New methods for investigating radiative corrections. Theory of the asymptotic operation[1)]

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The mathematical problem of constructing asymptotic expansions of multiloop Feynman diagrams with respect to the masses and external momenta is analyzed. This is the central problem in perturbative quantum field theory. The theory of the asymptotic operation, which is the most powerful tool for solving this problem, is reviewed. Its connection with the standard methods [Bogolyubov–Parasiuk–Hepp–Zimmermann (BPHZ) theory, leading logarithmic approximation, etc.] is explained. The problem of studying non-Euclidean asymptotic regions is discussed, and ways of solving it are indicated.

The main aim of this review is to explain what is the asymptotic operation (As operation) and how it is used in applications associated with the investigation and calculation of radiative corrections in the framework of applied perturbative quantum field theory.

In the first place, the relationship of the theory of asymptotic operation to elementary-particle physics can be explained by means of the following analogy (see Table I). In other words, in the theory of As operation one considers mathematical (computational) problems of applied perturbative quantum field theory, which is the formalism of modern high-energy physics. On the other hand, the theory of As operation can be regarded as a new applied branch in the theory of generalized functions (distributions), being distinguished by the specific formulations of the mathematical problems. Since in perturbative quantum field theory all amplitudes can be expressed explicitly in terms of Feynman diagrams, the situation here is in a certain sense simpler than in classical mechanics. However, the theory of the asymptotic operation is quite difficult and important.

In this review, we attempt to answer the following questions:

- What is the problem and what are its specific features?
- What is the As operation?
- What must be done in order to extend the theory to non-Euclidean asymptotic regions?

WHAT IS THE PROBLEM?

The answer to this question will be rather long, since the time spent on understanding the problem precisely will be well spent—there are more than enough examples of how damaging it is to begin too soon with calculations, proofs, the writing of papers, etc. The basis of the theory of the asymptotic operation is in fact the reexamination from a new point of view of the old but subtle and complicated problem of the asymptotic expansion of Feynman diagrams with respect to the masses and external momenta.

1. Feynman diagrams are still the main source of quantitative dynamical information in high-energy physics, and this situation is unlikely to change soon (see, for example, the proceedings of any conference on programs for new superaccelerators).

2. Multiloop diagrams must be calculated for the following reasons: Some effects arise entirely from radiative (loop) corrections; allowance for the higher corrections in QCD reduces the uncertainty associated with the truncation of the perturbation-theory series.[93,104]

3. The computational complexity of multiloop diagrams increases with increasing number of independent variables of integration in the diagram and the number of independent external dimensionless parameters (see Table II).

In Euclidean problems, the complexity of the integration is measured by the number l of loops if one needs to calculate the finite parts of l-loop diagrams. In the case of the calculation of a β function and other renormalization-group functions, one needs only the infinite parts of the diagrams, so that the complexity of the integration is measured by the number of loops -1. In non-Euclidean problems, the number of loops must be doubled, since here one must separate the timelike component, the integration with respect to which is performed independently of the spacelike components. This measure of complexity corresponds to the maximum depth of the hierarchy of singular manifolds of the integrand.

At the tree level, any amplitude can be calculated as a rational function of the kinematic variables. However, it must be borne in mind that one must often make integrations over the phase space, which are no better from the point of view of computational complexity than "actual" loops. At the single-loop level, it is possible to work in a number of cases with several parameters, even if the results will be very cumbersome in the most general case.[24,73] At the two-loop level, only a few problems with one parameter, or at the most two, are amenable to solution (cf. the recent calculations of the Leyden group[76]). At the three-loop level, similarly, only problems without dimensionless parameters admit systematic solution by analytic methods. One example is the problem of calculating g_e-2 in QED.[86] Another example is provided by massless integrals of propagator type, which admit effective systematic calculation in the three-loop approximation.[29,32] There are two types of problems that reduce to such integrals: the calculation of the coefficient functions of the Wilson operator expansion at short distances (the main application is to deep inelastic lepton–hadron scattering; see, for example, Ref. 85), and also the calculation of four-loop renormalization-group functions (see Refs. 23 and 89). An

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TABLE I.

Classical physics	High-energy physics
Newtonian mechanics	Perturbative quantum field theory
Theory of Hamiltonian systems	Theory of the As operation
General theory of differential equations	Theory of distributions (generalized functions)

effective decrease in the number of loops is made possible by the device described in Refs. 15, 20, 22, and 28, which retains only the leading terms of the expansion with respect to the momenta and masses of the corresponding diagrams (cf. the discussion below of the applications of the Euclidean asymptotic operation). In the third class of problems (which is related to those already mentioned), one considers integrals with masses.[102]

Finally, there are two examples of calculations of four-loop complexity: the impressive numerical calculation of g_e-2 in QED,[42] the accuracy of which still needs to be improved, and also the five-loop renormalization-group calculation of the β functions and anomalous dimensions in the $\varphi^4_{D=4}$ scalar model.[46] This last calculation was made by means of various analytic devices, and also numerical summation in one of the diagrams (which was then also calculated analytically[43]); it has an interesting application to the theory of phase transitions[25,51] (calculation of the critical exponents by the method of the Wilson–Fisher ε expansion). It is interesting that because of the somewhat slow convergence of the series that is summed for one of the exponents (η), it would also be helpful to calculate the six-loop correction to the anomalous dimension of the wave function. Such a calculation does not appear impossible if one uses both analytic and numerical methods (there are not that many diagrams, and most of them are trivial), but it is still extremely complicated.

4. It is clear that the only method that will really make it possible to advance to higher orders of perturbation theory is *reduction in the number of independent parameters*. Such a reduction can be achieved if one exploits the existence of large or small parameters (which are sometimes not obvious—examples are given below) and makes an asymptotic expansion with respect to them. In this way one can replace the original complicated function by a series in powers and logarithms of the expansion parameters. The coefficients of the resulting series are simpler functions (with fewer independent parameters) than the original one, and thus the achieved computational advantages are often great even if the number of such simpler functions is impressive—"actual" complexity is replaced by the sheer volume of algebra.

5. Finally, the problem of asymptotic expansions of Feynman diagrams contains *two logical levels*. The reason for this is that the physical quantities are represented by sums over the complete set of corresponding diagrams, which at the same time has a hierarchical structure. The heirarchical structure of the perturbation series is manifested in the existence of structural equations (for example, the renormalization-group equations[53]) which make it possible to sum the corrections. Under certain conditions, the asymptotic expansions also inherit a structure of this kind, so that it is possible to sum the higher corrections by means of some analog of the renormalization-group equations (for example, the Lipatov–Altarelli–Parisi equations, Kuraev–Lipatov–Fadin equations, etc.[18,105]). The derivation of such equations requires knowledge of the global structure of the expansions for the perturbation-theory series as a whole.

At the *lower level* of the problem of expansions in perturbative quantum field theory, we are concerned with expansions of individual diagrams. At this level, the problem has an *analytic* nature.

At the *higher level*, we are concerned with globally determined (i.e., in terms of complete sets of Feynman diagrams) amplitudes or Green's functions. Their expansions must be transformed to global form (of the type of an operator expansion) that admits a nonperturbative interpretation and the derivation of structural equations. The transition from the level of the individual diagrams to the global level[2)] has a *combinatorial* nature and is comparatively easy *if at the lower level a solution of the correct type has been found.*

"Correct" solutions are solutions that, first, satisfy the condition of complete factorization (see Refs. 39, 40, and 106 and the discussion later in this review). Second, the ex-

TABLE II.

