The method of variational perturbation theory in quantum field theory

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A nonperturbative approach to quantum field theory, variational perturbation theory, is reviewed. Its efficiency is demonstrated for the example of the quantum-mechanical anharmonic oscillator. The relationship to the method of the Gaussian effective potential is established, and the convergence properties of the series of variational perturbation theory are studied for the $\varphi^4_{(d)}$ model. The renormalization of the φ^4 model is studied and the nonperturbative β function is constructed. The result obtained agrees with the five-loop approximation. A nonperturbative expansion in quantum chromodynamics based on a new small parameter is formulated.

1. INTRODUCTION

At present, perturbation theory is the principal method of performing calculations in quantum field theory. Its use in combination with a renormalization procedure in quantum electrodynamics, the theory of electroweak interactions, and in the perturbative regime of quantum chromodynamics makes it possible to analyze a large number of problems. However, it is well known that the structure of a quantum-field model cannot be studied sufficiently completely using only perturbation theory. This is true not only for theories with large coupling constant, but also theories in which the coupling constant is relatively small, such as, for example, quantum electrodynamics. In quantum chromodynamics there are many problems whose solution requires nonperturbative approaches.

In recent years great efforts have been made to develop methods which somehow make it possible to go beyond perturbation theory. One class of such methods is the summation of perturbation series.[1–3] Here asymptotic expressions, obtained, for example, using the functional-integral method, are used for the higher-order terms.[4–8] The main difficulty of this approach is related to the asymptotic nature of the perturbation series. In general, the summation of such series involves a functional arbitrariness. The correct formulation of the problem involves the use of additional information about the sum of the series.[9] This information is known only for very simple model field theories.[10] Attempts to develop methods which are not directly related to the perturbation series have been made in many studies (see, for example, Refs. 11–18). Variational approaches have become widespread.[19–22] One example is the method of the Gaussian effective potential (GEP; Refs. 23–26). However, it should be noted that, as a rule, many variational methods encounter the difficulty of estimating the accuracy and stability of the results obtained using the variational procedure. In other words, in this method it is not always possible to construct an algorithm for calculating corrections to a quantity found variationally. The lack of such an algorithm makes it difficult to answer the question of the extent to which the so-called "principal contribution" adequately reflects the object studied and to find the region of validity of the expressions obtained. However, even if an algorithm for calculating the corrections, that is, the terms of some approximating series, exists, this is far from sufficient. The convergence properties of the series play a fundamental role. Actually, whereas for weak coupling even a divergent perturbation series approximates some quantity in a known manner as an asymptotic series, in the absence of a small parameter the approximating series must satisfy more rigorous requirements. In this case reliable information can be obtained using convergent series. In fact, it would be even better to deal not simply with a convergent series, but with a Leibniz series (an alternating series whose successive terms decrease in absolute value). In this case it would become possible to make two-sided estimates of the quantity in question on the basis of only the first few terms of the series. If in addition there were additional free parameters affecting the terms of the series, it would be possible to shrink the error bars the maximum amount.

For this purpose here we consider the method of variational perturbation theory (VPT; Refs. 27–33). In spite of the term "perturbation," the VPT method is nonperturbative. In it a quantity can be approximated by constructing series which are different from the perturbation series and which can be used to go beyond the weak-coupling regime. The calculability of the corrections in the VPT method is related to the fact that here, as in standard perturbation theory, only "calculable" Gaussian functional quadratures are used.[34] In addition, the VPT series can be written such that its terms correspond to ordinary Feynman diagrams. Here, of course, the VPT series will have a structure different from that in perturbation theory, and in general the diagrams will involve modified propagators and vertices.

The outline of this review is as follows. First we discuss the main ideas of the VPT method for a simple example. Then we demonstrate its use in the case of the quantum-mechanical anharmonic oscillator. In the fourth section we determine the relationship between this approach and the Gaussian effective potential. In the next section we study the convergence properties of the VPT series. In the sixth section we consider the question of renormalization and construct the nonperturbative β function for a scalar model. Then in the following section we discuss the use of this method in quantum chromodynamics. The results are summarized briefly in the conclusion.

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2. A SIMPLE EXAMPLE

Let us consider the integral

$$Z[g]=\int dx\ \exp(-S[x]), \tag{2.1}$$

where

$$S[x]=S_0[x]+gS_{\text{int}}[x],$$

$$S_0[x]=x^2=x_1^2+x_2^2,$$

$$S_{\text{int}}[x]=x_1^4+x_2^4, \quad dx=dx_1dx_2, \tag{2.2}$$

which is the zero-dimensional analog of the two-component φ^4 field theory. Here, as in field theory, we shall consider Gaussians, that is, integrals of the form

$$\int dxP(x)\exp\{-S_0[x]\}, \tag{2.3}$$

where $P(x)$ is some polynomial in the variables x_1 and x_2.

The first obvious possibility is the expansion of the integrand in (2.1) in a series in powers of the coupling constant g (were we shall use field-theoretic terminology). As a result, we arrive at the standard perturbation series:

$$Z[g]=\sum g^nC_n, \tag{2.4}$$

$$C_n=\frac{(-1)^n}{n!}\int d\vec{x}S_{\text{int}}^n\exp(-S_0[x]). \tag{2.5}$$

Whereas the expansion of the function (2.1) in the series (2.4) with coefficients (2.5) is unique, the inverse procedure of finding the sum of the series (2.4) without using additional information about the function $Z[g]$ is nonunique. For example, the same series will be obtained for the function $Z[g]+\exp(-1/g)$, which for large coupling constant g has asymptotic behavior different from that of (2.1). The reason for the incorrectness of the summation procedure in this example is the asymptotic nature of the series (2.4). Therefore, the perturbation series by itself without additional information about its sum cannot be used to evaluate the function $Z[g]$ for sufficiently large values of the coupling constant. The additional conditions needed for the unique summation of an asymptotic series are not known for realistic field-theoretic models. This problem requires special attention and so far has been solved only in a few simple cases.

The VPT method makes it possible to construct different expansions for $Z[g]$ using the same Gaussian quadratures. They differ from each other in the method of variation of the action (2.2) (choice of the trial VPT functional). In this section we consider two methods of constructing VPT expansions. The first and simplest is based on choosing a variation of harmonic form. The second is based on the use of an anharmonic VPT functional.

In the first case for the variational term we take an expression quadratic in the fields and rewrite the complete action as

$$S[x]=S_0^h[x]+S_{\text{int}}^h[x], \tag{2.6}$$

where

$$S_0^h[x]=S_0[x]+\chi S_0[x], \tag{2.7}$$

$$S_{\text{int}}^h[x]=gS_{\text{int}}[x]-\chi S_0[x], \tag{2.8}$$

and expand in powers of the new interaction (2.8). Obviously, here to calculate the terms of the series we need only Gaussian quadratures. As a result, the VPT series will have the form

$$Z[g]=\sum_n Z_n[g,\chi], \tag{2.9}$$

$$Z_n[g,\chi]=\frac{(-g)^n}{n!(1+\chi g)^{1+2n}}\int dx[S_{\text{int}}-\chi \times(1+\chi g)S_0]^n\exp(-S_0[x]). \tag{2.10}$$

Of course, the original quantity $Z[g]$ does not depend on the variational parameter χ, so when studying a finite number of terms of the series it is possible to choose χ freely, on the basis of considerations of optimization of the expansion (regarding this see, for example, Refs. 24, 27, 28, 31, and 35–38, in which various optimization methods and their use are studied). As a rule, in field theory we know only the first few terms of the series. The optimal values of the variational parameters are most often chosen on the basis of the first nontrivial order of the VPT. Here the results obtained will be stable only when the higher-order corrections to the principal contribution are sufficiently small. Let us consider the effect of the corrections for harmonic and anharmonic variations. We fix the variational parameter χ using the condition

$$\frac{\partial Z^{(N)}[g,\chi]}{\partial\chi}=0, \tag{2.11}$$

where $Z^{(N)}[g,\chi]$ is the Nth partial sum of the series (2.9)

$$Z^{(N)}[g,\chi]=\sum_{n=0}^{N} Z_n[g,\chi]. \tag{2.12}$$

For the first nontrivial order ($N=1$) Eq. (2.11) gives

$$\chi=(1/\tau-1)/g,$$

$$\tau=\frac{2}{9g}[\sqrt{1+9g}-1]. \tag{2.13}$$

In Fig. 1 we show the characteristic behavior of the Nth partial sums of the perturbation series and the VPT series for a relatively small value of the coupling constant $g=0.1$ and for the VPT parameter χ determined according to (2.13). In the case of the harmonic variation considered here the "beats" of the partial sums characteristic of asymptotic series begin somewhat later than in the case of perturbation theory. In this sense the harmonic procedure improves the convergence properties of the series. Nevertheless, if the variational parameter is held fixed and independent of the order, the resulting series is still divergent, as before. The nature of this divergence is exactly the same as that of the corresponding perturbation series. The point is that for fixed variational parameter the trial harmonic functional containing the field to a lower power than in the original interaction part of the action cannot compensate for higher-order terms

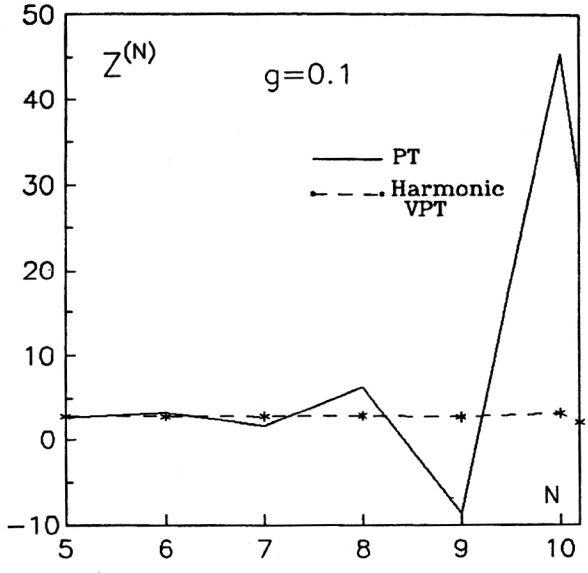

FIG. 1. Behavior of the Nth partial sums $Z^{(N)}[g]$ in the cases of perturbation theory and the VPT with the harmonic variational procedure with variational parameter fixed according to the first order.

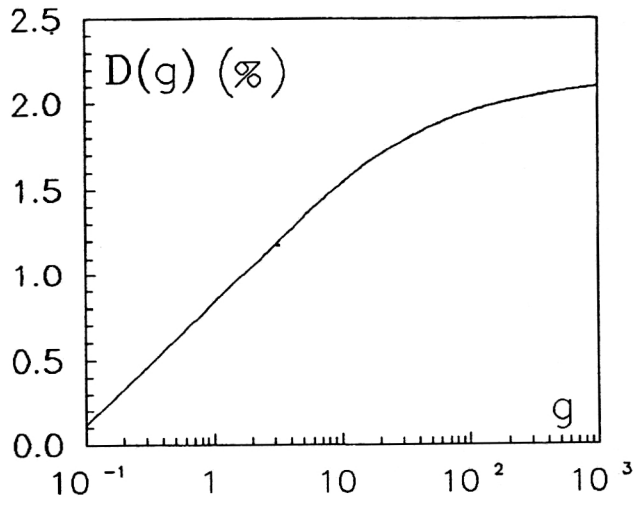

FIG. 2. Behavior of the relative error $D(g)=|Z^{(1)}(g)/Z_{\text{exact}}(g)-1|$ in percent as a function of the coupling constant g for the anharmonic variational procedure.

of the series as it must, because here the dominant contribution to the asymptotic expression is determined by large field configurations. However, as was noticed empirically in Ref. 18 and later discussed in Ref. 35, the results appear to be convergent if the variational parameter is "twisted" appropriately from order to order. A rigorous proof of this induced convergence in the zero- and one-dimensional cases is given in Refs. 39 and 40.

