diagrams. Therefore, in contrast to free πN scattering, in reactions of the $(p,n)_\Delta$ type and analogous processes we must have the manifestation of at least two characteristic scales corresponding to the two invariant momentum transfers:

$$t_d \equiv (P_1 - P_n)^2 = (P_2 - P_\Delta)^2 \equiv t,$$

$$t_{ex} \equiv (P_2 - P_n)^2 = (P_1 - P_\Delta)^2 \equiv u,$$

where t and u are the standard Mandelstam variables, which in the scale of distances correspond to

$$r_d \equiv 1/\sqrt{-t_d},$$

$$r_{ex} \equiv 1/\sqrt{-t_{ex}}.$$

As can be seen from Fig. 2, for all the main isobars [$\Delta(1232)$, $\Delta(1620)$, $\Delta(1950)$, and $\Delta(2420)$] the values of r_d and r_{ex} for the "forward" (at angle $\theta=0°$) charge-exchange reaction are close only in the neighborhood of the threshold for production of a Δ resonance [at the threshold, $u=t$, $s=s_{min}$, where $s=(P_1+P_2)^2$, the third Mandelstam variable, is the square of the invariant mass of the system]. With increasing energy of the incident proton, r_d increases relatively rapidly, whereas r_{ex} decreases smoothly. In going from one Δ isobar to another, the position of the r_d curve as a whole changes abruptly [roughly speaking, in accordance with a $1/n$ law with $n=1,2,3,4$ for $\Delta(1232)$, $\Delta(1620)$, $\Delta(1950)$, and $\Delta(2420)$, respectively]. The curve of r_{ex} as a function of the projectile kinetic energy T_p also changes abruptly in going from one Δ isobar to another. However, the amplitude of this jump is much smaller, and for all the

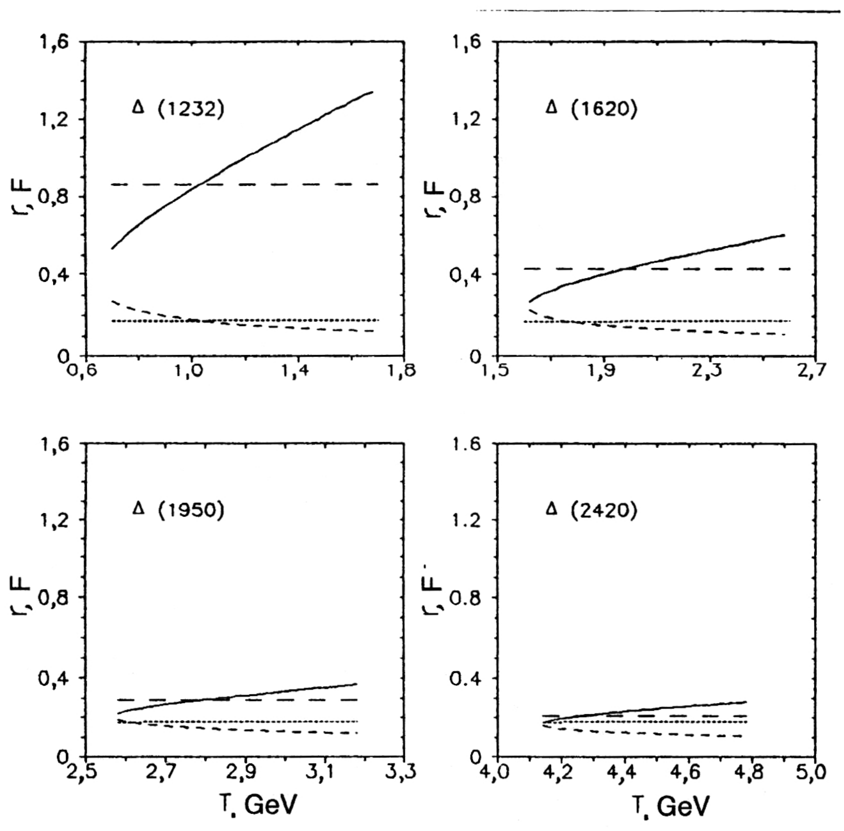

FIG. 2. Dependence of the scaling variables r_d and r_{ex} on the energy of the incident proton for typical Δ isobars belonging to the four main multiplets of baryons with isospin 3/2.

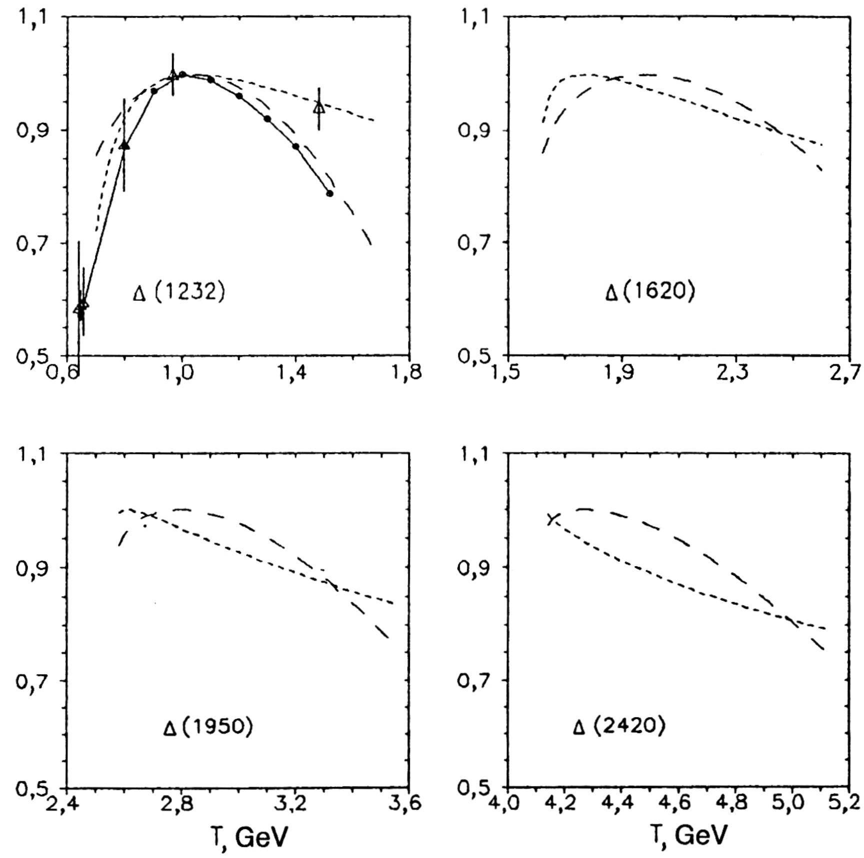

FIG. 3. Dependence of the characteristic functions δx_d and δx_{ex} on the energy of the incident proton for typical Δ isobars belonging to the four main multiplets of baryons with isospin 3/2. The experimental points in the graph for the Δ(1232) isobar describe the energy dependence of the cross sections of the reactions $pp \to n\Delta$ (circles) and $pp \to np\pi^+$ (triangles).

investigated cases r_{ex} is in the range $0.1 \leq r_{ex} \leq 0.3$ F, i.e., it is fairly close in order of magnitude to the characteristic "hard" radius $r_q \sim 0.2$ F of the nucleon. It is possible to interpret r_q as the radius of a hard core or of Jastrow correlations, as the radius of a constituent quark, as the nucleon Compton wavelength, or in some similar manner. In what follows, we shall speak of r_q as the radius of nucleon constituents without emphasizing any particular model interpretation.

In this connection, it is expedient to introduce the characteristic functions

$$\delta x_d \equiv 1 - \left[\frac{r_d - \langle r_0 \rangle}{\langle r_0 \rangle}\right]^2,$$

$$\delta x_{ex} \equiv 1 - \left[\frac{r_{ex} - \langle r_q \rangle}{\langle r_q \rangle}\right]^2.$$

As can be seen from Fig. 3, the energy dependence of δx_d and δx_{ex} has a definite resonance nature; for the Δ(1232) isobar, the positions of the maxima of δx_d and δx_{ex} coincide ($\langle r_0 \rangle = 0.86$ F, $\langle r_q \rangle = 0.18$ F). The $\delta x_d(T)$ and $\delta x_{ex}(T)$ resonance curves are well correlated, both with each other and with the following function, normalized to unity at the maximum:

$$\tilde{\sigma}_\Delta(T) \equiv \sigma_{p+p\to n+\Delta^{++}} / \sigma^{\max}_{p+p\to n+\Delta^{++}},$$

which describes the energy dependence of the total cross section for production of the P_{33} resonance.[77,78]

With increasing mass of the Δ isobar, the positions of the maxima of the functions $\delta x_d(T)$ and $\delta x_{ex}(T)$ move apart, the maximum of δx_{ex} rapidly approaches the threshold for production of a Δ resonance, and for Δ(2420) it enters the kinematically forbidden region.

This behavior of δx_d and δx_{ex} can be interpreted as follows. In the region of energies $T \approx 1$ GeV, a "direct" virtual pion resonates on the nucleon as a whole, and an "exchange" pion simultaneously resonates on a nucleon constituent. The constructive interference of the "direct" and "exchange" amplitudes leads to resonant enhancement of the cross section of the Δ isobar and also to an appreciable increase of the total and inelastic cross sections of the NN interaction.[77,78] With increasing beam energy, the "nucleon" and "constituent" resonances move apart, and therefore when $T_p > 3$ GeV the Δ(1232) isobar is weakly excited. In turn, the production of heavier Δ resonances in nucleon–nucleon collisions is always suppressed through the mismatch of the resonance conditions for the direct and exchange amplitudes.

The proposed two-scale model of the production of Δ isobars also makes it possible to explain qualitatively the termination of the spectrum of the Δ resonances. Indeed, at $n=5$ we see in accordance with Table I that there is no longer fulfillment of the condition

$$\frac{\langle r_0\rangle}{n+\gamma}\geqslant r_q, \tag{22}$$

i.e., the pion de Broglie wavelength becomes smaller than the nucleon Compton wavelength, and the wall of the "nucleon potential well" is screened. Therefore, the fact that the spectrum of baryonic resonances is bounded above is an additional indication of the viability of the two-scale model of resonant pion–nucleon scattering.

Thus, the selective excitation of the $\Delta(1232)$ resonance in the region of energies $T_p\sim 1$ GeV in the framework of the proposed approach can be regarded as a manifestation of a double shape resonance. From the point of view of formal scattering theory, such a resonance can be regarded as an anomalous enhancement of the process through the simultaneous fulfillment of the condition for interaction in the final (initial) state of two pairs of particles[79,80] (in our case, virtual pions with the nucleon and a constituent).

We regard a baryon as a resonating system that decays into a meson and a baryon with total mass less than the mass of the original baryon. The invariant mass of the original system is given by

$$\sqrt{s}=\sqrt{P_\mu^2+m_\mu^2}+\sqrt{P_b^2+m_b^2}, \tag{23}$$

where the indices μ and b correspond to the meson and baryon. In accordance with (21) and (23), the mass of the baryon resonance is determined by

$$M_n(B)=\sqrt{m_\mu^2+\left[\frac{n+\gamma}{r_0}\right]^2}+\sqrt{m_b^2+\left[\frac{n+\gamma}{r_0}\right]^2}+\Delta M_n, \tag{24}$$

where $\Delta M_n<\Gamma$ is a correction term to the mass formula and describes the dependence of $M_n(B)$ on the remaining quantum numbers $(I,l,J,P,S,...)$; by definition, it is equal to

$$\Delta M(I,l,J,P,S,C,....)=\sqrt{s}-\sqrt{m_\mu^2+\left[\frac{n+\gamma}{r_0}\right]^2}+\sqrt{m_b^2+\left[\frac{n+\gamma}{r_0}\right]^2}. \tag{25}$$

The formula (24) for the spectrum of baryonic masses is a physical analog of Balmer's formula and is also analogous to the von Weizsäcker formula for the nuclear masses in the sense that (24) contains a dominant term that describes the positions of the "centroids" of multiplets and also a correction term that depends on the particular quantum numbers of the resonance.

We emphasize once more that the main term in the mass formula (24) was obtained on the basis of a general condition for the occurrence of a resonance that does not depend on the particular form of the wave function, whereas model ideas must be invoked in order to calculate the correction ΔM_n.

All reactions in which the production and subsequent decay of a baryonic resonance occur by virtue of the strong interaction can be described by diagrams having the general form shown in Fig. 4. The symbols $m_{\rm in}$ ($m_{\rm out}$) and $B_{\rm in}$ ($B_{\rm out}$) denote the incident (outgoing) meson and baryon, while the thick line corresponds to the propagator of the baryonic resonance excited in the reaction

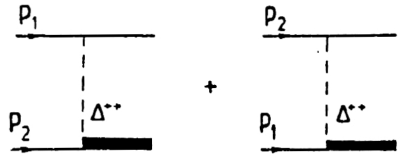

FIG. 4. Diagram of the production and subsequent decay of a baryonic resonance.

$$m_{\rm in}+B_{\rm in}\rightarrow\tilde{B}\rightarrow m_{\rm out}+B_{\rm out}. \tag{26}$$

The structure of (26) is universal in the sense that it is the same whether or not the incoming and outgoing particles are real or virtual and is independent of the method of detection of the final states. For example, in the case of the reaction $p+p\rightarrow n+\Delta^{++}\rightarrow n+p+\pi^+$ we can ignore the contribution of the diagrams in which the neutron is produced by the decay of the Δ isobar and describe the inclusive charge-exchange cross section by means of two diagrams (see Fig. 5).

Then, in accordance with Refs. 81 and 82,

$$\frac{d^2\sigma}{d\Omega dE_n}=\frac{2m^2P_n}{\lambda^{1/2}(s,m^2,m^2)(2\pi)^3}\Gamma_\Delta(s_\Delta)|G_\Delta(s_\Delta)|^2\times\langle 0|M(p+p\rightarrow n+\Delta^{++})|0\rangle^2, \tag{27}$$

where

$$\Gamma_\Delta(s_\Delta)=\frac{1}{6\pi}\left[\frac{f_{\pi N\Delta}}{m_\pi}\right]^2P_\pi^3\frac{m}{\sqrt{s_\Delta}}, \tag{28}$$

$$G_\Delta(s_\Delta)=\frac{2M_\Delta}{M_\Delta^2-s_\Delta-iM_\Delta\Gamma_\Delta(s_\Delta)}. \tag{29}$$

This formula is obtained by integrating over $d\Omega_\pi$ the analogous formula of Refs. 81 and 82 (we use below the notation, expressions, and terminology of Refs. 1, 2, 83, and 84), which is satisfied analytically if there are no decay diagrams in the Δ-isobar production amplitude $M(p+p\rightarrow n+\Delta^{++})$. At the same time, $\langle 0|M(p+p\rightarrow n+\Delta^{++})|0\rangle^2$ is the square of the modulus of the effective Δ-isobar production amplitude averaged over the spin projections; it depends both on the observed kinematic variables and on the kinematic variables of the virtual particles:

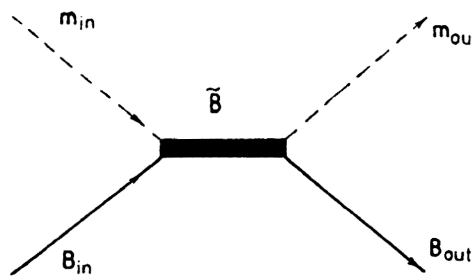

FIG. 5. Direct and exchange diagrams of the inclusive charge-exchange reaction $NN\rightarrow N\Delta$.

$$s_\Delta=3m^2+2mE_1-2mE_n-2E_1E_n+2P_1P_n\cos\theta_n, \tag{30}$$

$$t_1=-\mathbf{q}_1^2=2m^2-2E_1E_n+P_1P_n\cos\theta_n, \tag{31}$$

$$t_2=-\mathbf{q}_2^2=2m^2-2mE_n, \tag{32}$$

$$\cos\theta_{12}=\frac{m(m^2-s_\Delta)}{\sqrt{s_\Delta t_1 t_2}}, \tag{33}$$

$$s=2m(E_1+m), \tag{34}$$

where E_1 is the total energy of the incident proton, $P_1=\sqrt{E_1^2-m^2}$ is its momentum, and E_n and P_n are the energy and momentum of the detected neutron. As we have already noted, the expressions (27)–(34) were obtained in the spectator approximation[6] for the $\pi+\rho+g'$ model,[81,82] which is satisfactory for the reactions $p+p\to n+X$ and $n+p\to p+X$ only near the maximum of the Δ resonance.

