Inequivalent representations and phase structure of the $(\varphi^4)_d$ theory

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Nonperturbative methods of investigating the vacuum structure of quantum-field models (variational approach, constructive quantum field theory, the method of canonical transformations, etc.) are reviewed. The problem of unitarily inequivalent representations of the canonical commutation relations is discussed. The most detail is devoted to the method of investigating the phase structure of superrenormalizable theories of a self-interacting scalar field for arbitrary coupling constant and temperature based on canonical transformations and the renormalization-group approach. In this approach, canonical transformations are used to introduce a set of trial vacuum vectors (inequivalent representations of the canonical commutation relations). The leading dynamical contributions that form the ground state of the system are taken into account by means of the renormalization-group equations. The criteria for selecting the ground state are based on comparison of the free-energy densities and the effective coupling constants that characterize each representation. The mechanisms of rearrangement of the ground state investigated in the review can be used to analyze realistic quantum field theories.

1. INTRODUCTION

About 20 years ago. Coleman and Weinberg[1] established that radiative corrections can lead to spontaneous symmetry breaking (SSB) in theories in which the semiclassical (tree) approximation does not admit such breaking. At about the same time, Kirzhnits and Linde[2,3] showed that in some field theories with SSB postulated at zero temperature the symmetry is restored when the temperature is raised (see also Refs. 4–7).

These conclusions are based on the idea that a constant classical scalar field (a condensate) arises in the whole of space. The appearance of such a condensate signifies a rearrangement of the ground state of the system, as a result of which fields that interact with the scalar field change their mass. The nature of the interaction of the fields with each other is also changed. It was found that quantum field systems have a complicated phase (vacuum structure) and that at certain values of the coupling constants and temperature phase transitions can occur in them.

The methods proposed in Refs. 1–3 are based on a loop expansion of an effective potential, and they are therefore only valid under the condition of weak coupling. Further progress in the investigation of the phase structure of field systems is associated with the use of nonperturbative methods. The attention of investigators was mainly concentrated on theories of a scalar field with the Lagrangians

$$L(x)=\tfrac{1}{2}\,\varphi(x)(\Box-m^2)\varphi(x)-\frac{g}{4}\,\varphi^4(x), \tag{1.1}$$

$$L(x)=\frac{1}{2}\,\varphi(x)\left(\Box+\frac{1}{2}\,m^2\right)\varphi(x)-\frac{g}{4}\,\varphi^4(x), \tag{1.2}$$

$$L(x)=\frac{1}{2}\sum_i^N\,\varphi_i(x)(\Box-m^2)\varphi_i(x)-\frac{g}{4}\left(\sum_i^N\,\varphi_i^2(x)\right)^2 \tag{1.3}$$

in space–time R^d ($d=2$, 3, 4) at finite and zero temperature T. Here, $x=(\mathbf{x},t)$.

The Lagrangians (1.1) and (1.2) describe a single-component scalar field; they are invariant under the substitution $\varphi\to-\varphi$. The Lagrangian (1.3) describes a $O(N)$ multiplet of scalar fields and is invariant with respect to $O(N)$ transformations and also the substitution $\varphi_i\to-\varphi_i$. The parameters m and g are positive.

If the dimensionless parameters $G=g/2\pi m^{4-d}$ and $\theta=T/m$ are sufficiently small, then in the quantum theory the Lagrangians (1.1) and (1.3) describe a symmetric interaction, while the Lagrangian (1.2) corresponds to a spontaneously broken symmetry.

In the framework of constructive quantum field theory, Simon and Griffiths,[8,9] Glimm and Jaffe,[10,11] McBryan and Rosen,[12] and others obtained some rigorous theorems that prove the existence of nontrivial two-dimensional theories of a self-interacting scalar field. These theorems establish the existence of a phase transition in two-dimensional[9,10,11] and three-dimensional[12] φ^4 field theories at zero temperature. Indications were found that these are second-order transitions, but the proof of this is not complete.[11,12] At the same time, in the framework of constructive quantum field theory it did not prove possible to obtain any information about the critical value of the coupling constant or find the explicit dependence of the mass and order parameter on the coupling constant.

Mostly related to this subject are investigations of the so-called triviality problem of the $(\varphi^4)_d$ theory. It has been proved rigorously that for $d>4$ such a theory is either without interaction (the bare charge is equal to zero in the limit in which the regularization is lifted) or unstable (the charge is negative).[13] In the case $d=4$, it has not been possible to arrive at any final conclusions.[14]

From the physical point of view, a very attractive approach to the problem of phase structure is provided by the variational method of a Gaussian effective potential (GEP). This direction in field theory was initiated by Barnes and

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Ghandour,[15] Bardeen and Moshe,[16] Stevenson,[17] and Consoli *et al.*[18] Other investigations in this direction were made by Chang,[19] Magruder,[20] Baym and Grinstein,[21] and Grassi, Hakim, and Sivak.[22] Their studies differ in the renormalization methods, the methods used to take into account thermal effects, the use or not of the $1/N$ to study $O(N)$-invariant systems, etc. In their study of the $(\varphi^4)_2$ theory, Polley and Ritschel[23] went beyond the GEP (so-called post-Gaussian approximation).

In the framework of the variational approach, the critical values of the coupling constant and the temperature and the dependence of the mass and the order parameter (condensate) on the coupling constant and temperature were found approximately in some theories of a self-interacting scalar field. Polley and Ritschel[23] obtained a second-order phase transition in the $(\varphi^4)_2$ theory as predicted by the theorems mentioned above (see also Ref. 63).

At the same time, the specific features of the variational methods in quantum field theory make their results very unreliable if the theory contains divergences in the higher orders of perturbation theory (see Feynman's paper in Ref. 24). Just such a situation occurs in the models (1.1)–(1.3) for $d>2$. It has been noted by many authors[25–27] that this problem resides in the very formulation of the problem in the variational approach. On the one hand, the variational principle is applied to a fixed Hamiltonian with given set of bare parameters (masses and coupling constants), while the physically significant quantity is the effective potential with given set of renormalized parameters. After renormalization, the basic inequality

$$U^{+}_{\mathrm{eff}}(\varphi)\equiv\min_{\psi}\langle\psi|H|\psi\rangle\geqslant U_{\mathrm{eff}}(\varphi) \tag{1.4}$$

of the variational approach is useless, since the ultraviolet divergences of the variational estimate U^{+}_{eff} and the exact effective potential U_{eff} are different, and therefore the difference between U^{+}_{eff} and U_{eff} is infinitely large.[25] The inequality (1.4) can be made meaningful only when the Hamiltonian is an operator on the state space, i.e., if the theory contains only normally ordered divergences [(1.1)–(1.3) for $d=2$].

Another problem is associated with the impossibility of verifying the accuracy of the approximation in the variational method even when the inequality (1.4) does hold.[28]

We also mention the original approach of Chang[29] and Magruder[20] which is based directly on allowance for the structure of the renormalization of the exact potential in the framework of perturbation theory. Chang and Magruder investigated the phase structure of the model (1.2) in R^3 and found that with increasing coupling constant the symmetry is restored. This contradicts the results of the GEP approximation.[30]

In this paper we use a modification of the method of canonical transformations that enables us to take into account the structure of the renormalization in the higher orders and monitor the accuracy of the approximation. This approach was first used in Ref. 31 to investigate the models (1.1) and (1.3) in R^2. The generalization to the more complicated cases of three- and four-dimensional theories and to systems at finite temperature was made in Refs. 32–36. The final formulation of the method is given in Ref. 35.

The essence of the method consists of combination of two methods of quantum field theory: canonical transformations and the renormalization group (RG). The idea of such a combination arises from fundamental properties of local quantum field theory, namely, inequivalent representations of the canonical commutation relations (CCR) and ultraviolet divergences (see, for example, Refs. 47 and 57). From the physical point of view, the existence of inequivalent representations means that the vacuum state is not unique. At the same time, dynamical instability of the vacuum is associated with the radiative corrections to the physical parameters of the system. Renormalization (R) corresponds essentially to allowance for the leading radiative corrections. It is therefore to be expected that the R structure of the model will contain the basic (at least qualititative) information about its vacuum structure (see also Ref. 8).

In accordance with this intuitive motivation, we take as our point of departure the following:

- phases are manifested in quantum field theory as inequivalent representations;
- the basic information about the phase structure of the theory is contained in the renormalization structure.

If the renormalized coupling constant G is small and the temperature is equal to zero, then to quantize the models (1.1)–(1.3) one can use the canonical procedure with the Fock representation for particles with renormalized mass m. The procedure for constructing the S matrix presupposes a fixing of the renormalization scheme. Having this in mind, we wish to know what is the field system at other values of G and θ for a fixed renormalization scheme. We formulate the problem as follows:

What representation of the CCR is appropriate for different values of G and θ, and what physical picture corresponds to this representation?

By different phases of the system we shall understand inequivalent representations present in the theory for given G and θ.

The review is organized as follows. In the second and third sections we briefly present the method of canonical quantization and the renormalization-group formalism; the method used in the paper is based essentially on them. In the fourth section, we discuss the interconnection between the problem of the vacuum structure of a quantum field system an inequivalent representations of the CCR, and we consider the most important manifestations of inequivalent representations in quantum field theory. In the fifth section, the various aspects of quantization, inequivalent representations, canonical transformations, and the renormalization group that motivate the method of canonical transformations in quantum field theory are unified in a consistent formulation of this method. In the remaining sections, we investigate the phase structure of specific models.

2. CANONICAL QUANTIZATION AND THE *S* MATRIX

To study the behavior of a quantum field system when its parameters are varied, it is first of all necessary to describe the system for certain fixed values of the parameters. In other words, it is necessary to determine a kind of "boundary condition." Without this, the formulation of the problem will not be complete.

We shall assume that *at zero temperature in the weak-coupling limit* the self-interacting scalar field can be described in the framework of the *standard formalism of canonical quantization and construction of the S matrix by perturbation theory* (see Ref. 42). We write the classical Lagrangian of the system in the form

$$L(x)=\tfrac{1}{2}\varphi(x)(\Box-m^2(\mu))\varphi(x)-\frac{g(\mu)}{4}\varphi^4(x)$$
$$+\tfrac{1}{2}(Z_2-1)\varphi(x)\Box\varphi(x)-\tfrac{1}{2}\delta m^2(\mu)\varphi^2(x)$$
$$-\tfrac{1}{4}(Z_1-1)g(\mu)\varphi^4(x)-\delta E. \tag{2.1}$$

We have here used the standard notation for the renormalized field φ, mass $m(\mu)$, and coupling constant $g(\mu)$. The parameter μ characterizes the scale of the renormalization, which is implemented by means of the counterterms given in the second row of (2.1). Below we shall consider questions of renormalization specially.

As canonical variables, we choose the renormalized field φ and the canonically conjugate momentum, which is equal to

$$\pi(\mathbf{x},x_0)=\frac{\delta}{\delta\dot\varphi(\mathbf{x},x_0)}\int dyL(\mathbf{y},x_0)=Z_2\dot\varphi(\mathbf{x},x_0).$$

The canonical commutation relations have the form

$$[\varphi(x),\varphi(y)]_{x_0=y_0}=[\pi(x),\pi(y)]_{x_0=y_0}=0,$$
$$[\varphi(x),\pi(y)]_{x_0=y_0}=i\delta(\mathbf{x}-\mathbf{y}). \tag{2.2}$$

The Hamiltonian density is

$$H=\pi\dot\varphi-L=H_0+H_I+H_{ct},$$
$$H_0=\tfrac{1}{2}[\pi^2+(\nabla\varphi)^2+m^2(\mu)\varphi^2],\quad H_I=\tfrac{1}{4}g(\mu)\varphi^4,$$
$$H_{ct}=\frac{1}{2}\left[\left(\frac{1}{Z_2}-1\right)\pi^2+(Z_2-1)(\nabla\varphi)^2+\delta m^2(\mu)\varphi^2\right]$$
$$+\tfrac{1}{4}(Z_1-1)g(\mu)\varphi^4. \tag{2.3}$$

In the interaction representation, H is divided into the free part H_0 (with respect to which the state space is constructed) and the interaction Hamiltonian. The operators φ and π can be represented in the form

$$\varphi(x)=\int\frac{d\mathbf{k}}{2\pi}\frac{1}{\sqrt{2\omega}}[a(\mathbf{k})e^{i\mathbf{kx}-i\omega x_0}+a^+(\mathbf{k})e^{-i\mathbf{kx}+i\omega x_0}],$$
$$\pi(x)=\frac{1}{i}\int\frac{d\mathbf{k}}{2\pi}\sqrt{\frac{\omega}{2}}[a(\mathbf{k})e^{i\mathbf{kx}-i\omega x_0}$$
$$-a^+(\mathbf{k})e^{-i\mathbf{kx}+i\omega x_0}], \tag{2.4}$$
$$\omega(\mathbf{k})=\sqrt{\mathbf{k}^2+m^2(\mu)},\quad [a(\mathbf{k}),a^+(\mathbf{k}')]=\delta(\mathbf{k}-\mathbf{k}').$$

The creation, $a^+(\mathbf{k})$, and annihilation, $a(\mathbf{k})$ operators are defined on the Fock space of free particles with mass $m(\mu)$ and vacuum vector $|0\rangle$ satisfying the conditions

$$a(\mathbf{k})|0\rangle=0\forall\mathbf{k},\quad \langle 0|0\rangle=1. \tag{2.5}$$

In the framework of perturbation theory, the S matrix is constructed by iterations in powers of the interaction Hamiltonian (H_I+H_{ct}), so that the final expression takes the form (see, for example, Ref. 43)

$$S=\lim T\exp\left\{-i\int d^dx[H_I(x)+H_{ct}(x)]\right\}, \tag{2.6}$$

where the operators H_I and H_{ct} are expressed in the interaction representation, and the symbol lim means the lifting of the ultraviolet regularization. The operator H_{ct} is chosen in such a way that each term of the perturbation series for the S matrix is finite when the regularization is lifted. The regularization can be introduced directly in the perturbation series (one can use dimensional regularization,[45,49] the introduction of a form factor,[44,45,57] etc.).

Note that the formalism of thermo field dynamics (TFD)[58,64,65] makes it possible to include thermal effects in a natural manner in the framework of canonical quantization. We shall consider this in detail in the two final sections.

The dimensionless parameters of the model are the renormalized coupling constant $g(\mu)$ and (at nonzero temperature T) $\theta=T/m(\mu)$. For space–time dimension $d<4$, the dimensionless (perturbative) coupling constant is determined by the ratio $G=g/m^{4-d}(\mu)$.

For what follows, it is important to note that the scheme for constructing the S matrix described above is based, in particular, on the following assumptions:

1) the equal-time commutation relations (2.2) are satisfied, and the set of canonical variables is complete;

2) there exists a Poincaré-invariant normalizable vacuum;

3) the vacuum state is unique.

In addition, since the expression (2.6) for the S matrix is obtained by iteration, it is assumed that perturbation theory is valid and, therefore, the renormalized coupling constant is small: $g(\mu)\ll 1$.

The procedure given above also reflects the fact that for the construction of the S matrix it is not sufficient to specify the Lagrangian but one must also determine the "rules of calculation" of the coefficient functions in the perturbation series.[42] This need is due to the presence of ultraviolet divergences and can be reduced to a freedom in the definition of the time-ordered product of operators at coincident points. When the operator H_{ct} is introduced into the definition of the S matrix (2.6), this arbitrariness in the time-ordered product is replaced by an arbitrariness in the choice of the renormalization scheme (R scheme), *which must also be fixed.*

If the renormalization is realized on the mass shell, i.e., by construction the renormalized mass is equal to the physical (pole) mass and the residue of the two-point Green's function at the pole is equal to unity, then the Fock space mentioned above describes the asymptotic in-states and out-states.

3. THE RENORMALIZATION GROUP

3.1. Renormalization schemes

Determination of the S matrix (2.6) requires the renormalization scheme to be fixed. Different methods are used to eliminate ultraviolet divergences. They can be divided into two classes (see, for example, Ref. 48): the class of mass-independent R schemes and R schemes based on the subtraction from a divergent diagram of its value at certain values of the external momenta (for example, the canonical μ scheme, subtraction at zero external momenta, etc.).

An example of the mass-independent renormalization prescriptions is the minimal subtraction (MS) scheme (see, for example, Refs. 45, 49, and 50). Essentially, the renormalization in this scheme reduces to the prescription that in the framework of dimensional regularization one subtracts from the diagrams only the poles with respect to the variable $\varepsilon=d_{\mathrm{ph}}-d$, where d_{ph} is the dimension of physical space–time. Here, an arbitrary parameter μ having the dimensions of mass already arises on the regularization. Its value is not fixed by the MS scheme by itself. For this, it is necessary to specify some additional condition.

The canonical μ scheme that we just mentioned is determined by the following conditions on the renormalized propagator (a):

$$D(p^2)\to\frac{i}{p^2-m^2(\mu)+i0}\ \text{as}\ p^2\to\mu^2, \tag{3.1}$$

and the renormalized four-point vertex function (b):

$$\Gamma^{(4)}(s,t,u)=g(\mu)\quad\text{for}\ s=g=u=\tfrac{4}{3}\mu^2,$$

where $m(\mu)$ and $g(\mu)$ are the renormalized mass and renormalized coupling constant. The Mandelstam variables s, t, u are related to the external momenta in the usual manner:

$$s=(p_1+p_2)^2,\quad t=(p_1-p_3)^2,\quad u=(p_1-p_4)^2.$$

In the φ^4 theory, the subtraction point μ must satisfy the inequality

$$\mu^2\leqslant 9m^2(\mu),$$

and is otherwise arbitrary. Thus, as in the MS scheme, the renormalization scale μ must be fixed additionally.

In the determination of the renormalization scheme in the general case one chooses a certain one-parameter R_μ class of prescriptions and, in addition, fixes the renormalization scale μ. This last is done most naturally in terms of the parameters of the theory itself. In our case, such a method consists of fixing the ratio $m(\mu)/\mu(m(\mu)\neq 0)$.

Thus, we shall say that the renormalization scheme is fixed if:

- *a one-parameter R_μ class of renormalization prescriptions has been chosen;*
- *the renormalization scale has been fixed by the relation $m(\mu)/\mu=C$, where $m(\mu)$ is the renormalized mass, and C is some number.*

To conclude this section, we note that the renormalization condition on the mass shell corresponds to the μ scheme for $\mu+m$ $(C=1)$ [see (3.1)]. In this case, the renormalized and physical masses are equal: $m=m_{\mathrm{ph}}$. We shall use this simple fact to analyze the phase structure of the model (2.1) in R^4.

3.2. Renormalization-group equations

We rewrite the Lagrangian (2.1) in terms of the bare field φ_B, bare mass m_B, and bare coupling constant g_B:

$$L(x)=\tfrac{1}{2}\varphi_B(x)(\Box-m_B^2)\varphi_B(x)-\frac{g_B}{4}\,\varphi_B^4(x).$$

The bare and renormalized quantities are related by

$$\varphi_B=\sqrt{Z_2}\varphi,\quad m_B^2=Z_m m^2,\quad g_B=Z_g g.$$

The idea of renormalization invariance is that a change of the renormalization scheme, which affects the constants Z_i, is compensated by a change of the renormalized field, mass, and coupling constant, so that the bare quantities are unchanged. This has the consequence that the S matrix of the theory does not depend on the choice of the R scheme. This idea acquires a constructive significance it we have found the connection between the renormalized quantities in different renormalization schemes. Such a connection is established by the renormalization-group (RG) equations.[42,45] We give here expressions that we shall need later.