For constructing the anharmonic variation we rewrite the total action as

$$S[x]=S_0^a[x]+S_{\text{int}}^a[x], \tag{2.14}$$

where

$$S_0^a[x]=S_0[x]+\theta S_0^2[x], \tag{2.15}$$

$$S_{\text{int}}^a[x]=gS_{\text{int}}[x]-\theta S_0^2[x]. \tag{2.16}$$

Here the situation is somewhat more complicated than in the preceding case. After expansion in powers of the new interaction (2.16) non-Gaussian integrals arise, because the term $\theta S_0^2[x]$ remains in the exponent. However, the problem is easily solved by Fourier transformation:

$$\exp(-\theta S_0^2(x))=\int_{-\infty}^{\infty}\frac{du}{2\sqrt{\pi}}\exp\left\{-\frac{u^2}{4}+iu\sqrt{\theta}S_0(x)\right\}. \tag{2.17}$$

As a result, the VPT expansion can be written as[28]

$$Z[g]=\sum Z_n[g,\theta], \tag{2.18}$$

$$Z_n[g,\theta]=\int_0^{\infty}d\alpha(\alpha^2\theta)^n\exp(-\alpha-\alpha^2\theta)\times\sum_{k=0}^{n}\frac{a_k}{2k}\frac{(-g/\theta)^k}{(n-k)!}, \tag{2.19}$$

$$a_k=\sum_{l=0}^{k}\frac{\Gamma(2l+1/2)}{l!}\frac{\Gamma(2(k-l)+1/2)}{(k-l)!}. \tag{2.20}$$

Optimizing the first nontrivial order, for the variational parameter we find

$$\theta=\tfrac{3}{4}g. \tag{2.21}$$

The behavior of the relative error $D(g)=|Z^{(1)}(g)/Z_{\text{exact}}(g)-1|$ in percent as a function of the coupling constant g is shown in Fig. 2. We see from this figure that already the first nontrivial order ensures quite acceptable accuracy of the approximation of $Z[g]$ for all positive values of the coupling constant. The inclusion of higher-order corrections increases the accuracy of the VPT approximation even more.

Defining the convenient parameter $t=2\theta/g$, we rewrite Eq. (2.19) in a form convenient for studying the strong-coupling limit

$$Z_n[g,\theta]=\sqrt{\frac{2}{gt}}\int_0^{\infty}d\alpha,\alpha^{2n}\times\exp\left(-\alpha^2-\alpha\sqrt{\frac{2}{gt}}\right)\sum_{k=0}^{n}\frac{(-2/t)^k}{(2k)!(n-k)!}a_k. \tag{2.22}$$

It is interesting to observe that Eq. (2.22) can be used directly to determine the functional dependence of $Z[g]$ for $g\to\infty$. In fact, in Nth-order VPT for $g\to\infty$ we obtain

$$Z^{(N)}=A^{(N)}/\sqrt{g}, \tag{2.23}$$

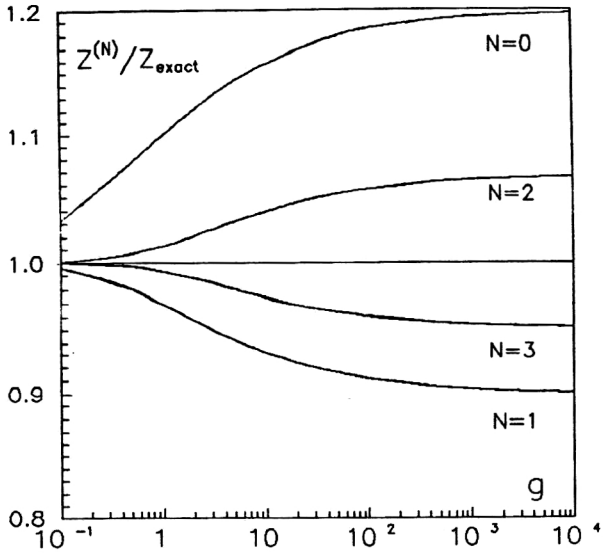

FIG. 3. Corridor of two-sided estimates of unity determined by the functions $Z^{(N)}/Z_{ex}$ for $N=0$, 1, 2, 3 and $t=1$.

where

$$A^{(N)}=\sqrt{\frac{2}{t}}\,\Gamma(N+3/2)\sum_{n=0}^{N}\frac{(-2/t)^k}{(2k+1)!(N-k)!}\,a_k. \tag{2.24}$$

Optimization of the first nontrivial order (the condition $\partial Z^{(1)}/\partial t=0$ for $Z_1=0$) gives $t_{opt}=3/2$, for which we find $A^{(1)}=3.212$, while the exact value is $A=\Gamma^2(1/4)/4=3.286$. The resulting error in the calculation is 2.3%. It can be shown that the approximations $Z^{(N)}$ for different values of the coupling constant give a stable result.[28] This is due to the convergence of the VPT series for $t>1/2$. This statement holds for any positive g.

The analog of the Sobolev inequality in this case is the relation

$$S_4[x,y]/S_2^2[x,y]\leqslant 1 \tag{2.25}$$

from which it follows that for $t>1$ the terms of the VPT series (2.18) all have positive sign, and at $t=1$ the regime changes and for $1/2<t<1$ the series is a Leibniz series. We note that the value of the variational parameter $t=1$ at which the alternating series becomes a series of terms of fixed sign corresponds to the value of t found from the criterion of asymptotic optimization of the VPT series according to which the contribution of higher-order terms of the series is minimized.[28]

In the regime of the Leibniz series two-sided estimates of the following form are valid for the exact value $Z_{ex}[g]$:

$$Z^{(2N+1)}<Z_{exact}<Z^{(2N)}, \tag{2.26}$$

where $Z^{(2N+1)}$ and $Z^{(2N)}$ are respectively odd and even partial sums of the VPT series. In Fig. 3 we show the corridor of such two-sided estimates of unity determined by the functions $Z^{(N)}/Z_{ex}$ for $N=0$, 1, 2, 3 and the parameter $t=1$. We see that already the first few partial sums give reasonable accuracy in the entire range of coupling constant.

3. THE ANHARMONIC OSCILLATOR

In this section we shall consider the use of the VPT method for obtaining the nonperturbative characteristics of the quantum-mechanical anharmonic oscillator (AO), which is a 1-dimensional φ^4 theory and often serves as a testing ground for various nonperturbative approaches. We write the Euclidean action functional in the form

$$S[\varphi]=S_0[\varphi]+\frac{m^2}{2}S_2[\varphi]+gS_4[\varphi], \tag{3.1}$$

where

$$S_0[\varphi]=\frac{1}{2}\int dx(\partial\varphi)^2, \tag{3.2}$$

$$S_2[\varphi]=\int dx\varphi^2, \tag{3.3}$$

$$S_4[\varphi]=\int dx\varphi^4. \tag{3.4}$$

Let us consider the Green function represented as a functional integral:

$$G_{2\nu}=\int D\varphi\{\varphi^{2\nu}\}\exp(-S[\varphi]), \tag{3.5}$$

where $\{\varphi^{2\nu}\}$ denotes the product of fields $\varphi(x_1)\cdots\varphi(x_{2\nu})$. We introduce dimensionless variables, setting $\varphi\to g^{-1/6}\varphi$ and $x\to g^{-1/3}x$, and we rewrite the action (3.1) as

$$S[\varphi]=S_0[\varphi]+\frac{\omega^2}{2}S_2[\varphi]+S_4[\varphi], \tag{3.6}$$

where

$$\omega^2=m^2g^{-2/3}. \tag{3.7}$$

The dimensionless Green functions $G_{2\nu}$ will be written in the form (3.5) with the action (3.6).

Just as in standard perturbation theory, we shall consider Gaussian functional integrals:

$$\int D\varphi\,\exp\left\{-\left[\frac{1}{2}\langle\varphi K\varphi\rangle+\langle\varphi J\rangle\right]\right\}$$
$$=\left(\det\frac{K}{-\partial^2+m^2}\right)^{-1/2}\exp\left(\frac{1}{2}\langle JK^{-1}J\rangle\right). \tag{3.8}$$

Any polynomial of the fields φ is as usual obtained by the corresponding number of differentiations with respect to the source $J(x)$.

We consider the following functional:

$$A[\varphi]=\theta S_0[\varphi]+\frac{\chi}{2}S_2[\varphi], \tag{3.9}$$

where θ and χ are variational parameters.

We rewrite the action (3.6) as (here we use the anharmonic variational procedure)

$$S[\varphi]=S_0^a[\varphi]+S_{int}^a[\varphi], \tag{3.10}$$

$$S_0^a[\varphi]=S_0[\varphi]+\frac{\omega^2}{2}S_2[\varphi]+A^2[\varphi], \tag{3.11}$$

$$S^a_{\rm int}[\varphi]=S_{\rm int}[\varphi]-A^2[\varphi]. \tag{3.12}$$

The trial functional $A^2[\varphi]$ has the same power of field as the original interaction $S_{\rm int}[\varphi]$. The exact Green functions do not depend on the variational parameters, so the freedom in choosing them can be used to improve the properties of the series. As a specific example, let us consider the problem of calculating the ground-state energy of the AO E_0 in the strong-coupling limit.[27] This problem is related to the calculation of the four-point Green function $G_4(0,0,0,0)\equiv G_4(0)$ owing to the relation

$$dE_0/dg=g^{-2/3}G_4(0). \tag{3.13}$$

Let us consider the asymptotic form of the higher-order terms of the VPT series. Here since in the end we shall approximate the desired quantity by the sum of the first few terms of the VPT expansion, it is natural to require that the contribution of higher-order terms of the series be minimal. As we shall see below, this asymptotic optimization leads to a relation between the variational parameters. The problem reduces to finding the asymptote of the functional integral

$$\int D\varphi[A^2-S_I]^n \exp[-(S_0-A^2)] \tag{3.14}$$

at large values of n. We shall use the method of steepest descents and write Eq. (3.14) as

$$n^n\int D\varphi \exp(-nS_{\rm eff}[\varphi]-n^{1/2}S_0[\varphi]), \tag{3.15}$$

where

$$S_{\rm eff}=A^2-\ln[A^2-S_I]. \tag{3.16}$$

The principal contribution is determined by the function φ_0 satisfying the equation $\delta S_{\rm eff}/\delta\varphi=0$, which in expanded form becomes

$$-\ddot{\varphi}_0+a\varphi_0-b\varphi_0^3=0, \tag{3.17}$$

where

$$a=\chi/\theta,\quad b=\{\theta A[\varphi_0](1-D[\varphi_0])\}^{-1},$$
$$D[\varphi_0]=A^2[\varphi_0]-S_I[\varphi_0]. \tag{3.18}$$

The corresponding solution with finite action has the form

$$\varphi_0=\pm\sqrt{\frac{2a}{b}}\,[\cosh\sqrt{a}(t-t_0)]^{-1}. \tag{3.19}$$

The arbitrary parameter t_0 reflects the translational invariance of the theory. By performing simple computations it is easily verified that the contribution of the higher-order terms of the VPT series will be a minimum if the variational parameters θ and χ are related as

$$\chi=(9/16\theta)^{1/3}. \tag{3.20}$$

The remaining independent parameter θ can be fixed to optimize the first few terms of the VPT expansion.

