The family problem in 4-dimensional fermionic grand unified string theories

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One of the main points of investigations in high-energy physics is to study the following chain: the law of quark and lepton mass spectra→the puzzles of the quark and lepton family mixing→a possible new family dynamics. The new family-symmetry dynamics might be connected to the existence of some exotic gauge or matter fields, or something else. For this it will be better to study the possible appearance of this gauge symmetry in the framework of the grand unified string theories. In the framework of the four-dimensional heterotic superstring with free fermions we investigate the rank-eight grand unified string theories (GUST) which contain the $SU(3)_H$ gauge family symmetry. We explicitly construct GUSTs with gauge symmetry $G=SU(5)\times U(1)\times(SU(3)\times U(1))_H$ and $G=SO(10)\times(SU(3)\times U(1))_H\subset SO(16)$ or $E(6)\times SU(3)_H\subset E(8)$ in a free complex fermion formulation. As the GUSTs originating from Kac–Moody algebras (KMA) contain only low-dimensional representations, it is usually difficult to break the gauge symmetry. We solve this problem by taking for the observable gauge symmetry the diagonal subgroup $G^{\rm sym}$ of the rank-16 group $G\times G\subset SO(16)\times SO(16)$ or $(E(6)\times SU(3)_H)^2\subset E(8)\times E(8)$. We discuss the possible fermion matter and Higgs sectors in these models. In these GUSTs there must exist "superweak" light chiral matter ($m_H^f<M_W$). The understanding of quark and lepton mass spectra and family mixing leaves a possibility for the existence of an unusually low mass breaking scale of the $SU(3)_H$ family gauge symmetry (several TeV).

1. THEORETICAL TRENDS BEYOND THE STANDARD MODEL

1.1. The family-mixing state in the SM and the origin of quark and lepton masses

There are no experimental indications which would compel one to go beyond the framework of the $SU(3)_C\times SU(2)_L\times U(1)_Y$ standard model (SM) with three generations of quarks and leptons. None of the up-to-date experiments contradicts, within their limits of accuracy, the validity of the SM predictions for low-energy phenomena. The fermion-mass origin, generation mixing, and CP-violation problems are among most exciting theoretical puzzles in the SM.

One has ten parameters in the quark sector of the SM with three generations: six quark masses, three mixing angles, and the Kobayashi–Maskawa (KM) CP-violation phase ($0<\delta^{\rm KM}<\pi$). The CKM (Cabibbo–Kobayashi–Maskawa) matrix in the Wolfenstein parametrization is determined by the four parameters—the Cabibbo angle $\lambda\approx 0.22$, A, ρ, and η:

$$\mathbf{V}_{\rm CKM}=\begin{pmatrix} V_{ud} & V_{us} & V_{ub}\\ V_{cd} & V_{cs} & V_{cb}\\ V_{td} & V_{ts} & V_{tb}\end{pmatrix} =\begin{pmatrix} 1-1/2\lambda^2 & \lambda & A\lambda^3(\rho-i\eta)\\ -\lambda & 1-1/2\lambda^2 & A\lambda^2\\ A\lambda^3(1-\rho-i\eta) & -A\lambda^2 & 1\end{pmatrix}. \tag{1}$$

In the complex plane the point (ρ,η) is a vertex of the unitarity triangle and describes the CP violation in the SM. The unitarity triangle is constructed from the following unitarity condition for $V_{\rm CKM}$: $V^*_{ub}+V_{cd}\approx A\lambda^3$.

Recently, the interest in the CP-violation problem was excited again by data on the search for direct CP-violation effects in neutral K mesons:[1,2]

$$\mathrm{Re}\left(\frac{\varepsilon'}{\varepsilon}\right)=(7.4\pm 6)\times 10^{-4}, \tag{2}$$

$$\mathrm{Re}\left(\frac{\varepsilon'}{\varepsilon}\right)=(23\pm 7)\times 10^{-4}. \tag{3}$$

The major contribution to the CP-violation parameters ε_K and ε'_K (K^0 decays), as well as to the $B_d^0-\bar{B}_d^0$ mixing parameter $x_d=\Delta m_{(B_d)}/\Gamma_{(B_d)}$, is due to the large t-quark mass contribution. The same statement holds also for some amplitudes of K- and B-meson rare decays. The CDF collaboration gives the following region for the top-quark mass: $m_t=174\pm 25$ GeV.[3] The complete fit, which is based on the low-energy data as well as on the latest LEP and SLC data and a comparison with the mass indicated by the CDF measurements, gives $m_t=162\pm 9$ GeV.[4]

The main drawbacks of the SM lie in our nonunderstanding of the generation problem, the mixing and the hierarchy of the quark and lepton mass spectra. For example, for quark masses $\mu\approx 1$ GeV we can obtain the following approximate relations:[5]

$$m_{i_k}\approx(q_H^u)^{2k}m_0,\quad k=0,1,2;\quad i_0=u,\quad i_1=c,\quad i_2=t,$$

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$$m_{i_k} \approx (q_H^d)^{2k} m_0, \quad k=0,1,2; \quad i_0=d, \quad i_1=s, \quad i_2=b, \tag{4}$$

where $q_H^u \approx (q_H^d)^2$, $q_H^d \approx 4-5 \approx 1/\lambda$, and $\lambda \approx \sin\theta_C$.

Here we used the conventional ratios of the "running" quark masses:[6]

$$m_d/m_s = 0.051 \pm 0.004,$$

$$m_u/m_c = 0.0038 \pm 0.0012,$$

$$m_s/m_b = 0.033 \pm 0.011,$$

$$m_c(\mu=1\ \text{GeV}) = 1.35 \pm 0.05\ \text{GeV},$$

$$m_t^{\text{phys}} \approx 0.6 m_t(\mu=1\ \text{GeV}). \tag{5}$$

This phenomenological formula predicts the following value for the t-quark mass:

$$m_t^{\text{phys}} \approx 180-200\ \text{GeV}. \tag{6}$$

In the SM these mass matrices and the mixing come from the Yukawa sector:

$$L_Y = QY_u\bar{q}_u h^* + QY_d\bar{q}_d h + LY_e\bar{l}_e h + \text{H.c.}, \tag{7}$$

where Q_i and L_i are three quark and lepton isodoublets, and q_{u_i}, q_{d_i}, and e_i are three right-handed antiquark and antilepton isosinglets, respectively; h is the ordinary Higgs doublet. In the SM the 3×3 family Yukawa matrices, $(Y_u)_{ij}$ and $(Y_d)_{ij}$, have no particular symmetry. Therefore it is necessary to find some additional mechanisms or symmetries beyond the SM which could reduce the number of independent parameters in the Yukawa sector L_Y. These new structures can be used for the determination of the mass hierarchy and family mixing.

To understand the origin of the generation mixing and the fermion mass hierarchy, several models beyond the SM suggest special forms for the mass matrix of the "up" and "down" quarks (Fritzsch ansatz, "improved" Fritzsch ansatz, "democratic" ansatz, etc.[7]). These mass matrices have less than ten independent parameters, or they could have some matrix elements equal to zero ("texture zeroes").[8] This allows us to determine the diagonalizing matrices U_L and D_L in terms of the quark masses:

$$Y_d^{\text{diag}} = D_L Y_d D_R^+, \quad Y_u^{\text{diag}} = U_L Y_d U_R^+. \tag{8}$$

For simplicity the symmetric form of the Yukawa matrices has been taken; therefore $D_L = D_R^*$, $U_L = U_R^*$. These ansatzes or zero "textures" can be checked experimentally in predictions for the mixing angles of the CKM matrix: $V_{\text{CKM}} = U_L D_L^+$. For example, one can consider the following approximate form at the scale M_X for the symmetric "texture" used in Ref. 8:

$$\mathbf{Y}_u = \begin{pmatrix} 0 & \lambda^6 & 0 \\ \lambda^6 & 0 & \lambda^2 \\ 0 & \lambda^2 & 1 \end{pmatrix} \quad \mathbf{Y}_d = \begin{pmatrix} 0 & 2\lambda^4 & 0 \\ 2\lambda^4 & 2\lambda^3 & 2\lambda^3 \\ 0 & 2\lambda^3 & 1 \end{pmatrix}. \tag{9}$$

Given these conditions, it is possible to follow the evolution down to low energies, via the renormalization-group equations of all the quantities, including the matrix elements of the Yukawa couplings $Y_{u,d}$, the values of the quark masses [see (4)], and the CKM matrix elements [see (1)]. Moreover, using these relations, we can compute U_L (or D_L) in terms of the CKM matrix and/or the quark masses.

In GUT extensions of the SM with the family gauge symmetry embedded, Yukawa matrices can acquire particular symmetry or an ansatz, depending on the Higgs multiplets to which they couple. The family gauge symmetry could help us to study in an independent way the origin of the "up" (U) and "down" (D) quark mixing matrices and, consequently, the structure of the CKM matrix $V_{\text{CKM}} = UD^+$. Owing to the local gauge family symmetry, a low-energy breaking scale gives us a chance to define the quantum numbers of quarks and leptons and thus establishes a link between them in families. For the mass fermion ansatz considered above in the extensions of the SM there could exist the following types of $SU(3)\times SU(2_L)$ Higgs multiplets: (1,2), (3,1), (8,1), (3,2), (8,2), (1,1),..., which in turn could exist in the spectra of string models.

In the framework of the rank-eight grand unified string theories we will consider an extension of the SM due to local family gauge symmetry, $G_H = SU(3)_H$, $SU(3)_H \times U(1)_H$ models, their developments, and their possible Higgs sector. Thus, for understanding the quark mass spectra and the difference between the origins of the "up" (or "down") quark and charged-lepton mass matrices in GUSTs we must study the Higgs content of the model, which we must use on the one hand for breaking the GUT, quark–lepton, $G_H = SU(3)_H, \ldots, SU(2)_L \times U(1)$ symmetries and, on the other, for Yukawa matrix constructions. The vital question arising here is the nature of the ν mass.

2. TOWARDS A LOW-ENERGY "EXACTLY SOLVABLE" GAUGE FAMILY SYMMETRY

2.1. The "bootstrap" gauge family models

The underlying analysis for this family symmetry-breaking scale relies on the modern experimental probability constraints for the typical rare flavor-changing processes. The estimates for the family symmetry-breaking scale have certain regularities, depending on the particular symmetry-breaking schemes and generation-mixing mechanisms (different ansatzes for quark and lepton mass matrices with 3_H or $3_H + 1_H$ generations were discussed in Ref. 5). As noted there, the current understanding of the quark and lepton mass spectra leaves room for the existence of an unusually low mass breaking scale of the non-Abelian gauge $SU(3)_H$ or $(SU(3)\otimes U(1))_H$ family symmetry at about several TeV. Some independent experiments for verifying the relevant hypotheses can been considered: light (π,K) and heavy (B,D) meson and charged-lepton flavor-changing rare decays (Refs. 5 and 24–26), and family symmetry-violation effects in e^+e^-, ep, and pp collider experiments (LEP, HERA, FNAL, and LHC).

The inclusion in the model of Higgs fields which transform under the $SU(3)_H \times SU(2)_L$ symmetry like $H^a = (\underline{8},\underline{1})$ [or $H_p^a = (\underline{8},\underline{2})$; $p=1,2$] and $X^i = (\underline{3},\underline{1})$ [or $X_p^i = (\underline{3},\underline{2})$; $p=1,2$] gives the following contribution to the family gauge-boson mass matrix:

$$(M_H^2)_{\underline{8}}^{ab}=g_H^2\sum_{d=1}^{8} f^{adc}f^{bdc'}\langle H^c\rangle\langle H^{c'}\rangle, \tag{10}$$

$$(M_H^2)_{\underline{3}}^{ab}=g_H^2\sum_{k=1}^{3} \frac{\lambda_{ik}^a}{2}\frac{\lambda_{kj}^b}{2}\langle X^i\rangle\langle X^j\rangle^*. \tag{11}$$

The lowest bound on M_H can be obtained from the analysis of the branching ratios of $\mu,\pi,K,D,B,$... rare decays ($\mathrm{Br}\geqslant 10^{-15-17}$).

In Ref. 5 we investigated samples of different scenarios of $SU(3)_H$ breaking down to the $SU(2)_H\times U(1)_{3H}$ and $U(1)_{3H}\times U(1)_{8H}$ and $U(1)_{8H}$ subgroups, as well as the mechanism of complete breaking of the base group $SU(3)_H$. We tried to realize the SUSY-conserving program on scales where the relevant gauge symmetry is broken. In the framework of these versions of gauge symmetry breaking, we searched for the spectra of horizontal gauge bosons and gauginos, and calculated the amplitudes of some typical rare processes. Theoretical estimates for the branching ratios of some rare processes obtained from these calculations were compared with the experimental data. Further, we obtained some bounds on the masses of H_μ bosons and the appropriate H gauginos. Of particular interest was the case of the $SU(3)_H$ group which breaks completely on the scale M_{H_0}. We calculated the splitting of the eight H-boson masses in a model-dependent fashion. This splitting, depending on the quark mass spectrum, allows us to reduce considerably the predictive ambiguity of the model—"an almost exactly solvable model."

We assume that when the $SU(3)_H$ gauge symmetry of the quark and lepton generations is violated, all eight gauge bosons acquire the same mass M_{H_0}. Such a breaking is not difficult to obtain by, say, introducing Higgs fields transforming in accordance with the triplet representation of the $SU(3)_H$ group. These fields are singlets under the standard-model symmetries [$z\in(3,1,10)$ and $\bar{z}\in(\bar{3},1,1,0)$; $\langle \bar{z}^{i\alpha}\rangle_0=\delta^{i\alpha}V$, $\langle z_i^\alpha\rangle_0=\delta_i^\alpha V$; $i,\alpha=1,2,3$, where $V = M_{H_0}$]. We understand that here we need a more elegant way to break this symmetry, like a dynamical one. But at this stage it is very important to establish a link between the mass spectra of the horizontal gauge bosons and the known heavy fermions like the t quark. The degeneracy of the masses of the eight gauge horizontal vector bosons is eliminated by using the VEVs of the Higgs fields violating the electroweak symmetry and determining the mass matrix of the "up" and "down" quarks (leptons). Thus, there is a set of Higgs fields (see Table I) $H(8,2)$, $h(8,2)$, $Y(\bar{3},2)$, $X(3,2)$, $\kappa_{1,2}(1,2)$ which could violate the $SU(2)\times U(1)$ symmetry and could determine the mass matrix of the "up" and "down" quarks. On the other hand, in order to calculate the splitting between the masses of the horizontal gauge bosons, one must take into account the VEVs of this set Higgs fields.

Now we can turn to the construction of the horizontal gauge-boson mass matrix M_{ab}^2 ($a,b=1,2,..,8$):

$$(M_H^2)_{ab}=M_{H_0}^2\delta_{ab}+(\Delta M_d^2)_{ab}+(\Delta M_u^2)_{ab}. \tag{12}$$

Here $(\Delta M_d^2)_{ab}$ and $(\Delta M_u^2)_{ab}$ are "known" functions of the heavy fermions, $(\Delta M_{u,d}^2)_{ab}=F_{ab}(m_t,m_b,\ldots)$, which mainly receive contributions due to the vacuum expectation values of the Higgs bosons that were used to construct the mass-matrix ansatzes for the $d(u)$ quarks.

For example, in the case of $N_g=\underline{3}+\underline{1}$ families with the Fritzsch ansatz for the quark mass matrices, using $SU(3)_H\times SU(2)$ Higgs fields (8,2),[5] we can write down some rough relations between the masses of the horizontal gauge bosons ("bootstrap" solution):

$$M_{H_1}^2\approx M_{H_2}^2\approx M_{H_3}^2\approx M_{H_0}^2+\frac{g_H^2}{4}\left[\frac{1}{\lambda^2}\frac{m_c m_t}{1-m_t/m_{t'}}\right]+\ldots,$$

$$M_{H_4}^2\approx M_{H_5}^2\approx M_{H_6}^2\approx M_{H_7}^2\approx M_{H_0}^2+\frac{g_H^2}{4}\left[\frac{1}{\tilde{\lambda}^2}m_t m_{t'}\right]+\ldots,$$

$$M_{H_8}^2\approx M_{H_0}^2+\frac{g_H^2}{3}\left[\frac{1}{\tilde{\lambda}^2}m_t m_{t'}\right]+\ldots, \tag{13}$$

where λ and $\tilde{\lambda}$ are Yukawa couplings.

We were interested in how the unitary compensation for the contributions of horizontal forces to rare processes[5] depend on the different versions of the $SU(3)_H$ symmetry breaking. The investigation of this dependence allows us first, to understand how low the horizontal symmetry-breaking scale M_H can be, and, second, how this scale is determined by a particular choice of a mass-matrix ansatz, both for quarks and for leptons.

We would like to stress the possible existence of a local family symmetry with a low-energy symmetry-breaking scale, i.e., the existence of rather light H bosons: $m_H\geqslant(1-10)$ TeV.[5] We have analyzed, in the framework of the "minimal" horizontal supersymmetric gauge model, the possibilities of obtaining a satisfactory hierarchy for quark masses and of relating it to the splitting of the horizontal gauge boson masses. We expect that with this approach the horizontal model will become more definite, since it will allow us to study the amplitudes of rare processes and the CP-violation mechanism more thoroughly. In this way we hope to get a deeper insight into the nature of the interdependence between the generation-mixing mechanism and the local horizontal symmetry-breaking scale.

2.2. The $N=1$ SUSY character of the $SU(3_H)$ gauge family symmetry

We will consider the supersymmetric version of the standard model, extended by the family (horizontal) gauge symmetry (and, if necessary, we will also extend this model by the $G_R=SU(2)_R$ right-hand gauge group). The supersymmetric Lagrangian of the strong, electroweak, and horizontal interactions, based on the $SU(3)_C\times SU(2)_L\times U(1)_Y\times SU(3)_H\ldots$ group [where the G_R gauge group and the Abelian gauge factor $U(1)_H$ can also be taken into consideration], has the general form

$$\mathscr{L}^{N=1}=\int d^2\theta\,\mathrm{Tr}(W^kW^k)+\int d^4\theta S_I^+ e^{\Sigma_k 2g_k\hat{V}_k}S_I$$

$$+\int d^4\theta\,\mathrm{Tr}(\Phi^+ e^{2g_H\hat{V}_H}\Phi e^{-2g_H\hat{V}_H})$$

$$+\int d^4\theta \, \mathrm{Tr}(H_y^+ e^{2g_2\hat{V}_2+y2g_1\hat{V}_1}e^{2g_H\hat{V}_H}H_y e^{-2g_H\hat{V}_H})$$

$$+\left(\int d^2\theta P(S_i,\Phi,H_y,\eta,\xi,\ldots)+\mathrm{H.c.}\right). \qquad (14)$$

(See, for comparison, $\mathscr{L}^{N=2}$ in Appendix A.) In Eq. (14) the index k runs over all the gauge groups: $SU(3)_C$, $SU(2)_L$, $U(1)_Y$, $SU(3)_H$, $V=\mathbf{T}^a V^a$, where V^a are real vector superfields, and $\mathbf{T}^a$ are the generators of the $SU(3)_C$, $SU(2)_L$, $U(1)_Y$, $SU(3)_H$ groups; S_I are left chiral superfields from fundamental representations, and $I=i,1,2$; $S_i=Q,u^c,d^c,L,e^c,\nu^c$ are matter superfields; $S_1=\eta$, $S_2=\xi$ are Higgs fundamental superfields; the Higgs left chiral superfield Φ transforms as the adjoint representation of the $SU(3)_H$ group, and the Higgs left chiral superfields H_y ($H_{Y=+1/2}=H$, $H_{Y=-1/2}=h$) transform nontrivially under the horizontal $SU(3)_H$ and electroweak $SU(2)_L$ symmetries (see Table I). The quantity P in (14) is a superpotential to be specified below. To construct it, we use the internal $U(1)_R$ symmetry which is customary for a simple $N=1$ supersymmetry.

In models with a global supersymmetry it is impossible to have simultaneously a SUSY breaking and a vanishing cosmological term. The reason is the semipositive definite nature of the scalar potential in the strict supersymmetry approach (in particular, in the case of broken SUSY we have $V_{\min}>0$). The problem of supersymmetry breaking with a vanishing cosmological term $\Lambda=0$ is solved in the framework of the $N=1$ SUGRA models. This may be done with an appropriate choice of the Kähler potential, in particular, in "mini–maxi" or "maxi"-type models.[27] In such approaches, the spontaneous breaking of the local SUSY is due to the possibility of obtaining nonvanishing VEVs for the scalar fields from the "hidden" sector of SUGRA.[27] The appearance in the observable sector of the so-called soft breaking terms is a consequence of this effect.

In the "flat" limit, i.e., neglecting gravity, one is left with the Lagrangian (14) and soft SUSY-breaking terms, which on the scales $\mu \ll M_{\mathrm{Pl}}$ have the form

$$\mathscr{L}_{SB}=\frac{1}{2}\sum_i m_i^2|\phi_i|^2+\frac{1}{2}m_1^2\,\mathrm{Tr}|h|^2+\frac{1}{2}m_2^2\,\mathrm{Tr}|H|^2$$

$$+\frac{1}{2}\mu_1^2|\eta|^2+\frac{1}{2}\mu_2^2|\xi|^2+\frac{1}{2}M^2\,\mathrm{Tr}|\Phi|^2$$

$$+\frac{1}{2}\sum_k M_k\lambda_k^a\lambda_k^a+\mathrm{H.c.}+\text{trilinear terms}, \qquad (15)$$

where $H_1=H$, $H_2=h$, and i runs over all the scalar matter fields $\tilde{Q},\tilde{u}^c,\tilde{d}^c,\tilde{L},\tilde{e}^c,\tilde{\nu}^c$, while k runs over all the gauge groups $SU(3)_H$, $SU(3)_C$, $SU(2)_L$, $U(1)_Y$. At energies close to the Planck scale all the masses, as well as the gauge couplings are correspondingly equal (this is true if the analytic kinetic function satisfies $f_{\alpha\beta}\sim\delta_{\alpha\beta}$),[27] but at low energies they have different values, depending on the corresponding renormalization-group equation (RGE). The squares of some masses may be negative, which permits spontaneous gauge symmetry breaking.

