Quantum groups and Yang–Baxter equations

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The principles of the theory of quantum groups are reviewed from the point of view of the possibility of using them for deformations of symmetries in physics models. The R-matrix approach to the theory of quantum groups is discussed in detail and taken as the basis of quantization of classical Lie groups and also some Lie supergroups. Rational and trigonometric solutions of the Yang–Baxter equation associated with the quantum groups $GL_q(N)$, $SO_q(N)$, and $Sp_q(2n)$ are presented. Elliptic solutions of the Yang–Baxter equation are also given. Applications of the theory of quantum groups and Yang–Baxter equations in different branches of theoretical physics are briefly discussed.

1. INTRODUCTION

In modern theoretical physics, the ideas of symmetry and invariance play a very important role. As a rule, symmetry transformations form groups, and therefore the most natural language for describing symmetries is the language of group theory.

About 15 years ago, in the study of two-dimensional integrable systems in the framework of the quantum inverse scattering method,[1] new algebraic structures arose, the generalizations of which were later called *quantum groups*. Similar structures also appeared in the solution of some models of statistical mechanics[2] and in the study of factorized scattering of solitons and strings.[3,4] So-called Yang–Baxter equations became a unifying basis of all these investigations.

The most important nontrivial examples of quantum groups are Lie quantum groups and quantum algebras, which can be regarded as deformations or quantizations of ordinary classical Lie groups and algebras (more precisely, one considers the quantization of the algebra of functions on a Lie group and the universal covering Lie algebra). The quantization is accompanied by the introduction of an additional parameter q (the deformation parameter), which plays a role analogous to the role of Planck's constant in quantum mechanics. In the limit $q \to 1$, the quantum Lie groups and algebras go over into the classical ones.

Although quantum groups are not groups in the mathematical sense, they nevertheless possess several properties that make it possible to speak of them as "symmetry groups." There are well-known examples of statistical systems (anisotropic Heisenberg magnets) and also systems of deformed oscillators with Hamiltonians invariant with respect to the special action of quantum groups (see, for example, Ref. 5). In this connection, the idea naturally arises of looking for and constructing other physical models possessing such quantum symmetries.

We list some of the existing approaches associated with realization of the idea of the quantization of symmetries in physics. Some of these approaches use the identity of the theory of representations of quantum and classical Lie groups and algebras (for q not equal to the roots of unity). As a result, we have, for example, identity of the dimensions of the irreducible representations (multiplets) for the group $SU(N)$ and for the quantum group $SU_q(N)$. Thus, we can use quantum Lie groups both for the classification of elementary particles and in nuclear spectroscopy investigations. Further, it is natural to wish to investigate the already existing field-theoretical models (for example, the Salam–Weinberg model or the standard model) with a view to establishing their connection with noncommutative geometry (see Ref. 6) and, in particular, the possibility of their being invariant with respect to quantum-group transformations. A very attractive idea is that of relating the deformation parameters of quantum groups to the mixing angles that occur in the standard model as free parameters. One of the possible realizations of this idea was proposed in Ref. 7. We also mention here the numerous attempts to deform the Lorentz and Poincaré groups and the construction of quantum versions of space–time corresponding to these deformations.[8,9]

The approaches that we have listed above associated with quantization of symmetries in physics represent only a small fraction of all the applications of the theory of quantum groups. Quantum groups and Yang–Baxter equations arise naturally in many problems of theoretical physics, and this makes it possible to speak of them and the theories of them as a new paradigm in mathematical physics. Unfortunately, the strict limits of the review make it impossible to discuss in detail all applications of quantum groups and Yang–Baxter equations. I have therefore restricted myself to a brief listing of certain areas in theoretical physics and mathematics in which quantum groups and Yang–Baxter equations play an important role. The list is given in the Conclusions. In Sec. 2, the mathematical foundations of the theory of quantum groups are presented. A significant proportion of Sec. 3, is a detailed exposition of the work of Faddeev, Reshetikhin, and Takhtadzhyan,[10] who have formulated the R matrix approach to the theory of quantum groups. In this section, we also consider questions of invariant Baxterization of R matrices, many-parameter deformations of Lie groups, and the quantization of some Lie supergroups. At the end of Sec. 3, we give elliptic solutions of the Yang–Baxter equation for which the algebraic basis (the type of quantum Lie groups in the case of trigonometric solutions) has not yet been clarified. Readers who are not fully acquainted with the theory of quantum groups are advised to begin reading the review with Subsection 3.3 in Sec. 3.

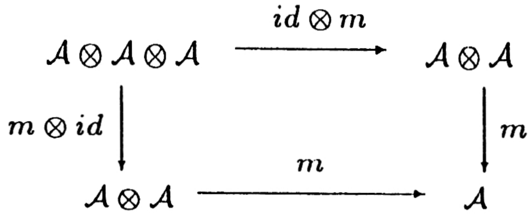


FIG. 1. Associativity axiom.

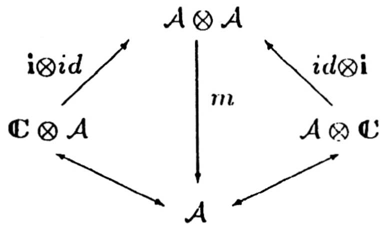


FIG. 2. Axioms for the identity.

As we have already mentioned, some applications of quantum groups and the Yang–Baxter equations are briefly considered in the Conclusions.

2. HOPF ALGEBRAS

This section of the review is based on the publications of Refs. 11–15.

We consider an associative algebra $\mathcal{A}$ with identity (over the field of complex numbers C; in what follows, all algebras that are introduced will also be understood to be over the field of complex numbers), each element of which can be expressed as a linear combination of basis elements $\{e_i\}$, where $i=1,2,3,...$ and $E^ie_i=I$ ($E^i\in C$) is the identity element. This means that for any two elements e_i and e_j we can define their multiplication in the form

$$\mathcal{A}\otimes\mathcal{A}\overset{m}{\rightarrow}\mathcal{A}\Rightarrow e_i\cdot e_j=m_{ij}^k e_k, \quad (2.1)$$

where m_{ij}^k is certain set of complex numbers that satisfy the condition

$$E^i m_{ij}^k=m_{ji}^k E^i=\delta_j^k \quad (2.2)$$

for the identity element, and also the condition

$$m_{ij}^l m_{lk}^n=m_{il}^n m_{jk}^l\equiv m_{ijk}^n, \quad (2.3)$$

which is equivalent to the condition of associativity for the algebra $\mathcal{A}$:

$$(e_ie_j)e_k=e_i(e_je_k). \quad (2.4)$$

The condition of associativity (2.4) for the multiplication (2.1) can obviously be represented in the form of a diagram (Fig. 1, in which we have in mind the commutativity of the diagram). In Fig. 1, m represents multiplication, and id denotes the identity mapping. The existence of the identity I means there exists a mapping $\mathbf{i}$:C$\rightarrow\mathcal{A}$ (embedding of C in $\mathcal{A}$)

$$k\overset{\mathbf{i}}{\rightarrow}k\cdot I, \quad k\in C. \quad (2.5)$$

For I, we have the condition (2.2), which is equivalent to the diagram of Fig. 2, in which the mappings C$\otimes\mathcal{A}\leftrightarrow\mathcal{A}$ and $\mathcal{A}\otimes$C$\leftrightarrow\mathcal{A}$ are natural isomorphisms. One of the advantages of the diagrammatic language used here is that it leads instantly to the definition of a new fundamental object—the coalgebra—if we reverse all the arrows in the diagrams.

Definition 1. A coalgebra $\mathcal{C}$ is a vector space (with basis $\{e_i\}$) equipped with a mapping $\Delta:\mathcal{C}\rightarrow\mathcal{C}\otimes\mathcal{C}$,

$$\Delta(e_i)=\Delta_i^{kj}e_k\otimes e_j, \quad (2.6)$$

which is called comultiplication, and also equipped with a mapping ε: $\mathcal{C}\rightarrow$C, which is called the coidentity. The coalgebra is called coassociative if the mapping Δ satisfies the condition of coassociativity (cf. the first diagram with arrows reversed)

$$(id\otimes\Delta)\Delta=(\Delta\otimes id)\Delta\Rightarrow\Delta_i^{nl}\Delta_l^{kj}=\Delta_i^{lj}\Delta_l^{nk}\equiv\Delta_i^{nkj}. \quad (2.7)$$

The coidentity ε must satisfy the following conditions (cf. the second diagram)

$$(\varepsilon\otimes id)\Delta=(id\otimes\varepsilon)\Delta=id\Rightarrow\varepsilon_i\Delta_k^{ij}=\Delta_k^{ji}\varepsilon_i=\delta_k^j. \quad (2.8)$$

Here the complex numbers ε_i are determined from the relations $\varepsilon(e_i)=\varepsilon_i$.

For algebras and coalgebras, the concepts of modules and comodules can be introduced. Thus, if $\mathcal{A}$ is an algebra, the left $\mathcal{A}$-module can be defined as a vector space N and a mapping ψ_L: $\mathcal{A}\otimes N\rightarrow N$ (action of $\mathcal{A}$ on N) such that the relations of Fig. 3 hold. In other words, the space N is the space of a representation for the algebra $\mathcal{A}$.

If N is a (co)algebra and the mapping ψ_L preserves the (co)algebraic structure of N (see below), then N is called a left $\mathcal{A}$-modular (co)algebra. The concepts of right module and modular (co)algebra are introduced similarly. If N is simultaneously a left and a right $\mathcal{A}$-module, then N is called a two-sided $\mathcal{A}$-module. It is obvious that the algebra $\mathcal{A}$ itself is a two-sided $\mathcal{A}$-module for which the left and right actions are given by the left and right multiplications in the algebra.

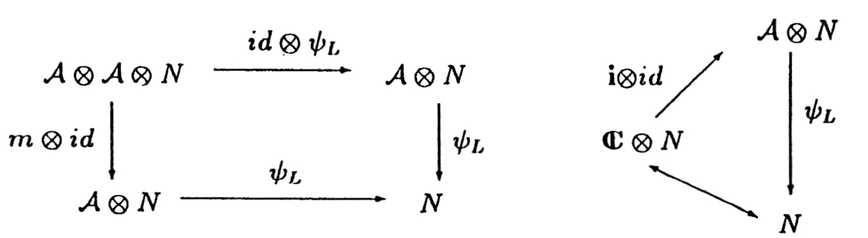


FIG. 3. Axioms for a left $\mathcal{A}$ module.

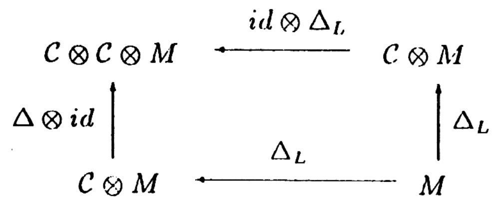


FIG. 4. Axioms for a left $\mathscr{A}$-comodule.

Now suppose that $\mathscr{C}$ is a coalgebra; then a left $\mathscr{C}$-comodule can be defined as a space M together with a mapping Δ_L: $M \to \mathscr{C} \otimes M$ (coaction of $\mathscr{C}$ on M) satisfying the axioms of Fig. 4 (in the diagrams in Fig. 3 defining modules it is necessary to reverse all the arrows).

If M is a (co)algebra and the mapping Δ_L preserves the (co)algebraic structure (for example, is a homomorphism; see below), then M is called a left $\mathscr{C}$-comodular (co)algebra. Right comodules are introduced similarly, after which two-sided comodules are defined in the natural manner. It is obvious that the coalgebra $\mathscr{C}$ is a two-sided $\mathscr{C}$-comodule.

Let $\mathscr{V}, \tilde{\mathscr{V}}$ be two vector spaces with bases $\{e_i\}$ and $\{\tilde{e}_i\}$. We denote by $\mathscr{V}^*, \tilde{\mathscr{V}}^*$ the corresponding dual linear spaces whose basis elements are linear functionals $\{e^i\}$: $\mathscr{V} \to \mathbb{C}$, $\{\tilde{e}^i\}$: $\tilde{\mathscr{V}} \to \mathbb{C}$. For the values of these functionals, we shall use the expressions $\langle e^i | e_j \rangle$ and $\langle \tilde{e}^i | \tilde{e}_j \rangle$. For every mapping L: $\mathscr{V} \to \tilde{\mathscr{V}}$ it is possible to define a unique mapping L^*: $\tilde{\mathscr{V}}^* \to \mathscr{V}^*$ induced by the equations

$$\langle \tilde{e}^i | L(e_j) \rangle = \langle L^*(\tilde{e}^i) | e_j \rangle. \tag{2.9}$$

In addition, for the dual objects there exists the linear injection

$$\rho: \mathscr{V}^* \otimes \tilde{\mathscr{V}}^* \to (\mathscr{V} \otimes \tilde{\mathscr{V}})^*,$$

which is given by the equations

$$\langle \rho(e^i \otimes \tilde{e}^j) | e_k \otimes \tilde{e}_l \rangle = \langle e^i | e_k \rangle \langle \tilde{e}^j | \tilde{e}_l \rangle.$$

A consequence of these facts is that for every coalgebra $(\mathscr{C}, \Delta, \varepsilon)$ it is possible to define an algebra $\mathscr{C}^* = \mathscr{A}$ (as dual object to $\mathscr{C}$) with multiplication $m = \Delta^* \cdot \rho$ and identity I that satisfy the relations

$$\langle a a' | c \rangle = \langle a \otimes a' | \Delta(c) \rangle = \langle a | c_{(1)} \rangle \langle a' | c_{(2)} \rangle,$$

$$\langle I | c \rangle = \varepsilon(c), \quad \forall a, a' \in \mathscr{A}, \quad \forall c \in \mathscr{C}.$$

We have here used the convenient notation of Ref. 11 for comultiplication in $\mathscr{C}$: $\Delta(c) = \Sigma_c c_{(1)} \otimes c_{(2)}$ [cf. (2.6)]. The summation symbol Σ_c will usually be omitted in the equations.

Thus, duality in the diagrammatic definitions of the algebras and coalgebras (reversal of the arrows) has in particular the consequence that the algebras and coalgebras are indeed duals of each other.

It is natural to expect that an analogous duality can also be traced for modules and comodules. Let $\mathscr{V}$ be a left comodule for $\mathscr{C}$. Then the left coaction of $\mathscr{C}$ on $\mathscr{V}$: $v \to \Sigma_v \bar{v}^{(1)} \otimes v^{(2)}$ $(\bar{v} \in \mathscr{C})$ induces a right action of $\mathscr{A} = \mathscr{C}^*$ on $\mathscr{V}$:

$$(v \otimes a) \overset{\psi_R}{\to} v \triangleleft a = \langle a | \bar{v}^{(1)} \rangle v^{(2)},$$

(here and in what follows, we omit the summation sign Σ_v), and therefore $\mathscr{V}$ is a right module for $\mathscr{A}$. Conversely, the right coaction of $\mathscr{C}$ on $\mathscr{V}$: $v \to v^{(1)} \otimes \bar{v}^{(2)}$ induces the left action of $\mathscr{A} = \mathscr{C}^*$ on $\mathscr{V}$:

$$(a \otimes v) \overset{\psi_L}{\to} a \triangleright v = v^{(1)} \langle a | \bar{v}^{(2)} \rangle.$$

From this we immediately conclude that the coassociative coalgebra $\mathscr{C}$ (which coacts on itself by the coproduct) is a natural module for its dual algebra $\mathscr{A}$. Indeed, the right action $\mathscr{C} \otimes \mathscr{A} \to \mathscr{C}$ is determined by the equations

$$(c, a) \to c \triangleleft a = \langle a | c_{(1)} \rangle c_{(2)}, \tag{2.10}$$

whereas for the left action $\mathscr{A} \otimes \mathscr{C} \to \mathscr{C}$ we have

$$(a, c) \to a \triangleright c = c_{(1)} \langle a | c_{(2)} \rangle. \tag{2.11}$$

Here $a \in \mathscr{A}$, $c \in \mathscr{C}$. The module axioms (shown in the form of the diagrams in Fig. 3) hold by virtue of the coassociativity of $\mathscr{C}$.

Finally, we note that the action of a certain algebra H on $\mathscr{C}$ from the left (from the right) induces an action of H on $\mathscr{A} = \mathscr{C}^*$ from the right (from the left). This obviously follows from relations of the type (2.9).

So-called bialgebras are the next important objects that are used in the theory of quantum groups.

Definition 2. An associative algebra with identity that is simultaneously a coassociative coalgebra with coidentity is called a bialgebra if the algebraic and coalgebraic structures are self-consistent. Namely, the comultiplication and coidentity must be homomorphisms of the algebras:

$$\Delta(e_i)\Delta(e_j) = \Delta(e_i e_j) = m_{ij}^k \Delta(e_k) \Rightarrow \Delta_i^{i'i''} \Delta_j^{j'j''} m_{i'j'}^{k'} m_{i''j''}^{k''} = m_{ij}^k \Delta_k^{k'k''},$$

$$\Delta(I) = I \otimes I, \quad \varepsilon(e_i e_j) = \varepsilon(e_i)\varepsilon(e_j), \quad \varepsilon(I) = 1. \tag{2.12}$$

Note that for every bialgebra we have a certain freedom in the definition of the multiplication (2.1) and the comultiplication (2.6). Indeed, all the axioms (2.3), (2.7), and (2.12) are satisfied if instead of (2.1) we take

$$e_i \cdot e_j = m_{ji}^k e_k,$$

or instead of (2.6) we choose

$$\Delta'(e_i) = \Delta_i^{jk} e_k \otimes e_j, \tag{2.13}$$

and at the same time the algebra is called noncommutative if $m_{ij}^k \neq m_{ji}^k$ and noncocommutative if $\Delta_k^{ij} \neq \Delta_k^{ji}$.

In quantum physics, it is usually assumed that all algebras of observables are bialgebras. Indeed, a coalgebraic structure is needed to define the action of the algebra of observables on the state $|\psi_1\rangle \otimes |\psi_2\rangle$ of the system that is the composite system formed from two independent systems with wave functions $|\psi_1\rangle$ and $|\psi_2\rangle$. In other words, it is only for bialgebras that it is possible to construct a theory of representations in which new representations can be obtained by multiplying old ones.

A classical example of a bialgebra is the universal covering Lie algebra, in particular, the spin algebra in three-dimensional space. To demonstrate this, we consider the Lie algebra g with generators J_α ($\alpha=1,2,3,...$) that satisfy the structure relations

$$J_\alpha J_\beta - J_\beta J_\alpha = t^\gamma_{\alpha\beta} J_\gamma.$$

Here $t^\gamma_{\alpha\beta}$ are structure constants. The covering of this algebra is the algebra U_g with basis elements consisting of the identity I and the elements $e_i = J_{\alpha_1} \dots J_{\alpha_n} \forall n \geq 1$, where the products of the generators J are ordered lexicographically, i.e., $\alpha_1 \leq \alpha_2 \leq \dots \leq \alpha_n$. The coalgebraic structure for the algebra U_g is specified by means of the mappings

$$\Delta(J_\alpha) = J_\alpha \otimes I + I \otimes J_\alpha, \quad \varepsilon(J_\alpha)=0, \quad \varepsilon(I)=1, \quad (2.14)$$

which satisfy all the axioms of a bialgebra. The mapping Δ in (2.14) is none other than the rule for addition of spins.

Considering exponentials of elements of a Lie algebra, one can arrive at the definition of a group bialgebra of the group G with structure mappings

$$\Delta(h) = h \otimes h, \quad \varepsilon(h)=1 \quad (\forall h \in G), \quad (2.15)$$

which obviously follow from (2.14). The next important example of a bialgebra is the algebra $\mathscr{A}(G)$ of functions on a group ($f: G \to \mathbb{C}$). This algebra is dual to the group algebra of the group G, and its structure mappings have the form ($f, f' \in \mathscr{A}(G)$; $h, h' \in G$):

$$(f \cdot f')(h) = f(h) f'(h), \quad (\Delta(f))(h,h') = f(h \cdot h'),$$

$$\varepsilon(f) = f(I_G),$$

where I_G is the identity element in the group G. In particular, if the functions T^i_j realize a matrix representation of the group G [the functions T^i_j can be regarded as generators in the algebra $\mathscr{A}(G)$], then we have

$$T^i_j(hh') = T^i_k(h) T^k_j(h') \Rightarrow \Delta(T^i_j) = T^i_k \otimes T^k_j.$$

Note that if g is non-Abelian, then U_g and G are noncommutative but cocommutative bialgebras, whereas $\mathscr{A}(G)$ is a commutative but noncocommutative bialgebra. Anticipating, we mention that the most interesting quantum groups are associated with noncommutative and noncocommutative bialgebras.