If the change in the renormalization prescription consists solely of a change of the renormalization scale μ in the framework of one R_μ class, then the RG equations have the form[45]

$$\begin{aligned}
&\nu\,\frac{dg(\nu)}{d\nu}=\beta\left(g(\nu),\,\frac{m(\nu)}{\nu}\right),\\
&\frac{\nu}{m^2(\nu)}\,\frac{dm^2(\nu)}{d\nu}=-\gamma_m\left(g(\nu),\,\frac{m(\nu)}{\nu}\right),\\
&\nu\,\frac{d\zeta(\nu)}{d\nu}=-\frac{1}{2}\,\gamma\left(g(\nu),\,\frac{m(\nu)}{\nu}\right),
\end{aligned} \tag{3.2}$$

where ζ is the constant of the finite renormalization of the field for change of the scale μ. The boundary conditions are

$$g(\nu)=g,\quad m(\nu)=m,\quad \zeta(\nu)=1\quad\text{for}\ \nu=\mu. \tag{3.3}$$

The renormalization-group functions $\bar\beta$, $\bar\gamma_m$, and $\bar\gamma$ are determined by a system of equations that follow from the requirement of invariance of the bare quantities:

$$\begin{aligned}
&\bar\beta\left(g(\nu),\,\frac{m(\nu)}{\nu}\right)=-\nu\,\frac{d}{d\nu}\ln Z_g,\\
&\bar\gamma_m\left(g(\nu),\,\frac{m(\nu)}{\nu}\right)=\nu\,\frac{d}{d\nu}\ln Z_m,\\
&\bar\gamma\left(g(\nu),\,\frac{m(\nu)}{\nu}\right)=\nu\,\frac{d}{d\nu}\ln Z_2.
\end{aligned} \tag{3.4}$$

The functions β, γ_m, and γ are obtained from the solutions of the system (3.4) by going to the limit $\beta=\lim\bar\beta$, $\gamma_m=\lim\bar\gamma_m$, $\gamma=\lim\bar\gamma$, which corresponds to lifting of the ultraviolet regularization. If we use a mass-independent R scheme, then Eqs. (3.2) can be readily solved in general form.

The traditional use of the RG formalism consists in an analysis of the asymptotic behaviors of the Green's functions with respect to the energy variables. In other words, one is interested in the dependence of the renormalized quantities (charge, mass, Green's functions) on the renormalization scale ν. This dependence is described precisely by Eqs. (3.2).

In this paper, we investigate a problem of a different kind. We are interested in the behavior of the ground state when we change the parameters g and m contained in the boundary conditions (3.3) while the renormalization scheme is kept fixed. The renormalization group is also helpful in the solution of this problem.

4. INEQUIVALENT REPRESENTATIONS OF THE CANONICAL COMMUTATION RELATIONS

The problem of unitarily inequivalent representations of the canonical commutation relations arose in van Hove's model.[51] A detailed mathematical analysis of this problem was first given by Friedrichs.[52] Further results were obtained by Haag,[54] and Wightman *et al.*[53,56] A detailed analysis of the whole group of questions relating to inequivalent representations is contained in the monographs of Refs. 74, 57, and 58, while in Ref. 59 there is a discussion of the axiomatics of a relativistically invariant canonical field theory with nonseparable Hilbert state space.

4.1. Infinite number of degrees of freedom

A characteristic feature that distinguishes a field from other physical systems is that it has an infinite (with the power of the continuum) number of degrees of freedom.

We shall consider a normalization box with periodic boundary conditions. A neutral scalar field can be expanded with respect to mutually independent normal oscillations of plane-wave type:

$$\varphi(x)=V^{-1/2}\sum_{\mathbf{k}}[2\omega(\mathbf{k})]^{-1/2}[a_{\mathbf{k}}e^{ikx}+a_{\mathbf{k}}^{+}e^{-ikx}],$$

where $\omega=(\mathbf{k}^2+m^2)^{1/2}$. The canonical commutation relations for a, a^+ have the form

$$[a_{\mathbf{k}},a_{\mathbf{k}'}^{+}]=\delta_{\mathbf{kk}'},$$

from which it can be seen that each normal oscillation has the structure of a harmonic oscillator. Each such oscillator has its own Hilbert space with ground state $|0;\mathbf{k}\rangle$, which is characterized by the relations $a_{\mathbf{k}}|0;\mathbf{k}\rangle=0$, $\langle 0;\mathbf{k}|0;\mathbf{k}\rangle=1$, and an infinite sequence of eigenstates of the particle number operator $N_{\mathbf{k}}=a_{\mathbf{k}}^{+}a_{\mathbf{k}}$. The Hilbert space of the complete field is the direct product of the Hilbert spaces of the simple oscillators. We shall denote the state vectors of the field by $|n_1,n_2,\dots,n_i,\dots\rangle$, where n_i is the number of quanta with momenta $\mathbf{k}_i$. Since each oscillator has a countable set (M) of states, and there exists a countable set (N) of oscillators, the number of dimensions of the complete Hilbert space is determined by the power of the set M^N, which is equal to the power of the continuum.

The set $\{|n_1,n_2,\dots,n_i,\dots\rangle\}$ cannot be used as the basis of a separable Hilbert space because it is uncountable. There exist infinitely many ways of choosing countable subspaces that can serve as basis of a separable Hilbert space. If two such subsets are bases of representations for the operators $(a_i, a_i^+;\ i=1,2,\dots)$, then these two representations are *unitarily inequivalent with respect to each other* in the sense that a vector of one representation is not a superposition of the basis vectors of the other representation.

If there is no interaction in the system, the complete number of particles is a conserved quantity. Therefore, the set $\{|n_1,\dots,n_i,\dots\rangle\}$ is too large to describe the free field; a restriction may be made to the subset

$$\{|n_1,\dots,n_i,\dots\rangle;\quad \sum_i n_i=\text{finite number}\}.$$

Such a subset is countable (see, for example, Ref. 58). After the introduction in it of the operation of the inner product, one obtains a separable Hilbert space, and this is called the Fock space.

In the case of interacting fields, the situation is changed, and a manifestation of this is Haag's theorem and its corollaries.[53–56] The proof of Haag's theorem is based essentially on the requirement of uniqueness of the vacuum, and this implicitly presupposes the absence of inequivalent representations and the possibility of adapting the Fock space to the representation of interacting fields.[47] Therefore, a radical way of correcting the situation could be to include inequivalent representations in the structure of the theory by defining the field as an operator in a nonseparable Hilbert space (at the same time, it will be necessary to give up uniqueness of the vacuum). A discussion of the axiomatics of such a theory can be found, for example, in Ref. 59. However, the realization of this idea is hindered already by the fact that there does not yet exist a systematic classification of the inequivalent representations of the CCR.

We pose a narrower problem. Following the tradition, we shall assume that the formalism introduced in the previous paragraphs gives a sufficient basis for calculating the elements of the S matrix provided the coupling constant is small: $g(\mu)\ll 1$. It is well known that for the description of the electromagnetic and weak interactions such an approach is successful. Of course, one must bear in mind that all the basic problems associated with Haag's theorem remain.

We shall nevertheless attempt to weaken the condition of uniqueness of the vacuum. The expression "the vacuum is not unique" means that the Hilbert space contains several normalizable and invariant vectors. These vectors are necessarily inequivalent (in the sense indicated above). In this case, identification of the vacuum and the ground state is no longer possible. It is necessary to find the true ground state of the system by explicit use of its dynamics. For this it is necessary to introduce, with allowance for the dynamics, the set of vacuum vectors (Fock spaces) and choose that one among them that for the given coupling constant and temperature is the best candidate for the role of the ground state. Note that for this there is no need to construct a complete (nonseparable) state space. It is sufficient to assume that it exists and that all the considered Fock spaces are subspaces of it. In each of these trial subspaces, the standard formalism of canonical quantization acts.

A method by which this program can be realized will be formulated in detail in the following section. Here we note that the set of trial vacuum vectors (inequivalent representations) is introduced by means of canonical transformations, the dynamics is taken into account by the renormalization-group equations, and the criteria for choosing the ground state are based on comparison of the effective coupling constants and free-energy densities that characterize each representation.

Before we turn to the formulation of the method, we consider some well-known examples of equivalent representations. Simultaneously, we shall clarify the motivation of our method and introduce expressions that will be used in subsequent calculations.

4.2. Canonical transformations of bosonic operators

We consider two sets of bosonic operators $[a(\mathbf{k}),a^+(\mathbf{k})]$ and $[b(\mathbf{k}),b^+(\mathbf{k})]$. We denote the Fock space on which these operators act by $H[a,b]$. The vacuum state $|0\rangle$ in it satisfies the relations

$$a(\mathbf{k})|0\rangle=0, \quad b(\mathbf{k})|0\rangle=0\forall\mathbf{k}.$$

The algebraic properties of the operators are given by

$$[a(\mathbf{k}),a^+(\mathbf{k}')]=\delta(\mathbf{k}-\mathbf{k}'), \quad [b(\mathbf{k}),b^+(\mathbf{k}')]=\delta(\mathbf{k}-\mathbf{k}'),$$

and the remaining commutators vanish. We introduce the operators $\alpha(\mathbf{k})$ and $\beta(\mathbf{k})$:

$$\begin{aligned}\alpha(\mathbf{k})&=c(\mathbf{k})a(\mathbf{k})-d(\mathbf{k})b^+(-\mathbf{k})\\ \beta(\mathbf{k})&=c(\mathbf{k})b(\mathbf{k})-d(\mathbf{k})a^+(-\mathbf{k}).\end{aligned} \tag{4.1}$$

Here, the numerical coefficients c and d are real functions of $\mathbf{k}^2$; they satisfy the condition $c^2-d^2=1$.

It guarantees that α and β have the same algebraic properties as a and b. In other words, the transformation (4.1) is canonical; it is called a Bogolyubov transformation.[60–62] We shall assume that the coefficient c is positive; we can then set $c-\cosh\xi$, $d=\sinh\xi$.

We introduce the operator

$$\begin{aligned}O[\xi]&=\exp\{A[\xi]\},\\ A[\xi]&=\int d^3k\xi(\mathbf{k})[a(\mathbf{k})b(-\mathbf{k})-b^+(\mathbf{k})a^+(-\mathbf{k})],\end{aligned} \tag{4.2}$$

so that

$$[a(\mathbf{k}),A(\xi)]=-\xi(\mathbf{k})b^+(-\mathbf{k}),$$

$$[b^+(-\mathbf{k}),A(\xi)]=-\xi(\mathbf{k})a(\mathbf{k}).$$

Applying these relations repeatedly, we obtain

$$O^{-1}(\xi)a(\mathbf{k})O(\xi)=a(\mathbf{k})\cosh\xi-b^+(-\mathbf{k})\sinh\xi=\alpha(\mathbf{k}). \tag{4.3}$$

Similarly

$$\beta(\mathbf{k})=O^{-1}(\xi)b(\mathbf{k})O(\xi). \tag{4.4}$$

The transformations (4.3), (4.4) appear to be unitary only in form. To verify this, we consider the matrix element

$$o_0(\xi)=\langle 0|0;\xi\rangle, \quad \text{where } |0;\xi\rangle=O^{-1}[\xi]|0\rangle.$$

It can be shown[58] that

$$o_0(\xi)=\exp\left\{-\delta^3(0)\int d^3k\ \ln[\cosh\xi(\mathbf{k})]\right\}\left(\delta^{(3)}(0)=\frac{V}{(2\pi)^3}\right),$$

from which it follows that in the limit of infinite volume V it vanishes irrespective of the convergence of the integral, so that $|0;\xi\rangle$ does not belong to $H[a,b]$. In other words, $O^{-1}[\xi]$ does not map $H[a,b]$ onto itself. On the other hand, by virtue of (4.3) and (4.4)

$$\alpha(\mathbf{k})|0;\ \xi\rangle=0, \quad \beta(\mathbf{k})|0;\ \xi\rangle=0,$$

from which it follows that $|0;\xi\rangle$ is the vacuum with respect to the action of the operators α and β, and a corresponding Fock space $H[a,b]$ can be constructed. The above considerations show that $H[a,b]$ and $H[\alpha,\beta]$ are two unitarily inequivalent representations of the CCR in the sense that there exists in $H[\alpha,\beta]$ a vector that cannot be represented as a superposition of the basis vectors of $H[a,b]$. Moreover, this is true for any vector in $H[\alpha,\beta]$. Such a situation is usually characterized intuitively by saying that the spaces $H[a,b]$ and $H[\alpha,\beta]$ are orthogonal.

There results do not mean that it is impossible to define the action of the operators α and β on vectors in $H[a,b]$. *Such a definition is given by the relations (4.11).* What we have said above reduces to the assertion that *the canonical transformation (4.1) is not realized by a unitary transformation, and the operators α and β are not annihilation operators in H[a,b], since in this space there is no vacuum corresponding to them.*

We give a further well-known example of a canonical transformation that generates inequivalent representations. Let the operator α be related to the annihilation operator a of the bosonic field by

$$\alpha(\mathbf{k})=a(\mathbf{k})+c(\mathbf{k}), \tag{4.5}$$

where c is a c-number function. Such a transformation generates a shift of the field by a c number. In operator form, the transformation (4.5) is

$$\alpha(\mathbf{k})=O^{-1}[c]a(\mathbf{k})O[c], \tag{4.6}$$

$$O[c]=\exp\left\{-\int d^3k[c^*(\mathbf{k})a(\mathbf{k})-c(\mathbf{k})a^+(\mathbf{k})]\right\}. \tag{4.7}$$

It is easy to verify the relation

$$|0;c\rangle\equiv O^{-1}[c]|0\rangle=\exp\left\{-\frac{1}{2}\int d^3k\left|c(\mathbf{k})\right|^2\right\}\times\exp\left\{-\int d^3kc(\mathbf{k})a^+(\mathbf{k})\right\}|0\rangle.$$

If we have [for example, for $c(\mathbf{k})=c\,\delta(\mathbf{k})$]

$$\int d^3k|c(\mathbf{k})|^2=\infty,$$

then the representations $H[a]$ and $H[\alpha]$ are unitarily inequivalent to each other.

4.3. Renormalization-group transformations

Among the transformations of the form (4.1), there are two important special cases. They are transition to a field with new mass and a scale transformation of the field.

Let $\{\varphi,\pi\}$ be canonical variables that describe the free scalar field with mass m, so that they satisfy the CCR (2.2) and can be expressed in terms of the creation, a^+, and annihilation, a operators by the relations (2.4) with $\omega=(\mathbf{k}^2+m^2)^{1/2}$.

We introduce the operator (α,α^+), which are related to (a,a^+) by

$$\begin{aligned}\alpha(\mathbf{k})&=\cosh\xi(\mathbf{k})a(\mathbf{k})-\sinh\xi(\mathbf{k})a^+(-\mathbf{k}),\\ \alpha^+(\mathbf{k})&=\cosh\xi(\mathbf{k})a^+(\mathbf{k})-\sinh\xi(\mathbf{k})a(-\mathbf{k}),\end{aligned}\tag{4.8}$$

or, in operator form,

$$\begin{aligned}&\alpha(\mathbf{k})=O^{-1}[\xi]a(\mathbf{k})O[\xi],\\ &O[\xi]=\exp\left\{\int d^3k\xi(\mathbf{k})[a(-\mathbf{k})a(\mathbf{k})-a^+(\mathbf{k})a^+(-\mathbf{k})]\right\}.\end{aligned}\tag{4.9}$$

If the parameter ξ is chosen in the form

$$\xi(\mathbf{k})=\frac{1}{2}\ln\left(\zeta^2\frac{\omega}{\Omega}\right),\quad \zeta=\text{const},\quad \Omega(\mathbf{k})=\sqrt{\mathbf{k}^2+M^2},\tag{4.10}$$

then (4.8) corresponds to transition to a new mass M and to a scale transformation of the field. In other words, (4.8) and (4.10) determine a canonical transformation of the form

$$\{\varphi,\pi\}\to\{\zeta\Phi,\zeta^{-1}\Pi\},\tag{4.11}$$

where the fields Φ and Π have the form

$$\begin{aligned}\Phi(x)&=\int\frac{d\mathbf{k}}{2\pi}\frac{1}{\sqrt{2\Omega}}[\alpha(\mathbf{k})e^{i\mathbf{kx}}+\alpha^+(\mathbf{k})e^{-i\mathbf{kx}}],\\ \Pi(x)&=\frac{1}{i}\int\frac{d\mathbf{k}}{2\pi}\sqrt{\frac{\Omega}{2}}[\alpha(\mathbf{k})e^{i\mathbf{kx}}-\alpha^+(\mathbf{k})e^{-i\mathbf{kx}}].\end{aligned}\tag{4.12}$$

The Fock spaces associated with the operators a and α are unitarily inequivalent.

If $M=m$, then the expressions (4.8) give only a scale transformation of the field, and for $\zeta=1$ they give only a transition to a new mass. The transformation (4.8) for the model (2.2)–(2.6) corresponds to a renormalization-group transformation provided we make simultaneously a finite renormalization of the coupling constant and M and ζ satisfy the RG equations (3.2).

On the other hand, we return to the quantized φ^4 theory determined by the relations (2.2)–(2.6) with fixed renormalization scheme, i.e., a definite R_μ class is chosen, and the renormalization scale is determined by $m/\mu=C$ (see Sec. 3.1). Further, by a canonical transformation we go over to fields with a certain new mass $M=tm(\mu)$, where t is a transformation parameter. It is clear that such a transformation changes the R scheme: $M/\mu\neq C$. It follows from this that *if we make canonical transformations that include transition to a new mass and wish at the same time to keep the renormalization scheme unchanged, then we must also make a compensating RG transformation $\mu\to\nu=t\mu$ in order to satisfy the relation $M/\nu=m/\mu$*, i.e., in order to ensure that the ratio of the mass to the renormalization scale is the same in the different representations. Such a RG transformation includes a scale transformation of the field [see (4.8)] and a finite renormalization of the coupling constant.

It is just such a situation that arises in the investigation of phase structure: By means of canonical transformations one introduces a set of inequivalent representations of the CCR and one requires that the R scheme be the same in all these representations.

5. THE METHOD OF CANONICAL TRANSFORMATIONS

In the foregoing, we have considered methods and problems that in some way or another are related to the method of canonical transformations. As we have proceeded, we have already announced some important aspects of this method. The various aspects of the method that is described below can be found in Refs. 31–36. Our point of departure is represented by the following assumptions:

1) in quantum field theory, different phases are manifested as unitarily inequivalent representations of the CCR;

2) the basic information about the phase structure of the theory is contained in the renormalization structure.

If the renormalized coupling constant G is small and the temperature is equal to zero, then for the quantization of the models (1.1)–(1.3) we can use the canonical procedure in a Fock representation for scalar particles with renormalized mass m. The procedure for constructing the S matrix presupposes the fixing of a renormalization scheme. Having this in mind, we wish to know what is the field system for other values of G and θ and for fixed renormalization scheme. We formulate the problem as follows:

What representation of the CCR is suitable for different values of G and θ, and what physical picture corresponds to this representation?

By different phases of the system we shall understand inequivalent representations present in the theory for given G and θ.

The method of canonical transformations reduces to the following.

1. One makes a canonical quantization of the theory in a representation that has a sensible physical interpretation in the weak-coupling limit, i.e., for $G\ll 1$. The renormalization scheme is fixed. This means that:

a) a definite one-parameter class R_μ of renormalization prescriptions has been chosen;

b) the renormalization scale μ is fixed by the relation $m/\mu=C$, where m is the renormalized mass, and C is some constant.

2. By means of canonical transformations, one introduces a set of inequivalent representations of the CCR such that the Hamiltonian has the *correct* form in each of these representations. This means that

$$H=H_0+H_I+H_{ct}+VE.$$

Here, H_0 is the standard free Hamiltonian. The interaction Hamiltonian H_1 contains field operators to a power greater than the second. The operator H_{ct} is determined by H_0 and H_1 and corresponds to equivalent R schemes in all representations.

We shall assume that R schemes in two representations with different masses m and M are equivalent if:

• in both representations the same R_μ class of renormalization prescriptions is used;

• the renormalization scales μ and ν in the first and second representations satisfy

$$\frac{m}{\mu}=\frac{M}{\nu}. \tag{5.1}$$

The fulfillment of the requirement of equivalence of the R schemes is ensured by a compensating RG transformation, by means of which the dynamics of the system if also taken into account.

The quantity E is related to the free-energy density F in accordance with $F=E-TS$, where S is the entropy density.

3. One chooses the representation with minimum free-energy density and smallest effective coupling constant $G_{\rm eff}(G,\theta)$. For $d<4$, this is determined by $G_{\rm eff}=g/2\pi M^{4-d}$. In the four-dimensional case, the definition is somewhat more complicated and we give it later.