TABLE I. The Higgs superfields with their $SU(3_H)$, $SU(3)_C$, $SU(2)_L$, $U(1)_Y$ (and possible $U(1)_H$ factor) quantum numbers.

	H	C	L	Y	Y_H
Φ	8	1	1	0	0
H	8	1	2	$-1/2$	$-y_{H1}$
h	8	1	2	1/2	y_{H1}
ξ	$\bar{3}$	1	1	0	0
η	3	1	1	0	0
Y	$\bar{3}$	1	2	1/2	$-y_{H2}$
X	3	1	2	$-1/2$	y_{H2}
κ_1	1	1	1 (2)	0 (1/2)	$-y_{H3}$
κ_2	1	1	1 (2)	0 ($-1/2$)	y_{H3}

Considering the SUSY version of the $SU(3)_H$ model, it is natural to ask why we need to supersymmetrize the model. On the basis of our present-day knowledge of the nature of supersymmetry,[27,28] the answer will be as follows:

(a) First, it is necessary to preserve the hierarchy of the scales: $M_{\mathrm{EW}}<M_{\mathrm{SUSY}}<M_H<...?...<M_{\mathrm{GUT}}$. Breaking the horizontal gauge symmetry, one has to preserve SUSY on that scale. Another hierarchy to be considered is $M_{\mathrm{EW}}<M_{\mathrm{SUSY}}\sim M_H$. In this case, the scale M_H should be rather low ($M_H\leqslant$ a few TeV).

(b) To use the SUSY $U(1)_R$ degrees of freedom to construct the superpotential and to forbid undesired Yukawa couplings.

(c) Super-Higgs mechanism—it is possible to describe Higgs bosons by means of massive gauge superfields.[28]

(d) To connect the vector-like character of the $SU(3)_H$ gauge horizontal model and $N=2$ SUSY (see Appendix A).

Since the expected scale of the horizontal symmetry breaking is sufficiently large, $M_H\gg M_{\mathrm{EW}}$, $M_H\gg M_{\mathrm{SUSY}}$ (where M_{EW} is the scale of the electroweak symmetry breaking, and M_{SUSY} is the value of the splitting into ordinary particles and their superpartners), it is reasonable to seek SUSY-preserving stationary vacuum solutions.

Let us construct the gauge-invariant superpotential P of the Lagrangian (14). With the fields given in Table I the most general superpotential will have the form

$$P=\lambda_0\left[\frac{1}{3}\mathrm{Tr}(\hat{\Phi}^3)+\frac{1}{2}M_I\,\mathrm{Tr}(\hat{\Phi}^2)\right]+\lambda_1[\eta\hat{\Phi}\xi+M'\eta\xi]$$

$$+\lambda_2\,\mathrm{Tr}(\hat{h}\hat{\Phi}\hat{H})+(\text{Yukawa couplings})$$

$$+(\text{Majorana terms}\quad \nu^c), \qquad (16)$$

where the Yukawa couplings can be constructed, for example, by using the Higg fields, H and h, transforming under $SU(3)_H\times SU(2_L)$ like (8,2):

$$P_Y=\lambda_3 Q\hat{H}d^c+\lambda_4 L\hat{H}e^c+\lambda_5 Q\hat{h}u^c. \qquad (17)$$

One can also consider other types of superpotential P_Y, using the Higgs fields from Table I.

Note that the fields Φ,H,h can be obtained at level 2 of the Kac–Moody algebra g or effectively at level 1 of the algebras g^{I}, g^{II} after "integration" over the heavy fields, when $G^{\mathrm{I}}\times G^{\mathrm{II}}\to G^{\mathrm{symm}}$ (see Sec. 2). The Higgs fields X and Y are very important in models with a fourth $SU(3)_H$-singlet generation. In the construction of the stationary solutions, only the following contributions of the scalar potential are taken into account:

$$V=\sum_i |F_i|^2+\sum_a |D^a|^2=V_F+V_D\geqslant 0, \tag{18}$$

where

$$V_F=\sum |\frac{\partial P_F}{\partial F_i}|^2=|\frac{\partial P_F}{\partial F_{\Phi^a}}|^2+|\frac{\partial P_F}{\partial F_{\xi_i}}|^2+|\frac{\partial P_F}{\partial F_{\eta_i}}|^2. \tag{19}$$

The case $\langle V\rangle=0$ of a supersymmetric vacuum can be realized within different gauge scenarios.[5]. By switching on the SUGRA, a vanishing scalar potential is no longer necessarily required to conserve the supersymmetry. The different gauge breaking scenarios do not result in obligatory vacuum degeneracy, as in the case of the global SUSY version. Let us write down each of the terms of (19):

$$\begin{aligned}P_F(\Phi,\xi,\eta)=&\lambda_0\left[\frac{i}{4\times 3} f^{abc}\Phi^a\Phi^b\Phi^c\right.\\&\left.+\frac{1}{4\times 3} d^{abc}\Phi^a\Phi^b\Phi^c+\frac{1}{4} M_I\Phi^c\Phi^c\right]_F\\&+\lambda_1[\eta_i(T^c)^i_j\xi^j\Phi^c+M'\eta_i\xi^i]_F\\&+\lambda_2\left[\frac{i}{4} f^{abc}h^a_i\Phi^b H^c_j\epsilon^{ij}\right.\\&\left.+\frac{d^{abc}}{4} h^a_i\Phi^b H^c_j\epsilon^{ij}\right]_F+\text{H.c.}\end{aligned} \tag{20}$$

The contribution of the D terms to the scalar potential will be

$$\begin{aligned}V_D=&g_H^2|\eta^+T^a\eta-\xi^+T^a\xi+(1/2)f^{abc}\Phi^b\Phi^{c+}\\&+(i/2)f^{abc}h^bh^{c+}+(i/2)f^{abc}H^bH^{c+}|^2\\&+g_2^2|h^+(\tau^i/2)h+H^+(\tau^i/2)H|^2\\&+(g')^2|(1/2)h^+h-(1/2)H^+H|^2.\end{aligned} \tag{21}$$

The SUSY-preserving condition for the scalar potential (18) is determined by the flat F_i and D^a directions: $\langle F_i\rangle=0=\langle D^a\rangle_0=0$. It is possible to remove the degeneracy of the supersymmetric value solutions by taking into account the interaction with supergravity, which was attempted in SUSY GUTs, e.g., in the $SU(5)$ one[27] [$SU(5)\to SU(5)$, $SU(4)\times U(1)$, $SU(3)\times SU(2)\times U(1)$].

The horizontal-symmetry spontaneous breaking to the intermediate subgroups in the first three cases of Ref. 5 can be realized by using the scalar components of the chiral complex superfields Φ, which are singlets under the standard gauge group. The Φ superfield transforms as the adjoint representation of $SU(3)_H$. The intermediate scale M_I can be sufficiently large: $M_I>10^5-10^6$ GeV. The complete breaking of the remnant symmetry group V_H on the scale M_H will occur as a result of the nonvanishing VEVs of the scalars from the chiral superfields $\eta(3_H)$ and $\xi(\bar{3}_H)$. The value $V_{\min}$ again corresponds to the flat directions: $\langle F_{\eta,\xi}\rangle_0=0$. The version (iv) corresponds to the minimum of the scalar potential in the case when $\langle\Phi\rangle_0=0$.

As for the electroweak breaking, it is due to the VEVs of the fields h and H, providing masses for the quarks and leptons. Note that the VEVs of the fields h and H must be of the order of M_W, as they determine the quark and lepton mass matrices. On the other hand, the masses of the physical Higgs fields h and H, which mix the generations, must be some orders higher than M_W, so as not to contradict the experimental bounds on FCNC. As a careful search for the Higgs potential shows, this is the picture that can be attained.

2.3 The superweak-like source of *CP* violation, and the baryon-stability and neutrino-mass problems in GUST with the non-Abelian gauge family symmetry

The existence of horizontal interactions might be closely connected to the CP-violation problem.[5] This interaction is described by the relevant part of the SUSY $SU(3)_H$ Lagrangian and has the form

$$\mathcal{L}_H=g_H\bar{\psi}_d\Gamma_\mu(D(\Lambda^a/2)D^+)\psi_d O_{ab}Z^b_\mu. \tag{22}$$

Here we have $(a,b=1,2,...,8)$. The matrix O_{ab} determines the relationship between the bare, H^b_μ, and physical, Z^b_μ, gauge fields and is calculated for the diagonalized mass matrix $(M^2_H)_{ab}$; $\psi_d=(\psi_d,\psi_s,\psi_b)$; g_H is the gauge coupling of the $SU(3)_H$ group.

After the calculations in the "bootstrap" model with the Higgs fields $H=(\lambda_a\varphi_a)/2$, $h=(\lambda_a\tilde{\varphi}_a)/2$ the expressions for the $(K^0_L-K^0_S)$, $(B^0_{dL}-B^0_{dS})$, $B(^0_{sL}-B^0_{sS})$, $(D^0_L-D^0_S)$,... meson mass differences (pure quark processes) at the tree level take the following general forms:

$$\begin{aligned}\left[\frac{(M_{12})^K_{ij}}{m_K}\right]_H=&\frac{1}{2}\frac{g_H^4}{M^4_{H_0}}\left\{\left[\tilde{\varphi}_a\left(D\frac{\lambda^a}{2}D^+\right)_{ij}\right]^2\right.\\&\left.+\left[\varphi_a\left(D\frac{\lambda^a}{2}D^+\right)_{ij}\right]^2\right\}f^2_{K_{ij}}R_K,\\\left[\frac{(M_{12})^D_{ij}}{m_D}\right]_H=&\frac{1}{2}\frac{g_H^4}{M^4_{H_0}}\left\{\left[\tilde{\varphi}_a\left(U\frac{\lambda^a}{2}U^+\right)_{ij}\right]^2\right.\\&\left.+\left[\varphi_a\left(U\frac{\lambda^a}{2}U^+\right)_{ij}\right]^2\right\}f^2_{D_{ij}}R_{D_{ij}},\end{aligned} \tag{23}$$

where $i,j=(1,2),(1,3),(2,3)$ for the K or D, B_d or T_u, and B_s or T_c meson systems.

The coefficients in (23) are calculated from (22), using (10) and the useful relation

$$\sum_a (DT^aD^+)_{ik}(DT^aD^+)_{mn}=\frac{1}{2}\left(\delta_{in}\delta_{km}-\frac{1}{3}\delta_{ik}\delta_{mn}\right). \tag{24}$$

For example, for K-meson systems we find the following contribution (if $D_L=D_R=D$):

$$(g_H^2/4)(D\lambda^aO^{ab}D^+)_{12}\frac{1}{M^2_0+\Delta M^2_b}(D\lambda^cO^{cb}D^+)_{12}$$

$$=\frac{g_H^2}{4M_0^2}(D\lambda^a O^{ab}D^+)_{12}(1-\Delta M_b^2/M_0^2)$$

$$\times(D\lambda^c O^{cb}D^+)_{12}$$

$$=-\frac{g_H^2}{4M_0^4}(D\lambda^a D^+)_{12}\Delta M_{ac}^2(D\lambda^c D^+)_{12}$$

$$=-\frac{g_H^4}{4M_0^4}(D\lambda^a+D^+)_{12}f^{kal}f^{kcl'}\varphi^l\varphi^{l'}(D\lambda^c D^+)_{12}$$

$$=\frac{g_H^4}{4M_0^4}(D[\lambda^k\lambda^m]D^+)_{12}\varphi^l\varphi^{l'}(D[\lambda^k\lambda^{m'}]D^+)_{12}$$

$$=\frac{g_H^4}{M_0^4}((D\lambda^a D^+)_{12}\varphi^a)^2. \tag{25}$$

It is interesting that if a difference between the gauge-boson masses is generated by Higgs fields in the representation (3,2) [see (11)], then the contribution in $[\Delta m/m]_H$ is equal to zero in the considered order (for the case $D_L=D_R$), since we will use Higgs fields (8,2) for these calculations. However, for processes including three equivalent indices (like $\mu\to 3e$) the Higgs fields (3,2) give nonzero contribution $\sim(\varphi D^+)_i\cdot(D\tilde{\varphi})_j$.

Note that (25) also holds for the case when D_L differs from D_R by a diagonal phase. For us the case $D_L=-D_R$, which corresponds to axial-vector terms, is important. In general, if $D_L\neq D_R$ (or $U_L\neq U_R$), then in (23) there is a quadratic term $g_H^2/M_0^2(D_L D_R^+)_{ij}(D_R D_L^+)_{ij}$, with $i\neq j$.

Substituting in (23) the expressions for φ, $\tilde{\varphi}$ and the elements d_{ij} of the D mixing matrix ("bootstrap" solution), we can obtain a lower limit for the value of M_{H_0} [$M_{H_0} < O$ (several TeV)]. Thus, we could analyze the ratios (similar ones for the $B_{d,s}$-meson system)

$$\left[\frac{\Delta m_K}{m_K}\right]_H=\frac{g_H^2}{M_{H_0}^2}\,\mathrm{Re}[C_K]f_K^2R_K<7\cdot 10^{-15} \tag{26}$$

and

$$\left[\frac{\mathrm{Im}M_{12}}{m_K}\right]_H=\frac{1}{2}\frac{g_H^2}{M_{H_0}^2}\,\mathrm{Im}[C_K]f_K^2R_K<2\cdot 10^{-17}. \tag{27}$$

In these formulas the expressions for $C_{K,D}$ are known functions in "bootstrap" models,[5] namely,

$$C_{K,D}=\frac{g_H^2}{2\lambda_Y^2}\frac{m_t^2}{M_{H_0}^2}\times f\left(\frac{m_i^{\mathrm{up}}}{m_j^{\mathrm{up}}};\frac{m_k^{\mathrm{down}}}{m_l^{\mathrm{down}}}\right), \tag{28}$$

where the f are known complex functions, whose forms depend on the quark fermion mass ansatzes.[5]

Here we note the following two points:

(a) The appearance of the phase in the CKM mixing matrix may be due to new dynamics operating at short distances ($r\ll 1/M_W$). Horizontal forces may be the source of this new dynamics.[5] Using this approach, we might have CP-violation effects due to both electroweak and horizontal interactions.

(b) CP is conserved in the electroweak sector ($\delta^{\mathrm{KM}}=0$), and its breaking is provided by the structure of the horizontal interactions. Let us consider the situation when $\delta^{\mathrm{KM}}=0$. In the SM, such a case might be realized just accidentally. The vanishing phase of the electroweak sector ($\delta^{\mathrm{KM}}=0$) might arise spontaneously as a result of some additional symmetry. Again, such a situation might occur within the horizontal extension of the electroweak model.

In particular, this model gives rise to a rather natural mechanism of superweak-like CP violation due to the $CP=-1$ part of the effective Lagrangian of the horizontal interactions: $(\epsilon'/\epsilon)_K\leqslant 10^{-4}$. That part of $\mathscr{L}_{\mathrm{eff}}$ includes the product of $SU(3)_H$ currents $I_{\mu i}I_{\mu j}$ ($i=1,4,6,3,8$; $j=2,5,7$, or vice versa, $i\leftrightarrow j$).[5] In the case of a vector-like $SU(3)_H$ gauge model the CP violation could be due solely to the charge symmetry breaking.

In the electroweak and horizontal interactions we might also have two CP-violating contributions to the amplitudes of B-meson decays. But it is possible to construct a scheme in which CP violation will occur only in the horizontal interactions. The last fact might lead to a very interesting CP-violation asymmetry $A_f(t)$ for the decays of neutral B_d^0 and $\bar{B}_d^0$ mesons to final hadron CP eigenstates, for example, to $f=(J/\Psi K_S^0)$ or $(\pi\pi)$:

$$A_f(t)\approx\sin(\Delta m_{B_d}t)\,\mathrm{Im}\left(\frac{p}{q}\times\rho_f\right),\quad \rho_f=\frac{A(\bar{B}\to f)}{A(B\to f)}. \tag{29}$$

In the standard model with the Kobayashi–Maskawa mechanism of CP violation the asymmetry of the decay of B_d^0 and $\bar{B}_d^0$ mesons to $J/\Psi K_S^0$, averaged over the time, is

$$A(J/\Psi K_S^0)\approx\eta_f\times\frac{x_d}{1+x_d^2}\times\sin 2\phi_3=-\frac{x_d}{1+x_d^2}$$

$$\times\frac{2\eta(1-\rho)}{(1-\rho)^2+\eta^2},$$

where $\eta_f=-1$ for a CP-odd $J/\Psi K_S^0$ final state; $\phi_3=\arg V_{td}$ is one of the angles (ϕ_i, $i=1,2,3$) of the unitary triangle. Let us compare this asymmetry with the analogous asymmetry of the B^0 and $\bar{B}^0$ decays to the CP-even final state (π^+,π^-), which is known to depend on the phase magnitudes of V_{ub} and V_{td}. Then

$$A(\pi^+\pi^-)\approx-\eta_f\times\frac{x_d}{1+x_d^2}\times\sin 2\phi_2=-\frac{x_d}{1+x_d^2}$$

$$\times\frac{2\eta[(\rho^2+\eta^2)-\rho]}{[(1-\rho)^2+\eta^2][\rho^2+\eta^2]},$$

where $\phi_2=\pi-\phi_1-\phi_3$ and $\phi_1=\arg V_{ub}^*=\delta_{13}$ ($\delta^{\mathrm{KM}}=\phi_1+\phi_3$).

The contributions of CP-violating horizontal interactions to the asymmetries for both B^0 decays could be very large (10–30%). They are identical for both decays, but the signs are opposite.

The space–time structure of the horizontal interactions depends on the $SU(3)_H$ quantum numbers of the quark and lepton superfields and their C-conjugate superfields. One can obtain vector (axial) horizontal interactions if the G_H particle quantum numbers are conjugate (equal) to those of the antiparticles. The question arising in these theories is how such horizontal interactions are related to the strong and electroweak ones. All these interactions can be unified within one gauge group, which would allow us to calculate the value of the coupling constant of the horizontal interactions. Thus, a unification of the horizontal, strong, and electroweak interactions might rest on the GUTs $\tilde{G}\equiv G\times SU(3)_H$ (where, for example, $\tilde{G}\equiv E(8)$, $G\equiv SU(5)$, $SO(10)$, or E_6), which may be further broken down to $SU(3)_H \times SU(3)_C\times SU(2)_L\times U(1)_Y$. To include "vector"-like horizontal gauge symmetry in GUTs we must introduce "mirror" superfields. More precisely, if we want to construct GUTs of the $\tilde{G}\equiv G\times SU(3)_H$ type, each generation must encompass double G-matter supermultiplets, mutually conjugate under the $SU(3)_H$ group. In this approach the first supermultiplet consists of the superfields f and $f^c_m\in 3_H$, while the second is constructed with the help of the supermultiplets f^c and $f_m\in \bar{3}_H$. In this scheme, proton decays are possible only in the case of mixing between ordinary and "mirror" fermions. In its turn, this mixing must, in particular, be related to the $SU(3)_H$ symmetry breaking.

The GUST spectra also predict the existence of new neutral neutrino-like particles interacting with the matter only by "superweak"-like coupling. It is possible to estimate the masses of these particles, and, as will be shown further, some of them have to be light (superlight) to be observed in modern experiments.

A variant for unusual nonuniversal family gauge interactions of the known quarks and leptons could be realized if for each generation we introduce new heavy quarks ($F=U,D$) and leptons (L,N) which are singlets (it is possible to consider doublets also) under $SU(2)_L$ and triplets under $SU(3)_H$. [This fermion matter could exist in string spectra. See all three models with $SU(3_H)\times SU(3_H)$ family gauge symmetry.] Let us consider for concreteness the case of charged leptons: $\Psi_l=(e,\mu,\tau)$ and $\Psi_L=(E,M,\mathcal{T})$. Primarily, for simplicity we suggest that the ordinary fermions do not take part in $SU(3)_H$ interactions ("white" color states). Then the interaction is described by the relevant part of the SUSY $SU(3)_H$ Lagrangian and takes the form

$$\mathscr{L}_H=g_H\tilde{\Psi}_{\mathscr{L}}\gamma_\mu \frac{\Lambda^a{}_{6\times 6}}{2}\Psi_{\mathscr{L}}O_{ab}Z^b_\mu, \tag{30}$$

where

$$\Lambda^a{}_{6\times 6}=\begin{pmatrix} S(L\lambda^a L^+)S & -S(L\lambda^a L^+)C \\ -C(L\lambda^a L^+)S & C(L\lambda^a L^+)C \end{pmatrix}.$$

Here we have $\Psi_{\mathscr{L}}=(\Psi_l;\Psi_L)$. The matrix O_{ab} $(a,b=1,2,3,...,8)$ determines the relationship between the bare, H^b_μ, and physical, Z^b_μ, gauge fields. The diagonal 3×3 matrices $S=\mathrm{diag}(s_e,s_\mu,s_\tau)$ and $C=\mathrm{diag}(c_e,c_\mu,c_\tau)$ define the nonuniversal character for lepton horizontal interactions, as the elements s_i depend on the lepton masses as $s_i \sim \sqrt{m_i/M_0}$ $(i=e,\mu,\tau)$. The same suggestion might be adopted for local quark family interactions.