The expression (27) contains the "nucleon" ($\langle r_0\rangle=0.86$ F) and "constituent" ($r_q\approx0.2$ F) radii in factorized form. Indeed, the resonance term $\Gamma_\Delta(s_\Delta)|G_\Delta(s_\Delta)|^2$ reaches a maximum at $s_\Delta=M_\Delta^2$, and in accordance with (23), (24), and Table I this is possible only in the case $n=1, \gamma=0, r_0=0.86$ F. Therefore

$$M_\Delta=\sqrt{m_\pi^2+\left[\frac{1}{r_0}\right]^2}+\sqrt{m^2+\left[\frac{1}{r_0}\right]^2}+\Delta M_\Delta, \tag{35}$$

and $\Delta M_\Delta \ll \Gamma_\Delta(M_\Delta)$. Thus, the "nucleon" radius $\langle r_0\rangle\approx0.86$ F is "hidden" in the propagator of the Δ isobar.

The Δ-isobar production amplitude $M(p+p\to n+\Delta^{++})$ contains the form factors $F_{\pi NN}$ and $F_{\pi N\Delta}$, in which the constituent scale parameter $r_q\approx0.2$ F is "hidden." (For example, for the JAIN set[82] the form factor $F_{\pi N\Delta}$ contains the cutoff parameter $\Lambda_\pi=1.2$ GeV, and the value $1/\Lambda_\pi\approx0.17$ F is in excellent agreement with our value $r_q\approx0.18$ F). Thus, the mathematical structure of the theory is such that significant enhancement of the cross section $d^2\sigma/d\Omega dE$ or of σ_{tot} is possible only when the resonance at the scale $\langle r_0\rangle=0.86$ F arises simultaneously with the resonance at the scale $r_q\approx0.2$ F. The expression (27) is the first term of the Mittag-Leffler expansion of the NN amplitude with respect to the resonances and can be generalized to the case of an arbitrary resonance by the substitution $M_\Delta\to M_n$ (n is the "principal quantum number").

We note that a resonance parametrization of nucleon electromagnetic form factors analogous to the vertex functions $F_{\pi NN}$ and $F_{\pi N\Delta}$ in both the form of the expression and physical meaning was widely used in the fifties.[85] In the same years, there was intensive investigation of a double-resonance model for electron–nucleon scattering.[86] However, in those studies the resonance form factors for the "core" and "resonance" were introduced additively. As a result, it was not possible to reconcile the existing experimental data with the first principles of quantum field theory.[87]

We summarize the results presented above. We have shown that the general physical (independent of the actual form of the wave equation) condition for the occurrence of resonance in wave systems makes it possible to describe the gross structure of the baryonic spectrum. We have also found that all baryons have approximately the same "radius" $\langle r_0\rangle=0.86$ F, which is close to the nucleon electromagnetic radius. This enables us to interpret them as shape resonances; moreover, only the $\Delta(1232)$ isobar can be regarded as a double shape resonance, and this eplains its unique distinction and selectivity in pion–nucleon and pion–nucleus physics.

The principal term in the mass formula (24) does not depend on the specific dynamical model. In the theory of low-energy potential scattering, there is the well-known effective-range approximation, which does not depend on the actual form of the interaction potential, and this enables us to regard Eq. (24) as a heuristic tool for searching for new low-energy approximations in quantum field theory (including QCD). We note in passing that Eq. (24) has a structure that is similar to the analogous formulas in the bag model,[88,89] and this may serve as a further argument for its having a deep physical basis.

4. ASYMPTOTIC METHOD FOR CALCULATING WIDTHS OF RESONANCES

The aim of this section is to generalize the formalism of Kadmenskiĭ to the case of the decay of short-lived quasistationary states in quantum mechanics and also to obtain asymptotic estimates of the widths of such states and to apply the new approach to the problem of estimating the widths of hadronic and dibaryonic resonances.[90] It should be noted that a semiclassical method for calculating the widths of short-lived quasistationary states was developed in Refs. 91 and 92. In what follows, we present the method of Ref. 90, which is based on analytic continuation of the solutions of the Schrödinger equation into the complex energy plane.

In the naive cluster model of α decay,[93] the width of a quasistationary level in the semiclassical approximation is determined by

$$\Gamma=\frac{P_\alpha\hbar^2K}{2mR}\exp\left[-2\int_R^{C/Q}dr\,k(r)\right], \tag{36}$$

where P_α is the probability of formation of an α particle, m is its reduced mass, R is the channel radius, $C=2(Z-2)e^2$ is the product of the charges of the α particle and of the daughter nucleus, Q is the energy of the α particle in the center-of-mass system of the parent nucleus, K and $k(r)$ are the wave vector of the α particle within the well and below the barrier, respectively,

$$K=\left[\frac{2m}{\hbar^2}\left(Q+V_N-\frac{C}{R}\right)\right]^{1/2},$$

$$k(r)=\left[\frac{2m}{\hbar^2}\left(\frac{C}{r}-Q\right)\right]^{1/2},$$

and V_N is the depth of the effective rectangular potential well. It should be noted that even this simplified approach makes it possible to obtain a sensible description of the experimental data by a suitable choice of the parameter R.

In the framework of the standard R-matrix formalism,[15] the width of a quasistationary state is calculated in accordance with the formula

$$\Gamma = 2\gamma_{CL}^2 P_{CL},$$

where λ_{CL}^2 is the reduced width of the level,

$$\gamma_{CL} = \left(\frac{\hbar^2}{2mR}\right)^{1/2} \int \varphi_C^* X_{LJM} dS,$$

φ_C is the surface wave function of the channel, X_{LJM} is the solution to the eigenvalue problem in the interior region of the decaying system, and P_{CL} is the barrier penetrability factor:

$$P_{CL} = \frac{kR}{F_L^2(\eta,kR) + G_L^2(\eta,kR)},$$

where F_L and G_L are the regular and irregular Coulomb functions, respectively. Strictly speaking, the R-matrix approach is the basis of most of the forms of the theory of resonance nuclear reactions popular in low-energy nuclear physics.

A very fruitful approach to the problem of α decay was proposed in studies of Kadmenskiĭ's group (see the monograph of Ref. 13 and the references in it). This approach is based on the use of the so-called integral formalism, which in the case of the naive cluster model leads to the following expression for the level width:

$$\Gamma = 2\pi \left| \int_0^R \tilde{F}_L(\eta,kr) V_{00}(r) \tilde{\varphi}_L(r) dr \right|^2. \quad (37)$$

The notation in this expression will be explained below.

In all the approaches listed above, a natural small parameter of the theory was the barrier penetrability factor:

$$P_{CL} \ll 1. \quad (38)$$

However, in the case of the α decay of highly excited nuclear states, and also in the physics of hadronic resonances, one continually comes up against a violation of the condition (38). We shall show how it is possible to calculate the width of a quasistationary state without making the assumption (38) of a small barrier penetrability.

For definiteness, we consider in the single-particle approximation the problem of the α decay of short-lived states of nuclei.

We denote by U_L the radial wave function of an α particle moving in a potential $V_{\alpha A}(r)$, and by F_L the analogous regular Coulomb function. The Schrödinger equations for U_L and F_L have the form

$$\frac{d^2U_L}{dr^2} + \frac{2m}{\hbar^2}\left(Q - V_c(r) - \frac{\hbar^2}{2m}\frac{L(L+1)}{r^2} - V_{\alpha A}(r)\right) U_L = 0,$$

$$\frac{d^2F_L}{dr^2} + \frac{2m}{\hbar^2}\left(Q - V_{cp}(r) - \frac{\hbar^2}{2m}\frac{L(L+1)}{r^2}\right) F_L = 0, \quad (39)$$

where L is the orbital angular momentum, V_c is the Coulomb potential calculated with allowance for the finite size of the nucleus, V_{cp} is the Coulomb potential of a point charge, and $V_{\alpha A}$ is the potential of the interaction between the α particle and the residual nucleus.

We use the standard Green's method and make the transformations

$$F_L \frac{d^2U_L}{dr^2} - U_L \frac{d^2F_L}{dr^2} = \frac{2m}{\hbar^2} V_{00}(r) U_L F_L, \quad (40)$$

where

$$V_{00}(r) = V_{\alpha A}(r) + V_c(r) - V_{cp}(r).$$

Integrating the expression (40) by parts, we obtain

$$F_L(\eta,kR) U_L'(R) - U_L(R) F_L'(\eta,kR) = \frac{2m}{\hbar^2} \int_0^R V_{00}(r) U_L(r) F_L(\eta,kr) dr. \quad (41)$$

Here and in what follows, the prime denotes the first derivative with respect to R.

If R_0 is the cutoff radius of the nuclear potential, then at $R > R_0$ we have

$$U_L(R) = \frac{i}{2}([G_L - iF_L] - S_L[G_L + iF_L]),$$

from which we immediately obtain

$$\frac{i}{2}([G_L' - iF_L'] - S_L[G_L' + iF_L'])F_L - \frac{i}{2}\Big([G_L - iF_L] - S_L[G_L + iF_L]F_L' = \frac{2m}{\hbar^2} \int_0^R F_L(\eta,kR) V_{00}(r) U_L(r) dr. \quad (42)$$

We use the following property of the Wronskian:

$$F_L' G_L - F_L G_L' = k,$$

where

$$k^2 = \frac{2m}{\hbar^2} Q.$$

We introduce the function

$$\varphi_L(r) = S_L^{-1} U_L(r), \quad (43)$$

which has the asymptotic behavior

$$\varphi_L(r) \propto \frac{i}{2}[S_L^{-1}(G_L(\eta,kr) - iF_L(\eta,kr)) - (G_L(\eta,kr) + iF_L(\eta,kr))]. \quad (44)$$

By definition, at resonance

$$S_L^{-1} = 0, \quad (45)$$

and the asymptotic behavior (44) simplifies considerably to

$$\tilde{\varphi}_L(r) \propto -\frac{i}{2}[G_L(\eta,kr) + iF_L(\eta,kr)]. \quad (46)$$

To emphasize that we are working only in the neighborhood of the resonance, we shall denote the wave function $\varphi_L(r)$ at

$S_L^{-1}=0$ in (46) and in what follows by $\bar{\varphi}_L(r)$. We express it in terms of the wave function $\tilde{\varphi}_L(r)$ normalized to unity in the interior region of the resonance:

$$\tilde{\varphi}_L(r)=A_L\bar{\varphi}_L(r),$$

$$\int_0^R|\tilde{\varphi}_L(r)|^2dr=1. \tag{47}$$

In what follows, to shorten the expressions, we shall denote by $G_L, F_L,...$ the values of the corresponding functions at the point R. To avoid confusion in the notation, we shall denote the derivative with respect to ρ by a dot. We introduce the correction function $\kappa(k,k^*)$:

$$\kappa^{-2}(k,k^*)=\frac{1}{2k}([G_L^*\dot{F}_Lk-F_L^*\dot{G}_Lk+G_L\dot{F}_L^*k^* -F_L\dot{G}_L^*k^*]+i[G_L\dot{G}_L^*k^*-G_L^*\dot{G}_Lk +F_L\dot{F}_L^*k^*-F_L^*\dot{F}_Lk]). \tag{48}$$

It is obvious that in the approximation of narrow widths

$$\kappa(k,k^*)=1. \tag{49}$$

We introduce a Coulomb function normalized to a δ function in the energy on the real axis:

$$\tilde{F}_L(\eta,kr)=\sqrt{\frac{2}{\pi\hbar v}}\,F_L(\eta,kr). \tag{50}$$

Collecting together the relations (42)–(50), we immediately obtain an exact expression for calculating the widths:

$$\Gamma=\frac{2\pi}{\kappa^2(k,k^*)}\left|\int_0^R\tilde{F}_L(\eta,kr)V_{00}(r)\tilde{\varphi}_L(r)dr\right|^2. \tag{51}$$

By virtue of the relation (49), the expression (51) goes over in the limit of narrow widths into Kadmenskiĭ's well-known expression. The relation (51) is the result of analytic continuation of Kadmenskiĭ's formalism into the complex plane of the energy.

The integrand in (51) has a good structure from the point of view of the possibility of constructing asymptotic estimates of the widths of quasistationary states. Indeed, the potential $V_{\alpha A}(r)$ decreases rapidly with increasing r, while, in contrast, the Coulomb function $F_L(\eta,kr)$ increases exponentially with increasing r, but to the right of the classical turning point it is bounded. The wave function $\tilde{\varphi}_L$ oscillates in the interior region of the nucleus and has a maximum near its surface. The product of these three functions has a sharp maximum in the surface region of the nucleus, and, therefore, the integral in (51) can be estimated by the method of steepest descent.

As a result, we obtain

$$\Gamma_{s.p.}=\xi_L(Q,r_0)\,\frac{kr_0}{2Q}\,[F_L(\eta,kr_0)V_{00}(r_0)]^2, \tag{52}$$

where $\Gamma_{s.p.}$ is the limit single-particle estimate of the width Γ, r_0 is the channel radius chosen on the basis of the condition of a maximum of the integrand in the expression (51), and ξ_L is a correction factor of the order of unity:

TABLE III. The α-decay widths of lead isotopes.

Nucleus	$\Gamma^{\rm int}$ (MeV)	$\Gamma^{\rm per}(R_0)$ (MeV)	$\Gamma_{\rm exp}$ (MeV) (Ref. 94)
^{182}Pb	$1.06\cdot10^{-19}$	$0.55\cdot10^{-19}$	$0.82\cdot10^{-20}$
^{184}Pb	$0.75\cdot10^{-20}$	$0.42\cdot10^{-20}$	$0.83\cdot10^{-21}$
^{186}Pb	$0.49\cdot10^{-21}$	$0.30\cdot10^{-21}$	$0.97\cdot10^{-22}$
^{188}Pb	$0.17\cdot10^{-22}$	$0.11\cdot10^{-22}$	$0.41\cdot10^{-23}$
^{190}Pb	$0.20\cdot10^{-24}$	$0.15\cdot10^{-24}$	$0.57\cdot10^{-25}$
^{192}Pb	$0.69\cdot10^{-27}$	$0.60\cdot10^{-27}$	$0.13\cdot10^{-27}$
^{194}Pb	$0.16\cdot10^{-30}$	$0.17\cdot10^{-30}$	$0.41\cdot10^{-31}$
^{210}Pb	$0.66\cdot10^{-37}$	$0.10\cdot10^{-36}$	$0.12\cdot10^{-37}$

$$\xi_L(Q,r_0)=\frac{8\pi|\tilde{\varphi}_L(r_0)|^2\sigma^2}{r_0\kappa^2(k,k^*)}. \tag{53}$$

In its turn, σ can be found from the expression

$$\sigma^{-2}=\frac{\partial}{\partial r}\left[\frac{1}{V_{00}}\frac{\partial V_{00}}{\partial r}+\frac{1}{F_L}\frac{\partial F_L}{\partial r}+\frac{1}{\tilde{\varphi}_L}\frac{\partial\tilde{\varphi}_L}{\partial r}\right]_{r=r_0}. \tag{54}$$

In accordance with the general philosophy of the approach developed here, the relations (52)–(54) have been obtained under the assumption that the channel radius is determined by the condition

$$\frac{\partial}{\partial r}[\tilde{F}_L(\eta,kr)V_{00}(r)\tilde{\varphi}_L(r)]_{r=r_0}=0, \tag{55}$$

and

$$\tilde{\varphi}_L'(r_0)\approx0. \tag{56}$$

The analyticity of the relations (52)–(54) makes it possible to use them to estimate the widths of hadronic resonances, i.e., in the cases when the interaction potential of the fragments of the decaying system in the interior region is not well known. In this case, the influence of the Coulomb barrier is weak, and the estimate (52) simplifies:

$$\Gamma_{s.p.}=\tilde{\xi}_L(Q,r_0)\,\frac{(kr_0)^{2L+1}}{2Q}\,[V_{00}(r_0)]^2. \tag{57}$$

A new correction factor arises when the Coulomb function is replaced by its asymptotic value. For estimates, we take

$$\tilde{\varphi}_L(r_0)\approx r_0^{-1/2},\quad \sigma\approx0.2\ \text{F},$$

and then $\tilde{\xi}\approx1.3$. It is difficult to estimate the value of $\tilde{\xi}$ more accurately, since σ determines the second derivative of $\ln(\tilde{\varphi}_L(r))$, which can be found only numerically.