It is obvious that for a bialgebra $\mathscr{H}$ it is also possible to introduce the concepts of left (co)modules and (co)modular (co)algebras [right (co)modules and (co)modular (co)algebras are introduced in exactly the same way]. Moreover, for the bialgebra $\mathscr{H}$ it is possible to introduce the concept of a left (right) bimodule B, i.e., a left (right) $\mathscr{H}$-module that is simultaneously a left (right) $\mathscr{H}$-comodule; at the same time, the modular and comodular structures must be self-consistent:

$$\Delta_L(\mathscr{H} \triangleright B) = \Delta(\mathscr{H}) \triangleright \Delta_L(B),$$

$$(\varepsilon \otimes id) \Delta_L(b) = b, \quad b \in B.$$

On the other hand, in the case of bialgebras the conditions of preservation of the (co)algebraic structure of (co)modules can be represented in a more explicit form. For example, for the left $\mathscr{H}$-modular algebra $\mathscr{A}$ we have ($a, b \in \mathscr{A}$; $h \in \mathscr{H}$)

$$h \triangleright (ab) = (h_{(1)} \triangleright a)(h_{(2)} \triangleright b), \quad h \triangleright I_A = \varepsilon(h) I_A.$$

In addition, for the left $\mathscr{H}$-modular coalgebra $\mathscr{A}$ we must have

$$\Delta(h \triangleright a) = \Delta(h) \triangleright \Delta(a) = h_{(1)} \triangleright a_{(1)} \otimes h_{(2)} \triangleright a_{(2)}, \quad \varepsilon(h \triangleright a) = \varepsilon(h) \varepsilon(a).$$

Similarly, the algebra $\mathscr{A}$ is a left $\mathscr{H}$-comodular algebra if

$$\Delta_L(ab) = \Delta_L(a) \Delta_L(b), \quad \Delta_L(I_A) = I_{\mathscr{H}} \otimes I_A,$$

and, finally, the coalgebra $\mathscr{A}$ is a left $\mathscr{H}$-comodular coalgebra if

$$(id \otimes \Delta) \Delta_L(a) = m_{\mathscr{H}}(\Delta_L \otimes \Delta_L) \Delta(a),$$

$$(id \otimes \varepsilon_A) \Delta_L(a) = I_{\mathscr{H}} \varepsilon_A(a), \quad (2.16)$$

where

$$m_{\mathscr{H}}(\Delta_L \otimes \Delta_L)(a \otimes b) = \bar{a}^{(1)} \bar{b}^{(1)} \otimes a^{(2)} \otimes b^{(2)}.$$

We now consider the bialgebra $\mathscr{H}$, which acts on a certain modular algebra $\mathscr{A}$. One further important property of bialgebras is that we can define a new associative algebra $\mathscr{A} \sharp \mathscr{H}$ as the cross product (smash product) of $\mathscr{A}$ and $\mathscr{H}$. Namely:

1) As a vector space, $\mathscr{A} \sharp \mathscr{H}$ is identical to $\mathscr{A} \otimes \mathscr{H}$.
2) The product is defined in the sense

$$(a \sharp g)(b \sharp h) = \sum_g a(g_{(1)} \triangleright b) \sharp (g_{(2)} h) \equiv (a \sharp I)(\Delta(g) \triangleright (b \sharp h)). \quad (2.17)$$

3) The identity element is $I \sharp I$.

If the algebra $\mathscr{A}$ is the bialgebra dual to the bialgebra $\mathscr{H}$, then the relations (2.17) and (2.11) determine the rules for interchanging the elements $(I \sharp g)$ and $(a \sharp I)$:

$$(I \sharp g)(a \sharp I) = (a_{(1)} \sharp I) \langle g_{(1)} | a_{(2)} \rangle (I \sharp g_{(2)}). \quad (2.18)$$

Thus, the subalgebras $\mathscr{A}$ and $\mathscr{H}$ in $\mathscr{A} \sharp \mathscr{H}$ do not commute with each other. The smash product depends on which action of the algebra $\mathscr{H}$ on $\mathscr{A}$ we choose. In addition, the smash product generalizes the concept of the semidirect product. In particular, if we take as bialgebra $\mathscr{H}$ the Lorentz group algebra [see (2.15)], and as module $\mathscr{A}$ the group of translations in Minkowski space, then the smash product $\mathscr{A} \sharp \mathscr{H}$ defines the structure of the Poincaré group.

The coanalog of the smash product, the smash coproduct $\mathscr{A} \natural \mathscr{H}$, can also be defined. For this, we consider the bialgebra $\mathscr{H}$ and its comodular coalgebra $\mathscr{A}$. Then on the space $\mathscr{A} \otimes \mathscr{H}$ it is possible to define the structure of a coassociative coalgebra:

$$\Delta(a \natural h) = (a_{(1)} \natural \bar{a}^{(1)}_{(2)} h_{(1)}) \otimes (a^{(2)}_{(2)} \natural h_{(2)}),$$

$$\varepsilon(a \natural h) = \varepsilon(a) \varepsilon(h). \quad (2.19)$$

The proof of the coassociativity reduces to verification of the identity

$$(m_{\mathscr{H}}(\Delta_L \otimes \Delta_{\mathscr{H}}) \otimes id)(id \otimes \Delta_L) \Delta_{\mathscr{A}}(a) = (id \otimes id \otimes \Delta_L) \times (id \otimes \Delta_{\mathscr{A}}) \Delta_L(a),$$

which is satisfied if we take into account the axiom (2.16) and the comodular axiom

$$(id\otimes\Delta_L)\Delta_L(a)=(\Delta_{\mathcal{H}}\otimes id)\Delta_L(a). \qquad (2.20)$$

Note that from the two bialgebras $\mathcal{A}$ and $\mathcal{H}$, which act and coact on each other in a special manner, it is possible to organize a new bialgebra that is simultaneously the smash product and smash coproduct of $\mathcal{A}$ and $\mathcal{H}$ (bismash product; see Ref. 14).

We can now introduce the main concept in the theory of quantum groups, namely, the concept of the Hopf algebra.

Definition 3. A bialgebra $\mathcal{A}$ equipped with an additional mapping S: $\mathcal{A}\to\mathcal{A}$ such that

$$m(S\otimes id)\Delta=m(id\otimes S)\Delta=\mathbf{i}\cdot\varepsilon \qquad (2.21)$$

is called a Hopf algebra. The mapping S is called the antipode and is an antihomomorphism with respect to both multiplication and comultiplication:

$$S(ab)=S(b)S(a), \quad (S\otimes S)\Delta(a)=\sigma\cdot\Delta(S(a)), \qquad (2.22)$$

where $a,b\in\mathcal{A}$, and σ denotes the operator of transposition, $\sigma(a\otimes b)=(b\otimes a)$. If we set

$$S(e_i)=S_i^je_j, \qquad (2.23)$$

then the axiom (2.21) can be rewritten in the form

$$\Delta_k^{ij}S_i^nm_{nj}^l=\Delta_k^{ij}S_j^nm_{in}^l=\varepsilon_kE^l. \qquad (2.24)$$

From the axioms for the structure mappings of a Hopf algebra, it is possible to obtain the useful equations

$$S_j^i\varepsilon_i=\varepsilon_j, \quad S_j^iE^j=E^i,$$

$$\Delta_k^{ji}(S^{-1})_i^nm_{nj}^l=\Delta_k^{ji}(S^{-1})_j^nm_{in}^l=\varepsilon_kE^l, \qquad (2.25)$$

which we shall use in what follows. Note that, in general, the antipode S is not necessarily invertible. An invertible antipode is said to be bijective.

The universal covering algebra U_g and the group bialgebra of the group G that we considered above can again serve as examples of cocommutative Hopf algebras. An example of a commutative Hopf algebra is the bialgebra $\mathcal{A}(G)$, which we also considered above. The antipodes for these algebras have the form

$$U_g:S(J_\alpha)=-J_\alpha, \quad S(I)=I, \quad G:S(h)=h^{-1},$$

$$\mathcal{A}(G):S(f)(h)=f(h^{-1}),$$

and satisfy the relation $S^2=id$, which holds for all commutative or cocommutative Hopf algebras.

From the point of view of the axiom (2.21), $S(a)$ looks like the inverse of the element a, although in the general case $S^2\neq id$. We recall that if a set of elements of $\mathcal{G}$ with associative multiplication $\mathcal{G}\otimes\mathcal{G}\to\mathcal{G}$ and with identity (semigroup) also contains all the inverse elements, then such a set $\mathcal{G}$ becomes a group. Thus, from the point of view of the presence of the mapping S, a Hopf algebra is a generalization of a group algebra [for which $S(h)=h^{-1}$], although by itself it obviously need not be a group algebra. In accordance with Drinfel'd's definition,[13] the concepts of a Hopf algebra and a quantum group are equivalent. Of course, the most interesting examples of quantum groups arise when one considers noncommutative and noncocommutative Hopf algebras.

We consider a noncommutative Hopf algebra $\mathcal{A}$ for which $\Delta\neq\Delta'$.

Definition 4. A Hopf algebra $\mathcal{A}$ for which there exists an inverse element $R\in\mathcal{A}\otimes\mathcal{A}$ such that $\forall a\in\mathcal{A}$

$$\Delta'(a)=R\Delta(a)R^{-1}, \qquad (2.26)$$

$$(\Delta\otimes id)(R)=R_{13}R_{23}, \quad (id\otimes\Delta)(R)=R_{13}R_{12} \qquad (2.27)$$

is called quasitriangular. Here the element

$$R=\sum_{ij}R^{(ij)}e_i\otimes e_j \qquad (2.28)$$

is called the universal R matrix, and the symbols $R_{12},...$ have the meaning

$$R_{12}=\sum_{ij}R^{(ij)}e_i\otimes e_j\otimes I, \quad R_{13}=\sum_{ij}R^{(ij)}e_i\otimes I\otimes e_j,$$

$$R_{23}=\sum_{ij}R^{(ij)}I\otimes e_i\otimes e_j.$$

The relation (2.26) shows that the noncocommutativity in a quasitriangular Hopf algebra can be kept "under control." It can be shown that for such a Hopf algebra the universal R matrix (2.28) satisfies the relations

$$(S\otimes id)R=R^{-1}, \quad (id\otimes S)R^{-1}=R,$$

$$(id\otimes\varepsilon)R=(\varepsilon\otimes id)R=I, \qquad (2.29)$$

and in addition the Yang–Baxter equation

$$R_{12}R_{13}R_{23}=R_{23}R_{13}R_{12}, \qquad (2.30)$$

to which an appreciable part of the review will be devoted, holds for it. The proof of Eq. (2.30) reduces to writing out the expression $(id\otimes\Delta')(R)$ in two different ways:

$$(id\otimes\Delta')(R)=R^{ij}e_i\otimes R\Delta(e_j)R^{-1}=R_{23}(id\otimes\Delta)$$
$$\times(R)R_{23}^{-1}=R_{23}R_{13}R_{12}R_{23}^{-1}. \qquad (2.31)$$

On the other hand, we have

$$(id\otimes\Delta')(R)=(id\otimes\sigma)(id\otimes\Delta)(R)$$
$$=(id\otimes\sigma)R_{13}R_{12}=R_{12}R_{13}, \qquad (2.32)$$

where σ is the transposition operator. Comparing (2.31) and (2.32), we readily obtain (2.30).

The next important concept that we shall need in what follows is the concept of the Hopf algebra $\mathcal{A}^*$ that is the dual of the Hopf algebra $\mathcal{A}$. We choose in $\mathcal{A}^*$ basis elements $\{e^i\}$ and define multiplication, the identity, comultiplication, the coidentity, and the antipode for $\mathcal{A}^*$ in the form

$$e^ie^j=m_k^{ij}e^k, \quad I=\tilde{E}_ie^i, \quad \Delta(e^i)=\Delta_{jk}^ie^j\otimes e^k,$$

$$\varepsilon(e^i)=\tilde{\varepsilon}^i, \quad S(e^i)=\tilde{S}_j^ie^j. \qquad (2.33)$$

Definition 5. Two Hopf algebras $\mathcal{A}$ and $\mathcal{A}^*$ with corresponding bases $\{e_i\}$ and $\{e^i\}$ are said to be dual to each other if there exists a nondegenerate pairing $\langle . | . \rangle$: $\mathcal{A}^*\otimes\mathcal{A}\to\mathbb{C}$ such that

$$\langle e^ie^j|e_k\rangle\equiv\langle e^i\otimes e^j|\Delta(e_k)\rangle=\langle e^i|e_{k'}\rangle\Delta_k^{k'k''}\langle e^j|e_{k''}\rangle,$$

$$\langle e^i|e_je_k\rangle \equiv \langle \Delta(e^i)|e_j\otimes e_k\rangle = \langle e^{i'}|e_j\rangle \Delta^i_{i'i''}\langle e^{i''}|e_k\rangle,$$
$$\langle S(e^i)|e_j\rangle = \langle e^i|S(e_j)\rangle, \quad \langle e^i|I\rangle = \varepsilon(e^i),$$
$$\langle I|e_i\rangle = \varepsilon(e_i). \tag{2.34}$$

By virtue of the nondegeneracy of the pairing $\langle .|.\rangle$ (2.34), we can always choose basis elements $\{e^i\}$ such that

$$\langle e^i|e_j\rangle = \delta^i_j. \tag{2.35}$$

Then from the axioms for the pairing (2.34) and from the definitions of the structure operations (2.1), (2.23), and (2.33) in the Hopf algebras $\mathscr{A}$ and $\mathscr{A}^*$ we readily deduce

$$m^{ij}_k = \Delta^{ij}_k, \quad m^k_{ij} = \Delta^k_{ij}, \quad \bar{S}^i_j = S^i_j, \quad \bar{\varepsilon}^i = E^i, \quad \bar{E}_i = \varepsilon_i. \tag{2.36}$$

Thus, the multiplication, identity, comultiplication, coidentity, and antipode in a Hopf algebra define, respectively, comultiplication, coidentity, multiplication, identity, and antipode in the dual Hopf algebra.

We denote by $\mathscr{A}^0$ the algebra $\mathscr{A}^*$ with oppositive comultiplication: $\Delta(e^i) = m^i_{kj}e^j\otimes e^k$. At the same time, it follows from (2.25) that the antipode for $\mathscr{A}^0$ will be not S but the skew antipode S^{-1}. Thus, the structure mappings for $\mathscr{A}^0$ have the form

$$e^ie^j = \Delta^{ij}_k e^k, \quad \Delta(e^i) = m^i_{kj}e^j\otimes e^k, \quad S(e^i) = (S^{-1})^i_j e^j. \tag{2.37}$$

The algebras $\mathscr{A}$ and $\mathscr{A}^0$ are said to be antidual, and for them we can introduce the antidual pairing $\langle\langle .|.\rangle\rangle$: $\mathscr{A}^0\otimes\mathscr{A}\to\mathrm{C}$, which satisfies the conditions

$$\langle\langle e^ie^j|e_k\rangle\rangle \equiv \langle\langle e^i\otimes e^j|\Delta(e_k)\rangle\rangle = \Delta^{ij}_k,$$
$$\langle\langle e^i|e_ke_j\rangle\rangle \equiv \langle\langle \Delta(e^i)|e_j\otimes e_k\rangle\rangle = m^i_{kj},$$
$$\langle\langle S(e^i)|e_j\rangle\rangle = \langle\langle e^i|S^{-1}(e_j)\rangle\rangle = (S^{-1})^i_j,$$
$$\langle\langle e^i|S(e_j)\rangle\rangle = \langle\langle S^{-1}(e^i)|e_j\rangle\rangle = S^i_j,$$
$$\langle\langle e^i|I\rangle\rangle = E^i, \quad \langle\langle I|e_i\rangle\rangle = \varepsilon_i. \tag{2.38}$$

Drinfel'd[13] showed that there exists a quasitriangular Hopf algebra $\mathscr{D}(\mathscr{A})$ that is a special smash product of the Hopf algebras $\mathscr{A}$ and $\mathscr{A}^0$: $\mathscr{D}(\mathscr{A}) = \mathscr{A}\bowtie\mathscr{A}^0$, which is called the quantum double. At the same time, the universal R matrix can be expressed in the form

$$R = (e_i\bowtie I)\otimes(I\bowtie e^i), \tag{2.39}$$

and the multiplication in $\mathscr{D}(\mathscr{A})$ is defined in accordance with (the summation signs are omitted)

$$(a\bowtie\alpha)(b\bowtie\beta) = a((\alpha_{(3)}\triangleright b)\triangleleft S(\alpha_{(1)}))\bowtie\alpha_{(2)}\beta, \tag{2.40}$$

where $\alpha,\beta\in\mathscr{A}^0$, $a,b\in\mathscr{A}$, $\Delta^2(\alpha) = \alpha_{(1)}\otimes\alpha_{(2)}\otimes\alpha_{(3)}$ and

$$\alpha\triangleright b = b_{(1)}\langle\langle\alpha|b_{(2)}\rangle\rangle, \quad b\triangleleft\alpha = \langle\langle\alpha|b_{(1)}\rangle\rangle b_{(2)}. \tag{2.41}$$

The coalgebraic structure on the quantum double is defined by the direct product of the coalgebraic structures on the Hopf algebras $\mathscr{A}$ and $\mathscr{A}^0$:

$$\Delta(e_i\bowtie e^j) = \Delta(e_i\bowtie I)\Delta(I\bowtie e^j) = \Delta^{nk}_i m^j_{lp}(e_n\bowtie e^p)$$
$$\otimes(e_k\bowtie e^l). \tag{2.42}$$

Finally, the antipode and coidentity for $\mathscr{D}(\mathscr{A})$ have the form

$$S(a\bowtie\alpha) = S(a)\bowtie S(\alpha), \quad \varepsilon(a\bowtie\alpha) = \varepsilon(a)\varepsilon(\alpha). \tag{2.43}$$

All the axioms of a Hopf algebra can be verified for $\mathscr{D}(\mathscr{A})$ by direct calculation. A simple proof of the associativity of the multiplication (2.40) and the coassociativity of the comultiplication (2.42) can be found in Ref. 15.

Taking into account (2.41), we can rewrite (2.40) as the commutator for the elements $(I\bowtie\alpha)$ and $(b\bowtie I)$:

$$(I\bowtie\alpha)(b\bowtie I) = \langle\langle S(\alpha_{(1)})|b_{(1)}\rangle\rangle(b_{(2)}\bowtie I)(I\bowtie\alpha_{(2)})$$
$$\times\langle\langle\alpha_{(3)}|b_{(3)}\rangle\rangle,$$

or, in terms of the basis elements $\alpha = e^t$ and $b = e_s$, we have[13]

$$(I\bowtie e^t)(e_s\bowtie I) = m^t_{klp}\Delta^{njk}_s(S^{-1})^p_n(e_j\bowtie I)(I\bowtie e^l)$$
$$\equiv(m^t_{ip}(S^{-1})^p_n\Delta^{nr}_s)(m^i_{kl}\Delta^{jk}_r)(e_j\bowtie I)$$
$$\times(I\bowtie e^l), \tag{2.44}$$

where m^t_{klp} and Δ^{njk}_s are defined in (2.3) and (2.7), and $(S^{-1})^p_n$ is the matrix of the skew antipode. It follows from Eqs. (2.3) and (2.7) and from the identities for the skew antipode (2.25) that

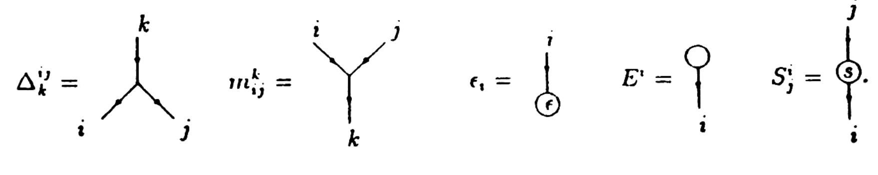
$$(m^q_{tk}\Delta^{ks}_m)(m^t_{ip}(S^{-1})^p_n\Delta^{nr}_s) = \delta^q_i\delta^r_m, \tag{2.45}$$

and this enables us to rewrite (2.44) in the form

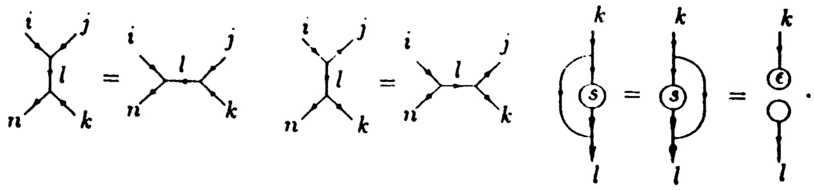
$$(m^q_{tk}\Delta^{ks}_m)(I\bowtie e^t)(e_s\bowtie I) = (m^q_{kl}\Delta^{jk}_m)(e_j\bowtie I)(I\bowtie e^l).$$

This equation is equivalent to R (2.39) satisfying the axiom (2.26). The relations (2.27) for R (2.39) are readily verified. Thus, $\mathscr{D}(\mathscr{A})$ is indeed a quasitriangular Hopf algebra with universal R matrix represented by (2.39).

In conclusion, we note that many relations for the structure constants of Hopf algebras [for example, the relation (2.45)] can be obtained and represented in a transparent form by means of the following diagrammatic technique:



For example, the axioms of associativity (2.3) and coassociativity (2.7) and the axioms for the antipode (2.24) can be represented in the form



We make two important remarks relating to the further development of the theory of Hopf algebras.

Remark 1. We consider a quasitriangular Hopf algebra $(\mathcal{A},\Delta,R)$ for which we can define an element

$$F=F^{ij}e_i\otimes e_j\in\mathcal{A}\otimes\mathcal{A}$$

that satisfies the conditions

$$(\Delta\otimes id)F=F_{13}F_{23},\quad (id\otimes\Delta)F=F_{13}F_{12},$$
$$F_{12}F_{13}F_{23}=F_{23}F_{13}F_{12},\quad F_{12}F_{21}=I\otimes I. \tag{2.46}$$

We define

$$\Delta^{(F)}(a)=F\Delta(a)F^{-1},\quad R^{(F)}=F_{21}RF_{12}^{-1},$$
$$U=m(id\otimes S)F.$$

Theorem (Ref. 16). $(\mathcal{A},\Delta^{(F)},R^{(F)})$ is a quasitriangular Hopf algebra with antipode and coidentity:

$$S^{(F)}(a)=US(a)U^{-1},\quad \varepsilon^{(F)}(a)=\varepsilon(a).$$

The new quasitriangular Hopf algebra $(\mathcal{A},\Delta^{(F)},R^{(F)})$ is called a twisted quasitriangular Hopf algebra.