In the theory of phase transitions, one usually employs a criterion based on comparison of the free energies. However, in quantum field theory the weak-coupling criterion appears to be more meaningful. From the physical point of view, F does not play any role, since it does not contribute to the elements of the S matrix. In addition, the free-energy density can never be found exactly or at least with equal accuracy in different phases, so that to a large degree the comparison becomes meaningless. At the same time, it is natural to assume that a large coupling constant in H means that the representation associated with H_0 does not describe real states and cannot be regarded as a suitable representation for the Hamiltonian H.

The requirement of equivalence of the R schemes in the different phases is due to the very formulation of the problem. Indeed, we are interested in *how the structure of the ground state of the system changes with changing values of the parameters g and m that occur in the boundary condition (3.3) for a fixed renormalization scheme.*

The requirement of a correct form of the Hamiltonian and the criterion of weak coupling apply to the usual picture of scattering in quantum field theory. The Hamiltonian H_0 describes the free asymptotic fields, and H_1 describes the interaction of the particles; it must not contain terms linear or quadratic in the fields, since they do not lead to any nontrivial interaction but merely redetermine the parameters of the free Hamiltonian. The usual perturbative expansion for the scattering amplitudes is meaningful if the effective (perturbative) coupling constant is sufficiently small. Therefore, we shall regard a representation as acceptable if the Hamiltonian has the correct form and the effective coupling constant is small. In addition, $G_{\rm eff}$ can be used to monitor the accuracy of the approximation.

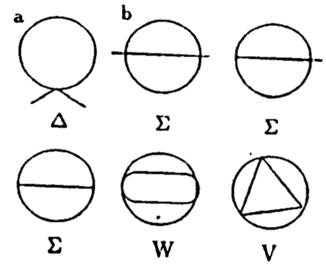

FIG. 1. Divergent diagrams in R^3.

6. THE φ^4 THEORY OF A SINGLE-COMPONENT FIELD

The three-dimensional models (1.1)–(1.3) are interesting above all in that, being superrenormalizable, they contain two-, three-, and four-loop divergences in the diagrams shown in Fig. 1. This circumstance makes its problematic to use variational methods (Refs. 25–27 and 38–30) and leads to different results in different approaches. We shall consider this in more detail.

Let us consider the case R^2. The GEP approxima tion,[19,37] Chang's original calculation[29] based on allowance for the renormalization structure, and the method of canonical transformations[31] lead to the same results for the model (1.1). The critical coupling constant G_c at which the first-order phase transition occurs and the mass of the particles in the phase with broken symmetry are the same in all these approaches.

The case $N>1$ (1.3) was investigated in the framework of the GEP approximation,[30] the $1/N$ expansion,[41] and the method of canonical transformations.[31] In accordance with Refs. 30 and 31, there is a first-order phase transition with symmetry breaking in the system. In the phase with broken symmetry, there is a $(N-1)$-multiplet of fields with nonvanishing masses. At the same time, the $1/N$ expansion does not exhibit any phase transition.[41] This discrepancy has a natural explanation, which is given in Ref. 30. The $1/N$ expansion is only valid provided $NG<1$ and although the GEP in the limit $N\gg 1$ is identical to the effective potential of the $1/N$ expansion the critical point G_c is not seen in the $1/N$ expansion ($NG_c\gg 1$ for $N\gg 1$).

Goldstone's theorem forbids spontaneous breaking of a continuous symmetry in R^2, since in this case particles with zero mass do not exist. Nevertheless, two different methods[30,31] indicate breaking of $O(N)$ symmetry, but the "Goldstone particles" have nonzero mass.

The situation is more complicated in the three-dimensional case. The significant difference between R^2 and R^3 is in the renormalization structures of the models. The different methods reflect this to unequal degrees.

The calculations of Chang[29] and Magruder[20] are based directly on the renormalization structure of the exact effective potential in the framework of perturbation theory. It is established in Refs. 29 and 20 that the discrete symmetry broken spontaneously at $G\ll 1$ in the model (1.2) in R^3 is restored with increasing G. Such a picture is the direct opposite of the situation in R^2 (Refs. 19 and 33).

At the same time, the GEP approximation in the models (1.1)–(1.3) in R^3 leads to a phase structure analogous to the two-dimensional case.[30] This contradicts the results of Chang and Magruder.

6.1. The φ^4 Hamiltonian in R^3

It is convenient to consider the model with the Lagrangian density

$$L(x)=\tfrac{1}{2}\,\varphi(x)(\Box-m^2)\varphi(x)-g_3\varphi^3(x)-\frac{g_4}{4}\,\varphi^4(x), \tag{6.1}$$

where g_3 and g_4 are coupling constants, and $x=(\mathbf{x},t)=(x_1,x_2,t)$.

The Hamiltonian density corresponding to the Lagrangian (6.1) has the form

$$H=H_0+H_I+H_{ct},$$
$$H_0[\varphi,\pi]=\tfrac{1}{2}{:}[\pi^2(\mathbf{x})+(\nabla\varphi(\mathbf{x}))^2+m^2\varphi^2(\mathbf{x})]; \tag{6.2}$$
$$H_I[\varphi,\pi]={:}\left[\frac{1}{4}\,g_4\varphi^4(\mathbf{x})+g_3\varphi^2(\mathbf{x})\right]{:},$$

$$\varphi(\mathbf{x})=\int\frac{d\mathbf{k}}{2\pi}\frac{1}{\sqrt{2\omega}}\,[a(\mathbf{k})e^{i\mathbf{kx}}+a^+(\mathbf{k})e^{-i\mathbf{kx}}],$$
$$\pi(\mathbf{x})=\frac{1}{i}\int\frac{d\mathbf{k}}{2\pi}\sqrt{\frac{\omega}{2}}\,[a(\mathbf{k})e^{i\mathbf{kx}}-a^+(\mathbf{k})e^{-i\mathbf{kx}}], \tag{6.3}$$
$$\omega(\mathbf{k})=\sqrt{\mathbf{k}^2+m^2},\quad [a(\mathbf{k}),a^+(\mathbf{k}')]=\delta(\mathbf{k}-\mathbf{k}').$$

The fields φ and π are canonical variables and satisfy the commutation relations

$$[\varphi(\mathbf{x}),\pi(\mathbf{y})]=i\delta(\mathbf{x}-\mathbf{y}).$$

The creation and annihilation operators (a^+,a) act on the Fock space with vacuum vector defined by the expression

$$a(\mathbf{k})|0\rangle=0,\quad \forall\mathbf{k}\in R^2. \tag{6.4}$$

All the operators in (6.2) are normally ordered with respect to the vacuum (6.4). The model is superrenormalizable and has a finite number of divergent diagrams (Fig. 1). To eliminate these divergences, it is necessary to introduce into the Hamiltonian an operator H_{ct} containing counterterms. In the scheme of subtractions at zero momentum, we have

$$H_{ct}(m)={:}\left[\frac{1}{2}\,A(m)\varphi^2(\mathbf{x})+C(m)\varphi(\mathbf{x})+\delta E(m)\right]{:},$$
$$A(m)=3!g_4^2\Sigma_0(m),\quad C(m)=3!g_3g_4\Sigma_0(m), \tag{6.5}$$
$$\delta E(m)=\tfrac{3}{4}g_4^2W_0(m)-\tfrac{9}{2}g_4^3V_0(m)+\tfrac{1}{2}3!g_3^2\Sigma_0(m).$$

Here and below, the subscript 0 denotes zero temperature. The functions that occur in (6.5) have the form

$$\Sigma_0(m)=\frac{1}{(4\pi)^2}\,\mathrm{reg}\int_0^\infty\frac{ds}{s}\,e^{-3ms},$$

$$W_0(m)=\frac{1}{(4\pi)^3}\,\mathrm{reg}\int_0^\infty\frac{ds}{s^2}\,e^{-4ms}, \tag{6.6}$$

$$V_0(m)=\frac{1}{4(2\pi)^5}\,\mathrm{reg}\int_0^\infty\frac{ds}{s}\,\tan^{-3}\left(\frac{s}{2m}\right).$$

Here, some regularization is understood.

6.2. Canonical transformations

We make the following canonical transformation:

$$\pi(\mathbf{x})\to\pi_t(\mathbf{x}),\quad \varphi(\mathbf{x})\to\varphi_t(\mathbf{x})+B, \tag{6.7}$$

where the fields φ_t and π_t have the form (6.3) but with new mass $M=mt$. The constant B has the meaning of a vacuum condensate. The transformation (6.7) can be represented in terms of creation and annihilation operators:

$$a(\mathbf{k})\to a(\mathbf{k},t)-2\pi mB\,\delta(\mathbf{k})$$
$$=U_2^{-1}(t)U_1^{-1}(B)a(\mathbf{k})U_1(B)U_2(t),$$
$$U_1(B)=\exp\left\{-2\pi mB\int d\mathbf{k}\delta(\mathbf{k})[a(\mathbf{k})-a^+(\mathbf{k})]\right\}, \tag{6.8}$$

$$U_2(t)=\exp\left\{\frac{1}{2}\int d\mathbf{k}\lambda(\mathbf{k},t)[a(-\mathbf{k})a(\mathbf{k})-a^+(\mathbf{k})a^+(-\mathbf{k})\right\}.$$

The transformation U_1 shifts the field φ by the constant B. The transformation that is the inverse of U_2 can be represented in the form

$$a(\mathbf{k})=a(\mathbf{k},\ t)\cosh(\lambda)+a^+(-\mathbf{k},\ t)\sinh(\lambda)$$
$$a^+(\mathbf{k})=a^+(\mathbf{k},\ t)\cosh(\lambda)+a(-\mathbf{k},\ t)\sinh(\lambda). \tag{6.9}$$

If the parameter λ is chosen in accordance with

$$\lambda(\mathbf{k},\ t)=\frac{1}{2}\ln\left(\frac{\omega(\mathbf{k})}{\omega(\mathbf{k},\ t)}\right),\quad \omega(\mathbf{k},t)=\sqrt{\mathbf{k}^2+m^2t^2},$$

then, using (6.9), we obtain a representation of the fields φ_t, π_t with mass M in terms of the operators $a(\mathbf{k},t)$, $a^+(\mathbf{k},t)$. These operators act on a Fock space with vacuum vector satisfying the relations

$$|0(t,B)\rangle=U_2^{-1}(t)U_1^{-1}(B)|0\rangle,\quad a(\mathbf{k},t)|0(t,B)\rangle=0\forall\mathbf{k}. \tag{6.10}$$

For $B\neq0$ and $t\neq1$, the representations of the CCR determined by Eqs. (6.4) and (6.10) are unitarily inequivalent.

We express the density of the Hamiltonian (6.2) in the new canonical variables, go over to normal ordering of the operators $(a(\mathbf{k},t),\ a^+(\mathbf{k},t))$, and introduce counterterms in accordance with the new representation of the free Hamiltonian in the framework of the scheme of subtractions at zero momentum. As a result, we obtain

$$H=H_0'+H_I'+H_{ct}'+H_1+E,$$
$$H_0'=\tfrac{1}{2}{:}[\pi_t^2(\mathbf{x})+(\nabla\varphi_t(\mathbf{x}))^2+M^2\varphi_t^2(\mathbf{x})]{:}\ , \tag{6.11}$$
$$H_I'={:}\left[\frac{1}{4}\,h_4\varphi_t^4(x)+h_3\varphi_t^3(\mathbf{x})\right]{:}\quad h_3=g_3+g_4B,$$
$$h_4=g_4.$$

The operator H'_{ct} has the structure of the expressions (6.5) and (6.6), in which it is necessary to make the substitutions

$$\varphi\to\varphi_t,\quad m\to M,\quad g_3\to h_3,\quad g_4\to h_4.$$

The operator H_1 has the form

$$H_1=:\left[\frac{1}{2}R(t,B)\varphi_t^2(\mathbf{x})+P(t,B)\varphi_t(\mathbf{x})\right]:,$$

$$\begin{aligned}R&=m^2-M^2+3g_4(B^2-D_0)+6g_3B+6g_4^2(\Sigma_0(m)\\&\quad-\Sigma_0(M)),\end{aligned}\tag{6.12}$$

$$\begin{aligned}P&=m^2B+g_4(B^3-3BD_0)+3g_3(B^2-D_0)+6g_4\\&\quad\times(g_3+g_4B)(\Sigma_0(M)-\Sigma_0(M)),\end{aligned}$$

and the vacuum energy density E the form

$$\begin{aligned}E&=E_0+E_1+E_2+E_3,\\E_0&=\tfrac{1}{2}m^2B^2+L_0(t),\\E_1&=\tfrac{1}{4}g_4(B^4-6D_0B^2+3D_0^2)+g_3(B^3-3D_0B),\\E_2&=\tfrac{3}{4}g_4^2(W_0(m)-W_0(M)-4D_0\Sigma_0(M))+3(g_4B\\&\quad+g_3)^2(\Sigma_0(m)-\Sigma_0(M)),\\E_3&=-\tfrac{9}{2}g_4^3(V_0(m)-V_0(M)),\end{aligned}\tag{6.13}$$

$$\begin{aligned}D_0(t)&=\Delta_0(0,m)-\Delta_0(0,M)=\int\frac{d\mathbf{k}}{(2\pi)^2}\left[\frac{1}{\omega(\mathbf{k})}\right.\\&\quad\left.-\frac{1}{\omega(\mathbf{k},t)}\right]=\frac{m}{4\pi}(t-1),\end{aligned}\tag{6.14}$$

$$\begin{aligned}L_0(t)&=\frac{1}{2}\int\frac{d\mathbf{k}}{(2\pi)^2}\left[\omega(\mathbf{k},t)-\omega(\mathbf{k})-\frac{M^2-m^2}{2\omega(\mathbf{k},t)}\right]\\&=\frac{m^3}{24\pi}(t-1)^2(2+t).\end{aligned}\tag{6.15}$$

The condition

$$H_1=0\Leftrightarrow\begin{cases}R(t,B)=0\\P(t,B)=0\end{cases}\tag{6.16}$$

ensures the correct form of the Hamiltonian H and determines the parameters t and B. From the physical point of view, this means that H describes scalar particles with mass M that depends on the coupling constant. This dependence is determined by Eqs. (6.16) and (6.12). Introducing the dimensionless variables

$$G_4=\frac{g_4}{2\pi m},\quad G_3=\frac{g_3}{m\sqrt{4\pi m}},\quad b=B\sqrt{\frac{4\pi}{m}}\tag{6.17}$$

and using (6.12), we represent (6.16) in the form

$$\begin{aligned}&-\tfrac{1}{2}t^2+\tfrac{1}{2}+\tfrac{3}{4}G_4(b^2-t+1)+3G_3b+\tfrac{3}{4}G_4^2\ln t=0,\\&b+\tfrac{1}{2}G_4b(b^2-3t+3)+3G_3(b^2-t+1)+3G_4\\&\quad\times\left(G_3+\frac{G_4}{2}b\right)\ln t=0.\end{aligned}\tag{6.18}$$

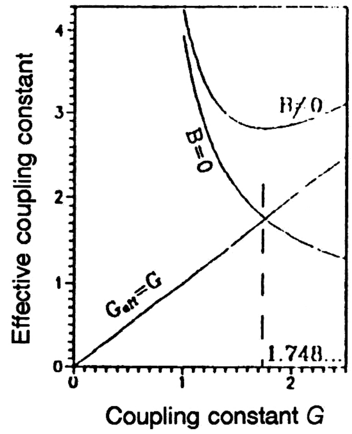

FIG. 2. Effective coupling constants in the model (1.1) in R^3.

The energy density (6.13) is finite, and its explicit form can be found in Refs. 32 and 33. The solutions of Eqs. (6.18) describe the possible phases of the theory. The system exists in the phase among them to which there correspond the minimum energy and the smallest effective coupling constant:

$$G_{\text{eff}}(G)=\frac{G}{t(G)}.\tag{6.19}$$

6.3. Symmetric model

We consider the phase structure of the theory with the symmetric Lagrangian (1.1). This case corresponds to the choice $G_4=G$, $G_3=0$. From (6.18), we obtain equations for the symmetric model:

$$\begin{aligned}&t^2-1-\tfrac{3}{2}G(b^2-t+1)-\tfrac{3}{2}G^2\ln t=0,\\&b\left[1+\frac{1}{2}G(b^2-3t+3)+\tfrac{3}{2}G^2\ln t\right]=0.\end{aligned}\tag{6.20}$$

Using the first equation, we obtain for the second two solutions: $b=0$ (symmetric), $b^2=t^2/G$ (asymmetric). We consider the phase with broken symmetry.

The phase with broken symmetry

In this case we have

$$\begin{aligned}&b^2=\frac{t^2}{G},\\&t^2-3Gt+3G^2\ln t+2+3G=0.\end{aligned}\tag{6.21}$$

For all G, the second of Eqs. (6.21) has a solution $t(G)<1$, so that the effective coupling constant G_{eff} (Fig. 2) satisfies the inequality

$$G_{\text{eff}}(G)>G\,\forall G.\tag{6.22}$$

The energy density in the phase with broken symmetry is positive for all values of G (see Fig. 3). This indicates that the phase with broken symmetry is not realized in the system.

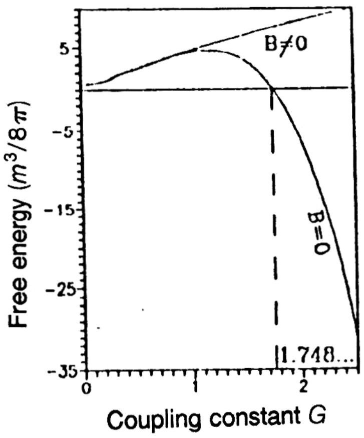

FIG. 3. Free-energy density in the model (1.1) in R^3.

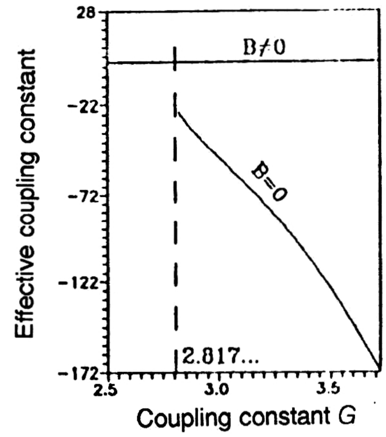

FIG. 4. Free-energy density in the model (1.2) in R^3.

The symmetric phase

Using (6.20) with $b=0$, we obtain the following equation for t:

$$2t^2+3Gt-3G^2 \ln t-2-3G=0. \tag{6.23}$$

We see first of all that (6.23) has two solutions: $t_1(G)\equiv 1$ [corresponding to the original representation (6.2)] and $t_2(G)$, and

$$t_2<1, \quad \text{if } G<G_c, \quad t_2\geqslant 1, \quad \text{if } G\geqslant G_c,$$

$$G_c=\frac{1}{2}\left(1+\sqrt{\frac{19}{3}}\right)=1.758\ldots .$$

The behavior of the energy as a function of G is shown in Fig. 3. The energy density for the solution t_2 is negative for $G>G_c$ and equal to zero for t_1. Thus, the point G_c corresponds to a phase transition from one symmetric phase to another. In the strong-coupling limit $G\gg 1$, the mass, effective coupling constant, and energy density behave as follows:

$$M(G)=mG\sqrt{\frac{3}{2}\ln G}\left[1+O\left(\frac{1}{\ln G}\right)\right],$$

$$G_{\text{eff}}(G)=\sqrt{\frac{2}{3\ln G}}\left[1+O\left(\frac{1}{\ln G}\right)\right]\ll 1, \tag{6.24}$$

$$\varepsilon(G)=-\frac{2}{3}\left(\frac{3}{2}G^2\ln G\right)^{3/2}\left[1+O\left(\frac{1}{\ln G}\right)\right].$$

It can be seen from the asymptotic behavior (6.24) that the second symmetric representation describes the system in the strong-coupling limit fairly accurately. However, when $G\sim 1$ none of the representations is acceptable since $G_{\text{eff}}\sim 1$. However, in any case the following conclusions hold:

- There is no symmetry breaking in the model (1.1) for any G.
- There are two symmetric phases, and a phase transition without rearrangement of the symmetry occurs at $G\approx 1.758$.