Complexity of integration	Number of parameters with which one can work effectively
tree approximation	arbitrary cross section
1 loop	many
2 loops (4 RG)	a few (usually one or two)
3 loops (4 RG)	zero (one dimensional parameter)
4 loops (5 RG) (record)	zero
more than 4 loops	?!

pansions of the diagrams must be composed of "pieces" of the original diagrams that constitute complete objects that themselves have the form of multiloop diagrams so that, after combinatorial regrouping of "pieces" that arise from different original diagrams, they can be added together to make objects of the type of matrix elements of certain composite operators. It is this circumstance that introduces the specific aspects of the problem of asymptotic expansions in applied quantum field theory as compared with the general mathematical problem.

It is a remarkable fact that the "correct" solutions can be described naturally only in the language of the theory of generalized functions[3)] (for more details, see below), whereas in the old approaches one works in the framework of ordinary integral calculus. This explains why it was not possible in the framework of the old approaches (BPHZ, etc.) to obtain results of practical value.[4)]

We summarize:

The central mathematical problem in perturbative quantum field theory is to obtain explicit expressions for the asymptotic expansions of multiloop diagrams with respect to the masses and external momenta that satisfy known additional requirements (the condition of complete factorization).

This problem still awaits its full solution despite the efforts of tens of theoreticians during the course of about 40 years and the presence of a number of partial results. What I really want to point out in this review is that the theory of the asymptotic operation now gives us a real hope of fundamental progress in the direction of the complete solution of the problem already in the near future.

Before explaining the basis of this hope, we must give some examples of specific physical problems in which asymptotic expansions of Feynman diagrams arise. This is necessary since, on the one hand, in phenomenological studies there is not always a sufficiently accurate description of the asymptotic regimes, small and large parameters in the problem, etc., and it is often not recognized that, in principle, for any asymptotic regime there is an analog of the operator expansion—even if the form of the expansion differs from the standard Wilson expansion with local operators.

On the other hand, specialists in mathematical physics usually restrict themselves to a small number of canonical problems in which a "rigorous" solution is possible, for example, problems in which one proves that the expansions are in powers and logarithms of a small parameter or proves the operator expansion at short distances or on the light cone.[5)] Unfortunately, rigor itself "beyond student studies does not have primary significance and in the presence of a genuine idea can always be introduced by any competent professional."[3] This remark was made by a first class "pure" mathematician and concerns "pure" mathematics, which studies "pure" models and examples. In real life, the great bulk of practical problems are so complicated that completely formalized proofs that would be genuinely accessible to universal survey by the theoretical community remain an aspiration. In such a situation, the word "rigor" acquires a mythological content and is often used with the aim of self-differentiation, being an indication of an absence of real ideas and results. The results in which we are above all interested are *correct* results—no matter whether they are accompanied by fully formalized proofs or not—and we are also looking for formalisms that lead to such new results. Correctness is not guaranteed by "rigor" but by a deep and exact understanding of the analytic essence of the studied objects and the structure of the problem, which must be adequately reflected in the methods used to find the solutions.

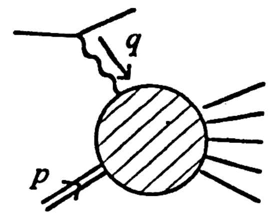

FIG. 1.

Examples of physical problems

First of all, the asymptotic regimes can be divided into those that are Euclidean and non-Euclidean. Euclidean regimes arise in problems that admit Wick rotations of the momenta of integration, so that all the diagrams become purely Euclidean. The class of Euclidean regimes includes two typical cases—the well-known Wilson short-distance regime, and also the heavy-mass regime, in which one considers low-energy amplitudes that include only light fields (for example, the effective low-energy Lagrangians of the weak interactions of light quarks[16,17]). There are also mixed cases. The first more or less complete phenomenologically meaningful solution to the problem of asymptotic expansions for general Euclidean regimes was given by the theory of the Euclidean As operation (Ref. 106 and references given there). The phrase "phenomenologically meaningful" is translated into the technical requirement of *complete factorization*, which the expansions must satisfy. This requirement will be discussed below.

The *deep inelastic scattering* of leptons by nucleons is well known (Fig. 1; see, for example, the review of Ref. 41). The cross section is parametrized by form factors that are functions $F(x,Q^2)$ of two kinematic variables: the dimensionless Bjorken variable x and the momentum transfer $Q^2=-q^2$. The standard asymptotic regime corresponds to $Q^2\to+\infty$ for fixed x:

$$F(x,Q^2) \underset{\substack{Q^2\to+\infty \\ x \text{ fixed}}}{\to} F_{\rm LAP}(x,Q^2)\times(1+O(Q^{-2})). \tag{1}$$

The leading term depends on Q^2 logarithmically, and the function $F_{\rm LAP}$ satisfies an integrodifferential equation known as the Lipatov–Altarelli–Parisi (LAP) equation. The "twist" corrections [i.e., corrections of order $O(Q^{-2})$] satisfy analogous equations. This is a rare case in which the structure of the twist corrections is known, the explanation for which is the direct connection between the problem and the well-studied operator expansion at short distances (in terms of moments of structure functions).

The corresponding asymptotic regime can be described by saying that p tends to a lightlike value:

$$p \to \tilde{p}, \quad \tilde{p}^2=0, \quad q \text{ fixed}, \quad q^2<0, \quad m_i=O(p-\tilde{p}). \tag{2}$$

The masses vanish at the same rate at which p approaches its lightlike limiting value. (In the coordinate representation, this corresponds to the light-cone limit.[44])

Note that p tends to a vector whose components are nonzero even though the Lorentz square is zero. This is a characteristic feature of non-Euclidean regimes.

Alternatively, one can consider the moments of the structure functions:

$$\int_0^1 dx\ x^n F(x,Q^2) \underset{\substack{Q^2 \to +\infty \\ x \text{ fixed}}}{\to} C_n(Q^2) \times M_n \times (1+O(Q^{-2})). \tag{3}$$

This asymptotic regime is essentially Euclidean. The coefficient functions $C_n(Q^2)$ are the Fourier transforms of the coefficients of the Wilson operator expansion.

The two asymptotic regimes are not equivalent. This circumstance is expressed by the fact that the problem of recovering the structure functions from their moments is improperly posed in the mathematical sense: Such recovery is possible only if *all* moments are known exactly (as in the recovery of the single-loop kernels of the LAP equations from the anomalous dimensions of the operators of the Wilson expansion), or if one uses additional information together with the values of some of the lowest moments.

The problem of small x and the Regge limit (see the review of Ref. 105). Here one considers the structure functions of deep inelastic scattering at small x, where the amplitude, as is expected, is most sensitive to gluon interaction:

$$F(x,Q^2) \text{as } x \to 0 \quad \text{for fixed } Q^2, \tag{4}$$

this being equivalent to the study of the Regge limit

$$s=(p+q)^2 \gg Q^2, p^2, m_i^2. \tag{5}$$

By the optical theorem, this is related to the amplitude of the elastic process:

$$F(s,t,u,m_i^2) \underset{s \to +\infty}{\approx} F(s) \times \ldots \times (1+O(s^{-1})). \tag{6}$$

The problem here is to determine the dependence on s, i.e., the form of $f(s)$. The ellipsis denotes a factor that does not depend on s.

The Regge limit has been studied in connection with the theory of dispersion relations and various bounds on cross sections since the fifties. The theoretical result obtained in the framework of perturbative quantum field theory is known as the Kuraev–Lipatov–Fadin (KLF) equation (see, for example, Ref. 18). This equation is deduced by means of the so-called technique of the leading logarithmic approximation,[2,72] and a complete understanding of the derivation is still a privilege of a narrow class of specialists. Moreover, the KLF equation is the analog of only the single-loop approximation of the renormalization-group equation for the coefficient functions of the Wilson operator expansion, and it is not known how it must be modified in order to go beyond the leading logarithmic approximation. The issue is very important, since the leading logarithmic approximation by no means always works well at accessible energies.[6)] It should also be noted that in the school of the leading logarithmic approximation considerable theoretical experience has been accumulated that is still not readily accessible to the uninitiated.