Remark 2. One can introduce a deformation of a Hopf algebra, called a quasi-Hopf algebra,[17] which is defined as an associative algebra $\mathcal{A}$ with identity with homomorphism Δ: $\mathcal{A}\to\mathcal{A}\otimes\mathcal{A}$, homomorphism ε: $\mathcal{A}\to\mathrm{C}$, antiautomorphism S: $\mathcal{A}\to\mathcal{A}$, and inverse element $\Phi\in\mathcal{A}\otimes\mathcal{A}\otimes\mathcal{A}$. At the same time, Φ and S satisfy the axioms

$$(id\otimes\Delta)\Delta(a)=\Phi\cdot(\Delta\otimes id)\Delta(a)\cdot\Phi^{-1},\quad a\in\mathcal{A}, \tag{2.47}$$

$$(id\otimes id\otimes\Delta)(\Phi)\cdot(\Delta\otimes id\otimes id)(\Phi)$$
$$=(I\otimes\Phi)\cdot(id\otimes\Delta\otimes id)(\Phi)\cdot(\Phi\otimes I), \tag{2.48}$$

$$(\varepsilon\otimes id)\Delta=id=(id\otimes\varepsilon)\Delta,\quad (id\otimes\varepsilon\otimes id)\Phi=I\otimes I,$$
$$S(a_{(1)})ba_{(2)}=\varepsilon(a)b,\quad a_{(1)}cS(a_{(2)})=\varepsilon(a)c,$$
$$\phi_{(1)}cS(\phi_{(2)})b\phi_{(3)}=I,\quad S(\bar\phi_{(1)})b\bar\phi_{(2)}cS(\bar\phi_{(3)})=I, \tag{2.49}$$

where b and c are certain fixed elements of $\mathcal{A}$, $\Delta(a)=a_{(1)}\otimes a_{(2)}$, and

$$\Phi=\phi_{(1)}\otimes\phi_{(2)}\otimes\phi_{(3)},\quad \Phi^{-1}=\bar\phi_{(1)}\otimes\bar\phi_{(2)}\otimes\bar\phi_{(3)}.$$

Thus, a quasi-Hopf algebra differs from an ordinary Hopf algebra in that the axiom of coassociativity is replaced by the weaker condition (2.47). In other words, a quasi-Hopf algebra is noncoassociative, although this noncoassociativity can be kept under control by means of the element Φ. In Ref. 17, an explicit example of a quasi-Hopf algebra associated with solutions of the Knizhnik–Zamolodchikov equation is given. On the other hand, it is natural to suppose that by virtue of the occurrence of the pentagonal relation (2.48) for the element Φ quasi-Hopf algebras will be associated with multidimensional generalizations of Yang–Baxter equations.

3. QUANTIZATION OF LIE GROUPS AND THE YANG–BAXTER EQUATION

In this section, we discuss the R-matrix approach to the theory of quantum groups,[10] on the basis of which we perform a quantization of classical Lie groups and also some Lie supergroups. We present trigonometric solutions of the Yang–Baxter equation invariant with respect to the adjoint action of the quantum groups $GL_q(N)$, $SO_q(N)$, and $Sp_q(2n)$. We briefly discuss the corresponding Yangian (rational) solutions, and also $Z_N\otimes Z_N$ symmetric elliptic solutions of the Yang–Baxter equation.

3.1. *RTT* algebras

We shall consider an algebra $\mathcal{A}$ whose generators are the identity element 1 and elements of an $N\times N$ matrix $T=\|T^i_j\|$, $i,j=1,\ldots,N$, that satisfy the following quadratic relations (RTT relations):

$$R^{i_1i_2}_{j_1j_2}T^{j_1}_{k_1}T^{j_2}_{k_2}=T^{i_2}_{j_2}T^{i_1}_{j_1}R^{j_1j_2}_{k_1k_2}\Leftrightarrow R_{12}T_1T_2$$
$$=T_2T_1R_{12}\Leftrightarrow \mathbf{RTT'}=\mathbf{TT'R}. \tag{3.1.1}$$

Here the indices 1 and 2 label the matrix spaces; $\mathbf{T}\equiv T_1\equiv T\otimes I$, $\mathbf{T}'\equiv T_2\equiv I\otimes T$; I is the $N\times N$ unit matrix; $\mathbf{R}\equiv\hat R_{12}\equiv P_{12}R_{12}\in\mathrm{Mat}(N)\otimes\mathrm{Mat}(N)$ is a numeric invertible matrix, and $P_{12}=\delta^{i_1}_{j_2}\delta^{i_2}_{j_1}$ is the transposition matrix. We shall assume that R_{12} is a lower triangular block matrix and satisfies the Yang–Baxter equation

$$R^{i_1i_2}_{j_1j_2}R^{j_1i_3}_{k_1j_3}R^{j_2j_3}_{k_2k_3}=R^{i_2i_3}_{j_2j_3}R^{i_1i_3}_{j_1k_3}R^{j_1j_2}_{k_1k_2}\Rightarrow R_{12}R_{13}R_{23}$$
$$=R_{23}R_{13}R_{12}\Leftrightarrow R_{12}R_{31}^{-1}R_{32}^{-1}$$
$$=R_{32}^{-1}R_{31}^{-1}R_{12} \tag{3.1.2}$$

or, in succinct notation,

$$\mathbf{RR'R}=\mathbf{R'RR'}, \tag{3.1.3}$$

where $\mathbf{R}'\equiv\hat R_{23}\equiv P_{23}R_{23}$, and the indices 1,2,3 label the matrix spaces in which the corresponding R matrices act nontrivially. Note that direct consequences of (3.1.3) are the equations

$$X(\mathbf{R})\mathbf{R'R}=\mathbf{R'R}X(\mathbf{R'}),\quad \mathbf{RR'}X(\mathbf{R})=X(\mathbf{R'})\mathbf{RR'}, \tag{3.1.4}$$

which make it possible to carry an arbitrary function $X(\mathbf{R})$ through the operators $\mathbf{RR'}$ and $\mathbf{R'R}$. The condition (3.1.2)–

(3.1.3) is sufficient to ensure that on monomials of third degree in T no relations additional to (3.1.1) arise. We shall consider the case when the R matrix depends on the numerical parameter $q=\exp(h)$, which is called the deformation parameter.

Suppose that the algebra $\mathcal{A}$ can be extended in such a way that it also contains all elements $(T^{-1})^i_j$:

$$(T^{-1})^i_k T^k_j = T^i_k (T^{-1})^k_j = \delta^i_j \cdot 1.$$

Then $\mathcal{A}$ becomes a Hopf algebra with structure mappings

$$\Delta(T^i_k) = T^i_j \otimes T^j_k, \quad \varepsilon(T^i_j) = \delta^i_j, \quad S(T^i_j) = (T^{-1})^i_j, \tag{3.1.5}$$

which, as is readily verified, satisfy the following axioms (see the previous section):

$$(id\otimes\Delta)\Delta(T^i_j) = (\Delta\otimes id)\Delta(T^i_j),$$
$$(\varepsilon\otimes id)\Delta(T^i_j) = (id\otimes\varepsilon)\Delta(T^i_j) = T^i_j,$$
$$m(S\otimes id)\Delta(T^i_j) = m(id\otimes S)\Delta(T^i_j) = \varepsilon(T^i_j)1. \tag{3.1.6}$$

The antipode S is not an involution, since instead of $S^2 = id$ we have

$$S^2(T^i_j) = D^i_k T^k_l (D^{-1})^l_j \tag{3.1.7}$$

(we shall prove this equation and determine the numeric matrix D very soon), which can be rewritten in the form

$$D^j_l T^l_k S(T^i_j) = D^i_k. \tag{3.1.8}$$

The relations (3.1.7) and (3.1.8) can be interpreted as the rules of transposition of the operations of taking the inverse matrix and the transpose (t):

$$D^t (T^{-1})^t = (T^t)^{-1} D^t.$$

It follows from the RTT relations (3.1.1) that the numeric matrix D is, apart from a constant c, given by the equations

$$\frac{1}{c} D^i_j = \tilde{R}^{ki}_{jk} = \mathrm{Tr}_{(2)}(P_{12}\tilde{R}_{12}), \quad \tilde{R}_{12} = ((R^{t_1}_{12})^{-1})^{t_1}, \tag{3.1.9}$$

where t_1 denotes the operation of transposition in the first matrix space, $\mathrm{Tr}_{(2)}$ is the trace over the second matrix space, and we assume that the matrix $R^{t_1}_{12}$ is invertible. Indeed, the relations (3.1.1) can be rewritten in the form

$$(T^{-1})^{i_2}_{j_2} T^{i_1}_{j_1} = [R^{t_1}_{12} T^{t_1}_1 T^{-1}_2 (R^{t_1}_{12})^{-1}]^{j_1 i_2}_{i_1 j_2}.$$

In this equation, we set $j_2 = i_1$, sum over i_1, and multiply the result from the left by $(R^{t_1}_{12})^{-1}$. After this, introducing the matrix D (3.1.9), we arrive at the relations (3.1.8). Note also that in accordance with the definition (3.1.9) we can obtain for the matrix D the equations

$$\frac{1}{c}\,\mathrm{Tr}_{(2)}(\mathbf{R}D_2)^{i_1}_{j_1} = (R_{12})^{i_2 i_1}_{j_1 j_2} \tilde{R}^{k j_2}_{i_2 k} \equiv \delta^{i_1}_{j_1}, \tag{3.1.10}$$

and in addition for the matrix D there always exists an inverse matrix D^{-1}, and it follows from the Yang–Baxter equation that $cD^{-1} = \mathrm{Tr}_{(2)}(P_{12}\widetilde{R^{-1}_{12}})$ (Refs. 15 and 18).

The matrix D^i_j (3.1.9) satisfying the conditions (3.1.7) and (3.1.8) defines the quantum trace.[10,18] To explain the concept of the quantum trace, we consider the N^2-dimensional adjoint $\mathcal{A}$-comodule E. We represent its basis elements in the form of an $N\times N$ matrix $E = \|E^i_j\|$, $i,j = 1,\ldots,N$. The adjoint coaction is

$$E^i_j \to T^i_{i'} S(T^{j'}_j) \otimes E^{i'}_{j'} \equiv (TET^{-1})^i_j, \tag{3.1.11}$$

where in the final part of the expression (3.1.11) we have introduced abbreviations that we shall use in what follows. We note that there is a different form of the adjoint coaction:

$$E^i_j \to E^{i'}_{j'} \otimes S(T^i_{i'}) T^{j'}_j \equiv (T^{-1}ET)^i_j. \tag{3.1.12}$$

It is clear that (3.1.11) and (3.1.12) are, respectively, left and right comodules. Both left and right comodules E are reducible, and the irreducible subspaces in E can be distinguished by means of the quantum traces. For the case (3.1.11), the quantum trace has the form

$$\mathrm{Tr}_q E \equiv \mathrm{Tr}(DE) \equiv \sum_{i,j=1}^{N} D^i_j E^j_i \tag{3.1.13}$$

and satisfies the following invariance property, which follows from Eqs. (3.1.7) and (3.1.8):

$$\mathrm{Tr}_q(TET^{-1}) = \mathrm{Tr}_q(E), \tag{3.1.14}$$

For the case (3.1.12), the definition of the quantum trace must be changed to

$$\overline{\mathrm{Tr}}_q E \equiv \mathrm{Tr}(D^{-1}E) \equiv \sum_{i,j=1}^{N} (D^{-1})^i_j E^j_i,$$
$$\overline{\mathrm{Tr}}_q(T^{-1}ET) = \overline{\mathrm{Tr}}_q(E), \tag{3.1.15}$$

this also following from (3.1.7) and (3.1.8). Thus, $\mathrm{Tr}_q(E)$ and $\overline{\mathrm{Tr}}_q(E)$ are, respectively, the scalar parts of the comodules E (3.1.11) and (3.1.12), whereas the q-traceless part of E forms (N^2-1)-dimensional (reducible in the general case and irreducible in the case of linear quantum groups) $\mathcal{A}$-adjoint comodules.

An important consequence of the definition of the quantum trace (3.1.14)–(3.1.15) and the RTT relations (3.1.1) is the fact that

$$T_1^{-1}\mathrm{Tr}_{q2}(f(\mathbf{R}))T_1 = \mathrm{Tr}_{q2}(f(\mathbf{R})),$$
$$T_2\overline{\mathrm{Tr}}_{q1}(f(\mathbf{R}))T_2^{-1} = \overline{\mathrm{Tr}}_{q1}(f(\mathbf{R}))$$

[here $f(.)$ is an arbitrary function, and Tr_{q1} and Tr_{q2} are the quantum traces over the first and second space, respectively], which indicates that the matrices $\mathrm{Tr}_{q2}(f(\mathbf{R}))$ and $\overline{\mathrm{Tr}}_{q1}(f(\mathbf{R}))$ must be proportional to the identity matrices if the R matrix acts in a nontrivial representation of the quantum group. In particular, we must have

$$\mathrm{Tr}_{q2}(\mathbf{R}^{\pm 1}) = c_{\pm} I_{(1)}, \quad \overline{\mathrm{Tr}}_{q1}(\mathbf{R}^{\pm 1}) = \bar{c}_{\pm} I_{(2)}, \tag{3.1.16}$$

where $c_{\pm}$ and $\bar{c}_{\pm}$ are certain constants related by $c_+\bar{c}_- = c_-\bar{c}_+$, and $I_{(k)}$ is the identity matrix in the kth space. Note that a direct consequence of (3.1.10) is

$$\mathrm{Tr}_{q2}(\mathbf{R}) = c\cdot I_{(1)}, \tag{3.1.17}$$

which holds for any nondegenerate representation of the R matrix. As we shall see below, for the quantum groups of the classical series the fact (3.1.16) does indeed hold. In what follows, we shall attempt to restrict consideration to either left or right adjoint comodules with quantum traces (3.1.13) or (3.1.15). The analogous relations for right or left comodules, respectively, can be considered in exactly the same way.

It can be seen from comparison of the relations (3.1.1) and (3.1.2) that for the generators T^i_j it is possible to choose the following finite-dimensional matrix representations:

$$(T^i_j)^k_l = R^{ik}_{jl} \equiv (R^{(+)})^i_j, \quad (T^i_j)^k_l = (R^{-1})^{ki}_{lj} \equiv (R^{(-)})^i_j. \tag{3.1.18}$$

Since the R matrix satisfies the Yang–Baxter equation, there exist linear functionals $(L^{\pm})^i_j$ that realize the homomorphisms (3.1.18), i.e., we have

$$\langle L^+_2, T_1\rangle = R_{12}, \quad \langle L^-_2, T_1\rangle = R^{-1}_{21}, \tag{3.1.19}$$

or, in general (matrix) form,

$$\langle L^{\pm}, T_1T_2\ldots T_k\rangle = R^{(\pm)}_1 R^{(\pm)}_2 \ldots R^{(\pm)}_k.$$

The Yang–Baxter equation (3.1.2) can now be reproduced from the RTT relations (3.1.1) by averaging them with the L operators.

From the requirement that the $(L^{\pm})^i_j$ form the algebra that is the dual of the algebra $\mathcal{A}$ (the definition of the dual algebra is given in Sec. 2), we obtain the following commutation relations for the generators $L^{(\pm)}$:

$$\mathbf{R}L^{\pm\prime}L^{\pm} = L^{\pm\prime}L^{\pm}\mathbf{R}, \quad \mathbf{R}L^{+\prime}L^{-} = L^{-\prime}L^{+}\mathbf{R}. \tag{3.1.20}$$

This algebra is obviously a Hopf algebra with comultiplication, antipode, and coidentity:

$$\Delta(L^{\pm})^i_j = (L^{\pm})^i_k \otimes (L^{\pm})^k_j, \quad S(L^{\pm}) = (L^{\pm})^{-1},$$

$$\varepsilon((L^{\pm})^i_j) = \langle (L^{\pm})^i_j, 1\rangle = \delta^i_j,$$

where we have assumed that the matrices $L^{\pm}$ are invertible. As was shown in Ref. 10, for the quantum groups of the classical series A_N, B_n, C_n, D_n [respectively, $SL_q(N)$, $SO_q(2n+1)$, $Sp_q(2n)$, $SO_q(2n)$], the relations (3.1.20) define quantum Lie algebras in which some of the generators $(L^{\pm})^i_j$ play the role of the quantum analog of the Cartan–Weyl basis. Note that the algebra (3.1.20) is a covariant algebra with respect to the left and right cotransformations

$$(L^{\pm})^i_j \to (T^{-1})^k_j \otimes (L^{\pm})^i_k \equiv (L^{\pm}T^{-1})^i_j,$$

$$(L^{\pm})^i_j \to (L^{\pm})^k_j \otimes (T^{-1})^i_k \equiv (T^{-1}L^{\pm})^i_j. \tag{3.1.21}$$

Thus, the matrices $L^i_j = (S(L^-)L^+)^i_j$ and $\bar{L} = L^+S(L^-)$ realize, respectively, the left and right adjoint comodules (3.1.11) and (3.1.12). It is readily verified that the coinvariants

$$C_M = \mathrm{Tr}_q((L)^M) = \frac{c_-}{\bar{c}_-}\overline{\mathrm{Tr}}_q((\bar{L})^M) \tag{3.1.22}$$

are central elements for the algebra (3.1.20). The last equation in (3.1.22) is proved as follows:

$$C_M = \mathrm{Tr}_q(S(L^-)(\bar{L})^M L^-) = \frac{1}{\bar{c}_-}\overline{\mathrm{Tr}}_{q1}\mathrm{Tr}_{q2}(S(L^{-\prime})$$

$$\times(\bar{L}')^M\mathbf{R}^{-1}L^{-\prime})$$

$$= \frac{1}{\bar{c}_-}\overline{\mathrm{Tr}}_{q1}\mathrm{Tr}_{q2}(S(L^{-\prime})\mathbf{R}^{-1}L^{-\prime}(\bar{L})^M)$$

$$= \frac{1}{\bar{c}_-}\overline{\mathrm{Tr}}_{q1}\mathrm{Tr}_{q2}(L^{-}\mathbf{R}^{-1}S(L^{-})(\bar{L})^M)$$

$$= \frac{c_-}{\bar{c}_-}\overline{\mathrm{Tr}}_q((\bar{L})^M),$$

where we have taken into account (3.1.16) and used Eqs. (3.1.20), from which, in particular, we deduce the relations

$$L^{\mp\prime}(\bar{L})^M = \mathbf{R}^{\pm1}(\bar{L}')^M\mathbf{R}^{\mp1}L^{\mp\prime},$$

$$(L')^M L^{\pm} = L^{\pm}\mathbf{R}^{\pm1}(L)^M\mathbf{R}^{\mp1},$$

which demonstrate the centrality of the elements (3.1.22).

Note also that the generators L^i_j and $\bar{L}^i_j$ satisfy the reflection equations

$$\mathbf{R}L\mathbf{R}L = L\mathbf{R}L\mathbf{R}, \quad \mathbf{R}\bar{L}'\mathbf{R}\bar{L}' = \bar{L}'\mathbf{R}\bar{L}'\mathbf{R}. \tag{3.1.23}$$

The first algebra in (3.1.23) (and similarly the second algebra) decomposes into the direct sum of two subalgebras, namely, into an Abelian algebra with generator $C_1 = \mathrm{Tr}_q(L)$ and an algebra with N^2-1 traceless generators:

$$\lambda\tilde{L}^i_j = L^i_j\frac{\mathrm{Tr}_q(I)}{C_1} - \delta^i_j, \tag{3.1.24}$$

where the factor $\lambda = q - q^{-1}$ is introduced to ensure that the operators $\tilde{L}$ have the correct classical limit as $q \to 1$. For the last algebra, it is easy to obtain the commutation relations

$$\mathbf{R}\tilde{L}\mathbf{R}\tilde{L} - \tilde{L}\mathbf{R}\tilde{L}\mathbf{R} = \frac{1}{\lambda}(\mathbf{R}^2\tilde{L} - \tilde{L}\mathbf{R}^2), \tag{3.1.25}$$

which can be regarded (for an arbitrary R matrix satisfying the Yang–Baxter equation) as a deformation of the commutation relations for Lie algebras. The relations (3.1.23) and (3.1.25) are extremely important and arise, for example, in the construction of a differential calculus on quantum groups as the commutation relations for invariant vector fields.

Note that from (3.1.14), (3.1.15), and (3.1.18) we can immediately obtain the following useful relations:

$$\mathrm{Tr}_{q2}(\mathbf{R}^{\pm1}\mathbf{E}\mathbf{R}^{\mp1}) = \mathrm{Tr}_q E \cdot I_{(1)},$$

$$\overline{\mathrm{Tr}}_{q1}(\mathbf{R}^{\pm1}\mathbf{E}'\mathbf{R}^{\mp1}) = \overline{\mathrm{Tr}}_q E \cdot I_{(2)}, \tag{3.1.26}$$

where $\mathbf{E} \equiv E_1 \equiv E \otimes I$ and $\mathbf{E}' \equiv E_2 \equiv I \otimes E$.