6.4. Model with initially broken symmetry

We consider the phase structure of the model (1.2). Equations for t and b can be obtained from (6.18) by the substitution $G_4=G$, $G_3=\sqrt{G}/2$. As a result, we obtain

$$t^2-1-\tfrac{3}{2}G(b^2-t+1)-3\sqrt{G}b-\tfrac{3}{2}G^2\ln t=0,$$

$$2b+Gb(b^2-3t+1)+3\sqrt{G}(b^2-t+1)+3G\sqrt{G}$$
$$\times(1+\sqrt{G}b)\ln t=0. \tag{6.25}$$

The same substitution must be made in (6.13) for the free-energy density. Two solutions for b follow from (6.25):

$$b=-\frac{1}{\sqrt{G}}\ (\text{symmetric}), \quad b=-\frac{1}{\sqrt{G}}$$
$$\pm\frac{t}{\sqrt{G}}\ (\text{asymmetric}).$$

Phase with broken symmetry

The second equation of (6.25) takes the form

$$t^2-3Gt+3G^2\ln t-1+3G=0. \tag{6.26}$$

This equation has the unique solution $t\equiv 1$, which leads to the original representation with SSB: $M=m$, $G_{\text{eff}}=G$, $E=0$.

The symmetric phase

For the symmetric case ($b=-1/\sqrt{G}$), we obtain

$$2t^2+3Gt-3G^2\ln t+1-3G=0. \tag{6.27}$$

It can be shown that Eq. (6.27) has a real solution only when $G\geqslant G_c=2.817\ldots$. Substituting this solution in the expression for the energy (6.13), we obtain the function shown in Fig. 4. For $G\gg 1$, the mass, effective coupling constant, and free-energy density have the same asymptotic behaviors (6.24) as in the case of the symmetric model. They show that the description is fairly accurate outside the critical region. However, the critical point G_c is found very approximately, since near this point $G_{\text{eff}}\sim O(1)$ (see Fig. 5). Since the order parameter $\sigma=\pm(t(G)/\sqrt{G})$ is discontinuous at the point G_c, the phase transition has first order. We summarize our conclusions:

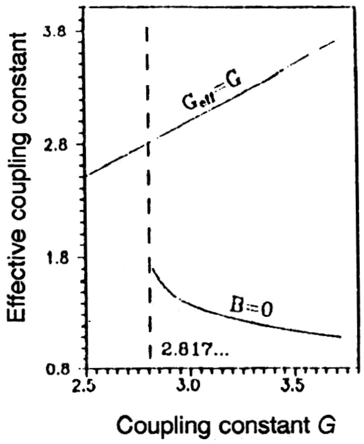

FIG. 5. Effective coupling constants in the model (1.2) in R^3.

1. Symmetry is restored in the system (1.2) if the coupling constant is sufficiently large.
2. There is a phase transition between the phase with broken symmetry and the symmetric phase at the point $G_c \sim 2.817\ldots$.

These conclusions agree with the results of Refs. 20 and 29 but contradict the GEP approximation. In the two-dimensional case, when the variational approach is valid, the method of canonical transformations leads to results that agree with those of the GEP approximation.[19] To demonstrate this, we consider the φ_2^4 model.[31]

6.5. Two-dimensional model (1.1)

Since in this case only the diagram of Fig. 1a and the corresponding vacuum diagram diverge, in (6.12) and (6.13) we only need to take into account the terms of first order in the coupling constant $g=g_4$ [$g_3=1$ for (1.1)]. In accordance with (6.16) and (6.13), we have

$$\begin{aligned} &m^2-M^2+3g(B^2-D_0(t))=0,\\ &m^2B+g(B^3-3BD_0(t))=0, \end{aligned} \tag{6.28}$$

$$E=\tfrac{1}{2}m^2B^2+L_0(t)+\tfrac{1}{4}g(B^4-6D_0(t)B^2+3D_0^2(t)), \tag{6.29}$$

where $t=M^2/m^2$. For the two-dimensional case, the functions D_0 and L_0 have the form

$$D_0(t)=\frac{1}{4\pi}\ln t, \quad L_0(t)=\frac{m^2}{8\pi}(t-1-\ln t).$$

Equations (6.28) determine the minimum of $E(t,B)$ with respect to the variables t and B and are identical to the conditions of a minimum of the GEP. The second equation of (6.28) has two solutions: $B=0$ (symmetric) and $B\neq 0$ (with broken symmetry).

The symmetric phase

Setting $B=0$ in the first equation of (6.28), we obtain

$$2(t-1)=-3G\ln t, \quad G=\frac{g}{2\pi m^2}.$$

There exists a unique solution $t\equiv 1$ with vacuum energy $E_S\equiv 0$.

The phase with broken symmetry

In this case, we rewrite (6.28) in the form

$$B^2=\frac{t}{2\pi G}, \quad t+2=3G\ln t.$$

With allowance for these equations, the energy density (6.29) takes the form

$$E_B=\frac{m^2}{8\pi}\left\{t-1-\frac{t^2+2}{2(t+2)}\ln t\right\}.$$

It can be shown that $E_B(G)\leqslant 0$ for $G\leqslant G_0=1.625\ldots$. The critical value of the coupling constant G_0 is identical to the GEP result.[19] In the strong-coupling limit $G\gg 1$, we obtain the asymptotic behaviors

$$B^2\to\frac{3}{4\pi}\ln G, \quad t\to 3G\ln G,$$

$$E_B\to-\frac{3m^2}{16\pi}G\ln G)^2,$$

$$G_{\rm eff}^{(1)}=\frac{g}{2\pi M^2}\to\frac{1}{3\ln G}, \quad G_{\rm eff}^{(2)}=\frac{g}{2\pi M^2}\sqrt{\sqrt{4\pi B}}$$

$$\to\frac{1}{\sqrt{3}\ln G}.$$

We shall discuss the two-dimensional models (1.1) and (1.2) in more detail in Sec. 10.

7. THE $O(N)$-INVARIANT THEORY

7.1. The $(\varphi^2)^2$ model in R^3

We investigate the phase structure of the model (1.3). The Hamiltonian density in the representation corresponding to $G\ll 1$ has the form

$$H=H_0+H_I+H_{ct},$$

$$H_0=\frac{1}{2}\sum_i^N :[\pi_i^2+(\nabla\varphi_i)^2+m^2\varphi_i^2]: ,$$

$$H_I=\frac{g}{4}:\left[\sum_i^N\varphi_i^2\right]^2: ,$$

$$H_{ct}=:\left[\frac{1}{2}A(m)\sum_i^N\varphi_i^2+\delta E(m)\right]: ,$$

$$A(m)=2(N+2)g^2\Sigma_0(m),$$

$$\delta E(m)=\tfrac{1}{4}N(N+2)g^2W_0(m)-\tfrac{1}{2}N(N+2)^2V_0(m). \tag{7.1}$$

The functions Σ_0, W_0, V_0 are the same as in (6.6). The fields φ_i and π_i have the form (6.3) and satisfy the usual CCR:

$$[\phi_i(\mathbf{x}),\pi_j(\mathbf{y})]=i\delta_{ij}\delta(\mathbf{x}-\mathbf{y}).$$

The states of the system are described by a Fock space with vacuum vector $|0\rangle$:

$$a_j(\mathbf{k})|0\rangle=0, \quad \forall j,\mathbf{k}, \quad \langle 0|0\rangle=1.$$

The Hamiltonian (7.1) is taken in the form of the normal product with respect to the vacuum $|0\rangle$.

7.2. Canonical transformation

In complete analogy with the manner with which we proceeded in the case of the single-component field, we make a canonical transformation of the form

$$(\varphi_i,\pi_i)\to(\Phi_i,\Pi_i),\quad i=1,\ldots,N-1,$$

$$(\varphi_N,\pi_N)\to(\Phi+B,\Pi).$$

Here, (Φ_i,Π_i) is a multiplet of fields with mass $M_0=mt_0$, (Φ,Π) are fields with mass $M=mt$, and B is a constant. The operator form of these transformations has a form analogous to (6.8)–(6.9). The fields (Φ_i,Π_i) and (Φ,Π) have the form (6.3) but with new masses M_0 and M and new creation and annihilation operators (α_i^+,α_i), which act on the Fock space with vacuum vector $|0\rangle\rangle$:

$$\alpha_i(\mathbf{k})|0\rangle\rangle=0,\quad \forall i,\mathbf{k}.$$

This new Fock space is unitarily inequivalent to the original one.

In the new variables, the Hamiltonian, normally ordered with respect to the vacuum $|0\rangle\rangle$, takes the following form in the scheme of renormalization at zero momentum:

$$H=H_0'+H_I'+H_{ct}'+H_1+E,$$

$$H_0'=\frac{1}{2}:\left[\sum_i^{N-1}(\Pi_i^2+(\nabla\Phi_i)^2+M_0^2\Phi_i^2)+\Pi^2+(\nabla\Phi)^2+M^2\Phi^2\right]:,$$

$$H_I'=\frac{g}{4}:\left[\Phi^4+4B\Phi^3+4B\Phi\sum_i^{N-1}\Phi_i^2+2\Phi^2\sum_i^{N-1}\Phi_i^2+\left(\sum_i^{N-1}\Phi^2\right)^2\right]:,$$

$$H_{ct}'=:\left[\frac{1}{2}A_\Phi(M,M_0)\Phi^2+\frac{1}{2}A_{\Phi_i}(M,M_0)\sum_i^{N-1}\Phi_i^2+C(M,M_0)\Phi+\delta E(M,M_0)\right]:,$$

$$H_1=:\left[\frac{1}{2}R(B,M,M_0)\Phi^2+\frac{1}{2}P(B,M,M_0)\sum_i^{N-1}\Phi^2+Q(B,M,M_0)\Phi\right]:,$$

$$R(B,M,M_0)=m^2-M^2-g(3D_0(t)+(N-1)D_0(t_0))+3gB^2+A(m)-A_\Phi(M,M_0),$$

$$P(B,M,M_0)=m^2-M_0^2-g(3D_0(t)+(N+1)D_0(t_0))+gB^2+A(m)-A_{\Phi_i}(M,M_0),$$

$$Q(B,M,M_0)=m^2B+gB^3-gB(3D_0(t)-(N+1)D_0(t_0))+BA(m)-C(M,M_0),$$

$$E=\frac{m^2}{2}B^2+\frac{g}{4}B^4+L_0(t_0)+\frac{g}{4}\{-2[3D_0(t)+(N-1)D_0(t_0)]B^2+3D_0^2(t)+(N-1)D_0^2(t_0)+2\times(N-1)D_0(t)D_0(t_0)\}+\tfrac{1}{2}A(m)B^2-\tfrac{1}{2}A(m)[D_0(t)+(N-1)D_0(t_0)]+\delta E(m)-\delta E(M,M_0).$$

Here, we have introduced the notation

$$A_\Phi=2g^2[3\Sigma_0(M)+(N-1)\Sigma_0^1(M,M_0)],$$

$$A_\Phi=2g^2[3\Sigma_0^1(M_0,M)+(N+1)\Sigma_0^1(M_0)],$$

$$C=2g^2B[3\Sigma_0(M)+(N+1)\Sigma_0^1(M,M_0)],$$

$$\delta E(M,M_0)=\delta E_1+\delta E_2+\delta E_3,$$

$$\delta E_1=g^2B^2[3\Sigma_0(M)+(N-1)\Sigma_0^1(M,M_0)],$$

$$\delta E_2=\tfrac{1}{4}g^2[3W_0(M)+(N^2-1)W_0(M_0)+2\times(N-1)W_0^1(M,M_0)],$$

$$\delta E_3=-\tfrac{1}{2}g^3[9V_0(M)+(N-1)(N+1)^2V_0(M_0)+3\times(N-1)V_0^1(M_0,M)+3(N-1)^2V_0^1(M,M_0)+8\times(N-1)V_2(M,M_0)],$$

D_0, L_0, Σ_0, W_0, and V_0 have already been encountered in (6.6) and (6.14), and the remaining divergent integrals are

$$\Sigma_0^1(M,M_0)=\frac{1}{(4\pi)^2}\,\mathrm{reg}\int_0^\infty\frac{ds}{s}\exp\{-s(M+2M_0)\},$$

$$W_0^1(M,M_0)=\frac{1}{(4\pi)^3}\,\mathrm{reg}\int_0^\infty\frac{ds}{s^2}\exp\{-2s(M+M_0)\},$$

$$V_0^1(M,M_0)=\frac{1}{4(2\pi)^5}\,\mathrm{reg}\int_0^\infty\frac{ds}{s}\tan^{-1}\left(\frac{s}{2M_0}\right)\tan^{-2}\left(\frac{s}{2M}\right),$$

$$V_0^2(M,M_0)=\frac{1}{4(2\pi)^5}\,\mathrm{reg}\int_0^\infty\frac{ds}{s}\tan^{-3}\left(\frac{s}{M+M_0}\right).$$

The correspondence between the divergent integrals and the diagrams is illustrated in Fig. 6. The form of the energy density in terms of the dimensionless variables (6.17) is given in Ref. 32.

We require that $H_1=0$. This ensures the correct form of the Hamiltonian and gives equations for the parameters t, t_0, B. In dimensionless variables, these equations have the form

$$3Gb^2-2f(1+2f)-3Gf-(N-1)Gf_0+3G^2\ln t+(N-1)G^2\ln\left(\frac{t+2t_0}{3}\right)=0,$$

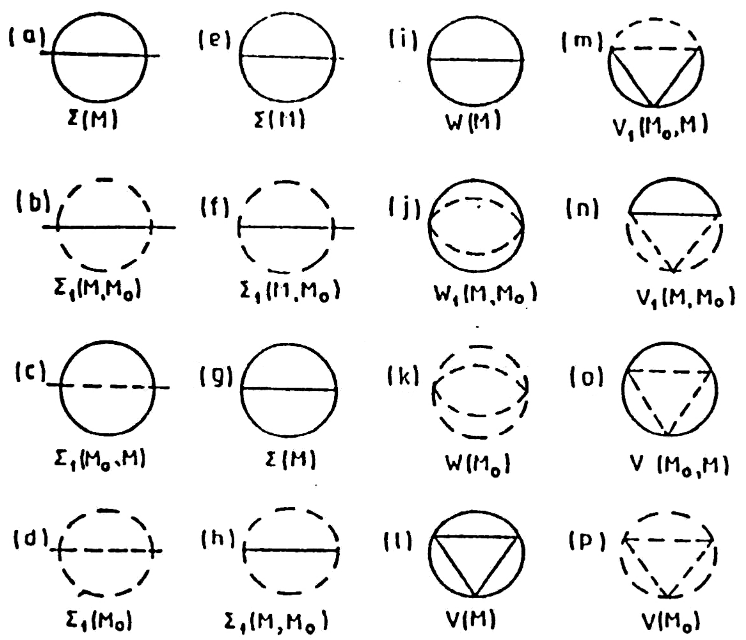

FIG. 6. Divergent vacuum diagrams: The solid line represents the field Φ, the broken line Φ_i; diagrams (a, b) contribute to A_Φ; (c, d) to A_{Φ_i}; (e, f) to C; (g, h) to δE_1; (i–k) to δE_2; and (l–p) to δE_3.

$$Gb^2-4f_0\left(1+\frac{f_0}{2}\right)-Gf-(N+1)Gf_0$$
$$+(N+1)G^2\ln t_0+G^2\ln\left(\frac{t_0+2t}{3}\right)=0, \tag{7.2}$$
$$b\left[2+Gb^2-3Gf-(N-1)Gf_0+3G^2\ln t\right.$$
$$\left.+(N-1)G^2\ln\left(\frac{t+2t_0}{3}\right)\right]=0,$$

where $f=t-1$, $f_0=t_0-1$. All the logarithmic terms in Eqs. (7.2) have appeared on account of the renormalization in the order G^2.

7.3. Phase structure

The following phases are possible:

$S_1(B=0,\ M=M_0=m,\ O(N)$ symmetric)

$S_2(B=0,\ M=M_0\neq m,\ O(N)$ symmetric)

$BS_1(B=0,\ M\neq M_0, O(N-1)$ and $\Phi\to-\Phi$ symmetric)

$BS_2(B\neq 0, M\neq M_0, O(N-1)$ symmetric).

These phases correspond to different solutions of the system (7.2). The $O(N)$-symmetric phases are determined by the conditions $b=0$, $t_0=t$. Using these conditions in (7.2), we obtain the following equation ($f=t-1$):

$$2f(2+f)+(N+2)Gf-(N+2)G^2\ln(1+f)=0. \tag{7.3}$$

S_1 phase with $M_0=M=m$

Equation (7.3) has the trivial solution $f\equiv 0$ ($t\equiv 1$) for all G, which corresponds to the original representation. The energy, mass, and effective coupling constant ($\equiv G$) are shown in Figs. 7–9 by the continuous curve.

S_2 phase with $M_0=M\neq m$

To eliminate the solution $t\equiv 1$, we rewrite (7.3) in the form

$$2+2t+(N+2)G=(N+2)G^2\,\frac{\ln t}{t-1}. \tag{7.4}$$

This equation has a unique solution for all G. The critical value $G=G_0$, at which there is a transition from one symmetric phase to another, is determined by the condition of vanishing of the energy in the phase with mass M. This occurs at $t=1$. Substituting $t=1$ in (7.4), we find

$$G_0=\frac{1}{2}\left[1+\sqrt{\frac{N+18}{N+2}}\right]. \tag{7.5}$$

The asymptotic behaviors for the strong-coupling regime $G\gg 1$ have the form

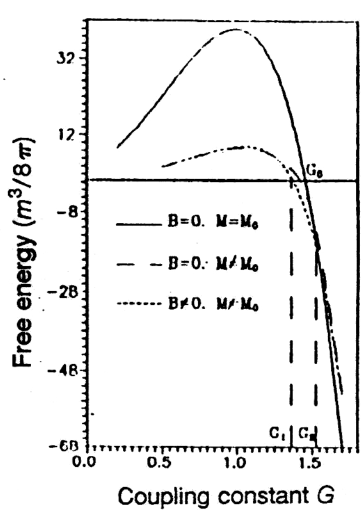

FIG. 7. Free-energy density in the model (1.3) in R^3 ($N=4$).

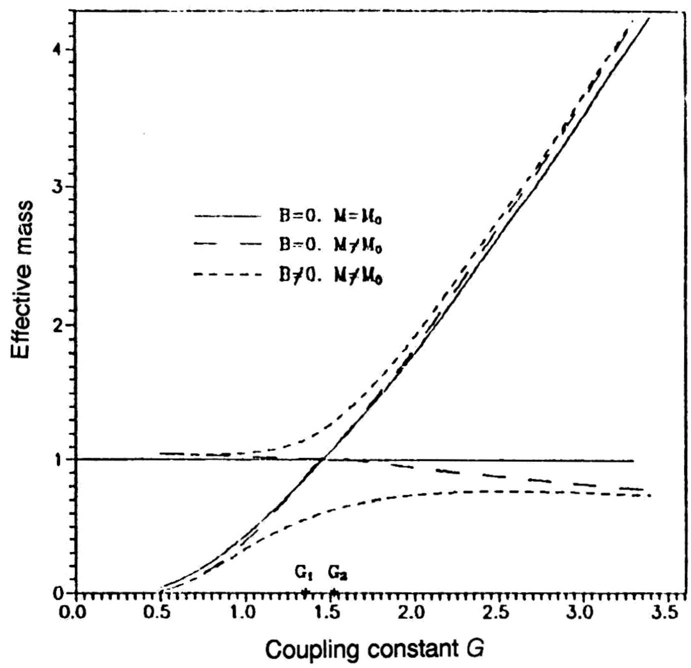

FIG. 8. Mass in different phases ($N=4$): the upper and lower broken curves correspond to M_0 and M in the phases with broken symmetry, respectively; the solid curve corresponds to M in the symmetric phases.

$$t(G)\to\sqrt{\frac{N+2}{2}}\,G\sqrt{\ln G},\quad G_{\rm eff}(G)\to\sqrt{\frac{2}{(N+2)\ln G}}\ll 1,$$
$$E(G)\to-\frac{m^3}{8\pi}\,N(N+2)\sqrt{\frac{N+2}{2}}\,G^3\ln^{3/2}(G). \tag{7.6}$$

The functions $E(G)$, $t(G)$, and $G_{\rm eff}(G)$ for intermediate G are shown in Figs. 7–9.