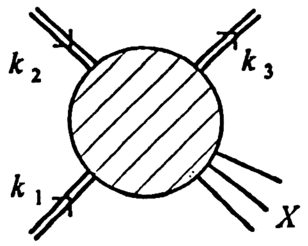

FIG. 2.

The *parton model with allowance for QCD* (see, for example, the review of Ref. 41) is a set of prescriptions having the aim of giving a description of the structure of the leading power-law terms in the asymptotic expansions of the amplitudes for a certain class of inclusive processes for a definite class of asymptotic regimes. The theoretical results on which the model is based are known as *factorization theorems*.[69] As a rule, they are obtained only for the leading power-law terms, and, as is emphasized in Ref. 69, their status cannot yet be regarded as satisfactory because of gaps in the derivation. For example, the proof of smallness of the remainder term of the expansion is technically very similar to the calculation of the following correction, and practically nothing is known about the power-law (twist) corrections in problems of the type of the Sudakov form factor. Here we do not yet have an effective formalism for dealing with either the analytic or the combinatorial aspects of the problem. For Euclidean regimes, an example of such a formalism is provided by the theory of the Euclidean asymptotic operation.

For example, in the framework of QCD, following Ref. 26, let us consider the process $H_1+H_2 \to H_3+X$, where X denotes an inclusive state (Fig. 2). Here there are three independent kinematic variables—the Mandelstam variables S and T and the invariant mass of the inclusive state M_X^2. The variable U is determined from the relation

$$S+T+U=M_X^2+m_1^2+m_2^2+m_3^2, \tag{7}$$

where m_j^2 are the masses of the three particles. There are also light parameters of the type of the quark masses, which are not indicated explicitly. Such processes are usually considered in the regime S, T, $M_X^2 \gg m_j^2$, which corresponds to the standard parton model with allowance for QCD. Then the cross section can be represented by a convolution of the cross section of a "hard parton subprocess" and parton distributions that are analogous to the distributions $F_{\text{LAP}}(x,Q^2)$ that arise in the study of deep inelastic scattering and satisfy the same integrodifferential equations. The essential dynamical information is concentrated in the hard cross section, which is the analog of the coefficient functions of the Wilson

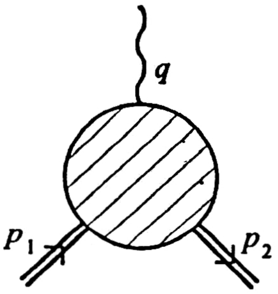

FIG. 3.

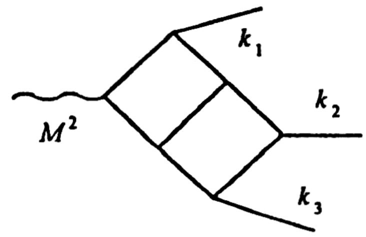

FIG. 4.

expansion. In the present case, it depends on two dimensionless parameters (S is usually removed from the expressions by scaling).

Example of a "hidden" small parameter. The calculation of the hard cross section in the previous example is rather cumbersome already at the single-loop level.[26] However, one can consider the quasielastic asymptotic regime $S,\ T \gg m_i^2,\ M_X^2$, and then it is only necessary to deal with one independent dimensionless parameter. (In practice, one works with the form of regime corresponding to $S,\ T \gg M_X^2 \gg m_i^2$.) Such a modification of the asymptotic regime is motivated by the following circumstance. As we have already noted, the expression in the parton model for the cross section contains the convolution of the hard cross section and the parton distributions $F_i(x,Q^2)$, which have power-law behavior of the form $(1-x)^n$ for $x=1$. It turns out that the region in which this suppression does not work corresponds to the quasielastic regime; moreover, the contributions that are dominant in the quasielastic regime are enhanced by singularities (see, for example, Ref. 91 and the references given there). It is found that the calculation of the single-loop corrections then becomes incomparably simpler, while the approximation based on retention of only the leading terms works remarkably well (with numerical accuracy 5%) all the way to $x_T \gtrsim 0.2$ (Ref. 91). This means that the problem contains the effective hidden small parameter M_X^2/S. Note that a "small" parameter is not necessarily numerically small; the only important thing is that an expansion is made with respect to such a parameter and that the expansion works with the necessary accuracy for values of the parameter that are of interest.

There is no general theory of the quasielastic regime; there are only certain results in the leading logarthmic approximation (see, for example, Ref. 92). Note that in the region of small x the parton model cannot work in any case. On the other hand, the quasielastic regime is in principle by no means worse than the standard regime corresponding to the parton model.

In the *problem of the Sudakov form factor*[2] one considers the behavior of the form factors of the electron, quark, and other elementary fields at large momentum transfers (Fig. 3). The results for the leading power-law term are described in Refs. 59 and 70. An interesting fact is that here new types of operator appear (string operators[100]), whereas in the ordinary Wilson operator expansion at short distances only local operators participate. String operators also appear in other problems.[100,7)]

Note that the Sudakov form factor decreases at large Q^2—this is the effect known as *Sudakov suppression.* The problem of the Sudakov form factor is important on account of the old conjecture (see, for example, Ref. 30) that for certain asymptotic regimes the Sudakov suppression could play a role like that of asymptotic freedom (cf. the recent paper of Ref. 96). This question remains to be clarified.

The asymptotic regime can here be described as follows:

$$p_{1,2} \to \bar{p}_{1,2}, \tag{8}$$

where $\bar{p}_{1,2}^2=0$, for fixed q and all $m=O(p-\bar{p})$ [cf. Eq. (2)].

Jets in QCD. The "jet" approach to processes at high energies represents a significant new paradigm in elementary-particle physics, which abounds in complicated interesting problems (see, for example, the review of Ref. 60). In practice, we are dealing here with cross sections of parton type with many soft and collinear singularities. We give below an example of a problem of asymptotic expansion whose essence is very different from what we have considered above.

The "double-box" problem.[8)] We consider a process with three hadron jets in the final state. In the $O(\bar{\alpha}_s^3)$ approximation, there are three-parton contributions of two-loop complexity, and all of these can be calculated analytically except for the diagrams of "double-box" type and their nonplanar analogs (Fig. 4; the nature of the virtual lines—gluon, quark, etc.—is unimportant to the extent that they are massless). The "double-box" diagram is a function of $M^2>0$ and the three invariant masses of the pairs of final partons $s_{ij}=(k_i+k_j)^2$, $\{i,j\}=\{1,2\}$, $\{2,3\}$, $\{3,1\}$. However, since $s_{12}+s_{23}+s_{31}=M^2$, there are only two independent dimensionless parameters in the problem, for example,

$$x=s_{12}/M^2>0, \quad y=s_{23}/M^2>0, \quad x+y<1. \tag{9}$$

Such an amplitude has infrared singularities (maximum singularity ε^{-4}). To ensure gauge-invariant canceling of all the infrared singularities in the final result, it is necessary to use dimensional regularization (other practical methods have not yet been developed, although they do not appear to be impossible). Another requirement is that it should be possible to use the result to generate by the Monte Carlo method $O(10^5)$ events, so that the answer must be an effective computational algorithm. This rules out a direct approach based on parametric representations.