For the family mixing we might suggest the following scheme. The primary 3×3 mass matrix for the light ordinary fermions is equal to zero: $M^0_{ff}\approx 0$. The 3×3 mass matrix for the heavy fermions is approximately proportional to unity $M^0_{FF}\approx M^Y_0\times 1$, where $M^Y_0\approx 0.5-1.0$ TeV and might be different for the $F_{\rm up}$ and $F_{\rm down}$ quarks and for the F_L leptons. We assume that the splitting between the new heavy fermions in each class F_Y (Y=up, down, L) is small and, at least in the quark sector, can be described by the t-quark mass. Thus, we think that in the first approximation it is possible to neglect the heavy-fermion mixing. The mixing in the light sector is completely explained by the coupling of the light fermions to the heavy fermions. As a result of this coupling the 3×3 mass matrix M^0_{fF} can be constructed in a "democratic" way, which can lead to the well known mass family hierarchy:

$$\mathbf{M}^0_{6\times 6}=\begin{pmatrix} M^0_{ff} & M^0_{fF} \\ M^0_{Ff} & M^0_{FF}\end{pmatrix},$$

where

$$M^0_{fF}\approx M^{\rm dem}_{fF}+M^{\rm corr}_{fF}. \tag{31}$$

The diagonalization of the M^0_{fF} mass matrix $XM^0_{fF}X^+$ ($X=L,D,U$ are mixing matrices) gives the eigenvalues, which define the family mass hierarchy $n^Y_1 \ll n^Y_2 \ll n^Y_3$ and the following relations between the masses of the known light fermions and a new heavy mass scale:

$$n^Y_i=\sqrt{m_i M^Y_0},\quad i=1_g,2_g,3_g;\quad Y=\text{up,down fermions}.$$

In this "see-saw" mechanism the common mass scale of the new heavy fermions might be not very far from the energy ~1 TeV, and, as a consequence, the mixing angles s_i might be not too small. There is another interesting relation between the mass scales n^Y_i that might be in this mechanism, at least for the quark case:

$$n_t/n_c=n_c/n_u=q^u_H,\quad q^u_H\approx 1/\lambda^2,$$

$$n_b/n_s=n_s/n_d=q^d_H,\quad q^d_H\approx 1/\lambda.$$

An explicit example of nonuniversal $SU(3_H)\times SU(3_H)$ local family interactions will be considered later (see model 3 in Sec. 2).

3. THE HETEROTIC SUPERSTRING THEORY WITH RANK 8 AND 16 GRAND UNIFIED GAUGE GROUPS

3.1. Conformal symmetry in the heterotic superstring

In the heterotic string theory in the left-moving (supersymmetric) sector there are $d-2$ (in the light-cone gauge) real fermions ψ^μ, their bosonic superpartners X^μ, and $3(10-d)$ real fermions χ^I. In the right-moving sector there are $d-2$ bosons $\bar{X}^\mu$ and $2(26-d)$ real fermions.

In the supersymmetric sector, world-sheet supersymmetry is realized nonlinearly via the supercharge

$$T_F=\psi^\mu\partial X_\mu+f_{IJK}\chi^I\chi^J\chi^K, \tag{32}$$

where f_{IJK} are the structure constants of a semisimple Lie group G of dimension $3(10-d)$.

The possible Lie algebras of dimension 18 for $d=4$ are $SU(2)^6$, $SU(3)\times SO(5)$, and $SU(2)\times SU(4)$. However, $N=1$ space–time supersymmetry cannot be attained in the last two cases.[34]

If we take the moments of the energy–momentum operator, we obtain the conformal generators with the following Virasoro algebra:

$$[L_n,L_m]=(n-m)L_{n+m}+\frac{c}{12}n(n^2-1)\delta_{n,-m}. \quad (33)$$

Using the Virasoro algebra, we can construct representations of the conformal group in which the highest-weight state is specified by two quantum numbers, the conformal weight h and the central charge c, such that

$$L_0|h,c\rangle=h|h,c\rangle$$
$$L_n|h,c\rangle=0, \quad n=1,2,3,\ldots . \quad (34)$$

For a massless state the conformal weight is $h=1$.

A Sugawara–Sommerfeld construction of the Virasoro algebra in terms of bilinear quantities in the Kac–Moody generators[17,18] allows us to obtain the following expression for the central Virasoro "charge":

$$c_g=\frac{2k\ \dim g}{2k+Q_\psi}=\frac{x\ \dim g}{x+\tilde{h}}. \quad (35)$$

In heterotic string theories[9,10] $(N=1\ \text{SUSY})_{\text{LEFT}}\times(N=0\ \text{SUSY})_{\text{RIGHT}}\oplus\mathcal{M}_{c_L;c_R}$ with $d\leq 10$, the conformal anomalies of the space–time sector are canceled by the conformal anomalies of the internal sector $\mathcal{M}_{c_L;c_R}$, where $c_L=15-3d/2$ and $c_R=26-d$ are the conformal anomalies in the left- and right-moving string sectors, respectively.

In the fermionic formulation of the four-dimensional heterotic string theory, in addition to the two transverse bosonic coordinates $X_\mu,\bar{X}_\mu$ and their left-moving superpartners ψ_μ, the internal sector $\mathcal{M}_{c_L;c_R}$ contains 44 right-moving $(c_R=22)$ and 18 left-moving $(c_L=9)$ real fermions (each real world-sheet fermion has $c_f=1/2$).

For a couple of years superstring theories, and particularly the heterotic string theory, have provided an efficient way to construct grand unified superstring theories (GUSTs) of all known interactions, despite the fact that it is still difficult to construct unique and fully realistic low-energy models resulting after decoupling of the massive string modes. This is the case because only 10-dimensional space–time allows the existence of two consistent (invariant under reparametrization, superconformal, modular, Lorentz, and SUSY transformations) theories with the gauge symmetries $E(8)\times E(8)$ or spin$(32)/Z_2$ (Refs. 9 and 10), which after compactification of the six extra space coordinates (into Calabi–Yau[11,12] manifolds or into orbifolds) can be used to construct GUSTs. Unfortunately, the process of compactification to four dimensions is not unique, and the number of possible low-energy models is very large. On the other hand, the construction of the theory directly in 4-dimensional space–time requires inclusion of a considerable number of free bosons or fermions in the internal string sector of the heterotic superstring.[13–16] This leads to a large internal symmetry group, such as a rank-22 group. The way of breaking this primordial symmetry is again not unique and leads to a huge number of possible models, each of them giving different low-energy predictions.

Because of the presence of the affine Kac–Moody algebra (KMA) $\hat{g}$ (which is a 2-dimensional manifestation of gauge symmetries of the string itself) on the world sheet, string constructions yield definite predictions concerning the representation of the symmetry group that can be used to construct low-energy models.[17,18] Therefore the following long-standing questions have a chance to be answered in this kind of unification scheme:

1. How are the chiral matter fermions assigned to the multiplets of the unifying group?

2. How is the GUT gauge symmetry breaking realized?

3. What is the origin and the form of the fermion mass matrices?

The first of these problems is, of course, closely connected to the quantization of the electromagnetic charge of matter fields. In addition, string constructions can shed some light on questions concerning the number of generations and the possible existence of mirror fermions which remain unanswered in conventional GUTs.[19]

There are not so many GUSTs describing the observable sector of standard models. They are well known: the SM gauge group, the Pati–Salam $[SU(4)\times SU(2)\times SU(2)]$ gauge group, the flipped $SU(5)$ gauge group, and the $SO(10)$ gauge group, which includes flipped $SU(5)$.[16]

There are good physical reasons for including the horizontal $SU(3)_H$ group in the unification scheme. First, this group naturally accommodates the three fermion families presently observed (explaining their origin); second, it can provide a correct and economical description of the fermion mass spectrum and mixing without invoking a high-dimensional representation of the conventional $SU(5)$, $SO(10)$, or $E(6)$ gauge groups. Construction of a string model (GUST) containing the horizontal gauge symmetry provides additional strong motivation for this idea. Moreover, the fact that in GUSTs high-dimensional representations are forbidden by the KMA is a very welcome feature in this context.

3.2. Possible ways of $E(8)$-GUST breaking leading to the $N_G=3$ or $N_G=3+1$ families

All this leads us naturally to consider possible forms for horizontal symmetry G_H, and the G_H quantum-number assignments for the quarks (antiquarks) and leptons (antileptons) which can be realized within the GUST framework. To include the horizontal interactions with three known generations in the ordinary GUST it is natural to consider rank-eight gauge symmetry. We can consider $SO(16)$ [or $E(6)\times SU(3)$], which is the maximal subgroup of $E(8)$ and contains the rank-eight subgroup $SO(10)\times(U(1)\times SU(3))_H$.[20] We will therefore be concerned with the following chains (see Fig. 1):

$$E(8)\to SO(16)\to \underline{SO(10)\times(U(1)\times SU(3))_H}$$
$$\to SU(5)\times(U(1)_{Y_5}\times(SU(3)\times U(1))_H$$

or

$$E(8)\to E(6)\times SU(3)\to(SU(3))^{\times 4}.$$

According to this scheme, one can obtain $SU(3)_H\times U(1)_H$ gauge family symmetry with $N_g=3+1$ [there are also other possibilities, e.g., $E(6)\times SU(3)_H\subset E(8)$ $N_g=3$ generations can be obtained by the second method of $E(8)$ gauge symmetry breaking via

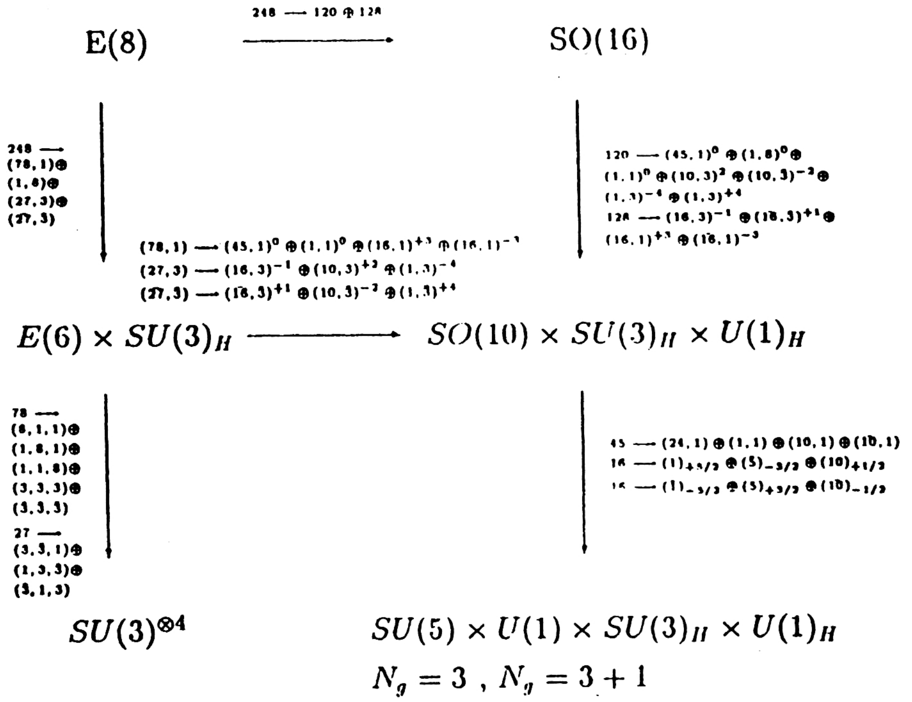

FIG. 1. Possible methods of $E(8)$ gauge symmetry breaking leading to the 3+1 or 3 generations.

$E(6)\times SU(3)_H$; see Fig. 1], where a possible additional fourth massive matter superfield could appear from $\underline{78}$ as a singlet of $SU(3)_H$ and transform as $\underline{16}$ under the $\overline{SO}(10)$ group.

In this note, starting from the rank-16 grand unified gauge group (which is the minimal rank allowed in strings) of the form $G\times G$ (Refs. 21 and 22) and making use of the KMA which selects the possible gauge-group representations, we construct string models based on the diagonal subgroup $G^{\text{symm}}\subset G\times G\subset SO(16)\times SO(16)(\subset E_8\times E_8)$.[21] We discuss $G^{\text{symm}}=SU(5)\times U(1)\times(SU(3)\times U(1))_H \subset SO(16)$, where the factor $(SU(3)\times U(1))_H$ is interpreted as the horizontal gauge family symmetry. We explain how the unifying gauge symmetry can be broken down to the standard-model group. Furthermore, the horizontal interaction predicted in our model can give an alternative description of the fermion mass matrices without invoking high-dimensional Higgs representations. In contrast with other GUST constructions, our model does not contain particles with exotic fractional electric charges.[21,23] This important virtue of the model is due to the symmetric construction of the electromagnetic charge Q_{em} from Q^{I} and Q^{II}, the two electric charges of each of the $U(5)$ groups:[21]

$$Q_{\text{em}}=Q^{\text{II}}\oplus Q^{\text{I}}. \tag{36}$$

We will consider the possible forms of the $G_H=SU(3)_H,\, SU(3)_H\times U(1),\, G_{HL}\times G_{HR},\ldots$ gauge family symmetries in the framework of the grand unification superstring approach. We will also study the matter spectrum of these GUSTs, and the possible Higgs sectors. The form of the Higgs sector is very important for GUST, G_H, and SM gauge symmetry breaking and for constructing Yukawa couplings.

3.3. Superstring-theory scale of unification and estimates of the horizontal coupling constant

Estimates of the M_{H_0} scale depend on the value of the family gauge coupling. These estimates can be made in GUST by using the string scale

$$M_{\text{SU}}\approx 0.73 g_{\text{string}}\times 10^{18}\ \text{GeV} \tag{37}$$

and the renormalization-group equations (RGE) for the gauge couplings, α_{em}, α_3, α_2, to the low energies:

$$\alpha_{\text{em}}(M_Z)\approx 1/128,$$
$$\alpha_3(M_Z)\approx 0.11,$$
$$\sin^2\theta_W(M_Z)\approx 0.233. \tag{38}$$

The string unification scale can be contrasted with the $SU(3^c)\times SU(2)\times U(1)$ naive unification scale, $M_{\text{GUT}}\approx 10^{16}$ GeV, obtained by running the SM particles and their SUSY partners to high energies. The simplest solution to this problem is the introduction of new heavy particles with SM quantum numbers, which can exist in string spectra.[16]

However, there are some other ways to explain the difference between the scales of string (M_{SU}) and ordinary (M_{GUT}) unifications. If one uses the breaking scheme $G^{\text{I}}\times G^{\text{II}}\to G^{\text{sym}}$ [where $G^{\text{I,II}}=U(5)\times U(3)_H\subset E_8$] described above, then the unification scale $M_{\text{GUT}}\sim 10^{16}$ GeV is the scale of breaking the $G\times G$ group, and string unification supplies the equality of the coupling constants of $G\times G$ on the string scale $M_{\text{SU}}\sim 10^{18}$ GeV. Otherwise, we can have an additional scale of symmetry breaking, $M_{\text{sym}}>M_{\text{GUT}}$. In any case, on the scale of breaking $G\times G\to G^{\text{sym}}$ the gauge coupling constants satisfy the equation

$$g_{\text{sym}}^2=(1/4)(g_{\text{I}}^2+g_{\text{II}}^2). \tag{39}$$

Thus, in this scheme acknowledge of the scales $M_{\rm SU}$ and $M_{\rm U}$ gives us a fundamental possibility of tracing the evolution of the coupling constant of the original group $G\times G$ to low energy and estimating the value of the horizontal gauge constant g_{3H}.

The agreement of $\sin^2\theta_W$ with experiment will show how realistic this model is.

Let us consider some relations which determine the value of $\sin^2\theta_W$ for different unification groups and for different methods of breaking.

First, let us consider the case of $SO(10)\times U(3)\to SU(5)\times U(1)\times[SU(3)\times U(1)]_H$ breaking, which can be illustrated by model 2. In this case matter fields are generated by world-sheet fermions with periodic boundary conditions. Consequently, all the representations of the matter fields can be considered as the result of destruction of the $\underline{16}$ and $\overline{16}$ representations of the $SO(10)$ group.

If we write the general expansion for a world-sheet fermion in the form

$$f(\sigma,\tau)=\sum_{n=0}^{\infty}\left[b^{+}_{n+1-\alpha/2}\exp\left[-i\left(n+\frac{1-\alpha}{2}\right)(\sigma+\tau)\right]\right.$$
$$\left.+d_{n+1+\alpha/2}\exp\left[i\left(n+\frac{1+\alpha}{2}\right)(\sigma+\tau)\right]\right],\quad(40)$$

where the quantization conditions are given by the anticommutation relations

$$\{b_a^+,b_b\}=\{d_a^+,d_b\}=\delta_{ab},$$

then the representation $\overline{16}$ of $SO(10)$ in terms of the creation (b_0^{i+}) and annihilation (b_0^i) operators will have the form

$$\overline{16}=\underline{1}+\overline{10}+\underline{5}=(1+b_0^{i+}b_0^{j+}+b_0^{i+}b_0^{j+}b_0^{k+}b_0^{l+})|0\rangle,\quad(41)$$

where $i,j,k,l=1,...,5$.

The Clifford algebra is realized via the γ matrix for the $SO(10)$ group, $\gamma_k=(b_k+b_k^+)$, and $\gamma_{5+k}=-i(b_k-b_k^+)$. The generators of the $U(5)$ subgroup can be written in terms of b_0^i as $T[U(5)]=(1/2)[b_i,b_j^+]$. Then the operator of the $U(1)_5$ hypercharge is

$$Y_5=1/2\sum_i[b_i,b_i^+]=5/2-\sum_i b_i^+,b_i.\quad(42)$$

But this generator is not normalized, since $Y_5(\underline{1},\overline{10},\underline{5})=5/2,1/2,-3/2$, respectively, and $\mathrm{Tr}_{\underline{16}}Y_5^2=20$.

In our scheme the electromagnetic charge is

$$Q_{\rm EM}=T_5-2/5Y_5,\quad(43)$$

where $T_5=\mathrm{diag}(1/15,1/15,1/15,2/5,-3/5)$. For the representation $\underline{5}$ of $SU(5)$ this means that

$$Q_{\rm EM}(\underline{5})=[\mathrm{diag}(0^3,1/2,-1/2)+\mathrm{diag}(2/30^3,-3/30^2)]$$
$$-(2/5)\cdot(-3/2)=\mathrm{diag}(0^3,1/2,-1/2)$$
$$+(1/2)[\mathrm{diag}(2/15^3,-3/15^2)+6/5]=t_3$$
$$+(1/2)[\tilde{t}_0-(4/5)Y_5]=t_3$$
$$+(1/2)\mathrm{diag}(4/3^3,1^2)=t_3+(1/2)y,\quad(44)$$

where y is the electroweak hypercharge.

Now let us write down the principal equation for the coupling constants:

$$g_5t_0A_\mu+(kg_5)YA'_\mu=g_1yB_\mu+g'_1y'B'_\mu.\quad(45)$$

In this equation (kg_5) is the $U(1)_5$ coupling constant on the scale where $U(5)$ is breaking down [on the $SO(10)\to U(5)$ scale, $k=1$]; the operators $t_0\sim\tilde{t}_0$, $Y\sim Y_5$ have equal norm; A and B are gauge fields.

The diagonal generators can be written in terms of creation and annihilation operators as

$$\mathrm{diag}(A_i)=\sum_{i=1}^{5}A_i(1-b_i^+b_i)=-\sum A_ib_i^+b_i.\quad(46)$$

Consequently, $\mathrm{Tr}_{16}\tilde{t}_0^2=8/15$. If we normalize the generators as $\mathrm{Tr}_{\underline{16}}t_0^2=\mathrm{Tr}_{\underline{16}}Y^2=8$, then $Y(\underline{5})=\sqrt{2/5}Y_5(\underline{35})=-3/\sqrt{10}$ and $t_0=(1/\sqrt{15})\mathrm{diag}(2^3,-3^2)$. Now after rewriting Eq. (45) separately for the three "up" components and the two "down" components, and the substitution $B_\mu=cA_\mu+sA'_\mu$, $B'_\mu=-sA_\mu+cA'_\mu$, where $c^2+s^2=1$, we find from Eq. (45) the relation

$$\sin^2\theta_W=\frac{g_1^2}{g_1^2+g_5^2}=\frac{15k^2}{16k^2+24}\bigg|_{k^2=1}=\frac{3}{8}.\quad(47)$$

Now let us consider the breaking $E_6\to U(5)\times U(1)$, which corresponds to models like model 4. The expansion of the matter representation $\underline{27}$ of the E_6 group under the group $SU(5)\times U(1)_5$ is

$$\underline{27}=(\overline{5}_{3/2}+\underline{10}_{-1/2}\underline{1}_{-5/2})+(\underline{5}_1+\overline{5}_{-1})+\underline{1}_0=[(b_0^{i+}$$
$$+b_0^{i+}b_0^{j+}b_0^{k+}+b_0^{i+}b_0^{j+}b_0^{k+}b_0^{l+}b_0^{m+})$$
$$+(d_{1/2}^{i+}+b_{1/2}^{i+})+1]|0\rangle.\quad(48)$$

The generalization of Eq. (42) to the case when the representation contains states from different sectors with different boundary conditions is

$$Y_5=\sum_i\left(\frac{\alpha_i}{2}+\sum_E^{\infty}[d_E^+(f_i)d_E(f_i)-b_E^+(f_i^*)b_E(f_i^*)]\right)\quad(49)$$

and, similarly for Eq. (46),

$$\mathrm{diag}(A_i)=\sum_i\left[A_i\cdot\sum_E^{\infty}(d_E^+(f_i)d_E(f_i)\right.$$
$$\left.-b_E^+(f_i^*)b_E(f_i^*))\right]\quad\left(\sum A_i=0\right).$$

Now we have $\mathrm{Tr}_{\underline{27}}Y_5^2=30$ and $\mathrm{Tr}_{\underline{27}}\tilde{t}_0^2=4/5$. By comparison with the preceding case, both norms are 1.5 times greater; hence (47) holds for this case too. But now B'_μ is some linear combination of gauge fields.