BS_1: phase mass splitting for zero condensate $b=0$

Breaking of the $O(N)$ invariance solely as a result of splitting of the masses in the multiplet, $M\neq M_0$, corresponds to $b=0$, $t\neq t_0$. The Hamiltonian in this phase is invariant with respect to the group $O(N-1)$ and the discrete transformation $\Phi\to-\Phi$. This phase is described by the system (7.3) with $b=0$. In order to eliminate from (7.3) the solution $t=t_0$, it is convenient to introduce the variable $r=t_0/t$, subtract the second equation from the first, and divide the result by $1-r$. We then obtain the system of equations

$$2(r+1)t^2+2Gt-(N+1)G^2\,\frac{\ln r}{1-r}$$
$$+(N-1)G^2\,\frac{\ln\left(\frac{1+2r}{3}\right)}{1-r}-G^2\,\frac{\ln\left(\frac{2+r}{3}\right)}{1-r}=0,$$

$$2+(N+2)G-rt(2rt+(N+1)G)-Gt$$

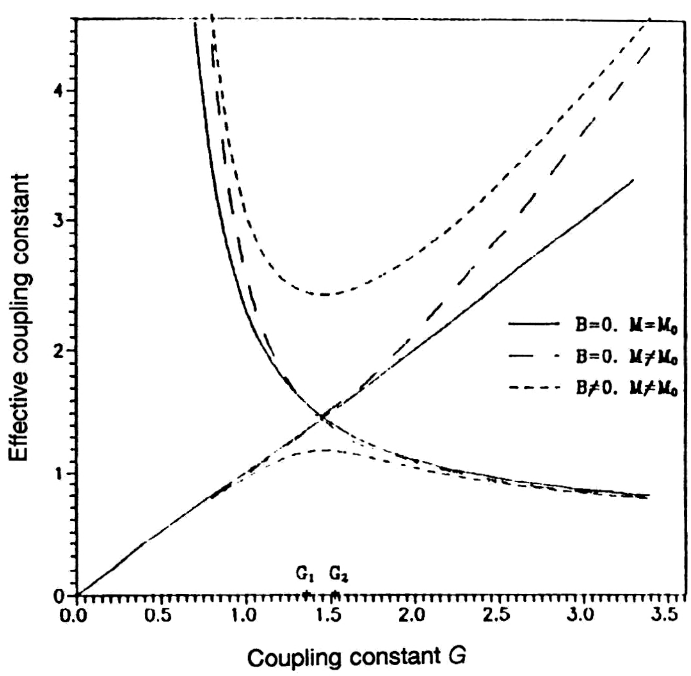

FIG. 9. Effective coupling constants in different phases ($N=4$): The upper and lower broken curves correspond to $G_{\rm eff}$ and $G_{\rm eff}^{(0)}$ in the phases with broken symmetry, respectively; the solid curve corresponds to $G_{\rm eff}$ in the symmetric phases.

$$+(N+2)G^2 \ln (rt)+G^2 \ln \left(\frac{r+2}{3r}\right)=0.$$

From these equations we find the asymptotic behaviors at $G \gg 1$:

$$\begin{aligned}
&t_0 \to \sqrt{\frac{N+2}{2}}\, G\sqrt{\ln G}, \quad t \to \sqrt{\frac{N+2}{2}}\, G^{\frac{1-N}{3}}\sqrt{\ln G},\\
&G_{\text{eff}}^0=\frac{G}{t_0(G)} \to \frac{\sqrt{2}}{\sqrt{(N+2)\ln G}} \ll 1,\\
&G_{\text{eff}}=\frac{G}{t(G)} \to \frac{\sqrt{2}}{\sqrt{(N+2)\ln G}}\, G^{N+2/3} \gg 1,\\
&E \to -\frac{m^3}{24\pi}(N^2+3N-4)\sqrt{\frac{N+2}{2}}\, G^3\sqrt{\ln^3 G}.
\end{aligned} \tag{7.7}$$

The energy, masses, and effective coupling constants are shown as functions G in Figs. 7–9.

BS_2: phase with mass splitting and $b \neq 0$

The last equation of the system (7.2) has a nonzero solution for b:

$$Gb^2=-2+(N-2)G+3Gt+(N-1)Gt_0-3G^2 \ln t$$

$$-(N-1)G^2 \ln \left(\frac{t+2t_0}{3}\right). \tag{7.8}$$

The phase corresponding to this solution is $O(N-1)$ invariant and is characterized by mass splitting, $M \neq M_0$, and a nonvanishing condensate: $B \neq 0$. It is described by the first two equations of the system (7.2) and by Eq. (7.8); these are conveniently represented in the form

$$b^2=\frac{t^2}{G},$$

$$-2r^2t^2+2G(1-r)t+3G^2 \ln r+G^2 \ln \left(\frac{2+r}{3r}\right)$$

$$-(N-1)G^2 \ln \left(\frac{1+2r}{3r}\right)=0,$$

$$(1-2r^2)t^2-(N+1)Grt-Gt+2+(N+2)G^2 \ln (rt)$$

$$+G^2 \ln \left(\frac{2+r}{3r}\right)=0,$$

where $r=t_0/t$. The asymptotic behavior of the functions t, t_0, G_{eff}, G_{eff}^0, and E (see Figs. 7–9) is the same as in the previous case (7.7). The condensate $b(G)$ for $G \gg 1$ has the form

$$b(G) \to \sqrt{\frac{N+2}{2}}\, G^{-(2N+1)/6}\sqrt{\ln G}.$$

An important point concerning both phases with broken symmetry is that the mass M_0 of the fields Φ_i belonging to the $O(N-1)$ multiplet is greater than the mass M of the field Φ (see Fig. 8). This contradicts Goldstone's theorem: The "Goldstone bosons" Φ_i should have zero mass. Note that Eqs. (7.2) also have solutions satisfying $M_0<M$, but the energies corresponding to these solutions are positive and increase with increasing G.

It follows from the asymptotic behaviors (7.6) and (7.7) that in the strong-coupling regime the phase S_2 is realized, since only in it does the effective coupling constant decrease with increasing G. Comparison of the asymptotic behaviors of the energies confirms this conclusion. For any N the energy of the phase S_2 is less than the energies of the remaining phases. The transition between the first (original and second symmetric phases occurs at the point G_0 [see (7.5)].

More detailed although approximate (the effective coupling constants are of order unity) information follows from comparison of the energies. It can be seen from Fig. 7 that phase transitions with the following rearrangement of the symmetry are possible:

$$O(N) \to O(N-1) \to O(N).$$

The system goes over from the original phase S_1 to the phase BS_2 with $B \neq 0$, and then to the phase S_2 with $M \neq m$. For example, for $N=4$ the critical coupling constants are $G_1=1.357...$, $G_2=1.525...$. Qualitatively, this picture does not depend on N: $G_1<G_0<G_2 \forall N$.

In any case, the following is true. For $G \ll 1$ and $G \gg 1$ the system exists in the first and second symmetric phases, respectively. The vacuum and Hamiltonian are $O(N)$ invariant, and the effective interaction is weak.

7.4. The $(\varphi^2)^2$ model in R^2

As in the case of a single-component field, the GEP approximation[30] and the method of canonical transformations[31] lead to the same result for the model (1.3) in R^2.

In this case, only the diagrams of first order in the coupling constant diverge (Fig. 1a), so that in two-dimensional space–time we obtain from the requirement $H_1=0$ the equations

$$\begin{aligned}
&m^2-M^2-g(3D_0(t)+(N-1)D_0(t_0))+3gB^2=0,\\
&m^2-M_0^2-g(3D_0(t)+(N+1)D_0(t_0))+gB^2=0,\\
&m^2B+gB^3-gB(3D_0(t)-(N+1)D_0(t_0))=0,
\end{aligned} \tag{7.9}$$

and the expression for the energy density has the form

$$\begin{aligned}
E=&\frac{m^2}{2}B^2+\frac{g}{4}B^4+(N-1)L_0(t_0)+L_0(t)\\
&+\frac{g}{4}\{-2[3D_0(t)+(N-1)D_0(t_0)]B^2+3D_0^2(t)\\
&+(N-1)D_0^2(t_0)+2(N-1)D_0(t)D_0(t_0)\},\\
D_0(t)=&\frac{1}{4\pi}\ln t, \quad L_0(t)=\frac{m^2}{8\pi}(t-1-\ln t),
\end{aligned} \tag{7.10}$$

where $t=M^2/m$, $t_0=M_0^2/m^2$. Equations (7.9) determine a minimum of the energy (7.10) as a function of (t,t_0,B) and are identical to the equations that minimize the GEP.

The solution $B=0$ of the first equation of (7.9) leads to the original symmetric representation with $M=M_0=m$ and

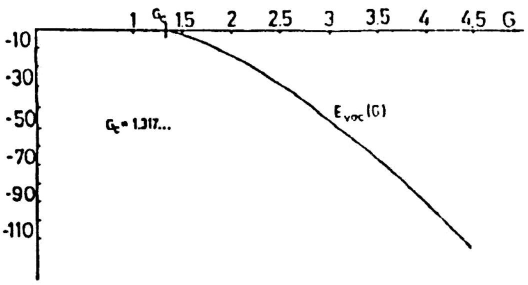

FIG. 10. Effective coupling constants for the model (1.3) in R^2.

$E\equiv 0$. In contrast to the three-dimensional case, there are no other symmetric solutions. The nontrivial solution of the last equation of (7.9) has the form

$$B^2=3D_0(t)+(N-1)D_0(t_0)-\frac{m^2}{g}.$$

Substituting it in the remaining equations, we find

$$t+2=G(3\ \ln t+(N-1)\ln t_0),\quad t_0=G\ \ln\left(\frac{t}{t_0}\right),$$

where $G=g/2\pi m^2$. The critical coupling constant is determined from the condition $E(G_c)=0$. For $N=4$ it is equal to $G_c=1.317\ldots$. The dependence of the energy on G is shown in Fig. 10. For $G\gg 1$, we have

$$t(G)\rightarrow(N+2)G\ \ln G,\quad t_0(G)\rightarrow G\ \ln\ \ln G,$$

$$E_B(G)\rightarrow-\frac{m^2}{8\pi}\,\tfrac{3}{4}G\ \ln^2 G,$$

$$G_{\text{eff}}(G)=\frac{g}{2\pi M^2}\rightarrow\frac{1}{(N+2)\ln G},$$

$$G^0_{\text{eff}}(G)=\frac{g}{2\pi M_0^2}\rightarrow\frac{1}{\ln\ \ln G}.$$

The effective coupling constants as function of G for $N=4$ are shown in Fig. 11.

It can be seen from Figs. 10 and 11 that the point G_c is critical and corresponds to a phase transition with symmetry breaking. Since the order parameter B is discontinuous at the point G_c, there is a first-order phase transition. All these conclusions agree with the results of the GEP approximation (see Ref. 30).

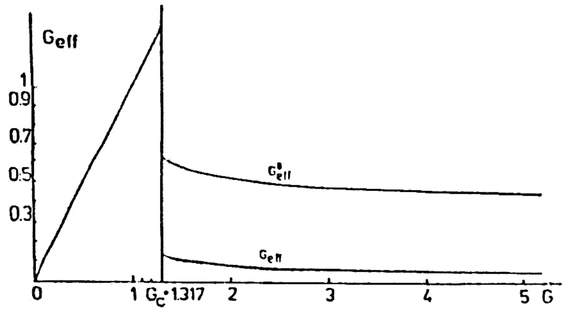

FIG. 11. Vacuum energy density for the model (1.3) in R^2.

8. STABILITY WITH RESPECT TO THE CHOICE OF THE RENORMALIZATION SCHEME IN THE ORIGINAL REPRESENTATION

In this section, we shall investigate the stability of the results obtained in Sec. 6 with respect to the choice of the R scheme. First of all, we reduce the problem to a form convenient for this purpose. We write the Lagrangian (1.1) in the form

$$L=\tfrac{1}{2}\varphi(x)(\Box-m_B^2)\varphi(x)-\tfrac{1}{4}g\varphi^4(x),$$

where $m_B^2=m^2(\mu)+\delta m^2(\mu)$, and $m(\mu)$ and $\delta m(\mu)$ are the renormalized mass and the mass counterterm in some fixed R scheme (see Sec. 3.1). A parameter of the theory is here the dimensionless coupling constant $G=g/2\pi m(\mu)$. The Hamiltonian density in the representation with mass $m(\mu)$ has the form

$$\begin{aligned}&H=H_0+H_I+H_{ct},\\&H_0=\tfrac{1}{2}[\pi^2+(\nabla\varphi)^2+m^2(\mu)\varphi^2],\quad H_I=\tfrac{1}{4}g\varphi^4,\\&H_{ct}=\tfrac{1}{2}\delta m^2(\mu)\varphi^2.\end{aligned}\tag{8.1}$$

The operators φ and π satisfy the usual CCR.

We make a canonical transformation to the representation with new mass $M=m(\mu)t$ and condensate $B=$const:

$$(\varphi,\pi)\rightarrow(\Phi+B,\Pi),\tag{8.2}$$

accompanying it with a compensating renormalization-group transformation (see Sec. 4.4): $\mu\rightarrow\nu=\mu t$ in order to ensure the condition of equivalence of the R schemes in the different representations:

$$\frac{m(\mu)}{\mu}=\frac{M}{\nu}.\tag{8.3}$$

In the new representation, the Hamiltonian density takes the form

$$\begin{aligned}&H=H_0'+H_I'+H_{ct}'+H_1,\\&H_0'=\tfrac{1}{2}[\Pi^2+(\nabla\Phi)^2+M^2\Phi^2],\quad H_I'=\tfrac{1}{4}g(\Phi^4+4B\Phi^3),\\&H_{ct}'=\tfrac{1}{2}\ \delta m^2(\mu t)\Phi^2+\delta m^2(\mu t)B\Phi,\\&H_1=\tfrac{1}{2}[m^2(\mu t)+3gB^2-M^2]\Phi^2+[m^2(\mu t)+gB^2]B\Phi.\end{aligned}\tag{8.4}$$

To ensure the correct form of the Hamiltonian, we set $H_1=0$, and this leads to the following equations for B and t:

$$\begin{aligned}&m^2(\mu t)+3gB^2-m^2(\mu)t^2=0,\\&B[m^2(\mu t)+gB^2]=0.\end{aligned}\tag{8.5}$$

We shall investigate the dependence of Eqs. (8.5) on the R scheme. In the case of the symmetric solution, $B=0$, the first equation of (8.5) takes the form

$$m^2(\mu t)=t^2m^2(\mu).\tag{8.6}$$

We solve this equation by calculating the mass $m(\mu t)$ in different R schemes.

8.1. The scheme of subtractions at zero momentum with arbitrary "mass"

In this scheme, the mass counterterms are given by diagrams (see Figs. 1a and 1b) with zero external momentum and arbitrary "mass" μ in the propagators. This is one of the possible ways of introducing the scale μ in the $(\varphi^4)_3$ theory. We set

$$m_B^2=\bar{m}^2(\mu)+\delta\bar{m}_a^2(\mu)+\delta\bar{m}_b^2(\mu), \tag{8.7}$$

where $\delta\bar{m}_a^2(\mu)$ and $\delta\bar{m}_b^2(\mu)$ correspond to the diagrams of Figs. 1a and 1b. These counterterms are readily calculated:

$$\delta\bar{m}_a^2(\mu)=-3g\Delta_{\text{reg}}(\mu), \quad \delta\bar{m}_b^2(\mu)=3!g^2\Sigma_{\text{reg}}(\mu),$$
$$\Delta_{\text{reg}}=\frac{1}{(2\pi)^2}\,\text{reg}\int_0^\infty \frac{du\,u^2}{u^2+\mu^2}, \tag{8.8}$$
$$\Sigma_{\text{reg}}=\frac{1}{(4\pi)^2}\,\text{reg}\int_0^\infty \frac{dt}{t}\exp\{-3\mu t\}.$$

Suppose that in the original representation we use the usual renormalization scheme at zero momentum corresponding to a particular choice of the parameter μ equal to the renormalized mass m, i.e., the condition

$$\bar{m}(m)=m \tag{8.9}$$

fixes the standard scheme of subtractions at zero momentum in the framework of the R scheme with arbitrary μ. Equation (8.6) takes the form

$$\bar{m}^2(mt)=t^2m^2. \tag{8.10}$$

Using the R invariance of the bare mass m_B and (8.9), we find

$$\bar{m}^2(\mu)=m^2\left[1+\frac{3}{2}G\left(1-\frac{\mu}{m}\right)+\frac{3}{2}G^2\ln\left(\frac{\mu}{m}\right)\right], \tag{8.11}$$

where $G=g/2\pi m$. Using (8.11) with $\mu=mt$ in (8.10), we obtain Eq. (6.23).

8.2. Dimensional regularization: the MS scheme

We introduce the notation

$$\varepsilon=3-d, \quad \alpha=\frac{g}{2\pi}, \quad m_B^2=m^2(\mu)+\delta m_a^2+\delta m_b^2, \tag{8.12}$$

where $m(\mu)$ is the running mass in the MS scheme. Making the standard calculations, we find for the diagram of Fig. 1a

$$\delta m_a^2=-3gm(\mu)\,\frac{1}{(2\pi)^{d/2}}\left(\frac{2\pi\mu}{m(\mu)}\right)^\varepsilon\Gamma\left(1-\frac{d}{2}\right).$$

Setting $d=3$, we obtain the final result

$$\delta m_a^2=\tfrac{3}{2}\,\alpha m(\mu), \tag{8.13}$$

which is natural for dimensional regularization in the case of odd dimension of the "physical" space–time. The calculation of the diagram of Fig. 1b for zero external momentum gives

$$\Sigma_{\text{reg}}=\frac{3}{4}\,\alpha^2\left[\frac{1}{\varepsilon}+\ln\left(\frac{4\pi\mu^2}{m^2(\mu)}\right)-\gamma_E+O(\varepsilon)\right]. \tag{8.14}$$

In the MS scheme, only the divergent part of this expression is included in the counterterm:

$$\delta m_b^2=\tfrac{3}{4}\alpha^2\,\frac{1}{\varepsilon}\,. \tag{8.15}$$

Substituting (8.13) and (8.15) in (8.12), we obtain

$$m_B^2=m^2(\mu)+\tfrac{3}{2}\alpha m(\mu)+\tfrac{3}{4}\,\alpha^2\,\frac{1}{\varepsilon}\,. \tag{8.16}$$

We go over in (8.16) to the new scale ν. In the limit $\varepsilon\to 0$, we obtain

$$m^2(\nu)+\tfrac{3}{2}\alpha m(\nu)-\tfrac{3}{2}\alpha^2\ln\left(\frac{\nu}{\mu}\right)-m^2(\mu)-\tfrac{3}{2}\alpha m(\mu)=0 \tag{8.17}$$

with the obvious condition

$$m(\nu)|_{\nu=\mu}=m(\mu). \tag{8.18}$$

The solution of this equation that satisfies (8.18) has the form

$$m(\nu)=-m(\mu)-\tfrac{3}{2}\,\alpha+\sqrt{\left[2m(\mu)+\frac{3}{2}\,\alpha\right]^2+6\alpha^2\ln\left(\frac{\nu}{\mu}\right)}. \tag{8.19}$$

Setting $\nu=\mu t$ and $m(\nu)=m(\mu)t$ in (8.17), we obtain an equation identical to (6.23). Thus, in the two cases the function $t(\cdot)$ is identical, whereas the running masses $\bar{m}(\nu)$ and $m(\nu)$ are completely different [see (8.11) and (8.19)].

8.3. Dimensional regularization: subtractions at zero momentum

We include in the counterterm δm_b^2 not only the pole term of the expression (8.14) but also its finite part. The bare mass takes the form

$$m_B^2=\bar{m}^2(\mu)+\frac{3}{2}\,\alpha\bar{m}(\mu)+\frac{3}{4}\,\alpha^2\left[\frac{1}{\varepsilon}-\gamma_E+\ln\left(\frac{4\pi\mu}{\bar{m}^2(\mu)}\right)\right]. \tag{8.20}$$

Going over in the standard manner in (8.20) to the new scale ν, we obtain the equation

$$\bar{m}^2(\nu)-\bar{m}^2(\mu)+\tfrac{3}{2}\alpha[\bar{m}(\nu)-\bar{m}(\mu)]-\tfrac{3}{2}\alpha^2\ln\left(\frac{\nu}{\mu}\right)+\tfrac{3}{4}\alpha^2\ln\left(\frac{\bar{m}^2(\mu)\nu^2}{\bar{m}^2(\nu)\mu^2}\right)=0. \tag{8.21}$$

We see that the running mass $\bar{m}$ determined by Eq. (8.21) differs from $\bar{m}$ and m [see (8.11) and (8.19)]. Nevertheless, substitution of

$$\nu=\mu t, \quad \bar{m}(\nu)=\bar{m}(\mu)t$$

in (8.21) gives an equation identical to (6.23).