The multiparticle nature of the problem can be taken into account by the introduction of a spectroscopic factor w:

$$\Gamma=w\Gamma_{s.p.}.$$

As an illustration of the viability of the approach, we give numerical examples. Thus, in a consideration of the α decay of lead isotopes the width of the individual states may vary by 18 orders of magnitude. Despite this, it can be seen from Table III that the steepest-descent estimates of the widths agree reasonably well with exact calculations in the framework of Kadmenskiĭ's integral formalism.

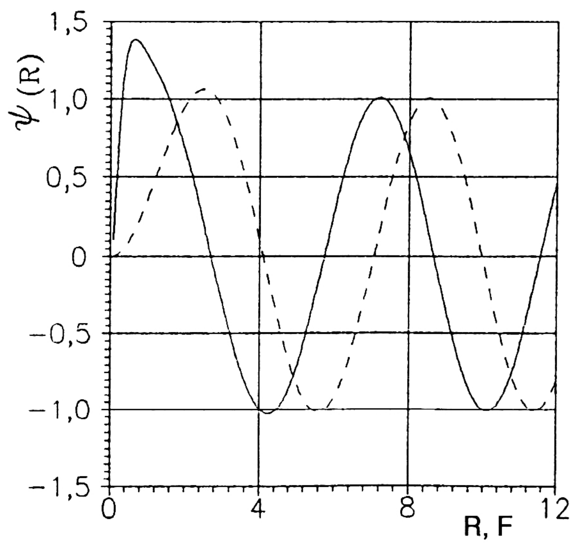

FIG. 6. Pion wave function in the center-of-mass system of the decaying $\Delta(1232)$ isotope. The dashed curve corresponds to the wave function of a free pion with orbital angular momentum $L=1$.

In the calculations, we set $R_0=1.25A^{1/3}$ F and used the Woods–Saxon potential with diffuseness parameter $a=0.55$ F. The depth of the potential was chosen in accordance with the α-particle energy Q, using the well-depth fitting procedure.

In numerous studies made in the framework of Kadmenskiĭ's approach (see the monograph of Ref. 13) it was established that the integral in (37) accumulates in the surface region of the nucleus. Moreover, neither the presence of channel coupling (see Ref. 95) nor allowance for superfluid correlations and renormalization of the interaction in going from free nucleons to bound intranuclear nucleons (see Ref. 96) changes the fundamental conclusion that the α-decay width is formed in the cluster (surface) region of the nucleus.

We use the above generalization of Kadmenskiĭ's formalism for a numerical estimate of the widths of nonstrange baryonic resonances. We first consider the problem of resonant P-wave πN scattering according to the diagrammatic technique presented in Ref. 6.

For this, we calculate in the first Born approximation the long-range part of the effective potential of the πN interaction:[38]

$$U_{\rm eff}^{\pi N}(r)=-f_{\pi NN}^2\,\frac{m+m_\pi}{mm_\pi}\left[\frac{m}{m_\pi}\right]^2\frac{P_\pi^2}{\sqrt{s}}\,\frac{\exp(-\alpha r)}{r},\qquad(58)$$

where m (m_π) is the nucleon (pion) mass, $f_{\pi NN}=1$ is the strong coupling constant, $\alpha=\sqrt{2mE_\pi}$; $E_\pi=\sqrt{P_\pi^2+m_\pi^2}$; $P_\pi=\lambda^{1/2}(s,m^2,m_\pi^2)/(2\sqrt{s})$, and the triangle function λ is defined by the standard relation $\lambda(x,y,z)=x^2+y^2+z^2-2xy-2yz-2zx$.

To analyze the problem of the widths of the hadronic resonances, we use the correspondence principle.

We calculate the wave function $\psi(r)$ of a pion produced by decay of a Δ isobar by integrating numerically the Klein–Gordon equation linearized with respect to the potential $U_{\rm eff}^{\pi N}$. It can be seen from Fig. 6 that the potential $U_{\rm eff}^{\pi N}(r)$ affects the pion wave function in the interior region of the nucleon; moreover, the position of the maximum of the wave function virtually coincides with the first Bohr radius $r_0=0.86$ F for $L=1$.

TABLE IV. Masses and widths of πN resonances (in mega-electron-volts).

	Resonance $n+\gamma$	$\Delta(1232)$ 1	$N^*(1440)$ $1+1/2$
M	Theory	1234	1370
	Exp.[24]	1230–1234	1430–1470
Γ	Theory	100	260
	Exp.[24]	115–125	250–450

It follows from this that the resonances are formed at relatively large separations between the separating decay products. This enables us to establish an analogy between the physics of hadronic resonances and α decay (i.e., between the physics of quark and nucleon clusters), in which the α particle is formed in the surface region of the nucleus.[13,95,96] In Table IV, we give the widths of hadronic resonances calculated in the approximation (57), the physical basis of which is the analogy noted above between α decay and the decay of hadronic resonances. It can be seen from Table IV that the agreement between the theory and experiment is good.

The fact that the position of the outer maximum of the wave function $\psi(r)$ of the pion produced by the decay of the Δ isobar [or $N^*(1440)$ resonance] practically coincides with $r_0=0.86$ F indicates that in the physics of hadronic resonances we are dealing with a quantum-mechanical analog of open resonators.[50] In the well-known monograph of Ref. 17, some general properties of quantum analogs of open resonators were analyzed.

A very serious test of the validity of the proposed approach is the test of its applicability in the analysis of dibaryon properties. The history of the narrow dibaryonic resonances is dramatic. The existing literature on this question is very rich (see, for example, the reviews of Refs. 27, 28, and 97–99 and the references in them) but is contradictory. Although there are discrepancies between the experimental data from different groups, we decided to use the extensive results obtained by the Dubna collaboration.[100]

Calculations of the widths of the dibaryonic resonances were made using the Hulthén potential

$$V_{00}(r)=V_0\,\frac{\exp(-\mu r)}{1-\exp(-\mu r)},\qquad(59)$$

and for proton–neutron resonances the depth of the potential was taken to be 35 MeV, and for proton–proton resonances 49 MeV. We took the parameter value $\mu=1.1$ F^{-1}. It can be seen from Tables V and VI that there is good agreement between the theory and the experiment of Ref. 100.

Thus, Tables IV–VI demonstrate the good agreement between the theoretical and experimental values of the widths. Moreover, in all the calculations of the masses and widths of the baryonic and dibaryonic resonances we used one and the same parameter value $r_0=0.86$ F.

TABLE V. Invariant masses and widths (mega-electron-volts) of diproton resonances. The observed experimental widths are given without correction for the instrumental resolution.

	$n+\gamma$	1/2	1	1+1/2	2	2+1/2	3	3+1/2
M	Theory	1890	1932	1998	2088	2198	2326	2468
	Exp.[100]	1886	1937	1999	2087	2172		
Γ	Theory	4	9	12	17	22		
	Exp.[100]	4 ∓ 1	7 ∓ 2	9 ∓ 4	12 ∓ 7	7 ∓ 3		

It must be especially emphasized that in the framework of one and the same approach, without fitting parameters, it has been possible to describe both the broad hadronic resonances (Δ isobar and Roper resonance) and the narrow dibaryonic resonances, which have widths that are 1–2 orders of magnitude smaller. The origin of this smallness of the widths of the dibaryonic resonances is well illustrated by Fig. 7, in which we have plotted on the same scale the πN potential (58) and the Hulthén potential for the neutron–proton interaction. The small depth of the latter in the neighborhood of the channel radius is due to the fact that at low energies the NN interaction is determined by the exchange of a pion far from the mass shell. In accordance with (52), it is then the smallness of the interaction potential that leads to the narrow width of the corresponding resonance.

In the case of the πN interaction, the pion is on the mass shell, and the corresponding potential has a large depth. Therefore, all the known baryons that decay as a result of strong processes have characteristic widths of the order of 100 MeV.

Thus, the widths of the hadronic and dibaryonic resonances can be fairly well reproduced in the framework of the correspondence principle on the basis of the deep physical analogy with the theory of α decay. The experience accumulated in the process of the solution of numerous problems in the physics of α decay unambiguously indicates that the results of the calculation are insensitive to the choice of the potentials of the αA interaction in the interior region of the nucleus. The theoretical α-decay width is also virtually independent of the behavior of the effective wave function of the α particle in the interior region of the nucleus. The physics of α decay is cluster physics. All the observable quantities are determined by the potential of the αA interaction on the surface of the nucleus.

The transfer of this approach to the physics of hadronic resonances is especially important, since a potential description of such resonances is possible only in the surface region of the decaying hadrons. Therefore, systematic investigation of the widths of hadronic and dibaryonic resonances is capable of yielding useful information about the interaction potentials in the neighborhood of the radius of the channel responsible for the formation of the resonance. In our opinion, the smallness of the widths of the dibaryonic resonances compared with the widths of the hadronic resonances is due to the smallness of the NN interaction relative to the πN interaction in the resonance formation region.

5. PRINCIPLES OF SIMILITUDE AND DIMENSION IN NUCLEAR PHYSICS

The use of the principles of similitude, dimension, and automodelity proved to be very fruitful in the mechanics of continuous media, hydrodynamics, the theory of combustion, etc.[101–103] In particle physics, the automodelity hypothesis was introduced by Matveev, Muradyan, and Tavkhelidze.[41] A further development of the principles of similitude, dimension, and automodelity was achieved in the studies of Baldin[42] devoted to the relativistic theory of dynamical systems.

We illustrate the viability of similitude methods in well-known examples in nonrelativistic nuclear physics.

According to von Weizsäcker,[104] there is a dependence

$$R=R_0A^{1/3} \tag{60}$$

of the radius of a nucleus on the number of nucleons A in the nucleus. It follows directly from the expression (60) that the volume of the nucleus is proportional to the number of nucleons. This property makes nuclei like ordinary homogeneous materials. On the basis of the theory of a Fermi liquid that he had developed, Landau introduced the concept of quasiparticles.[105] The most important characteristic of the Fermi liquid was the de Broglie wavelength $\lambda_f=1/k_f$ of a quasiparticle on the Fermi surface. The introduction of the concept of quasiparticles appears well justified only if

$$k_fR\gg1, \quad R\gg\lambda_f, \tag{61}$$

and this condition is identical to the condition of applicability of the semiclassical approximation in quantum mechanics.

In the framework of the theory of finite Fermi systems, the ground-state energy of a nucleus can be represented as a sum

TABLE VI. Invariant masses and widths of neutron–proton resonances (in mega-electron-volts).

	$n+\gamma$	1/2	1	1+1/2	2	2+1/2	3	3+1/2
M	Theory	1892	1933	2000	2089	2200	2327	2469
	Exp.[100]			1998	2084			
Γ	Theory	2	4	6	8	10		
	Exp.[200]			14 ∓ 4	11 ∓ 5			

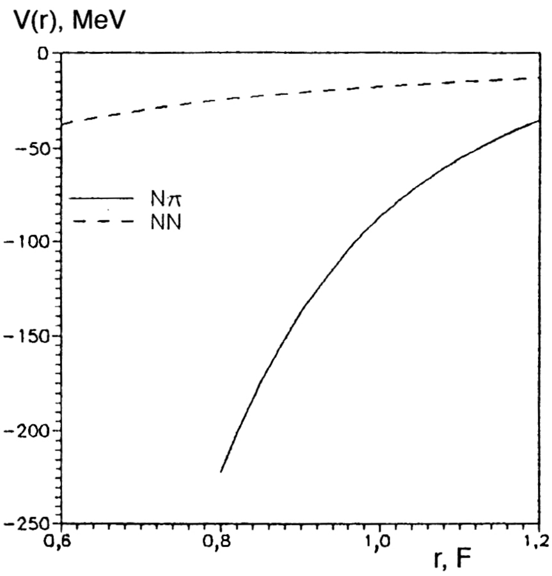

FIG. 7. Hulthén potential for the neutron–proton interaction (dashed curve) and effective potential of the πN interaction (solid curve).

$$E=E_0+E_1,$$

where E_0 and E_1 are smooth and oscillating functions of the number of quasiparticles, respectively. In accordance with Ref. 106, the energy E_1 has period $\pi/2$ in the k_fR scale. Indeed, for quasiparticles in a nucleus the condition of applicability of the semiclassical approximation holds. Therefore, it is possible to use the Bohr–Sommerfeld radial quantization rule. The upshot is that we obtain a quantization rule for the Fermi level:[106]

$$k_fR=\frac{\pi}{2}n, \quad n=2,3,4,..., \tag{62}$$

which reproduces rather well the magic numbers of nucleons in a nucleus. This result can be interpreted as follows: The greatest stability is possessed by the nuclei for which an integer number of de Broglie half-wavelengths of a quasiparticle on the Fermi surface fits into the diameter $d=2R$.

For nuclear resonances of collective nature,[18] there is also the relation

$$k=\beta\frac{1}{R}=\beta_1A^{-1/3},$$

which is analogous to (62). Thus, in the given examples the observed quantum effects are explained by the existence of resonant eigenfrequencies for the de Broglie wavelengths within a spherical cavity of radius R.

In this section, we have briefly considered the gross structure of nuclei, disregarding the fine details. The nucleus was regarded as a coarse system, the dynamics of which can be described in its general features according to the theory of a Fermi liquid. Such an approach was used by Nosov in the monograph of Ref. 106. The applicability of such an approach requires fulfillment of the relations (60) and (61). At the same time, it turns out that the most important characteristic of a nucleus as a coarse system is the value of k_fR, which by definition is equal to the angular momentum $L=k_fR$ of a quasiparticle on the Fermi surface.

The automodelity (scaling) hypothesis introduced into particle physics by Matveev, Muradyan, and Tavkhelidze[41] presupposes invariance with respect to the transformation $P_i\rightarrow\xi P_i$ of the momentum space.