Further, let us consider model 1. This case corresponds to the breaking $SO(16)\to U(8)\to U(5)\times U(3)$. The matter fields arise from sectors with $\alpha=\pm1/2$ and correspond to chips of the $SU(8)$ representations

$$\left.\begin{array}{l}8\to[(1,3)]+(5,1)\\ 56\to[(1,1)+(10,3)]+(\overline{10},1)+(5,\overline{3})\\ \overline{56}\to[(10,1)+(\overline{5},3)]+(1,1)+(\overline{10},\overline{3})\\ \overline{8}\to[(\overline{5},1)]+(1,\overline{3})\end{array}\right\}\sim 128_{SO(16)},$$

where only the fields in square brackets survive after the GSO projection.

For this model it is necessary to make the replacement $Y_5\to\tilde{Y}_5=-(1/4)(Y_5+5Y_3)$ in Eq. (44). Now we can calculate the norms of the operators $\tilde{t}_0$ and $\tilde{Y}_5$ for this model:

$$\mathrm{Tr}_{128}\tilde{Y}_5^2=160=20\times 8,$$

$$\mathrm{Tr}_{128}\tilde{t}_0^2=\frac{64}{15}=\frac{8}{15}\times 8.$$

Hence we find again the formula (47), but now A'_μ is a linear combination of gauge fields, which corresponds to $\tilde{Y}_5$ hypercharge, and kg_5 is its coupling constant.

Finally, let us consider the model with following chain of gauge-group breaking: $E_6\times SU(3)_H\to SU(3)^3\times SU(3)_H$. Then the representation $\underline{27}$ of the E_6 group takes the form $27=(1,3,\overline{3})+(3,\overline{3},1)+(\overline{3},1,3)$.

Let us write down the quantum numbers for the breaking: $SU(3)_C\times SU(3)_L\times SU(3)_R\to SU(3)_C\times[SU(2)\times U(1)]_L\times U(1)_R^n$ (we consider one generation). For this construction the electromagnetic charge is

$$Q_{\mathrm{EM}}=t_3^L+(1/2)(Y_L+Y_R)=t_3^L+(1/2)y_{\mathrm{EW}},$$

where $t_3^L=\mathrm{diag}(1/2,-1/2,0)$, $Y_L=\mathrm{diag}(-1/3,-1/3,2/3)$, $Y_R=\mathrm{diag}(2/3,2/3,-4/3)$. This corresponds to the following set of fields:

$$(1,3,\overline{3})\sim\begin{pmatrix}(\nu & l)_L & \nu_R^c\\ L_1^U & L_1^D & L\\ L_2^U & L_2^D & e_R^c\end{pmatrix},\quad (3,\overline{3},1)\sim 3\times[(d$$

$$-u)_L D_L],\quad (\overline{3},1,3)\sim\overline{3}\times\begin{pmatrix}D_R^c\\ d_R^c\\ u_R^c\end{pmatrix},$$

where capital letters correspond to additional heavy quarks and leptons.

Now we can calculate the norms of Y_L, Y_R,

$$\mathrm{Tr}_{27}Y_L^2=4,\quad \mathrm{Tr}_{27}Y_R^2=16,$$

and write down the equation for the gauge coupling constants,

$$g_LY_LA_L^\mu+(1/2)g_RY_RA_R^\mu=g_1y_{EW}B^\mu+g_1'y'B'^\mu, \tag{50}$$

where $g_R=kg_L$ and, on the scale of the breaking $SU(3)_L\times SU(3)_R\to SU(2)_L\times U(1)_L\times U(1)_R$, $k=1$.

For example, for (ν,e,ν_R^c) we have $Y_L=\mathrm{diag}(-1/3,-1/3,2/3)$, $Y_R=-2/3$, and $y_{\mathrm{EW}}=\mathrm{diag}(-1,-1,0)$. Then from Eq. (50) [by analogy with (45) and (47)] we find

$$\sin^2\theta_W=\frac{g_1^2}{g_1^2+g_L^2}=\frac{k^2}{2k^2+4}. \tag{51}$$

The analysis of the RG equations allows us to state that the horizontal coupling constant g_{3H} does not exceed the electroweak one g_2.

For example, if below the M_{GUT} scale in the nonhorizontal sector we have effectively the standard model with four generations and two Higgs doublets (as in models 1 and 2), then the evolution of the gauge coupling constants is described by the equations

$$\alpha_S^{-1}(\mu)=\alpha_5^{-1}(M_{\mathrm{GUT}})+8\pi b_3\ln(\mu/M_{\mathrm{GUT}}), \tag{52}$$

$$\alpha^{-1}(\mu)\sin^2\theta_W=\alpha_5^{-1}(M_{\mathrm{GUT}})+8\pi b_2\ln(\mu/M_{\mathrm{GUT}}), \tag{53}$$

$$\frac{15}{k^2+24}\alpha^{-1}(\mu)\cos^2\theta_W=(k^2\alpha_5(M_{\mathrm{GUT}}))^{-1}+8\pi b_1\ln(\mu/M_{\mathrm{GUT}}), \tag{54}$$

where

$$b_3=\frac{1}{16\pi^2},\quad b_2=-\frac{3}{16\pi^2},\quad b_1=-\frac{21}{40\pi^2}.$$

From these equations and from (38) we find

$$M_{\mathrm{GUT}}=1.3\cdot 10^{16}\ \mathrm{GeV},\quad k^2=0.9,\quad \alpha_5^{-1}=14. \tag{55}$$

Now we can obtain the relation between $g_{\mathrm{str}}\equiv g$ and M_{sym} from the RG equations for the gauge running constants $g_5^{\mathrm{sym}}\equiv g_5$, g_5^{I}, and g_5^{II} on the M_{GUT}–M_{SU} scale. For example, for model 1,

$$b_5^{\mathrm{sym}}=-\frac{3}{4\pi^2},\quad b_5^{\mathrm{I}}=-\frac{5}{16\pi^2},\quad b_5^{\mathrm{II}}=\frac{3}{16\pi^2},$$

and from the RGE we find the relation

$$\frac{g_5^2}{4\pi^2+6g_5^2\ln(M_{\mathrm{GUT}}/M_{\mathrm{sym}})}=g^2\times\frac{8\pi^2-g^2\ln(M_{\mathrm{sym}}/M_{\mathrm{SU}})}{[8\pi^2-5g^2\ln(M_{\mathrm{sym}}/M_{\mathrm{SU}})]\cdot[8\pi^2+3g^2\ln(M_{\mathrm{sym}}/M_{\mathrm{SU}})]}. \tag{56}$$

According to this equation, we find that if $M_{\mathrm{SU}}\approx 10^{18}$ GeV and if the scale of breaking down to the symmetric subgroup changes in the region $M_{\mathrm{sym}}=1.5\cdot 10^{16}$–$10^{18}$ GeV, then $g_{\mathrm{str}}\sim O(1)$. Note that these values agree with (37).

Using the RG equations for the running constant g_{3H} and the value of the string coupling constant g_{str}, we can estimate the value of the horizontal coupling constant at low energies. For model 1 we have

$$b_{3H}^{\mathrm{sym}}=-\frac{5}{2\pi^2},\quad b_{3H}^{\mathrm{I}}=-\frac{21}{16\pi^2},\quad b_5^{\mathrm{II}}=-\frac{13}{16\pi^2},$$

and, taking into account the relation (39), we find from the RGE for g_{3H} that

$$g_{3H}^2(O(1\ \mathrm{TeV}))\approx 0.05,$$

and this value depends very slightly on the scale M_{sym}. However, note that for all our estimates the presence of the breaking of the $G\times G$ group down to the diagonal subgroup G_{sym} played a crucial role.

The above calculations show that for evaluation of the strength of processes involving gauge horizontal bosons at low energies we can use the inequality

$$\alpha_{3H}(\mu)\leqslant\alpha_2(\mu).$$

4. WORLD-SHEET KAC–MOODY ALGEBRA AND MAIN FEATURES OF RANK-EIGHT GUST

4.1. Representations of the Kac–Moody algebra and vertex operators

Let us begin with a short review of the KMA results.[17,18] In the heterotic string the KMA is constructed by means of the operator product expansion (OPE) of the fields J^a of the conformal dimension (0,1):

$$J^a(z)J^b(w)=\frac{1}{z-w^2}k\delta^{ab}+\frac{1}{z-w}if^{abc}J^c+\dots. \quad (57)$$

The structure constants f^{abc} for the group g are normalized so that

$$f^{acd}f^{bcd}=Q_\psi\delta^{ab}=\tilde{h}\psi^2\delta^{ab}, \quad (58)$$

where Q_ψ and ψ are the quadratic Casimir and the highest weight of the adjoint representation, and $\tilde{h}$ is the dual Coxeter number. The quantity ψ/ψ^2 can be expanded as an integer linear combination of the simple roots of g:

$$\frac{\psi}{\psi^2}=\sum_{i=1}^{\text{rank }g}m_i\alpha_i. \quad (59)$$

The dual Coxeter number can be expressed in terms of the integers m_i in the form

$$\tilde{h}=1+\sum_{i=1}^{\text{rank }g}m_i, \quad (60)$$

and for the simply laced groups (all roots are equal, and $\psi^2=2$) A_n, D_n, E_6, E_7, and E_8 they are equal to $n+1$, $2n-2$, 12, 18, and 30, respectively.

The KMA $\hat{g}$ allows us to grade the representations R of the gauge group by a level number x (a non-negative integer) and by a conformal weight $h(R)$. An irreducible representation of the affine algebra $\hat{g}$ is characterized by the vacuum representation of the algebra g and the value of the central term k, which is connected to the level number by the relation $x=2k/\psi^2$. The value of the level number of the KMA determines the highest-weight unitary representations which can be present in the spectrum by the relation

$$x=\frac{2k}{\psi^2}\geqslant\sum_{i=1}^{\text{rank }g}n_im_i, \quad (61)$$

where the sets of non-negative integers $\{m_i=m_1,\dots,m_r\}$ and $\{n_i=n_1,\dots,n_r\}$ define the highest root and the highest weight of a representation R, respectively:[17,18]

$$\mu_0=\sum_{i=1}^{\text{rank }g}n_i\alpha_i. \quad (62)$$

In fact, the KMA at level 1 is realized in the 4-dimensional heterotic superstring theories with free world-sheet fermions which allow a complex fermion description.[14–16] One can obtain the KMA at a higher level by working with real fermions and using some tricks.[29] For these models the level of the KMA is equal to the Dynkin index of the representation M to which the free fermions are assigned,

$$x=x_M=\frac{Q_M}{\psi^2}\frac{\dim M}{\dim g} \quad (63)$$

(Q_M is the quadratic Casimir eigenvalue of the representation M) and is equal to 1 in cases when real fermions form the vector representation M of $SO(2N)$, or when the world-sheet fermions are complex and M is the fundamental representation of $U(N)$.[17,18]

Thus, in strings with a KMA at level 1 realized on the world-sheet, only a very restricted set of unitary representations can arise in the spectrum:

1. Singlet and totally antisymmetric tensor representations of $SU(N)$ groups, for which $m_i=(1,\dots,1)$.
2. Singlet, vector, and spinor representations of $SO(2N)$ groups with $m_i=(1,2,2,\dots,2,1,1)$.
3. Singlet, $\underline{27}$, and $\underline{\overline{27}}$-plets of $E(6)$ corresponding to $m_i=(1,2,2,3,2,1)$.
4. Singlet of $E(8)$ with $m_i=(2,3,4,6,5,4,3,2)$.

Therefore only these representations can be used to incorporate matter and Higgs fields in GUSTs with a KMA at level 1.

In principle it might be possible to construct explicitly an example of a level-1 KMA representation of the simply laced algebra $\hat{g}$ (A,D,E types) from the level-1 representations of the Cartan subalgebra of g. This construction is achieved by using the vertex operator of the string, where these operators are assigned to a set of lattice points corresponding to the roots of a simply-laced Lie algebra g. In the heterotic string approach the vertex operator for a gauge boson with momentum p and polarization ζ is a primary field of conformal dimension (1/2,1) and can be written in the form

$$V^a=\zeta_\mu\psi_\mu(\bar{z})J^a\exp(ipX),\quad p_\mu p^\mu=\zeta_\mu p^\mu=0. \quad (64)$$

Here X_μ is the string coordinate, and ψ^μ is a conformal dimension-(1/2,0) Ramond–Neveu–Schwartz fermion.

4.2. Features of the level-1 KMA in matter and Higgs representations in rank 8 and 16 GUST constructions

For example, to describe chiral matter fermions in GUSTs with the gauge symmetry group $SU(5)\times U(1)\subset SO(10)$, one can use the sum of level-1 complex representations $\underline{1}(-5/2)+\underline{\bar{5}}(+3/2)+\underline{10}(-1/2)=\underline{16}$. On the other hand, as real representations of $SU(5)\times U(1)\subset SO(10)$, from which Higgs fields can arise, one can take, for example, $\underline{5}$ + $\underline{\bar{5}}$ representations arising from the real representation $\underline{10}$ of $SO(10)$. Real Higgs representations like $\underline{10}(-1/2)+\overline{\underline{10}}(+1/2)$ of $SU(5)\times U(1)$ originating from $\underline{16}+\overline{\underline{16}}$ of $SO(10)$, which has been used in Ref. 6 for further symmetry breaking, are also allowed.

Another example is provided by the decomposition of $SO(16)$ representations under $SU(8)\times U(1)\subset SO(16)$. Here, only singlet, $v=\underline{16}$, $s=\underline{128}$, and $s'=\underline{128}'$ representations of $SO(16)$ are allowed by the KMA [$s=\underline{128}$ and $s'=\underline{128}'$ are the two nonequivalent, real spinor representations with the highest weights $\pi_{7,8}=(1/2)(\varepsilon_1+\varepsilon_2+...+\varepsilon_7\mp\varepsilon_8)$, $\varepsilon_i\varepsilon_j=\delta_{ij}$]. From item 2 we can obtain the following $SU(8)\times U(1)$ representations: singlet, $\underline{8}+\overline{\underline{8}}(=\underline{16})$, $\underline{8}+\underline{56}+\overline{\underline{56}}+\overline{\underline{8}}(=\underline{128})$, and $\underline{1}+\underline{28}+\underline{70}++\overline{\underline{28}}+\overline{\underline{1}}(=\underline{128}')$. The highest weights of the $SU(8)$ representations $\pi_1=8$, $\pi_7=\overline{\underline{8}}$ and $\pi_3=\underline{56}$, $\pi_5=\overline{\underline{56}}$ are

$$\pi_1=1/8(\underbrace{7\varepsilon_1-\varepsilon_2-\varepsilon_3-\varepsilon_4-\varepsilon_5}-\underbrace{\varepsilon_6-\varepsilon_7-\varepsilon_8}),$$

$$\pi_7=(1/8\underbrace{\varepsilon_1+\varepsilon_2+\varepsilon_3+\varepsilon_4+\varepsilon_5}+\underbrace{\varepsilon_6+\varepsilon_7-7\varepsilon_8}),$$

$$\pi_3=1/8(\underbrace{5\varepsilon_1+5\varepsilon_2+5\varepsilon_3-3\varepsilon_4-3\varepsilon_5}-\underbrace{3\varepsilon_6-3\varepsilon_7-3\varepsilon_8}),$$

$$\pi_5=1/8(\underbrace{-3\varepsilon_1-3\varepsilon_2-3\varepsilon_3-3\varepsilon_4-3\varepsilon_5}+\underbrace{5\varepsilon_6+5\varepsilon_7+5\varepsilon_8}). \tag{65}$$

Similarly, the highest weights of the $SU(8)$ representations $\pi_2=\underline{28}$, $\pi_6=\overline{\underline{28}}$, and $\pi_4=\underline{70}$ are

$$\pi_2=1/4(\underbrace{3\varepsilon_1+3\varepsilon_2-\varepsilon_3-\varepsilon_4-\varepsilon_5}-\underbrace{\varepsilon_6-\varepsilon_7-\varepsilon_8}),$$

$$\pi_6=1/4(\underbrace{\varepsilon_1+\varepsilon_2+\varepsilon_3+\varepsilon_4+\varepsilon_5}+\underbrace{\varepsilon_6-3\varepsilon_7-3\varepsilon_8}),$$

$$\pi_4=1/2(\underbrace{\varepsilon_1+\varepsilon_2+\varepsilon_3+\varepsilon_4+\varepsilon_5}-\underbrace{\varepsilon_6-\varepsilon_7-\varepsilon_8}). \tag{66}$$

However, as we will demonstrate, in each of the string sectors the generalized Gliozzi–Scherk–Olive projection (the GSO projection in particular guarantees the modular invariance and supersymmetry of the theory and also gives some nontrivial restrictions on the gauge group and its representations) necessarily eliminates either $\underline{128}$ or $\underline{128}'$. It is therefore important that, in order to incorporate chiral matter in the model, only one spinor representation is sufficient. Moreover, if one wants to solve the chirality problem by applying further GSO projections (which break the gauge symmetry), the representation $\overline{\underline{10}}$ which otherwise, together with $\underline{10}$, could form a real Higgs representation, also disappears from this sector. Therefore, the existence of $\overline{\underline{10}}_{-1/2}+\underline{10}_{1/2}$, needed for breaking $SU(5)\times U(1)$, is incompatible (in our opinion) with the possible solution of the chirality problem for the family matter fields.

Thus, in the rank-8 group $SU(8)\times U(1)\subset SO(16)$ with Higgs representations from the level-1 KMA only, one cannot arrange for further symmetry breaking. Moreover, construction of realistic fermion mass matrices seems to be impossible. In old-fashioned GUTs (see, e.g., Ref. 19), not originating from strings, the representations of level 2 were commonly used to solve these problems.

The way out of this difficulty is based on the following important observations. First, all higher-dimensional representations of (simply-laced) groups like $SU(N)$, $SO(2N)$, or $E(6)$, which belong to the level-2 representation of the KMA [according to Eq. (61)], appear in the direct product of the level-1 representations:

$$R_G(x=2)\subset R_G(x=1)\times R'_G(x=1). \tag{67}$$

For example, the level-2 representations of $SU(5)$ will appear in the corresponding direct products of

$$\underline{15},\underline{24},\underline{40},\underline{45},\underline{50},\underline{75}\subset\underline{5}\times\underline{5},\underline{5}\times\overline{\underline{5}},\underline{5}\times\underline{10},\text{ etc.} \tag{68}$$

In the case of $SO(10)$ the level-2 representations can be obtained by suitable direct products:

$$\underline{45},\underline{54},\underline{120},\underline{126},\underline{210},\underline{144}\subset\underline{10}\times\underline{10},\overline{\underline{16}}\times\underline{10},\overline{\underline{10}}\times\underline{16},\underline{16}\times\underline{16},\overline{\underline{16}}\times\underline{16}. \tag{69}$$

The level-2 representations of $E(6)$ are the corresponding factors of the decomposition of the direct products:

$$\underline{78},\underline{351},\underline{351}',\underline{650}\subset\overline{\underline{27}}\times\underline{27}\text{ or }\underline{27}\times\underline{27}. \tag{70}$$

The only exception for this rule is the $E(8)$ group, two level-2 representations ($\underline{248}$ and $\underline{3875}$) of which cannot be constructed as a product of level-1 representations.[20]

Second, the diagonal (symmetric) subgroup G^{symm} of $G\times G$ effectively corresponds to the level-2 KMA $g(x=1)\oplus g(x=1)$ (Refs. 21 and 22) because by taking the $G\times G$ representations in the form (R_G, R'_G) of $G\times G$, where R_G and R'_G belong to level 1 of G, one obtains representations of the form $R_G\times R'_G$ when one considers only the diagonal subgroup of $G\times G$. This observation is crucial, because such a construction allows one to obtain level-2 representations. This construction was used implicitly in Ref. 22 [see also Ref. 21 where we constructed some examples of GUSTs with gauge symmetry realized as a diagonal subgroup of a direct product of two rank-eight groups $U(8)\times U(8)\subset SO(16)\times SO(16)$].

In strings, however, not all level-2 representations can be obtained in that way because, as we will demonstrate, some of them become massive (with masses of the order of the Planck scale). The condition ensuring that states in the string spectrum transforming as a representation R are massless is

$$h(R)=\frac{Q_R}{2k+Q_{ADJ}}=\frac{Q_R}{2Q_M}\leqslant 1, \tag{71}$$

where Q_i is the quadratic Casimir invariant of the corresponding representations, and M has already been defined before [see Eq. (63)]. Here the conformal weight is defined by $L_0|0\rangle=h(R)|0\rangle$,

$$L_0=\frac{1}{2k+Q_\psi}\times\left(\sum_{a=1}^{\dim g}\left(T_0^aT_0^a+2\sum_{n=1}^{\infty}T_{-n}^aT_n^a\right)\right),\qquad(72)$$

where $T_n^a|0\rangle=0$ for $n>0$. The condition (71), when combined with (61), gives a restriction on the rank of the GUT groups ($r\leqslant 8$) whose representations can accommodate chiral matter fields. For example, for antisymmetric representations of $SU(n=l+1)$ we have the values $h=p(n-p)/(2n)$. More precisely, for the $SU(8)$ group, $h(\underline{8})=7/16$, $h(\underline{28})=3/4$, $h(\underline{56})=15/16$, $h(\underline{70})=1$; for $SU(5)$, $h(\underline{5})=2/5$ and $h(\underline{10})=3/5$; for the $SU(3)$ group, $h(\underline{3})=1/3$, although for the adjoint representation of $SU(3)$, $h(\underline{8})=3/4$; for the $SU(2)$ doublet representation, we have $h(\underline{2})=1/4$. For the vector representation of the orthogonal series D_l, $h=1/2$, and for the spinor representation $h(\text{spinor})=l/8$.