Let us return to the calculation of the ground-state energy of the AO. The VPT series for the Green function $G_4(0)$ can be written as

$$G_4(0)=\sum_{n=0}^{\infty}\sum_{m=0}^{\infty}\frac{1}{(n-m)!}\left(\frac{d}{d\alpha}\right)^{n-m}\cdot\langle g_m(z^2)$$
$$\times[1+iu\theta\sqrt{1-\alpha}]^{-2-2m}\rangle, \tag{3.21}$$

where

$$g_m(z^2)=\frac{(-1)^m}{m!}\int D\varphi\varphi^4(0)\exp[-(S_0+z^2/2S_2)],$$
$$z^2=[\omega^2+iu\chi\sqrt{1-\alpha}][1+iu\theta\sqrt{1-\alpha}]^{-1},$$
$$\langle\cdots\rangle=\int_{-\infty}^{\infty}\frac{du}{2\sqrt{\pi}}\exp\left\{-\frac{u^2}{4}\right\}(\cdots). \tag{3.22}$$

After differentiation with respect to α in (3.21) we must set $\alpha=0$. The functions $g_m(z^2)$ are ordinary coefficients of the perturbation expansion of $G_4(0)$ and can be found by the standard diagrammatic technique. We see from (3.21) that the Nth-order VPT is generated by the same diagrams as the Nth order of ordinary perturbation theory. Of course, here the structure of the series is changed and the form of the propagator is modified. The functions $g_m(z^2)$ are easily expressed in terms of the coefficients A_n of the perturbation expansion of the ground-state energy of the AO E_0. The corresponding relation is given by

$$g_m(z^2)=\frac{(1+m)A_{1+m}}{z^{2+3m}}. \tag{3.23}$$

The numerical values of the coefficients A_n can be found in, for example, Refs. 41 and 42. For the first nontrivial order considered here we need the following coefficients: $A_1=3/4$ and $A_2=-21/8$.

In addition, it is convenient, using the expression

$$a^{-\nu}=\frac{1}{\Gamma(\nu)}\int d\alpha\alpha^{\nu-1}\exp(-a\alpha),$$

to write (3.21) as

$$G_4(0)=\sum_{n=0}^{\infty}\sum_{m=0}^{\infty}\frac{(1+m)A_{1+m}}{(n-m)!}[\Gamma(1+m/2)\Gamma$$
$$\times(1+3m/2)]^{-1}\left(\frac{d}{d\alpha}\right)^{n-m}F_m(\theta,\chi,\alpha), \tag{3.24}$$

where

$$F_m(\theta,\chi,\alpha)=\int_0^{\infty}dx\,x^{m/2}\exp(-x)\int_0^{\infty}dy\,y^{3m/2}$$
$$\times\exp[-\omega^2y-(1-\alpha)\cdot(x\theta+y\chi)]. \tag{3.25}$$

In the strong-coupling limit we can take $\omega^2=0$ in (3.25). However, we note that by expanding the expression $\exp(-\omega^2y)$ in powers of ω^2 it is also possible to calculate the corresponding corrections to the principal contribution.

From (3.24) and (3.25) in the strong-coupling limit in Nth-order VPT we find

TABLE I. Ground-state energy of the φ^{2k} anharmonic oscillator.

k	θ	$E_0^{exact}[g]$	$E_0^{(1)}[g]$
2	0.027926	$0.668\,g^{1/3}$	$0.663\,g^{1/3}$
3	0.038009	$0.680\,g^{1/4}$	$0.698\,g^{1/4}$
4	0.040149	$0.704\,g^{1/5}$	$0.709\,g^{1/5}$

$$E_0^{(N)}=3g^{1/3}\sum_{n=0}^{N}\sum_{m=0}^{n}\frac{(1+m)A_{1+m}}{(n-m)!}\left(\frac{16}{9}\,\theta\right)^{1/3+m/2}$$
$$\times[\Gamma(1+m/2)\Gamma(1+3m/2)]^{-1}R_{n,m}(\theta), \quad (3.26)$$

where

$$R_{n,m}(\theta)=\int_0^\infty dx\; x^{m/2}\exp(-x)\int_0^\infty dy\; y^{3m/2}$$
$$\times(\theta x+y)^{2(n-m)}\exp[-(\theta x+y)^2]. \quad (3.27)$$

The optimal value of the parameter θ in different versions of the optimization procedure (see Refs. 27 and 28) is $\theta_{1,2}\ll 1$. Using this fact, in first-order VPT we find

$$E_0^{(1)}=g^{1/3}(\varepsilon_0+\varepsilon_1), \quad (3.28)$$

where

$$\varepsilon_0=\tfrac{3}{2}A_1\sqrt{\pi}x^2, \quad (3.29)$$

$$\varepsilon_1=\tfrac{3}{4}A_1\sqrt{\pi}x^2+\frac{4\Gamma(5/4)}{\sqrt{\pi}}A_2x^5, \quad (3.30)$$

$$x=(\tfrac{16}{9}\theta)^{1/6}. \quad (3.31)$$

Using the "principle of minimal sensitivity" (Refs. 24 and 35) $\partial E_0^{(1)}/\partial x=0$, we find $x_{opt}=0.6062$ and $E_0^{(1)}(x_{opt})=0.660g^{1/3}$. The numerical value is $E_{exact}=0.668g^{1/3}$ (Ref. 42). A more accurate calculation for the φ^{2k} AO (Ref. 29) not based on smallness of θ is given in Table I, where the same optimal values of the parameter θ are quoted.

To find the corrections to the principal contribution we expand in ω^2 in (3.25). As a result, we obtain

$$E_0^{(1)}=g^{1/3}(0.663+0.1407\omega^2-0.0085\omega^4+\cdots).$$

We give the numerical result for comparison:[42]

$$E_0^{ex}=g^{1/3}(0.668+0.1437\omega^2-0.0088\omega^4+\cdots).$$

Let us now calculate the mass parameter μ^2 related to the two-point Green function as $\mu^{-2}=G_2(p=0)$, where

$$G_2(p=0)=\int dt\int D\varphi\varphi(t/2)\varphi(-t/2)\exp(-S[\varphi]). \quad (3.32)$$

We shall compare the result with the corresponding numerical value:[43]

$$\mu^2_{exact}=3.009g^{2/3}. \quad (3.33)$$

In this case the VPT expansion has the form

$$G_2(0)=g^{-2/3}\frac{1}{2}\sum_{n=0}^{\infty}\sum_{m=0}^{n}\frac{\Gamma(n+1/2-m/4)}{(n-m)!}$$
$$\times\frac{B_m}{\Gamma(1+3m/2)}x^{2+3m}, \quad (3.34)$$

where B_m are the dimensionless perturbation coefficients. Here we shall use the values $B_0=1$ and $B_1=-6$. In first-order VPT from (3.34) we find

$$G_2^{(1)}=g^{-2/3}(G_{20}+G_{21}),$$

where

$$G_{20}=\frac{\sqrt{\pi}}{2}x^2, \quad (3.35)$$

$$G_{21}=\frac{\sqrt{\pi}}{4}x^2-4\cdot\Gamma(5/4)\cdot x^5, \quad (3.36)$$

Using the principle of minimal sensitivity $[\partial G_2^{(1)}/\partial x=0]$, we obtain the value $\mu^2=3.078g^{2/3}$, which should be compared to (3.33).

The vacuum energy can also be calculated using the propagator $G_2(p)$ (Refs. 28 and 29). In addition, it is possible to estimate the ground-state energy. For this we define the quantity $\mu_1=E_1-E_0$, for which, using the spectral representation

$$G_2(p)=2\sum_{n=0}^{\infty}\frac{\mu_n}{p^2+\mu_n^2}|\langle 0|\hat{x}|n\rangle|^2, \quad (3.37)$$

where the matrix elements of the coordinate operator are calculated for eigenstates of the full Hamiltonian, we obtain the estimate

$$\mu_1\leqslant\mu_1^{(+)}, \quad \mu_1^{(+)}=2\frac{G_2(t=0)}{G_2(p=0)}. \quad (3.38)$$

By analogy with the sum rules in quantum mechanics,[44] we can expect fairly rapid saturation of the spectral representation (3.37). In this case the quantities μ_1 and $\mu_1^{(+)}$ should be close to each other. In first-order VPT we find $\mu_1^{(+)}=1.763g^{1/3}$, while the numerical value is $\mu_1^{exact}=1.726g^{1/3}$ (Ref. 42).

We conclude this section by noting that the VPT method can be used to construct the nonperturbative effective potential.[27,28] The generalization to the case of the φ^{2k} AO can be found in Refs. 29 and 32.

4. THE GAUSSIAN EFFECTIVE POTENTIAL

Among the widely used nonperturbative methods is the method of the Gaussian effective potential (GEP; Refs. 23–26). As in many other nonperturbative approaches, in the GEP method the questions of the region of applicability and the reliability of the results are important. In other words, it is necessary to know the degree to which the principal contribution calculated by the variational method adequately describes the quantity in question. Therefore, the question of the existence of an algorithm for calculating the corrections to the principal variational contribution is of primary impor-

tance. In this respect the GEP method is superior (see Refs. 24 and 25) to a number of other nonperturbative approaches of the variational type.

However, it should be noted that the possibility of calculating corrections by itself is not sufficient for resolving the question of the stability of the results. Here the convergence properties of the series play a fundamental role. In fact, if the theory contains a small parameter—the coupling constant, even a divergent perturbation series can give useful information at small coupling constant. A different picture arises when there is no small parameter and no effective one arises. In this case we can count on the results to be reliable only when we are dealing with convergent series. In other words, in nonperturbative approaches the problem of calculating the corrections to the principal contribution and the problem of analyzing the series convergence properties must go hand in hand.

In this section we shall consider the VPT method for $\lambda\varphi^4$ theory in a space of n dimensions. As explained above, in the VPT approach the desired quantity is initially written as a series, which determines the algorithm for calculating the corrections of any order. A fact of technical importance is that it is possible to construct VPT series such that the Nth-order VPT approximation uses only those Feynman diagrams which form the same Nth-order standard perturbation theory. The presence of free variational parameters in the method means that the convergence properties of the VPT series can be influenced by choosing them. We shall deal with the relationship between the VPT and the GEP methods. We shall propose several methods of deriving the GEP within the framework of the VPT. In all the variants considered the GEP arises as the first nontrivial order of the VPT. Here, however, the corresponding VPT series possesses considerably different convergence properties.

First we shall use the VPT method for the variational correction of the semiclassical approximation. The action functional has the form (pseudo-Euclidean metric)

$$S[\varphi]=S_0[\varphi]-\frac{m^2}{2}S_2[\varphi]-\lambda S_4[\varphi], \tag{4.1}$$

where

$$S_0[\varphi]=\tfrac{1}{2}\int dx(\partial\varphi)^2, \tag{4.2}$$

$$S_p[\varphi]=\int dx\varphi^p. \tag{4.3}$$

Henceforth we shall have in mind dimensional regularization, setting $n=d-2\varepsilon$, where d is an integer. As usual, we isolate the classical contribution in the Green-function generating functional $W[J]$, writing

$$W[J]=\int D\varphi\ \exp\{i[S[\varphi]+\langle J\varphi\rangle]\}=\exp\{i[S[\varphi_c]+\langle J\varphi_c\rangle]\}D[J], \tag{4.4}$$

where

$$D[J]=\int D\varphi\ \exp\{-iP[\varphi]\}, \tag{4.5}$$

$$P[\varphi]=\int dx\left[\frac{1}{2}\varphi(\partial^2+m^2+12\lambda\varphi_c^2)\varphi+4\lambda\varphi_c\varphi^3+\lambda\varphi^4\right]. \tag{4.6}$$

The function φ_c obeys the classical equation of motion

$$\delta S/\delta\varphi_c=-J.$$

In the standard semiclassical approximation only terms quadratic in the fields φ remain in the expression for $P[\varphi]$. In this case the functional integral for $D[J]$ becomes Gaussian, and the ordinary one-loop representation arises for $W[J]$ (Ref. 45).