We shall now assume that the R matrix satisfies the characteristic equation

$$(\mathbf{R}-\lambda_1)(\mathbf{R}-\lambda_2)\ldots(\mathbf{R}-\lambda_M) = 0 \quad (\lambda_i \neq \lambda_j \text{ if } i \neq j). \tag{3.1.27}$$

In this case, the pairings (3.1.19) are automatically degenerate. For R matrices satisfying (3.1.27), we can introduce a set of M projectors:

$$\mathbf{P}_k = \prod_{j\neq k}\frac{(\mathbf{R}-\lambda_j)}{(\lambda_k-\lambda_j)}, \tag{3.1.28}$$

which can be used for the spectral decomposition

$$F(\mathbf{R})=\sum_{k=1}^{M} F(\lambda_k)\mathbf{P}_k \quad (3.1.29)$$

of an arbitrary function F of the R matrix. In particular, for $F=1$ we obtain the completeness condition. Finally, we note that it is sometimes helpful to use in place of the projectors (3.1.28) the operators

$$\sigma_k=\mathbf{1}-2\mathbf{P}_k, \quad \sigma_k^2=\mathbf{1}, \quad (3.1.30)$$

which are related by

$$(M-2)\mathbf{1}=\sum_{k=1}^{M} \sigma_k. \quad (3.1.31)$$

3.2. The semiclassical limit

We assume that the R matrix introduced in (3.1.1) has the following expansion in the limit $h\to 0$ ($q\to 1$):

$$R_{12}=\mathbf{1}+hr_{12}+O(h^2). \quad (3.2.1)$$

Here $\mathbf{1}=I\otimes I$ denotes the $(N^2\times N^2)$ unit matrix. One says that such R matrices have quasiclassical behavior, and r_{12} is called a quasiclassical r matrix. It is readily found from the quantum Yang–Baxter equation (3.1.3) that r_{12} satisfies the so-called classical Yang–Baxter equation

$$[r_{12},r_{13}+r_{23}]+[r_{13},r_{23}]=0. \quad (3.2.2)$$

Substituting the expansion (3.2.1) in the RTT relations (3.1.1), we obtain

$$[T_1,T_2]=h[T_1T_2,r_{12}]+O(h^2). \quad (3.2.3)$$

This equation demonstrates the fact that the RTT relations (3.1.1) can be interpreted as a quantization (deformation) of the classical Poisson bracket (Sklyanin bracket[19]):

$$\{T_1,T_2\}=[T_1T_2,r_{12}]. \quad (3.2.4)$$

The classical Yang–Baxter equation (3.2.2) guarantees fulfillment of the Jacobi identity for the bracket (3.2.4). From the requirement of antisymmetry of the Poisson bracket (3.2.4), we obtain

$$\{T_1,T_2\}=[T_1T_2,-r_{21}]. \quad (3.2.5)$$

Thus, the semiclassical r matrix $r_{12}^{(-)}=-r_{12}$ corresponding to the representation $R^{(-)}$ (3.1.18) must also be a solution of Eq. (3.2.2), as is readily shown by making the substitution $3\leftrightarrow 1$ in (3.2.2). On the other hand, comparing (3.2.4) and (3.2.5), we obtain

$$T_1T_2(r_{12}+r_{21})=(r_{12}+r_{21})T_1T_2. \quad (3.2.6)$$

Thus,

$$t_{12}=\frac{1}{2}(r_{12}+r_{21}) \quad (3.2.7)$$

is an invariant with respect to the adjoint action of the matrix T_1T_2 (it is an ad-invariant). We introduce the new semiclassical r matrix

$$\tilde{r}_{12}=\frac{1}{2}(r_{12}-r_{21}). \quad (3.2.8)$$

Then the Sklyanin bracket can be represented in the manifestly antisymmetric form

$$\{T_1,T_2\}=[T_1T_2,\tilde{r}_{12}], \quad (3.2.9)$$

and the matrix $\tilde{r}$ (3.2.8) satisfies the modified classified Yang–Baxter equation

$$[\tilde{r}_{12},\tilde{r}_{13}+\tilde{r}_{23}]+[\tilde{r}_{13},\tilde{r}_{23}]=(1/4)[r_{23}+r_{32},r_{13}+r_{31}] =[t_{23},t_{13}]. \quad (3.2.10)$$

Note that the algebra (3.1.23) can also be regarded as the result of quantization of a certain Poisson structure. For example, for the first of these algebras we have

$$\{L_2,L_1\}=[L_1,[L_2,\tilde{r}_{12}]]+L_1t_{12}L_2-L_2t_{12}L_1,$$

where again we must assume that $[L_1L_2,t_{12}]=0$ [cf. (3.2.6)]. On the other hand, the relations (3.1.25) in the zeroth order in h give the equations

$$[\tilde{L}_1,\tilde{L}_2]=[t_{12},\tilde{L}_1],$$

and this enables us to regard (3.1.25) as a deformation of the relations of a Lie algebra.

3.3. The quantum groups $GL_q(N)$ and $SL_q(N)$ and the corresponding quantum hyperplanes

In this subsection, we discuss the simplest quantum groups, which are the quantizations (deformations) of the linear Lie groups $GL(N)$ and $SL(N)$. We begin with the definition of a quantum hyperplane. We recall that the Lie group $GL(N)$ is the set of nondegenerate $N\times N$ matrices T^i_j that act on an N-dimensional vector space, whose coordinates we denote by $\{x^i, i=1,\ldots,N\}$. Thus, we have the transformations

$$x^i\to\tilde{x}^i=T^i_jx^j, \quad (3.3.1)$$

which we can regard from a different point of view. Namely, let $\{T^i_j\}$ and $\{x^i\}$ $(i,j=1,\ldots,N)$ be the generators of two Abelian (commuting) algebras

$$[x^i,x^j]=[T^i_j,T^k_l]=[T^i_j,x^k]=0. \quad (3.3.2)$$

Then the transformation (3.3.1) can be regarded as an action of the algebra $\{T\}$ on the algebra $\{x\}$ that preserves the Abelian structure of the latter, i.e., we have $[\tilde{x}^i,\tilde{x}^j]=0$.

We introduce a deformed N-dimensional "vector space" whose coordinates $\{x^i\}$ commute as follows:

$$x^ix^j=qx^jx^i, \quad i<j \quad (3.3.3)$$

where q is some number (the deformation parameter). In other words, we now have a noncommutative associative algebra with N generators $\{x^i\}$. In accordance with (3.3.3), any element of this algebra, which is a monomial of arbitrary degree

$$x^{i_1}x^{i_2}\cdots x^{i_K}, \quad (3.3.4)$$

can be uniquely ordered lexicographically, i.e., in such a way that $i_1\leq i_2\leq\ldots\leq i_K$. Of such algebras, one says that they possess the Poincaré–Birkhoff–Witt (PBW) property. An algebra with N generators satisfying (3.3.3) is called an N-dimensional quantum hyperplane.[20,21] The relations (3.3.3) can be written in the matrix form

$$R^{i_1 i_2}_{j_1 j_2} x^{j_1} x^{j_2} = q x^{i_2} x^{i_1} \Leftrightarrow R_{12} x_1 x_2 = q x_2 x_1 \Leftrightarrow \mathbf{R} x x' = q x x'. \tag{3.3.5}$$

Here the indices 1 and 2 label the vector spaces on which the R matrix, realized in the tensor square $\mathrm{Mat}(N)_1 \otimes \mathrm{Mat}(N)_2$, act. Thus, the indices 1 and 2 of the R matrix show how the R matrix acts on the direct product of the first and second vector spaces. We emphasize that the R matrix depends on the parameter q and, generally speaking, its explicit form is recovered nonuniquely from the relations (3.3.3). However, if we require that the R matrix (3.3.5) be constructed by means of two $GL(N)$-invariant tensors $\mathbf{1}_{12}$ and P_{12}, i.e.,

$$R^{i_1 i_2}_{j_1 j_2} = (\delta^{i_1}_{j_1} \delta^{i_2}_{j_2}) \cdot a_{i_1 i_2} + (\delta^{i_1}_{j_2} \delta^{i_2}_{j_1}) \cdot b_{i_1 i_2},$$

and also satisfy the Yang–Baxter equation (3.1.1) and have lower-triangular block form ($R^{i_1 i_2}_{j_1 j_2} = 0,\ i_1 < j_1$), then we obtain the explicit representation

$$\begin{aligned} R_{12} = R^{i_1 i_2}_{j_1 j_2} &= q \sum_i (e_{ii})^{i_1}_{j_1} \otimes (e_{ii})^{i_2}_{j_2} + \sum_{i \neq j} (e_{ii})^{i_1}_{j_1} \otimes (e_{jj})^{i_2}_{j_2} \\ &+ \lambda \sum_{i>j} (e_{ij})^{i_1}_{j_1} \otimes (e_{ji})^{i_2}_{j_2} = \delta^{i_1}_{j_1} \delta^{i_2}_{j_2} (1 + (q-1) \delta^{i_1 i_2}) \\ &+ \lambda \delta^{i_1}_{j_2} \delta^{i_2}_{j_1} \Theta_{i_1 i_2}, \end{aligned} \tag{3.3.6}$$

$$\Theta_{ij} = \{1 \ \text{ if } i > j, \quad 0 \ \text{ if } i \leqslant j\}.$$

Here $i,j = 1,\dots,N$, $(e_{ij})^k_l = \delta^{ik} \delta_{jl}$, and $\lambda = q - q^{-1}$. It can be verified (using the diagrammatic technique of Sec. 3.4) that this R matrix satisfies the Hecke relation [a special case of (3.1.27)]

$$\mathbf{R}^2 = \lambda \mathbf{R} + \mathbf{1} \Leftrightarrow \mathbf{R} - \mathbf{R}^{-1} - \lambda \mathbf{1} = 0. \tag{3.3.7}$$

The following helpful relations also follow from the explicit form (3.3.6) for the $GL_q(N)$ R matrix:

$$R^{t_1 t_2}_{12} = R_{21}, \quad R_{12}\left(\frac{1}{q}\right) = R^{-1}_{12}(q).$$

In the semiclassical limit (3.2.1), the relation (3.3.7) can be rewritten in the form

$$r_{12} + r_{21} = 2 P_{12}. \tag{3.3.8}$$

Thus, for the Lie–Poisson structure on the group $GL(N)$ the transposition matrix $t_{12} = P_{12}$ can be taken as the ad-invariant tensor. For the $\tilde{r}$ matrix (3.2.8) determining the Sklyanin bracket, we obtain from (3.3.6) the expression

$$\begin{aligned} \tilde{r}_{12} &= \sum_{i>j} [(e_{ij})^{i_1}_{j_1} \otimes (e_{ji})^{i_2}_{j_2} - (e_{ji})^{i_1}_{j_1} \otimes (e_{ij})^{i_2}_{j_2}] \\ &\in gl(N) \wedge gl(N). \end{aligned} \tag{3.3.9}$$

In accordance with (3.3.7), (3.1.28), and (3.1.29) for $q^2 \neq -1$ the matrix $\mathbf{R}$ has the spectral decomposition

$$\mathbf{R} = q \mathbf{P}^+ - q^{-1} \mathbf{P}^-, \tag{3.3.10}$$

with projectors

$$\mathbf{P}^{\pm} = (q + q^{-1})^{-1} \{q^{\mp 1} \mathbf{1} \pm \mathbf{R}\}, \tag{3.3.11}$$

which are the quantum analogs of the symmetrizer ($\mathbf{P}^+$) and antisymmetrizer ($\mathbf{P}^-$), as can be seen by setting $q=1$ in (3.3.11). Using the projector $\mathbf{P}^-$, we can represent the definition (3.3.3) of the quantum hyperplane in the form

$$\mathbf{P}^- x x' = 0. \tag{3.3.12}$$

Note that the relations

$$\mathbf{P}^+ x x' = 0 \Leftrightarrow (x^i)^2 = 0, \quad x^i x^j = -q^{-1} x^j x^i \quad (i<j) \tag{3.3.13}$$

define a fermionic N-dimensional quantum hyperplane that is a deformation of the algebra of N fermions: $x^i x^j = -x^j x^i$.

For the given R matrix, the quantum trace (3.1.13) and the matrix D (3.1.9) can be chosen in the form

$$\begin{aligned} \mathrm{Tr}_q A &\equiv \mathrm{Tr}(DA) \equiv \sum_{i=1}^{N} q^{-N-1+2i} A^i_i, \\ D &\equiv q^N \mathrm{Tr}_{(2)}(P_{12} \tilde{R}_{12}) = \mathrm{diag}\{q^{-N+1}, q^{-N+3}, \dots, q^{N-1}\}. \end{aligned} \tag{3.3.14}$$

We also note the useful relations [cf. (3.1.16)]

$$\mathrm{Tr}_{q(2)} \mathbf{R}^{\pm 1} = q^{\pm N} \cdot I_{(1)}, \quad \mathrm{Tr}_q(I) = \mathrm{Tr}(D) = [N]_q, \tag{3.3.15}$$

where $[N]_q = (q^N - q^{-N})/(q - q^{-1})$. Further, by virtue of the diagonality of the D matrix, we can readily obtain the cyclic property

$$\mathrm{Tr}_{q1} \mathrm{Tr}_{q2} (\mathbf{R} E_{12}) = \mathrm{Tr}_{q1} \mathrm{Tr}_{q2} (E_{12} \mathbf{R}) \tag{3.3.16}$$

for any quantum matrix $E_{12} \in \mathrm{Mat}(N) \otimes \mathrm{Mat}(N)$.

A natural question now is that of the properties that must be satisfied by the elements of the $N \times N$ matrix T^i_j that determine the transformations (3.3.1) of the quantum bosonic, (3.3.3) and (3.3.12), and fermionic, (3.3.13), hyperplanes in order that the transformed coordinates $\tilde{x}^i$ form the same quantum algebras (q hyperplanes) (3.3.12) and (3.3.13). It is readily seen that the elements of the $N \times N$ matrix T^i_j must satisfy the conditions

$$\mathbf{P}^{\pm} \mathbf{T} \mathbf{T}' \mathbf{P}^{\mp} = 0, \tag{3.3.17}$$

which are equivalent to the RTT relations (3.1.1).

Definition. A Hopf algebra with N^2 generators T^i_j that satisfy the relations (3.1.1), where the R matrix is defined in (3.3.6), is called the algebra of functions on the quantum group $GL_q(N)$ and is denoted by $\mathrm{Fun}(GL_q(N))$.

For the quantum group $GL_q(N)$, we can define the quantum determinant $\det_q(T)$, which is a deformation of the ordinary determinant, and is also a central element for the algebra $\mathrm{Fun}(GL_q(N))$:

$$\begin{aligned} \det_q(T) \mathcal{E}^q_{j_1 j_2 \dots j_N} &= \mathcal{E}^q_{i_1 i_2 \dots i_N} T^{i_1}_{j_1} \cdot T^{i_2}_{j_2} \cdots T^{i_N}_{j_N}, \\ \det_q(T) \mathcal{E}^{q\, i_1 i_2 \dots i_N} &= T^{i_1}_{j_1} \cdot T^{i_2}_{j_2} \cdots T^{i_N}_{j_N} \mathcal{E}^{q\, j_1 j_2 \dots j_N}. \end{aligned} \tag{3.3.18}$$

Here the q-deformed antisymmetric tensor $\mathcal{E}^q_{j_1 j_2 \dots j_N} = \mathcal{E}^{q\, j_1 j_2 \dots j_N}$ is defined as follows:

$$\mathcal{E}^q_{12\dots N} = 1, \quad \langle \mathcal{E}^q |_{12\dots N} \mathbf{P}^+_{k,k+1} = 0, \quad 1 \leqslant k < N, \tag{3.3.19}$$

where in the second equation we understand the indices $1,2,...,N$ as the numbers of the vector spaces, and $\mathbf{P}^{+}_{k,k+1}=I^{\otimes(k-1)}\otimes\mathbf{P}^{+}\otimes I^{\otimes(N-k-1)}$ defines the symmetrizer (3.3.10) acting in the spaces k and $k+1$. Using such a form of expression, we can represent the definitions of the quantum determinant (3.3.18) in the form

$$\det_q(T)\langle\mathscr{E}^q|_{12...N}=\langle\mathscr{E}^q|_{12...N}T_1\cdot T_2\cdots T_N,$$

$$\det_q(T)|\mathscr{E}^q\rangle_{12...N}=T_1\cdot T_2\cdots T_N|\mathscr{E}^q\rangle_{12...N}. \tag{3.3.20}$$

Here $T_m=I^{\otimes(m-1)}\otimes T\otimes I^{\otimes(N-m)}$. The fact that $\det_q(T)$ is indeed a central element in the algebra $GL_q(N)$ can be obtained by using the definition (3.3.20), the RTT relations, and the equations

$$qI_{N+1}\langle\mathscr{E}^q|_{12...N}=\langle\mathscr{E}^q|_{12...N}R_{1,N+1}\cdot R_{2,N+1}\cdots R_{N,N+1},$$

$$q^{-1}I_{N+1}\langle\mathscr{E}^q|_{12...N}=\langle\mathscr{E}^q|_{12...N}R^{-1}_{N+1,1}\times R^{-1}_{N+1,2}\cdots R^{-1}_{N+1,N}, \tag{3.3.21}$$

where the indices $1,2,...,N+1$ are understood as the ordinal numbers of the spaces. The relations (3.3.21) follow from the expressions for the quantum determinants

$$\det_q(R^{(\pm)})=q^{\pm 1}, \tag{3.3.22}$$

where the $R^{(\pm)}$-matrix representations for T^i_j are given in (3.1.18). In their turn, the relations (3.3.22) follow from the fact that $R^{(+)}$ and $R^{(-)}$ are, respectively, upper and lower triangular block matrices with diagonal blocks of the form

$$(R^{(\pm)i}{}_i)^k_l=\delta^k_l q^{\pm\delta_{ik}}.$$

The dual relations for the tensor $|\mathscr{E}^q\rangle_{12...N}$ can be readily obtained from (3.3.21) and the identity for the $GL_q(N)$ R matrix $(\mathbf{R}_{12})^{t_1t_2}=\mathbf{R}_{12}$. Note that it is sometimes convenient to use Eqs. (3.3.21) and their duals in the form[22]

$$q^{\pm1}\langle\psi|_1\langle\mathscr{E}^q|_{23...N+1}=\langle\psi|_{N+1}\langle\mathscr{E}^q|_{12...N}\mathbf{R}^{\pm1}_N\cdots\mathbf{R}^{\pm1}_2\cdot\mathbf{R}^{\pm1}_1,$$

$$q^{\pm1}|\psi\rangle_1|\mathscr{E}^q\rangle_{23...N+1}=\mathbf{R}^{\pm1}_1\cdot\mathbf{R}^{\pm1}_2\cdots\mathbf{R}^{\pm1}_N|\psi\rangle_{N+1}|\mathscr{E}^q\rangle_{12...N},$$

where

$$\mathbf{R}_k=I^{\otimes(k-1)}\otimes\mathbf{R}\otimes I^{\otimes(N-k)}\in\mathrm{Mat}(N)^{\otimes(N+1)}. \tag{3.3.23}$$

The algebra $\mathrm{Fun}(SL_q(N))$ can now be obtained from the algebra $\mathrm{Fun}(GL_q(N))$ by imposing the subsidiary condition $\det_q(T)=1$ and, in accordance with (3.3.22), the matrix representations (3.1.19) for $T^i_j\in\mathrm{Fun}(SL_q(N))$ have the form

$$\langle L^+_2,T_1\rangle=\frac{1}{q^{1/N}}R_{12},\quad \langle L^-_2,T_1\rangle=q^{1/N}R^{-1}_{21}.$$

We now consider how it is possible to define the complexification of the linear quantum groups. We first consider the case of the group $GL_q(N)$ and assume that q is a real number.

We must define an involution operation $*$ on the algebra $\mathrm{Fun}(GL_q(N))$ or, in other words, we must define the dual algebra $\mathrm{Fun}(\widetilde{GL}_q(N))$ with generators $\tilde{T}=(T^\dagger)^{-1}$ $(T^\dagger=(T^*)^t)$ and structure relations identical to (3.1.1):

$$R_{12}\tilde{T}_1\tilde{T}_2=\tilde{T}_2\tilde{T}_1R_{12}. \tag{3.3.24}$$

We now introduce the extended algebra with generators $\{T,\tilde{T}\}$ that is the smash product of the algebras (3.1.1) and (3.3.24) with subsidiary smash commutation relations (see, for example, Refs. 9 and 10)

$$\mathbf{R}T\tilde{T}'=\tilde{T}T'\mathbf{R}. \tag{3.3.25}$$

It is natural to relate this algebra to $\mathrm{Fun}(GL_q(N,\mathbb{C}))$.

The case of $SL_q(N,\mathbb{C})$ can be obtained from $GL_q(N,\mathbb{C})$ by imposing two subsidiary conditions on the central elements:

$$\det_q(T)=1,\quad \det_q(\tilde{T})=1. \tag{3.3.26}$$

The real form $U_q(N)$ is separated from $GL_q(N,\mathbb{C})$ if we require

$$T=\tilde{T}=(T^\dagger)^{-1}, \tag{3.3.27}$$

and if in addition to this we impose the conditions (3.3.26), then the group $SU_q(N)$ is distinguished.

In the case $|q|=1$, the definition of involutions on the linear quantum groups is a nontrivial problem that can be solved[23] only after extension of the algebra of functions on the quantum groups to the algebra of functions on their tangent bundles.