There are some other important cases. The values of the conformal weights for $G=SO(16)$ or $E(6)\times SU(3)$, the representations $\underline{128}$, $(\underline{27},\underline{3})$ ($h(\underline{128})=1, h(\underline{27},\underline{3})=1$), respectively, satisfy both conditions. Obviously, these (important for incorporation of chiral matter) representations will exist in the level-2 KMA of the symmetric subgroup of the group $G\times G$.

In general, the condition (71) severely constrains massless string states transforming as $(R_G(x=1),R'_G(x=1))$ of the direct product $G\times G$. For example, for $SU(8)\times SU(8)$ and for $SU(5)\times SU(5)$ constructed from $SU(8)\times SU(8)$ only representations of the form

$$R_{N,N}=((\underline{N},\underline{N})+\text{H.c.}),\quad((\underline{N}\,\overline{\underline{N}})+\text{H.c.})\qquad(73)$$

with $h(R_{N,N})=(N-1)/N$, where $N=8$ or 5, respectively, can be massless. For $SO(2N)\times SO(2N)$, massless states are contained only in the representations

$$R_{v,v}=(\underline{2N},\underline{2N})\qquad(74)$$

with $h(R_{v,v})=1$. Thus, for the GUSTs based on a diagonal subgroup $G^{\text{symm}}\subset G\times G$, G^{symm}, the high-dimensional representations, which are embedded in $R_G(x=1)\times R'_G(x=1)$, are also severely constrained by the condition (71).

For spontaneous breaking of $G\times G$ gauge symmetry down to G^{symm} (rank G^{symm}=rank G) one can use the direct product of representations $R_G(x=1)\times R_G(x=1)$, where $R_G(x=1)$ is the fundamental representation of $G=SU(N)$ or the vector representation of $G=SO(2N)$. Furthermore, $G^{\text{symm}}\subset G\times G$ can subsequently be broken down to a smaller-dimensional gauge group (of the same rank as G^{symm}) through the VEVs of the adjoint representations which can appear as a result of $G\times G$ breaking. Alternatively, the real Higgs superfields (73) or (74) can directly break $G\times G$ gauge symmetry down to $G_{\text{I}}^{\text{symm}}\subset G^{\text{symm}}$ (rank $G_{\text{I}}^{\text{symm}}\leqslant$rank G^{symm}). For example, when $G=SU(5)\times U(1)$ or $SO(10)\times U(1)$, $G\times G$ can be directly broken in this way down to $SU(3^c)\times G_{\text{EW}}^{\text{I}}\times G_{\text{EW}}^{\text{II}}\times\ldots$.

The above examples show clearly that within the framework of GUSTs with the KMA one can get interesting gauge symmetry-breaking chains, including realistic ones when the $G\times G$ gauge symmetry group is considered. However, the lack of the higher-dimensional representations [which are forbidden by (71)] for the level-2 KMA prevents the construction of realistic fermion mass matrices. That is why we consider an extended grand unified string model of rank eight, $SO(16)$ or $E(6)\times SU(3)$ of $E(8)$.

The full chiral $SO(10)\times SU(3)\times U(1)$ matter multiplets can be constructed from the $SU(8)\times U(1)$ multiplets.

$$(\underline{8}+\underline{56}+\overline{\underline{8}}+\overline{\underline{56}})=\underline{128}\qquad(75)$$

of $SO(16)$. In the 4-dimensional heterotic superstring with free complex world-sheet fermions, in the spectrum of the Ramond sector there can appear also representations which are factors in the decomposition of $\underline{128}'$, in particular, $SU(5)$ decuplets ($\underline{10}+\overline{\underline{10}}$) from ($\underline{28}+\overline{\underline{28}}$) of $SU(8)$. However, their $U(1)_5$ hypercharge does not allow us to use them for $SU(5)\times U(1)_5$ symmetry breaking. Thus, in this approach we have only singlet and ($\underline{5}+\overline{\underline{5}}$) Higgs fields which can break the grand unified $SU(5)\times U(1)$ gauge symmetry. Therefore it is necessary (as we already explained) to construct rank-eight GUSTs based on a diagonal subgroup $G^{\text{symm}}\subset G\times G$ of the primordial symmetry group, where in each rank-eight group G the Higgs fields will appear only in singlets and in the fundamental representations [see (73)].

A comment concerning $U(1)$ factors can be made here. Since the available $SU(5)\times U(1)$ decuplets have nonzero hypercharges with respect to $U(1)_5$ and $U(1)_H$, these $U(1)$ factors may remain unbroken down to low energies in the model considered, which seems very interesting.

5. MODULAR INVARIANCE IN GUST CONSTRUCTION WITH NON-ABELIAN GAUGE FAMILY SYMMETRY

5.1. Spin basis in the free world-sheet fermion sector

The GUST model is completely defined by a set Ξ of spin boundary conditions for all these world-sheet fermions (see Appendix B). In a diagonal basis the vectors of Ξ are determined by the values of the phases $\alpha(f)\in(-1,1)$ that the fermions f acquire $[f\rightarrow-\exp(i\pi\alpha(f))f]$ when they are parallel-transported around the string.

To construct the GUST according to the scheme outlined at the end of the previous section we consider three different bases, each with six elements $B=b_1,b_2,b_3,b_4\equiv S,b_5,b_6$ (see Tables II, V, and VIII).

Following Ref. 15 (see Appendix B), we construct the canonical basis in such a way that the vector $\overline{1}$, which belongs to Ξ, is the first element b_1 of the basis. The basis vector $b_4=S$ is the generator of supersymmetry[16] responsible for the conservation of the space–time SUSY.

In this section we have chosen a basis in which all left movers ($\psi_\mu;\chi_i,y_i,\omega_i$; $i=1,\ldots,6$) (on which the world-sheet supersymmetry is realized nonlinearly) as well as the 12 right movers ($\bar{\varphi}_k$; $k=1,\ldots,12$) are real, whereas the (8+8) right movers $\overline{\Psi}_A$, $\overline{\Phi}_M$ are complex. Such a construction corresponds to the $SU(2)^6$ group of automorphisms of the left supersymmetric sector of a string. Right- and left-moving real fermions can be used for breaking the G^{comp} symmetry.[16] In order to have a possibility of reducing the rank of the compactified group G^{comp}, we must select the spin boundary conditions for the maximal possible number, $N_{LR}=12$, of left-moving, $\chi_{3,4,5,6},y_{1,2,5,6},\omega_{1,2,3,4}$, and right-moving, $\bar{\phi}^{1,\ldots,12}$

TABLE II. Basis of the boundary conditions for all world-sheet fermions. Model 1.

Vectors	$\psi_{1,2}$	$\chi_{1,...,6}$	$y_{1,...,6}$	$\omega_{1,...,6}$	$\bar{\varphi}_{1,...,12}$	$\Psi_{1,...,8}$	$\Phi_{1,...,8}$
b_1	11	111111	111111	111111	1^{12}	1^8	1^8
b_2	11	111111	000000	000000	0^{12}	$1/2^8$	0^8
b_3	11	111100	000011	000000	$0^4 1^8$	0^8	1^8
$b_4=S$	11	110000	001100	000011	0^{12}	0^8	0^8
b_5	11	001100	000000	110011	1^{12}	$1/4^5 - 3/4^3$	$-1/4^5\ 3/4^3$
b_6	11	110000	000011	001100	$1^2 0^4 1^6$	1^8	0^8

$(\bar{\phi}^p=\bar{\varphi}_p, p=1,...,12)$ real fermions. The KMA based on 16 complex right-moving fermions gives rise to the "observable" gauge group G^{obs} with

$$\text{rank}(G^{\text{obs}})\leq 16. \tag{76}$$

The study of the Hilbert spaces of the string theories is connected to the problem of finding all possible choices of the GSO coefficients $\mathscr{C}[^{\alpha}_{\beta}]$ (see Appendix B) such that the one-loop partition function

$$Z=\sum_{\alpha,\beta}\mathscr{C}\begin{bmatrix}\alpha\\ \beta\end{bmatrix}\prod_f Z\begin{bmatrix}\alpha_f\\ \beta_f\end{bmatrix} \tag{77}$$

and its multiloop counterparts are all modular-invariant. In this formula $\mathscr{C}[^{\alpha}_{\beta}]$ are GSO coefficients, α and β are $(k+l)$-component spin vectors $\alpha=[\alpha(f_1^r),..., \alpha(f_k^r);\alpha(f_1^c),...,\alpha(^l_c)]$, the components α_f,β_f specify the spin structure of the fth fermion, and $Z[...]$ is the corresponding one-fermion partition function on the torus: $Z[...]=\text{Tr}\exp[2\pi i H_{(\text{sect.})}]$.

The physical states in the Hilbert space of a given sector α are obtained by acting on the vacuum $|0\rangle_\alpha$ with the bosonic and fermionic operators with frequencies

$$n(f)=1/2+(1/2)\alpha(f),\quad n(f^*)=1/2-(1/2)\alpha(f^*) \tag{78}$$

and subsequently applying the generalized GSO projections. The physical states satisfy the Virasoro condition:

$$M_L^2=-1/2+(1/8)(\alpha_L\cdot\alpha_L)+N_L=-1+(1/8)(\alpha_R\times\alpha_R)+N_R=M_R^2, \tag{79}$$

where $\alpha=(\alpha_L,\alpha_R)$ is a sector in the set Ξ, $N_L=\Sigma_L(\text{freq})$, and $N_R=\Sigma_R(\text{freq})$.

We keep the same sign convention for the fermion-number operator F as in Ref. 16. For complex fermions we have $F_\alpha(f)=1$, $F_\alpha(f^*)=-1$, with the exception of the periodic fermions, for which we find $F_{\alpha=1}(f)=-(1/2)(1-\gamma_{5f})$, where $\gamma_{5f}|\Omega\rangle=|\Omega\rangle$, $\gamma_{5f}b_0^+|\Omega\rangle=-b_0^+|\Omega\rangle$.

TABLE III. Choice of the GSO basis $\gamma[b_i,b_j]$. Model 1. (i labels the rows, and j labels the columns.)

	b_1	b_2	b_3	b_4	b_5	b_6
b_1	0	1	1	1	1	0
b_2	1	1/2	0	0	1/4	1
b_3	1	−1/2	0	0	1/2	0
b_4	1	1	1	1	1	1
b_5	0	1	0	0	−1/2	0
b_6	0	0	0	0	1	1

The full Hilbert space of the string theory is constructed as a direct sum of different sectors $\Sigma_i m_i b_i$ $(m_i=0,1,...,N_i)$, where the integers N_i define additive groups $Z(b_i)$ of the basis vectors b_i. The generalized GSO projection leaves in sectors α those states whose b_i fermion number satisfies the relation

$$\exp(i\pi b_i F_\alpha)=\delta_\alpha \mathscr{C}^*\begin{bmatrix}\alpha\\ b_i\end{bmatrix}, \tag{80}$$

where the space–time phase $\delta_\alpha=\exp(i\pi\alpha(\psi_\mu))$ is equal to -1 for the Ramond sector and $+1$ for the Neveu–Schwarz sector.

5.2. $SU(5)\times U(1)\times SU(3)\times U(1)$—Model 1

Model 1 is defined by the six basis vectors given in Table II, which generates the $Z_2\times Z_4\times Z_2\times Z_2\times Z_8\times Z_2$ group under addition.

In our approach the basis vector b_2 is constructed as a complex vector with the 1/2 spin boundary conditions for the right-moving fermions Ψ_A, with $A=1,...,8.$ Initially, it generates chiral matter fields in the $\underline{8}+\underline{56}+\overline{\underline{56}}+\overline{\underline{8}}$ representations of $SU(8)\times U(1)$, which subsequently are decomposed under $SU(5)\times U(1)\times SU(3)\times U(1)$, to which $SU(8)\times U(1)$ becomes broken by applying the b_5 GSO projection.

Generalized GSO projection coefficients are originally defined with an ambiguity of 15 signs, but some of them are fixed by the supersymmetry conditions. Above, in Table II, we present the set of numbers

$$\gamma\begin{bmatrix}b_i\\ b_j\end{bmatrix}=\frac{1}{i\pi}\log\mathscr{C}\begin{bmatrix}b_i\\ b_j\end{bmatrix},$$

which we use as a basis for our GSO projections.

In our case of the $Z_2^4\times Z_4\times Z_8$ model, we initially have 256×2 sectors. After applying the GSO projections we have only 49×2 sectors containing massless states, which, depending on the vacuum energy values, E_L^{vac} and E_R^{vac}, can be naturally divided into certain classes and which determine the GUST representations.

In general, the RNS (Ramond–Neveu–Schwarz) sector (built on the vectors b_1 and $S=b_4$) has high symmetry, including $N=4$ supergravity and gauge $SO(44)$ symmetry. The corresponding gauge bosons are constructed as follows:

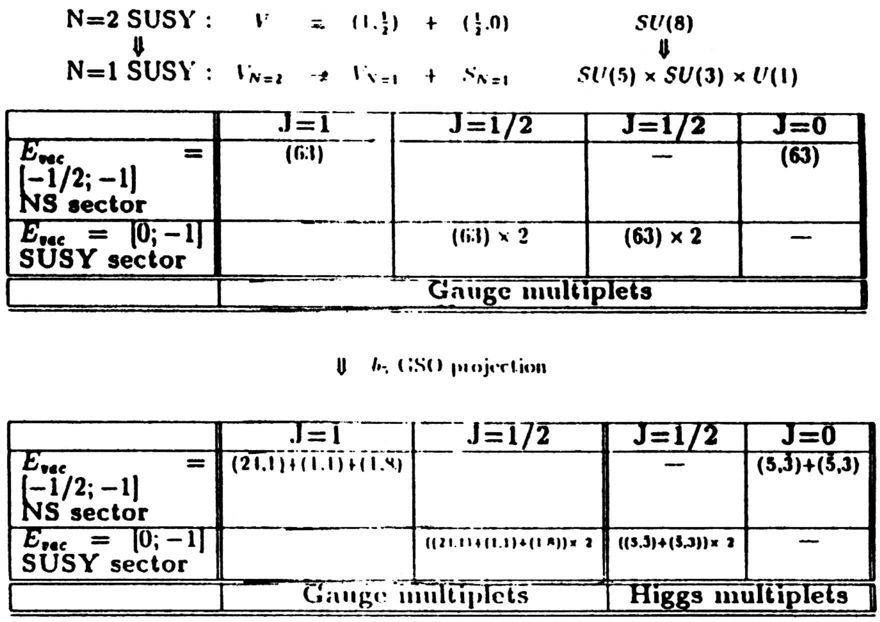

FIG. 2. Supersymmetry breaking.

TABLE IV. List of quantum numbers of the states. Model 1.

No.	b_1,b_2,b_3,b_4,b_5,b_6	SO_{hid}	$U(5)^{\text{I}}$	$U(3)^{\text{I}}$	$U(5)^{\text{II}}$	$U(3)^{\text{II}}$	$\bar{Y}_5^{\text{I}}$	$\bar{Y}_3^{\text{I}}$	$\bar{Y}_5^{\text{II}}$	$\bar{Y}_3^{\text{II}}$
1	RNS		5	3	1	1	−1	−1	0	0
			1	1	5	$\bar{3}$	0	0	−1	−1
$\hat{\Phi}$	0 2 0 1 2(6) 0		5	1	5	1	−1	0	−1	0
			1	3	1	3	0	1	0	1
			5	1	1	3	−1	0	0	1
			1	3	5	1	0	1	−1	0
2	0 1 0 0 0 0		1	3	1	1	5/2	−1/2	0	0
			$\bar{5}$	3	1	1	−3/2	−1/2	0	0
$\hat{\Psi}$			10	1	1	1	1/2	3/2	0	0
	0 3 0 0 0 0		1	1	1	1	5/2	3/2	0	0
			$\bar{5}$	1	1	1	−3/2	3/2	0	0
			10	3	1	1	1/2	−1/2	0	0
3	0 0 1 1 3 0	−1±2	1	1	1	3	0	−3/2	0	−1/2
	0 0 1 1 7 0	−1±2	1	$\bar{3}$	1	1	0	1/2	0	3/2
$\hat{\Psi}^H$										
	0 2 1 1 3 0	+1±2	1	$\bar{3}$	1	3	0	1/2	0	−1/2
	0 2 1 1 7 0	+1±2	1	1	1	1	0	−3/2	0	+3/2
4	1 1 1 0 1 1	∓1±3	1	1	1	$\bar{3}$	0	−3/2	0	1/2
	1 1 1 0 5 1	∓1±3	1	$\bar{3}$	1	1	0	1/2	0	−3/2
$\hat{\Phi}^H$										
	1 3 1 0 1 1	∓1±3	1	$\bar{3}$	1	$\bar{3}$	0	1/2	0	1/2
	1 3 1 0 5 1	∓1±3	1	1	1	1	0	−3/2	0	−3/2
5	0 1(3) 1 0 2(6) 1	−1±3	1	$3(\bar{3})$	1	1	±5/4	±1/4	±5/4	∓3/4
		+1±3	$5(\bar{5})$	1	1	1	±1/4	∓3/4	±5/4	∓3/4
$\hat{\phi}$	0 1(3) 1 0 4 1	−1±3	1	1	1	$3(\bar{3})$	±5/4	∓3/4	±5/4	±1/4
		+1±3	1	1	$5(\bar{5})$	1	±5/4	∓3/4	±1/4	∓3/4
6	1 2 0 0 3(5) 1	±1−4	1	1	1	1	±5/4	±3/4	∓5/4	∓3/4
$\hat{\sigma}$	1 1(3) 0 1 5(3) 1	+1∓4	1	1	1	1	±5/4	±3/4	±5/4	±3/4
	0 0 1 0 2(6) 0	∓3+4	1	1	1	1	±5/4	∓3/4	±5/4	∓3/4

TABLE V. Basis of the boundary conditions for model 2.

Vectors	$\psi_{1,2}$	$\chi_{1,\ldots,6}$	$y_{1,\ldots,6}$	$\omega_{1,\ldots,6}$	$\bar{\varphi}_{1,\ldots,12}$	$\Psi_{1,\ldots,8}$	$\Phi_{1,\ldots,8}$
b_1	11	1^6	1^6	1^6	1^{12}	1^8	1^8
b_2	11	1^6	0^6	0^6	0^{12}	$1^5\ 1/3^3$	0^8
b_3	11	$1^2 0^2 1^2$	0^6	$0^2 1^2 0^2$	$0^8\ 1^4$	$1/2^5\ 1/6^3$	$-1/2^5\ 1/6^3$
$b_4=S$	11	$1^2\ 0^4$	$0^2 1^2 0^2$	$0^4\ 1^2$	0^{12}	0^8	0^8
b_5	11	$1^4\ 0^2$	$0^4\ 1^2$	0^6	$1^8\ 0^4$	$1^5\ 0^3$	$0^5\ 1^3$
b_6	11	$0^2 1^2 0^2$	$1^2\ 0^4$	$0^4\ 1^2$	$1^2 0^2 1^6 0^2$	1^8	0^8

$$\psi^{\mu}_{1/2}|0\rangle_L \otimes \Psi^I_{1/2}\Psi^J_{1/2}|0\rangle_R,$$

$$\psi^{\mu}_{1/2}|0\rangle_L \otimes \Psi^I_{1/2}\Psi^{*J}_{1/2}|0\rangle_R, \quad I, \quad J=1,\ldots,22. \tag{81}$$

While the $U(1)_J$ charge for the Cartan subgroups is given by the formula $Y=\alpha/2+F$ [where F is the fermion number; see (80)], it is obvious that the states (81) generate the root lattice for $SO(44)$:

$$\pm\epsilon_I\pm\epsilon_J(I\neq J); \quad \pm\epsilon_I\mp\epsilon_J. \tag{82}$$

The other vectors break $N=4$ SUSY down to $N=1$, and the gauge group $SO(44)$ down to $SO(2)^3_{1,2,3}\times SO(6)_4 \times[SU(5)\times U(1)\times SU(3)_H\times U(1)_H]^2$; see Fig. 1.

In general, additional basis vectors can generate extra vector bosons and extend the gauge group that remains after applying the GSO projection to the RNS sector. In our case the dangerous sectors are $2b_2+nb_5$, $n=0,2,4,6$; $2b_5$: $6b_5$. But our choice of GSO coefficients cancels all the vector states in these sectors. Thus, gauge bosons in this model appear only from the RNS sector.

In the NS sector the b_3 GSO projection leaves $(5,\overline{3})+(\overline{5},3)$ Higgs superfields (see Fig. 2):

$$\chi^{1,2}_{1/2}|\Omega\rangle_L\otimes\Psi^a_{1/2}\Psi^{i*}_{1/2}; \quad \Psi^{a*}_{1/2}\Psi^i_{1/2}|\Omega\rangle_R \text{ and } \Psi\to\Phi, \tag{83}$$

where $a,b=1,\ldots,5$, with $i,j=1,2,3$.