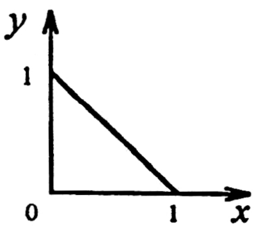

FIG. 5.

A possible scenario for systematic work with such diagrams, which is also valid in many other cases, is to represent them as expansions in powers and logarithms of x, y, $1-x$, $1-y$.

To explain this point, we note the following. It is often incorrently assumed that exact expressions for studied integrals in terms of special functions are the end result to which one should strive in analytic calculations. In fact, what one should regard as the final result are asymptotic expressions at singular points (including infinity) and effective numerical algorithms for regular regions. "Explicit" representations in terms of special functions are helpful only to the extent that there is complete information about the behavior of such functions that makes it possible to obtain asymptotic expressions and carry out numerical calculations. On the other hand, guaranteed universal methods of obtaining such explicit representations in all cases do not exist, and even if they do they may not be as helpful as one would wish. One example is provided by the expressions in terms of infinite multiple series with respect to generalized hypergeometric functions obtained by Mellin transformation (see Ref. 103 and the references given there). Another example is provided by the numerical calculations of the type described in Ref. 99, in which it is sometimes necessary to calculate dilogarithms to an accuracy of 15 places in order to obtain a numerical result with a reasonable (not high) accuracy.

However, if there is a possibility of obtaining directly and systematically asymptotic expansions at singular points for the studied expressions, then in conjunction with numerical algorithms (using, for example, Feynman parameters) this makes it possible to obtain directly all the necessary information, avoiding the cumbersome step of "analytic calculation."

In an actual case, say, expansion with respect to x leads to a series whose coefficients are calculable functions of y. To obtain such an expansion, it is necessary to use the non-Euclidean form of the technique of the asymptotic operation. Of course, one can also make expansions for the asymptotic regimes $0<x\ll y\ll 1$, etc. In this way one can obtain representations of the amplitude in the form of expansions with respect to x and y near the boundaries of the kinematic triangle (Fig. 5). The convergence of such expansions is geometric, and the radius of convergence is determined by the singularities of the function, which are localized along the boundaries of the kinematic triangle. If the convergence is not sufficiently rapid near the center of the triangle, one can always construct a simple interpolation formula, using several values obtained numerically; in the central region, the function is analytic.

As regards computational complexity, measured by the number of poles with respect to $\varepsilon=D-4$, the scenario just described corresponds to the record five-loop renormalization-group (Euclidean) calculations of Refs. 46. However, the non-Euclidean complications make the problem much more cumbersome, and at the same time the non-Abelian nature of QCD considerably increases the amount of algebra. Nevertheless, at the present time this is the only realistic scenario for calculating diagrams of double-box type.

There is a remarkable recent analytic result[95] for a scalar double box, but only for the planar case; unfortunately, the result is available only for four dimensions, so that one must take $k_i^2>0$ in order to regularize the infrared divergences. To reduce the four-dimensional results with nonzero k_i^2 to the necessary form (dimensional regularization with $k_i^2=0$), one can make an expansion for $k_i^2\to 0$. (Note that the lifting of the regularization for singular products is similar to implementation of an asymptotic expansion in the sense of the theory of distributions; this emphasizes the fundamental role of this concept—see the discussion below). It appears unlikely that a combination of devices like those used in the calculation of the planar double box in Ref. 95 could also lead to success in the nonplanar case (cf. Ref. 94). One would then have to return to the complete scenario described above.

Finally, at some point it may be necessary to make a more careful analysis of the singular boundaries of the regions. It is then necessary to use global results of the type of factorization theorems and the corresponding equations for such asymptotic regimes. This is one further variation on the theme of asymptotic expansions.

There are also many other physical problems, including asymptotic expansions of Feynman diagrams, for the discussion of which we do not have sufficient space here, for example, the effective low-energy theory of heavy quarks,[88] etc.

General formulation of the problem of asymptotic expansions in perturbative quantum field theory

First of all, we must describe a specific asymptotic regime in terms of the masses and external momenta instead of the scalar (for example, Mandelstam) kinematic variables, i.e., we must determine which masses and components of the momenta are small compared with the others. Suppose Q, $M\gg k$, m. For technical reasons, it is simpler to work with small parameters of an expansion that is made than with ones that tend to infinity. Because of the homogeneity of the amplitudes with respect to the dimensional parameters, the two methods of formulating the problem are equivalent:

$Q, M\gg k, m$	
Q, M fixed large parameters	$k, m\to 0$ small parameters

It should be emphasized that the expression $Q, M\gg k, m$, which describes the asymptotic regime, need not hold in the naive numerical sense. It merely means that the original function will be replaced by an expansion with respect to the ratio of two scales. The radius of numerical usefulness of the obtained expansion may be quite large (cf. the examples above).

Ultraviolet renormalization introduces an additional dimensional parameter μ_{ren}. Without serious loss of generality, it can be assumed that all the amplitudes are renormalized in the MS scheme (or in any other massless renormalization scheme). Then the renormalized diagrams are polynomials in $\log \mu_{\text{ren}}$ and it is unimportant whether one regards μ_{ren} as a small or large parameter. In what follows, μ_{ren} will be ignored.

If $A(Q,M,k,m)$ denotes the investigated amplitude (Green's function), then it is necessary to find an expansion of the form

$$A(Q,M,k,m) \cong \sum_i C_i(Q,M,\mu_{\text{fact}}) D_i(k,m,\mu_{\text{fact}}), \quad (10)$$

where C_i are usually called "coefficient functions," and D_i "matrix elements" (the terminology is inherited from the Wilson operator expansion at short distances).

In the expression (10) we also give explicitly one further parameter, μ_{fact}, which arises in the process of factorization of the large and small parameters:

$$\log(Q^2/m^2) \to \log(Q^2/\mu_{\text{fact}}^2) + \log(\mu_{\text{fact}}^2/m^2).$$

It may (or may not) be equated to the renormalization parameter.

Complete factorization

An extremely important requirement that was completely omitted in the old formulation of the expansion problem is the complete factorization of the large and small parameters.[39] It can be explained as follows. First, C_i must contain only contributions proportional to one and the same power of a large parameter. Second, they must not depend on the small parameters k and m. Because of the homogeneity of A, the same assertions will, *mutatis mutandis*, also hold for D_i. As a consequence, μ_{fact} cannot be naively interpreted as a cutoff in the momentum representation separating regions of integration, since the use of such cutoffs leads to the appearance of power-law dependences on μ_{fact}. The expression "flabby cutoff"[9] determines more precisely the nature of μ_{fact} (cf. the example below). Its global significance can be elucidated as follows. The expressions for D_i are the matrix elements of certain complicated operators that are automatically equipped with a suitable ultraviolet renormalization. If the requirement of complete factorization is satisfied, then it turns out automatically that such a renormalization uses one of the massless schemes, for example, the MS scheme, and μ_{fact} is the renormalization parameter of the scheme (cf. the construction of the asymptotic operation given in Refs. 97 and 98).