The above calculations enable us to conclude that although the running mass depends very strongly on the R scheme the equation for the parameter t is more stable with respect to the choice of the renormalization scheme.

9. THE FOUR-DIMENSIONAL φ^4 THEORY

9.1. The $(\varphi^4)_4$ Hamiltonian and canonical transformations

The Hamiltonian density that describes the system (1.1) and (2.1) in the strong coupling regime has the form (2.3). The canonical variables φ and π can be represented in the form (2.4). They satisfy the usual CCR. The operators of creation $a^+(\mathbf{k})$ and annihilation, $a(\mathbf{k})$, in terms of which φ and π are expressed are defined on a Fock space of particles with mass $m(\mu)$ and vacuum vector $|0\rangle$ satisfying the conditions

$$a(\mathbf{k})|0\rangle=0\forall\mathbf{k}, \quad \langle 0|0\rangle=1.$$

The renormalization scheme in (2.3) is fixed, i.e., a definite choice of the R_μ class has been made, and the ratio $C=m(\mu)/\mu$ is fixed.

The representation (2.3) corresponds to $g(\mu)\ll 1$. Bearing this in mind, we want to know what is the system in the strong-coupling regime $g(\mu)\gg 1$.

We make the canonical transformation

$$(\varphi,\pi)\rightarrow\left(\frac{1}{\zeta}\Phi+\frac{1}{\zeta}B,\zeta\Pi\right). \tag{9.1}$$

Here, (Φ, Π) are fields with mass $M=tm(\mu)$, and B is a constant. In accordance with the requirement of equivalence of the R schemes in different representations (see Sec. 4.4) the canonical transformation to the new mass M is accompanied by a compensating change $\mu\rightarrow\nu=t\mu$ of the renormalization point, as a result of which the constant ζ of a finite renormalization of the field appears in (9.1). The explicit form of the canonical transformation (9.1) in terms of the creation and annihilation operators is given in (4.8) and (4.9). The fields (Φ, Π) satisfy the CCR. For $t\neq 1$ and $B\neq 0$ they are defined on a Fock space that is unitarily inequivalent to the original one.

The Hamiltonian density in the new representation takes the form

$$\begin{aligned}
&H=H_0'+H_I'+H_{ct}'+H_1,\\
&H_0'=\tfrac{1}{2}[\Pi^2+(\nabla\Phi)^2+M^2\Phi^2], \quad H_I'=\tfrac{1}{4}g(\nu)\\
&\qquad\times[\Phi^4+4B\Phi^3],\\
&H_{ct}'=\frac{1}{2}\left[\left(\frac{1}{Z_2'}-1\right)\Pi^2+(Z_2'-1)(\nabla\Phi)^2\right]+\frac{1}{2}[\delta m^2(\nu)\\
&\qquad+3(Z_1'-1)g(\nu)B^2]\Phi+\tfrac{1}{4}(Z_1'-1)g(\nu)\\
&\qquad\times(\Phi^4+4B\Phi^3)+[\delta m^2(\nu)+(Z_1'-1)g(\nu)B^2]B\Phi,
\end{aligned} \tag{9.2}$$

$$\begin{aligned}
&H_1=\tfrac{1}{2}[m^2(\nu)+3g(\nu)B^2-M^2]\Phi^2+[m^2(\nu)\\
&\qquad+g(\nu)B^2]B\Phi.
\end{aligned}$$

Here, $\nu=t\mu$, $M=tm(\mu)$. To obtain the correct form of the Hamiltonian, we set $H_1=0$, and this leads to the equations

$$\begin{aligned}
&m^2(\mu t)+3g(\mu t)B^2-m^2(\mu)t^2=0,\\
&B[m^2(\mu t)+g(\mu t)B^2]=0.
\end{aligned} \tag{9.3}$$

Here, $m(\mu t)$ and $g(\mu t)$ are related to $m(\mu)$ and $g(\mu)$ by a scale transformation and are determined by the RG equations (3.2) with $\nu=t\mu$ and boundary conditions

$$g(\mu t)=g(\mu) \quad \text{for } t=1, \quad m(\mu t)=m(\mu) \quad \text{for } t=1. \tag{9.4}$$

Equations (9.3) and (9.4) describe in general form the phase structure of the $(\varphi^4)_4$ theory in an arbitrary renormalization scheme. These equations reduce the problem of the phase structure to the properties of the RG functions.

9.2. The symmetric phase

Setting $B=0$, we obtain for t the equation

$$m^2(\mu t)=t^2m^2(\mu). \tag{9.5}$$

Note that (9.5) has the same form as the corresponding equation (8.6) in R^3.

It is remarkable that in conjunction with Eq. (9.5) the system (3.2) can be solved in general form (without knowing the γ_m and β functions) in an arbitrary R scheme, whereas for the system (3.2) itself this can be done only in mass-independent renormalization schemes. Indeed, the second equation of (3.2) and (9.5) mean that the parameter t is a function of the coupling constant, which in these equations can be regarded as an independent variable. As a result, the system (3.2), (9.5) can be rewritten in the form

$$\begin{aligned}
&\frac{1}{t}\frac{dt}{d\bar{g}}=\beta^{-1}\left(\bar{g},\frac{m(\mu t)}{\mu t}\right),\\
&\frac{2}{t}\frac{dt}{d\bar{g}}=-\gamma_m\left(\bar{g},\frac{m(\mu t)}{\mu t}\right)\beta^{-1}\left(\bar{g},\frac{m(\mu t)}{\mu t}\right),\\
&m(\mu t)=tm(\mu),
\end{aligned} \tag{9.6}$$

where $\bar{g}$ denotes the renormalized coupling constant. By virtue of the boundary conditions (9.5), we have

$$t(\bar{g})=1 \quad \text{for } \bar{g}=g. \tag{9.7}$$

The first two equations of (9.6) with allowance for the third take the form

$$\frac{d\ln t}{d\bar{g}}=\frac{1}{\beta(\bar{g},C)}, \quad 2\frac{d\ln t}{d\bar{g}}=-\frac{\gamma_m(\bar{g},C)}{\beta(\bar{g},C)}. \tag{9.8}$$

The constant C was fixed in the construction of the original representation (2.3).

A free parameter in the system (9.8) with the boundary condition (9.7) is the boundary value of the coupling constant g. Integrating (9.8) over $\bar{g}$ and taking into account (9.7), we obtain

$$\ln t=\int_g^{G_{\text{eff}}}\frac{dx}{\beta(x,C)}, \quad \int_g^{G_{\text{eff}}}dx\,\frac{2+\gamma_m(x,C)}{\beta(x,C)}=0, \tag{9.9}$$

where $G_{\text{eff}}=\bar{g}(g)$ is the effective coupling constant. By virtue of (9.7), these equations have the solution $t\equiv 1$, $G_{\text{eff}}\equiv g$. The existence of other solutions depends on the form of the RG equations. To remove the problem of the scheme dependence, we shall assume that in the original representation (2.3) we use the canonical μ scheme and $C=1$ (for more details, see Sec. 3.1). We recall that in the μ scheme the two-point Green's function is normalized by the condition

$$\tilde{G}(p^2) \overset{p^2\to\mu^2}{\longrightarrow} \frac{i}{p^2-m^2(\mu)}, \tag{9.10}$$

and for $C=m(\mu)/\mu=1$ we obtain a renormalization scheme on the mass shell, i.e., $m(\mu)$ is equal to the physical mass $m_{\rm ph}$ by construction. The system (9.9) takes the form

$$\ln t=\int_g^{G_{\rm eff}}\frac{dx}{\beta(x)},\quad \int_g^{G_{\rm eff}}dx\,\frac{2+\gamma_m(x)}{\beta(x)}=0,$$

$$M=m_{\rm ph}t, \tag{9.11}$$

where

$$\gamma_m(g)\equiv\gamma_m(g,1),\quad \beta(g)\equiv\beta(g,1),$$

and $\gamma_m(g,m(\mu)/\mu)$ and $\beta(g,m(\mu)/\mu)$ are calculated in the framework of the canonical μ scheme.

Note that (9.5) in this case has the form

$$\frac{m(m_{pt}t)}{m_{\rm ph}t}\equiv\frac{m(M)}{M}=1, \tag{9.12}$$

so that M, like $m_{\rm ph}$, satisfies the condition of renormalization on the mass shell,

$$\tilde{G}(p^2) \overset{p^2\to M^2}{\longrightarrow} \frac{i}{p^2-M^2}, \tag{9.13}$$

and, therefore, has the meaning of the physical mass in the new representation (9.2).

We return to the system (9.11). Since the exact γ_m and β functions are unknown, we simply consider different possibilities. The behavior of $\gamma_m(x)$ and $\beta(x)$ at small x can be found by perturbation theory. The integrand in the second equation of (9.11) behaves as

$$F(x)=\frac{2+\gamma_m(x)}{\beta(x)} \overset{x\to0}{\longrightarrow} \frac{2}{\beta_1\alpha^2}, \tag{9.14}$$

where $\alpha=3!x/(4\pi)^2$, $\beta_1=3/2$. It is known that the β function is positive for $x\in(0,g^*)$ where the ultraviolet-stable point g^* may be either finite or infinite. If the function $F(x)$ does not change sign in the interval $(0,g^*)$, then (9.11) has only the trivial solution $G_{\rm eff}\equiv g$, $t\equiv1$. Another possibility is illustrated in Fig. 12. A second solution of (9.11) exists if $F(x)$ changes sign at a certain point $g_c\in(0,g^*)$.

For example, suppose

$$\gamma_m=-ax,\quad \beta=bx^2(g^*=\infty),$$

where $a>0$, $b>0$.
After integration, we find from (9.11)

$$-\frac{1}{G_{\rm eff}}+\frac{1}{g}-\frac{a}{2}\ln\left(\frac{G_{\rm eff}}{g}\right)=0,\quad b\ln t=\frac{1}{g}-\frac{1}{G_{\rm eff}}.$$

The asymptotic behaviors in the strong-coupling regime have the form

$$t(g)\overset{g\to\infty}{\longrightarrow} g^{-a/2b}\ll1,\quad G_{\rm eff}(g)\overset{g\to\infty}{\longrightarrow}\frac{2}{a\ln g}\ll1. \tag{9.15}$$

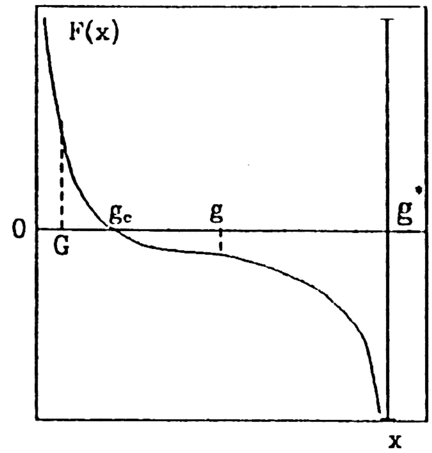

FIG. 12. Possible behavior of the integrand.

This example illustrates the general picture. The effective coupling constant depends only g, and

$$G_{\rm eff}(g)\overset{g\to0}{\longrightarrow} g^*,\quad G_{\rm eff}(g_c)=g_c,\quad G_{\rm eff}(g)\overset{g\to g^*}{\longrightarrow}0, \tag{9.16}$$

and, since $\beta(x)>0$,

$$t(g)\overset{g\to0}{\longrightarrow}\infty,\quad t(g_c)=1,\quad t(g)\overset{g\to g^*}{\longrightarrow}0. \tag{9.17}$$

From comparison of the asymptotic behaviors of the effective coupling constants we conclude that in the strong-coupling regime ($g\ll1$) the system exists in a phase with mass $m_{\rm ph}$, while in the strong-coupling limit ($g\to g^*$) a different symmetric phase is realized with physical massed $M\ll m_{\rm ph}$ [see (9.13)] and coupling constant $G_{\rm eff}\ll1$. The phase transition occurs at a value g_c of the coupling constant g at which the anomalous dimension of the operator φ^2 compensates its canonical dimension: $2+\gamma_m(g_c)=0$.

9.3. Dynamical symmetry breaking

The Hamiltonian H'_{ct} reflects the well-known fact that the counterterms for the $(\varphi^4)_4$ theory with spontaneous symmetry breaking are completely determined by the counterterms for the symmetric case (for example, see Ref. 45). Thus, $m(\mu t)$ and $g(\mu t)$ in (9.3) are determined by the same equations (3.2) for $B=0$ and $B\neq0$.

Equations (9.3) can be readily rewritten in the form

$$B^2=-\frac{m^2(\mu t)}{g(\mu t)},\quad t^2=-2\,\frac{m^2(\mu t)}{m^2(\mu)}. \tag{9.18}$$

Equations (9.18) do not have real solutions if $m^2(\mu t)>0$, $\forall g(\mu)$, t. Just such a situation occurs for any mass-independent R scheme. Thus, at least in this case there are no representations in the system with $B\neq0$ and, therefore, there is no dynamical symmetry breaking.

9.4. Correlation between the phase structure and ultraviolet divergences

We can now compare the phase structures of the models (1.1), (1.2), (1.3) in R^4, R^3, and R^2 (Ref. 31). It can be well

TABLE I. Phases in the strong-coupling regime.

	$G \ll 1$	$G \gg 1$
R^2	$\frac{1}{2}m^2\varphi^2+\frac{1}{4}g\varphi^4$	$\frac{1}{2}M^2\Phi^2+\frac{1}{4}g\Phi^4+gB(g)\Phi^3$ (BS)
R^3		$\frac{1}{2}M^2\Phi^2+\frac{1}{4}g\Phi^4$ (S)
R^4	$\frac{1}{2}m^2\varphi^2+\frac{1}{4}g\varphi^4$	$\frac{1}{2}M^2\Phi^2+\frac{1}{4}G_{\mathrm{eff}}\Phi^4$, (S) if $\exists g_c \in (0, g^*)$: $2+\gamma_m(g_c)=0$
		?, if $\forall g \in (0, g^*) 2+\gamma_m(g) > 0$
R^2	$\frac{1}{2}m^2\varphi^2+\frac{1}{4}g\varphi^4+m\sqrt{\frac{g}{2}}\varphi^3$	$\frac{1}{2}M^2\Phi^2+\frac{1}{4}g\Phi^4+gB(g)\Phi^3$ (S)
R^3		$\frac{1}{2}M^2\Phi^2+\frac{1}{4}g\Phi^4$ (S)
R^2	$\frac{1}{2}m^2\sum_i^N\varphi_i^2+\frac{1}{4}g\left[\sum_i^N\varphi_i^2\right]^2$	$\frac{1}{2}M^2\Phi^2+\frac{1}{2}M_0^2\sum_i^{N-1}\Phi_i^2$ (BS) $+\frac{1}{4}g\left[\sum_i^{N-1}\Phi_i^2+\Phi^2\right]+gB(g)\Phi\left[\sum_i^{N-1}\Phi_i^2+\Phi^2\right]$
R^3		$\frac{1}{2}M^2\sum_i^N\Phi_i^2+\frac{1}{4}g\left[\sum_i^N\Phi_i^2\right]^2$ (S)

seen from Table I that the behaviors of the systems are quite different for different d. Irrespective of the symmetry of the initial Lagrangian, for $G \gg 1$ the phase with broken symmetry is realized in R^2, whereas in R^3 we have the symmetric phase. It may be concluded that the different ultraviolet behavior leads to a different phase structure. The following heuristic argument (see, for example, Simon's monograph of Ref. 8) to some extent explains the correlation between the phase structure of the theory and the nature of its ultraviolet divergences. An intuitively clear reason for the breaking of the symmetry in the $(\varphi^4)_2$ theory is the normal ordering of the Hamiltonian. In other words, the breaking of the symmetry in this case is explained by the contribution to the mass renormalization of the diagram of Fig. 1a, which in the strong-coupling regime changes the sign of the bare mass m_B. The opposite situation occurs in the $(\varphi^4)_3$ theory, since now two diagrams with different signs contribute to m_B. The bare mass is positive at large g and there is no symmetry breaking. In the $(\varphi^4)_4$ theory, the picture is much more complicated since the bare mass is represented by a series of varying sign. This series can be positive for any g, so that reasons for the appearance of a phase with broken symmetry are altogether absent.

10. THE $(\varphi^4)_2$ THEORY AND FINITE TEMPERATURE

The very formulation of the problem of the dynamical rearrangement of the ground state of a quantum-field system when a coupling constant (such as the fine structure constant) is changed appears rather artificial, although it is of interest from the purely theoretical point of view. Much closer to real physics is the investigation of the behavior of a field system when the temperature changes. Such a problem has a direct bearing on several problems in solid-state physics,[6,58] on systems of the type of a quark–gluon plasma, or on the problem of the evolution of the universe at early stages.[3]

Active investigation of dynamical temperature-dependent effects in quantum field theory began with Kirzhnits's paper of Ref. 2. Most studies have been of four-dimensional theories in the one- and two-loop approximations to the effective potential. The phase structure of the $(\varphi^4)_2$ theory was investigated in the framework of a high-temperature expansion in Refs. 4–6.

We shall obtain the phase diagrams in the (G,θ) plane for the series (1.1) and (1.2). We recall that $G=g/2\pi m^{4-d}$ and $\theta=T/m$ are dimensionless parameters of the theory. Whereas the behaviors of the systems in R^2 and R^3 with respect to the variable G are quite different (see Table I), the dependence on the temperature is qualitatively the same in the two cases, namely, irrespective of the initial symmetry the system is in the symmetric phase if the temperature is sufficiently high. This is the most general conclusion from the form of the phase diagrams.

10.1. Hamiltonian at zero temperature and canonical transformations

From the physical point of view, any representation of the CCR will be adequate only if the mass of the particles depends on the temperature. *A priori*, this dependence is unknown even in the weak-coupling limit. We therefore begin the construction of the finite-temperature theory with the representation of the CCR at $T=0$, and then, using the method of canonical transformations, we introduce a temperature dependence. As in Sec. 6, we shall deal with the Lagrangian (6.1), which combines the two models (1.1) and (1.2).

Initial representation for $G \ll 1$, $T=0$

To eliminate all the divergences, it is sufficient to write the Hamiltonian density in normal form:

$$H=H_0+H_I, \quad H_0[\varphi,\pi]=\tfrac{1}{2}:[\pi^2(x)+[\nabla\varphi(x)]^2+m^2\varphi^2(x)]:,$$
$$H_I[\varphi,\pi]=:\left[\frac{1}{4}g_4\varphi^4(x)+g_3\varphi^3(x)\right]:. \tag{10.1}$$

The operators φ and π satisfy the usual commutation relations. The creation and annihilation operators $a^+(k)$ and $a(k)$ act on a Fock space with mass m. The vacuum vector $|0\rangle$ satisfies the condition $a(k)|0\rangle=0$, $\forall k$.

Canonical transformation

We make a canonical transformation

$$\pi(x)\to\pi_t(x), \quad \varphi(x)\to\varphi_t(x)+B, \tag{10.2}$$

where φ_t and π_t are fields with mass $M=mt$. The new creation and annihilation operators $a^+(k,t)$ and $a(k,t)$ act on the Fock space with vacuum vector

$$|0(t,B)\rangle=U_2^{-1}(t)U_1^{-1}(B)|0\rangle, \quad a(k,t)|0(t,B)\rangle=0, \quad \forall k. \tag{10.3}$$

Here, U_1 and U_2 are represented by the expressions (6.8).