3.4. Many-parameter deformation $GL_{q,r_{ij}}(N)$

In this subsection, we consider a many-parameter deformation of the linear group $GL(N)$ (Refs. 16, 21, and 24–27). A many-parameter quantum hyperplane is defined by the relations

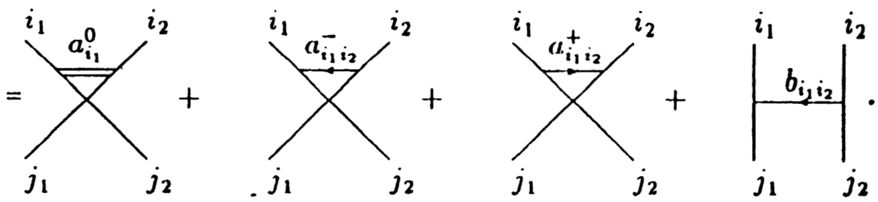
$$x^ix^j=r_{ij}x^jx^i,\quad i<j, \tag{3.4.1}$$

which can be written in the R-matrix form (3.3.5) if we introduce an additional parameter q. Thus, we have $N(N-1)/2+1$ deformation parameters: r_{ij}, $i<j$, and q. The corresponding R matrix has the form[27]

$$R_{12}=R^{i_1,i_2}_{j_1,j_2}=\delta^{i_1}_{j_1}\delta^{i_2}_{j_2}\left(q\delta^{i_1i_2}+\Theta_{i_2i_1}\frac{q}{r_{i_1i_2}}+\Theta_{i_1i_2}\frac{r_{i_2i_1}}{q}\right)+(q-q^{-1})\delta^{i_1}_{j_2}\delta^{i_2}_{j_1}\Theta_{i_1i_2}, \tag{3.4.2}$$

where Θ_{ij} is defined in (3.3.6). By direct calculation we can show that the R matrix (3.4.2) satisfies the Yang–Baxter equation (3.1.3) and the Hecke condition (3.3.7), which is the same as in the one-parameter case. In these calculations, it is convenient to use the diagrammatic technique,

$$\mathbf{R}=\hat{R}^{i_1,i_2}_{j_1,j_2}=\delta^{i_1}_{j_2}\delta^{i_2}_{j_1}(a^0_{i_1}\delta^{i_1i_2}+\Theta_{i_2i_1}a^-_{i_1i_2}+\Theta_{i_1i_2}a^+_{i_1i_2})+b_{i_1i_2}\delta^{i_1}_{j_1}\delta^{i_2}_{j_2}\Theta_{i_2i_1} \tag{3.4.3}$$



It turns out that not all solutions of the Yang–Baxter equation (3.1.3) that can be represented in the form (3.4.3) are exhausted by the many-parameter R matrices (3.4.2). Indeed, if we substitute the matrix (3.4.3) in the Yang–Baxter equation (3.1.3), we obtain the following conditions on the coefficients $a_i^0, a_{ij}^{\pm}, b_{ij}$:

$$b_{ij}=b, \quad a_{ij}^{+}a_{ji}^{-}=c, \quad (a_i^0)^2-ba_i^0-c=0 \quad (\forall i,j). \tag{3.4.4}$$

We choose for convenience, in place of the parameter c, a different parameter q, setting $c=q(q-b)$. After this, we orthonormalize (3.4.3) in such a way that $b=q-q^{-1}$. Then $c=1$, and a_i^0 can take the two values $\pm q^{\pm 1}$. For such a normalization, the R matrix (3.4.3) satisfies the Hecke relation (3.3.7). If we set $a_i^0=q$ (or $a_i^0=-q^{-1}$) for all i, then we arrive at the many-parameter case $GL_{q,r_{ij}}(N)$ (3.4.2). If, however, we set

$$a_i^0=q \quad (1\leqslant i\leqslant K), \quad a_i^0=-q^{-1} \quad (K+1\leqslant i\leqslant N), \tag{3.4.5}$$

then the R matrix (3.4.3) does not reduce to (3.4.2) and will correspond to a many-parameter deformation of the supergroup $GL(K|N-K)$ (we consider this case below in Sec. 3.6).

By virtue of the fulfillment of the Hecke identity (3.3.7) for the many-parameter case, we can introduce the same projectors $\mathbf{P}^-$ and $\mathbf{P}^+$ as in the one-parameter case (3.3.10), the first of them defining the bosonic quantum hyperplane (3.4.1) [the relations (3.3.5) with R matrix (3.4.2)], and the second defining the fermionic quantum hyperplane

$$\mathbf{P}^+x_1x_2=0 \Leftrightarrow (x^i)^2=0, \quad q^2x^ix^j=-r_{ij}x^jx^i \quad (i>j). \tag{3.4.6}$$

Regarding (3.4.1) and (3.4.6) as comodules for the many-parameter quantum group $GL_{q,r_{ij}}(N)$, we find that the generators T^i_j of the algebra $\mathrm{Fun}(GL_{q,r_{ij}}(N))$ satisfy the same RTT relations (3.1.1) but with R matrix (3.4.2). Note, however, that the quantum determinant $\det_q(T)$ (3.3.18) is not central in the many-parameter case.[26] This is due to the fact that for the many-parameter R matrix equations of the type (3.3.21) do not hold. Therefore, reduction to the SL case by means of the relation $\det_q(T)=1$ is possible only under certain restrictions on the parameters q and r_{ij}. A detailed discussion of these facts can be found in Refs. 26 and 27. Note that by an appropriate twisting of the R matrix (3.4.2) it is possible to reduce the many-parameter case to the one-parameter case (see Refs. 16 and 27 and Remark 1 at the end of Sec. 2).

3.5. $GL_q(N)$-invariant Baxterized R matrix

By Baxterization, we mean the construction of an R matrix that depends not only on a deformation parameter q but also on an additional complex spectral parameter x. If we wish to find a solution $\mathbf{R}(x)$ of the Yang–Baxter equation (with spectral parameter x) satisfying the condition of quantum invariance

$$T_1T_2\mathbf{R}(x)(T_1T_2)^{-1}=\mathbf{R}(x), \quad (T^i_j\in \mathrm{Fun}(GL_q(N))),$$

then we must seek it in the form

$$\mathbf{R}(x)=b(x)(1+a(x)\mathbf{R}) \tag{3.5.1}$$

[here $a(x)$ and $b(x)$ are certain functions of x], since by virtue of the Hecke condition (3.3.7) there exist only two basis matrices $\mathbf{1}$ and $\mathbf{R}$ that are invariants in the sense of the relations (3.1.1). The Yang–Baxter equation with dependence on the spectral parameter is chosen in the form

$$\mathbf{R}(x)\mathbf{R}'(xy)\mathbf{R}(y)=\mathbf{R}'(y)\mathbf{R}(xy)\mathbf{R}'(x). \tag{3.5.2}$$

Only the function $a(x)$ is fixed by this equation. Indeed, we substitute here (3.5.1) and take into account (3.1.3) and the Hecke condition (3.3.7); we then obtain the equation

$$a(x)+a(y)+\lambda a(x)a(y)=a(xy), \tag{3.5.3}$$

which is readily solved by the change of variables $a(x)=(1/\lambda)(\tilde{a}(x)-1)$. After this, we obtain for a the general solution

$$a(x)=(1/\lambda)(x^{\xi}-1), \tag{3.5.4}$$

where for simplicity the arbitrary parameter ξ can be set equal to -2. For convenience, we choose the normalizing function $b(x)=\lambda x$. Then the Baxterized R matrix satisfying the Yang–Baxter equation (3.5.2) will have the form

$$\mathbf{R}(x)=b(x)(1+(1/\lambda)(x^{-2}-1)\mathbf{R})=x^{-1}\mathbf{R}-x\mathbf{R}^{-1}. \tag{3.5.5}$$

It is a remarkable fact that the relations (3.1.20) can be represented as follows:

$$\mathbf{R}(x)L'(xy)L(y)=L'(y)L(xy)\mathbf{R}(x), \tag{3.5.6}$$

where the spectral parameters x and y are arbitrary, and

$$L(x)=x^{-1}L^+-xL^-. \tag{3.5.7}$$

Moreover, if we average the relation (3.5.6) with matrix T^i_j acting in the third space, we obtain the Yang–Baxter equation (3.5.2). Thus, in a certain sense (3.5.6) generalizes (3.5.2). We also recall that (3.5.2) is the condition of unique ordering of the monomials of third degree $L_1(x)L_2(y)L_3(z)$ for the algebra (3.5.6) ("diamond" condition):

$$
\begin{array}{ccccc}
 & L(y)L(x)L(z) & \longrightarrow & L(y)L(z)L(x) & \\
\nearrow & & & & \searrow \\
L(x)L(y)L(z) & & & & L(z)L(y)L(x). \\
\searrow & & & & \nearrow \\
 & L(x)L(z)L(y) & \longrightarrow & L(z)L(x)L(y) &
\end{array}
$$

We now note that from the algebra (3.5.6), disregarding the particular representation (3.5.7) for the $L(x)$ operator, we can obtain a realization for the Yangian $Y(gl(N))$ (see Ref. 13). Indeed, in (3.5.2) and (3.5.6) we make the change of spectral parameters

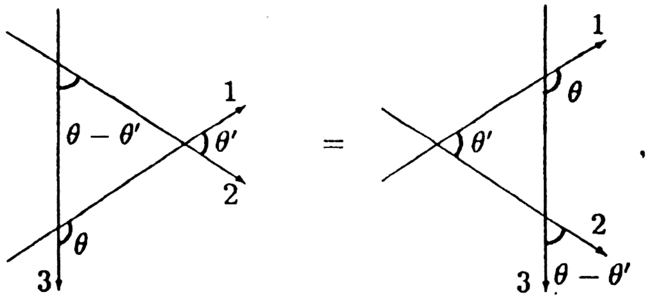
$$x=\exp\left(-\frac{1}{2}\lambda(\theta-\theta')\right),\quad y=\exp\left(-\frac{1}{2}\lambda\theta'\right). \tag{3.5.8}$$

Then the relations (3.5.2) and (3.5.6) can be rewritten in the form

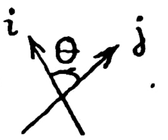
$$\mathbf{R}(\theta-\theta')\mathbf{R}'(\theta)\mathbf{R}(\theta')$$
$$=\mathbf{R}'(\theta')\mathbf{R}(\theta)\mathbf{R}'(\theta-\theta')\Rightarrow R_{23}(\theta-\theta')R_{13}(\theta)R_{12}(\theta')=R_{12}(\theta')R_{13}(\theta)R_{23}(\theta-\theta'), \tag{3.5.9}$$

$$\mathbf{R}(\theta-\theta')L'(\theta)L(\theta')=L'(\theta')L(\theta)\mathbf{R}(\theta-\theta'). \tag{3.5.10}$$

Note that Eqs. (3.5.9) have a beautiful graphical representation in the form of the triangle equation[3]



(3.5.9′)

where $R_{ij}(\theta) =$ 

In (3.5.10) we now go to the limit $\lambda=q-q^{-1}\to 0$. On the basis of (3.5.5), choosing $b(x)=1$, we readily find that in this limit the R matrix is proportional to the Yang matrix:

$$\mathbf{R}(\theta)=(1+\theta P_{12}). \tag{3.5.11}$$

For the operators $L(\theta)$, we shall assume the validity of the expansion

$$L(\theta)^i_j=\delta^i_j+\sum_{k=1}^{\infty} T^{(k)^i}_j\theta^{-k}, \tag{3.5.12}$$

where $T^{(k)^i}_j$ are the generators of the Yangian $Y(gl(N))$ (Ref. 13). The defining relations for the Yangian $Y(gl(N))$ are obtained from (3.5.10) by substituting (3.5.11) and (3.5.12). The comultiplication for $Y(gl(N))$ obviously has the form

$$\Delta(L(\theta)^i_j)=L(\theta)^i_k\otimes L(\theta)^k_j. \tag{3.5.13}$$

The Yangian $Y(sl(N))$ can be obtained from $Y(gl(N))$ after the imposition of a subsidiary condition on the generators $T^{(k)^i}_j$:

$$\det_q(L(\theta))=1.$$

The relations (3.5.10) play an important role in the quantum inverse scattering method.[1] The matrix representations for the operators (3.5.7) satisfying (3.5.6) lead to the formulation of lattice integrable systems (see, for example, Ref. 28). Equations (3.5.9) are the conditions of factorization of the S matrices in certain exactly solvable two-dimensional models of quantum field theory (see Ref. 3). These questions will be discussed in more detail in the final section of the review.

3.6. The quantum supergroups $GL_q(N|M)$ and $SL_q(N|M)$

We choose the R matrix (3.4.3) in the form (cf. Ref. 29)

$$\mathbf{R}=\hat{R}^{i_1,i_2}_{j_1,j_2}=(-1)^{(i_1)(i_2)}\delta^{i_1}_{j_2}\delta^{i_2}_{j_1}(q^{1-2(i_1)(i_2)\delta^{i_1i_2}}+\Theta_{i_2i_1}$$
$$+\Theta_{i_1i_2})+(q-q^{-1})\delta^{i_1}_{j_1}\delta^{i_2}_{j_2}\Theta_{i_2i_1}, \tag{3.6.1}$$

i.e., we have set

$$a_i^0=(-)^{(i)}q^{1-2(i)}, \quad a_{ij}^+=(a_{ij}^-)^{-1}=(-1)^{(i)(j)},$$
$$b=q-q^{-1}.$$

Here $(i)=0,1$, and, therefore, a_i^0 can take the two values $\pm q^{\pm 1}$. Thus, as we assumed in Sec. 3.4, the R matrix (3.6.1) must correspond to some supergroup. Indeed, suppose that the R matrix acts in the space of the direct product of the two supervectors $x^{j_1}\otimes y^{j_2}$ and $(j)=0,1$ (mod(2)) denotes the parity of the components x^j of the supervector. For definiteness, we will assume that $(j)=0$ $(1\leq j\leq N)$ and $(j)=1$ $(N+1\leq j\leq N+M)$. In the limit $q\to 1$, we find that $\mathbf{R}$ tends to the supertransposition operator:

$$\mathbf{R}^{i_1 i_2}_{j_1 j_2}\to(-)^{(i_1)(i_2)}\delta^{i_1}_{j_2}\delta^{i_2}_{j_1}\equiv\mathscr{P}_{12}.$$

As was noted in Sec. 3.4, the R matrix (3.6.1) satisfies the Yang–Baxter equation (3.1.3) and the Hecke relation (3.3.7). In place of the matrix $\mathbf{R}$, we introduce the new R matrix

$$R_{12}=\mathscr{P}_{12}\mathbf{R}\Rightarrow\mathbf{R}=\mathscr{P}_{12}R_{12}=(-)^{(1)(2)}P_{12}R_{12}.$$

Then from the Yang–Baxter equation (3.1.3) there follows the graded form of the Yang–Baxter equation[32] for the new R matrix:

$$R_{12}(-)^{(2)(3)}R_{13}(-)^{(2)(3)}R_{23}$$
$$=R_{23}(-)^{(2)(3)}R_{13}(-)^{(2)(3)}R_{12}. \quad (3.6.2)$$

Here we have set $(-)^{(1)(2)}=(-1)^{(i_1)(i_2)}\delta^{i_1}_{j_1}\delta^{i_2}_{j_2}$ and taken into account the fact that R_{12} is an even R matrix, i.e.,

$$R^{i_1 i_2}_{j_1 j_2}\neq 0 \quad \text{if } (i_1)+(j_1)+(i_2)+(j_2)=0\Rightarrow$$
$$(-)^{(3)((1)+(2))}R_{12}=R_{12}(-)^{(3)((1)+(2))}.$$

Finally, the quantum multidimensional superplanes for the R matrices that we have introduced have the form (see, for example, Refs. 30 and 31)

$$(\mathbf{R}-q)x_1x_2=0\Leftrightarrow x^ix^j=(-)^{(i)(j)}qx^jx^i \quad (i<j),$$
$$(x^i)^2=0 \quad \text{if } (i)=1,$$
$$(\mathbf{R}+q^{-1})x_1x_2=0\Leftrightarrow qx^ix^j=-(-)^{(i)(j)}x^jx^i \quad (i<j),$$
$$(x^i)^2=0 \quad \text{if } (i)=0. \quad (3.6.3)$$

The second hyperplane can be interpreted as the exterior algebra of differentials dx^i of the coordinates x^i for the first hyperplane.

We consider the coaction (3.3.1) of the quantum supergroup on the quantum superspaces (3.6.3). From the condition of covariance of the supermodules (3.6.3), we readily deduce the graded form of the RTT relations:

$$\mathbf{R}T_1(-)^{(1)(2)}T_2(-)^{(1)(2)}=T_1(-)^{(1)(2)}T_2(-)^{(1)(2)}\mathbf{R}\Leftrightarrow$$
$$R_{12}T_1(-)^{(1)(2)}T_2(-)^{(1)(2)}=(-)^{(1)(2)}T_2(-)^{(1)(2)}T_1R_{12}, \quad (3.6.4)$$

which are the defining relations for the generators T^i_j of the algebra $\mathrm{Fun}(GL_q(N|M))$. The matrix $\|T^i_j\|$ can be represented in the block form

$$T^i_j=\begin{pmatrix} A & B \\ C & D\end{pmatrix}, \quad (3.6.5)$$

where the elements of the $N\times N$ matrix A and of the $M\times M$ matrix D form the algebras $\mathrm{Fun}(GL_q(N))$ and $\mathrm{Fun}(GL_q(M))$, respectively. It follows from this that the noncommutative matrices D and $A-BD^{-1}C$ are invertible, and therefore so is the matrix $\|T^i_j\|$, as follows from the Gauss decomposition

$$\begin{pmatrix} A & B \\ C & D\end{pmatrix}=\begin{pmatrix} 1 & BD^{-1} \\ 0 & 1\end{pmatrix}\begin{pmatrix} A-BD^{-1}C & 0 \\ 0 & D\end{pmatrix}$$
$$\times\begin{pmatrix} 1 & 0 \\ D^{-1}C & 1\end{pmatrix}. \quad (3.6.6)$$

Thus, the algebra $\mathrm{Fun}(GL_q(N|M))$ with defining relations (3.6.4) is a Hopf algebra with structure mappings (3.1.5), where in the definition of Δ the tensor product is understood as a graded tensor product.

We now compare the relations (3.6.4) with the graded Yang–Baxter equation (3.6.2). From this comparison we readily see that the finite-dimensional matrix representations for the generators T^i_j of the quantum algebra $\mathrm{Fun}(GL_q(N|M))$ [the superanalogs of the representations (3.1.18)] can be chosen in the form

$$T_1=(-)^{(1)(3)}R_{13}(-)^{(1)(3)}\equiv R^{(+)},$$
$$T_1=(R^{-1})_{31}\equiv R^{(-)}. \quad (3.6.7)$$

From this we obtain in an obvious manner definitions of the quantum superalgebras of the dual algebras $\mathrm{Fun}(GL_q(N|M))$ [cf. Eqs. (3.1.19)]:

$$\langle L_2^+,T_1\rangle=(-)^{(1)(2)}R_{12}(-)^{(1)(2)}, \quad \langle L_2^-,T_1\rangle=R_{21}^{-1}. \quad (3.6.8)$$

Note that for $GL_q(N|M)$ we can define the quantum supertrace (see Ref. 31) and the quantum superdeterminant.[33] The algebra $\mathrm{Fun}(SL_q(N|M))$ is distinguished by the relation $\mathrm{sdet}_q(T)=1$.

The quantum supergroup $GL_q(N|M)$ was studied in detail from somewhat different positions in Ref. 33. The simplest example of a quantum supergroup, $GL_q(1|1)$, has been investigated in many studies (see, for example, Refs. 31 and 34). The R matrices (3.4.3) can be used to construct the supersymmetric Baxterized solutions of the Yang–Baxter equation (3.5.5) obtained for the first time in Ref. 35. The Yangian limits of these solutions have been used to formulate integrable supersymmetric magnets.[36] The universal R matrices for the linear quantum supergroups were constructed in Ref. 37.

3.7. The quantum groups $SO_q(N)$ and $Sp_q(2n)$ (B, C, and D series)

In Ref. 10, quantum groups with the defining relations (3.1.1), which are quantum deformations of the Lie groups $SO(N)$, where $N=2n+1$ (B_n series) and $Sp(N)$, $SO(N)$, where $N=2n$ (C_n and D_n series), were studied. It was shown that the R matrices for the groups $SO_q(N)$ and

$Sp_q(N)$ (their explicit form[10] is given below in Sec. 3.8) satisfy the cubic characteristic equation (3.1.27):

$$(\mathbf{R}-q\mathbf{1})(\mathbf{R}+q^{-1}\mathbf{1})(\mathbf{R}-\varepsilon q^{\varepsilon-N}\mathbf{1})=0, \quad (3.7.1)$$

where the case $\varepsilon=+1$ corresponds to the orthogonal groups $SO_q(N)$ (B and D series), while the case $\varepsilon=-1$ corresponds to the symplectic groups $Sp_q(2n)$ (C series). The projectors (3.1.28) corresponding to the characteristic equation (3.7.1) can be written as follows:[10]

$$\mathbf{P}^{\pm}=\frac{(\mathbf{R}\pm q^{\mp1}\mathbf{1})(\mathbf{R}-\nu\mathbf{1})}{(q+q^{-1})(q^{\pm1}\mp\nu)}$$

$$\equiv\frac{1}{q+q^{-1}}(\pm\mathbf{R}+q^{\mp1}\mathbf{1}+\mu_{\pm}\mathbf{K}), \quad (3.7.2)$$

$$\mathbf{P}^{0}=\frac{(\mathbf{R}-q\mathbf{1})(\mathbf{R}+q^{-1}\mathbf{1})}{(\nu-q)(q^{-1}+\nu)}\equiv\mu^{-1}\mathbf{K}.$$

Here $\nu=\varepsilon q^{\varepsilon-N}$,

$$\mu=\frac{(q-\nu)(q^{-1}+\nu)}{\lambda\nu}=\frac{\lambda+\nu^{-1}-\nu}{\lambda}=(1+\varepsilon[N-\varepsilon]_q),$$

$$\mu_{\pm}=\pm\frac{\lambda}{(1\mp q^{\pm1}\nu^{-1})}=\mp\frac{\nu\pm q^{\mp1}}{\mu}.$$

We also give the relations between the parameters ν, μ, $\mu_{\pm}$ that we introduced:

$$q\mu_{+}-q^{-1}\mu_{-}=\nu(\mu_{+}+\mu_{-}), \quad \mu_{+}+\mu_{-}=-\frac{q+q^{-1}}{\mu},$$

which are very convenient in various calculations that use the projector (3.7.2). For convenience, we have introduced in (3.7.2) the renormalized projector $\mathbf{K}$ ($\mathbf{K}^2=\mu\mathbf{K}$), which projects $\mathbf{R}$ onto the "singlet" eigenvalue ν:

$$\mathbf{KR}=\mathbf{RK}=\nu\mathbf{K}. \quad (3.7.3)$$

Note that for $\mathbf{K}$ we have the equation [cf. (3.3.7)]

$$\mathbf{R}-\mathbf{R}^{-1}-\lambda+\lambda\mathbf{K}=0, \quad (3.7.4)$$

which is none other than the characteristic equation (3.7.1) rewritten in a different form. Note also that the projector $\mathbf{P}^{\pm}$ (3.7.2) can be represented in the convenient form

$$\mathbf{P}^{\pm}=\frac{1}{q+q^{-1}}(\pm\tilde{\mathbf{R}}+q^{\mp1}\mathbf{1})-\frac{1}{2\mu}(1\pm\varepsilon)\mathbf{K}, \quad (3.7.5)$$

where the matrix

$$\tilde{\mathbf{R}}=\mathbf{R}-\frac{1}{2}[\mu_{-}(1+\varepsilon)+\mu_{+}(\varepsilon-1)]\mathbf{K}$$

satisfies the Hecke condition (3.3.7).