Since the expansions for the perturbative Green's functions are obtained by combinatorial regrouping of the expansions for the individual diagrams, the meaning of complete factorization must be made precise at the level of the individual diagrams. At this level, the requirement of complete factorization means that the expansions must be with respect to pure powers and logarithms, and, at the first glance, such a requirement may appear to have no content. Indeed, it is well known to specialists in calculations of multiloop diagrams that for all the combinatorial complexity the analytic nature of multiloop diagrams is rather simple—they are integrals of rational functions of the variables of integration. Computational experience shows that their asymptotic expansions with respect to the masses and external momenta as extracted from "explicit" expressions in terms of special functions or by division of the regions of integration, etc., are always in powers and logarithms of a small parameter. Formal proofs of this fact for different asymptotic regimes were given in Refs. 13, 27, 31, 34, and 55.[10)] However, the essence of the problem of the expansion of multiloop diagrams is to break up the original integral (before the explicit calculation, which is a separate problem with its own specific features) into "pieces" in such a way that some "pieces" have a power-logarithmic dependence on the expansion parameter, while the others have the same dependence on the large parameters. The "pieces" must be complete objects that again have the form of multiloop diagrams, so that they can, after combinatorial regrouping of "pieces" that derive from different original diagrams, be added to make objects of the type of matrix elements of certain composite operators. In the studies cited above, this was not done (moreover, such a problem was not apparently posed). At the same time, in the papers of Zimmermann,[10,12] in which expansions were constructed in operator form for the short-distance regime, the "pieces" did not have a purely power-logarithmic dependence on the small parameter.

As an illustration, we consider the following one-dimensional model of a single-loop integral. It depends on a large "momentum" Q, a small "momentum" k, and also on a small nonvanishing mass m:

$$I(Q,k,m) = \int_0^{+\infty} dp \, \frac{1}{p+k+m} \times \frac{1}{p+Q+m}.$$

The problem is to construct an expansion of this integral in the limit $Q \to \infty$. Three constructions are possible.

1) The most direct approach is to break up the region of integration:

$$\int_0^{+\infty} dp = \int_0^{\mu} dp + \int_{\mu}^{+\infty} dp, \quad (11)$$

and in the first subregion expand the integrand with respect to $Q \to \infty$, and in the second with respect to k, $m \to 0$. Then all the integrals can be calculated, and it is easy to see that in the expansion there will be only powers and logarithms of the parameters; there will also be powers of μ that cancel after consideration of similar such integrals over the other two regions. This last fact indicates that in such an approach certain important properties are ignored.

2) The standard solution, of the type obtained in Ref. 10, takes the following form. One first makes the preliminary calculation

$$I(Q,k,m)=\int_0^{+\infty}dp\left[\frac{1}{p+k+m}-\frac{1}{p+m}\right]\times\frac{1}{p+Q+m}$$

$$+\int_0^{+\infty}dp\,\frac{1}{p+m}\times\frac{1}{p+Q+m}.$$

In the first term it is now possible to replace the propagator that depends on Q by the value corresponding to the asymptotic behavior at large Q as follows:

$$I(Q,k,m)=\left\{\frac{1}{Q+m}\right\}\int_0^{+\infty}dp\left[\frac{1}{p+k+m}-\frac{1}{p+m}\right]$$

$$+\left\{\int_0^{+\infty}dp\,\frac{1}{p+m}\times\frac{1}{p+Q+m}\right\}+O(Q^{-2}). \quad (12)$$

The expression in the square brackets corresponds to the renormalized matrix element of a local operator (subtraction at zero k in the MOM scheme[53]). In the curly brackets we have taken the pieces that depend on Q (which corresponds to "short distances" after Fourier transformation) and not on k ("large distances")—the coefficient functions are ultimately made up of such expressions.

Although formally the problem has been solved in Wilson's formulation[8]—the large and small distances are factored out—for phenomenological applications this is not sufficient for the following reasons.

First, physical problems are formulated directly in the momentum representation, and the correct asymptotic regime is $Q\gg k$, m, i.e., the mass must be a small parameter (unless specifically stipulated otherwise).

Second, by explicit calculation of the integrals one can readily show that the expressions for the "coefficient functions" contain in the asymptotic limit of large Q terms with "twist" corrections, i.e., corrections suppressed by factors $O(m/Q)$ compared with the leading power. In asymptotically free theories, the coefficients of such terms cannot be calculated in perturbation theory.[40]

Third, the presence of a rather complicated dependence on m in the coefficient functions makes calculations of the radiative corrections to them practically impossible.

Fourth, the method does not give satisfactory results for theories with massless particles (for example, QCD)—this is manifested in the impossibility of setting $m=0$ in Eq. (12) on account of the unintegrable singularity at $p=0$.

Finally, for non-Euclidean regimes, in which the singularities are localized on nonlinear manifolds (the light cone), the approach with presubtractions is most inflexible. In essence, if one is to make a preliminary subtraction like the one made above, it is necessary to know the answer in advance. This circumstance—the absence of sufficiently powerful heuristics for more complicated problems—is decisive for the evaluation of the standard approach.

3) Solutions for problems of this type free of the listed shortcomings were first obtained in the framework of the theory of the As-operation[39] (see also Ref. 106 and the references given there). To compare the expressions obtained here with the ones given above, we restrict ourselves to the final answer, since the method of the arguments, which leads automatically to such expressions, will be discussed below:

$$I(Q,k,m)=\left\{\frac{1}{Q}\right\}\int_0^{+\infty}dp\left[\frac{1}{p+k+m}-\left\{\frac{1}{p}\right\}^{\tilde{r}}\right]$$

$$+\left\{\int_0^{+\infty}dp\left\{\frac{1}{p}\right\}^{\tilde{r}}\times\frac{1}{p+Q}\right\}+O(Q^{-2}). \quad (13)$$

The expression $\{p^{-1}\}^{\tilde{r}}$ is the distribution defined by means of the equation

$$\int_0^{+\infty}dp\left\{\frac{1}{p}\right\}^{\tilde{r}}\varphi(p)=\int_0^{+\infty}dp\,\frac{1}{p}\left[\varphi(p)-\theta\left(\frac{p}{\mu_{\text{fact}}}\right)\varphi(0)\right],$$

where $\theta(p/\mu)$ is a cutoff factor equal to 1 for $p\lesssim\mu$ and 0 for $p\gtrsim\mu$. In the more general case, one obtains distributions that, as in our example have simple scaling properties, and this results in a power—logarithmic dependence on Q.[97,106] In the example given above, this can be demonstrated by direct calculation.

The subtraction in the square brackets already corresponds to a renormalization scheme of the type of the MS scheme.[97] If an intermediate regularization of dimensional type is used, then we obtain representations of the type

$$\left\{\frac{1}{p}\right\}^{\tilde{r}}=\frac{1}{p}+\frac{\text{const}}{\varepsilon}\,\delta(p),$$

i.e., compared with the "bare" expression obtained from the original one by expansion in m we obtain "counterterms," which are localized at the points of the singularities. If such representations are used (general expressions for this case are given in Refs. 97, 106, and 107), then in the calculations it is sufficient to deal with integrals without masses, ignoring the simple terms containing δ functions. For the coefficient functions of the Wilson operator expansion, massless integrals of propagator type are obtained,[39,45] and for such integrals there exists an effective algorithm for calculating up to three loops inclusively (Refs. 29, 32, 38, 66, 67, and 84).

Although in this simple example it is possible to rewrite the result in such a way as to eliminate expressions containing distributions, in more complicated cases (when there are more momenta of integration) it is more difficult to do this, and then the heuristic connection with the conclusion will be lost. It should also be noted that in mathematics it is no less, indeed it is even more (much more) important to study how to obtain new correct results than to "prove rigorously" ones already found.

Thus, it may be concluded, first, that expansions satisfying the requirement of complete factorization can be obtained in a form suitable for carrying out combinatorial factorization. Second, such expansions are most naturally described in the language of the theory of generalized functions.