Four (3_H+1_H) generations of chiral matter fields from the $(SU(5)\times SU(3))_I$ group forming $SO(10)$ multiplets $(\underline{1},\underline{3})+(\underline{\overline{5}},\underline{3})+(\underline{10},\underline{3})$ and $(\underline{1},\underline{1})+(\underline{\overline{5}},\underline{1})+(\underline{10},\underline{1})$ are contained in the b_2 and $3b_2$ sectors. Applying b_3 GSO projection to the $3b_2$ sector, we obtain the massless states

$$b^+_{\psi 12}b^+_{\chi 34}b^+_{\chi 56}|\Omega\rangle_L$$

$$\otimes\{\Psi^{i*}_{3/4},\Psi^a_{1/4}\Psi^b_{1/4}\Psi^c_{1/4},\Psi^a_{1/4}\Psi^i_{1/4}\Psi^j_{1/4}\}|\Omega\rangle_R,$$

$$b^+_{\chi 12}b^+_{\chi 34}b^+_{\chi 56}|\Omega\rangle_L$$

$$\otimes\{\Psi^{a*}_{3/4},\Psi^a_{1/4}\Psi^b_{1/4}\Psi^i_{1/4},\Psi^i_{1/4}\Psi^j_{1/4}\Psi^k_{1/4}\}|\Omega\rangle_R \tag{84}$$

TABLE VI. Choice of the GSO basis $\gamma[b_i,b_j]$. Model 2. (i labels the rows, and j labels the columns.)

	b_1	b_2	b_3	b_4	b_5	b_6
b_1	0	1	1/2	0	0	0
b_2	0	2/3	−1/6	1	0	1
b_3	0	1/3	5/6	1	0	0
b_4	0	0	0	0	0	0
b_5	0	1	−1/2	1	1	1
b_6	0	1	1/2	1	0	1

with the space–time chirality $\gamma_{5\psi_{12}}=-1$ and $\gamma_{5\psi_{12}}=1$, respectively. In these formulas the Ramond creation operators $b^+_{\psi_{1,2}}$ and $b^+_{\chi_{\alpha,\beta}}$ of the zero modes are constructed from a pair of real fermions (as indicated by the double indices): $\chi_{\alpha,\beta}$, with $(\alpha,\beta)=(1,2),(3,4),(5,6)$. Here, as in (83), the indices take the values $a,b=1,\ldots,5$ and $i,j=1,2,3$.

We stress that without using the b_3 projection we would get matter supermultiplets belonging to real representations only, i.e., "mirror" particles would remain in the spectrum. The b_6 projection instead eliminates all chiral matter superfields from the $U(8)^{\mathrm{II}}$ group.

Since the matter fields form the chiral multiplets of $SO(10)$, it is possible to write down the $U(1)_{Y_5}$ hypercharges of the massless states. In order to construct the right electromagnetic charges for the matter fields, we must define the hypercharge operators for the observable $U(8)^{\mathrm{I}}$ group as

$$Y_5=\int_0^{\pi}d\sigma\sum_a \Psi^{*a}\Psi^a, \quad Y_3=\int_0^{\pi}d\sigma\sum_i \Psi^{*i}\Psi^i, \tag{85}$$

and analogously for the $U(8)^{\mathrm{II}}$ group.

Then the orthogonal combinations

$$\tilde{Y}_5=\tfrac{1}{4}(Y_5+5Y_3), \quad \tilde{Y}_3=\tfrac{1}{4}(Y_3-3Y_5) \tag{86}$$

play the role of the hypercharge operators of the $U(1)_{Y_5}$ and $U(1)_{Y_H}$ groups, respectively. In Table IV we give the hypercharges $\tilde{Y}^{\mathrm{I}}_5,\tilde{Y}^{\mathrm{I}}_3,\tilde{Y}^{\mathrm{II}}_5,\tilde{Y}^{\mathrm{II}}_3$.

The full list of states in this model is given in Table IV. For fermion states only the sectors with positive (left) chirality are written. Superpartners arise from sectors with the $S=b_4$ component changed by 1. The chirality under the hidden $SO(2)^3_{1,2,3}\times SO(6)_4$ is defined as $\pm_1,\pm_2,\pm_3,\pm_4$, respectively. The lower signs in items 5 and 6 correspond to the sectors with the components given in brackets.

In the next section we discuss the problem of rank-eight GUST gauge symmetry breaking. The point is that according to the results of Sec. 4 the Higgs fields $(\underline{10}_{1/2}+\underline{\overline{10}}_{-1/2})$ do not appear.

5.3. $SU(5)\times U(1)\times SU(3)\times U(1)$—Model 2

We consider now another $[U(5)\times U(3)]^2$ model which, after breaking the gauge symmetry by means of the Higgs mechanism, leads to a spectrum similar to that of model 1.

This model is defined by the basis vectors given in Table V with the $Z_2^4\times Z_6\times Z_{12}$ group under addition.

The GSO coefficients are given in Table VI.

TABLE VII. List of quantum numbers of the states. Model 2.

No.	b_1,b_2,b_3,b_4,b_5,b_6	SO_{hid}	$U(5)^{\text{I}}$	$U(3)^{\text{I}}$	$U(5)^{\text{II}}$	$U(3)^{\text{II}}$	Y_5^{I}	Y_3^{I}	Y_5^{II}	Y_3^{II}
1	RNS	$6_1\ 2_2$	1	1	1	1	0	0	0	0
		$2_3\ 2_4$	1	1	1	1	0	0	0	0
			5	1	$\bar{5}$	1	1	0	−1	0
	0 0 4 1 0 0		1	3	1	3	0	−1	0	−1
	0 0 8 1 0 0		1	$\bar{3}$	1	$\bar{3}$	0	1	0	1
2	0 1 0 0 0 0		5	$\bar{3}$	1	1	−3/2	−1/2	0	0
			1	$\bar{3}$	1	1	5/2	−1/2	0	0
	0 3 0 0 0 0		$\overline{10}$	1	1	1	1/2	3/2	0	0
3	0 1 10 0 0 0		1	1	$\overline{10}$	3	0	0	1/2	1/2
	0 3 6 0 0 0		1	1	5	1	0	0	−3/2	−3/2
			1	1	1	1	0	0	5/2	−3/2
4	0 2 3 0 0 0	−3±4	1	3	1	1	−5/4	−1/4	5/4	3/4
5	0 0 3 0 0 0	+3±4	1	1	$\bar{5}$	1	−5/4	3/4	1/4	3/4
6	0 0 9 0 0 0	+3±4	1	1	5	1	5/4	−3/4	−1/4	−3/4
7	0 4 9 0 0 0	−3±4	1	$\bar{3}$	1	1	5/4	1/4	−5/4	−3/4
8,9	0 5 0 1 0 1	−1±3	1	3	1	1	0	−1	0	0
	0 3 0 1 0 1	+1+3	5	1	1	1	1	0	0	0
		+1−3	$\bar{5}$	1	1	1	−1	0	0	0
		−1+3	1	1	5	1	0	0	1	0
		−1−3	1	1	$\bar{5}$	1	0	0	−1	0
	0 5 8 1 0 1	+1+3	1	1	1	$\bar{3}$	0	0	0	1
10	0 3 3 0 0 1	+1±4	1	1	1	1	−5/4	3/4	5/4	3/4
11	1 0 3 0 0 1	±2−3	1	1	5	1	−1/4	3/4	−5/4	−3/4
	1 2 11 0 0 1	±2−3	1	1	1	$\bar{3}$	−5/4	3/4	−5/4	1/4
12	1 0 9 0 0 1	±2+3	$\bar{5}$	1	1	1	1/4	−3/4	5/4	3/4
	1 4 9 0 0 1	±2+3	1	$\bar{3}$	1	1	5/4	1/4	5/4	3/4
13	0 0 0 1 1 1	±2+3	1	1	1	1	0	−3/2	0	3/2
	0 2 0 1 1 1	±2−3	1	3	1	1	0	1/2	0	3/2
	0 2 8 1 1 1	±2−3	1	1	1	$\bar{3}$	0	−3/2	0	−1/2
	0 4 8 1 1 1	±2+3	1	3	1	$\bar{3}$	0	1/2	0	−1/2
	1 0 3 1 1 1	+1+3	1	1	1	1	5/4	3/4	−5/4	3/4
	1 0 9 1 1 1	+1+3	1	1	1	1	−5/4	−3/4	5/4	−3/4
	1 3 3 0 1 1	−1−3	1	1	1	1	−5/4	−3/4	−5/4	3/4
	1 3 9 0 1 1	−1+3	1	1	1	1	5/4	3/4	5/4	−3/4

The model corresponds to the following chain of gauge symmetry breaking:

$$E_8^2 \to SO(16)^2 \to U(8)^2 \to [U(5)\times U(3)]^2.$$

TABLE VIII. Basis of the boundary conditions for model 3.

Vectors	$\psi_{1,2}$	$\chi_{1,\ldots,6}$	$y_{1,\ldots,6}$	$\omega_{1,\ldots,6}$	$\bar{\varphi}_{1,\ldots,12}$	$\Psi_{1,\ldots,8}$	$\Phi_{1,\ldots,8}$
b_1	11	111111	111111	111111	1^{12}	1^8	1^8
b_2	11	111111	000000	000000	0^{12}	$1^5 1/3^3$	0^8
b_3	11	000000	111111	000000	$0^8 1^4$	$0^5 1^3$	$0^5 1^3$
$b_4 = S$	11	110000	001100	000011	0^{12}	0^8	0^8
b_5	11	111111	000000	000000	0^{12}	0^8	$1^5 1/3^3$
b_6	11	001100	110000	000011	$1^2 0^2 1^6 0^2$	1^8	0^8

Here the breaking of the $U(8)^2$ group down to $[U(5)\times U(3)]^2$ is determined by the basis vector b_5, and the breaking $N=2$ SUSY→$N=1$ SUSY is determined by the basis vector b_6.

It is interesting to note that in the absence of the vector b_5 the $U(8)^2$ gauge group is restored by the sectors $4b_3$, $8b_3$, $2b_2$+c.c., and $4b_2$+c.c.

The full massless spectrum for the given model is given in Table VII. By analogy with Table IV only fermion states with positive chirality are written, and obviously vector supermultiplets are absent. The hypercharges are determined by the formula

TABLE IX. Choice of the GSO basis $\gamma[b_i,b_j]$. Model 3. (i labels the rows, and j labels the columns.)

	b_1	b_2	b_3	b_4	b_5	b_6
b_1	0	1	0	0	1	0
b_2	0	2/3	1	1	1	1
b_3	0	1	0	1	1	1
b_4	0	0	0	0	0	0
b_5	0	1	1	1	2/3	0
b_6	0	1	0	1	1	1

$$Y_n=\sum_{k=1}^{n}(\alpha_k/2+F_k).$$

The given model possesses the hidden gauge symmetry $SO(6)_1\times SO(2)^3_{2,3,4}$. The corresponding chirality is given in the column labeled $SO_{\rm hid}$. The sectors are divided by horizontal lines and without the b_5 vector form $SU(8)$ multiplets.

For example, let us consider row 2. In the sectors b_2 and $5b_2$, in addition to the states $(1,\bar{3})$ and $(5,\bar{3})$, the state (10,3) appears, and in the sector $3b_2$, besides $(\overline{10},1)$, the states (1,1) and $(\bar{5},1)$ survive too. All these states form $\bar{8}$ + 56 representations of the $SU(8)^{\rm I}$ group.

Similarly, we can obtain the full structure of the theory according to the $U(8)^{\rm I}\times U(8)^{\rm II}$ group. (For correct restoration of the $SU(8)^{\rm II}$ group we must invert the 3 and $\bar{3}$ representations.)

In model 2, matter fields appear in both the $U(8)^{\rm I}$ and $U(8)^{\rm II}$ groups. This is the main difference between this model and model 1. However, note that in model 2, as in model 1, all the gauge fields appear in the RNS sector only, and the $10+\overline{10}$ representation (which can be the Higgs field for gauge symmetry breaking) is absent.

5.4. $SO(10)\times SU(4)$— Model 3

As an illustration, we can consider the GUST construction involving $SO(10)$ as a GUT gauge group. We consider the set of six vectors $B=b_1,b_2,b_3,b_4\equiv S,b_5,b_6$ given in Table VIII.

The GSO projections are given in Table IX. It is interesting to note that in this model the horizontal gauge symmetry $U(3)$ extends to $SU(4)$. Vector bosons needed for this appear in the sectors $2b_2$ $(4b_2)$ and $2b_5$ $(4b_5)$. For further breaking of $SU(4)$ down to $SU(3)\times U(1)$ we need an additional basis spin vector.

Thus, the given model possesses the gauge group $G^{\rm comp}\times[SO(10)\times SU(4)]^2$, and matter fields appear both in the first and in the second group symmetrically. The sectors $3b_2$ and $5b_2$+c.c. give the matter fields $(\underline{16},\underline{4};\underline{1},\underline{1})$ (first group), and the sectors $3b_5$ and $5b_5$+c.c. give the matter fields $(\underline{1},\underline{1};\ \underline{16},\underline{4})$ (second group).

Of course, for a realistic model we must add some basis vectors which give additional GSO projections.

The condition for generation chirality in this model results in the choice of Higgs fields as vector representations $SO(10)$ ($\underline{16}$ + $\overline{\underline{16}}$ are absent). According to (74), the only Higgs fields $(\underline{10},\underline{1};\underline{10},\underline{1})$ of $(SO(10)\times SU(4))^{\times 2}$ appear in the model (from the RNS sector), which can be used for GUT gauge symmetry.

6. MORE EXPLICIT METHODS OF MODEL-BUILDING

In previous models we had to guess how to obtain certain algebraic representations and select boundary-condition vectors and GSO coefficients solely according to basis-building rules. Below, we will develop some methods that help to build models for more complicated cases such as $E_6\times SU(3)$ and $SU(3)\times SU(3)\times SU(3)\times SU(3)$.

6.1. Normalization of algebra roots

It is known that the $U(1)$ eigenvalue when acting on a state in the sector α is $\alpha_i/2+F_i$. Thus, the square of a root represented by a state in the sector α is $\Sigma_i(\alpha_i/2+F_i)$.

Consider now the mass condition. It reads (for the right mass only)

$$M_R^2=-1+\tfrac{1}{8}(\alpha_R\cdot\alpha_R)+N_R=0.$$

In general we can write n_f as

$$n(f)=F^2\,\frac{1+F\alpha(f)}{2}=\frac{F^2}{2}+F\,\frac{\alpha}{2}$$

for any $F=0,\pm1$ ($F^3=F$ for those values).

Now the formula for M_R^2 reads

$$M_R^2=-1+\frac{1}{8}\sum_{i=1}^{22}(\alpha_i^2)+\sum_{i=1}^{22}\left(\frac{F_i^2}{2}+F_i\,\frac{\alpha_i}{2}\right).$$

Hence

$$2=\sum_{i=1}^{22}\left(\frac{\alpha}{2}+F\right)^2.$$

Clearly, this is the square of the algebra root and is equal to 2 for any massless state. Obviously, for massive states the normalization will differ from this value.

On the other hand, if we wish to obtain a gauge group like $G^{\rm I}\times G^{\rm II}\times G^{\rm hid}$, then the sectors that give the gauge group should be

$(0^{10}|\alpha^{\rm I}0^80^6)$ for $G^{\rm I}$,

$(0^{10}|0^8\alpha^{\rm II}0^6)$ for $G^{\rm II}$,

$(0^{10}|0^80^8\alpha^{\rm hid})$ for $G^{\rm hid}$,

where we assume that both of $G^{\rm I,II}$ are rank-eight groups (the symbol | divides the left and right movers). With other vectors we will get roots that mix some of our algebras.

With all this in mind, we can develop some methods of building GSO projectors (vectors that apply appropriate GSO projections on the states in order to obtain certain representations) for several interesting cases.

6.2. Building GSO projectors for a given algebra

When we follow a certain breaking chain of E_8, it is very natural to take the E_8 construction as a starting point. Note that the root lattice of E_8 arises from two sectors: the

NS sector gives $\underline{120}$ of $SO(16)$, while the sector with 1^8 gives $\underline{128}$ of $SO(16)$. This corresponds to the following choice of simple roots:

$$\pi_1=-e_1+e_2,$$

$$\pi_2=-e_2+e_3,$$

$$\pi_3=-e_3+e_4,$$

$$\pi_4=-e_4+e_5,$$

$$\pi_5=-e_5+e_6,$$

$$\pi_6=-e_6+e_7,$$

$$\pi_7=-e_7+e_8,$$

$$\pi_8=\tfrac{1}{2}(e_1+e_2+e_3+e_4+e_5-e_6-e_7-e_8).$$

On the basis of this choice of roots it is very clear how to build a basis of simple roots for any subalgebra of E_8. One can just find appropriate vectors π_i of the same form as in E_8 with the necessary scalar products or build a weight diagram and break it in the desired fashion to find roots corresponding to certain representations in terms of the E_8 roots.

After the basis of simple roots is written down, one can build GSO projectors in the following way.

A GSO projection is defined by the projector $(b_i \cdot F)$ acting on a given state. The goal is to find those b_i that allow only states from the algebra lattice to survive. Note that $F_i=\gamma_i-\alpha_i/2$ (γ_i are the components of a root in the basis of e_i), so that the value of the GSO projector for the sector α depends only on γ_i. Thus, if the scalar products of all the simple roots that arise from a given sector with the vector b_i are equal, mod 2, they will certainly survive GSO projection. Taking several such vectors b_i, one can eliminate all the extra states that do not belong to a given algebra.

Suppose that the simple roots of the algebra have the form

$$\pi_i=\tfrac{1}{2}(\pm e_1\pm e_2\pm e_3\pm e_4\pm e_5\pm e_6\pm e_6\pm e_7\pm e_8),$$

$$\pi_j=(\pm e_k\pm e_m).$$

With this choice we must find vectors b which give 0 or 1 in a scalar product with all the simple roots. Note that $(b\cdot\pi_i)=(b\cdot\pi_j)\bmod 2$ for all i,j, so that $c_i=(b\cdot\pi_i)$ are either all equal to $0 \bmod 2$ or equal to $1 \bmod 2$ [for $\pi_j=(\pm e_k\pm e_m)$ it should be $0 \bmod 2$ because they arise from the NS sector]. Values 0 or 1 are taken because if root $\pi\in$ algebra lattice, then $-\pi$ is also a root. With such a choice of simple roots and scalar products with b, all the states from a sector like 1^8 will have the same projector value. Roots like $\pm e_i\pm e_j$ arise from the NS sector and are sums of roots like $\pi_i=1/2(\pm e_1\pm e_2\pm e_3\pm e_4\pm e_5\pm e_6\pm e_7\pm e_8)$, and therefore they have scalar products equal to 0 mod 2, as required for the NS sector.

Now the vectors b are obtained very easily. Consider

$$c_i=(b\cdot\pi_i)=b_jA_{ji},$$

where $A_{ji}=(\pi_i)_j$ is the matrix of root components in the e_j basis. Hence $b=A^{-1}\cdot c$, where either all $c_j=0 \bmod 2$ or $c_j=1 \bmod 2$. One must try some combination of c_j to obtain an appropriate set of b. The next task is to combine those b_i that satisfy the modular-invariance rules and do not give extra states in the spectrum.

6.3. Breaking a given algebra by using GSO projectors

It appears that this method of constructing GSO projectors allows us to break a given algebra down to its subalgebra.

Consider the root system of a simple Lie algebra. It is well known that if $\pi_1,\pi_2\in\Delta$, where Δ is a set of positive roots, then $(\pi_1-\pi_2(\pi_1\cdot\pi_2))\in\Delta$ also. For simply laced algebras this means that if π_i, $\pi\in\Delta$ and $(\pi_i\cdot\pi)=-1$, where π_i is a simple root, then $\pi+\pi_i$ is also a root. This rule holds automatically in the string construction: if a sector gives some simple roots, then all the roots of the algebra and only these roots also exist (but some of them may be found in another sector). Because the square of every root represented by a state is 2, it follows that if $(\pi_i\cdot\pi)\neq-1$, then $(\pi+\pi_i)^2\neq 2$. Thus, one must construct GSO projectors by checking only the simple roots. On the other hand, if one cuts out some of the simple roots, the algebra will be broken. For example, if a vector b has a noninteger scalar product with a simple root π_1 of E_6, we obtain the algebra $SO(10)\times U(1)$ [$(b\cdot\pi_1)$ can even be 1 if the other products are equal to 0 mod 2].

More complicated examples are $E_6\times SU(3)$ and $SU(3)\times SU(3)\times SU(3)\times SU(3)$. For the former we must forbid the π_2 root but permit it to form the $SU(3)$ algebra. Note that in the E_8 root system there are two roots with $3\pi_2$. We will use them for $SU(3)$. Thus, the product $(b\cdot\pi_2)$ must be 2/3, while the others must be 0 mod 2.

We can also obtain GSO projectors for all the interesting subgroups of E_8 in this way, but so far the choice of the constant for the scalar products (c_i in the previous subsection) is rather experimental, so that it is more convenient to follow a certain breaking chain.

Below, we will give some results for $E_6\times SU(3)$, $SU(3)\times SU(3)\times SU(3)\times SU(3)$, and $SO(10)\times U(1)\times SU(3)$. We will give the algebra basis and the vectors that give the GSO projection needed for obtaining this algebra.