In the semiclassical limit, the characteristic equation (3.7.4) can be rewritten as follows:

$$\tfrac{1}{2}(r_{12}+r_{21})=P_{12}-\varepsilon\mathbf{K}_{12}^{(0)}, \quad (3.7.6)$$

i.e., as in the $GL_q(N)$ case (3.3.8), the semiclassical limit (3.7.6) of the characteristic equation fixes the ad-invariant part of the semiclassical r matrix. We have here used an expansion of the matrix $\mathbf{K}=\mathbf{K}^{(0)}+h\mathbf{K}^{(1)}+O(h^2)$, the first term of which is

$$(\mathbf{K}^{(0)})^{i_1i_2}_{j_1j_2}=(C_0)^{i_1i_2}(C_0^{-1})_{j_1j_2}\Rightarrow\mathbf{K}_{12}^{(0)}=|C_0\rangle_{12}\langle C_0^{-1}|_{12},$$

where the matrices $(C_0)^{ij}$: $(C_0)^2=\varepsilon I$, $(C_0)^t=\varepsilon C_0$ are the metric (symmetric) and symplectic (antisymmetric) matrices, respectively, for the groups $SO(N)$ and $Sp(2n)$. The semiclassical expansion for the projectors (3.7.2) and (3.7.5) has the form

$$\mathbf{P}_{cl}^{\pm}=\frac{1}{2}((1\pm P)\pm hP\tilde{r}-(1\pm\varepsilon)\mathbf{P}_{cl}^{0}),$$

$$\mathbf{P}_{cl}^{0}=\frac{\varepsilon}{N}(\mathbf{K}^{(0)}+h\mathbf{K}^{(1)}), \quad (3.7.7)$$

where $P=P_{12}$, and the semiclassical matrix $\tilde{r}$, which satisfies the modified classical Yang–Baxter equation, is defined in (3.2.8) and is equal to

$$\tilde{r}=r_{12}-P_{12}+\varepsilon K_{12}^{(0)}=-r_{21}+P_{12}-\varepsilon K_{12}^{(0)}.$$

The ranks of the quantum projectors (3.7.2) are equal (for q not equal to the roots of unity) to the ranks of the projectors (3.7.7), which are readily calculated in the classical limit $h=0$. Accordingly, we have:[10]

1) for the groups $SO_q(N)$

$$\mathrm{rank}(P^{(+)})=\frac{N(N+1)}{2}-1, \quad \mathrm{rank}(P^{(-)})=\frac{N(N-1)}{2},$$

$$\mathrm{rank}(P^{(0)})=1; \quad (3.7.8)$$

2) for the groups $Sp_q(2n)$

$$\mathrm{rank}(P^{(+)})=\frac{N(N+1)}{2}, \quad \mathrm{rank}(P^{(-)})=\frac{N(N-1)}{2}-1,$$

$$\mathrm{rank}(P^{(0)})=1. \quad (3.7.9)$$

The number of generators for the algebras Fun($SO_q(N)$) and Fun($Sp_q(2n)$) must be equal to the number of generators in the undeformed case, since for the generators T^i_j (3.1.1) the following subsidiary conditions are imposed in the quantum case:

$$TCT^tC^{-1}=CT^tC^{-1}T=1I\Rightarrow \quad (3.7.10)$$

$$T_1T_2|C\rangle_{12}=|C\rangle_{12}, \quad \langle C^{-1}|_{12}T_1T_2=\langle C^{-1}|_{12}, \quad (3.7.11)$$

these being a direct generalization of the classical conditions on the elements of the groups $SO(N)$ and $Sp(2n)$. The matrices C^{ij} and C^{-1}_{kl}, which are understood in (3.7.11) as objects in Vect(N)⊗Vect(N) (1 and 2 label the spaces) are the q analogs of the metric and symplectic matrices C_0 for $SO(N)$ and $Sp(N)$, respectively. The explicit form of these matrices, which is given in Ref. 10 (see also Sec. 3.8) is for us as yet unimportant, but we note the equation

$$C^{-1}=\varepsilon C. \quad (3.7.12)$$

Substituting the matrix representations (3.1.18) for T^i_j in the relations (3.7.10), we obtain the following conditions on the R matrices:

$$R_{12}=C_1(R_{12}^{t_1})^{-1}C_1^{-1}=C_2(R_{12}^{-1})^{t_2}C_2^{-1}, \quad (3.7.13)$$

where, as usual, $C_1=C\otimes I$ and $C_2=I\otimes C$. As consequences of (3.7.13) we have subsidiary conditions on the generators of the dual algebra (3.1.20):

$$L_2^{\pm}L_1^{\pm}|C\rangle_{12}=|C\rangle_{12}, \quad \langle C^{-1}|_{12}L_2^{\pm}L_1^{\pm}=\langle C^{-1}|_{12},$$

and also the equation

$$R_{12}^{t_1t_2}=C_1^{-1}C_2^{-1}R_{12}C_1C_2, \tag{3.7.14}$$

which we shall need below. The semiclassical analogs of the conditions (3.7.13) and (3.7.14) have the form

$$r_{12}=-(C_0)_1r_{12}^{t_1}(C_0)_1^{-1}=-(C_0)_2r_{12}^{t_2}(C_0)_2^{-1}$$
$$=(C_0)_1(C_0)_2r_{12}^{t_1t_2}(C_0)_1^{-1}(C_0)_2^{-1}.$$

It follows from Eqs. (3.7.10) and (3.7.12) that the antipode $S(T)=CT^tC^{-1}$ for the Hopf algebras Fun($SO_q(N)$) and Fun($Sp_q(N)$) satisfy the relation

$$S^2(T)=(CC^t)T(CC^t)^{-1}, \tag{3.7.15}$$

which is analogous to (3.1.7). Thus, the matrix D that defines the quantum trace for the quantum groups of the B, C, and D series can be chosen in the form

$$D=CC^t \Leftrightarrow D_j^i=C^{ik}C^{jk}. \tag{3.7.16}$$

We now note that the matrix $|C\rangle_{12}\langle C^{-1}|_{12}\in \mathrm{Mat}(N)\otimes\mathrm{Mat}(N)$ projects any vector $|X\rangle_{12}$ onto the vector $|C\rangle_{12}$, i.e., the rank of the projector $|C\rangle\langle C^{-1}|$ is 1. In addition, from (3.7.11) we have

$$|C\rangle\langle C|TT'=TT'|C\rangle\langle C|.$$

Therefore $|C\rangle\langle C^{-1}|\sim\mathbf{P}^0$, and, as was established in Ref. 10,

$$|C\rangle\langle C^{-1}|\equiv\mathbf{K}. \tag{3.7.17}$$

Using this relation, we can represent Eqs. (3.7.10) and (3.7.11) in the equivalent form

$$TT'\mathbf{K}=\mathbf{K}TT'=\mathbf{K}. \tag{3.7.18}$$

We now give some important relations for the matrices $\mathbf{R}$ and $\mathbf{K}$; many of them are given, in some form or other, in Ref. 10. We note first that in accordance with (3.1.4)

$$\mathbf{KR'R}=\mathbf{R'RK'}\Leftrightarrow\mathbf{RR'K}=\mathbf{K'RR'}. \tag{3.7.19}$$

Further, from Eqs. (3.7.13) and (3.7.17) [or substituting the matrix representations (3.1.18) in (3.7.18)], we obtain

$$\mathbf{R}^{\pm1}\mathbf{R}'^{\pm1}\mathbf{K}=PP'\mathbf{K}=\mathbf{K}'PP',$$
$$\mathbf{K}\mathbf{R}'^{\pm1}\mathbf{R}^{\pm1}=\mathbf{K}P'P=P'P\mathbf{K}'. \tag{3.7.20}$$

where $P=P_{12}$ and $P'=P_{23}$. A consequence of these relations is the equations

$$\mathbf{R}'^{\pm1}\mathbf{K}\mathbf{R}'^{\pm1}=\mathbf{R}^{\mp1}\mathbf{K}'\mathbf{R}^{\mp1}\Leftrightarrow\mathbf{RR'K}=\mathbf{K'R}^{-1}\mathbf{R}'^{-1},$$
$$\mathbf{R'RK'}=\mathbf{KR}'^{-1}\mathbf{R}^{-1}. \tag{3.7.21}$$

In particular, taking into account the characteristic equation (3.7.4), we obtain the identity

$$\mathbf{RK'R}=\mathbf{R}'^{-1}\mathbf{K}\mathbf{R}'^{-1}=\mathbf{R'KR'}+\lambda(\mathbf{RK'}-\mathbf{KR'}-\mathbf{R'K}$$
$$+\mathbf{K'R})+\lambda^2(\mathbf{K}-\mathbf{K}'), \tag{3.7.22}$$

which will be used in Sec. 3.9. Equation (3.7.17) leads to the identities

$$\mathbf{KK'}=\mathbf{K}P'P=P'P\mathbf{K}', \quad \mathbf{K'K}=PP'\mathbf{K}=\mathbf{K}'PP', \tag{3.7.23}$$

from which we immediately obtain

$$\mathbf{KK'K}=\mathbf{K}, \quad \mathbf{K'KK'}=\mathbf{K}'. \tag{3.7.24}$$

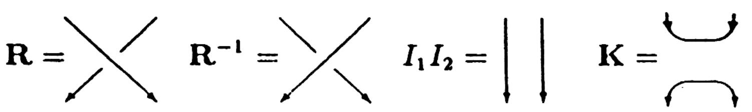
We now compare the relations (3.7.20) and (3.7.23). The result of this comparison is the equations

$$\mathbf{R}'^{\pm1}\mathbf{R}^{\pm1}\mathbf{K}'=\mathbf{KK}'=\mathbf{K}\mathbf{R}'^{\pm1}\mathbf{R}^{\pm1},$$
$$\mathbf{R}^{\pm1}\mathbf{R}'^{\pm1}\mathbf{K}=\mathbf{K'K}=\mathbf{K}'\mathbf{R}^{\pm1}\mathbf{R}'^{\pm1}. \tag{3.7.25}$$

We now apply to the first of the chain of equations in (3.7.25) the matrix $\mathbf{K}$ from the right (or $\mathbf{K}'$ from the left) and take into account (3.7.3) and (3.7.24). We then obtain

$$\mathbf{K}'\mathbf{R}^{\pm1}\mathbf{K}'=\nu^{\mp1}\mathbf{K}', \quad \mathbf{K}\mathbf{R}'^{\pm1}\mathbf{K}=\nu^{\mp1}\mathbf{K}. \tag{3.7.26}$$

As we shall see in Sec. 3.9, the relations (3.7.19)–(3.7.26) will be sufficient for the construction of $SO_q(N)$- and $Sp_q(2n)$-symmetric Baxterized $\mathbf{R}(x)$ matrices. The relations (3.7.19), (3.7.21), and (3.7.24)–(3.7.26) have a natural graphical representation if we use the diagrammatic technique



(3.7.26′)

We now give some important relations for the quantum trace (3.1.13) corresponding to the quantum groups $SO_q(N)$ and $Sp_q(N)$. Similar relations for the q trace (3.1.15) can be derived in exactly the same way. From the definitions of the matrix $\mathbf{K}$ (3.7.17) and the matrix D (3.7.16), we obtain

$$\mathrm{Tr}_{q2}(\mathbf{K})=\varepsilon I_{(1)}. \tag{3.7.27}$$

We use the relations (3.7.14) and the definition of the quantum trace (3.1.13) with matrix D (3.7.16); then, for an arbitrary quantum matrix E_j^i, we obtain the relations ($E\equiv E_1$)

$$\mathbf{R}^nE\mathbf{K}=\varepsilon\mathrm{Tr}_{q2}(\mathbf{K}E\mathbf{R}^n)\mathbf{K},$$
$$\mathbf{K}E\mathbf{R}^n=\varepsilon\mathbf{K}\mathrm{Tr}_{q2}(\mathbf{R}^nE\mathbf{K}), \quad \forall n. \tag{3.7.28}$$

$$\mathbf{K}E\mathbf{K}=\varepsilon\,\mathrm{Tr}_q(E)\mathbf{K}. \tag{3.7.29}$$

Calculating the Tr_{q2} of (3.7.28), we deduce the equation

$$\mathrm{Tr}_{q2}(\mathbf{R}^n E\mathbf{K})=\mathrm{Tr}_{q2}(\mathbf{K}E\mathbf{R}^n),\quad \forall n. \tag{3.7.30}$$

Further, from the first identity of (3.7.26), averaging it by means of Tr_{q2}, we readily obtain for the algebras $\mathrm{Fun}(SO_q(N))$ and $\mathrm{Fun}(Sp_q(N))$ analogs of (3.1.16). These take the form

$$\mathrm{Tr}_{q2}(\mathbf{R}^{\pm})\equiv\mathrm{Tr}_2(CC'\mathbf{R}^{\pm})=q^{\pm(N-\varepsilon)}I_{(1)}=\varepsilon\nu^{\mp1}I_{(1)}. \tag{3.7.31}$$

Using this relation and Eq. (3.7.4), we can calculate

$$\mathrm{Tr}_q(I)=\varepsilon+[N-\varepsilon]_q=\varepsilon\mu. \tag{3.7.32}$$

We now separate irreducible representations for the left adjoint comodules (3.1.11). For an arbitrary $N\times N$ quantum matrix E^i_j, we have

$$\begin{aligned}E&=\varepsilon\,\mathrm{Tr}_{q2}(E\mathbf{K})=\varepsilon\,\mathrm{Tr}_{q2}(\mathbf{P}^0E\mathbf{K}+\mathbf{P}^+E\mathbf{K}+\mathbf{P}^-E\mathbf{K})\\&\equiv E^{(1)}+E^{(+)}+E^{(-)},\\E&=\varepsilon\,\mathrm{Tr}_{q2}(\mathbf{K}E)=\varepsilon\,\mathrm{Tr}_{q2}(\mathbf{K}E\mathbf{P}^0+\mathbf{K}E\mathbf{P}^++\mathbf{K}E\mathbf{P}^-)\\&\equiv E^{(1)}+E^{(+)}+E^{(-)}.\end{aligned} \tag{3.7.33}$$

It is obvious that the tensors $E^{(i)}$, $(i=\pm,1)$ are invariant with respect to the adjoint coaction (3.1.11) and $\mathrm{Tr}_{q2}(\mathbf{P}^{(j)}E^{(i)}\mathbf{K})=0$ (if $i\neq j$) by virtue of (3.7.28). Thus, (3.7.33) is the required decomposition of the adjoint comodule E into irreducible components. It is clear that the component $E^{(1)}$ is proportional to the unit matrix, $(E^{(1)})^i_j=E^{(1)}\cdot\delta^i_j$, and, thus, applying Tr_{q1} to (3.7.33), we obtain

$$\mathrm{Tr}_q(E)=E^{(1)}\mathrm{Tr}_q(I)=\varepsilon\mu E^{(1)}, \tag{3.7.34}$$

where we have used the property (3.3.16), which also holds for the case of the quantum groups $SO_q(N)$ and $Sp_q(2n)$.

To conclude this subsection, we note that, as in the case of the linear quantum groups, we can define fermionic and bosonic quantum hyperplanes covariant with respect to the coactions of the groups $SO_q(N)$ and $Sp_q(N)$. Taking into account the ranks of the projectors (3.7.8) and (3.7.9), we can formulate definitions of the hyperplanes for $SO_q(N)$ ($\varepsilon=1$) and for $Sp_q(N)$ ($\varepsilon=-1$) in the form

$$(\mathbf{P}^-+(\varepsilon-1)\mathbf{K})xx'=0 \tag{3.7.35}$$

for the bosonic hyperplane [number of relations $N(N-1)/2$] and

$$(\mathbf{P}^++(\varepsilon+1)\mathbf{K})xx'=0 \tag{3.7.36}$$

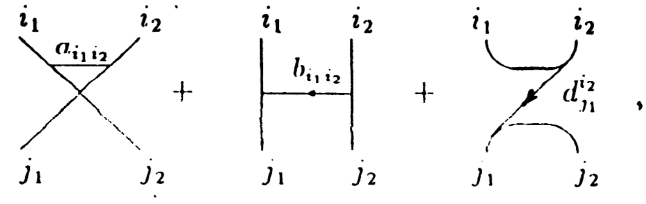
for the fermionic hyperplane [number of relations $N(N+1)/2$]. For all these algebras, the elements $\mathbf{K}xx'$ are central elements, and it is obvious that for $Sp_q(N)$ bosons and $SO_q(N)$ fermions we have $\mathbf{K}xx'=0$. Note that the conditions (3.7.10) and (3.7.11) can be understood as conditions of invariance of the quadratic forms $x_{(1)}C^{-1}x_{(2)}$ and $y_{(1)}Cy_{(2)}$ with respect to left and right transformations of the hyperplanes $x_{(k)}$ and $y_{(k)}$:

$$x^i_{(k)}\to T^i_j\otimes x^j_{(k)},\quad y_{(k)_i}\to y_{(k)_j}\otimes T^j_i.$$

3.8. The many-parameter case of the $SO_{q,a_{ij}}(N)$ and $Sp_{q,a_{ij}}(N)$ groups and the quantum supergroups $Osp_q(N|2m)$

In this subsection we show that it is possible to define many-parameter deformations of the groups $SO(N)$ and $Sp(2n)$ and also the quantum supergroups $Osp_q(N|2m)$ if we consider for the R matrix the ansatz[27]

$$\mathbf{R}=\hat{R}^{i_1,i_2}_{j_1,j_2}=\delta^{i_1}_{j_2}\delta^{i_2}_{j_1}a_{i_1i_2}+\delta^{i_1}_{j_1}\delta^{i_2}_{j_2}b_{i_1i_2}\Theta_{i_2i_1}+\delta^{i_1i_2'}\delta_{j_1j_2'}d^{i_2}_{j_1}\Theta^{i_2}_{j_1}$$

$=$  , (3.8.1)

where $\Theta^i_j=\Theta_{ij}$, $j'=K+1-j$, $K=N$ for the groups $SO(N)$, $Sp(N)$, and $K=N+2m$ for the groups $Osp(N|2m)$. The expression (3.8.1) is a natural generalization of the expression (3.4.3) for the many-parameter R matrix corresponding to the linear quantum groups. Namely, the third term in (3.8.1) is constructed from the SO-invariant tensor $\delta^{i_1i_2'}\delta_{j_1j_2'}$, which takes into account the presence in the invariant metrics for the considered groups. The functions Θ are introduced in (3.8.1) in order to ensure that the matrix $R_{12}=P_{12}\mathbf{R}$ has lower triangular block form. This is necessary for the correct definition of the operators $L^{(\pm)}$ by means of the expressions (3.1.19).