We apply the above principles to the analysis of the spectrum of hadronic (and other subatomic) resonances. We use the fact that for spherically symmetric systems the angular momentum L is an integral of the motion. In addition, it remains unchanged under the dilatation transformation

$$P\rightarrow\xi P, \quad r\rightarrow\xi^{-1}r, \tag{63}$$

where $P=\hbar k$. In this case, the de Broglie wavelength transforms as $\lambda_D\rightarrow\xi^{-1}\lambda_D$. Thus, in the semiclassical approximation $L=R/\lambda_D$, i.e., the integral of the motion L is determined by the characteristic scale of the system and by the corresponding de Broglie wavelength. It was shown in Refs. 36–39 in actual examples that the quantization condition (21) can apply to hadronic resonances that decay through both the strong and the weak interaction. Therefore, we adopt the following working hypothesis: The quantization condition (21) is valid for resonances of any nature, irrespective of the form of the interaction, provided that there is fulfillment of the semiclassical condition (61). In other words, we are dealing with an automodelity principle of the second kind. That is, to the principles of the theory of dimension and automodelity there are added certain properties of the asymptotic behavior of the considered quantities. In our case, this is the requirement of fulfillment of the semiclassical condition (61).

We rewrite the formula

$$m_n(R)=\sqrt{m_1^2+P^2}+\sqrt{m_2^2+P^2}, \tag{64}$$

using the relations $\lambda_D=\hbar/P$, $\lambda_C=\hbar/mc$:

$$m_n(R)=\hbar c\sqrt{\lambda(1)_C^{-2}+\lambda_D^{-2}}+\hbar c\sqrt{\lambda(2)_C^{-2}+\lambda_D^{-2}}, \tag{64a}$$

where λ_C is the Compton wavelength for the products of binary decay of the resonance. Here λ_D/λ_C and r_0/λ_D are similarity parameters, and for fixed values of them the invariant mass remains self-similar under variation of all the remaining parameters of the problem.

Under scaling transformations of the form

$$P\rightarrow\xi P, \quad m_i\rightarrow\xi m_i, \tag{63a}$$

the invariant masses of the resonances vary as homogeneous functions of the parameter ξ of the scaling transformation:

$$m_n(R)\rightarrow\xi m_n(R). \tag{63b}$$

Thus, we have formulated an automodelity principle for the distribution of the resonance masses.

It should be noted that in the above formulation of the automodelity principle for the distribution of the resonance masses the de Broglie and Compton wavelengths play equally important roles. It is only for definite relationships between them that resonances arise, namely

$$\frac{r_0}{\lambda_D} \propto \frac{\lambda_C}{\lambda_D} \propto n.$$

Despite the approximate nature of our relations [in establishing them, we have used the asymptotic semiclassical condition (61)], their accuracy proved to be remarkably high (see Refs. 36–39). Below, we shall use these relations for a systematic investigation of resonances in the (pp), $(\pi\pi)$, and (e^+e^-) systems, and on the basis of a comparative analysis of these systems we shall give an interpretation of the "Darmstadt effect."

6. NONSTRANGE DIBARYONIC RESONANCES

The history of dibaryonic resonances is dramatic and contradictory. The present status of dibaryonic resonances has been presented in recent reviews,[27–29] in which the experimental data on the narrow dibaryonic resonances in the range of invariant masses up to 2300 MeV are analyzed. A characteristic feature of the dibaryonic resonances is their anomalously narrow width. The explanation of this feature is a very serious test for any theoretical model. Theoretical studies are reviewed in Ref. 107 (see also the recent publications of Refs. 108 and 109). The existing theoretical models are not capable of explaining why the dibaryonic resonances are so narrow. The most recent study available to us[110] gives an estimate $\Gamma \leqslant 40$ MeV for the width. As was noted above, the dibaryonic resonances are narrow compared with the hadronic resonances because the NN interaction is weak compared with the πN interaction in the resonance formation region (for more details, see above). The invariant masses of the dibaryonic resonances calculated in the existing theoretical models[107] are systematically greater than the experimental values by 300 MeV.

However, it should be noted that the experimental spectrum of dibaryonic resonances is very rich, and, naturally, our approach is not capable of describing all the observed dibaryonic states, since from the beginning it is intended only to reproduce the positions of the centroids of the corresponding multiplets. Therefore, we shall analyze below the experimental spectrum of the diproton resonances, using for this purpose all the experimental data available to us. For this, we shall use the relation (64), from which we can find the momentum P of the observed proton (antiproton) in the center-of-mass system of the decaying dibaryonic resonance. We use the momentum distribution obtained for different diproton and antiproton–proton states to calculate the mass spectrum of the e^+e^- system, using the principle of scale invariance. The experimental data in conjunction with the results of our calculations on the basis of (21) and (24) are presented in Table VII (Ref. 111). (Note that in our approach the masses of the diproton and proton–antiproton resonances must be equal and, as can be seen from Table VII, such equality does indeed hold to a good accuracy.)

7. THE DARMSTADT EFFECT

The present status of the experiments and theoretical models relating to the Darmstadt effect are reviewed by Pokotilovskiĭ.[117] (The literature on this problem is exceptionally rich; see Ref. 117.) The essence of the Darmstadt effect is that in a collision of heavy ions at an energy below the Coulomb barrier there are observed to be narrow positron lines and electron–positron pairs in the range of effective masses 1.4–1.9 MeV with widths 2–40 keV. The existence of the effect is now widely accepted. The fact that narrow lines are continually being observed in independent experiments at the statistical significance level 5σ–6σ makes it impossible to interpret them as statistical fluctuations. We may also mention the widely accepted conclusion that the hypothesis of two-particle decay into a positron and an electron of a free elementary or composite particle at rest in the center-of-mass system of the colliding ions cannot be an adequate representation of the physics of the considered effect. A similar effect is also observed in the elastic scattering of positrons at low energies by atomic electrons. However, searches for resonance states in the scattering of positrons by electrons at the corresponding energies give much stronger bounds on their widths: $\Gamma < 10^{-3}$ eV (see the references to the experimental data in Ref. 24 and also the theoretical study of Ref. 118).

The attraction of the effect arises from the surprising nature of the situation: The extremely well-known e^+e^- sys-

TABLE VII. Mass spectrum of diproton and proton–antiproton resonances (in mega-electron-volts). The experimental data for the proton–antiproton resonances are taken from the compilation of Ref. 24.

Theory	NN experiment	$N\bar{N}$ experiment
	1877.5±0.5 [Ref. 116]	
	1886±1 [Ref. 100]	
1890	1892 [Ref. 100]	
	1898±1 [Ref. 100], 1902 [Ref. 28]	1897±1
	1916±2 [Refs. 28,100], 1918±3 [Ref. 112]	1920
1932	1937±2 [Ref. 100],1932±3 [Ref. 112]	1930±2
	1941 [Ref. 28]	1942±5, 1949±10
	1955±2 [Ref. 100], 1956±3 [Ref. 113]	
	1965±2 [Ref. 100], 1969 [Ref. 28]	1968
	1980±2 [Ref. 100]	
1999	1999±2 [Ref. 100]	
	2008±3 [Ref. 100], 2016 [Ref. 28], 2017±3 [Ref. 100]	2022±6
	2035±8 [Ref. 113]	
	2046±3 [Ref. 100], 2052 [Ref. 28]	
2089	2087±3 [Ref. 28,100]	2080±10, 2090±20
	2106±2 [Ref. 100], 2122 [Ref. 28]	2110±10
	2129±5 [Ref. 100], 2155 [Ref. 28]	
	2172±5 [Ref. 100]	2180±10
2199	2194 [Ref. 28]	2207±13
	2220 [Ref. 114], 2236 [Ref. 28], 2238±3 [Ref. 100]	2229
	2282±4 [Ref. 28, 100]	2260
2327		2307±6
	2350 [Ref. 114]	2380
2470		2480±30, 2450±10
		≈2500
2625		
2790	2735 [Ref. 115]	2710±20
		2850±5
2964		
3145		
3332		3370±10
3524		3600±20
3720		

tem exhibits properties that previously had been in no way suspected or ever predicted.

During the last decade, the situation has become even more intriguing in connection with the problem of the hypothetical axion, a neutral ($J^{\pi}=0^{-}$) particle with a mass that cannot be predicted by the theory. The axion problem has stimulated many unsuccessful experiments aimed at its detection. Therefore, models often based on fantastic ideas have been created to explain the Darmstadt effect. As a conclusion, we give the summary of the review of Ref. 117: The main features of the Darmstadt effect present a serious problem for interpretation and have not found an explanation.

We emphasize once more that resonance peaks in positron scattering by free electrons are not observed, but they are observed in the $e^{+}e^{-}$ system in the case of heavy-ion scattering and also in the scattering of positrons by (bound) atomic electrons. In other words, the Darmstadt effect is observed only in the presence of an external force center.

We believe that it is extremely important to draw attention to certain well-known analogous cases in different branches of physics:

1) The ABC "particle," or the ABC effect. Observation of the ABC "particle" was first reported in Ref. 31 by Abashian, Booth, and Crowe (the particle gets its name from the initial letters of the authors' surnames) in the reaction

$$p+d/d+p\rightarrow{}^{3}\mathrm{He}+X,$$

at energies of the incident protons from 624 to 743 MeV. In the distribution of the ^{3}He momentum in the laboratory system a sharp peak was observed, and this was subsequently interpreted as a manifestation of resonance structure in the two-pion system at the mass 315 MeV. Since no such clear effect was observed in the reaction

$$p+d/d+p\rightarrow{}^{4}\mathrm{He}+X,$$

Abashian, Booth, and Crowe concluded that the ABC "particle" has spin zero. The literature on the experimental study of the ABC effect is fairly rich, and the present status of the question is discussed in the review of Ref. 32. We mention that this effect has been investigated experimentally in numerous reactions (the authors of the review of Ref. 32 mention 54 reactions). The ABC effect is observed only when the reaction products include not only the two pions but also a third particle (or many particles). According to this review, it is not possible to rule out the existence of four resonance-like $I=0$, $J^{\pi}=0^{+}$ states with large widths at energies 315, 455, 550, and 750 MeV. It should be noted especially that (in contrast to the case of the Darmstadt effect, which stimulated a flood of theoretical studies) the ABC effect has essentially escaped notice.

In Table VIII we present a compilation of experimental data available to us and results of their theoretical analysis.

The indications contained in Table VIII of the possible existence of new resonances can open up an extremely interesting field of the physics of resonances slightly above the thresholds for the production of n particles, for example, two, three, four,... pions, mesons, nucleons, etc.

We now turn to the history of the discovery of the resonances. The ρ meson with mass 770 MeV and width $\Gamma=152$ MeV was discovered[124] in a study of the angle and energy dependences of $\pi\pi$ scattering. The ρ meson decays into two pions ($\approx 100\%$) and is the only firmly established two-pion resonance with minimum mass 770 MeV. Study of three-pion systems led to the discovery of the $\omega(782)$ meson with $\Gamma=8.4$ MeV in the scattering of antiprotons by protons[125] and of the $\eta(547)$ meson[126] with $\Gamma=1.2$ keV in the $\pi^{+}+d\rightarrow p+p+\pi^{+}\pi^{-}\pi^{0}$ reaction. The three-pion decay channel is in fact dominant for these mesons: 55% for the $\eta(547)$ meson and 89% for the $\omega(782)$ meson. Just two years ago there was a report[127] of the observation of a 0^{-} resonance with mass 749 ± 30 MeV and $\Gamma=32\pm17$ MeV in the three-pion system in the $\pi^{-}+A\rightarrow\pi^{+}\pi^{-}\pi^{-}+A$ reaction. The vector mesons ρ and ω were predicted (see the references in the review of Ref. 32), and there was also a prediction[128] of the existence of a 0^{-} resonance with mass 749 ± 30 MeV in the three-meson system, but the possible existence of the η meson was not suspected by anyone.

TABLE VIII. Mass spectrum of dipion resonances (in mega-electron-volts).

Theory	$\pi\pi$ experiment
	315, $\Gamma=25$ [Ref. 32], 313 ± 3 [Ref. 100]
	332 ± 3 [Ref. 100]
361	350 ± 10 [Ref. 120]
	395 [Ref. 119], 400 [Ref. 122], 388 ± 2 [Ref. 100]
	425 [Ref. 121]
	455 [Ref. 32], 470 ± 7 [Ref. 100]
537	550 [Ref. 32], 520 [Ref. 119]
	583 ± 3 [Ref. 100]
	600, $\Gamma=600$ [Ref. 123]
	652 ± 2 [Ref. 100]
743	750 [Ref. 32], 769.9 ± 0.8, $\Gamma=151.2\pm1.2$ [Ref. 24]
959	980 ± 10, Γ from 40 to 400 [Ref. 24]
1180	1275 ± 5, $\Gamma=185\pm20$ [Ref. 24]
1405	1465 ± 25, $\Gamma=310\pm60$ [Ref. 24]

Thus, the lowest resonance in the three-pion system, the η meson, has an invariant mass that is 225 MeV smaller than the mass of the lowest resonance in the two-pion system (the ρ meson), and at the same time $\Gamma_{\rho}/\Gamma_{\eta}\approx10^{5}$ (Ref. 24). Moreover, all the resonances mentioned above for which the three-pion decay channels are dominant live much longer than the ρ meson. Further, the $\eta'(958)$ meson, which decays predominantly (65%) into $\pi\pi\eta$, has width 200 keV, i.e., it lives longer than the ρ meson by a factor 760.

We give one further interesting example. The dipion mass distributions observed in the decay of the Υ mesons cannot be explained in the framework of the existing theoretical models (see the review of Ref. 129). For example, in the $\Upsilon(3S)\rightarrow\Upsilon(2S)+\pi^{+}\pi^{-}$ decay a peak is observed[120] in the two-pion mass distribution at energy 350 ± 10 MeV, and in the $\Upsilon(2S)\rightarrow\Upsilon(1S)+\pi^{+}\pi^{-}$ decay a peak is observed[130] at ≈310 MeV. Moreover, in the $\Upsilon(3S)\rightarrow\Upsilon(1S)+\pi^{+}\pi^{-}$ decay two peaks are observed—at ≈400 and at ≈800 MeV, respectively.[130] Therefore, the two-pion system in the presence of a third particle exhibits resonance properties at values of the invariant mass of the two-pion subsystem that are lower than in the case of the free $\pi\pi$ interaction.

To summarize: The ABC effect, like the Darmstadt ef-

TABLE IX. Distributions of the momenta P (MeV/c) and masses (MeV) for diproton, dipion, and e^+e^- resonances.

$P(NN)$	$m(NN)$	$P(\pi\pi)$	$m(\pi\pi)$	$P(e^+e^-)$	$420\,P(e^+e^-)$	$m(e^+e^-)$
20	1877	71	313			
29	1877.5	90	332			
94	1886	105	350			
120	1892	140	395			
155	1902	160	425			
193	1916	206	497	0.4637	195	1.380
230	1932	237	550	0.5476	230	1.498
250	1942	255	583	0.5992	252	1.575
274	1955			0.6553	275	1.662
298	1969	295	652	0.7029	295	1.738
316	1980			0.7566	318	1.826
344	1999	348	750	0.8109	341	1.917
368	2016	359	770			
394	2035					
415	2052					
457	2087					
477	2106	470	980			
502	2129					
529	2155					
547	2172					
568	2194					
593	2220					
609	2238	619	1270			
649	2282					
670	2307					
707	2350	711	1450			

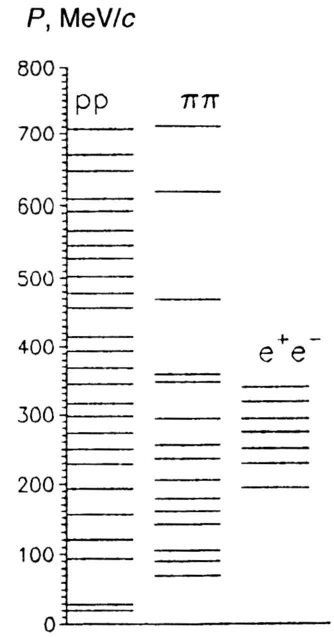

FIG. 8. Values of the momenta P for the decay products of resonances in the pp, $\pi\pi$, and e^+e^- systems calculated using the relation (64) for the experimental values of the resonance masses.

fect, is observed only in the presence of a third particle (or particles).