To calculate $D[J]$ by the VPT method we first use a harmonic variation. We write the functional $P[\varphi]$ as

$$P[\varphi]=\int dx\left[\frac{1}{2}\varphi(\partial^2+z^2)\varphi+\lambda\left(4\varphi_c\varphi^3+\varphi^4-\frac{\chi^2}{2}\varphi^2\right)\right], \tag{4.7}$$

where

$$z^2=m^2+12\lambda\varphi_c^2+\lambda\chi^2.$$

Then the VPT for $D[J]$ is written as

$$D[J]=\left[\det\frac{\partial^2+z^2}{\partial^2}\right]^{-1/2}\sum_{n=0}^{\infty}\frac{(-i\lambda)^n}{n!}\left[\int dx\left(4\varphi_c\hat{\varphi}^3+\hat{\varphi}^4-\frac{\chi^2}{2}\hat{\varphi}^2\right)\right]^n\exp\left[-\frac{i}{2}\langle j\Delta j\rangle\right]_{j=0}, \tag{4.8}$$

where

$$\Delta(p)=(p^2-z^2+i0)^{-1},$$

$$\hat{\varphi}(x)=i\frac{\delta}{\delta j(x)}.$$

We restrict ourselves to the first two terms in (8). Their contributions to the effective potential are respectively

$$V_0=\frac{1}{n}z^2\Delta_0(z^2), \tag{4.9}$$

$$V_1=\lambda\left[3\Delta_0^2(z^2)-\frac{\chi^2}{2}\Delta_0(z^2)\right], \tag{4.10}$$

where

$$\Delta_0(z^2)=\mu^{2\varepsilon}\frac{\Gamma(1-n/2)}{(4\pi)^{n/2}}(z^2)^{n/2-1} \tag{4.11}$$

is the Euclidean propagator $\Delta(x=0,z^2)$ in dimensional regularization. The optimization condition

$$\frac{d(V_0+V_1)}{dz^2}=0$$

gives the equation for the variational parameter z^2:

$$z^2=m^2+12\lambda\varphi^2+12\lambda\Delta_0(z^2). \tag{4.12}$$

Taking into account (4.12), for the effective potential in this order of VPT we find

$$V_{\text{eff}}(\varphi)=V_{cl}+V_0+V_1=\tfrac{1}{2}m^2\varphi^2+\lambda\varphi^4+\frac{1}{n}z^2\Delta_0(z^2)$$
$$+\tfrac{1}{2}(m^2-z^2)\Delta_0(z^2)+\lambda[3\Delta_0^2(z^2)$$
$$+6\varphi^2\Delta_0(z^2)]. \tag{4.13}$$

It is easily checked that Eq. (4.13) taking into account the optimization condition (4.12) coincides with the GEP in a space of n dimensions.[26]

Let us now calculate $D[J]$ using anharmonic variation of the action functional. We choose the variation in the form $R^2[\varphi]$, where

$$R[\varphi]=\frac{\chi}{2\Omega^{1/2}}\int dx\varphi^2(x).$$

The appearance here of the volume of coordinate space Ω is related to the fact that constant field configurations are required for obtaining V_{eff} from the effective action. The above choice of the VPT functional ensures that the variational parameter χ is independent of Ω.

As a result, we obtain

$$D[J]=\sum_{n=0}^{\infty}\frac{(-i)^n}{n!}\int D\varphi\left[\lambda\int dx(\varphi^4+4\varphi_c\varphi^3)\right.$$
$$\left.-R^2[\varphi]\right]^n\exp\left\{-i\left[\frac{1}{2}\int dx\varphi(\partial^2+m^2\right.\right.$$
$$\left.\left.+12\lambda\varphi_c^2)\varphi+R^2[\varphi]\right]\right\}. \tag{4.14}$$

Any power of $R^2[\varphi]$ in Eq. (4.14) can be selected by performing the required number of differentiations of the expression $\exp(-i\varepsilon R^2[\varphi])$ with respect to the parameter ε and then setting $\varepsilon=1$ Regarding the term $R^2[\varphi]$ in the exponent making the functional integral non-Gaussian, the problem is again easily solved by Fourier transformation, as a result of which the first power $R[\varphi]$ appears in the exponential.

Therefore, the VPT series is written as

$$D[J]=\sum_{n=0}^{\infty}(-1)^n\sum_{k=0}^{n}\frac{(-i)^{n-k}}{(n-k)!n!}\left[\frac{d}{d\varepsilon}\right]^{n-k}\sqrt{\Omega}\int_{-\infty}^{\infty}\frac{dv}{2\sqrt{\pi}}$$
$$\times\exp\left\{i\Omega\frac{v^2}{4}-i\frac{\pi}{4}\right\}\left[\det\frac{\partial^2+M^2}{\partial^2}\right]^{-1/2}\left[\lambda\right.$$
$$\left.\times\int dx(4\varphi_c\hat{\varphi}^3+\hat{\varphi}^4)\right]^k\exp\left[-\frac{i}{2}\langle j\Delta j\rangle\right]_{j=0}, \tag{4.15}$$

where

$$M^2=m^2+12\lambda\varphi_c^2+\sqrt{\varepsilon}\chi v.$$

The integral over v in (4.15) contains the large parameter Ω and can therefore be calculated by the stationary-phase method. As a result, the effective potential in the first nontrivial order of the VPT has the form

$$V_{\text{eff}}=V_c+V_0+V_1,$$

$$V_0=\frac{1}{n}M^2\Delta_0-\frac{\chi^2}{4}\Delta_0^2,$$

$$V_1=-\frac{\chi^2}{4}\Delta_0^2+3\lambda\Delta_0^2. \tag{4.16}$$

Here M^2 is the mass parameter for $\varepsilon=1$ and $v=v_0$, where v_0 is the stationary-phase point in the integral (4.15). The equation which it obeys has the form

$$M^2=m^2+12\lambda\varphi^2+\chi^2\Delta_0(M^2). \tag{4.17}$$

Different versions of the optimization, both the requirement on $\min|V_1|$ (in this case a solution to the equation $V_1=0$ exists) and the principle of minimal sensitivity $\partial V_{\text{eff}}/\partial\chi^2=0$, lead to the same value of the variational parameter χ^2:

$$\chi^2=12\lambda.$$

As a result, the effective potential (4.16) together with the condition (4.17) gives the GEP.

Let us obtain the GEP by yet another method, which is not based on the representation (4.4) but directly on the original functional $W[j]$. We shall consider an anharmonic two-parameter variation of the action:

$$Q[\varphi]=\frac{a^2}{\Omega}S_2^2[\varphi]+\frac{b^4}{\Omega^3}S_1^4[\varphi]. \tag{4.18}$$

The VPT series for the generating functional $W[j]$ is written as

$$W[j]=\sum_{n=0}^{\infty}\frac{i^n}{n!}\int D\varphi[Q[\varphi]-\lambda S_4[\varphi]]^n$$
$$\times\exp\left\{i\left[S_0-m^2S_2-\varepsilon\frac{a^2}{\Omega}S_2^2-\theta\frac{b^4}{\Omega^3}S_1^4+\langle j\varphi\rangle\right]\right\}. \tag{4.19}$$

The parameters ε and θ are introduced here so that the expressions in the preexponent involving S_1 and S_2 can be selected by differentiation with respect to these parameters. In the final result it is necessary to set $\varepsilon=1$ and $\theta=1$. In this case only powers of the interaction term λS_4 remain in the preexponent in the functional integral. The expression in the exponential in (4.19) is reduced to one quadratic in the fields by means of Fourier transformation. As a result, (4.19) is written as

$$W[j]=\Omega^2\int_{-\infty}^{\infty}dx\frac{dp}{2\pi}\int_{-\infty}^{\infty}dy\frac{dq}{2\pi}$$
$$\times\exp\{i\Omega[px-qy-p^2-q^4]\}$$
$$\times\sum_{n=0}^{\infty}\sum_{k=0}^{n}\sum_{m=0}^{n-k}\frac{i^{n-k}}{m!(n-k-m)!}$$
$$\times\left(i\frac{\partial}{\partial\varepsilon}\right)^m\left(i\frac{\partial}{\partial\theta}\right)^{n-k-m}\left[\det\frac{\partial^2+M^2}{\partial^2}\right]^{-1/2}w_k[J,M^2], \tag{4.20}$$

where

$$M^2=m^2+\sqrt{\varepsilon}ax,\quad J=j+\theta^{1/4}by,$$

and $w_k[J,M^2]$ are the perturbation series coefficients for the generating functional $W[j]$. The standard Feynman rules with the mass parameter M^2 in the propagator can be used to calculate the $w_k[J,M^2]$.

In the first nontrivial order for the generating functional of connected Green functions

$$Z[j]=(i\Omega)^{-1}\ln W[j]$$

we find

$$Z^{(1)}=jx-\tfrac{1}{2}m^2x^2+\left(\frac{1}{2}-\frac{1}{n}\right)m^2y\Delta_0(m^2y)-\tfrac{1}{2}m^2\Delta_0(m^2y)$$
$$-\lambda[3\Delta_0^2(m^2y)+6\Delta_0(m^2y)x^2+x^4], \qquad (4.21)$$

where

$$x=\frac{J}{M^2},\quad y=\frac{M^2}{m^2}.$$

The optimization condition $\partial Z^{(1)}/\partial x=0$ leads to the equation

$$m^2x+4\lambda x(3\Delta_0+x^2)=j. \qquad (4.22)$$

Similarly, for the parameter y, requiring that $\partial Z^{(1)}/\partial y=0$, we obtain

$$m^2(y-1)=12\lambda(\Delta_0+x^2). \qquad (4.23)$$

Using (4.22) and (4.23), we find

$$\varphi=\frac{dZ^{(1)}}{dj}=\frac{\partial Z^{(1)}}{\partial j}=x.$$

For the effective potential $V_{\text{eff}}=j\varphi-Z$ we finally obtain

$$V_{\text{eff}}=\tfrac{1}{2}m^2\varphi^2+\left(\frac{1}{n}-\frac{1}{2}\right)M^2\Delta_0(M^2)+\tfrac{1}{2}m^2\Delta_0(M^2)$$
$$+\lambda[3\Delta_0^2(M^2)+6\Delta_0(M^2)\varphi^2+\varphi^4]. \qquad (4.24)$$

It is easily seen that Eq. (4.24) coincides with the GEP.

All the methods of constructing VPT series for the effective potential described in this section have been chosen such that the first nontrivial VPT correction gives the GEP. However, the properties of the series are different, in spite of the same result in first order. We shall discuss this question in the following section. We conclude the present section by noting that a new viewpoint on the problem of renormalization of the GEP can be found in Refs. 46–50.