$E_6\times SU(3)$. This case follows from E_8, using the root basis from the previous subsection and choosing

$$c_i=\left(-2,-\frac{2}{3};0,2,-2,-2,2,0\right).$$

This gives a GSO projector of the form

$$b_1=\left(1,1,\frac{1}{3},\frac{1}{3},\frac{1}{3},\frac{1}{3},\frac{1}{3}\right).$$

The basis of simple roots arises from the sector with 1^8 in the right part and has the form

$$\pi_1=\tfrac{1}{2}(+e_1+e_2+e_3+e_4+e_5+e_6+e_7+e_8),$$

$$\pi_2=\tfrac{1}{2}(+e_1+e_2-e_3-e_4-e_5-e_6-e_7-e_8),$$

$$\pi_3=\tfrac{1}{2}(+e_1-e_2-e_3-e_4+e_5+e_6-e_7+e_8),$$

$$\pi_4=\tfrac{1}{2}(-e_1+e_2+e_3-e_4-e_5+e_6+e_7-e_8),$$

TABLE X. Basis of the boundary conditions for Model 4.

Vectors	$\psi_{1,2}$	$\chi_{1,\dots,9}$	$\omega_{1,\dots,9}$	$\bar{\varphi}_{1,\dots,6}$	$\Psi_{1,\dots,8}$	$\Phi_{1,\dots,8}$
b_1	11	1^9	1^9	1^6	1^8	1^8
b_2	11	$\hat{\frac{1}{3}},1;-\hat{\frac{2}{3}},0,0,\hat{\frac{2}{3}}$	$\hat{\frac{1}{3}},1;-\hat{\frac{2}{3}},0;0,\hat{\frac{2}{3}}$	$\frac{2}{3}^3-\frac{2}{3}^3$	$0^2-\frac{2}{3}^6$	$1^2\ \frac{1}{3}^6$
b_3	00	0^9	0^9	0^6	1^8	0^8
b^4	11	$\hat{1},1;\hat{0},0,0,\hat{0}$	$\hat{1},1;\hat{0},0,0,\hat{0}$	0^6	0^8	0^8

$$\pi_5=\tfrac{1}{2}(+e_1-e_2+e_3+e_4+e_5-e_6-e_7-e_8),$$
$$\pi_6=\tfrac{1}{2}(-e_1+e_2-e_3+e_4-e_5+e_6-e_7+e_8),$$
$$\pi_7=\tfrac{1}{2}(+e_1-e_2+e_3-e_4-e_5-e_6+e_7+e_8),$$
$$\pi_8=\tfrac{1}{2}(-e_1+e_2-e_3-e_4+e_5-e_6+e_7+e_8). \quad (87)$$

$SO(10)\times U(1)\times SU(3)$. This case follows from $E_6\times SU(3)$. In addition to b_1 we must find a vector that cuts out π_3. Using

$$c_i=(0,0,1,0,0,0,0,0)$$

and the inverse matrix of the $E_6\times SU(3)$ basis, we obtain a GSO projector of the form

$$b_2=\left(0,0,\frac{1}{3},-\frac{2}{3},\frac{1}{3},\frac{1}{3},-\frac{2}{3},\frac{1}{3}\right).$$

The basis of simple roots is the same as for $E_6\times SU(3)$, excluding π_3.

$SU(3)\times SU(3)\times SU(3)\times SU(3)$. Using the $E_6\times SU(3)$-basis inverse matrix with

$$c_i=\left(1,-1,-1,\frac{1}{3},1,\frac{1}{3},-1,-1\right),$$

we obtain a GSO projector of the form

$$b_2=\left(-\frac{1}{3},\frac{1}{3},1,1,\frac{1}{3},\frac{1}{3},-\frac{1}{3},-\frac{1}{3}\right).$$

It is easy to see that this c_i cuts out the π_4 and π_6 roots, but owing to an appropriate combination in the E_6 root system two $SU(3)$ group will remain. The basis of simple roots is

$$\pi_1=\tfrac{1}{2}(+e_1+e_2+e_3+e_4+e_5+e_6+e_7+e_8),$$
$$\pi_2=\tfrac{1}{2}(+e_1+e_2-e_3-e_4-e_5-e_6-e_7-e_8),$$
$$\pi_3=\tfrac{1}{2}(+e_1-e_2+e_3-e_4-e_5-e_6+e_7+e_8),$$
$$\pi_4=\tfrac{1}{2}(-e_1+e_2+e_3-e_4+e_5+e_6-e_7-e_8),$$
$$\pi_5=\tfrac{1}{2}(+e_1-e_2+e_3+e_4+e_5-e_6-e_7-e_8),$$
$$\pi_6=\tfrac{1}{2}(-e_1+e_2-e_3-e_4+e_5-e_6+e_7+e_8),$$
$$\pi_7=\tfrac{1}{2}(-e_1+e_2+e_3+e_4-e_5-e_6+e_7-e_8),$$
$$\pi_8=\tfrac{1}{2}(+e_1-e_2-e_3-e_4+e_5+e_6+e_7-e_8). \quad (88)$$

Using all these methods, we can construct the model described in the next section.

TABLE XI. Choice of the GSO basis $\gamma[b_i,b_j]$. Model 4. (i labels the rows, and j labels the columns.)

	b_1	b_2	b_3	b_4
b_1	0	1/3	1	1
b_2	1	1	1	1
b_3	1	1	0	1
b_4	1	1/3	1	1

6.4. $E_6\times SU(3)$ model with three generations—Model 4

This model illustrates a branch of E_8 breaking $E_8\to E_6\times SU(3)$ and is an interesting result on a way to obtain three generations with gauge horizontal symmetry. The basis of the boundary conditions (see Table X) is rather simple, but there are some subtle points. In Ref. 34 the possible left parts of the basis vectors were worked out; see that paper for details. We just use the notation given in Ref. 34 (a hat on the left part means a complex fermion, the other fermions on the left sector are real, and all the right movers are complex) and an example of a commuting set of vectors.

The construction of the $E_6\times SU(3)$ group compelled us to use rational values for the left boundary conditions. This seems to be the only way to obtain such a gauge group with appropriate matter content.

The model has $N=2$ SUSY. We can also construct a model with $N=0$, but, according to Ref. 34, by using vectors that can give rise to $E_6\times SU(3)$ (with realistic matter fields) one cannot obtain $N=1$ SUSY.

Let us give a brief review of the model content. First, we note that all the superpartners of the states in the sector α are found in the sector $\alpha+b_4$, as in all previous models. However, the same sector may contain, say, matter fields and gauginos simultaneously.

The observable gauge group $(SU(3)_H^{\mathrm{I}}\times E_6^{\mathrm{I}})\times(SU(3)_H^{\mathrm{II}}\times E_6^{\mathrm{II}})$ and the hidden group $SU(6)\times U(1)$ arise from the sectors NS, b_3, and $3b_2+b_4$. Matter fields in the representations $(3,27)+(\overline{3},\overline{27})$ for each $SU(3)_H\times E_6$ group are found in the sectors $3b_2$, b_3+b_4, and b_4. There are also some interesting states in the sectors b_2, b_2+b_3, $2b_2+b_3+b_4$, $2b_2+b_4$ and $5b_2$, $5b_2+b_3$, $4b_2+b_3+b_4$, $4b_2+b_4$ that form the representations $(\overline{3},3)$ and $(3,\overline{3})$ of the $SU(3)_H^{\mathrm{I}}\times SU(3)_H^{\mathrm{II}}$ group. These states are singlets under both E_6 groups but have nonzero $U(1)_{\mathrm{hidden}}$ charge.

We believe that the model permits further breaking of E_6 down to other grand-unification groups, but the problem of supersymmetry breaking $(N=2)\to(N=1)$ is a great obstacle for this approach.

7. GUST SPECTRUM (MODEL 1)

7.1. Gauge symmetry breaking

Let us consider model 1 in detail. In model 1 there exists a possibility of breaking the GUST group $(U(5)\times U(3))^{\mathrm{I}}\times(U(5)\times U(3))^{\mathrm{II}}$ down to the symmetric group by the ordinary Higgs mechanism:[9]

$$G^{\mathrm{I}}\times G^{\mathrm{II}}\rightarrow G^{\mathrm{symm}}\rightarrow\ldots\,. \tag{89}$$

To achieve such breaking one can use nonzero vacuum expectation values of the tensor Higgs fields (see Table IV, row 1) contained in the $2b_2+2(6)b_5(+S)$ sectors, which transform under the $(SU(5)\times U(1)\times SU(3)\times U(1))^{\mathrm{symm}}$ group in the following way:

$$(\underline{5},\underline{1};\underline{5},\underline{1})_{(-1,0;-1,0)}\rightarrow(\underline{24},\underline{1})_{(0,0)}+(\underline{1},\underline{1})_{(0,0)};$$
$$(\underline{1},\underline{3};\underline{1},\underline{3})_{(0,1;0,1)}\rightarrow(\underline{1},\underline{8})_{(0,0)}+(\underline{1},\underline{1})_{(0,0)}, \tag{90}$$
$$(\underline{5},\underline{1};\underline{1},\underline{3})_{(-1,0;0,1)}\rightarrow(\underline{\bar{5}},\underline{3})_{(1,1)};$$
$$(\underline{1},\underline{3};\underline{5},\underline{1})_{(0,1;-1,0)}\rightarrow(\underline{5},\underline{\bar{3}})_{(-1,-1)}. \tag{91}$$

The diagonal vacuum expectation values for the Higgs fields (90) break the GUST group $(U(5)\times U(3))^{\mathrm{I}}\times(U(5)\times U(3))^{\mathrm{II}}$ down to the "skew"-symmetric group with generators Δ_{symm} of the form

$$\Delta_{\mathrm{symm}}(t)=-t^{*}\times 1+1\times t. \tag{92}$$

The corresponding hypercharge of the symmetric group has the form

$$\bar{Y}=\tilde{Y}^{\mathrm{II}}-\tilde{Y}^{\mathrm{I}}. \tag{93}$$

Similarly, for the electromagnetic charge we get

$$Q_{\mathrm{em}}=Q^{\mathrm{II}}-Q^{\mathrm{I}}=(T_5^{\mathrm{II}}-T_5^{\mathrm{I}})+\tfrac{2}{5}(\tilde{Y}_5^{\mathrm{II}}-\tilde{Y}_5^{\mathrm{I}})=\bar{T}_5+\tfrac{2}{5}\bar{Y}_5, \tag{94}$$

where $T_5=\mathrm{diag}(1/15,1/15,1/15,2/5,-3/5)$. Note that this charge quantization does not lead to exotic states with fractional electromagnetic charges (e.g., $Q_{\mathrm{em}}=\pm 1/2,\pm 1/6$).

Thus, in the breaking scheme (92) it is possible to avoid color-singlet states with fractional electromagnetic charges, to achieve the desired GUT breaking, and, moreover, to obtain the usual value for the weak mixing angle at the unification scale [see (47)].

The adjoint representations which appear on the right-hand side of (90) can be used for further breaking of the symmetric group. This can lead to the final physical symmetry

$$(SU(3^c)\times SU(2_{\mathrm{EW}})\times U(1_Y)\times U(1)')$$
$$\times(SU(3_H)\times U(1_H)) \tag{95}$$

with low-energy gauge symmetry of the quark–lepton generations with an additional $U(1)'$ factor.

Note that when we use the same Higgs fields as in (90) there exists also another interesting way to break the $G^{\mathrm{I}}\times G^{\mathrm{II}}$ gauge symmetry:

$$G^{\mathrm{I}}\times G^{\mathrm{II}}\rightarrow SU(3^c)\times SU(2)^{\mathrm{I}}_{\mathrm{EW}}\times SU(2)^{\mathrm{II}}_{\mathrm{EW}}\times U(1_{\bar{Y}})$$
$$\times SU(3_H)^{\mathrm{I}}\times SU(3_H)^{\mathrm{II}}\times U(1_{\bar{Y}_H})\rightarrow\ldots\,. \tag{96}$$

This is attractive because it naturally solves the Higgs doublet–triplet mass-splitting problem with a rather low energy scale of GUST symmetry breaking.[30]

In turn, the Higgs fields $\hat{h}_{(\Gamma,N)}$ from the NS sector

$$(\underline{5},\underline{\bar{3}})_{(-1,-1)}+(\underline{\bar{5}},\underline{3})_{(1,1)} \tag{97}$$

are obtained from the $N=2$ SUSY vector representation $\underline{63}$ of $SU(8)^{\mathrm{I}}$ [or $SU(8)^{\mathrm{II}}$] by applying the b_5 GSO projection (see Fig. 2 and Appendix A). These Higgs fields [and the fields (91)] can be used to construct chiral fermion (see Table IV, row 2) mass matrices.

The b spin boundary conditions (Table II) generate chiral matter and Higgs fields with the GUST gauge symmetry $G_{\mathrm{comp}}\times(G^{\mathrm{I}}\times G^{\mathrm{II}})_{\mathrm{obs}}$ [where $G_{\mathrm{comp}}=U(1)^3\times SO(6)$ and $G^{\mathrm{I,II}}$ have already been defined]. The chiral matter spectrum, which we denote by $\hat{\Psi}_{(\Gamma,N)}$ with $\Gamma=\underline{1},\underline{\bar{5}},\underline{10}$ and $N=\underline{3},\underline{1}$, consists of $N_g=3_H+1_H$ families. See Table IV, row 2 for the $((SU(5)\times U(1))\times(SU(3)\times U(1))_H)^{\mathrm{symm}}$ quantum numbers.

The $SU(3_H)$ anomalies of the matter fields (row 2) are naturally canceled by the chiral "horizontal" superfields forming two sets: $\hat{\Psi}^H_{(1;N;1,N)}$ and $\hat{\Phi}^H_{(1,N;1,N)}$, with $\Gamma=\underline{1}$, $N=\underline{1},\underline{3}$ [with both $SO(2)_2$ chiralities; see Table IV, rows 3 and 4].

The horizontal fields (rows 3 and 4) cancel all the $SU(3)^{\mathrm{I}}$ anomalies introduced by the chiral matter spectrum (row 2) of the $(U(5)\times U(3))^{\mathrm{I}}$ group [owing to the b_6 GSO projection, the chiral fields of the $(U(5)\times U(3))^{\mathrm{II}}$ group disappear from the final string spectrum]. Performing the decomposition of the fields (rows 3 and 4) under $(SU(5)\times SU(3))^{\mathrm{symm}}$, we obtain (among others) the three "horizontal" fields

$$(\underline{1},\underline{\bar{3}})_{(0,-1)},\quad(\underline{1},\underline{1})_{(0,-3)},\quad(\underline{1},\underline{\bar{6}})_{(0,1)}, \tag{98}$$

coming from $\hat{\Psi}^H_{(\underline{1},\underline{3};\underline{1},\underline{1})}$ (or $\hat{\Psi}^H_{(\underline{1},\underline{1};\underline{1},\underline{\bar{3}})}$), $\hat{\Psi}^H_{(\underline{1},\underline{1};\underline{1},\underline{1})}$, and $\hat{\Psi}^H_{(\underline{1},\underline{1};\underline{1},\underline{\bar{3}})}$, respectively, which make the low-energy spectrum of the resulting model (96) $SU(3_H)^{\mathrm{symm}}$-anomaly free. The other fields arising from rows 3 and 4 of Table IV form anomaly-free representations of $(SU(3_H)\times U(1_H))^{\mathrm{symm}}$:

$$2(\underline{1},\underline{1})_{(0,0)},\quad(\underline{1},\underline{\bar{3}})_{(0,-1(2))}$$
$$+(\underline{1},\underline{3})_{(0,1(-2))},\quad(\underline{1},\underline{8})_{(0,0)}. \tag{99}$$

The superfields $\hat{\phi}_{(\Gamma,N)}+\mathrm{H.c.}$, where $\Gamma=\underline{1},\underline{5}$ and $N=\underline{1},\underline{3}$, from Table IV, row 5, forming representations of $(U(5)\times U(3))^{\mathrm{I,II}}$, have either Q^{I} or Q^{II} exotic fractional charges. Because of the strong G^{comp} gauge forces, these fields may develop the double scalar condensate $\langle\hat{\phi}\hat{\phi}\rangle$, which can also serve for $U(5)\times U(5)$ gauge symmetry breaking. For example, the composite condensate $\langle\hat{\phi}_{(5,1;1,1)}\hat{\phi}_{(1,1;\bar{5},1)}\rangle$ can break the $U(5)\times U(5)$ gauge symmetry down to the symmetric diagonal subgroup with generators of the form

$$\Delta_{\mathrm{symm}}(t)=t\times 1+1\times t, \tag{100}$$

so that for the electromagnetic charges we would have the form

$$Q_{\mathrm{em}}=Q^{\mathrm{II}}+Q^{\mathrm{I}}, \tag{101}$$

leading again to no exotic, fractionally charged states in the low-energy string spectrum.

The superfields which transform nontrivially under the compactified group $G^{\text{comp}}=SO(6)\times SO(2)^{\times 3}$ (denoted by $\hat{\sigma}$+H.c.), and which are singlets of $(SU(5)\times SU(3))\times(SU(5)\times SU(3))$, arise in three sectors (see Table IV, row 6). The superfields $\hat{\sigma}$ form the spinor representations $\underline{4}+\overline{\underline{4}}$ of $SO(6)$ and are also spinors of one of the $SO(2)$ groups. They have following hypercharges $\tilde{Y}_5^{\text{I,II}},\tilde{Y}_3^{\text{I,II}}$:

$$\tilde{Y}=(5/4,\mp 3/4;5/4,\mp 3/4),$$
$$\tilde{Y}=(5/4,3/4;-5/4,-3/4). \tag{102}$$

With regard to the diagonal G^{symm} group with generators given by (92) or (100), the fields $\hat{\sigma}$ from sets (a), (b), or (c) are of zero hypercharge and can therefore be used to break the $SO(6)\times SO(2)^{\times 3}$ group.

Note that for the fields $\hat{\phi}$ and for the fields $\hat{\sigma}$ any other electromagnetic charge quantization different from (94) or (101) would lead to "quarks" and "leptons" with exotic fractional charges; for example, for the $\underline{5}$ and $\underline{1}$ multiplets, according to the values of the hypercharges [see (102)] the generator Q^{II} (or Q^{I}) has the eigenvalues $(\pm 1/6,\pm 1/6,\pm 1/6,\pm 1/2,\mp 1/2)$ or $\pm 1/2$, respectively.

The scheme of breaking of the gauge group to the symmetric subgroup, which is similar to the scheme of model 1, works for model 2 as well. In this case vector-like multiplets $(\underline{5},\underline{1};\overline{\underline{5}},\underline{1})$ from the RNS sector and $(\underline{1},\underline{3};\underline{1},\underline{3})$ from $4b_3$ $(8b3)$ play the role of Higgs fields. Then the generators of the symmetric subgroup and the electromagnetic charges of the particles are determined by the expressions

$$\Delta_{\text{sym}}^{(5)}=t^{(5)}\times 1\oplus 1\times t^{(5)},$$
$$\Delta_{\text{sym}}^{(3)}=(-t^{(3)})\times 1\oplus 1\times t^{(3)},$$
$$Q_{\text{em}}=t_5^{(5)}-2/5Y^5,$$
$$\text{where } t_5^{(5)}=(1/15,1/15,1/15,2/5,-3/5). \tag{103}$$

After this symmetry breaking the matter fields (see Table VII, rows 2 and 3), as usual for flip models, occur in representations of the $U(5)$ group and form four generations $(\underline{1}+\underline{5}+\overline{\underline{10}};\ \overline{\underline{3}}+\underline{1})_{\text{sym}}$. The Higgs fields form the adjoint representation of the symmetric group, as in model 1, which is necessary for breaking of the gauge group to the standard group. Moreover, owing to quantization of the electromagnetic charge according to (103), states with exotic charges in the low-energy spectrum also do not appear in this model.

7.2. Superpotential and nonrenormalizable contributions

The ability to give a correct description of the fermion masses and mixings will, of course, constitute the decisive criterion for selection of a model of this kind. Therefore, within our approach one must:

1) study the possible nature of the G_H horizontal gauge symmetry ($N_g=3_H$ or 3_H+1_H);

2) investigate the possible cases for the G_H quantum numbers for the quarks (antiquarks) and leptons (antileptons), i.e., whether one can obtain vector-like or axial-like structure (or even chiral $G_{HL}\times G_{HR}$ structure) for the horizontal interactions;

3) find the structure of the sector of the matter fields which are needed for the $SU(3)_H$ anomaly cancellation (chiral neutral "horizontal" or "mirror" fermions);

4) write out all possible renormalizable and relevant nonrenormalizable contributions to the superpotential W and their consequences for the fermion mass matrices.

All these questions are currently under investigation. Here we restrict ourselves to some general remarks.

With the chiral matter and "horizontal" Higgs fields available in model 1 constructed in this paper, the possible form of the renormalizable (trilinear) part of the superpotential responsible for the fermion mass matrices is well restricted by the gauge symmetry:

$$W_1=g\sqrt{2}[\hat{\Psi}_{(1,3)}\hat{\Psi}_{(\bar{5},1)}\hat{h}_{(5,\bar{3})}+\hat{\Psi}_{(1,1)}\hat{\Psi}_{(\bar{5},3)}\hat{h}_{(5,3}$$
$$+\hat{\Psi}_{(10,3)}\hat{\Psi}_{(\bar{5},3)}\hat{h}_{(\bar{5},3)}+\hat{\Psi}_{(10,3)}\hat{\Psi}_{(10,1)}\hat{h}_{(5,\bar{3})}. \tag{104}$$

From the above form the Yukawa couplings it follows that two (chiral) generations have to be very light (in comparison with the M_W scale).