To complete the picture, we give some examples that are well known in low-energy nuclear physics. Two protons do not have a bound state, and proton–proton elastic scattering at low energies does not manifest any singular features, though there are experimental indications that in the $p+n\rightarrow(pp)+\pi^-$ reaction a narrow statistically reliable irregularity is observed[116] at the two-proton invariant mass 1877.5±0.5 MeV. A diproton resonance was observed in Ref. 131 in the $dd\rightarrow 2n+(2p)$ reaction: The peak value of the resonance decay energy was 0.43±0.09 MeV, its mean value was 0.45±0.05 MeV, and the total width was 0.14 ±0.13 MeV. When a neutron is added to the two protons, a bound system (the triton) is formed with total binding energy 8.48 MeV. In the $^{11}\mathrm{Li}\rightarrow{}^{9}\mathrm{Li}+(nn)$ reaction, a resonance-like structure is observed[132] in the two-neutron system at energy 50 keV of the relative motion, or at momentum $P=5$ MeV/c in the center-of-mass system of the two neutrons. Therefore, in this case the de Broglie wavelength of the neutron is ≈40 F. Thus, the effective distance between the two neutrons is ≈40 F. It is well known that the triton t does not have resonance states (the cross section for elastic $p+d$ scattering does not exhibit any singular features), but in the $^{6}\mathrm{He}+\mathrm{H}\rightarrow\alpha+t$ reaction a resonance of the triton with $E^*=7.0\pm0.3$ MeV and width $\Gamma=0.6\pm0.3$ MeV is observed.[133]

Intensive investigations of the Darmstadt effect began immediately after the discovery[33] of narrow positron peaks in heavy-ion scattering. The experimental values of the e^+e^- mass spectrum are given in Table IX. Narrow resonances in the total positron–electron spectrum were observed by the ORANGE group[134] for the following systems: U+Ta with energy of the peaks $E_{e^+}+E_{e^-}=634\pm5.803\pm6$ keV, U+Pb with energy of the peaks $E_{e^+}+E_{e^-}=575\pm6.787\pm8$ keV, and U+U with energy of the peaks $E_{e^+}+E_{e^-}=555\pm8.630\pm8.815\pm8$ keV. The EPOS group[135] observed peaks at the energies 620, 750, and 810 keV. For the resonance with energy 634 keV observed in U+Ta collisions, measurements of the angular correlations of the leptons were made. The results of these measurements contradict the assumption of decay of a neutral free particle with zero spin (the hypothetical axion). Some e^+e^- peaks indicate emission of particles at 180°, and this is usually interpreted as agreeing with the scenario of the decay of a free particle moving with the center-of-mass velocity of the colliding ions. It was concluded in Ref. 134 that an electron–positron pair cannot be emitted from an individual moving nucleus and that the pair is emitted in the neighborhood of a third heavy positively charged partner moving with low transverse velocity ($|v_\pm|<0.02c$). Finally, there are indications that the cross section of the process, as a function of the energy of the incident ions, has a resonance nature.

Table IX gives the momenta P for diproton, dipion, and e^+e^- resonances (see also Fig. 8). For the diproton and e^+e^- resonances, a clear regular structure with interval ≈25 MeV/c and ≈53 keV/c, respectively, is observed in the P distribution. With regard to the dipion resonances, it is here difficult to arrive at a definite conclusion because of the large uncertainty in the masses and widths of these resonances, though if one wanted, one could also recognize in this case some regularity in the P distribution. It can be concluded from Table IX and Fig. 8 that similarity between the $P(pp)$ and $P(e^+e^-)$ distributions is indeed observed for the dipro-

TABLE X. Mass spectrum of e^+e^- resonances (in kilo-electron-volts). The similitude parameter is $\xi=420$.

Theory	Experiment
1026	1043 [Ref. 138]
1031	1062 [Ref. 138], 1067 [Ref. 137]
1116	
1172	
1260	
1375	1380 [Ref. 134]
1497	1498
1568	1532 [Ref. 136], 1575 [Ref. 134]
1657	1662 [Refs. 136,139], 1652 [Ref. 134]
1749	1738 [Ref. 136]
1818	1831,1837 [Ref. 134]
1932	1917 [Ref. 136]
2029	
2134	
2225	
2402	

ton and e^+e^- resonances. From comparison of the experimental data on $P(pp)$ and $P(e^+e^-)$ it follows that

$$P(pp)=420P(e^+e^-). \tag{65}$$

Thus, the values of $P(e^+e^-)$ can be determined from the experimental data on the proton–proton and antiproton–proton resonances. The invariant masses for the e^+e^- resonances can be readily calculated from the momenta P of the e^+e^- pairs determined in such a manner. The results and a comparison with the experimental data on the invariant masses are presented in Table X. As can be seen from Table X, the error in the description of the experimental data is less than 1%. Moreover, in the approach that we have presented there is just one adjustable parameter—the parameter ξ of the scale transformation [see the definition (63a)]. In the considered case, it was found to be 420.

We also observe similarity between the momentum distributions $P(pp)$ and $P(\pi\pi)$ for the diproton and dipion resonances (see Table IX and Fig. 8) with parameter $\xi\approx 1$ of the scale transformation.

Thus, there is invariance of the diproton, dipion, and e^+e^- resonances with respect to the transformation (63). In the first two cases, it holds for strongly interacting systems, and in the final case it holds for a system with electromagnetic interaction. In other words, the invariance with respect to the transformation (63) and the automodelity principle hold for all the resonances considered in the present review, independently of the form of the interaction.

Thus, on the basis of the above we arrive at the conclusion that the Darmstadt effect must be interpreted as a resonance in the e^+e^- system in the presence of a third (heavy) partner, in complete analogy with dipion physics. Using the Heisenberg uncertainty principle, we can estimate the characteristic dimensions of the e^+e^- resonances: 200–4000 F. Therefore, the e^+e^- resonances are large, having dimensions of the order of or greater than the electron Compton wavelength (386 F), in complete agreement with the concept of quantization of the asymptotic momentum.

Such an interpretation of the Darmstadt effect permits a quantitative estimate of an important characteristic such as the momentum transfer. Experimentally, the mean value of the momentum transfer to the leptons is 800–1000 keV/c (Ref. 134) for the resonance with mass 1652 keV, agreeing well in order of magnitude with the theoretical estimate 650 keV/c.

Therefore, the Darmstadt effect can be explained on the basis of traditional physical ideas without introducing the hypothesis of the existence of axions or any other new "elementary particles." We mention that in nuclear and elementary-particle physics there are many other resonance effects similar to the Darmstadt and ABC effects.

We summarize what we have said above as follows:

1) Resonances in subatomic systems satisfy the principle of similitude, by virtue of which the de Broglie and Compton wavelengths are commensurate.

2) In all the investigated cases, resonances arose under the condition of commensurability of the de Broglie wavelength with the geometrical dimensions of the system, independently of the physical nature of the interaction.

3) The Darmstadt effect and the ABC effect have a simple and natural explanation on the basis of the proposed concept. They are explained by the occurrence of resonances in a wave system that has been localized in space by the presence of a force center (a third particle or several additional particles).

8. EXOTIC BARYONIC RESONANCES

The simplest possible multiquark state is the $q^2\bar{q}^2$ system introduced in Ref. 140. Even for such simple systems, calculations in the framework of lattice or other QCD-inspired models (for references, see Ref. 26) are not sufficiently accurate to give a satisfactory result. The recent calculations[26] are restricted to the $L=0$ $q^2\bar{q}^2$ case. Here we wish to present the results of the calculations of Ref. 141 and make some predictions concerning new resonance candidates and also compare them with existing experimental data that are available to us. It should be noted that multiparticle decay can be regarded as a chain of binary decays: two-particle decay of the primary resonance into two clusters, then the decay of these clusters again into two particles, etc. This agrees with the generally accepted model of multiparticle production through the decay of intermediate resonances. Therefore, multiparticle decay can be interpreted as a

TABLE XI. Spectrum of invariant masses (MeV) for $p\pi^+\pi^-\pi^0 \equiv p\omega^0$ resonances.

	$n+\gamma$	1/2	1	1+1/2	2	2+1/2	3	3+1/2	4
M	Theory	1736	1781	1854	1951	2069	2205	2355	2517
	Exp.	1700 [Ref. 143]	1780 [Ref. 142]						

TABLE XII. Spectrum of invariant masses (MeV) for $\Sigma(1385)K$ resonances.

	$n+\gamma$	1/2	1	1+1/2	2	2+1/2	3	3+1/2	4
M	Theory	1897	1948	2029	2133	2255	2392	2542	2702
	Exp.		1956 [Ref. 144]	2050 [Ref. 145]					

branching process in which intermediate resonances play an important role. Such an approach indicates a way in which the mass formula (24) can be used to study multiparticle decays of resonances. This approximation gives a very simple way of estimating the invariant masses of resonances that decay into multiparticles.

It is well known that baryonic resonances do not decay through the $N\omega^0$ channel, but a resonance structure is observed (see Ref. 142) in the $p\pi^+\pi^-\pi^0$ system with mass 1780±40 MeV and width 250±80 MeV in the region of the ω^0 resonance in the $\pi^+\pi^-\pi^0$ system. This resonance was explained as a manifestation of a cryptoexotic $uuds\bar{s}$ structure. We propose to interpret the structure of the $N^*(1780)$ resonance as a molecular state of the $p\omega^0$ system (see Table XI). An N^* resonance with mass 1700 MeV and width 150 MeV was given in Ref. 143. It decays through the $p\eta$, $\Delta 2\pi$, and $p3\pi$ channels.

The present status of the narrow exotic baryonic resonances is discussed in the review paper of Ref. 25. Therefore, we give below only the results of calculations together with the experimental data of Ref. 25 and brief remarks.

A resonance in the $\Sigma(1385)K$ system with invariant mass 1956^{+8}_{-6} MeV and width 27±15 MeV was first observed by the authors of Ref. 144 and was interpreted as the $N_\phi(1960)$ baryon. The structure of this resonance was considered in Ref. 144 as a candidate for an exotic baryonic resonance with hidden strangeness ($udds\bar{s}$). The resonance with mass 2050±6 MeV and width ≈120 MeV in Ref. 145 was regarded as a new baryon with hidden strangeness. These conclusions should be regarded as preliminary, requiring confirmation by further measurements with improved statistics.

Some resonance-like structures with mass 2170 MeV and $\Gamma\approx110$ MeV were observed in the experiment of Ref. 145 for the $p\phi$ and $\Lambda(1520)K^+$ systems. These two experimental spectra have a similar nature, and the model considered in this review describes them well. The $p\phi$ and $\Lambda(1520)K^+$ systems have resonances at the masses 2160 and 2162 MeV, respectively (see Table XIII), and as a result the coherent sum of the amplitudes of the two resonances gives a narrow peak in the experimental spectrum. It is completely obscure how these resonances should be interpreted in the framework of the other existing models. Karnaukhov *et al.*[146] observed a resonance-like structure in the $K_s^0K^+p\pi^-\pi^-$ system with mass 3521±3 MeV and width 6^{+21}_{-6} MeV. The statistical significance of the observed structure is 10σ. This structure has zero strangeness and was called the $R(3520)$ baryon. The $R(3520)$ baryon has a significantly larger mass than the threshold for the $K_s^0K^+p\pi^-\pi^-$ system and was considered in Ref. 146 as a candidate for a five-quark state. We can interpret it as a manifestation of a quasimolecular resonance for the $K_s^0K^+p\pi^-\pi^-$ many-particle system. There exist many combinations that, in principle, could be verified experimentally. For example, the combinations $(((\bar{K}^0p)_{1520}K^+)_{2031}(\pi^-\pi^-)_{361})$ with estimated mass 3532 MeV or $(((\bar{K}^0\pi^-)_{892}K^+)_{1406}p)_{2446}\pi^-$ with estimated mass 3539 MeV can be regarded as candidates for such a structure. Therefore, some quasimolecular states may overlap at the mass 3520 MeV and give a sharp peak as a result of coherent superposition of amplitudes. Most experimental data in the field of strange baryons were obtained using the propane bubble chamber of the Dubna collaboration (see Ref. 147 and the references in it). The results of calculations in the framework of our model agree well with these experimental data.

There was noted earlier (see the references in Ref. 148) the possible existence of narrow baryonium (denoted by M_S for the systems $\Lambda\bar{p}\pi$, $\Lambda\bar{p}\pi\pi$ with $S=-1$ and by $\bar{M}_S$ for the systems $\bar{\Lambda}p\pi$, $\bar{\Lambda}p\pi\pi$ with $S=+1$) with mass 3060±5(st.) ±20(syst.) MeV and width not greater than 35±5 MeV. This result is confirmed by the WA-62 data.[149] In our model, the invariant masses of the resonances do not depend on the strangeness. Therefore, the $\Lambda\bar{p}\pi$, $\bar{\Lambda}p\pi$, and $\Lambda p\pi$ systems, for example, must have almost the same invariant masses. The mass predicted in our model for such systems is 3030 MeV (see Table XIV). It is close to the value given in Ref. 148: 3060±5(st.)±20(syst.) MeV.

The narrow baryonium M_Φ was observed[148] in the mass spectrum of $\Lambda\bar{p}K\pi$, $\bar{\Lambda}pK\pi$, $\Lambda\bar{p}\bar{K}$, $\bar{\Lambda}pK$, $p\bar{p}K\bar{K}$ with mass 3260±5(st.)±20(syst.) MeV and width $\Gamma\leqslant35\pm5$ MeV. There exist many possible quasimolecular resonances with

TABLE XIII. Spectrum of invariant masses (MeV) for $\Lambda(1520)K$ resonances.

	$n+\gamma$	1/2	1	1+1/2	2	2+1/2	3	3+1/2	4
M	Theory	2031	2081	2160	2261	2380	2514	2660	2816
	Exp.			2170 [Ref. 145]					
Spectrum of invariant masses (MeV) for ϕp resonances.									
	$n+\gamma$	1/2	1	1+1/2	2	2+1/2	3	3+1/2	4
M	Theory	1972	2011	2075	2162	2269	2393	2532	2683
	Exp.				2170 [Ref. 145]				

TABLE XIV. Masses (MeV) of resonances that decay through the channels $p\Lambda$, $(p\Lambda)_{2067}\pi$, $((p\Lambda)_{2067}\pi)_{2250}\pi$, $(((p\Lambda)_{2067}\pi)_{2250}\pi)_{2434}\pi$.

	$p\Lambda$ decay		$p\Lambda\pi$ decay		$p\Lambda\pi\pi$ decay		$p\Lambda\pi\pi\pi$ decay	
$n+\gamma$	Exp.	Theor.	Exp.	Theor.	Exp.	Theor.	Exp.	Theor.
1/2	2095 [Ref. 147]	2067		2250		2434		2617
1	2129 [Ref. 147]	2105		2348		2530		2713
1+1/2	2181 [Ref. 147]	2166	2495 [Ref. 147]	2466		2647		2829
2	2224 [Ref. 147]	2250		2595		2775		2956
	2263 [Ref. 147]							
2+1/2	2357 [Ref. 147]	2353		2734		2912	3100 [Ref. 149]	3090
3	2500 [Ref. 24]*	2473		2879	3060 [Ref. 148]	3054		3230
3+1/2		2608	3060 [Ref. 148]	3031		3203		3376
4		2755		3188		3357		3528

*Note that the mass 2500 MeV is taken from Ref. 24 for the $p\bar{\Lambda}$ system.

very nearly equal masses. For example, there are the state $(\Lambda\bar{p})_{2166}\bar{K}$ with mass 3251 MeV, the state $((p\bar{p})_{1890}K)_{2448}\bar{K}$ with mass 3270 MeV, and the state $((\bar{\Lambda}p)_{2067}K)_{2624}\pi$ with mass 3275 MeV. All these quasimolecular states of subsystems can have resonances with almost the same masses, and it is precisely this that is responsible for the large peak of the considered resonance. It is very interesting to investigate multibaryonic resonances. In particular, we would like to mention the resonance observed[151] in the three-proton system with mass 3.27±0.02 GeV and width 0.07±0.04 GeV. The mass calculated in the framework of our model is 3288 MeV for the quasimolecular $((pp)_{1890}p)$ state.