5. CONVERGENCE PROPERTIES OF VPT SERIES

Let us now consider massless φ^4 theory in four-dimensional Euclidean space with the action

$$S[\varphi]=S_0[\varphi]+\lambda S_4[\varphi]. \qquad (5.1)$$

We construct the VPT series for the vacuum functional

$$W[0]=\int D\varphi\ \exp\{-S[\varphi]\}. \qquad (5.2)$$

It is not difficult to generalize the method to the case of Green functions. As the variation we shall use a functional of anharmonic form

$$\tilde{S}[\varphi]=\theta^2S_0^2[\varphi]. \qquad (5.3)$$

Then the VPT series for (5.2) is written as

$$W[0]=\sum_{n=0}^{\infty}W_n[0,\theta],$$

$$W_n[0,\theta]=\frac{(-1)^n}{n!}\int D\varphi\ \exp\{-S_{\text{eff}}[\varphi,n]\} \qquad (5.4)$$

where

$$S_{\text{eff}}[\varphi,n]=S_0[\varphi]+\theta^2S_0^2[\varphi]-n\ \ln\{\lambda S_4[\varphi]-\theta^2S_0^2[\varphi]\}. \qquad (5.5)$$

The principal contribution to the asymptotic form of the higher-order terms of the series comes from field configurations satisfying the equation

$$\frac{\delta S_{\text{eff}}[\varphi_0,n]}{\delta\varphi_0(x)}=0. \qquad (5.6)$$

Varying (5.5), we obtain

$$-\partial^2\varphi_0+\frac{a}{3!}\varphi_0^3=0, \qquad (5.7)$$

where

$$a=\frac{4!\lambda n}{D[\varphi_0]+2\theta^2S_0[\varphi_0](n+D[\varphi_0])}, \qquad (5.8)$$

$$D[\varphi]=\lambda S_4[\varphi]-\theta^2S_0^2[\varphi]. \qquad (5.9)$$

The solution of Eq. (5.7) that we need has the form

$$\varphi_0(x)=\pm\sqrt{\frac{48}{a}}\frac{\mu}{(x-x_0)^2+\mu^2}. \qquad (5.10)$$

The arbitrary parameters x_0 and μ reflect the translational and scale invariance of this model.

For what follows it is convenient to define new variables, setting

$$g=4C_s\lambda,\quad \theta^2=g\chi. \qquad (5.11)$$

Here $C_s=4!/(16\pi)^2$ is the constant involved in the Sobolev inequality (see, for example, Refs. 51 and 52):

$$S_4[\varphi]\leqslant 4C_sS_0^2[\varphi]. \qquad (5.12)$$

For the functional (5.9) on functions (5.10) we find

$$D[\varphi_0]=4\frac{(16\pi^2)^2}{a^2}g(1-\chi). \qquad (5.13)$$

Substituting $S_0[\varphi]$ and (5.13) into (5.8), we obtain an equation for the parameter a, the solution of which has the form

$$a=\{\sqrt{b^2/4+nb}-b/2\}^{-1},$$
$$b=[(32/\pi^2)^2g\chi]^{-1}. \qquad (5.14)$$

In the limit of large n we find

$$D[\varphi_0]\sim n(1-\chi)/\chi. \qquad (5.15)$$

Then in leading order in n we obtain

$$W_n[0,\theta]\sim\frac{(-1)^n}{n!}n^n\left(\frac{1-\chi}{\chi}\right)^n\exp\{-n\}. \qquad (5.16)$$

The inclusion of higher orders in n, including the functional determinant, leads to the appearance in (5.16) of an n-dependent multiplicative factor, which is not dominant and does not affect the convergence properties of the series.

We see from Eq. (5.16) that independently of the value of the coupling constant g the VPT series is absolutely convergent for $\chi>1/2$. Here, as follows from the Sobolev inequality (5.12) for $\chi>1$, the VPT series is a series of positive terms. For $1/2<\chi<1$ the terms of the series at large n form a Leibniz series. Here again the value $\chi=1$ corresponds to a change of both the regime of the series and its asymptotic optimization.

Let us now consider the two-parameter VPT in a space of arbitrary number of dimensions. It is clear from the preceding analysis that the term containing the source in the action, the preexponential polynomial factor in the functional integral of the type $\{\varphi^{2\nu}\}$, and the mass term do not introduce anything fundamental from the viewpoint of studying the convergence properties. We therefore restrict ourselves here to consideration of the vacuum functional, for which the terms of the VPT series have the form

$$J_k=\frac{1}{k!}\int D\varphi(A^2[\varphi]-S_4[\varphi])^k \exp[-(S_0[\varphi]+A^2[\varphi])], \tag{5.17}$$

where $A[\varphi]$ is defined as in (3.9). Making the substitution $\varphi\to k^{1/4}\varphi$, we obtain

$$J_k=\frac{k^k}{k!}I_k, \tag{5.18}$$

$$I_k=\int D\varphi \exp(-kS_{\text{eff}}[\varphi]-k^{1/2}S_0[\varphi]), \tag{5.19}$$

where

$$S_{\text{eff}}[\varphi]=A^2[\varphi]-\ln D[\varphi], \tag{5.20}$$

$$D[\varphi]=A^2[\varphi]-S_4[\varphi]. \tag{5.21}$$

The dominant contribution to the functional integral (5.19) comes from field configurations φ_0 which minimize the effective action (5.20). The corresponding equation has the form

$$-\partial^2\varphi_0+a\varphi_0-b\varphi_0^3=0, \tag{5.22}$$

where

$$a=\chi/\theta, \tag{5.23}$$

$$b=2[\theta A[\varphi_0](1-D[\varphi_0])]^{-1}. \tag{5.24}$$

It is convenient to transform to the function $f(x)$ satisfying the equation

$$[-\partial^2+1]f(x)-f^3(x)=0 \tag{5.25}$$

and related to the function $\varphi_0(x)$ as

$$\varphi_0(x)=\sqrt{\frac{a}{b}}f(\sqrt{a}x). \tag{5.26}$$

Let us define a constant depending on the space dimension:[2]

$$C=\int dx f^4(x). \tag{5.27}$$

In Ref. 2 it was shown that the absolute minimum of the total action is attained on spherically symmetric solutions of the equation of motion. Using this result and also equation (5.22), it can be shown that in this case the minimum of S_{eff} is also attained on spherically symmetric solutions. The constant (5.27) can be calculated in principle (see Ref. 2), but its value is not important for our purposes.

For the functionals $S_4[\varphi_0]$ and $A^2[\varphi_0]$ we find

$$S_4[\varphi_0]=\alpha/b^2, \tag{5.28}$$

$$A^2[\varphi_0]=\alpha^2\tau/b^2, \tag{5.29}$$

where α and τ are defined as

$$\alpha=Ca^{2-n/2}, \tag{5.30}$$

$$\tau=\theta^2/4. \tag{5.31}$$

It follows from (5.24) that the three parameters α, b, and τ are related as

$$\alpha\tau(1-D[\varphi_0])=1, \tag{5.32}$$

where

$$D[\varphi_0]=\alpha(\alpha\tau-1)/b^2. \tag{5.33}$$

Therefore, as before, only two parameters remain independent. In leading order in k for (5.17) we find

$$J_k\sim k^{-1/2}D^k[\varphi_0]\exp\{-k[A^2[\varphi_0]-1]\}. \tag{5.34}$$

Using Eq. (5.22) it can be checked that $A[\varphi_0]=1$. The region of parameters in which the VPT series converges is determined by the inequality

$$|D[\varphi_0]|<1. \tag{5.35}$$

The best choice of variational parameters for which the contribution of the higher-order terms of the series is a minimum (the method of asymptotic optimization; Refs. 28 and 30) corresponds to the condition $D[\varphi_0]=0$, which leads to the relation $\alpha\tau=1$. The remaining independent parameter is fixed by optimizing the sums of the first few terms of the VPT series. The condition of asymptotic optimization for the original parameters θ and χ is written as

$$\chi=\left(\frac{16}{\theta^n C^2}\right)^{1/4-n}. \tag{5.36}$$

In particular, in the one-dimensional case corresponding to the anharmonic oscillator $C=16/3$ and the condition (5.36) becomes (3.20).

6. THE NONPERTURBATIVE β FUNCTION IN THE $\varphi^4_{(4)}$ MODEL

In this section we consider the renormalization procedure in the VPT method for the example of a scalar field theory. Our discussion is based on that of Ref. 33. For the massless φ^4 model in four dimensions the Euclidean action has the form

$$S[\varphi]=S_0[\varphi]+S_I[\varphi], \tag{6.1}$$

where

$$S_0[\varphi]=\tfrac{1}{2}\int dx\varphi(-\partial^2)\varphi, \tag{6.2}$$

$$S_I[\varphi]=\frac{(4\pi)^2}{4!}g\int dx\varphi^4. \tag{6.3}$$

As is well known, the perturbation series for the Green-function generating functional

$$W[J]=\int D\varphi\ \exp\left\{-S[\varphi]+\int dx\ J\cdot\varphi\right\} \tag{6.4}$$

diverges and the function $W[J]$ as a function of the coupling constant g is not an analytic function near $g=0$. The actual asymptotic behavior of the higher-order terms of the perturbation series can be found using the method of steepest descents.[4] Here the number labeling the term of the series can serve as the large saddle-point parameter. The principal contribution is determined by configurations which correspond to large fields proportional to a positive power of the saddle-point parameter. Clearly, in this case the interaction term (6.3) cannot be viewed as a small perturbation of the free action (6.2).

As we have seen in the preceding sections, in the VPT method a new interaction term S_I' and, correspondingly, a new free action S_0', are constructed within the framework of the original dynamics. This splitting is accomplished using several variational parameters. These parameters are fine-tuned such that the convergence properties of the VPT series are improved. As shown in the preceding section, there exist methods of constructing the VPT in which the resulting series converge in a finite region of parameters.

Here we shall consider a VPT functional of the form

$$\tilde{S}[\varphi]=\theta^2 S_0^2[\varphi] \tag{6.5}$$

and rewrite the total action (6.1) as

$$S[\varphi]=S_0'[\varphi]+\eta S_I'[\varphi], \tag{6.6}$$

where

$$S_0'[\varphi]=S_0[\varphi]+\tilde{S}[\varphi], \tag{6.7}$$

and

$$S_I'[\varphi]=S_I[\varphi]-\tilde{S}[\varphi]. \tag{6.8}$$

We have introduced the parameter η for convenience. Its power will correspond to the order of the VPT in which we are working. We shall set $\eta=1$ in the final expressions. The parameter θ^2 in Eq. (6.5) is the variational parameter. The original functional (6.4) is independent of this parameter, so it can be used to optimize the VPT expansion. In the normalization of the interaction term used here it is convenient to replace the parameter θ^2 by the parameter t using the relation

$$\theta^2=4C_s\frac{(4\pi)^2}{4!}g\cdot t, \tag{6.9}$$

where $C_s=4!/(16\pi)^2$ is the Sobolev constant involved in the inequality (see, for example, Refs. 2, 3, 51, and 52 regarding this)

$$\int dx\varphi^4\leq C_s\left[\int dx\ \varphi(-\partial^2)\varphi\right]^2. \tag{6.10}$$

The parameter t is fixed on the basis of the principle of asymptotic optimization discussed in the preceding section. As a result, we find $t=1$ (Ref. 30).

After expansion in the parameter η the functional integral is brought to Gaussian form by Fourier transformation. As a result, for the Green function $G_{2\nu}$ in Nth-order VPT we obtain

$$G_{2\nu}^{(N)}=\int_0^\infty d\alpha\alpha^{\nu-1}\exp(-\alpha-\theta^2\alpha^2)$$
$$\times\sum_{n=0}^{N}\eta^n\alpha^{2n}\sum_{k=0}^{n}\frac{(\theta^2)^{n-k}}{(n-k)!}\frac{g_{2\nu}^k}{\Gamma(2k+\nu)}. \tag{6.11}$$

Here the functions $g_{2\nu}^k$ are the usual coefficients of the perturbation series for the Green function $G_{2\nu}$. They can be calculated using the standard Feynman rules.