We substitute the R matrix (3.8.1) in the Yang–Baxter equation (3.1.3). It is obvious that the first two terms in (3.8.1) make contributions to the Yang–Baxter equation that are analogous to the contributions to the Yang–Baxter equation for the many-parameter case of the linear quantum groups (see Sec. 3.4). It is therefore clear that for the parameters a_{ij} and b_{ij} we reproduce the conditions (3.4.4), which in the convenient normalization $c=1$, $b=q-q^{-1}$ have the form

$$b_{ij}=b=\lambda, \quad a_{ii}=a_i^0=\pm q^{\pm 1} \quad (\forall i,j),$$
$$a_{ij}a_{ji}=1 \quad (\text{for} \quad i\neq j, i\neq j'). \tag{3.8.2}$$

Note that the final condition in (3.8.2) is somewhat weaker than in (3.4.4) (because of the restriction $i\neq j'$). This is due to the fact that the contributions to the Yang–Baxter equation proportional to a_{ii} begin to be canceled against the contributions from the third term in (3.8.1). The corresponding condition on a_{ii} fulfilling the Yang–Baxter equation can be expressed as follows:

$$a_{jj'}=a_{j'j}=\kappa_j(a_j^0-b)(j\neq j')\Leftrightarrow a_j^0 a_{jj'}$$
$$=a_j^0 a_{j'j}=\kappa_j(j\neq j'), \tag{3.8.3}$$

where for the constants κ_i we have $\kappa_j=\pm 1$. With allowance for Eqs. (3.8.2), the relations (3.8.3) are equivalent to two possibilities ($j\neq j'$):

$$1)\ a_j^0=q\rightarrow \frac{a_{jj'}}{\kappa_j}=q^{-1},$$

$$2)\ a_j^0=-q^{-1}\rightarrow \frac{a_{jj'}}{\kappa_j}=-q. \tag{3.8.4}$$

We shall see below that if we restrict consideration to just the first possibility (or only the second), then we obtain the R matrices for the quantum groups $SO_q(N)$ and $Sp_q(2n)$. If, however, we consider the mixed case, when both possibilities are satisfied (for different j), we are justified in expecting (by analogy with the linear quantum groups; see Sec. 3.6) that the corresponding R matrix will be associated with the supergroups $Osp_q(N|2m)$. The case $j=j'$ is obviously realized only for groups of the series B $[SO_q(2n+1)]$ and for the supergroups $Osp_q(2n+1|2m)$, and it follows from the Yang–Baxter equation (3.1.3) that

$$a_{jj'}|_{j=j'}=\frac{\kappa+1}{2}=1. \tag{3.8.5}$$

For the groups $SO_q(2n)$ and $Sp_q(2n)$, the parameter $a_{jj'}$ ($j=j'$) is simply absent. Further allowance for the contributions to the Yang–Baxter equation from the third term in (3.8.1) leads to the equations

$$a_{ij}a_{i'j}=\kappa_2, \quad a_{ji}a_{ji'}=\kappa_j, \quad \kappa_i=(\kappa_{i'})^{-1}=\pm 1, \tag{3.8.6}$$

$$\lambda d_k^j\kappa_i+d_i^j d_k^i=0 \tag{3.8.7}$$

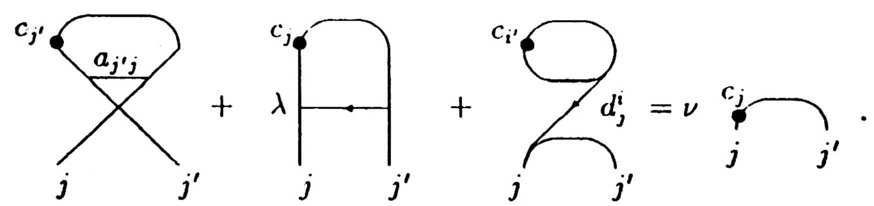
(there is no summation over repeated indices). The general solution of Eq. (3.8.7) has the form

$$d_j^i=-\lambda\kappa_i\frac{c_j}{c_i}, \tag{3.8.8}$$

where c_k are as yet arbitrary parameters. The remaining terms in the Yang–Baxter equation that do not cancel when the conditions (3.8.2)–(3.8.8) hold give recursion relations for the coefficients c_i:

$$c_{j'}a_{j'j}+\lambda c_j\Theta_{j'j}-\lambda c_j\sum_{i>j}\varkappa_i\frac{c_{i'}}{c_i}=\nu c_j. \tag{3.8.9}$$

These relations can be represented graphically in the form



Note that the relations (3.8.9) automatically lead to the characteristic equation (3.7.4) and are equivalent to Eqs. (3.7.3) if we define the metric (symplectic) matrices C [see (3.7.17)] in the form

$$C^{ij}=\varepsilon\delta^{ij'}\frac{1}{c_j}, \quad (C^{-1})_{ij}=\varepsilon\delta_{ij'}c_i. \tag{3.8.10}$$

Here the parameter $\varepsilon=\pm 1$ (see Sec. 3.7) is introduced in order to match the definition of the matrices C to the study of Ref. 10. Note also that the constant ν is fixed by the relations (3.8.9) uniquely.

We now consider the solution of Eqs. (3.8.9), which we write in the form

$$\gamma_j a_{j'j}+\lambda\Theta_{j'j}-\lambda\sum_{i=j+1}^{K}\kappa_i\gamma_i=\nu, \tag{3.8.11}$$

where

$$\gamma_j=\frac{1}{\gamma_{j'}}=\frac{c_{j'}}{c_j}. \tag{3.8.12}$$

In what follows, we shall consider only the case $\kappa_i=+1$, since the case $\kappa_i=-1$ gives the same R matrices. Equation (3.8.11) is readily solved by the ansatz

$$X_j=q^{2j}\sum_{i>j}\gamma_i,$$

after which we find the parameters $\gamma_j=q^{-2j}(q^2X_{j-1}-X_j)$ and fix ν by taking into account the properties (3.8.12).

A) For the groups $SO_q(N)$ ($\varepsilon=+1$) and $Sp_q(N)$ ($\varepsilon=-1$), we use possibility 1 in (3.8.4) (possibility 2 gives an analogous result except for the substitution $q\to-q^{-1}$). The corresponding solution of (3.8.11) has the form

$$\gamma_j\equiv\frac{c_{j'}}{c_j}=\nu q^{2(N-j)+1}\quad(j>j'),\quad \nu=\varepsilon q^{\varepsilon-N}. \quad (3.8.13)$$

B) For the groups $Osp(N|2m)$, we choose a grading in accordance with the rules $(j)=0$ for $m+1\leqslant j\leqslant m+N$ and $(j)=1$ for $1\leqslant j\leqslant m$, $m+N+1\leqslant j\leqslant N+2m$. Thus, for $(j)=0$, possibility 1 is realized, and for $(j)=1$ possibility 2 [we are referring to the possibilities in (3.8.4)]. Accordingly, we obtain

$$\gamma_j=(-1)^{(j)}\nu q^{(-1)^{(j)}2(N-j)+1-(j)4m},\quad \nu=q^{1+2m-N}. \quad (3.8.14)$$

It is obvious that for the groups $SO_q(2n+1)$ and $Osp_q(2n+1|2m)$ we have $\gamma_j=\gamma_{j'}=1$ for $j=j'$. Note that if in (3.8.14) we set $m=0$ or $N=0$, $q\to-q^{-1}$, then we reproduce (3.8.13).

In order to determine from the conditions (3.8.13) the parameters c_j and, thus, to fix the matrices C (3.8.10), we require fulfillment of the relation (3.7.12). Substitution of (3.8.10) in (3.7.12) gives the equation $c_jc_j'=\varepsilon$, which together with (3.8.13) enables us to choose c_j in the form[10]

$$c_j=-q^{j-(1/2)(N+\varepsilon+1)}(j>j')\Rightarrow c_j=\varepsilon_j q^{-\rho_j}, \quad (3.8.15)$$

where $\varepsilon_i=+1\ \forall i$ [the groups $SO_q(N)$], $\varepsilon_i=+1$ ($1\leqslant i\leqslant n$), $\varepsilon_i=-1$ ($n+1\leqslant i\leqslant 2n$) [the groups $Sp_q(2n)$], and

$$(\rho_1,\ldots,\rho_N)=\begin{cases}(n-\frac{1}{2},n-\frac{3}{2},\ldots,\frac{1}{2},0,-\frac{1}{2},\ldots,-n+\frac{1}{2})B:(SO_q(2n+1)),\\ (n,n-1,\ldots,1,-1,\ldots,-n)C:(Sp_q(2n)),\\ (n-1,n-2,\ldots,1,0,0,-1,\ldots,-n+1)D:(SO_q(2n)).\end{cases}$$

The analog of the relation (3.7.12) for the groups $Osp_q(N|M)$ is the equation

$$C^{ij}=(-1)^{(i)}(C^{-1})_{ij}\Rightarrow(-1)^{(i)}c_ic_{i'}=1,$$

and with allowance for (3.8.14) we have

$$c_j=-q^{(-1)^{(j)}(j-m-N/2)-1+(j)(N+1)}\quad(j>j'),$$

where $(j)=0,1$.

To conclude this subsection, we give the final expression for the R matrix (3.8.1) corresponding to the many-parameter deformation of the groups $SO(N)$ and $Sp(2n)$ (Ref. 27):

$$R_{12}=R^{i_1,i_2}_{j_1,j_2}=\delta^{i_1}_{j_1}\delta^{i_2}_{j_2}\left(q\,\delta^{i_1i_2}\Big|_{i_1\neq i_2'}+q^{-1}\delta^{i_1i_2'}\Big|_{i_1\neq i_2}+\Theta_{i_1i_2}a_{i_2i_1}\Big|_{i_1\neq i_2'}+\Theta_{i_2i_1}\frac{1}{a_{i_1i_2}}\Big|_{i_1\neq i_2'}+\delta^{i_1i_1'}\delta^{i_2i_2'}\right)+\lambda\,\delta^{i_1}_{j_1}\delta^{i_2}_{j_2}\Theta_{i_1i_2}-\lambda\,\delta^{i_1i_2'}\delta_{j_1j_2'}\Theta^{i_1}_{j_1}\varepsilon_{i_1}\varepsilon_{j_1}q^{\rho_{i_1}-\rho_{j_1}}. \quad (3.8.16)$$

We have here taken into account the relations (3.8.2), (3.8.4), (3.8.5), (3.8.8), and (3.8.15) and set $\kappa_i=+1$. It is also necessary to take into account the conditions (3.8.6), which show that the independent parameters are q and a_{ij} for $i<j\leqslant j'$ (the numbers of these parameters are $n(n-1)/2+1$ and $n(n+1)/2+1$, respectively, for the groups of the series C, D and B). Note that the last term in the round brackets in the expression (3.8.16) is present only for the groups of the series B. If we set $a_{ij}=1$, then the R matrices (3.8.16) are identical to the one-parameter R matrices given in Ref. 10.

The R matrices constructed in this subsection for the quantum supergroups can be obtained on the basis of the results of Ref. 35, in which Baxterized trigonometric solutions of the Yang–Baxter equation associated with the classical supergroups $Osp(N|2m)$ were obtained. Rational solutions, some special cases, and other questions relating to the subject of the quantum supergroups $Osp_q(N|2m)$ are also discussed in Refs. 36 and 38.

3.9. $SO_q(N)$- and $Sp_q(N)$-invariant Baxterized R matrices

Arguing as in Sec. 3.5, we conclude that the $SO_q(N)$- and $Sp_q(N)$-invariant Baxterized matrices $\mathbf{R}(x)$ must be sought [by virtue of the fact that the characteristic equation (3.7.1) is cubic] in the form of a linear combination of the three basis matrices $\mathbf{1}$, $\mathbf{R}$, $\mathbf{R}^2$. Expressing $\mathbf{R}^2$ in terms of $\mathbf{K}$, we can represent $\mathbf{R}(x)$ in the form

$$\mathbf{R}(x)=c(x)(\mathbf{1}+a(x)\mathbf{R}+b(x)\mathbf{K}), \quad (3.9.1)$$

where $a(x)$, $b(x)$, and $c(x)$ are certain functions that depend on the spectral parameter x. We determine the functions $a(x)$ and $b(x)$ from the Yang–Baxter equation (3.5.2), for which we consider the expression

$$X(a_i,\bar{a}_i,b_j,\bar{b}_j)=(\mathbf{1}+a_1\mathbf{R}+b_1\mathbf{K})(\mathbf{1}+a_2\mathbf{R}'+b_2\mathbf{K}')\times(\mathbf{1}+a_3\mathbf{R}+b_3\mathbf{K})-(\mathbf{1}+\bar{a}_3\mathbf{R}'+\bar{b}_3\mathbf{K}')\times(\mathbf{1}+\bar{a}_2\mathbf{R}+\bar{b}_2\mathbf{K})(\mathbf{1}+\bar{a}_1\mathbf{R}'+\bar{b}_1\mathbf{K}'). \quad (3.9.2)$$

Here $\mathbf{1}=I^{\otimes 3}$, $\mathbf{K}=\mathbf{K}_{12}$, $\mathbf{K}'=\mathbf{K}_{23}$. Using Eqs. (3.7.19)–(3.7.26), we obtain for $X(a,\bar{a},b,\bar{b})$ (3.9.2) the expression

$$\begin{aligned}X(a,\bar{a},b,\bar{b})=&(a_1a_3-\overline{a_1a_3})+(\kappa+\lambda^2\eta)\mathbf{K}-(\bar{\kappa}+\lambda^2\eta)\mathbf{K}'\\&+(a_1+a_3+\lambda a_1a_3-\bar{a}_2)\mathbf{R}\\&-(\bar{a}_1+\bar{a}_3+\lambda\overline{a_1a_3}-a_2)\mathbf{R}'+(\beta-\bar{\beta})\mathbf{KK}'\\&+(\gamma-\bar{\gamma})\mathbf{K}'\mathbf{K}+(a_1a_2-\overline{a_1a_2})\mathbf{RR}'\\&+(a_2a_3-\overline{a_2a_3})\mathbf{R}'\mathbf{R}+(\rho-\lambda\eta)\mathbf{KR}'\\&-(\bar{\rho}-\lambda\eta)\mathbf{RK}'+(\sigma-\lambda\eta)\mathbf{R}'\mathbf{K}-(\bar{\sigma}\\&-\lambda\eta)\mathbf{K}'\mathbf{R}+(a_1a_2a_3-\overline{a_1a_2a_3})\\&\times\mathbf{RR}'\mathbf{R}+(\eta-\bar{\eta})\mathbf{R}'\mathbf{KR}'.\end{aligned} \quad (3.9.3)$$

Here

$$\kappa=b_3-\bar{b}_2+a_1(\nu b_3-\lambda\nu a_3-\lambda b_2b_3)+b_1(1+\nu a_3-\lambda a_3b_2)+b_1b_3(\mu+\nu^{-1}a_2+b_2),$$

$$\beta = b_1 b_2 + a_2 a_3 b_1 + \lambda a_3 b_1 b_2,$$

$$\gamma = b_2 b_3 + a_1 a_2 b_3 + \lambda a_1 b_2 b_3,$$

$$\rho = a_2 b_1 + a_3 b_1 b_2 - \overline{a_1 b_2}, \quad \sigma = a_2 b_3 + a_1 b_2 b_3 - \overline{a_3 b_2},$$

$$\eta = a_1 a_3 b_2. \tag{3.9.4}$$

and we set $\overline{\kappa(a,\bar{a},b,\bar{b})} = \kappa(\bar{a},a,\bar{b},b)$, etc.

We consider the equation $X(a,\bar{a},b,\bar{b})=0$, where we shall assume that $\bar{a}_i = a_i$, $\bar{b}_i = b_i$ and set

$$a_1 = a(x), \quad a_2 = a(xy), \quad a_3 = a(y), \quad b_1 = b(x),$$

$$b_2 = b(xy), \quad b_3 = b(y).$$

Then the equation $X=0$ reduces to the Yang–Baxter equation (3.5.2), and the following relations arise for the variables a_i and b_i:

$$a_1 + a_3 + \lambda a_1 a_3 = a_2,$$

$$b_3 - b_2 - \lambda \nu a_1 a_3 + \nu a_1 b_3 - \lambda a_1 b_2 b_3$$
$$+ b_1 (1 + \nu a_3 - \lambda a_3 b_2 + \mu b_3 + \nu^{-1} a_2 b_3 + b_2 b_3)$$
$$+ \lambda^2 a_1 a_3 b_2 = 0,$$

$$a_2 b_1 + a_3 b_1 b_2 = a_1 b_2 + \lambda a_1 a_3 b_2,$$

$$a_2 b_3 + a_1 b_2 b_3 = a_3 b_2 + \lambda a_1 a_3 b_2. \tag{3.9.5}$$

These four relations are equivalent to the three functional equations

$$a(x) + a(y) + \lambda a(x) a(y) = a(xy), \tag{3.9.6}$$

$$b(y) - b(xy) + a(x)[\nu b(y) - \lambda \nu a(y) - \lambda b(xy) b(y)$$
$$+ \lambda^2 a(y) b(xy)] + b(x)[1 + \nu a(y) - \lambda a(y) b(xy)$$
$$+ \mu b(y) + \nu^{-1} a(xy) b(y) + b(xy) b(y)] = 0, \tag{3.9.7}$$

$$a(xy) b(y) + a(x) b(xy) b(y)$$
$$= b(xy)(a(y) + \lambda a(x) a(y)), \tag{3.9.8}$$

since the third and fourth relations in (3.9.5) give the same equation (3.9.8). As was to be expected, Eq. (3.9.6) is identical to Eq. (3.5.3) obtained in the $GL_q(N)$ case, and its general solution is given in (3.5.4). By means of (3.9.6), we can transform the right-hand side of Eq. (3.9.8) in such a way that (3.9.8) reduces to the equation

$$\frac{a(x)}{a(xy)} = \frac{b(xy) - b(y)}{b(xy)(b(y)+1)} \equiv 1 - \frac{b(y)(1+b(y))^{-1}}{b(xy)(1+b(xy))^{-1}}. \tag{3.9.9}$$

We now note that Eq. (3.9.6) can be rewritten in the form

$$\frac{a(x)}{a(xy)} = 1 - \frac{a(y)(\lambda a(y)+1)^{-1}}{a(xy)(\lambda a(xy)+1)^{-1}} \tag{3.9.10}$$

and, comparing (3.9.9) and (3.9.10), we arrive at the result

$$\frac{a(y)(b(y)+1)}{(\lambda a(y)+1) b(y)} = \text{const} \equiv \frac{\alpha+1}{\lambda}, \tag{3.9.11}$$

where α denotes an arbitrary parameter. The specific choice of the constant on the right-hand side of (3.9.11) is made for convenience in what follows. Substituting the solution (3.5.4) in (3.9.11), we obtain the following general expression for $b(y)$:

$$b(y) = \frac{y^{\xi} - 1}{\alpha y^{\xi} + 1}. \tag{3.9.12}$$

It is a remarkable fact that Eq. (3.9.7) is satisfied identically on the functions (3.5.4) and (3.9.12) if the constant α satisfies the quadratic equation

$$\alpha^2 - \frac{\lambda}{\nu} \alpha - \frac{1}{\nu^2} = 0, \tag{3.9.13}$$

the two solutions of which are readily found:

$$\alpha_{\pm} = \pm \frac{q^{\pm 1}}{\nu} = \pm \varepsilon q^{N - \varepsilon \pm 1}. \tag{3.9.14}$$

Thus, the solutions of the Yang–Baxter equation (3.5.2) can be represented in the form

$$\mathbf{R}(x) = c(x) \left(1 + \frac{1}{\lambda} (x^{\xi} - 1) \mathbf{R} + \frac{x^{\xi} - 1}{\alpha x^{\xi} + 1} \mathbf{K} \right), \tag{3.9.15}$$

and we have the two possibilities $\alpha = \alpha_{\pm}$ (3.9.14), which are inequivalent [both for the $SO_q(N)$ case and for the $Sp_q(N)$ case], since these solutions cannot be reduced to each other by any functional transformations of the spectral parameter. For convenience we choose $c(x) = \lambda x$ and $\xi = -2$ in (3.9.15); then for the R matrices (3.9.15) we can propose four equivalent forms of expression:

$$\mathbf{R}(x) = x^{-1} \mathbf{R} - x \mathbf{R}^{-1} + \lambda \frac{\alpha_{\pm} + 1}{\alpha_{\pm} x^{-1} + x} \mathbf{K} = \frac{1}{\alpha_{\pm} x^{-1} + x}$$
$$\times (-\mathbf{R}^{-1} x^2 + (\mathbf{R} - \alpha_{\pm} \mathbf{R}^{-1} + \lambda(\alpha_{\pm} + 1) \mathbf{K})$$
$$+ \alpha_{\pm} \mathbf{R} x^{-2})$$
$$= \frac{x - x^{-1}}{x + \alpha_{\pm} x^{-1}} \left(-x \mathbf{R}^{-1} - \alpha_{\pm} x^{-1} \mathbf{R} + \frac{\lambda(\alpha_{\pm} + 1)}{x - x^{-1}} \right)$$
$$= (x^{-1} q - x q^{-1}) \mathbf{P}^{+} + (xq - (xq)^{-1}) \mathbf{P}^{-}$$
$$+ \frac{(\alpha_{\pm} + x^{-2})(\nu \alpha_{\pm} + (\lambda - \nu \alpha_{\pm}) x^2)}{x + \alpha_{\pm} x^{-1}} \mathbf{P}^0. \tag{3.9.16}$$

The last expression determines the spectral decomposition of $\mathbf{R}(x)$, from which, for example, we can readily obtain

$$\mathbf{R}(1) = \lambda \mathbf{1}, \quad \mathbf{R}(i) = \pm i (q + q^{-1})(1 - 2\mathbf{P}^{\pm})$$
$$= \pm i (q + q^{-1}) \sigma^{\pm},$$
$$\mathbf{R}(x) \mathbf{R}(x^{-1}) = (\lambda^2 - (x - x^{-1})^2). \tag{3.9.17}$$

The relations (3.9.17) agree with the Yang–Baxter equation (3.5.2). The second relation in (3.9.17) gives a connection between $\mathbf{R}(i)$ and the operators introduced in (3.1.30) (the signs $\pm$ correspond to the parameter choice $\alpha = \alpha_{\pm}$).

The Baxterized $\mathbf{R}$ matrices (3.9.15) and (3.9.16) must determine algebras with the defining relations (3.5.6). However, a realization of the operators $L(x)$ in terms of the generators $L^{(\pm)}$ of the quantum algebras $U_q(so(N))$ and

$U_q(sp(N))$ [analogous to (3.5.7)] is, unfortunately, not known to me. Such a realization would be extremely helpful for many applications.