At the present time, there exist experimental data[152] on the occurrence of quasistationary states for multinucleon (clusters containing up to six nucleons) systems. Therefore, the study of clusters as multiparticle resonances is a wide and interesting field in physics.

9. DENSITY OF MESONIC STATES

To calculate the number and density of the mesonic states that have been fairly well established experimentally[24] (except for the strange mesons on account of their sparse statistics) as functions of their invariant masses, it is possible to use a very simple version of the method of strength functions[40] (for details, see the monograph of Ref. 18).

We regard the density of mesonic resonance states as a sum of normalized Breit–Wigner distributions:

$$\frac{dN}{dm}=\frac{1}{2\pi}\sum_i (2J_i+1)(2I_i+1)\frac{\Gamma_i}{(m-m_i)^2+\Gamma_i^2/4}, \quad (66)$$

where m_i and Γ_i are the masses and widths of the experimentally observed mesonic resonance states.[24] Figure 9 shows the density of the mesonic resonance states as a function of the mass m (continuous curve). It can be seen that this density has a regular, periodically oscillating structure with period $\Delta m \approx 200$ MeV (predicted in Ref. 38) in the region of light colorless mesons ($S=C=B=0$) and the ψ and Υ mesons.

The results of our calculations are correlated with the experimental data (see Fig. 9; the asterisks are the calculated "centroid" positions of the n-pion resonances), except in the region of dipion mass $m(\pi\pi)_S=360$ MeV and the region of the six-pion $\eta\eta$ mass, $m(\eta\eta)_S=1118$ MeV. All the indications are that these regions require more careful experimental study (see above). It can be clearly seen from Fig. 9 that the mass spectrum of the low resonances has an almost equidistant nature and that the separation between the resonances (between the "centroids" of the resonances) is ≈200 MeV if these resonances are created by the successive addition of pions. Addition of K mesons gives a separation of about 500 MeV [$m_{\text{theor}}(K^+K^-)_S=1014$ MeV, $m_{\text{theor}}((K^+K^-)_SK)_S=1533$ MeV, $m_{\text{theor}}((K^+K^-)_S(K^+K^-)_S)_S=2052$ MeV]; the addition of D mesons gives a separation of about 1900 MeV; the addition of B mesons, a separation of about 5300 MeV, etc. In the framework of the scheme presented above, the high-lying rotation-like states must be based on the lower resonances.[38]

The present status of the ψ mesons was considered in the review of Ref. 153. The ψ systems can be studied by using the following approximations:

1. The properties of this system are usually described as radially excited $c\bar{c}$ states (for details, see Ref. 153).

2. The mass distribution of such a system can be estimated in the framework of the concept considered above by using Eq. (24) to calculate the mass of an n-pion resonance. In this case, the $((8\pi)_S(8\pi)_S)_P$ state with mass 3100 MeV appears as a suitable candidate for the $J/\psi(3097)$ state; the states $(J/\psi(3097)\pi)_S$, $(J/\psi(3097)(2\pi)_S)_S$, and $(J/\psi(3097)(6\pi)_S)_S$ with masses 3280, 3478, and 4240 MeV, respectively, have not been observed in experiments. The $(J/\psi(3097)(4\pi)_S)_S$ state with mass 3866 MeV may have been observed: There are certain indications[154] of the existence of a state with 3836 MeV. The states

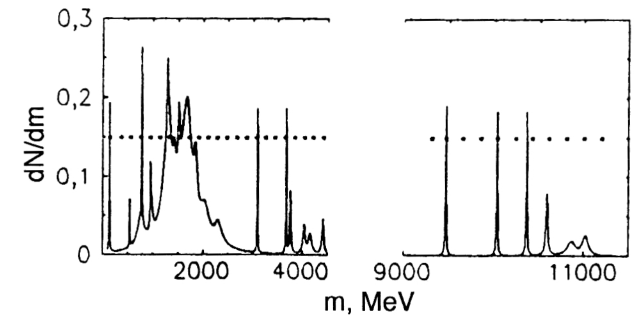

FIG. 9. Density distribution of the experimentally well-established mesonic states (solid curve) and positions of the "centroids" of n-pion resonances calculated in accordance with the formula (24).

TABLE XV. Masses (MeV) of resonances that decay through the channels $\Lambda\Lambda,(p\Lambda)_{2067}\Lambda,\Lambda(\pi\pi)_{361},\Sigma p$.

	$\Lambda\Lambda$ decay		$p\Lambda\Lambda$ decay		$\Lambda\pi\pi$ decay		Σp decay	
$n+\gamma$	Exp.	Theor.	Exp.	Theor.	Exp.	Theor	Exp.	Theor.
1/2		2243		3192		1500		2148
1	2290 [Ref. 150]	2278		3219		1566	2173 [Ref. 147]	2185
1+1/2	2365 [Ref. 147]	2335		3263	1704 [Ref. 147]	1666	2218 [Ref. 147]	2245
2		2412		3323		1789		2326
2+1/2		2508		3393		1931	2408 [Ref. 147] 2384 [Ref. 147]	2426
3		2621		3489	2071 [Ref. 147]	2087		2544
3+1/2		2748	3568 [Ref. 147]	3591		2253		2675
4		2887		3705		2428		2819
4+1/2		3038		3829	2604 [Ref. 147]	2611		2974
5		3198		3962		2800		3138

$(J/\psi(3097)(3\pi)_S)_S$, $(J/\psi(3097)(5\pi)_S)_S$, and $(J/\psi(3097)(7\pi)_S)_S$ with masses 3668, 4053, and 4438 MeV could be decay products of the states $\psi(3685)$, $\psi(4040)$, and $\psi(4415)$, respectively. The results of our calculations are shown in Fig. 9.

3. The ψ mesons (at least some of them) can be regarded as dimesons in the P state, $(D\bar{D})_P,(D\bar{D}^*)_P,(D^*\bar{D}^*)_P$, or as molecular states in accordance with the proposals made by Voloshin and Okun'[155] like the $K\bar{K}$ molecular description of the $f_0(975)$ and $a_0(980)$ resonances proposed in Ref. 156. The experimental data and our calculations in accordance with Eq. (24) show that the $\psi(3770)$ and $\psi(4040)$ states are candidates for such dimeson molecular states. Our model successfully describes the masses of these resonances (see Table XVI); the $(D\bar{D}^*)_P$ state with mass 3906 MeV (our estimate) has not been observed experimentally.

Finally, the ψ mesons may have molecular states in the three-meson decay channel (1^{--}). In this connection, we propose the following candidates:

- for the $\eta_c(2980)$ decay, the system

TABLE XVI. Calculated masses (MeV) of some ψ mesons.

Decay modes	Fraction Γ_i/Γ	P(exp)	P(theory)	m(theory)	$n+\gamma$
		η_c (2980), $I^G(J^{PC})=0^+(0^{-+})$			
$(\omega(1390)\omega(1600))_S$	-		115	2999	1/2
$((\phi\phi)_S\eta'(958))_S$	-	-	115	3021	1/2
		$J/\psi(3097)$, $I^G(J^{PC})=0^-(1^{--})$			
Decay modes	Fraction Γ_i/Γ	P(exp)	P(theory)	m(theory)	$n+\gamma$
$((8\pi)_S(8\pi)_S)_P$	-		229	3100	1
$(\Xi(1530)\bar{\Xi}(1530))_P$	-	-	229	3098	1
		$\psi(3686)$, $I^G(J^{PC})=?^?(1^{--})$			
Decay modes	Fraction Γ_i/Γ	P(exp)	P(theory)	m(theory)	$n+\gamma$
$(J/\psi(3097)\eta(547))_S$	2.7%	196	115	3658	1/2
$(J/\psi(3097)(\pi^0\pi^0)_S)_D$	18.4%	481	-	3704	2
$(J/\psi(3097)(\pi^+\pi^-)_S)_D$	32.4%	477	-	3714	2
		$\psi(3770)$, $I^G(J^{PC})=?^?(1^{--})$			
Decay modes	Fraction Γ_i/Γ	P(exp)	P(theory)	m(theory)	$n+\gamma$
$(D\bar{D})_P$	dominant	242	229	3766	1
		$\psi(4040)$, $I^G(J^{PC})=?^?(1^{--})$			
Decay modes	Fraction Γ_i/Γ	P(exp)	P(theory)	m(theory)	$n+\gamma$
$D^0\bar{D}^0$	seen	774	803	4059	3+1/2
$D^*(2010)^0\bar{D}^0$	seen	577	574	4040	2+1/2
$(D^*(2010)^0\bar{D}^*(2010)^0)_P$	seen	228	229	4040	1
		$\psi(4160)$, $I^G(J^{PC})=?^?(1^{--})$			
Decay modes	Fraction Γ_i/Γ	P(exp)	P(theory)	m(theory)	$n+\gamma$
$(\psi(3770)(\pi^-\pi^+)_S)_S$	not seen	-	115	4150	1/2
		$\psi(4415)$, $I^G(J^{PC})=?^?(1^{--})$			
Decay modes	Fraction Γ_i/Γ	P(exp)	P(theory)	m(theory)	$n+\gamma$
$(23\pi)_S$	not seen	-	-	4416	1/2

$[[\eta'(958)\eta'(958)]_P\eta'(958)]_P$ with mass 2970 MeV,

• for $J/\psi(3097)$, the system $[[\phi(1020)\phi(1020)]_S \phi(1020)]_S$ with mass 3082 MeV,

• for $\psi(3686)$, the system $[[\phi(1020)\phi(1020)]_S\omega(1600)]_S$ with mass 3660 MeV.

The present status of the Υ mesons was discussed in the review of Ref. 157; see also the results of the CLEO collaboration.[158] Bottomonium is usually regarded as a radially excited $b\bar{b}$ system. Our results, obtained in the framework of the model presented above, are given in Fig. 9. They correlate well with the experimental data, except in the regions of masses 9640, 9840, and 10227 MeV. As in the case of charmonium, the dimeson $(B\bar{B})_P$ molecular state with mass 10568 MeV can be associated with Υ(10580). The dimeson states $(B\bar{B}^*)_P$ and $(B^*\bar{B}^*)_P$ with masses 10614 and 10660 MeV, respectively, have not been observed experimentally.

Finally, the Υ mesons can have molecular states in the channel of three-charmonium–meson (1^{--}) decay. We propose the following candidates:

• for Υ(10023) decay, the system $[[J/\psi(3097)J/\psi(3097)]_S\psi(3770)]_S$ with mass 9971 MeV,

• for Υ(10355) decay, the system $[[J/\psi(3097)J/\psi(3097)]_S\psi(4160)]_S$ with mass 10361 MeV,

• for Υ(10580) decay, the systems $[[J/\psi(3097)J/\psi(3097)]_S\psi(4415)]_S$ and $[[J/\psi(3097)\psi(3685)]_S\psi(3770)]_S$ with masses 10615 and 10559 MeV, respectively,

• for Υ(10860) decay, the system $[[J/\psi(3097)\psi(3685)]_S\psi(4040)]_S$ with mass 10829 MeV.

So far, the following systems have not been observed:

1. $[[\eta_c(2980)\eta_c(2980)]_P\eta_c(2980)]_P$ with mass 8971 MeV.
2. $[[J/\psi(3097)J/\psi(3097)]_SJ/\psi(3097)]_S$ with mass 9298 MeV.
3. $[[J/\psi(3097)J/\psi(3097)]_S\psi(3685)]_S$ with mass 9886 MeV (there are some indications of the existence of such a state[159]).
4. $[[J/\psi(3097)J/\psi(3097)]_S\psi(4040)]_S$ with mass 10241 MeV.
5. $[[J/\psi(3097)\psi(3685)]_S\psi(4415)]_S$ with mass 11203 MeV (there are some indications of the existence of such a state[160]).

We have demonstrated that three-meson (1^{--}) molecular states in the decay channels must play an important role in the study of the mass distributions of the ψ and Υ mesons. It is interesting that the ψ particles are approximately three times heavier than the ϕ and ω particles. The Υ particles are approximately nine times heavier than the ϕ and ω particles and three times heavier than the ψ particles.

The main characteristic feature of such states of molecular type is that the relative momentum in any binary system is very small: $\sim$100 MeV/c. Therefore, we are dealing here with low-energy resonances that require further investigation.

This gives grounds for making the following assumption: Three-meson (1^{--}) molecular states can exist in the regions 27–33 GeV, 81–100 GeV (the region of the W and Z bosons), 240–300 GeV, etc.

Finally, it is interesting to note that molecular states in three-particle decay channels also exist for various mesonic and baryonic resonances. As an example, we give some of them here:

• for $\eta(547)$ decay, the system $[[\pi\pi]_S\pi]_S$ with mass 556 MeV,

• for $\eta'(958)$ decay, the system $[[\pi\pi]_S\eta]_S$ with mass 938 MeV,

• for $N(1440)$ P_{11} decay, the system $[[N\pi]_P\pi]_S$ with mass 1420 MeV,

• for $N(1520)$ D_{13} decay, the system $[[N\pi]_P\pi]_P$ with mass 1522 MeV,

• for $(PP\pi)(2065)$ decay,[161,162] the system $[[pp]_S\pi]_S$ with mass 2076 MeV.

10. PARADIGMS OF SIMILITUDE

It can be seen from the material presented above that many properties of hadronic resonances can be understood on the basis of the deep physical analogy with the theory of α decay and the theory of resonance effects in nuclear physics. We regard resonances as the result of excitation of natural modes of interacting systems in the case of infinite motion of them. The behavior (21) is related to the concept of "automodelity" in that symmetries of solutions and properties of boundary conditions are considered. If the wavelengths of the scattered particles are commensurate with the characteristic dimensions of the interacting systems or systems that are being formed (or, in other words, the de Broglie and Compton wavelengths are commensurate, this leads to quantization of the momentum and to a simple formula for the mass spectrum of the resonances like that of the series of atomic and nuclear lines. This establishes a similarity in the spectra of hadronic resonances and the nuclear and atomic spectra.

We give one further very interesting example that is hardly known in high-energy physics. We have in mind the physics of the so-called metallic clusters, which were discovered comparatively recently—in 1984. It was found that the atoms of some metals form bound systems consisting of a huge number of atoms (up to 20000). It was shown in experiments[163] that in the metallic clusters there exist shells, with, moreover, the same magic numbers as in nuclei and atoms. This discovery means that in nature there exists another small system possessing an average field. It can now be regarded as widely accepted (at the least, the majority of calculations are made in such a representation) that the metallic clusters represent a system of valence electrons moving in the average field of positively charged ions (see the reviews of Refs. 164–166 and the original studies of Refs. 167 and 168). The presence of the average field and, as a result, the presence of the shell effects has the consequence that the metallic clusters have some general properties (spectra) in common with hadronic resonances and atoms, nuclei, and solids. Thus, the metallic clusters constitute a unique phenomenon at the meeting place of elementary-particle physics, atomic and nuclear physics, and also the physics of the solid state, crystallography, thermodynamics, quantum and classical statistical physics, etc. Therefore, we must expect to find here new interesting results (see the reviews of Refs.