Expressing the Hamiltonian in the new variables and going over to normal ordering with respect to the vacuum $|0(t,B)\rangle$ we obtain

$$H=H_0'+H_I'+H_1+E,$$
$$H_0'=\tfrac{1}{2}:[\pi_t^2(x)+(\nabla\varphi_t(x))^2+M^2\varphi_t^2(x)]:, \tag{10.4}$$
$$H_I'=:\left[\frac{1}{4}h_4\varphi_t^4(x)+h_3\varphi_t^3(x)\right]:, \quad h_3=g_3+g_4B,$$
$$h_4=g_4,$$
$$H_1=:\left[\frac{1}{2}R(t,B)\varphi_t^2(x)+P(t,B)\varphi_t(x)\right]:,$$
$$R=m^2-M^2+3g_4(B^2-D_0)+6g_3B, \tag{10.5}$$
$$P=m^2B+g_4(B^3-3BD_0)+3g_3(B^2-D_0),$$
$$E=\tfrac{1}{2}m^2B^2+L_0+\frac{g_4}{4}[b^4-6B^2D_0+D_0^2]+g_3B\times[B^2-3D_0],$$
$$D_0(t)=\frac{1}{4\pi}\ln t, \quad L_0(t)=\frac{m^2}{8\pi}[t-1-\ln t]. \tag{10.6}$$

To introduce a temperature dependence, we make one further canonical transformation, which is the basic idea of the method of thermo field dynamics (TFD).

10.2. Thermo field dynamics

A detailed description of TFD can be found in the review of Ref. 64 or in the monograph of Ref. 58. We here restrict ourselves to a brief exposition of the basic idea.

The basic idea of TFD is to double the number of field variables when $T>0$. The intuitive justification for this reduces to the following. The presence of a thermal reservoir leads to the existence of a large number of excited quanta and free energy levels, which are called holes. Therefore, the absorption of energy by the system occurs in two ways—either by the excitation of additional quanta or by filling of free positions in the space of the particles, i.e., by the annihilation of holes with negative energy (we emphasize the difference from antiparticles in the usual understanding, which are holes in a state space with negative energy). These two absorption mechanisms are independent of each other, and it is this that leads to the doubling in the number of the field variables.

In accordance with this motivation, we construct the operators

$$\alpha(k)=a(k,t)\otimes\mathbf{1}, \quad \tilde{\alpha}(k)=\mathbf{1}\otimes a(k,t), \tag{10.7}$$

which satisfy the usual CCR:

$$[\alpha(k),\alpha^+(k')]=\delta(k-k'), \quad [\tilde{\alpha}(k),\tilde{\alpha}^+(k')]=\delta(k-k')$$

and act on the Fock space with vacuum

$$|0(t,B)\rangle\rangle=|0(t,B)\rangle\otimes|0(t,B)\rangle. \tag{10.8}$$

The operator $\alpha(k)$ and $\tilde{\alpha}(k)$ describe the annihilation of particles and holes. At a nonzero temperature, the ground state of the system is not a state without particles. The mean number of particles is described by a distribution function (bosonic or fermionic). The next step is to go over to a quasiparticle picture. Quasiparticles correspond to excitations above the physical ground state, the thermal vacuum.

The temperature-dependent operators $\alpha(k,\beta)$, $\tilde{\alpha}(k,\beta)$ $(\beta=1/T)$ are introduced by means of a temperature-dependent Bogolyubov transformation

$$\begin{aligned}\alpha(k,\beta)&=\alpha(k)\cosh(\xi)-\tilde{\alpha}^+(k)\sinh(\xi)\\ \tilde{\alpha}(k,\beta)&=\tilde{\alpha}(k)\cosh(\xi)-\alpha^+(k)\sinh(\xi),\end{aligned} \tag{10.9}$$

or equivalently, in operator form,

$$\alpha(k,\beta)=U^{-1}(\beta)\alpha(k)U(\beta),$$
$$\tilde{\alpha}(k,\beta)=U^{-1}(\beta)\tilde{\alpha}(k)U(\beta),$$
$$U(\beta)=\exp\left\{\int dk\,\xi(k,\beta)[\tilde{\alpha}(k)\alpha(k)-\alpha^+(k)\tilde{\alpha}^+(k)]\right\}.$$

The operators $\alpha(k,\beta)$, $\tilde{\alpha}(k,\beta)$ are defined on a Fock space with thermal vacuum:

$$|0(\beta,t,B)\rangle=U^{-1}(\beta)|0(t,B)\rangle\rangle, \tag{10.10}$$

and

$$\alpha(k,\beta)|0(\beta,t,B)\rangle=\tilde{\alpha}(k,\beta)|0(\beta,t,B)\rangle=0 \quad \forall k,\beta.$$

The parameter $\xi(k,\beta)$ is determined by the requirement that the mean number of particles in the state $|0(\beta,t,B)\rangle$ be equal to the statistical distribution function:

$$\langle 0(\beta,t,B)|\alpha^+(k)\alpha(k)|0(\beta,t,B)\rangle=n(\omega(k,t))=[\exp\{\beta\omega(k,t)\}-1]^{-1}.$$

Using (10.7) and the transformation that is the inverse of (10.9), we find

$$\sinh^2(\xi)=[\exp\{\beta\omega(k,t)\}-1]^{-1}. \tag{10.11}$$

The condition (10.11) guarantees that $T=1/\beta$ has the meaning of a statistical temperature. In accordance with (10.7), we construct the fields (Φ, Π) and $(\tilde{\Phi},\tilde{\Pi})$, which are defined on the Fock space with vacuum (10.8). The expressions for the tilde-conjugate variables $\tilde{\Phi}$, $\tilde{\Pi}$ are obtained from Φ, Π by the substitution $(\alpha,\alpha^+)\rightarrow(\tilde{\alpha},\tilde{\alpha}^+)$.

To determine the fields on the space with the thermal vacuum (10.10) we use the transformation that is the inverse of (10.9), by means of which the operators Φ, Π, $\tilde{\Phi}$, $\tilde{\Pi}$ can be expressed in terms of $\alpha(k,\beta)$, $\alpha^+(k,\beta)$, $\tilde{\alpha}(k,\beta)$, $\tilde{\alpha}^+(k,\beta)$.

The density of the total Hamiltonian in the TFD formalism is determined by

$$\hat{\mathscr{H}}=\mathscr{H}-\tilde{\mathscr{H}},\quad \mathscr{H}=H\otimes\mathbf{1},\quad \tilde{\mathscr{H}}=\mathbf{1}\otimes H. \tag{10.12}$$

Here, $\mathscr{H}$ describes the particles and $\tilde{\mathscr{H}}$ the holes with negative energy, this explaining the appearance of the minus sign in front of $\tilde{\mathscr{H}}$. The Hamiltonian density $\mathscr{H}$ in the representation of the thermal vacuum (10.10) takes the form

$$\mathscr{H}=\mathscr{H}_0''+\mathscr{H}_I''+\mathscr{H}_1'+E',$$

$$\mathscr{H}_0''=\tfrac{1}{2}:[\Pi^2(x)+(\nabla\Phi(x))^2+M^2\Phi^2(x)]: ,$$

$$\mathscr{H}_I''=:\left[\frac{1}{4}h_4\Phi^4(x)+h_3\Phi^3(x)\right]: ,$$

$$\mathscr{H}_1'=:\left[\frac{1}{2}\mathscr{R}(t,B;\beta)\Phi^2(x)+\mathscr{S}(t,B;\beta)\Phi(x)\right]: , \tag{10.13}$$

$$\mathscr{R}=m^2-M^2+3g_4(B^2-D)+6g_3B,$$

$$\mathscr{S}=m^2B+g_4(B^3-3BD)+3g_3(B^2-D),$$

$$D(t,\beta)=\frac{1}{4\pi}\ln t-\frac{1}{\pi}d(\theta/\sqrt{t}).$$

The energy density E' is obtained from E by the substitution $D_0\rightarrow D$, $L_0\rightarrow L$, where

$$L(t,\beta)=\frac{m^2}{8\pi}\{t-1-D(t,\theta)+4t[2s(\theta/\sqrt{t})+d(\theta/\sqrt{t})]\},$$

$$d(z)=\int_0^\infty\frac{du}{\sqrt{1+u^2}}\left(\exp\left\{\frac{1}{z}\sqrt{1+u^2}\right\}-1\right)^{-1}, \tag{10.14}$$

$$s(z)=\int_0^\infty\frac{duu^2}{\sqrt{1+u^2}}\left(\exp\left\{\frac{1}{z}\sqrt{1+u^2}\right\}-1\right)^{-1}.$$

In accordance with the requirement of equivalence of the renormalization schemes, the normal product in these expressions relates to $\alpha(k,\beta)$, $\alpha^+(k,\beta)$. The operator $\tilde{\mathscr{H}}$ is constructed in accordance with the rule $\tilde{\mathscr{H}}=\mathscr{H}^*[\tilde{\Phi},\tilde{\Pi}]$.

The density E' of the internal energy of the state $|0(t,B;\beta)\rangle$ is related to the free-energy density F by

$$F=E'-TS, \tag{10.15}$$

where S is the entropy density:

$$S=-\int\frac{dk}{\sqrt{2\pi}}[n,(k,t)\ln n(k,t)-(1-n(k,t))\times\ln(1-n(k,t))]$$
$$=\frac{m^2}{\pi}\frac{t}{T}[2s(\theta/\sqrt{t})+d(\theta/\sqrt{t})],$$
$$n(k,t)=[\exp(\beta\omega(k,t)-1]^{-1}. \tag{10.16}$$

From (10.6), (10.14), (10.15), and (10.16) we obtain

$$F=\tfrac{1}{2}m^2B^2+\frac{g_4}{4}[B^4-6B^2D(t;\beta)+3D^2(t;\beta)]+g_3B[B^2-3D]+\frac{m^2}{8\pi}\{t-1-4\pi D(t;\theta)-4t[2s(\theta/\sqrt{t})+d(\theta/\sqrt{t})]\}. \tag{10.17}$$

We require that

$$\mathscr{R}(t,B;\theta)=0,\quad \mathscr{S}(t,B;\theta)=0. \tag{10.18}$$

It is easy to verify the equivalence of (10.18) and the equations

$$\frac{\partial F(t,B)}{\partial B}=0,\quad \frac{\partial^2F(t,B)}{\partial B^2}=M^2=m^2t,$$

which are analogous to the conditions of minimality and stability of the effective potential.[3] On the other hand, (10.18) determines the minimum of $F(t,B)$ as a function of the two variables t and B only for $\theta=0$. Thus, it is only in this case that our results should be compared with the results of the GEP approximation.

10.3. The symmetric model

For $g_3=0$ and $g_4=g$ we obtain from (10.18) and (10.13) the following equations for B and t $(G=g/2\pi m^2)$:

$$B[gB-3gD(t;\theta)+m^2]=0,$$
$$3gB^2-3gD(t;\theta)-m^2(t-1)=0. \tag{10.19}$$

Symmetric phase

Setting $B=0$ in the second equation of (10.19), we obtain

$$\frac{2}{3G}(t-1)=-\ln t+4d(\theta/\sqrt{t}). \tag{10.20}$$

This equation has a unique solution for all G and θ, and $t(G,\theta)\equiv1$ only for $\theta=0$. The free-energy density has the form

$$F_S=\frac{m^2}{8\pi}\left\{\left(\frac{2}{3G}+1\right)(t-1)+\frac{(t-1)^2}{3G}-4t[2s(\theta/\sqrt{t})+d(\theta/\sqrt{t})]\right\}. \tag{10.21}$$

Phase with broken symmetry

Using the nontrivial solution for B, we rewrite (10.19) in the form

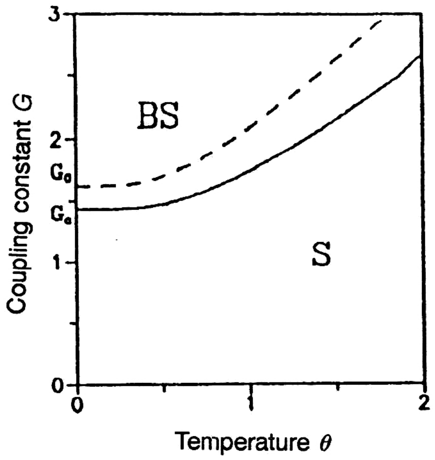

FIG. 13. Phase diagram for the symmetric $(\varphi^4)_2$ theory.

$$B^2=\frac{t}{4\pi G}, \quad \frac{t}{3G}+\frac{2}{3G}=\ln t-4d(\theta/\sqrt{t}).$$

The second equation has a solution only for G and θ such that $G \geqslant G_c(\theta)$. The function $G_c(\theta)$ is shown in Fig. 13. The free-energy density is given by

$$F_B=\frac{m^2}{8\pi}\left\{-\frac{1}{2G}+\left(1-\frac{1}{3G}\right)(t-1)-\frac{(t-1)^2}{6G}\right.$$
$$\left.-4t[2s(\theta/\sqrt{t})+d(\theta/\sqrt{t})]\right\}. \tag{10.22}$$

Comparing the effective coupling constants $G_{\rm eff}=G/t(G,\theta)$ in the symmetric phase and phase with broken symmetry, we find that the boundary of the phases is given by the function $G_c(\theta)$ (the solid curve in Fig. 13). Comparing the free energies F_S and F_B, we obtain the boundary shown by the broken curve in Fig. 13. It can be seen that these two boundaries do not contradict each other. The value of $G_0=1.625\ldots$ coincides with the critical coupling constant in the GEP approximation, since for $\theta=0$ Eq. (10.19) are identical to the equations that minimize the Gaussian effective potential.

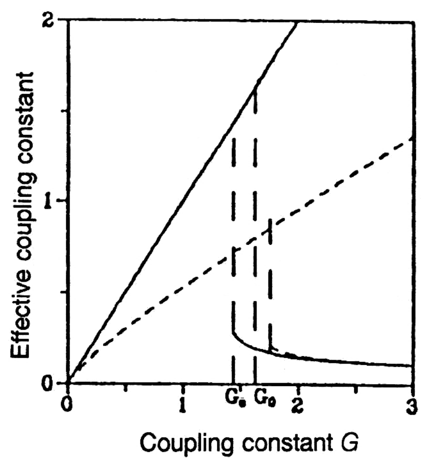

FIG. 14. Effective coupling constants for the symmetric $(\varphi^4)_2$ theory: The upper curves correspond to the symmetric phase with the broken curve for $\theta=1$ and the solid curve for $\theta=0$.

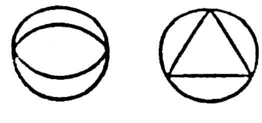

FIG. 15. The $O(G^2)$ and $O(G^3)$ diagrams in the symmetric phase.

In the critical region, the effective coupling constant is fairly large in both phases (see Fig. 14), so that the perturbative corrections will be large and can change the boundary represented by the broken curve. To estimate this change, we calculate the corrections to the free energy at zero temperature. We here restrict ourselves to order $O(G^3)$ for the symmetric phase and $O(G^2_{\rm eff})$ for the phase with broken symmetry. The necessary diagrams are shown in Figs. 15 and 16. As a result, we obtain

$$\Delta F_S=\frac{m^2}{8\pi}(-1.671G^2+4.039G^3+O(G^4)),$$
$$\Delta F_B=\frac{M^2}{8\pi}(-1.758G_{\rm eff}-4.316G^2_{\rm eff}-O(G^3_{\rm eff})). \tag{10.23}$$

It can be seen that the (asymptotic) series for ΔF_B has constant sign. This is a usual property of systems with degenerate vacuum (see, for example, Ref. 66). Making a Borel summation of the expansion (10.23), we find

$$\Delta F_S=\frac{m^2}{8\pi}\left(\int_0^\infty dte^{-t}\,\frac{1+0.806Gt}{1+0.806Gt+0.836(Gt)^2}-1\right),$$

$$\Delta F_B=-\frac{M^2}{8\pi}\left(\mathrm{VP}\int_0^\infty dte^{-t}\,\frac{1+0.531G'_{\rm eff}}{1-1.228G'_{\rm eff}}-1\right).$$

The symbol VP denotes the principal value of the integral in the Cauchy sense (on methods of summing asymptotic series of constant sign, see Ref. 66). The solid and broken curves in Fig. 17 show the free energy without and with allowance for the corrections, respectively. The corrections shift the vertical point from $G_0\sim1.625\ldots$ to $G_c(0)\sim1.44\ldots$. A similar picture must also hold for $\theta\neq0$.

The order parameter

$$\sigma=\pm\sqrt{\frac{t(G,\theta)}{4\pi G}} \tag{10.24}$$

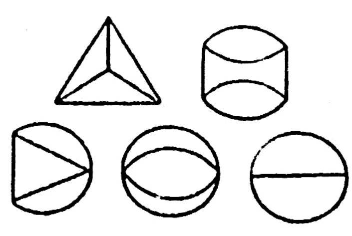

FIG. 16. The $O(G_{\rm eff})$ and $O(G^2_{\rm eff})$ diagrams in the phase with broken symmetry.

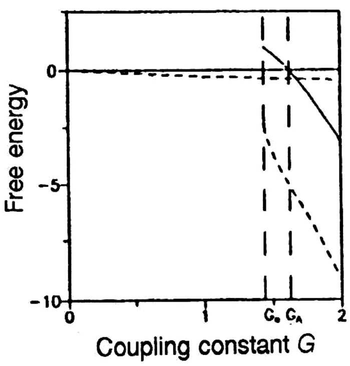

FIG. 17. The free-energy density: The broken curves correspond to F_S and F_B, the solid curves to $F_S+\Delta D_S$ and $F_B+\Delta F_B$.

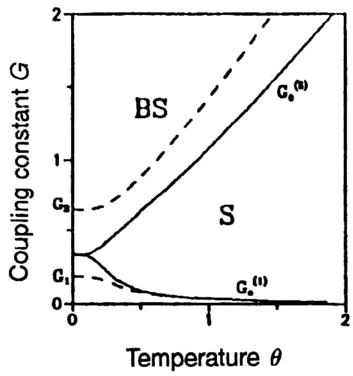

FIG. 19. Phase diagram for the $(\varphi^4)_2$ theory with initially broken symmetry.

and the mass $M^2=m^2t(G,\theta)$ (Fig. 18) are discontinuous at the phase boundaries, so that we have a first-order transition. However, this result is not reliable because of the large value of $G_{\rm eff}$ near the boundary. In the regimes of strong and weak coupling, the description is fairly accurate, since the effective coupling constant in these cases is small.

10.4. Model with initially broken symmetry

Substituting $g_4=g$ and $g_3=m(g/2)^{1/2}$ in (10.18) and (10.13), we obtain the equations

$$\begin{gathered} gB^3+3m\sqrt{g/2}B^2+B[m^2-3gD(t,\theta)] \\ -3m\sqrt{g/2}D(t,\theta)=0, \\ 3gB^2+3m\sqrt{2g}B-3gD(t,\theta)-M^2+m^2=0. \end{gathered} \tag{10.25}$$

In accordance with the solutions of this system, there are two phases with broken symmetry and one symmetric phase.