We should stress the fact that the expansion (6.11) in powers of the coupling constant g contains all powers of g. The first N coefficients of this expansion coincide with the corresponding coefficients calculated in Nth-order perturbation theory.

Let us now turn directly to the realization of the renormalization program. Instead of the field φ and the coupling constant g, we introduce the bare quantities φ_0 and g_0. The field φ_0 is related to the field φ as $\varphi_0=Z^{1/2}\varphi$. Here we shall calculate the divergent constants Z and g_0 on the basis of the VPT. The constant Z is calculated using the propagator G_2. For the approximation considered here it is sufficient to write it out in first order. Using (6.11), we find

$$Z^{(1)}=\Gamma(1)J_1(\theta_0^2)+\eta\theta_0^2\Gamma(3)J_3(\theta_0^2), \tag{6.12}$$

where we have defined

$$J_\nu(\theta^2)=\frac{1}{\Gamma(\nu)}\int_0^\infty d\alpha\ \alpha^{\nu-1}\exp(-\alpha-\alpha^2\theta^2). \tag{6.13}$$

The function $J_\nu(\theta^2)$ is normalized by the condition $J_\nu(0)=1$.

For the connected part of the four-point Green function in second-order VPT we obtain

$$-G_4^{(2)}(\mu^2)=\eta g_0 J_4(\theta_0^2)+\eta^2\left[g_0\frac{\theta_0^2}{1!}\frac{\Gamma(6)}{\Gamma(4)}J_6(\theta_0^2)\right.$$
$$\left.-\frac{3}{2}g_0^2 J_6(\theta_0^2)\ln\frac{\Lambda^2}{\mu^2}\right]. \tag{6.14}$$

In this expression we have written out only the divergent part which we shall need below, and we have used the scheme with symmetric normalization point μ^2. For the bare coupling constant g_0 we write the VPT expansion in the form $g_0=g(1+\eta\alpha+\cdots)$. Analogous expansions must be written down for the parameter θ_0^2 and $J_\nu(\theta_0^2)$. The divergent coefficient α is determined from Eqs. (6.12), (6.14), and the requirement that $-Z^2G_4(\mu^2)$ be finite. If we now change the normalization point $\mu\to\mu'$ and use the fact that the bare constant g_0 is independent of this point, we arrive at a relation between the constants g and g':

$$g' = g + \eta\beta(g)\ln\frac{\mu'^2}{\mu^2}, \tag{6.15}$$

where the β function is expressed as

$$\beta(g) = \frac{3}{2}g^2\frac{J_6(\theta^2)/J_4(\theta^2)}{1-\theta^2\{[\Gamma(6)J_6(\theta^2)/\Gamma(4)J_4(\theta^2)]-2[\Gamma(3)J_3(\theta^2)/\Gamma(1)J_1(\theta^2)]\}}. \tag{6.16}$$

The parameter θ^2 in (6.16) is defined in terms of the coupling constant g according to (6.9) with the optimal value $t=1$.

The expansion of the β function (6.16) in a series in powers of the coupling constant contains all powers of g. It is interesting to compare the resulting coefficients of the VPT of the β function (6.16) with the known coefficients of perturbation theory. From (6.16) we find

$$\beta(g) = 1.5g^2 - 2.25g^3 + 14.63g^4 - 134.44g^5 + 1456.6g^6 - 17627.2g^7 + \cdots . \tag{6.17}$$

In performing the renormalization the counterterms contained only divergent expressions. In dimensional regularization this corresponds to the case where the counterterms include only the pole parts (see Refs. 53 and 54, and also the review of Ref. 55). The corresponding perturbative β function in the five-loop approximation has the form[56]

$$\beta_{\text{perturb}}(g) = 1.5g^2 - 2.83g^3 + 16.27g^4 - 135.80g^5 + 1420.69g^6 + \cdots . \tag{6.18}$$

We note that in the calculation of the β function (6.16) we have used only the lowest nontrivial order of the VPT. For this approximation the agreement between Eqs. (6.17) and (6.18) is quite acceptable. In Ref. 56 a prediction is made, on the basis of summation of the perturbation series, for the six-loop coefficient of the perturbative β function: $\beta_6 = 17\,200 \pm 50$. This value also agrees with the expansion (6.17).

It follows from (6.16) that the β function grows monotonically with increasing coupling constant g and does not have an ultraviolet-stable point (Fig. 4). For large coupling constants the asymptotic behavior of the β function (6.16) is

$$\beta(g) \simeq \frac{3}{10}\frac{\sqrt{\pi}}{\frac{3\pi}{8}-1}g^{3/2}. \tag{6.19}$$

In the literature one can find various predictions for the asymptotic behavior of the β function obtained by different methods. For example, we note that the rate of increase of the β function in (6.19) is larger than the linear growth found in Ref. 57, and smaller than the quadratic growth obtained in Ref. 58.

7. APPLICATION TO QUANTUM CHROMODYNAMICS

Many problems in quantum chromodynamics require the use of nonperturbative methods for their solution. Great efforts based on a wide variety of approaches have been and are being made in this area. In this section we shall consider the use of the VPT method in quantum chromodynamics as a possible method of performing calculations in the nonperturbative regime. Our discussion will be based on Ref. 59. We shall proceed via the construction of the VPT series on the basis of the harmonic variational procedure (we discussed a somewhat different possibility earlier in Ref. 60). It has not yet been possible to prove the convergence of the VPT series for quantum chromodynamics in this case. Nevertheless, there are promising results obtained for simpler models[39,40] which suggest induced convergence.[35] An important feature of this approach is the possibility of using the ideas of VPT in quantum chromodynamics to construct a new expansion parameter which is always smaller than unity for any value of the coupling constant.

To explain the basic idea behind the method, let us consider pure Yang–Mills theory. The inclusion of quarks does not present any problem. The corresponding Lagrangian has the form

$$L_{\text{YM}} = -\tfrac{1}{4}(F_{\mu\nu})^2 - \tfrac{1}{2}gF_{\mu\nu}[A_\mu \times A_\nu] - \tfrac{1}{4}g^2[A_\mu \times A_\nu]^2 + L_{\text{g.f.}} + L_{\text{F.P.}} = L_0(A) + gL_3(A) + g^2L_4(A), \tag{7.1}$$

where $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$, $L_{\text{g.f.}}$ is the gauge addition, and $L_{\text{F.P.}}$ is the Faddeev–Popov Lagrangian.

The expression $L_3(A)$ generates three-gluon and ghost-gluon-ghost vertices. For us it will be important that these interactions are Yukawa interactions. The expression $L_4(A)$ generates four-gluon vertices. We introduce the fields $\chi_{\mu\nu}$ and transform $L_4(A)$ such that the gluon fields and the fields $\chi_{\mu\nu}$ interact in Yukawa fashion. The corresponding transformation has the form

$$\exp\left\{-i\frac{g^2}{4}\int dx[A_\mu \times A_\nu]^2\right\} = \int D\chi \exp\left\{\frac{i}{2}\int dx\chi_{\mu\nu}^2 + i\frac{g}{\sqrt{2}} \times \int dx\chi_{\mu\nu}[A_\mu \times A_\nu]\right\}. \tag{7.2}$$

The functional integral in (7.2) is normalized by the condition

$$\int D\chi \exp\left\{\frac{i}{2}\int dx\chi^2_{\mu\nu}(x)+i\int dx\chi_{\mu\nu}(x)J_{\mu\nu}(x)\right\}$$
$$=\exp\left\{\frac{i}{2}\int dx\,dy\,J_{\mu\nu}(x)\Delta(x,y)_{\mu\nu;\,\mu_1\nu_1}J_{\mu_1\nu_1}(y)\right\}, \tag{7.3}$$

where $\Delta(x,y)$ is the propagator of the field χ

$$\Delta(x,y)^{ab}_{\mu\nu;\mu_1\nu_1}=\delta(x-y)\,\delta_{ab}\delta_{\mu\mu_1}\delta_{\nu\nu_1}. \tag{7.4}$$

Therefore, the action functional can be written as

$$S=S_0(\chi)+S(A,\chi)+S^{\text{Yuk}}_{YM}(A), \tag{7.5}$$

where

$$S(A,\chi)=\tfrac{1}{2}\int dx\,dy\,A^a_\mu(x)[D^{-1}(x,y|\chi)]^{ab}_{\mu\nu}A^b_\nu(y), \tag{7.6}$$

and $D(x,y|\chi)$ is the propagator of the gauge field in the field χ

$$[D^{-1}(x,y|\chi)]^{ab}_{\mu\nu}=[-\partial^2 g_{\mu\nu}\delta_{ab}+g\sqrt{2}f_{abc}\chi^c_{\mu\nu}$$
$$+\text{gauge terms}]\delta(x-y). \tag{7.7}$$

For the Green function we can write

$$G(\cdots)=\langle G_{\text{Yuk}}(\cdots|\chi)\rangle, \tag{7.8}$$

where

$$G_{\text{Yuk}}(\cdots|\chi)=\int DA[\cdots]\exp\{i[S(A,\chi)+S^{\text{Yuk}}_{\text{YM}}(A)]\}, \tag{7.9}$$

$$\langle\cdots\rangle=\int D\chi[\cdots]\exp[iS_0(\chi)]. \tag{7.10}$$

The Green functions $G_{\text{Yuk}}(\cdots|\chi)$ are determined only by Yukawa-type interactions with the gluon propagator $D(x,y|\chi)$. In Fig. 5a we show the full gluon propagator $\langle D_{\text{full}}(x,y|\chi)\rangle$. Diagrams with four-gluon vertices arise in the expansion of $D(x,y|\chi)$ in a perturbation series (Fig. 5b). They are added to the Yukawa diagrams and the standard diagrammatic representation of the perturbation series arises (Fig. 5c).

Following the ideas of the VPT method, we write the Lagrangian in the form

$$L(A,\chi)=L_0(A,\chi)+L_I(A,\chi),$$
$$L_0(A,\chi)=\zeta^{-1}L(A,\chi)+\xi^{-1}L(\chi),$$
$$L_I(A,\chi)=\eta[gL^{\text{Yuk}}_{\text{YM}}(A)-(\zeta^{-1}-1)L(A,\chi)$$
$$-(\xi^{-1}-1)L(\chi)], \tag{7.11}$$

where ζ and ξ are variational parameters. The initial complete Lagrangian $L(A,\chi)$ is of course independent of them. Therefore, the freedom in choosing ζ and ξ can be used to obtain a VPT expansion with specified properties. We recall that here we are trying to construct a new small expansion parameter.

We see from (7.11) that if the parameters introduced satisfy $0<\zeta<1$ and $0<\xi<1$, by writing the Lagrantian in the form (7.11) we "weaken" the interaction term and "strengthen" the free action. As before, we set $\eta=1$ after all the calculations. This parameter will be provided for also in the propagator $D(x,y|\chi)$ in combination with the coupling constant. As a result, for the Green function we obtain

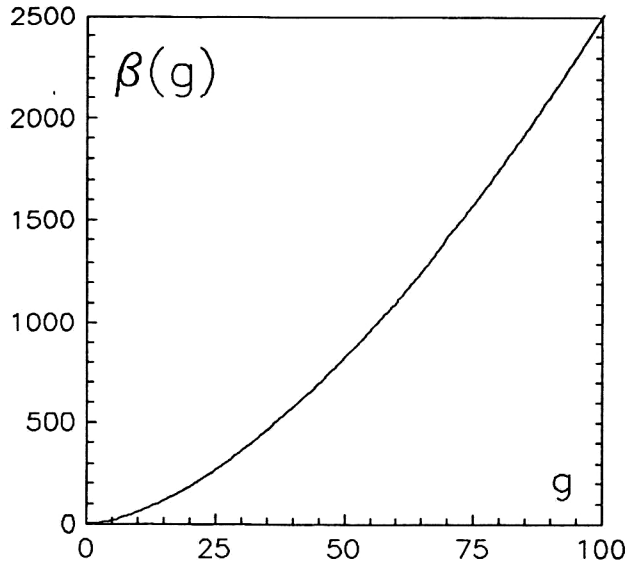

FIG. 4. Graph of the β function given by (6.16).