164–166). For example, in Ref. 165 an attempt was already made to trace the transition from atoms to solids and to understand the stage at which a metallic cluster represents an "embryon" of a solid, carrying the basic properties of a solid.

The physical nature of this similarity of the spectra of a large number of diverse systems is now understood. Shells in nuclei, atoms, and metallic clusters are a common characteristic property of closed quantum systems, and it is now a widely accepted point of view that shells arise as a result of the occurrence of closed periodic orbits of the motion of a particle in a closed cavity. These orbits are smeared and entangled by the quantum fluctuations of the system, and only the simplest orbits survive. These orbits can be interpreted as vibrations with nearly equal frequencies and amplitudes. The coherent interference of these amplitudes leads to beats of the resulting amplitude. This is the physical essence of the shells.

In connection with what we have said above, the question arises of the possible existence of shell effects in the mass distribution of the resonances of elementary particles. By all appearances, the experimental data presented above (see Fig. 9) do not contradict such a hypothesis. However, this hypothesis requires further verification.

The original theory of the hydrogen atom was essentially constructed by Bohr by analogy with the planetary solar system and was called the "planetary theory of the hydrogen atom." This was the starting point for the development of modern quantum theory, or "wave mechanics" in the old terminology. Conversely, by analogy with the theory of hydrogen-like atoms one now begins to regard the solar system as a wave dynamical system.[169] It was found that the spectrum of wave frequencies of the solar system can be represented in a form that resembles the scheme of spectroscopic levels of a hydrogen-like atom. The concept of wave resonance of the solar system proved to be very successful in the description of astrodynamical rhythms (for details, see Ref. 169).

The method of analogy is very widely and successfully used in the investigation of the most varied effects. The "triple analogy" is well known. The related nature of the processes of transport of heat (heat transfer), matter (mass transfer), and momentum (hydrodynamic resistance) is manifested under certain conditions as similitude of velocity, temperature, and concentration fields. Many textbooks give examples of electrohydrodynamic and electrothermal analogies. Electrical analogy is an exceptionally effective method of experimental study of slow thermal or hydrodynamic effects. Replacement of an investigated process by its electrical analog gives great advantages. For example, the time required to study the annual variations of soil temperature is reduced by nine orders of magnitude in the method of electrical analogy. Moreover, the method gives an excellent picture of all the characteristic properties of the annual variations of the soil temperature.

Naturally, the method of analogy can be applied only to effects that are described by the same equation and the same boundary conditions in a dimensionless representation. In other words, the method of analogy is used to investigate the properties of a little studied effect by comparison with a well-known effect, and in a dimensionless representation the boundary-value problems are identical for the two cases. Identity of the mathematical form of a boundary-value problem can reflect real unity of the physical mechanism of studied effects as in the case of the "triple analogy," in which all three compared effects are due to the same process of displacement of a medium[170] if the medium is matter in the limiting gaseous state. Under these conditions, a pure form of molecular transport is realized. The carriers are molecules that in the process of their displacement are not subject to any influences and interact only in collisions. Under such circumstances, all transport processes (heat flow, matter flow, and momentum flow) develop in exactly the same way. Therefore there must be a rigorous physical analogy. Identity of the dimensionless laws is a reflection of the identity of the physical conditions and the unity of the physical mechanism of the compared processes.

More attractive for us are the cases when the compared processes have different physical essences. These include the cases mentioned above of the electrothermal and electrohydrodynamic analogies (for more details, see the monographs of Refs. 170 and 171). Thus, the compared processes can include phenomena of different physical natures. The only criterion in accordance with which the phenomena are reduced to one and the same group for investigation by means of the method of analogy is identity of the dimensionless representations. There arise in this way special correspondences (identity of all dimensionless characteristics and their ratios) between phenomena that are arbitrarily different in their physical nature. Processes are similar (or are not similar) to each other irrespective of whether they are the same or different in their physical nature.

Therefore, when the method of analogy is used the question arises of the criterion of similitude. In other words, the question arises of the quantitative characteristics of the phenomenon and of the scales of the relationship. Identity of the mathematical form of the equations for compared effects in a dimensionless representation need not ensure similitude of the effects. The corresponding boundary conditions must also be identical. In many cases, one can identify a particular value of a complex of parameters (variables) of the problem, which is called the characteristic value, whereas the directly specified value is distinguished as the typical value. Equality of the characteristic value of the parameter complex to its typical value is a necessary condition for similitude of different effects. We illustrate this by the example of the Biot dimensionless number[170,171] (the case of heat transfer from a solid to a liquid and vice versa), which is obtained from the continuity equation on the surface of the system:

$$Bi=\frac{l}{\delta}, \tag{67}$$

where l is a geometrical parameter, the typical dimension of the system. The parameter δ, which is a combination of thermophysical parameters ($\delta=\lambda/\alpha$), is called the characteristic length. This dimensionless number combines one parameter l characterizing the geometrical properties of the system and two thermophysical parameters characterizing the intensity of heat transfer: α from the liquid to the solid and λ within the solid. An important circumstance for us is the possibility of reducing the definition of the physical conditions on the boundary of the body to the single characteristic length δ. Comparison of this length with the typical scale l of the body leads to the new form of the Biot dimensionless number.

The most important feature of this problem is that it is formulated in terms of boundary conditions of the third kind. Therefore, in accordance with the formulation of the problem the temperature conditions are assumed to be known only for the ambient medium.

In what follows, we shall attempt to take the ideas presented in the above heat-transfer problem to solve quantum-mechanical problems. The boundary conditions usually employed in quantum theory are such that it is difficult to use here directly the methods of classical physics, in particular, methods of similitude. We shall consider some examples from quantum theory.

We begin with the example of the hydrogen atom. The electrostatic force between the proton and the electron leads to the formation of bound states of the hydrogen atom. Following Bohr, we require equality of the Coulomb and centrifugal forces (Ehrenfest's theorem for writing down the equation of motion of the electron along a semiclassical trajectory):

$$\frac{e^2}{r^2}=\frac{mv^2}{r}=\frac{L^2}{mr^3}=\frac{n^2\hbar^2}{mr^3}. \tag{68}$$

In obtaining the last relation, we have used Bohr's quantization condition (4). From equality of the first and last terms in (68), we obtain the admissible values of r, or the Bohr radii:

$$r=\frac{L^2}{me^2}=\frac{n^2\hbar^2}{me^2}=n^2a_0, \tag{69}$$

where by definition a_0 is the radius of the first Bohr orbit:

$$a_0=\frac{\hbar^2}{me^2}=\alpha^{-1}\lambda_C=\lambda_D. \tag{70}$$

Therefore, the radius of the first Bohr orbit is equal to the corresponding electron de Broglie wavelength, and the ratio of the Compton wavelength to the de Broglie wavelength is equal to the fine-structure constant $\alpha=e^2/\hbar c$. From (68) we can also obtain the useful relations

$$\alpha=n\frac{\lambda_C}{\lambda_D}=n\frac{v}{c}, \tag{71}$$

$$\frac{r}{\lambda_D}=n. \tag{72}$$

Thus, the Compton and de Broglie wavelengths play a fundamental role in the quantization of the radius of the orbits of the hydrogen atom; moreover, such quantization is possible only when the ratio of the Compton wavelength to the de Broglie wavelength (v/c) is comparable with the fine-structure constant α. The ratio (72) can be interpreted as a parameter of similitude for the hydrogen atom, in which r is the radius of the Bohr orbit, playing the role of the geometrical dimension of the system (typical radius), while $\lambda_D=\hbar/P=\hbar/mv$, which has the dimensions of a length, plays the role of the characteristic radius. Therefore, the ratio of the typical radius to the characteristic radius is a rational (in the present case, integer) number that plays the role of a parameter of similitude in the quantization problem for the hydrogen atom. The parameter of similitude can be rewritten in the different form $\alpha\lambda_D/\lambda_C$, as is obvious.

To conclude this example, we find the energy of the hydrogen atom, which is equal to the sum of the kinetic and potential energies:

$$E=T+V=\frac{mv^2}{2}-\frac{e^2}{r}=-\frac{e^2}{2r}=-\frac{e^2}{2n^2a_0}. \tag{73}$$

In deriving this relation, we have used Eq. (68).

We have deliberately repeated Bohr's prescription for quantization of the hydrogen atom (now known as the correspondence principle), emphasizing the possibility of obtaining a parameter of similitude. We can then readily observe an analogy with the problem of heat transfer (continuity equation on the surface of the solid $\Leftrightarrow$ equality of the Coulomb and centrifugal forces on the Bohr orbit). The difference is that, in his calculations, Bohr used an Ehrenfest adiabatic invariant (in the given case, the angular momentum of the system) and the hypothesis of its quantization. Further, a solid has a clearly defined physical boundary surface, whereas in the case of the hydrogen atom there is no such physical surface. However, Bohr defined the "surface" of the hydrogen atom as quantized orbits on which there is equality of the Coulomb and centrifugal forces. Later, de Broglie interpreted the appearance of stable quantized orbits as the result of occurrence of standing waves. As we see, Bohr quantized the hydrogen atom on the basis of the method of analogy. We have briefly discussed above the modern development of Bohr's ideas for the study of regular and chaotic motion of closed systems. We note once more the well-known fact that Bohr solved the problem of quantizing the hydrogen atom in 1913, long before the creation of quantum mechanics.

To illustrate the theses presented above, we use Bohr's method to quantize open systems, in particular, to study the properties of hadronic resonances. We shall here make an analogy with the theory of α decay and the theory of open electromagnetic resonators.

We consider P-wave pion–nucleon scattering. As was noted above, the long-range part of the effective pion–nucleon attractive potential is described by the expression (58). Using Newton's second law to write down the equation of motion along the semiclassical trajectory (actually, Ehrenfest's theorem),

$$\frac{m_\pi v^2}{r}=\frac{\partial U_{\rm eff}^{\pi N}}{\partial r}, \tag{74}$$

and Bohr's quantization condition for the angular momentum,

$$m_\pi vr=L, \tag{75}$$

and also taking into account the relativistic corrections ($m_\pi\to E_\pi$) in (75), we obtain the equation

$$L^2=f^2_{\pi NN}\frac{m+m_\pi}{\sqrt{s}}\frac{m}{m_\pi}\frac{P^2_\pi}{E_\pi}r_0(1+\alpha r_0)e^{-\alpha r_0}\equiv F^2(s,r_0), \tag{76}$$

which establishes a relationship between the constant $f_{\pi NN}$, the parameter $r_0=0.86$ F of the theory, and the mass of the P resonance: $M(R)=\sqrt{s}$. At the same time, the masses of the ground states of the hadrons (i.e., the stable pion and proton particles) are assumed given. In this case it turns out that if for the two lowest P-wave resonances [the $\Delta(1232)$ resonance and the Roper resonance $N^*(1440)$] we fix in accordance with the general philosophy the parameter $r_0=0.86$ F, we immediately reproduce the peak ($\sqrt{s}=M_\Delta=1232$ MeV)

$$F(M^2_\Delta,r_0=0.86)=0.97\approx 1=L. \tag{77}$$

Similarly, for the Roper resonance $N^*(1440)$ (i.e., P_{11}, $L=1$ and $M_{N^*}=1440$ MeV) we obtain

$$F(M^2_{N^*(1440)},r_0=0.86)=0.91\approx 1=L. \tag{78}$$

This means that the parameter $r_0=0.86$ F has the physical meaning of an analog of the first Bohr orbit for the strong interaction. It is interesting to note that if we regard the parameter r_0 as a function of the coupling constant $f_{\pi NN}$ on the basis of Eq. (76), then we immediately arrive at the asymptotic estimate $r_0\propto\ln(f_{\pi NN})$, which unexpectedly illustrates the impossibility of constructing a perturbation theory for the standard effective interaction Hamiltonian

$$H_{\pi NN}=-\frac{f_{\pi NN}}{m_\pi}(\boldsymbol{\sigma}\cdot\Delta)(\boldsymbol{\tau}\cdot\boldsymbol{\varphi}). \tag{79}$$

The spectrum of hadronic resonances is quasimolecular in the sense that its gross structure is formed in the surface region of the parent hadrons, where the final products have already been formed, and the quasistationary states of molecular type are formed by final-state interaction. Thus, the wave properties of matter are the most important characteristics of the macroscopic and microscopic world. The resonances (standing waves) that arise in the corresponding systems ensure their stability and determine their "lifetime."

11. CONCLUSIONS

As we noted in the Introduction, a relatively small number of "coarse" systems have been well studied in physics. The integral properties of these systems can be described with reasonable accuracy in the simplest approaches, but the adequacy of description of the physical picture becomes worse as the picture is made more detailed. In physics, a classical example of correct treatment of coarse systems is the theory of diffraction, in which the determining parameters of the problem are the target diameter and the wavelength of the incident particle. The experimentally observed diffraction picture is practically independent of the physical characteristics of the target material and the actual physical nature of the wave.

Diffraction is one of the most general physical effects and is observed in all quasilinear wave systems in both classical and quantum physics. Moreover, in all the studied cases of diffraction the relations of the theory of similitude and automodelity are satisfied: Apart from the scale of the problem, the final result is determined solely by the ratio of the target diameter and the wavelength, and these quantities are always commensurate.

In Refs. 36–39, the diffraction concept of describing hadronic resonances proposed in this review was formulated as follows: Periodic motion, refraction, and interference of de Broglie waves in a bounded volume of space are responsible for the formation of resonances in any quantum system. If the wavelength of the incident particle is such that it excites characteristic vibrations of the target, then the phenomenon of resonance occurs. As a result, there arise well-localized waves concentrated at the interface of two media (the vacuum and the decaying hadron). The boundary conditions play a decisive role in the occurrence of such localization. In this review, we have restricted ourselves to studying the case of a radiation boundary condition, which led to asymptotic quantization of the momentum and made it possible to reproduce the experimentally observed gross structure of the mass spectrum of hadronic resonances.

Thus, in the framework of the diffraction theory of hadronic resonances presented here we in fact encounter the problem of finding the discrete spectrum of strongly interacting elementary particles (i.e., according to de Broglie, a discrete set of characteristic vibrations in a wave system) for the case of infinite motion. In the case, well-studied in nonrelativistic quantum mechanics, of finite motion, the problem of finding the discrete spectrum is formulated in the form of the Schrödinger equation for bound states. For the case of infinite motion, to which in wave mechanics there corresponds a boundary condition in the form of a plane-wave asymptotic behavior, the wave operator has a discrete spectrum only if the radiation condition is imposed at the channel radius. Such a generalization of the theory of diffraction to the case of the physics of intermediate energies is equivalent from the formal point of view to R-matrix theory.[15]

In the framework of this diffraction approach, the widths of hadronic resonances have been found on the basis of the deep physical analogy with the theory of α decay and have been found to be in good agreement with the existing experimental data.

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Finally, we should like to mention that many of the ideas presented in the review were born under the influence of the papers of A. M. Baldin.