It remains to note the following. The requirement of complete factorization is extremely important from both the technical and the conceptual points of view. This is due to the circumstance that such expansions possess the property of uniqueness (see Ref. 106). It follows from this, in particular, that such expansions inherit properties of the type of gauge invariance, uniqueness, etc., from the original amplitude in an orderly manner, automatically, and one does not need to make a special investigation to confirm that the final expressions do not violate, say, unitarity. *All that one must ensure is that the expansions are in powers and logarithms of the expansion parameter.* On the other hand, the logarithmic–power nature of the expansions is preserved in the recursive constructions typical of the theory of the asymptotic operation because a product of logarithmic–power expansions will again be an expansion of the same type.

THE PROBLEM OF THE EXPANSION OF MULTILOOP DIAGRAMS FROM THE MATHEMATICAL POINT OF VIEW

...the perturbative form of the Wilson
expansion at short distances was
fully understood more than ten years ago.
...A new theory is not needed.
Referee of the Journal Nuclear Physics B
Do not make haste to understand me.
Chinese proverb

Why distribution theory?

Let us consider the following integral, which it is required to expand in the limit $m \to 0$ (for brevity, we omit the infinitesimally small imaginary corrections in the denominator):

$$\int dp\, \frac{1}{(p^2-m^2)(p-Q)^2}\,. \tag{14}$$

The well-known difficulty associated with the expansion of such an integral is that the m-dependent propagator acquires in a formal expansion in m singularities whose power increases with increasing expansion order:

$$\frac{1}{p^2-m^2} = \frac{1}{p^2} + \frac{m^2}{p^4} + O(m^4). \tag{15}$$

The usual method adopted to avoid this difficulty is to divide the region of integration in such a way as to isolate the dangerous points and/or make suitable subtractions at the point $p=0$ for the factors that do not participate in the expansion with respect to m in order to neutralize these singularities. This idea is the basis of the usual approaches—either in Zimmermann's form[10] or in the form of the technique of "principal regions of integration."[2,69]

In itself, such an approach is completely correct. However, in the multiloop case the number of additional subtractions increases, and it becomes difficult to work with it. This is the reason why the so-called forest formula[10,21] (which gives a closed formal description of the resulting piling up of subtractions for the simplest case of Euclidean expansions) is usually regarded as a model for solving the problem in the general case. However, as is explained below, the expected extension of the theory of the As-operation to the non-Euclidean case would be incorrectly regarded as the construction of a non-Euclidean form of the forest formula, since this last may not exist in a form amenable to study.

Bearing in mind that it has not proved possible to find a non-Euclidean forest formula for more than 20 years, we shall use the method of close examination—the most powerful of the known methods of solving nontrivial problems.

First of all, difficult problems should not be considered in isolation, and one must also not lose sight of their essential aspects. Thus, it may be recalled that in the context of applied quantum field theory there is little sense in considering an individual diagram, since we are here concerned with infinite heirarchically (recursively[11]) organized sets of such diagrams. Moreover, there are many models in which similar integrals arise. For example, let us consider the following form of the expression (14), which differs from (14) only in the presence of a heavy mass:

$$\int \mathrm{d}p\, \frac{1}{(p^2-m^2)((p-Q)^2-M^2)}\,. \tag{16}$$

It may be noted that the m-dependent factor that leads to the difficulties is the same here as in (14), and the entire difference is all in the "passive" factor $((p-Q)^2-M^2)^{-1}$, which is smooth at the singular point. It is natural to assume that the exact form of the "passive" factor is unimportant to the extent that it is smooth, and we can write:

$$\int \mathrm{d}p\, \frac{1}{(p^2-m^2)}\, \varphi(p), \tag{17}$$

where $\varphi(p)$ is an everywhere smooth function that decreases at large p sufficiently rapidly to ensure convergence (to avoid secondary complications due to ultraviolet divergences).[12)] It is clear that both the expression (17) and its expansion with respect to m—whatever the form of such an expansion—are linear in $\varphi(p)$. Expanding (17) for all "good" $\varphi(p)$ is exactly the same as *expanding the propagator* $(p^2-m^2)^{-1}$ *in the sense of distribution theory.*

It is here convenient to clarify what is the difference from the practical point of view between expansions in the sense of distributions and formal Taylor expansion. The key result here is the so-called *extension principle*,[33,106] which is a constructive though abstract (actually, formally rather simple) proposition like the classical Hahn–Banach theorem on the extension of functionals. The extension principle states that if one is given an m-dependent functional defined on a linear space, and also another functional that approximates the first on a subspace, then the approximating functional can be extended to the complete space preserving the approximation property. From the practical point of view, the prescription of the extension principle consists, roughly speaking, of the addition to the formal expansion of counterterms that are localized at the singular points and are therefore proportional to δ functions and their derivatives. As in the case of the R operation in the coordinate representation, the coefficients of such counterterms diverge in order to compensate the unintegrable singularities of the formal expansion. However, in contrast to the R operation, *the finite parts of the coefficients are not arbitrary*, because they must ensure the approximation properties of the obtained expansion in the sense of distributions. In reality, if the expansion must have a definite analytic form (in our case, the condition of complete factorization requires that it contain only powers and logarithms of the expansion parameter), then the *coefficients of the counterterms are uniquely determined.* The practical prescription for determining the coefficients is given by *self-consistency conditions.*[33,106]

In the present case, the m-dependent functional is a distribution corresponding to the unexpanded propagator (17); the space is the space of test functions; the approximating functional is the formal Taylor expansion (15); the subspace on which this last is a well-defined approximating functional consists of test functions such that $\varphi(0)=0$. It is clear that the only way of constructing an extended functional is to add to the formal expansion a delta function with suitable coefficient. (This argument is similar to the arguments of Bogolyubov that led him to the discovery of the correct form of the R operation.[1,6] In reality, it was Bogolyubov's construction that from the very beginning served as the model for the theory of the As operation.[33]) Thus, the complete expansion takes the form

$$\frac{1}{p^2-m^2}\approx \text{As}\,\frac{1}{p^2-m^2}+O(m^4)$$
$$=\frac{1}{p^2}+\frac{m^2}{p^4}+c(m)\delta(p)+O(m^4), \qquad (18)$$

where the coefficient $c(m)$ is uniquely determined by the condition of self-consistency. In the present case, its explicit expression (for the case of dimensional regularization) has the form

$$c(m)=\int \mathrm{d}p\,\frac{1}{p^2-m^2}. \qquad (19)$$

Explicit expressions for the general Euclidean case can be found in Ref. 106 (dimensionally regularized form) and in Ref. 87 (in a form that does not depend on the regularization). Here, it is sufficient to emphasize that the counterterms introduced by the As operation serve two aims. First, they contain divergent parts that compensate the divergences of the formal expansion in the same way that the R operation eliminates the ultraviolet divergences in the coordinate representation; second, the counterterms contain finite parts that are uniquely determined (in contrast to the case of the R operation), and their role is to "rearrange" the expansion in such a way as to guarantee the approximation properties.

The above reformulation of the problem in the language of distributions is so important (essentially, the entire theory of the asymptotic operation is a logical development of this idea) that it warrants several remarks of a general nature.

1. The problem (17) is a generalization of the original problem (14). There are two forms of generalization. In one we ignore some important aspect of the problem, so that its nature is essentially changed; the absence of important structures may make a solution difficult or even impossible. In the other generalization one ignores secondary although possibly cumbersome details. As a result, it is possible to concentrate on the things that are truly important, and at the same time

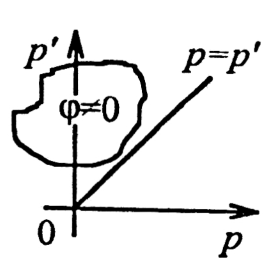

FIG. 6.

the problem is genuinely simplified, even if new concepts are brought into play. It is this kind of generalization that we have in our case.