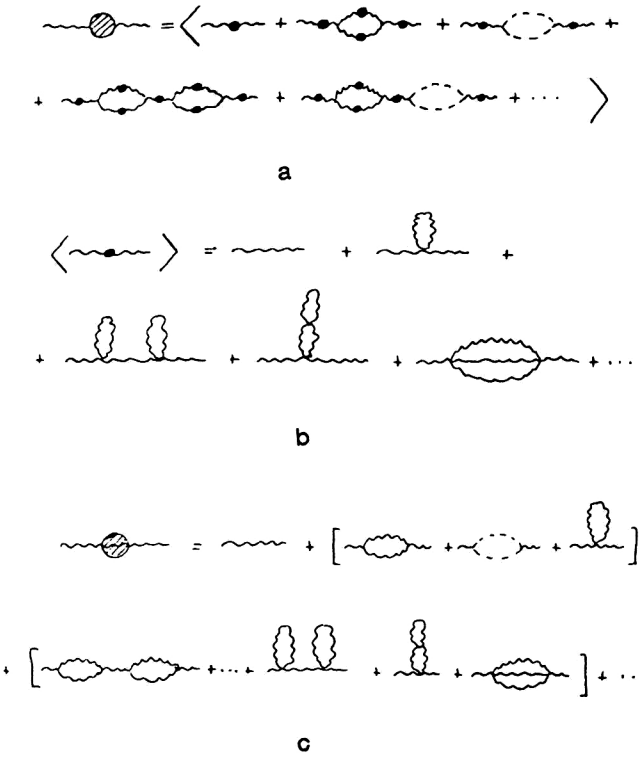

FIG. 5. Diagrammatic representation of the complete gluon propagator according to perturbation theory using the χ transformation. The wavy line with the point corresponds to the function $D(\chi)$.

$$G(\cdots)=\sum_n G_n(\cdots),$$

$$G_n(\cdots)=\frac{1}{n!}\,\eta^n\int D\chi DA[\cdots]$$
$$\times[iS_I(A,\chi)]^n\exp[iS_0(A,\chi)],$$
$$=(i\eta)^n\sum_{k=0}^{n}\frac{1}{(n-k)!k!}\int D\chi DA[\cdots]$$
$$\times[gS_{\rm YM}^{\rm Yuk}(A)]^k[(\zeta^{-1}-1)S(A,\chi)$$
$$+(\xi^{-1}-1)S(\chi)]^{n-k}\exp[iS_0(A,\chi)]. \quad (7.12)$$

To simplify the calculations we redefine $L_0(A,\chi)$ as

$$L_0(A,\chi)\Rightarrow L_0'(A,\chi)=[1+\kappa(\zeta^{-1}-1)]L(A,\chi)$$
$$+[1+\kappa(\xi^{-1}-1)]L(\chi). \quad (7.13)$$

After this any power of

$$[(\zeta^{-1}-1)S(A,\chi)+(\xi^{-1}-1)S(\chi)]$$

in (7.12) can be obtained using the corresponding number of differentiations of the expression $\exp[iS_0'(A,\chi,\kappa)]$ with respect to the parameter κ and then setting $\kappa=1$.

Therefore, from (7.12) and (7.13) we find

$$G_n=\eta^n\sum_{k=0}^{n}\frac{1}{(n-k)!}\left(-\frac{\partial}{\partial\kappa}\right)^{n-k}\langle g_k(\kappa)\rangle, \quad (7.14)$$

where the functions

$$g_k(\kappa)=\frac{i^k}{k!}\int DA[\cdots][gS_{\rm YM}^{\rm Yuk}(A)]^k$$
$$\times\exp\left\{i[1+\kappa(\zeta^{-1}-1)]\int dx\,L(A,\chi)\right\} \quad (7.15)$$

are constructed on the basis of the Yukawa diagrams of Yang–Mills theory with the propagator

$$[1+\kappa(\zeta^{-1}-1)]^{-1}D(x,y|\chi)\to\zeta D(x,y|\chi)$$

for $\kappa=1$. The χ-field propagator contains the factor $[1+\kappa(\xi^{-1}-1)]^{-1}$, which becomes ξ for $\kappa=1$.

The differentiation operator

$$\frac{1}{l!}\left(-\frac{\partial}{\partial\kappa}\right)^l$$

gets rid of the factor $(1-\zeta)^l$ for the gluon propagator and $(1-\xi)^l$ for the χ-field propagator.

It is easy to show that the Nth order of the VPT series coincides with Nth-order perturbation theory up to $O(g^{N+1})$

$$G_{\rm VPT}^{(N)}=\sum_{n=0}^{N}G_n=G_{PT}^{(N)}+O(g^{N+1}). \quad (7.16)$$

Therefore, for small coupling constants the VPT expansion leads to the same results as standard perturbation theory.

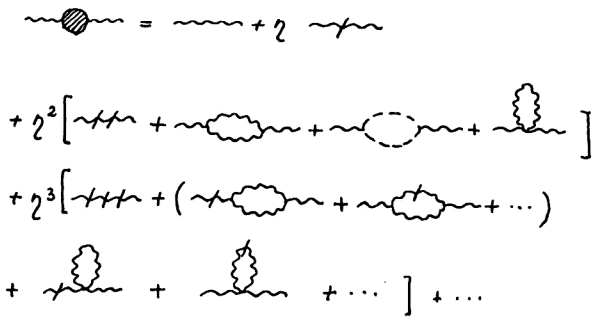

FIG. 6. Diagrams corresponding to the VPT expansion of the complete gluon propagator.

In Fig. 6 we show the diagrams for the complete gluon propagator corresponding to the new expansion. The slashes on the gluon line denote differentiation with respect to the parameter κ and, consequently, presence of the factor $(1-\zeta)$. In the case where the gluon line arose owing to the χ field the corresponding factor is $(1-\xi)$.

The resulting expansion has the following structure:

$$1+\eta(1-\zeta)+\eta^2[(1-\zeta)^2+g^2\zeta^3+g^2\xi]+\eta^3[(1-\zeta)^3$$
$$+g^2\zeta^3(1-\zeta)+g^2\xi(1-\zeta)+g^2\xi(1-\xi)]+\cdots. \quad (7.17)$$

It is easy to verify that this structure is preserved also for other Green functions.

We now see that if we choose

$$\xi=\zeta^3, \quad (1-\zeta)^2\sim g^2\zeta^3 \quad (7.18)$$

we find that the nth term of the VPT series contains the usual factor $(1-\zeta)^n$, and the second condition in (7.18) guarantees that the inequality $(1-\zeta)<1$ is satisfied for all positive values of the coupling constant.

Using dimensional regularization with $d=4-2\varepsilon$, for the renormalization constants $Z_{1\rm YM}$ and $Z_{3\rm YM}$ (see Ref. 61 for the corresponding notation) in leading order VPT we find

$$Z_{1\rm YM}=1+\lambda\zeta^3[N(\tfrac{17}{6}-\tfrac{3}{2}\alpha_G)-\tfrac{4}{3}N_f]\,\frac{1}{2\varepsilon}, \quad (7.19)$$

$$Z_{3\rm YM}=1+\lambda\zeta^3[N(\tfrac{13}{3}-\alpha_G)-\tfrac{4}{3}N_f]\,\frac{1}{2\varepsilon}. \quad (7.20)$$

Here we have already introduced quarks, which are easily incorporated into the scheme described above owing to their Yukawa interaction with gluons. The quantity $\lambda=\alpha_S/(4\pi)$ and the parameter ζ are related to each other as

$$(1-\zeta)^2=C\lambda\zeta^3 \quad (7.21)$$

with positive constant C.

From (7.19) and (7.20) we find

$$\lambda_0=\mu^{2\varepsilon}Z_{1\rm YM}^2Z_{3\rm YM}^{-3}\lambda=\lambda\mu^{2\varepsilon}\left[1-\lambda\zeta^3\,\frac{b_0}{\varepsilon}\right], \quad (7.22)$$

where $b_0=11-2/3N_f$. From this for the β function we obtain

$$\beta(\lambda)=\lim_{\varepsilon\to 0}\mu^2\frac{\partial\lambda}{\partial\mu^2}=-b_0\left(\lambda\frac{\partial}{\partial\lambda}-1\right)(\lambda^2\zeta^3)=$$

$$-2b_0\lambda^2\frac{\zeta^4}{3-\zeta}. \tag{7.23}$$

Solving the corresponding renormalization group equation, we find

$$\ln\frac{Q^2}{\Lambda^2}=\frac{C}{2b_0}f(\zeta), \tag{7.24}$$

where the function $f(\zeta)$ has the form

$$f(\zeta)=\frac{2}{(1-\zeta)^2}+\frac{12}{1-\zeta}+21\ln\frac{\zeta}{1-\zeta}-\frac{9}{\zeta}. \tag{7.25}$$

It is easily seen that in the perturbative region, when $Q^2\gg\Lambda^2(\zeta\sim 1)$ from (7.21), (7.24), and (7.25) we obtain the well known one-loop result:

$$\lambda(Q^2)=\frac{1}{b_0\ln(Q^2/\Lambda^2)}.$$

For decreasing Q^2 and, accordingly, increasing $\lambda(Q^2)$ the logarithmic growth is replaced by power-law growth $\lambda(Q^2)\sim Q^{-2}$.

To construct the nonrelativistic quark-antiquark potential we use the expression

$$V(r)=-\frac{16\pi}{3}\int\frac{dQ}{(2\pi)^3}\exp(iQr)\frac{\alpha_S(Q^2)}{Q^2}. \tag{7.26}$$

In order to find $V(r)$ we approximate $\alpha_S(Q^2)$ by the expression

$$\alpha_S^{\text{appr}}(Q^2)=\frac{a_1}{Q^2}+\frac{4\pi}{b_0\ln(a_2+Q^2/\Lambda^2)} \tag{7.27}$$

with two parameters a_1 and a_2, which is the minimal form allowing the expression found for $\alpha_S(Q^2)$ to approach the appropriate region of Q^2. The result of this approximation for $\Lambda_{\overline{\text{MS}}}=140$ MeV, $N_f=3$, and $C=0.82$ with the parameters $a_1=0.260$ GeV^2 and $a_2=897.0$ is shown in Fig. 7. The conversion of $\Lambda_{\overline{\text{MS}}}$ to the MS scheme used here was accomplished using the usual two-loop expression. Substituting (7.27) into (7.26), we obtain the nonrelativistic quark-antiquark potential shown by the solid line in Fig. 8. The dashed line shows the phenomenological potential taken from Ref. 62 and agreeing with meson spectroscopy.