12. APPENDIX

To obtain the asymptotic quantization condition (21), we consider the boundary condition (18). The spherical Riccati–Hankel functions are related to the Hankel functions by

$$h_L^{(1)}(Pr)=\sqrt{\frac{\pi Pr}{2}}\,H_{L+1/2}^{(1)}(Pr).$$

Using the well-known asymptotic expansions for a Hankel function and its derivative,[172] we can obtain expressions that are uniform for $|\arg Pr|\leqslant\pi-\epsilon,\ \epsilon>0$:

$$H_{L+1/2}^{(1)}(Pr)\approx 2e^{-\pi i/3}\left(\frac{4\zeta}{1-z^2}\right)^{1/4}\frac{Ai(\xi)}{\left(L+\frac{1}{2}\right)^{1/3}},$$

$$[H_{L+1/2}^{(1)}(Pr)]'\approx -4\frac{e^{\pi i/3}}{z}\left(\frac{1-z^2}{4\zeta}\right)^{1/4}\frac{Ai'(\xi)}{\left(L+\frac{1}{2}\right)^{2/3}},$$

where

$$z=Pr/\left(L+\frac{1}{2}\right),\quad \xi=e^{2\pi i/3}\left(L+\frac{1}{2}\right)^{2/3}\zeta,$$

$$\frac{2}{3}\zeta^{3/2}=\log\frac{1+\sqrt{1-z^2}}{z}-\sqrt{1-z^2}.$$

Taking into account the asymptotic behavior for the Airy function at large ξ,

$$Ai(\xi)\approx\frac{1}{2}\pi^{-1/2}\xi^{-1/4}e^{-2/3\xi^{3/2}}(1+O(\xi^{-1})),$$

$$Ai'(\xi)\approx-\frac{1}{2}\pi^{-1/2}\xi^{1/4}e^{-2/3\xi^{3/2}}(1+O(\xi^{-1})),$$

we obtain

$$Ai'(\xi)/Ai(\xi)\approx-\xi^{1/2}.$$

The logarithmic derivatives in this case have the form

$$[H_{L+1/2}^{(1)}(Pr)]'/H_{L+1/2}^{(1)}(Pr)|_{r=r_0+0}\approx-\sqrt{1-z^2}/z. \quad \text{(A1)}$$

Following Refs. 36–39, we assume that for well-isolated resonances $f=0$. We consider this case. For $|Pr_0|\gg 1$, we obtain

$$\left.\frac{[H_{L+1/2}^{(1)}(Pr)]'}{H_{L+1/2}^{(1)}(Pr)}\right|_{r=r_0+0}\approx\left.\frac{1}{h_L^{(1)}(Pr)}\frac{dh_L^{(1)}(Pr)}{dr}\right|_{r=r_0+0}=0. \quad \text{(A2)}$$

It follows from (A1)–(A2) that $Pr_0\approx L+1/2$, and we must consider asymptotic expansions near the caustic surface. For $z=O(1)$ and $(L+1/2)\gg 1$, we can use the following asymptotic behaviors:

$$H_{L+1/2}^{(1)}(Pr)\approx(Ai(-\tau)-iBi(-\tau))/\nu,$$

$$[H_{L+1/2}^{(1)}(Pr)]'\approx-(Ai(-\tau)-iBi(-\tau))/\nu^2,$$

where $\tau=(Pr-L-1/2)/\nu$, $\nu=((L+1/2)/2)^{1/3}$. Therefore

$$\left.\frac{1}{h_L^{(1)}(Pr)}\frac{dh_L^{(1)}(Pr)}{dr}\right|_{r=r_0+0}\approx\frac{1}{2r}-\left.\frac{P}{\nu}\frac{Ai'(-\tau)-iBi'(-\tau)}{Ai(-\tau)-iBi(-\tau)}\right|_{r=r_0+0}.$$

Bearing in mind that $(L+1/2)\gg 1$ and $z=O(1)$, we obtain

$$\left.\frac{1}{h_L^{(1)}(Pr)}\frac{dh_L^{(1)}(Pr)}{dr}\right|_{r=r_0+0}\approx-\left.\frac{2\nu^2}{r}\frac{Ai'(-\tau)-iBi'(-\tau)}{Ai(-\tau)-iBi(-\tau)}\right|_{r=r_0+0}.$$

Using the relations between the Airy functions, we obtain

$$\left.\frac{1}{h_L^{(1)}(Pr)}\frac{dh_L^{(1)}(Pr)}{dr}\right|_{r=r_0+0}\approx-\frac{2\nu^2}{r}e^{2\pi i/3}\left.\frac{Ai'(\tau e^{-\pi i/3})}{Ai(\tau e-^{\pi i/3})}\right|_{r=r_0+0}\approx 0.$$

Therefore, $Ai'(\tau e^{-\pi i/3})=0$. From this we obtain

$$Pr_0=L+\frac{1}{2}+e^{\pi i/3}\nu a_s',\quad s=1,2,\ldots,$$

where a_s' are the zeros of Ai' (a_s' are real negative numbers).

The problems of the quantization conditions for resonance states and the functional behavior of the resonance wave functions in space have not been investigated in general form. In the initial stage, these questions can be studied in model examples. In this Appendix, we assume for simplicity that the interior wave functions of the resonance states are solutions of the Schrödinger equation with an optical potential in the form of a complex rectangular well:

$$V(r)=\begin{cases}-(V_0+iW_0) & r\leqslant r_0\\ 0 & r>r_0\end{cases}.$$

Then $u_{in}(r)=j_L(Qr)=\sqrt{\pi Qr/2}J_{L+1/2}(Qr)$. Using the asymptotic expansion $J_{L+1/2}(L+1/2+\tau\nu)\approx Ai(-\tau)/\nu$ and the well-known[173] estimate $|Ai(x)|\leqslant(2\sqrt{\pi})^{-1}\times x^{-1/4}e^{-(2/3)x^{3/2}}$ for the Airy function when $x>0$, we can readily find that $j_L(Qr)$ decreases exponentially when Qr varies from $L+1/2$ to 0. Thus, within the sphere $r\leqslant r_c$ [$r_c=(L+1/2)/Q$ is the radius of the caustic surface] the wave function is concentrated within a ring $L+1/2\leqslant Qr\leqslant Qr_0$, the width of which is approximately $\nu|a_s'|$. This corresponds to the phenomenon of whispering-gallery waves (see also Refs. 50 and 59).

Further, we require the resonance wave functions to have a maximum at the well radius $r=r_0$, i.e., we impose the boundary conditions

$$h_L^{(1)'}(Pr_0)=0. \quad \text{(A3)}$$

Then from the condition of equality, on the boundary, of the logarithmic derivatives of the interior and exterior wave functions,

TABLE XVII. Values of $P_{(k)}r_0$, $-V_{(k,l)}$, and $\bar{P}_{(k,l)}r_0$ in the case $h_L^{(1)\prime}(Pr_0)=0$.

L	Pr_0	V_0+iW_0	$\bar{P}r_0$
1	$\pm 0.866-i0.500$	$394.3\pm i48.6$	$0.979-i0.356$
2	$\pm 1.807-i0.702$	$684.7\pm i142.4$	$1.991-i0.426$
2	$-i1.60$	983.3	$1.191-i0.081$
3	$\pm 2.758-i0.843$	$1000.8\pm i260.8$	$2.991-i0.468$
3	$\pm 0.871-i2.157$	$1606.2\pm i210.7$	$1.028-i2.143$
4	$\pm 3.715-i0.954$	$1338.5\pm i397.7$	$3.998-i0.500$
4	$\pm 1.752-i2.571$	$2260.3\pm i505.6$	$2.024-i2.539$
4	$-i2.949$	2549.4	$-i2.949$
5	$\pm 4.676-i1.048$	$1695.0\pm i549.7$	$4.938-i0.527$
5	$\pm 2.644-i2.908$	$2942.4\pm i862.8$	$3.003-i2.859$
5	$\pm 0.869-i3.544$	$3522.7\pm i345.5$	$0.991-i3.546$
6	$\pm 5.641-i1.129$	$2068.2\pm i714.6$	$5.977-i0.550$
6	$\pm 3.545-i3.195$	$3650.1\pm i1270.9$	$3.976-i3.132$
6	$\pm 1.743-i4.034$	$4524.7\pm i788.9$	$1.956-i4.039$
6	$-i4.285$	4812.3	$-i4.285$
7	$\pm 6.610-i1.201$	$2455.8\pm i891.7$	$6.972-i0.571$
7	$\pm 4.452-i3.448$	$4381.3\pm i1722.4$	$4.944-i3.371$
7	$\pm 2.623-i4.454$	$5553.7\pm i1311.0$	$2.907-i4.463$
7	$\pm 0.868-i4.897$	$6129.9\pm i477.2$	$0.970-i4.902$

$$j_L'(Qr_0)h_L^{(1)}(Pr_0)-j_L(Qr_0)h_L^{(1)\prime}(Pr_0)=0, \tag{A4}$$

it follows that

$$j_L'(Qr_0)=0. \tag{A5}$$

Here $Q=\sqrt{-(2m/\hbar^2)V(r)+P^2}$. We can regard the boundary condition (A3) as a quantization condition for the asymptotic momentum P. The number of roots of Eq. (A3) is $L+1$. We denote them by $P_{(k)}$, $k=1,2,\ldots,L+1$. On the other hand, the number of roots of Eq. (A5) is infinite; we denote them by $Q_{(n)}$, $n=1,2,3,\ldots$. Further, we can determine the values of the potentials using the formula

$$V_{(k,n)}(r)=\frac{\hbar^2}{2m}(P_{(k)}^2-Q_{(n)}^2). \tag{A6}$$

Thus, proceeding solely from the boundary conditions (A3) and (A5), we have the possibility of calculating an unbounded discrete sequence of complex potentials $V_{(k,n)}$. In the considered example, we have for simplicity used rectangular-well potentials, and in this case the specification of the boundary conditions is sufficient to determine the parameters of the potential. At the same time, we obtain an equivalent unbounded discrete sequence of complex potentials. Physically, these potentials can be chosen on the basis of the number of zeros of the interior wave functions $u_{in}(r)$ in the region $0<r\leqslant r_0$. For example, the wave function $u_{in}(r)$ of the lowest-energy resonance state must not have zeros, and so forth. From the general point of view, one can specify any physically sensible functional form for the potential and then determine some of its parameters from the boundary conditions. In other words, the specification of the boundary conditions imposes restrictions on the parameters of the employed potentials, and in this sense a self-consistent problem for describing resonances arises. Such a procedure is well developed in the theory of direct nuclear reactions; we refer the reader to the reviews of Refs. 174–176.

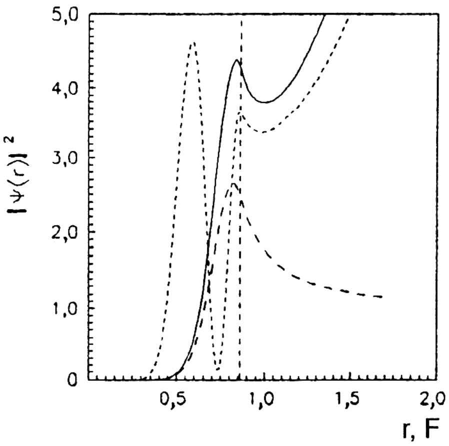

FIG. 10. Wave functions $\psi_{L=7}^{\rm orb}$ (solid curve), $\psi_{L=7}^{\rm rad}$ (short dashes), and $\psi_{L=7}^{\rm cau}$ (long dashes).

Further, we consider real potentials, taking the real parts of the potentials $V_{(k,n)}(r)$ calculated above, i.e,

$$\bar{V}_{(k,n)}(r)=\mathrm{Re}V_{(k,n)}(r). \tag{A7}$$

We find for them eigenvalues $\bar{P}_{(k,n)}$ satisfying the condition of equality of the logarithmic derivatives (A4). The values of $P_{(k)}r_0$, $-V_{(k,l)}$, and $\bar{P}_{(k,l)}r_0$ are given in Table XVII. It can be seen from the table that

$$\bar{P}_{(1,1)}r_0\approx L. \tag{A8}$$

Thus, proceeding solely from the boundary conditions, we have obtained a condition of quantization of the asymptotic momentum P (analog of Bohr–Sommerfeld quantization) for the resonance states. The corresponding wave functions $\psi_{L=7}^{\rm orb}(\bar{P}_{(1,1)}r)$ are localized near the surface $r=r_0$ (see Fig. 10), and we may interpret $r=r_0$ as the first Bohr orbit of the resonance state with angular momentum L.

TABLE XVIII. Values of $P_{(k)}r_0$, $-V_{(k,l)}$, and $\bar{P}_{(k,l)}r_0$ in the case $h_{L+1}^{(1)}\times(Pr_0)=0$.

L	Pr_0	V_0+iW_0	$\bar{P}r_0$
1	$\pm 0.866-i1.500$	$1947.8\pm i145.8$	$1.465-i0.606$
2	$\pm 1.1754-i1.839$	$2756.6\pm i362.0$	$2.484-i0.555$
2	$-i2.322$	3042.1	$1.715-i0.230$
3	$\pm 2.657-i2.104$	$3608.5\pm i625.7$	$3.481-i0.530$
3	$\pm 0.867-i2.896$	$4184.7\pm i281.8$	$2.280-i0.134$
4	$\pm 3.572-i2.325$	$4498.5\pm i931.5$	$4.472-i0.518$
4	$\pm 1.746-i3.352$	$5370.7\pm i655.4$	$3.003-i0.103$
4	$-i3.647$	5656.9	$2.369-i0.075$
5	$\pm 4.493-i2.516$	$5423.2\pm i1268.3$	$5.463-i0.507$
5	$\pm 2.626-i3.736$	$6596.5\pm i1100.9$	$3.815-i0.092$
5	$\pm 0.868-i4.248$	$7170.8\pm i413.5$	$2.689-i0.0075$
6	$\pm 5.421-i2.686$	$6379.8\pm i1633.5$	$6.454-i0.502$
6	$\pm 3.517-i4.070$	$7859.0\pm i1606.3$	$4.678-i0.088$
6	$\pm 1.739-i4.758$	$8719.6\pm i926.9$	$3.229-i0.0037$
6	$-i4.971$	9010.4	$2.504-i0.00027$
7	$\pm 6.364-i2.839$	$7365.9\pm i2024.0$	$7.445-i0.500$
7	$\pm 4.414-i4.368$	$9155.9\pm i2163.7$	$5.571-i0.0865$
7	$\pm 2.616-i5.205$	$10314.5\pm i1527.9$	$3.895-i0.0026$
7	$\pm 0.868-i5.588$	$10888.2\pm i544.0$	$2.582-i0.168e-4$

Using the recursion relations for the Riccati–Hankel and Riccati–Bessel functions, we transform the expression (A4) to the form

$$j_L(Qr_0)h_{L+1}^{(1)}(Pr_0)-j_{L+1}(Qr_0)h_L^{(1)}(Pr_0)=0. \quad \text{(A9)}$$

We impose the following boundary condition at $r=r_0$:

$$h_{L+1}^{(1)}(Pr_0)=0. \quad \text{(A10)}$$

Then

$$j_{L+1}(Qr_0)=0. \quad \text{(A11)}$$

Repeating the above procedure, we obtain values of $P_{(k)}$, $Q_{(n)}$, $V_{(k,n)}$, $k=1,2,\ldots,L+1$, $n=1,2,\ldots$, satisfying the conditions (A9)–(A11), and we calculate $\bar{P}_{(k,n)}$ for the real potentials Re $\bar{V}_{(k,n)}(r)$. The results of the calculations are given in Table XVIII. It can be seen from the table that to good accuracy the following quantization condition holds:

$$\bar{P}_{(1,1)}r_0\approx L+\frac{1}{2}. \quad \text{(A12)}$$

The corresponding wave function $\psi^{\text{rad}}_{L=7}(\bar{P}_{(1,1)}r)$ for $L=7$ is given in Fig. 10; compared with the wave function $\psi^{\text{orb}}_{L=7}(\bar{P}_{(1,1)}r)$, it has one additional zero within the range of the potential. Therefore, we call the quantization condition (A12) radial quantization.

Finally, there is the interesting special case in which the caustic surface for free motion of the decay products of the resonance coincides with r_0; we then obtain the quantization condition

$$P_{\text{cau}}r_0=\sqrt{L(L+1)}\approx L+\frac{1}{2}. \quad \text{(A13)}$$

The corresponding wave function $\psi^{\text{cau}}_{L=7}(\bar{P}_{\text{cau}}r)$ is shown in Fig. 10.

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