Symmetric phase

The first equation of (10.25) has the solution

$$B=-\frac{1}{\sqrt{4\pi G}}. \tag{10.26}$$

From (10.26) and the second equation of (10.25) we obtain for t the equation

$$\frac{2}{3G}t+\frac{1}{3G}=-\ln t+4d(\theta\sqrt{t}). \tag{10.27}$$

There exists a unique solution for all G and θ. With allowance for the relations (10.26) and (10.27), we obtain for the energy density the expression

$$F_S=\frac{m^2}{8\pi}\left\{\frac{1}{2G}+\left(1+\frac{2}{3G}\right)(t-1)+\frac{(t-1)^2}{3G}\right. \\ \left.-4t[2s(\theta/\sqrt{t})+d(\theta/\sqrt{t})]\right\}. \tag{10.28}$$

Phases with broken symmetry

Using the remaining solutions of the first equation of (10.25),

$$B=-\frac{1\pm\sqrt{t}}{4\pi G},$$

we obtain from the second equation

$$\frac{1}{3G}t-\frac{1}{3G}=\ln t-4d(\theta/\sqrt{t}). \tag{10.29}$$

This equation has solutions only for (G,θ) such that

$$G\leqslant G_c^{(1)}(\theta)\text{or } G\geqslant G_c^{(2)}(\theta).$$

The functions $G_c^{(1)}(\theta)$ and $G_c^{(1)}(\theta)$ are shown in Fig. 19. There are two solutions, and they are equal to each other at $G=G_c^{(1)}(\theta)$ or $G=G_c^{(2)}(\theta)$. These solutions are two different phases with broken symmetry. It can be seen from Fig. 19 that $G_c^{(1)}(0)=G_c^{(2)}(0)=G_c$. Substituting $t=1$ in (10.29), we find $G_c=1/3$. The region on the phase plane below $G_c^{(1)}(\theta)$ corresponds to the first phase with broken symmetry, while the region above $G_c^{(2)}(\theta)$ corresponds to the second such phase. The free-energy density has the form

$$F_B=\frac{m^2}{8\pi}\left\{\left(1-\frac{1}{3G}\right)(t-1)-\frac{(t-1)^2}{6G}-4t[2s(\theta/\sqrt{t})\right. \\ \left.+d(\theta/\sqrt{t})]\right\}.$$

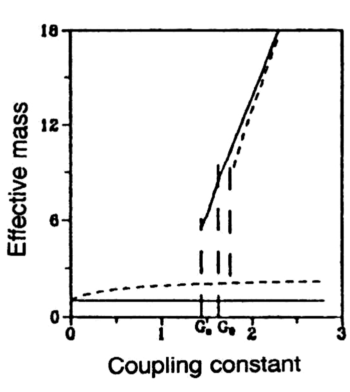

FIG. 18. Masses in the symmetric model: The broken curve corresponds to $\theta=1$, the solid curve to $\theta=0$: the upper curves represent the phase with broken symmetry.

As can be seen from Fig. 19, the boundaries of the phases, found by comparing the effective coupling constants (Fig.

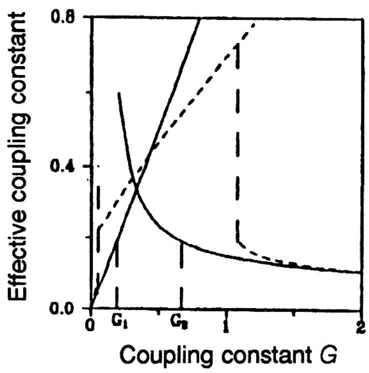

FIG. 20. Effective coupling constants for the $(\varphi^4)_2$ theory with initially broken symmetry: The solid curve corresponds to $\theta=0$, the broken curve to $\theta=1$; the symmetric phase is represented by the upper broken curve.

20) and the free energies, are mutually consistent. The values $G_1=0.19...$ and $G_2=0.64...$ in Fig. 19 agree with the critical points of the GEP approximation at zero temperature.[19]

Thus, there are two phases with broken symmetry and one symmetric phase. At zero temperature, the symmetry is broken for all G, although at $G=1/3$ there is a phase transition without change of symmetry. At the same time, for any fixed G the symmetry is restored if the temperature is sufficiently high. The phase transitions have first order, since the mass and order parameter (10.24) are discontinuous on the phase boundaries (see Fig. 21, $\theta=1$). The effective coupling constant is small everywhere except in the critical regions, where $G_{\text{eff}}\sim O(1)$, so that our description is sufficiently accurate only outside the region of the phase transitions.

The boundary $G_c^{(1)}(\theta)$ lies in the region of applicability of the high-temperature expansion ($\theta \gg G$). Its form agrees with the results of this method.[4–6]

11. THREE-DIMENSIONAL φ^4 MODEL AT FINITE TEMPERATURE

11.1. Representation of thermal vacuum

A canonical transformation to fields with a new mass and nonzero condensate was made in Sec. 6 [see (6.2) and (6.11)]. Therefore, we shall not reproduce here the calculations associated with this transformation. In addition, the transition from the representation (6.11) to the expressions that follow below is made in complete analogy with the two-dimensional case [see (10.7)–(10.21)]. The only fundamental difference is associated with the additional counterterms that eliminate the divergence of the second-order diagram (see Fig. 1b). Therefore, we begin directly with the expression for the Hamiltonian density $\mathscr{H}$ in the representation of the thermal vacuum $|0(\beta,t,B)\rangle$ (10.10):

$$\mathscr{H}=\mathscr{H}_0''+\mathscr{H}_I''+\mathscr{H}_{ct}''+\mathscr{H}_1',$$

$$\mathscr{H}_0''=\tfrac{1}{2}:[\Pi^2(\mathbf{x})+(\nabla\Phi(\mathbf{x}))^2+M^2\Phi^2(\mathbf{x})]:, \tag{11.1}$$

$$\mathscr{H}_I''=:\left[\frac{1}{4}h_4\Phi^4(\mathbf{x})+h_3\Phi^3(\mathbf{x})\right]:.$$

The normal ordering in these expressions applies to the temperature-dependent operators $\alpha(\mathbf{k},\beta)$ and $\alpha^+(\mathbf{k},\beta)$. The counterterm operator $\mathscr{H}_{ct}''$ in this representation takes the form

$$\mathscr{H}_{ct}''(M,\beta)=:\left[\frac{1}{2}A(M)\Phi^2(\mathbf{x})+C(M)\Phi(\mathbf{x})\right]:, \tag{11.2}$$

where the temperature-dependent functions A and C have the form[36]

$$A(M)=3!g_4^2\Sigma(M),\quad C(M)=3!g_3g_4\Sigma(M),$$

$$\Sigma(M)=\Sigma_0(M)+3\Sigma_\beta(M)+3\Sigma_{\beta\beta}(M),$$

$$\Sigma_\beta(M)=\frac{1}{2(2\pi)^2}\sigma_\beta(t,\theta),$$

$$\Sigma_{\beta\beta}(M)=\frac{1}{2(2\pi)^2}\sigma_{\beta\beta}(t,\theta), \tag{11.3}$$

$$\sigma_\beta(t,\theta)=-\ln 3\cdot\frac{\theta}{t}\ln\left(1-\exp\left\{-\frac{t}{\theta}\right\}\right),$$

$$\sigma_{\beta\beta}(t,\theta)=\int_1^\infty\int_1^\infty dxdy\left[\exp\left\{\frac{xt}{\theta}\right\}-1\right]^{-1}\times\left[\exp\left\{\frac{yt}{\theta}\right\}-1\right]^{-1}\times\left[\frac{1}{\sqrt{4(x^2+y^2+xy)-3}}+\frac{1}{\sqrt{4(x^2+y^2-xy)-3}}\right]. \tag{11.4}$$

In the scheme of subtractions at zero momentum, the operator $\mathscr{H}_1'$ takes the form

$$\mathscr{H}_1'=:\left[\frac{1}{2}\mathscr{R}(t,B,\beta)\Phi^2(\mathbf{x})+\mathscr{S}(t,B,\beta)\Phi(\mathbf{x})\right]:,$$

$$\mathscr{R}=m^2-M^2+3g_4(B^2-D)+6g_3B+6g_4^2(\Sigma_0(m)-\Sigma(M)), \tag{11.5}$$

$$\mathscr{S}=m^2B+g_4(B^3-3BD)+3g_3(B^2-D)+6g_4$$

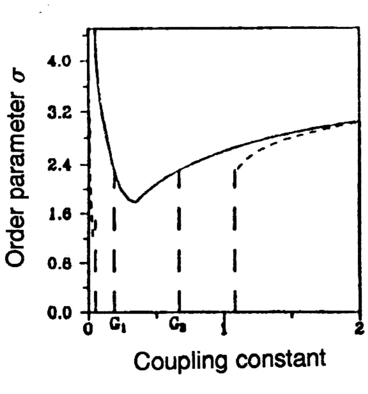

FIG. 21. Order parameter: The solid curve corresponds to $\theta=0$, the broken curve to $\theta=1$.

$$\times(g_3+g_4B)(\Sigma_0(m)-\Sigma(M)),$$

$$D(t,\beta)=\frac{m}{4\pi}\left[t-1+2\theta\ \ln\left(1-\exp\left\{-\frac{t}{\theta}\right\}\right)\right]. \tag{11.6}$$

The Hamiltonian density $\tilde{\mathscr{H}}$ is constructed in accordance with the rule $\tilde{\mathscr{H}}=\mathscr{H}^*[\tilde{\Phi},\tilde{\Pi}]$ to ensure that the Hamiltonian has the correct form, we require that

$$\mathscr{H}_1'=0\Leftrightarrow\begin{cases}\mathscr{R}(t,B,\beta)=0\\ \mathscr{S}(t,B,\beta)=0.\end{cases} \tag{11.7}$$

From the physical point of view, this means that $\tilde{\mathscr{H}}$ describes scalar particles with mass M that depends on the coupling constant G and the temperature θ. This dependence is determined by Eqs. (11.7). It is convenient to go over to the dimensionless quantities

$$G_4=\frac{g_4}{2\pi m},\quad G_3=\frac{g_3}{m\sqrt{4\pi m}},\quad b=B\sqrt{\frac{4\pi}{m}}. \tag{11.8}$$

Using the definitions (11.8), we represent (11.5) in the form

$$\begin{aligned}&-\tfrac{1}{2}t^2+\tfrac{1}{2}+\tfrac{3}{4}G_4(b^2-d(t,\theta))+3G_3b+\tfrac{3}{4}G_4^2\\&\quad\times(\ln t-6\sigma_\beta(t,\theta)-6\sigma_{\beta\beta}(t,\theta))=0,\\&b+\tfrac{1}{2}G_4b(b^2-3d(t,\theta))+3G_3(b^2-d(t,\theta))+3G_4\\&\quad\times\left(G_3+\frac{G_4}{2}b\right)(\ln t-6\sigma_\beta(t,\theta)-6\sigma_{\beta\beta}(t,\theta))=0,\end{aligned} \tag{11.9}$$

$$d(t,\theta)=t-1+2\theta\ \ln\left(1-\exp\left\{-\frac{t}{\theta}\right\}\right). \tag{11.10}$$

The different solutions of these equations describe the possible phases of the system.

11.2. The symmetric model

We consider the model with the Lagrangian (1.1). For this, we set $G_4=G$, $G_3=0$. Using (11.9), we obtain

$$\begin{aligned}&t^2-1-\tfrac{3}{2}G(b^2-d(t,\theta))-\frac{3}{2}G_4^2(\ln t-6\sigma_\beta(t,\theta)\\&\quad-6\sigma_{\beta\beta}(t,\theta))=0,\\&b[1+\tfrac{1}{2}G(b^2-3d(t,\theta))+\tfrac{3}{2}G^2(\ln t-6\sigma_\beta(t,\theta)\\&\quad-6\sigma_{\beta\beta}(t,\theta))]=0.\end{aligned} \tag{11.11}$$

Symmetric phase

From (11.11), (11.10), and (11.3) we obtain for t the equation

$$\begin{aligned}&2t^2+3Gt-2-3G-3G^2\ln t+18G^2\sigma_{\beta\beta}(t,\theta)+6G\\&\quad\times\left(\theta-3\ln 3G\frac{\theta}{t}\right)\ln\left(1-\exp\left\{-\frac{t}{\theta}\right\}\right)=0,\end{aligned} \tag{11.12}$$

where the function $\sigma_{\beta\beta}$ is defined in (11.4). Equation (11.12) has two solutions in the regions S_1 and S_2 in Fig. 22, while in the region BS there are no solutions, and

$$t_2(G,\theta)<t_1(G,\theta)\neq 1,\quad \text{if } (\theta,G)\in S_1$$

$$t_2(G,\theta)>t_1(G,\theta)\neq 1,\quad \text{if } (\theta,G)\in S_2.$$

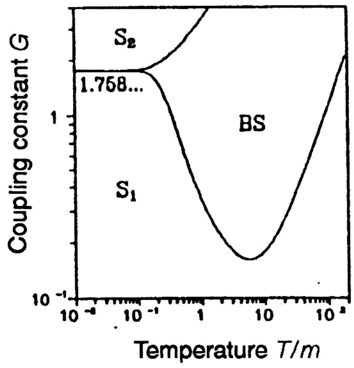

FIG. 22. Phase diagram for the symmetric $(\varphi^4)_3$ theory.

In accordance with the criterion for selecting phases on the basis of the effective interaction, we must set

$$t(G,\theta)=\begin{cases}t_1(G,\theta), & \text{if } (\theta,G)\in S_1\\ t_2(G,\theta), & \text{if } (\theta,G)\in S_2.\end{cases} \tag{11.13}$$

We emphasize that *at nonzero temperature neither t_1 nor t_2 corresponds to the initial representation* (6.2).

The high-temperature asymptotic behavior ($\theta\gg 1$, $t\gg 1$) is determined by the term linear in G. This becomes obvious if we note that [see (11.4)]

$$\begin{aligned}&\sigma_{\beta\beta}(t,\theta)\overset{\theta\gg t\gg 1}{\longrightarrow}C\frac{\theta^2}{t^2}+O\left(\frac{\theta}{t}\ln\frac{\theta}{t}\right),\\&C=\int_1^\infty\int_1^\infty\frac{dxdy}{xy}\left[\frac{1}{\sqrt{4(x^2+y^2+xy)-3}}\right.\\&\qquad\left.+\frac{1}{\sqrt{4(x^2+y^2-xy)-3}}\right]\end{aligned} \tag{11.14}$$

and

$$\theta\gg\frac{\theta}{t},\quad \theta\ln\theta\gg\frac{\theta^2}{t^2}.$$

The asymptotic behaviors of the mass $M=mt$ and the effective coupling constant $G_{\rm eff}=G/t$ have the form

$$\begin{aligned}&t\overset{\theta\gg G}{\longrightarrow}\sqrt{3G\theta\ln\theta},\quad t\overset{\theta\gg G}{\longrightarrow}\sqrt{\frac{3}{2}G^2\ln G},\\&G_{\rm eff}\overset{\theta\gg G}{\longrightarrow}\sqrt{\frac{G}{3\theta\ln\theta}}\ll 1,\quad G_{\rm eff}\overset{\theta\gg G}{\longrightarrow}\sqrt{\frac{2}{3\ln G}}\ll 1.\end{aligned} \tag{11.15}$$

Phase with symmetry breaking

In this case we have the equations

$$\begin{aligned}&b^2=\frac{t^2}{G},\\&t^2-3Gt+3G^2\ln t-18G^2\sigma_{\beta\beta}(t,\theta)\\&\quad-6G\left(\theta-3\ln 3G\frac{\theta}{t}\right)\ln\left(1-\exp\left\{-\frac{t}{\theta}\right\}\right)=0,\end{aligned} \tag{11.16}$$

for the derivation of which we have used the expressions (11.11), (11.10), and (11.3).

For all (θ,G), the second equation of (11.16) has a unique solution. It can be shown that the solution with asymptotic behaviors of the type $1 \ll t \ll \theta$ or $t \gg \theta$ for $\theta \gg 1$ does not exist. This means that at a high temperature $G_{\rm eff}(G,\theta)$ in the symmetric phase is less than in the phase with broken symmetry. We conclude from this that the system is symmetric at high temperature. Numerical solution of Eqs. (11.12) and (11.16) shows that the same is true for all $(\theta,G) \in S_1, S_2$.

It is convenient to represent the result in the form of the phase diagram in Fig. 22. The boundaries of the phases correspond to first-order transitions, since the order parameter is discontinuous at the critical points. The asymptotic behaviors of the effective coupling constant show [see (11.15)] that our approach is sufficiently accurate outside the critical region.

We summarize the conclusions of this subsection:

- There is no symmetry breaking in the three-dimensional model (1.1) for all θ if $G \ll 1$.
- There are two symmetric phases and one phase with broken symmetry, and the transition with rearrangement of the symmetry of the system occurs at intermediate values of G and θ (Fig. 22).
- The system is symmetric if the temperature θ or the coupling constant G is sufficiently large.

11.3. Initially broken symmetry

We consider the phase structure of the model (1.2) with initially broken symmetry. The equations for the parameters t and b are obtained from (11.9) by the substitution

$$G_4 = G, \quad G_3 = \tfrac{1}{2}\sqrt{G}.$$

As a result, we obtain

$$t^2 - 1 - \tfrac{3}{2}G(b^2 - d(t,\theta)) - 3\sqrt{G}b - \tfrac{3}{2}G^2(\ln t - 6\sigma_\beta(t,\theta) - 6\sigma_{\beta\beta}(t,\theta)) = 0, \tag{11.17}$$

$$2b + Gb(b^2 - 3d(t,\theta)) + 3\sqrt{G}(b^2 - d(t,\theta)) + 3G\sqrt{G}(1 + \sqrt{G}b)(\ln t - 6\sigma_\beta(t,\theta) - 6\sigma_{\beta\beta}(t,\theta)) = 0.$$

Two solutions for b follow from (11.17):

$$b = -\frac{1}{\sqrt{G}}\ (\text{symmetric}), \quad b = -\frac{1}{\sqrt{G}} \pm \frac{t}{\sqrt{G}}\ (\text{asymmetric}).$$

Using their asymmetric solution, we obtain the equation

$$t^2 - 3Gt - 1 + 3G + 3G^2 \ln t - 18G^2\sigma_{\beta\beta}(t,\theta) - 6G \times \left(\theta - 3\ln 3G\,\frac{\theta}{t}\right) \ln\left(1 - \exp\left\{-\frac{t}{\theta}\right\}\right) = 0, \tag{11.18}$$

whereas for the symmetric solution ($b = -1/\sqrt{G}$)

$$2t^2 + 3Gt + 1 - 3G - 3G^2 \ln t + 18G^2\sigma_{\beta\beta}(t,\theta) + 6G \times \left(\theta - 3\ln 3G\,\frac{\theta}{t}\right) \ln\left(1 - \exp\left\{-\frac{t}{\theta}\right\}\right) = 0. \tag{11.19}$$

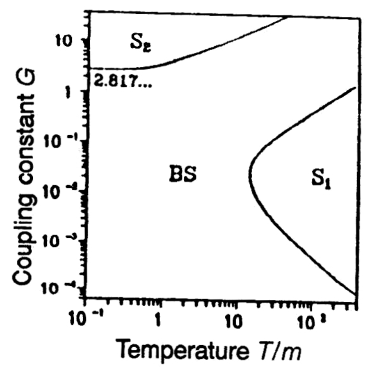

FIG. 23. Phase diagram for the $(\varphi^4)_3$ theory with initially broken symmetry.

Symmetric phase

There are two solutions of (11.19) in the regions S_1 and S_2 in Fig. 23, while in the region BS there are no solutions. The asymptotic behaviors of the mass $M = mt$ and the effective coupling constant $G_{\rm eff} = G/t$ are the same as in (11.15).

Phase with broken symmetry

Equation (11.18) has a unique solution for all (G,θ). Analysis of the asymptotic behaviors, numerical solution of Eqs. (11.19) and (11.18), and comparison of the effective coupling constants give the phase diagram shown in Fig. 23. The boundary of the phases for $G \ll 1$ agrees with what one would expect from perturbative calculation of the effective potential.[3] The temperature is fairly accurate outside the critical regions [see (11.15)]. At the same time, the phase boundaries are determined only approximately, since the effective coupling constant is fairly large in the neighborhood of the phase transitions.

The order parameter is discontinuous at the boundary, so that we have first-order phase transitions. We emphasize that this result cannot be regarded as well established in our approach. We summarize.

- In the system (1.2), the symmetry is restored if the temperature or the coupling constant is sufficiently large.
- There are phase transitions between the broken-symmetry and symmetric phases, the boundaries between which are shown approximately in Fig. 23.
- The method of canonical transformations makes it possible to determine the temperature dependence of the mass. The procedure for determining it is fairly accurate outside the critical regions.
- If $G_{\rm eff}(G,\theta) \ll 1$, then one can make the usual perturbative calculations, using the Hamiltonian (11.1)–(11.2).

11.4. Systems in R^2 and R^3

We compare the phase structure of the models (1.1) and (1.2) in R^3 and R^2 at finite temperature. The phase diagrams for the two-dimensional models are shown in Fig. 13 and 19. The behaviors of the systems with respect to the variable G are completely different in R^2 and R^3 (see also Table I). We have a phase with broken symmetry in space–time R^2 and a symmetric phase in R^3 when $G \gg 1$ irrespective of the symmetry of the initial Lagrangians (1.1) and (1.2). At the same time, the behavior with respect to the variable θ is qualitatively the same in R^2 and R^3. The systems are symmetric if the temperature is sufficiently high.

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