Let us also consider the effective quark mass. The corresponding anomalous dimension in our approximation has the form

$$\gamma_m=4\lambda\frac{\partial}{\partial\lambda}(\lambda\zeta^3). \tag{7.28}$$

Using (7.21) and (7.24) and solving the equation

$$\frac{d\ln m(Q^2)}{d\ln Q^2}=-\gamma_m, \tag{7.29}$$

we find

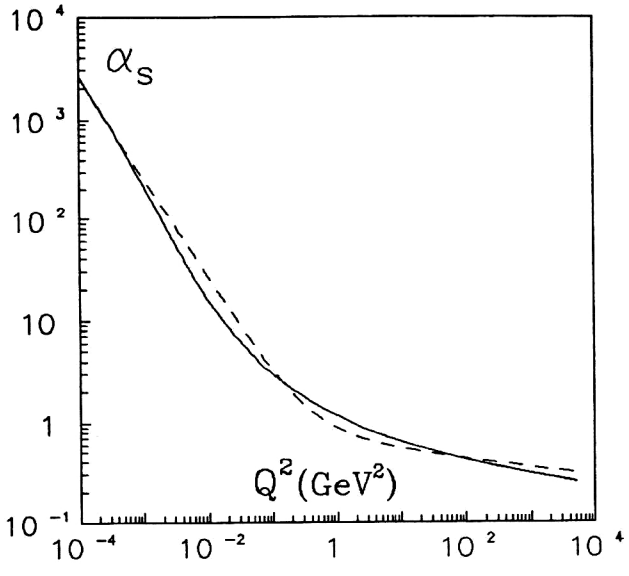

FIG. 7. Result of approximating $\alpha_S(Q^2)$ by Eq. (7.27). The solid line corresponds to $\alpha_S(Q^2)$ obtained using the VPT. The dashed line shows the result of the approximation using Eq. (7.27).

$$m(Q^2)=\hat{m}\left[\frac{b_0}{2\pi}\alpha_S(Q^2)\right]^{4/b_0}, \tag{7.30}$$

where $\hat{m}$ is the renormalization-invariant mass parameter. The quantity $m(Q^2)$ for the same parameters used earlier and for $\hat{m}=7$ MeV is shown in Fig. 9.

The singular infrared behavior of the invariant charge

$$\alpha_S(Q^2)\sim Q^{-2} \tag{7.31}$$

has been discussed often in the literature (see, for example, the review of Ref. 63). If in momentum space the quark interaction potential is written as

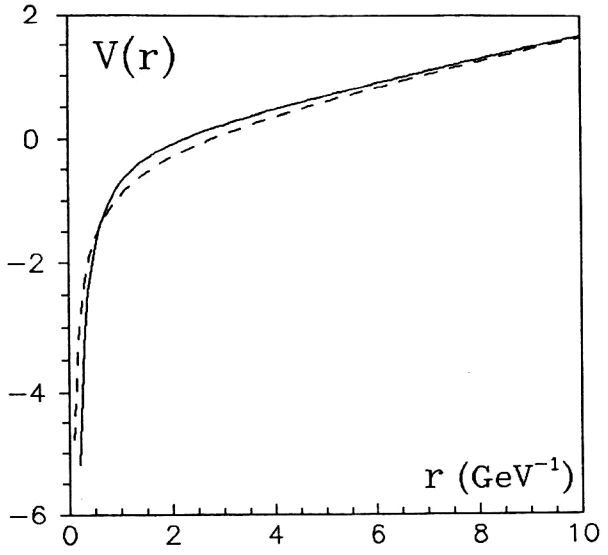

FIG. 8. Nonrelativistic quark interaction potential. The solid line is the potential obtained here and the dashed line is the phenomenological potential taken from Ref. 62.

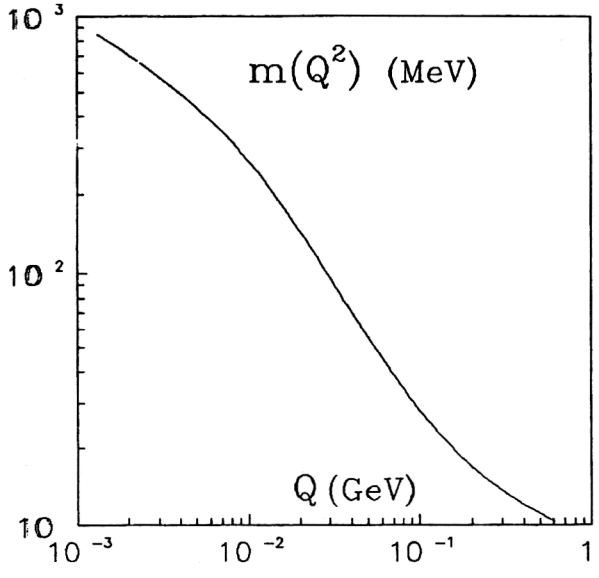

FIG. 9. Running quark mass $m(Q^2)$ as a function of Q for $\hat{m}=7$ MeV.

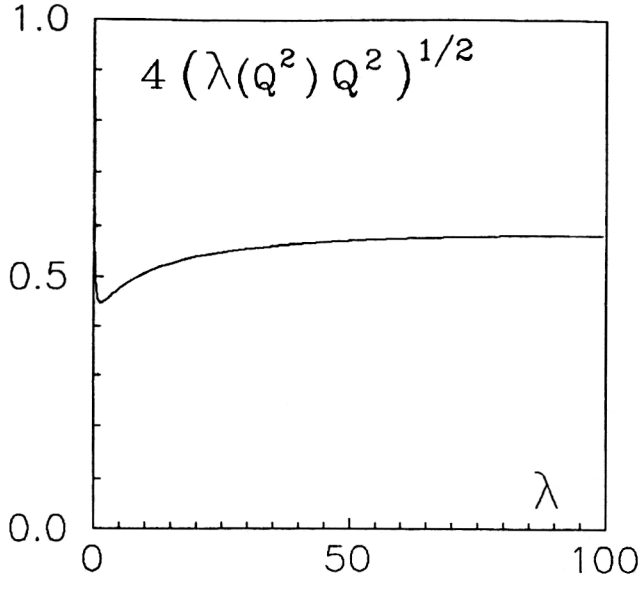

FIG. 11. The quantity $4\sqrt{Q^2\lambda(Q^2)}$ as a function of λ.

$$V(q^2)=-\frac{16\pi}{3}\frac{\alpha_S(q^2)}{q^2}, \tag{7.32}$$

this behavior ensures the linear growth of the potential in coordinate space $V(r)\sim r$ at large distances.

The asymptotic behavior (7.31) is related to the asymptotic form of the β function $\beta(\lambda)\to-\lambda$ at large coupling constants. This infrared picture of quantum chromodynamics is consistent with the Schwinger-Dyson equations[64] and with lattice calculations.[65] The behavior of $-\beta(\lambda)/\lambda$ for the parameters quoted above is shown in Fig. 10, from which we see that at large λ its value is close to unity. In Fig. 11 we show the quantity $4\sqrt{Q^2\lambda(Q^2)}$, for which the phenomenological value at large coupling constants following from meson spectroscopy is 0.59 GeV (Ref. 63). We see that this value appears quite clearly in the graph.

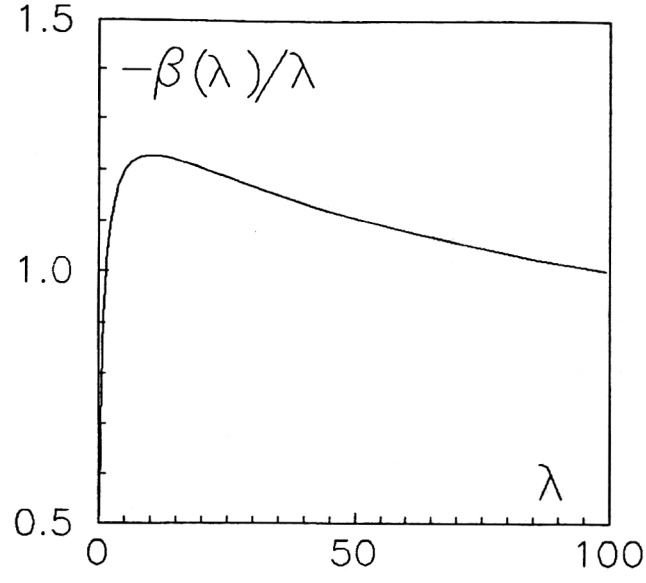

FIG. 10. The quantity $-\beta(\lambda)/\lambda$ as a function of λ.

8. CONCLUSION

Here we have considered an approach to quantum field theory based on variational perturbation theory. The original action functional is rewritten using some variational addition and an expansion in the effective interaction is made. Therefore, in contrast to many nonperturbative approaches, in the VPT method the quantity under study from the beginning is written in the form of a series which makes it possible to calculate the needed corrections. The VPT method thereby allows for the possibility of determining the degree to which the principal contribution found variationally using some optimization principle adequately reflects the problem in question and of determining the region of applicability of the results obtained.

The possibility of performing calculations using this approach is based on the fact that the VPT method, like standard perturbation theory, uses only Gaussian functional quadratures. Here, of course, the VPT series possesses a different structure and, in addition, some of the Feynman rules are modified at the level of the propagators and vertices. The form of the diagrams themselves does not change, which is very important technically. The diagrams contributing to the Nth order of the VPT expansion are of the same form as those contributing to the Nth order of ordinary perturbation theory.

The variational parameters arising in the VPT method allow the convergence properties of the VPT series to be controlled. In the case of the anharmonic variational procedure for φ^4 field theory there is a finite region of parameter values in which the VPT series converges for all positive values of the coupling constant. For the harmonic variational procedure there are indications, as discussed above, that the

VPT series will also converge in the sense of so-called induced convergence, when the variational parameters are "twisted" from order to order. The possibility of constructing Leibniz series in field theory is also interesting. In this case the first few terms of the series can be used to obtain two-sided estimates of the sum of the series, and the existence of variational parameters makes it possible to narrow these estimates the maximum amount in a given order of VPT.

Here we have presented different methods for using VPT to derive the nonperturbative Gaussian effective potential. For example, it arises as the first nontrivial approximation in the variational correction of the one-loop contribution both at the level of the harmonic variational procedure and for the anharmonic one. However, the properties of the resulting series are different. If we leave the mass parameter optimizing the effective potential fixed according to the first order, which can be convenient owing to the relative simplicity of the equation which it satisfies, convergence of the series can be ensured only by the anharmonic method of introducing the trial functional. Convergence for the harmonic variational procedure can only be obtained in the sense of induced convergence, by fine-tuning the variational parameter from order to order. Nevertheless, the harmonic method of constructing the VPT is very attractive owing to its simplicity and the possibility of generalization to other field theoretic models. This is why we have selected it for the application of VPT to quantum chromodynamics.

This approach we have proposed to quantum chromodynamics is based on an expansion in which a new small parameter is used. This parameter obeys an equation whose solution is always smaller than unity for any value of the coupling constant. Therefore, while remaining within the limits of applicability of this expansion it is possible to deal with considerably lower energies than in the case of perturbation theory. In the present study we have considered only the first nontrivial order. The results obtained for the nonperturbative β function and the quark interaction potential look promising. Of course, the question remains of the stability of these results and the use of this scheme to describe many other quantities and processes. We are in the process of studying this. Without getting ahead of ourselves, we can make the following two remarks. First, for the method described here we can count on the induced convergence of the VPT series. As noted above, a rigorous proof of this type of convergence exists only for simple models.[39,40] In the case of quantum chromodynamics it is not yet possible to suggest a rigorous demonstration of convergence. However, our preliminary calculations for the next highest order indicate that the picture described above remains valid. Second, an important feature of this approach is the fact that for sufficiently small α_S it reproduces the standard perturbation theory. Therefore, all the high-energy physics is preserved in our method. In going to lower energies where standard perturbation theory ceases to be valid $[\alpha_S(Q^2)\sim 1]$, our expansion parameter remains small and we do not find ourselves outside the region of applicability of our approach.

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