The construction of realistic quark and lepton mass matrices depends, of course, on the nature of the horizontal interactions. In the construction described in Sec. 5 there is a freedom in choosing the spin boundary conditions for the $N_{LR}=12$ left and right fermions in the basis vectors b_3, b_5, b_6,..., which in the Ramond sector $2b_2$ may yield other Higgs fields, denoted by $\tilde{h}_{(\Gamma,\underline{N})}$ and transforming as $(\underline{5},\underline{3})_{(-1,1)}+(\overline{\underline{5}},\overline{\underline{3}})_{(1,-1)}\subset\underline{28}\pm\overline{\underline{28}}$ of $SU(8)$. Using these Higgs fields, we obtain the following alternative form of the renormalizable part of the superpotential W:

$$W_1'=g\sqrt{2}[\hat{\Psi}_{(1,3)}\hat{\Psi}_{(\bar{5},3)}\tilde{h}_{(5,3)}+\hat{\Psi}_{(10,1)}\hat{\Psi}_{(\bar{5},3)}\tilde{h}_{(\bar{5},\bar{3})}$$
$$+\hat{\Psi}_{(10,3)}\hat{\Psi}_{(10,3)}\tilde{h}_{(5,3)}+\hat{\Psi}_{(10,3)}\hat{\Psi}_{(\bar{5},1)}\tilde{h}_{(\bar{5},\bar{3})}]. \tag{105}$$

To construct realistic fermion mass matrices one must also use the Higgs fields (90) an (91) and Table IV, row 5, and also take into account all relevant nonrenormalizable contributions.[16]

The Higgs fields (90) can be used to construct Yukawa couplings of the horizontal superfields (rows 3 and 4). The most general contribution of these fields to the superpotential is

$$W_2=g\sqrt{2}[\hat{\Phi}^H_{(1,1;1,\bar{3})}\hat{\Phi}^H_{(1,\bar{3};1,1)}\hat{\Phi}_{(1,3;1,3)}$$
$$+\hat{\Phi}^H_{(1,1;1,1)}\hat{\Phi}^H_{(1,\bar{3};1,\bar{3})}\hat{\Phi}_{(1,3;1,3)}$$
$$+\hat{\Phi}^H_{(1,\bar{3};1,\bar{3})}\hat{\Phi}^H_{(1,\bar{3};1,\bar{3})})\hat{\Phi}_{(1,\bar{3};1,\bar{3})}$$
$$+\hat{\Psi}^H_{(1,3;1,1)}\hat{\Psi}^H_{(1,3;1,\bar{3})}\hat{\Phi}_{(1,3;1,3)}$$
$$+\hat{\Psi}^H_{(1,1;1,\bar{3})}\hat{\Psi}^H_{(1,3;1,\bar{3})}\hat{\Phi}_{(1,\bar{3};1,\bar{3})}. \tag{106}$$

From this expression it follows that some of the horizontal fields in (99) (rows 3 and 4) remain massless at the tree level. This is a remarkable prediction: the fields (99) interact with the ordinary chiral matter fields only through the $U(1_H)$ and $SU(3_H)$ gauge boson and therefore are very interesting in the context of experimental searches for new gauge bosons.

The superfields $\hat{\Phi}^H_{(1,3;1,1)}$ and $\hat{\Psi}^H_{(1,\bar{3};1,1)}$ (see rows 3 and 4) can be used to construct the nonrenormalizable contributions in the superpotential W. For example, the term

$$\Delta W_1=\frac{cg^3}{M_{Pl}^2}\Psi_{(10,1)}\Psi_{(10,1)}\Phi_{(1,3;5,1)}\hat{\Phi}^H_{(1,3;1,1)}\hat{\Phi}^H_{(1,3;1,1)} \tag{107}$$

can give a contribution to the mass of the fourth-generation down-type quark [$c=O(1)$; see Ref. 16). To get a reasonable value of the mass for this quark we must arrange for $SU(3_H)^{\rm I}$ gauge symmetry breaking at an energy scale near the Planck scale, i.e., $\langle\hat{\Phi}^H_{(1,3;1,1)}\rangle=\langle\hat{\Phi}^H_{(1,\bar{3};1,1)}\rangle\sim M_{\rm Pl}$. In this case one can get the $SU(3_H)^{\rm II}$ family gauge group with a low-energy symmetry-breaking scale. Finally, we remark that the Higgs sector of our GUST allows conservation of the G_H gauge family symmetry down to low energies ($\sim O(1$ TeV).[5] Thus, in this energy region we can expect new interesting physics (new gauge bosons, new chiral matter fermions, and superweak-like CP violation in K-, B-, and D-meson decays with $\delta_{\rm KM}\langle 10^{-4}$ (Ref. 5)).

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APPENDIX A. SHORT INTRODUCTION TO $N=2$ SUSY MODELS

Let us consider a gauge interaction of the vector (gauge) hypermultiplet and the Fayet–Sohnius (matter) hypermultiplet. In the language of $N=1$ SUSY these hypermultiplets consist of $N=1$ superfields as follows: vector hypermultiplet—vector superfield $\mathbf{V}=(V_m;\lambda)$ and chiral superfield $\mathbf{\Phi}=(N;\phi)$; Fayet–Sohnius hypermultiplet—two chiral superfields: $\mathbf{X}=(X;\psi)$ and mirror $\mathbf{Y}=(Y;\chi)$. (In brackets we have written the bosonic components in the first place.)

Suppose that we have matter multiplets in some representations of a gauge group G with generators t_a:

$$\mathrm{Tr}(t_at_b)=k\delta_{ab},\quad [t_at_b]=if_{abc}t_c.$$

Then the $N=2$ gauge Lagrangian in the language of $N=1$ superfields takes the form

$$\begin{aligned}\mathscr{L}^{N=2}=&\left[\frac{1}{16kg^2}\,\mathrm{Tr}\,W^\alpha W_\alpha\right]_F+\mathrm{H.c.}+[(1/k)\\ &\times\mathrm{Tr}(\mathbf{\Phi}^+e^{2g\mathbf{V}}\mathbf{\Phi}e^{-2g\mathbf{V}})+\mathbf{X}^+_je^{2g\mathbf{V}}\mathbf{X}^j\\ &+\mathbf{Y}^+_je^{-2g\mathbf{V}^T}\mathbf{Y}^j]_D+[i\sqrt{2}g\mathbf{Y}^T_j\mathbf{\Phi}\mathbf{X}^j]_F+\mathrm{H.c.}\end{aligned} \tag{108}$$

Here the superstrength is $W_\alpha=-(1/4)\bar{D}_{\dot\alpha}\bar{D}^{\dot\alpha}e^{-2g\mathbf{V}}D_\alpha e^{2g\mathbf{V}}$, $\mathbf{V}=\mathbf{V}^at_a$, and similarly for $\mathbf{\Phi}$. It is interesting that the Yukawa couplings of $N=2$ theories are entirely determined by the gauge structure.

Under the following gauge transformations this Lagrangian is invariant:

$$\begin{aligned}&e^{2g\mathbf{V}}\to\exp(ig\Lambda^+_at_a)e^{2g\mathbf{V}}\exp(-ig\Lambda_bt_b),\\ &\mathbf{\Phi}\to\exp(ig\Lambda_at_a)\mathbf{\Phi}\exp(-ig\Lambda_bt_b),\\ &\mathbf{X}\to\exp(ig\Lambda_at_a)\mathbf{X},\quad \mathbf{Y}\to\exp(-ig\Lambda_at^*_a)\mathbf{Y}.\end{aligned} \tag{109}$$

We can see that the ordinary fields (superfields $\mathbf{X}$) and the mirror fields (superfields $\mathbf{Y}$) are transformed as mutually conjugate representations of a gauge group.

After excluding the auxiliary fields the Lagrangian $\mathscr{L}^{N=2}$ takes the form (we suppose that $k=1/2$)

$$\begin{aligned}\mathscr{L}^{N=2}=&\mathrm{Tr}(-1/2V_{mn}V^{mn}-2i\lambda\sigma^m\nabla_m\bar\lambda+2N^+\nabla^2N\\ &+2i\nabla_m\phi\sigma^m\bar\phi+2\sqrt{2}ig\lambda[\phi N^+]+2\sqrt{2}ig\bar\lambda[\bar\phi N]\\ &-g^2[NN^+]^2)+X^+\nabla^2X+i\nabla_m\bar\psi\bar\sigma^m\psi+Y^+\nabla^2Y^+\\ &\times i\nabla_m\bar\chi\bar\sigma^m\chi+\sqrt{2}ig(X^+\lambda\psi-Y^+\lambda^T\chi-\chi^T\phi X\\ &-Y^T\phi\psi-\chi^TN\psi)+\mathrm{H.c.}-g^2X^+\{N^+N\}X\\ &-g^2Y^T\{NN^+\}Y^*-(g^2/4)(X^+_iX_j)(X^+_jX_i)-(g^2/4)\\ &\times(Y^+_iY_j)(Y^+_jY_i)+(g^2/2)(Y^T_jX_i)(X^+_iY^*_j)\\ &-g^2(Y^T_iY^*_j)(X^+_jX_i),\end{aligned} \tag{110}$$

where i and j label the matter hypermultiplets. Note that the covariant derivative for the $\mathbf{X}$ fields is $\nabla^X_m=\partial_m+igV_m$, but for the $\mathbf{Y}$ fields $\nabla^Y_m=\partial_m-igV^T_m$. Since the $N=2$ SUSY has been presented, this Lagrangian possesses a hidden global internal $SU(2)$ symmetry. The component fields $(\lambda,-\phi)$ and (X,Y^*) are doublets under the internal $SU(2)$ group, and the remaining fields are singlets.

The attractive feature of the $N=2$ SUSY theory is that the β function is nonzero at the one-loop level only,

$$\beta(g_A)=\frac{g^3}{8\pi^2}\left(\sum_\sigma T_A(R_\sigma)-C_2(G_A)\right), \tag{111}$$

where

$$f_{ijk}f_{ljk}=C_2(G)\delta_{il}, \quad \mathrm{Tr}\, t_a^{(\sigma)}t_b^{(\sigma)}=T(R_\sigma)\delta_{ab},$$

and we suppose that gauge groups like $G=\Pi_A\otimes G_A$ and R_σ are representations for chiral superfields.

We can see that this theory can be made finite for some gauge group through a certain choice of representations of the matter fields if the following relation holds:[31]

$$C_2(G_A)=\sum_\sigma T_A(R_\sigma). \tag{112}$$

Let us write down the representations of some subgroups of E_8 which guarantee the finiteness of the $N=2$ thoery:[32]

$SU(5)$: p, q, and r matter multiplets in the representations $(5+\overline{5})$, $(10+\overline{10})$, and $(15+\overline{15})$, respectively, for which $p+3q+7r=10$.

$SO(10)$: p and q matter multiplets in the representations $(10+10)$ and $(16+\overline{16})$, respectively, for which $p+2q=8$.

E_6: four multiplets in the representation $(27+\overline{27})$.

E_7: three multiplets in the representation (56+56).

E_8: one multiplet in the lowest (adjoint) representation (248+248). This implies the existence of $N=4$ SUSY.

There are five types of soft SUSY-breaking operators, and addition of them to the Lagrangian does not destroy the finiteness of the theory.[32,33]

1) Any gauge-invariant $N=1$ supersymmetric mass addition. For example, $m\,\mathrm{Tr}\,\mathbf{\Phi}^2|_F$+H.c., $m\mathbf{Y}_i^T\mathbf{X}_i|_F$+H.c. The first addition can be written in component fields as

$$\mathrm{Tr}(-m\phi\phi-m^*\bar\phi\bar\phi-2|m|^2N^+N)+i\sqrt{2}gmX^+NY^* \\ -i\sqrt{2}gm^*Y^TN^+X. \tag{113}$$

This addition breaks $N=2$ down to $N=1$ SUSY.

2) Any gauge-invariant masses for scalar fields of the form A^2-B^2. We assume that the scalar is $(A+iB)/\sqrt{2}$.

3) Certain mass terms of the form A^2+B^2.

$$U_1N_a^*N_a+\sum_i (U_2^iX_i^+X_i+U_3^iY_i^+Y_i) \tag{114}$$

if for each i we have $U_1+U_2^i+U_3^i=0$.

4) Certain combinations of mass addition and trilinear scalar addition:

$$\mathrm{Tr}(-m\lambda\lambda-m^*\overline{\lambda\lambda}-2|m|^2N^+N)-i\sqrt{2}gmY^TNX \\ +i\sqrt{2}gm^*X^+N^+Y^*. \tag{115}$$

This combination is simply the addition of type 1 under the transformation $\exp(i\pi t_2)$ of the internal $SU(2)$ group. It also breaks $N=2$ down to $N=1$ SUSY.

5) Gauge-invariant scalar trilinear operators of the form:

a. $k^{ijk}(X_iX_jX_k+Y_i^*Y_j^*Y_k^*)$ + H.c.;

b. $\varkappa^{ijk}(X_iX_jY_k+Y_i^*Y_j^*X_k^*)$ + H.c., together with a certain scalar mass A^2+B^2 and scalar trilinear addition $\mathrm{Tr}\,N^3$+H.c., where X and Y do not lie in the adjoint representation, and the gauge group must be $SU(n)$ with $n\geqslant 3$ or their direct product. However, these terms lead to a potential that is unbounded from below.

APPENDIX B. RULES FOR CONSTRUCTING CONSISTENT STRING MODELS FROM FREE WORLD-SHEET FERMIONS

The partition function of the theory is a sum over terms corresponding to world-sheets of different genera g. For consistency of the theory we must require that the partition function be invariant under modular transformations, which are reparametrizations not continuously connected to the identity. For this we must sum over the different possible boundary conditions for the world-sheet fermions with appropriate weights.[35]

If the fermions are parallel-transported around a nontrivial loop of a genus-g world-sheet M_g, they must transform into themselves:

$$\chi^I\to L_g(\alpha)^I_J\chi^J, \tag{116}$$

and similarly for the right-moving fermions. The only constraints on $L_g(\alpha)$ and $R_g(\alpha)$ that they be orthogonal matrix representations of $\pi_1(M_g)$ in order to leave the energy–momentum current invariant and the supercharge (32) invariant, apart from a sign. This means that

$$\psi^\mu\to-\delta_\alpha\psi^\mu, \quad \delta_\alpha=\pm1, \tag{117}$$

$$L^I_{gI'}L^J_{gJ'}L^K_{gK'}f_{IJK}=-\delta_\alpha f_{I'J'K'} \tag{118}$$

and, consequently, $-\delta_\alpha L_g(\alpha)$ is an automorphism of the Lie algebra of G.

Further, the following restrictions on $L_g(\alpha)$ are imposed:

(a) $L_g(\alpha)$ and $R_g(\alpha)$ are Abelian matrix representations of $\pi_1(M_g)$. Thus, all the $L_g(\alpha)$ and all the $R_g(\alpha)$ can be simultaneously diagonalized in some basis.

(b) There is commutativity between the boundary conditions on surfaces of different genera.

When all the $L(\alpha)$ and $R(\alpha)$ have been simultaneously diagonalized, transformations like (116) can be written as

$$f\to-\exp(i\pi\alpha_f)f. \tag{119}$$

Here and in Eqs. (117) and (118) the minus signs are conventional.

The boundary conditions (116) and (117) are specified in this basis by a vector of phases

$$\alpha=[\alpha(f_1^L),\ldots,\alpha(f_k^L)|\alpha(f_1^R),\ldots,\alpha(f_l^R)]. \tag{120}$$

For complex fermions and $d=4$, we have $k=10$ and $l=22$. The phases in this formula are reduced mod(2) and are chosen to be in the interval $(-1,+1]$.

Modular transformations mix the spin structures amongst one another within a surface of given genus. Thus, by requiring modular invariance of the partition function we impose constraints on the coefficients $\mathscr{C}\left[\begin{smallmatrix}\alpha_1\ldots\alpha_g\\ \beta_1\ldots\beta_g\end{smallmatrix}\right]$ [weights in the partition-function sum; see, for example, Eq. (77)], which in turn imposes constraints on what spin structures are allowed in a consistent theory. According to the assumptions (a) and (b), these coefficients factorize:

$$\mathscr{C}\begin{bmatrix}\alpha_1\ldots\alpha_g\\ \beta_1\ldots\beta_g\end{bmatrix}=\mathscr{C}\begin{bmatrix}\alpha_1\\ \beta_1\end{bmatrix}\mathscr{C}\begin{bmatrix}\alpha_2\\ \beta_2\end{bmatrix}\ldots\mathscr{C}\begin{bmatrix}\alpha_g\\ \beta_g\end{bmatrix}. \tag{121}$$

The requirement of modular invariance of the partition function thus gives rise to constraints on the one-loop coefficients $\mathscr{C}$ and hence on the possible spin structures (α,β) on the torus.

For rational phases $\alpha(f)$ (we consider only this case) the possible boundary conditions α form a finite addition group $\Xi = \Sigma_{i=1}^{k} \oplus Z_{N_i}$ which is generated by a basis $(b_1,\dots,b_k)$, where N_i is the smallest integer for which $N_i b_i = 0$ mod(2). Multiplication of two vectors from Ξ is defined by

$$\alpha\cdot\beta=(\alpha_L^i\beta_L^i-\alpha_R^j\beta_R^j)_{\text{complex}}+(1/2)(\alpha_L^k\beta_L^k-\alpha_R^l\beta_R^l)_{\text{real}}. \tag{122}$$

The basis satisfies the following conditions, derived in Ref. 15:

(A1) The basis $(b_1,\dots,b_k)$ is chosen to be canonical:

$$\sum m_i b_i=0 \Leftrightarrow m_i=0 \mod(N_i), \quad \forall i.$$

Then an arbitrary vector α from Ξ is a linear combination $\alpha=\Sigma a_i b_i$.

(A2) The vector b_1 satisfies $(1/2)N_1 b_1=\mathbf{1}$. This is clearly satisfied by $b_1=\mathbf{1}$.

(A3) $N_{ij} b_i\cdot b_j=0$ mod(4), where N_{ij} is the least common multiple of N_i and N_j.

(A4) $N_i b_i^2=0$ mod(4); however, if N_i is even, we must have $N_i b_i^2=0$ mod(8).

(A5) The number of real fermions that are simultaneously periodic under any four boundary conditions b_i, b_j, b_k, b_l is even, where i, j, k, and l are not necessarily distinct. This implies that the number of periodic real fermions in any b_i must be even.

(A6) The boundary-condition matrix corresponding to each b_i is an automorphism of the Lie algebra that defines the supercharge (32). All such automorphisms must commute with one another, since they must be simultaneously diagonalizable.

For each group of boundary conditions Ξ there are a number of consistent choices for the coefficients $\mathscr{C}[\dots]$, which are determined from the requirement of invariance under modular transformations. The number of such theories corresponds to the number of different choices of $\mathscr{C}[^{b_i}_{b_j}]$. This set must satisfy the following equations:

(B1)

$$\mathscr{C}\begin{bmatrix} b_i \\ b_j \end{bmatrix} = \delta_{b_i} e^{2\pi i n/N_j} = \delta_{b_j} e^{i\pi(b_i\cdot b_j)/2} e^{2\pi i m/N_i}.$$

(B2)

$$\mathscr{C}\begin{bmatrix} b_1 \\ b_1 \end{bmatrix} = \pm e^{i\pi b_1^2/4}.$$

The values of $\mathscr{C}[^{\alpha}_{\beta}]$ for arbitrary $\alpha, \beta\in\Xi$ can be obtained by means of the following rules:

(B3)

$$\mathscr{C}\begin{bmatrix} \alpha \\ \alpha \end{bmatrix} = e^{i\pi(\alpha\cdot\alpha+1\cdot 1)/4}\mathscr{C}\begin{bmatrix} \alpha \\ b_1 \end{bmatrix}^{N_1/2}.$$

(B4)

$$\mathscr{C}\begin{bmatrix} \alpha \\ \beta \end{bmatrix} = e^{i\pi(\alpha\cdot\beta)/2}\mathscr{C}\begin{bmatrix} \beta \\ \alpha \end{bmatrix}^*.$$

(B5)

$$\mathscr{C}\begin{bmatrix} \alpha \\ \beta+\gamma \end{bmatrix} = \delta_\alpha \mathscr{C}\begin{bmatrix} \alpha \\ \beta \end{bmatrix}\mathscr{C}\begin{bmatrix} \alpha \\ \gamma \end{bmatrix}.$$

The relative normalization of all the $\mathscr{C}[\dots]$ is fixed in these expressions conventionally to be $\mathscr{C}[^0_0]\equiv 1$.

For each $\alpha\in\Xi$ there is a corresponding Hilbert space of string states $\mathscr{H}_\alpha$ that potentially contribute to the one-loop partition function. If we write $\alpha=(\alpha_L|\alpha_R)$, then the states in $\mathscr{H}_\alpha$ are those that satisfy the Virasoro condition:

$$M_L^2=-c_L+(1/8)\alpha_L\cdot\alpha_L+\sum_{L-\text{mov.}}(\text{freq.})=-c_R+(1/8)\alpha_R\cdot\alpha_R+\sum_{R-\text{mov.}}(\text{freq.})=M_R^2. \tag{123}$$

Here $c_L=1/2$ and $c_R=1$ in the heterotic case. In the $\mathscr{H}_\alpha$ sector the fermion $f(f^*)$ has the oscillator frequencies

$$\frac{1\pm\alpha(f)}{2}+\text{integer}. \tag{124}$$

The only states $|s\rangle$ in $\mathscr{H}_\alpha$ that contribute to the partition function are those that satisfy the generalized GSO conditions

$$\left\{e^{i\pi(b_i\cdot F_\alpha)}-\delta_\alpha\mathscr{C}\begin{bmatrix} \alpha \\ \beta_i \end{bmatrix}^*\right\}|s\rangle=0 \tag{125}$$

for all b_i. Here $F_\alpha(f)$ is the fermion-number operator. If α contains periodic fermions, then $|0\rangle_\alpha$ is degenerate, transforming as a representation of an $SO(2n)$ Clifford algebra.

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