2. Our reformulation represents an embedding of the original problem in a more general one. Here there is a certain freedom, which we have the right to choose on grounds of convenience. In our case, the freedom includes the choice of the type of allowed test functions $\varphi(p)$, the choice of topology (for its absence) in the set of linear functionals, etc. However, such technical details are less important than the heuristics of the general scenario and must be chosen on the grounds of convenience as the problem is solved. In our case, the most convenient choice[97] consists of using test functions and functionals in the Schwartz spaces $\mathcal{S}(P)$ and $\mathcal{S}'(P)$, respectively.

3. Working initially with test functions that decrease for all sufficiently large p corresponds to the introduction of a smooth ultraviolet cutoff. This corresponds to the observation that what happens around $p=0$ is independent of the behavior of the integral at large p. We shall see below how the transition to integrations over infinite volume occurs. It is here sufficient to note that the introduction of localized test functions enables us to concentrate on the neighborhood of the studied singularity. Essentially, this is only a more refined realization of the basic idea of the method of principal regions of integration.

Localization property and recursive structure of the asymptotic operation

We consider a multiloop diagram in which the denominator contains the same m-dependent propagator as before:

$$\int \mathrm{d}p\,\frac{1}{(p^2-m^2)}\int \mathrm{d}p'\,\frac{1}{(p-p')^2\ldots}. \qquad (20)$$

(see Fig. 6). The singularities of the formal expansion of the m-dependent propagator are localized on the subspace $p=p'$ (vertical axis in Fig. 6). In accordance with the philosophy of the method of principal regions of integration, we consider the contributions that arise from the neighborhood of this subspace. From the point of view of distribution theory, this means that we consider an expansion of test functions $\varphi(p,p'\ldots)$ that are equal to zero around other singularities of the integrand. The support of φ [i.e., the set at which $\varphi(p,p'\ldots)\neq 0$] can have the form as in Fig. 6. It is clear that the product of such $\varphi(p,p'\ldots)$ and the remaining part of the integrand, $\varphi(p,p'\ldots)\times 1/(p-p')^2\ldots$, can be regarded as a smooth test function, so that the problem degenerates to the

TABLE III.

Alternative	BPHZ, etc.: to sacrifice recursion in order to eliminate singularities.	*As* operation: use recursion by employing distribution theory.
Aim	Formal rigor.	New results; computational methods; heuristically nontrivial proofs.
For	Uses only methods of ordinary integral calculus.	The advantages of recursion are fully used; deterministic prescription for constructing expansion.
Against	Extremely cumbersome expressions; no connection with heuristic arguments; the answer must be known in advance; not every recursion can be resolved!	Nonstandard technique for working with singularities (new branch of distribution theory).

case of the single propagator (17). We introduce the notation **As** for the operation that is applied to the m-dependent products and gives as a result their expansion in the sense of distributions. Then the above arguments can be formally represented in the form

$$\mathrm{As}\circ\left(\frac{1}{(p^2-m^2)}\times\frac{1}{(p-p')^2\ldots}\right)$$
$$=\left(\mathrm{As}\circ\frac{1}{(p^2-m^2)}\right)\times\left(\mathrm{T}\circ\frac{1}{(p-p')^2\ldots}\right),$$

which is valid for the test functions described above. The operation **T** on the right-hand side is the ordinary Taylor expansion with respect to m. Its presence reflects the fact that the factors whose expansion does not give singularities in the given region do not require special treatment.

More generally: Let G be a product of m-dependent propagators that must be expanded with respect to m in the sense of distributions, and O be a region of the space of integration in which only some of the propagators in G are singular (we denote the product of such propagators by $G^{\rm sing}$), so that all the other propagators in G are regular (we denote their product by $G^{\rm reg}\equiv G/G^{\rm sing}$). Then we obtain the following, which is fundamental:

Localization property of the As operation

$$\mathrm{As}\circ G|_O=\mathrm{T}\circ G^{\rm reg}\times\mathrm{As}\circ G^{\rm sing}.$$

This property reveals the key recursive structure present in our expansion problem. Its role here is similar to the role of the microcausality condition in the construction of the R operation.[53]

The recursive organization of the problem has manifest advantages, since it enables one to break up the original problem into simpler subproblems and reduce the arguments—both at the heuristic stage and in the formal proof—to the study of one inductive step. All this is completely obvious and is precisely the aim that the theory of the As operation follows. Moreover, a paradigmatic example of such an approach is already long known—it is the construction by Bogolyubov of the ultraviolet R operation.[1,53] In this connection, it is helpful to recall the microcausality condition for the S matrix, which in a simple special case has the form

$$T[\mathcal{L}(x)\mathcal{L}(y)\mathcal{L}(z)(w)]$$
$$=T[\mathcal{L}(x)\mathcal{L}(y)]T[\mathcal{L}(z)\mathcal{L}(w)],\quad x^0,y^0>z^0,w^0.$$

In this relation, T denotes time ordering, and $\mathcal{L}(x)$ is the density of the interaction Lagrangian at the space—time point x. Whereas in the expansion problem the recursive structure became apparent only after the point of view of distribution theory had been adopted, in the theory of the R operation it was at the forefront from the very beginning—Bogolyubov's original argument was manifestly recursive and of a generalized functional form.[13)] It was not fully understood and mastered by the theoretical community (note that even in the book of Ref. 53 itself the key paper of Ref. 1 is not cited completely accurately)—until it was used as model for the theory of the As operation.

The fundamental dilemma: singular factors and recursion

Thus, the fundamental recursion in the expansion problem (the localization property) concerns a product of singular factors. Accordingly, the dilemma is how to handle singularities (see Table III).

Traditional solution and forest formula

For the reasons already discussed, it was assumed for rather a long time as self-evident that all expressions containing singularities must be reduced to ordinary absolutely convergent integrals by recursive resolution. This approach was initiated by Bogolyubov and Parasiuk[4 14)] who followed the aim of "rigorous" expression of their formula for the R operation and with time developed into what is known as BPHZ (Bogolyubov–Parasiuk–Hepp–Zimmermann) theory and became decisive for the "rigorous" study of Feynman diagrams, the reasons for which it is instructive to consider. First, the formalized realization of the alternative approach requires the creation of a special mathematical technique, which does not exist in a finished form in standard distribution theory (cf., for example, the so-called d inequalities, which make it possible to describe the analytic structure of singularities at an isolated point in the presence of singularities localized on manifolds that pass through this point[56,97]). It was therefore natural that one first attempted to get by using the methods of ordinary integral calculus, even to the detriment of the heuristic aspects and intuition.

A second reason was the popularity of parametric representations. It is sufficient to recall that parametric representations were used in the fifties to study the analytic properties of amplitudes (cf., for example, Landau's equations[5]). The use and "study" of parametric representations creates the illusion that the integrand is simplified (but this is deceptive, because its fundamental multiplicative structure is then destroyed; we recall also that the most successful methods of analytic calculation of multiloop diagrams avoid parametric representations—cf., for example, the widely used algorithm of integration by parts for massless integrals in Refs. 29, 32, 38, 66, 67, and 84). Another illusion is that in such a study one gets the impression that there is more than the simple creation of a systematic notation for the formal description of integrands and that the difficulties of iatrogenic nature (the already mentioned destruction of the multiplicative structure) are overcome.

Finally, Hepp, who refined Bogolyubov and Parasiuk's proof and introduced their result into the main stream of the world theoretical tradition, was obviously not interested in the generalized-functional aspects of the derivation and the mechanism of the R operation but regarded "Bogolyubov's method" (i.e., the final expression for the R operation and not the derivation at all) rather formally as a "phenomenological procedure," that needed to be "proved" [cf. his book Ref. 9, especially Chap. 6 and the remarks before Eq. (6.26)].