To conclude this subsection, we give the expressions for the R matrices of the Yangians $Y(so(N))$ and $Y(sp(N))$. These R matrices can be obtained from (3.9.15) after the ansatz

$$\xi=-2, \quad x=\exp(-\lambda\theta/2),$$

an appropriate choice of the normalization function $c(x)$, and passage to the limit $h\to 0$ [$q=\exp(h)\to 1$]. Further, it is easy to see that the cases $\alpha=\alpha_+$, $\varepsilon=1$ ($SO_q(N)$) and $\alpha=\alpha_-$, $\varepsilon=-1$ ($Sp_q(2n)$) reduce to Yang's R matrix (3.5.11). The nontrivial $SO(N)$- and $Sp(N)$-symmetric Yangian R matrices for $Y(so(N))$ and $Y(sp(N))$ correspond to the choice in (3.9.15) of $c(x)=2-\varepsilon(N+2\theta)$, and

$$\alpha=\alpha_-, \quad \varepsilon=1(SO_q(N));$$
$$\alpha=\alpha_+,$$
$$\varepsilon=-1(Sp_q(N)), \tag{3.9.18}$$

and have the form

$$\mathbf{R}(\theta)=(2-\varepsilon(N+2\theta))\mathbf{1}+\theta(2-\varepsilon(N+2\theta))P_{12} + \varepsilon 2\theta K_{12}^{(0)}. \tag{3.9.19}$$

Then all the expressions for the Yangians $Y(so(N))$ and $Y(sp(N))$ are identical to the expressions (3.5.10), (3.5.12), and (3.5.13).

The Yangian R matrix (3.9.19) for the $SO(N)$ case was found in Ref. 3, and that for the $Sp(2n)$ case in Ref. 39. These R matrices were used in Ref. 28 to construct and investigate exactly solvable $SO(N)$- and $Sp(2n)$-symmetric magnets. Baxterized trigonometric R matrices (3.9.15) corresponding to the parameter values (3.9.18) were first found by Bazhanov in 1984 and were published in Ref. 40. The same R matrices were independently constructed in Ref. 41.

3.10. Elliptic solutions of the Yang–Baxter equation

In this subsection, we consider $Z_N\otimes Z_N$-symmetric solutions of the Yang–Baxter equation (3.5.9) (Ref. 42). The elements $R^{i_1i_2}_{j_1j_2}(\theta)$ of the corresponding R matrix will be expressed in terms of elliptic functions of the spectral parameter θ.

We construct this solution explicitly, following the method of Ref. 42. We consider two matrices g and h such that $g^N=h^N=1$:

$$g=\begin{pmatrix} 1 & 0 & 0 & \dots & 0 \\ 0 & \omega & 0 & \dots & 0 \\ \vdots & & & & \vdots \\ 0 & 0 & 0 & \dots & \omega^{N-1} \end{pmatrix},$$
$$h=\begin{pmatrix} 0 & 1 & 0 & \dots & 0 \\ 0 & 0 & 1 & \dots & 0 \\ \vdots & & & & \\ 1 & 0 & 0 & \dots & 0 \end{pmatrix}, \tag{3.10.1}$$

where $\omega=\exp(2\pi i/N)$ and $hg=\omega gh$. The matrices g and h are Z_N-graded generators of the algebra Mat(N), the graded basis for which can be chosen in the form

$$I_\alpha=I_{\alpha_1\alpha_2}=g^{\alpha_1}h^{\alpha_2}, \quad \alpha_{1,2}=0,1,\dots,N-1. \tag{3.10.2}$$

On the other hand, the matrices (3.10.2) realize a projective representation of the group $Z_N\otimes Z_N: I_\alpha I_\beta=\omega^{\alpha_2\beta_1}I_{\alpha+\beta}$. Any matrix $R_{12}(\theta)=R^{i_1i_2}_{j_1j_2}(\theta)$ can now be written in the form

$$R_{12}(\theta)=W_{\alpha,\beta}(\theta)I_\alpha\otimes I_\beta.$$

We consider the $Z_N\otimes Z_N$-invariant subset of such matrices:

$$R_{12}(\theta)=W_\alpha(\theta)I_\alpha\otimes I_\alpha^{-1}, \tag{3.10.3}$$

where $I_\alpha^{-1}=h^{-\alpha_2}g^{-\alpha_1}=\omega^{\alpha_1\alpha_2}I_{-\alpha}$. The invariance of the matrices (3.10.3) is expressed by the relations

$$R_{12}(\theta)=I_\gamma^{\otimes 2}R_{12}(\theta)(I_\gamma^{\otimes 2})^{-1}, \tag{3.10.4}$$

which obviously follow from the identity

$$I_\gamma I_\alpha I_\gamma^{-1}=\omega^{\langle\alpha,\gamma\rangle}I_\alpha, \quad \langle\alpha,\gamma\rangle=\alpha_1\gamma_2-\alpha_2\gamma_1.$$

It was noted in Ref. 42 that the relations

$$R_{12}(\theta+1)=g_1^{-1}R_{12}(\theta)g_1=g_2R_{12}(\theta)g_2^{-1},$$
$$R_{12}(\theta+\tau)=\exp(-i\pi\tau)\exp(-2\pi i\theta)h_1^{-1}R_{12}(\theta)h_1 =\exp(-i\pi\tau)\exp(-2\pi i\theta)h_2R_{12}(\theta)h_2^{-1},$$
$$R_{12}(\theta)=I_\alpha\otimes I_\alpha^{-1}\equiv P_{12}, \tag{3.10.5}$$

where τ is some complex parameter (period), are consistent with the Yang–Baxter equation (3.5.9) and can be regarded as subsidiary conditions to these equations. Moreover, for the $Z_N\otimes Z_N$-invariant R matrix (3.10.3) the conditions (3.10.5) determine the solution of the Yang–Baxter equation uniquely. Indeed, substitution of (3.10.3) in (3.10.5) leads to the equations

$$W_\alpha(\theta+1)=\omega^{\alpha_1}W_\alpha(\theta),$$
$$W_\alpha(\theta+\tau)=\exp(-i\pi\tau)\exp(-2\pi i\theta)\omega^{-\alpha_2}W_\alpha(\theta),$$
$$W_\alpha(0)=1, \tag{3.10.6}$$

the solution of which can be found by means of an expansion in a Fourier series and has the form

$$W_\alpha(\theta)=\frac{\Theta_\alpha(\theta+\eta)}{\Theta_\alpha(\eta)}, \tag{3.10.7}$$

where

$$\Theta_\alpha(u)=\sum_{m=-\infty}^{\infty}\exp\left[i\pi\tau\left(m+\frac{\alpha_2}{N}\right)^2 +2\pi i\left(m+\frac{\alpha_2}{N}\right)\left(u+\frac{\alpha_1}{N}\right)\right]. \tag{3.10.8}$$

The parameter η in (3.10.7) is arbitrary. For $N=2$, the solution (3.10.7) is identical to the solution obtained by Baxter[2] in connection with the investigation of the so-called eight-vertex lattice model.

Direct substitution of the expression (3.10.3) in the Yang–Baxter equation (3.5.9) shows that the functions $W_\alpha(\theta)$ must satisfy the relations

$$\sum_\gamma W_\gamma(\theta-\theta')W_{\alpha-\gamma}(\theta)W_{\beta+\gamma}(\theta') \times(\omega^{\langle\gamma,\beta\rangle}-\omega^{\langle\alpha-\gamma,\beta\rangle})=0. \quad (3.10.9)$$

It is interesting to note that apparently there does not yet exist a direct proof of the fact that the identity (3.10.9) holds when the functions (3.10.7) and (3.10.8) are substituted. All proofs[43] known to the author use, in some form or other, indirect approaches.

4. CONCLUSIONS

In the previous sections of the review, we have presented the fundamentals of the theory of quantum groups. We have also considered how it is possible to obtain trigonometric and rational (Yangian) solutions of the Yang–Baxter equation on the basis of the theory of quantum Lie groups. Unfortunately, in the previous sections it was not possible for us to discuss in detail the numerous applications of the theory of quantum groups and the Yang–Baxter equation in both theoretical physics and mathematics. In this final section, we shall merely give a brief list of such applications that, in the author's opinion, have definite interest.

Before we do this, we recall that in the physics of condensed media two-dimensional exactly solvable models are used to describe various layered structures, contact surfaces in electronics, surfaces of superconducting liquids like He II, etc. Two-dimensional integrable field theories are used to describe dynamical effects in one-dimensional spatial systems (such as light tubes, nerve fibers, etc.). In addition, such field theories (and also integrable systems on one-dimensional chains) can also arise on the reduction of multidimensional field theories (see, for example, Ref. 44).

1. We have already mentioned that the quantum inverse scattering method[1] (an introduction to this method that can be readily understood by a wide circle of readers can be found in Ref. 45) is designed as a constructive procedure for solving quantum two-dimensional integrable systems. In addition, the quantum inverse scattering method makes it possible to construct quantum integrable systems on one-dimensional chains (see, for example, Refs. 28, 36, and 46). The point of departure is the relation (3.5.10) for the L operators, which can be rewritten in the form

$$R_{12}(\theta-\theta')L_{2k}(\theta)L_{1k}(\theta') =L_{1k}(\theta')L_{2k}(\theta)R_{12}(\theta-\theta'). \quad (4.1)$$

Here $L_{ik}(\theta)$ are $N\times N$ matrices in the auxiliary space V_i with matrix coefficients that are the operators in the state space of the kth site of a chain consisting of M sites:

$$L_{ik}(\theta)=I^{\otimes(k-1)}\otimes L_i(\theta)\otimes I^{\otimes(M-k)}\rightarrow[L_{ik},L_{i'k'}] =0 \ (k\neq k'). \quad (4.2)$$

In (4.2), the symbol $\otimes$ denotes the direct product of the operator spaces. It is clear that from the Yang–Baxter equation (3.5.9) there always follow representations for the L operators in the form of R matrices:

$$L_{ik}(\theta)=(R_{ik}(\theta))^{-1}, \quad L_{ik}(\theta)=R_{ki}(\theta). \quad (4.3)$$

In the given case, $L_{ik}(\theta)$ act nontrivially in the space $V_i\otimes V_k$. To construct an integrable system, we introduce the monodromy matrix

$$T_i(\theta)=D_i^{(M)}L_{iM}(\theta)D_i^{(M-1)}L_{iM-1}\cdots D_i^{(1)}L_{i1}(\theta). \quad (4.4)$$

If the matrices $D^{(k)}$ satisfy the relations

$$R_{ji}(\theta)D_i^{(k)}D_j^{(k)}=D_j^{(k)}D_i^{(k)}R_{ji}(\theta),$$
$$[D_i^{(k)},D_j^{(r)}]=[D_i^{(k)},L_{jr}]=0,$$

then it follows from (4.1) that

$$R_{ij}(\theta-\theta')T_i(\theta)T_j(\theta')=T_j(\theta')T_i(\theta)R_{ij}(\theta-\theta'). \quad (4.5)$$

The trace of the monodromy matrix (4.4) over the auxiliary space i forms the transfer matrix $t(\theta)=\mathrm{Tr}_{(i)}(T_i(\theta))$, which gives a commuting family of operators: $[t(\theta),t(\theta')]=0$, as follows directly from (4.5). From this family, we choose a certain local operator H, which is interpreted as the Hamiltonian of the system. Locality of the Hamiltonian is a natural physical requirement and means that H describes the interaction of only nearest-neighbor sites of the chain. The remaining operators in the commuting set $t(\theta)$ give an infinite set of integrals of the motion, this indicating the integrability of the constructed system. In many well-known cases, the local Hamiltonians are identical to the logarithmic derivatives of the transfer matrices:

$$H=\frac{d}{d\theta}\ln(t(\theta))|_{\theta=0}.$$

For example, if we choose as L operators (4.3) the Yangian R matrices (3.9.19), then we obtain $SO(N)$ ($\varepsilon=+1$) and $Sp(N)$ ($N=2n$, $\varepsilon=-1$) invariant models of magnets with Hamiltonians[28]

$$H=\sum_{l=1}^{M}\left(P_{l,l+1}-\frac{2}{N-2\varepsilon}K_{l,l+1}\right)+M,$$
$$O_{M,M+1}=O_{M,1},$$

where $K_{l,l+1}=C^{i_l i_{l+1}}C_{j_l j_{l+1}}$, and $P_{l,l+1}=\delta^{i_l}_{j_{l+1}}\delta^{i_{l+1}}_{j_l}$ are the transposition matrices. These models are generalizations of the XXX models of Heisenberg magnets.

2. The Yang–Baxter equation (3.5.9):

$$S_{23}(\theta-\theta')S_{13}(\theta)S_{12}(\theta')=S_{12}(\theta')S_{13}(\theta)S_{23}(\theta-\theta'), \quad (4.6)$$

in conjunction with the subsidiary relations of unitarity and crossing symmetry

$$S_{12}(\theta)S_{21}(-\theta)=I_{12}, \quad S_{12}(\theta)=(S_{21}(i\pi-\theta))^{t_1}, \quad (4.7)$$

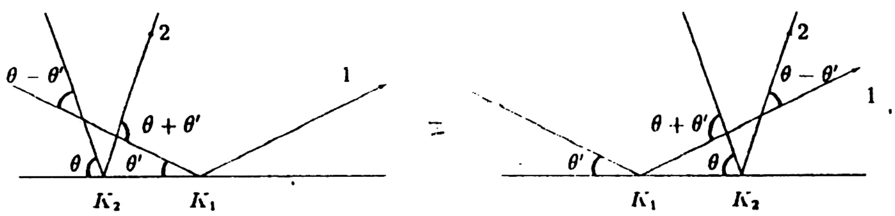
uniquely determine factorized S matrices (with a minimal set of poles) describing the scattering of particle-like excitations in (1+1)-dimensional integrable relativistic models.[3] The matrix $S^{i_1 i_2}_{j_1 j_2}(\theta)$ is interpreted as the S matrix for the scattering of two particles with isotopic spins i_1 and i_2 to two particles with spins j_1 and j_2, and the spectral parameter θ is none other than the difference of the rapidities of these particles. The many-particle S matrices decompose into products of two-particle matrices (factorization). In this sense, the Yang–Baxter equation (4.6) is the condition of uniqueness of the determination of the many-particle S matrices.

The reflection equation,[47,48] which depends on the spectral parameters,

$$S_{12}(\theta-\theta')K_2(\theta)S_{21}(\theta+\theta')K_1(\theta')$$

$$=K_1(\theta')S_{12}(\theta+\theta')K_2(\theta')S_{21}(\theta-\theta'), \tag{4.8}$$

determines in conjunction with the relations (4.6) and (4.7) factorized scattering of particles (solitons) on a half-line.[47,48] In this case, the operator matrix $K_1(\theta)=K^{i_1}_{j_1}(\theta)$ describes reflection of a particle with rapidity θ at a finite point of the half-line. Graphically, the relation (4.8) can be represented in the form



In (4.8), we now go to the limit θ, $\theta'\to\pm\infty$ in such a way that $\theta-\theta'\to\pm\infty$, and at the same time we set

$$K(\theta)|_{\theta\to\infty}=L, \quad S_{12}(\theta)|_{\theta\to\infty}=R_{21}.$$

$$K(\theta)|_{\theta\to-\infty}=L^{-1}, \quad S_{12}(\theta)|_{\theta\to-\infty}=(R_{12})^{-1}.$$

Then (4.8) goes over into (3.1.23), and this is the reason why all algebras with defining relations of the type (3.1.23) are called algebras of the reflection equation.[48]

Note that every solution of the Yang–Baxter equation (4.6) with the conditions (4.7) determines an equivalence class of relativistic integrable systems with the given factorized S matrix. Thus, every classification of solutions of the Yang–Baxter equation is, to some degree, a classification of the integrable systems with the properties indicated above.

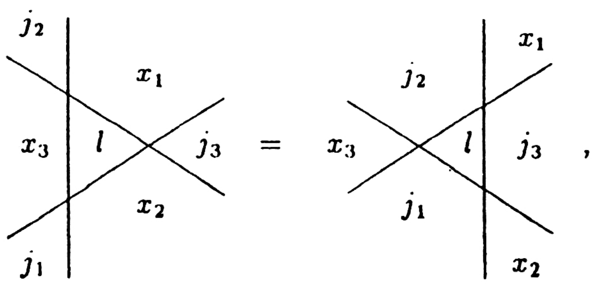
3. The Yang–Baxter equation expressed in the form (3.1.3) shows that the matrices $\mathbf{R}_k$ (3.3.23) satisfying the locality conditions

$$[\mathbf{R}_k,\mathbf{R}_{k'}]=0, \quad |k-k'|>1$$

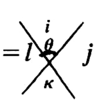
realize representations of the braid algebra B_{N+1} (see, for example, Refs. 10 and 49). An arbitrary braid (with $N+1$ filaments) can be constructed by multiplying the matrices $\mathbf{R}_k$ and their inverses. Graphically, a braid can be represented by using the diagrammatic representation of the R matrices (3.7.26′). After this, the given braid can be closed to a site by means of the $(N+1)$-th quantum trace Tr_q and $\overline{\mathrm{Tr}}_q$. The resulting expression will obviously be an invariant of the corresponding site.

Considering besides the R matrices (3.3.23) R matrices acting in the dual spaces (see Ref. 18), we can generalize the construction given above and construct invariants of knotted ribbons.[18]

4. We mention the application of the Yang–Baxter equation in multiloop calculations in quantum field theory. There is a form of the Yang–Baxter equation (see Refs. 2, 50, and 51) that can also be represented in the form of the triangle equation (3.5.9′), but the indices are ascribed, not to the "lines," but to the "faces":



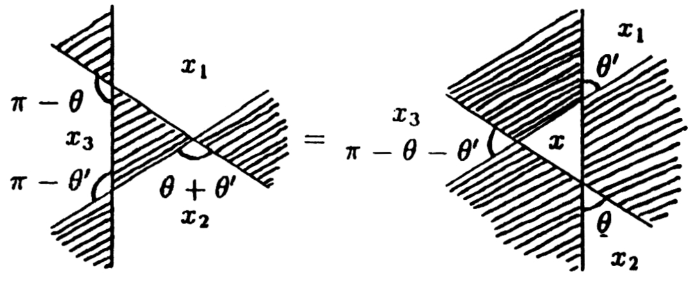
(4.8′)

where $R^{ij}_{kl}(\theta)=$  $=R^{kl}_{ij}(\theta)$, and there is a summation over the index l.

The relation (4.8′), like (3.5.9′), gives the conditions of integrability of two-dimensional lattice statistical systems with weights determined by the R matrices $R^{ij}_{kl}(\theta)$. We now note that the Yang–Baxter equation (4.8′) has a solution in the form $R^{ij}_{kl}(\theta)=G^{i}_{k}(\theta)G^{j}_{l}(\pi-\theta)$, where the matrices $G^{x}_{x'}=G^{x'}_{x}$ satisfy the star triangle relation (see, for example, Refs. 2 and 50)

$$f(\theta,\theta')G^{x_1}_{x_2}(\theta+\theta')G^{x_2}_{x_3}(\pi-\theta')G^{x_3}_{x_1}(\pi-\theta)=\sum_{x} G^{x_1}_{x}(\theta')G^{x_2}_{x}(\theta)G^{x_3}_{x}(\pi-\theta-\theta'), \tag{4.9}$$

where $f(.,.)$ is an arbitrary function. The relations (4.9) for $f=1$ can be represented graphically in the form



We now consider the massless Feynman propagator

$$G_D(x-x'|\alpha)=\frac{\Gamma(\alpha)}{(x-x')^{2\alpha}}=\frac{\Gamma(\alpha)}{((x-x')_\mu(x-x')^\mu)^\alpha}, \tag{4.10}$$

where $\alpha=D/2-1+\beta$; $D=4-2\varepsilon$ is the dimension of space–time, x_μ are its coordinates, and ε and β are, respectively, the parameters of the dimensional and analytic regularizations. The propagator (4.10) satisfies the relation

$$G_D\left(x_1-x_2\left|\frac{D}{2}-\alpha_3\right.\right)G_D\left(x_2-x_3\left|\frac{D}{2}-\alpha_1\right.\right)\times G_D\left(x_3-x_1\left|\frac{D}{2}-\alpha_2\right.\right)^{\Sigma\alpha_i=D}=\int\frac{d^Dx}{\pi^{D/2}}\prod_{i=1}^{3}G_D(x-x_i|\alpha_i) \tag{4.11}$$

which can be readily obtained if on the right-hand side of (4.11) we choose $x_3=0$ and make a simultaneous inversion transformation of the variables of integration, $x^\mu\to x^\mu/x^2$, and of the coordinates $x^\mu_{1,2}$. The relations (4.9) and (4.11) are equivalent if we set

$$G^{x}_{x'}(\theta)=G_D\left(x-x'\left|\frac{D}{2}-\frac{D\theta}{2\pi}\right.\right),\quad f(\theta,\theta')=1. \tag{4.12}$$

Thus, the analytically and dimensionally regularized massless propagator (4.10) satisfies the infinite-dimensional star triangular relation (4.9) and accordingly, on the basis of (4.10) and (4.12), we can construct solutions of the Yang–Baxter equation (4.8′). This remark was made in Ref. 50, in which calculations were made of vacuum diagrams with an infinite number of vertices corresponding to a planar square lattice (ϕ^4 theory, $D=4$), a planar triangular lattice (ϕ^6 theory, $D=3$), and a honeycomb lattice (ϕ^3 theory, $D=6$). The star triangular relation (4.11) (known also as the uniqueness relation) was used in addition for analytic calculation of the diagrams that contribute to the 5-loop β function of the $\phi^4_{D=4}$ theory and of massless ladder diagrams, and also to investigate the symmetry groups of dimensionally and analytically regularized massless Feynman diagrams (see Ref. 52). We emphasize that an extremely interesting problem is that of massive deformation of the propagator function (4.10) and the corresponding deformation of the star triangular relation (4.11).

5. Note that we have not considered at all the numerous applications of quantum Lie groups with deformation parameters q satisfying the conditions $q^N=1$, i.e., when the parameters q are equal to the roots of unity. These applications (see, for example, Ref. 53) are associated with the specific theory of representations of the quantum groups that, generally speaking, can no longer be regarded as the deformation of classical Lie groups.

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