With regard to the attitude of Bogolyubov himself, it may be supposed that the correctness of the expression for the R operation had to be clear to him from the set of heuristic arguments based on the coordinate and parametric representations, and only the problem of the "rigorous" expression of the result was posed. In addition, Bogolyubov was obviously fascinated (judging from the appreciable attention devoted in the monograph of Ref. 6 to the parametric representation, which, as is now quite clear does not warrant it), like many others, by the intriguing (but inessential) interaction of the combinatorial and graph-theoretical aspects of the α-parameter representation.

Whatever the reason, the BPHZ tradition has remained dominant for 30 years. The corresponding formula with multiple summation, representing the solution of the corresponding recursion in the Euclidean case, is known as the "forest formula."[10,21]

Of course, in distribution theory there is nothing supernatural, and its general theorems guarantee that any expression containing distributions can be rewritten in the language of ordinary integrals, and the procedure of deriving expansions in the sense of distributions can, at least in principle, be reduced (at the lowest logical level) to subtractions, regrouping of terms, division of a region of integration into sectors of Hepp type, etc. But it is just as true, for example, that any mathematical expression can in practice be reduced to finite manipulations with integers. However, it would be not at all sensible to ignore the advantages of working with concepts of a higher logical level (rational and complex numbers, mathematical analysis, etc.) merely to avoid studying the rules of rational arithmetic and the infinitesimally small.

On the other hand, not all recursions can be reduced to a transparent nonrecursive formula (in our case, the forest formula contains multiple summations over special sets of subgraphs—forests). It is already difficult to claim that the Euclidean forest formula is transparent.[15] In the non-Euclidean case the situation is much worse (this must be already clear from the fact that the non-Euclidean form of the problem is still unsolved). The formal reason is as follows. Strictly speaking, the localization property by itself is insufficient. It merely enables one to determine the structure of the counterterms [cf. (18)]. Their coefficients [cf. (19)] must be found by means of additional arguments based on the extension principle. The expressions for the coefficients close the recursion. In the Euclidean case, such expressions are fairly simple, especially in dimensional regularization [cf. (19)]. In the non-Euclidean case, the expressions for the coefficients contain additional expansions, so that the recursion as a whole is complicated in a fundamental manner (it becomes a two-story edifice). Therefore, even if a non-Euclidean forest formula in the exact sense of the word is constructed, it is not to be expected that it will be helpful even for the purposes of formal verification of the results obtained by more refined methods.

This leads us to the following. There is another specific reason for the heuristic worthlessness of the standard approach based on the use of the forest formula. The typical structure of the R operation or the As operation (from the point of view of distribution theory) has the form

$$\mathbf{r}_G\circ\left(G/\Gamma\sum_{\Gamma\subset G}\ldots\mathbf{r}_\Gamma\circ\Gamma\right), \tag{21}$$

where $\mathbf{r}_\Gamma$ is the operator that "treats" the singularity associated with the subproduct Γ. The construction (21) reflects the simple fact that before we consider the singularity associated with the complete product G (and localized, say, at an isolated point), we must "treat" the singularities associated with subproducts $\Gamma\supset G$ (and localized on manifolds that pass through this point). It must be emphasized that the explicit form of the final operator $\mathbf{r}_G$ can be determined only after the operators $\mathbf{r}_\Gamma$ for all the subproducts have been constructed. However, the procedure for resolving the recursion includes evaluation of subtractional operators in the presence of test functions. This can be represented in the symbolic form

$$\int(\mathbf{r}\circ\Gamma)\varphi\to\int\Gamma\times(\varphi-\mathbf{T}\varphi),$$

where $\mathbf{T}$ is some type (not necessarily exactly) of Taylor expansion. This means that in the obtained integrals *the order of the subtractions is reversed* (in exact similarity to the reversal of the order of operators when the adjoint is taken). In the more general case (21), the effect can be represented in the form

$$\int X\mathbf{r}_G\circ\sum_{\Gamma\subset G}Y\mathbf{r}_\Gamma\circ Z\to\int\sum_{\Gamma\subset G}Z\times(1-\mathbf{T}_\Gamma)[Y\times(1-\mathbf{T}_G)X].$$

(Note that the structure of the right-hand side is explicitly similar to the structure of the ladder formula; cf. the three-point products in §30 of the book of Ref. 53.)

Thus, if we work with expressions corresponding to resolved recursions, the subtraction that must be made first, $(1-\mathbf{T}_G)$, corresponds to a singularity that must be analyzed last—after all the subtractions for the subgraphs have been determined.

The formal reversal of the order of the subtractions in the subgraphs is the technical reason why the ordinary methods do not permit effective use of recursion. *This explains their failure in the non-Euclidean case.*

On the other hand, in the framework of the traditional approach attempts were never made to study (and develop) the heuristic aspects of the considered problems. In this connection, we may recall that the heuristics used by Bogolyubov to find the correct expression for the R operation was completely ignored in BPHZ theory. There is nothing remarkable in this; for whereas the proof given by Bogolyubov and Parasiuk was based on the first approach to the resolution of the dilemma formulated above, Bogolyubov's original argument[1] corresponds to the second approach.

The solution proposed by the theory of the As operation

It is well known to the specialists that any formal proof concerning the properties of multiloop diagrams ultimately reduces to an ordinary count of powers. As we have already noted, the traditional approach attempts to consider the complete problem in its entirety, nondestructively, so that in a count of the powers one must consider the entire expression in such a way that all the terms generated by the subtractions are considered simultaneously (for example, the remainder term in the expansion of a renormalized l-loop diagram).

The key analytic idea in the theory of As operation consists of using the localization condition for structuring of the problem to the form of the iteration of one and the same elementary step, in which one considers *only a singularity localized at an isolated point.*

Once this has been understood, the problem is essentially reduced to finding and analyzing a limited number of representative examples with small number of loops in order to understand the mechanism of the central recursive step (we recall that the constructive prescription for the case of an isolated singularity is given by the extension principle discussed above).

In the Euclidean case, it is sufficient to consider two-loop diagrams.[16)] The following two-loop example can illustrate this point.

Consider the two-loop Euclidean diagram shown in Fig. 7. The two upper heavy lines correspond to heavy particles with mass M, which has the order Q. The three other propagators contain masses $m \ll Q$, M. We direct Q along the heavy lines. Then the entire dependence on the heavy parameters will be conveniently localized in the two heavy propagators. The problem is to expand the diagram with respect to $m/M \to 0$.

For simplicity, we consider a two-dimensional theory. Then there are no ultraviolet divergences, whereas nontrivial

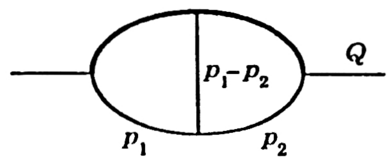

FIG. 7.

singularities arise already in the terms of the leading order $O(m^0)$ of the formal expansion in m.

The integral as a whole has the form

$$\int d^2p_1 d^2p_2 \frac{1}{p_1^2+m^2} \times \frac{1}{p_2^2+m^2}$$
$$\times \frac{1}{(p_1-p_2)^2+m^2} \frac{1}{(p_1-Q)+M^2} \times \frac{1}{(p_2-Q)+M^2}.$$

It can be seen that the two heavy propagators do not depend on m and form a "test function," the precise form of which is not important. Clearly, it is sufficient to expand the product of the three m-dependent factors in the sense of distribution theory and substitute the result in the integral.

The construction of the expansion of the m-dependent factors in the sense of distribution theory is an iteration of the following sequence of steps:

•formal Taylor expansion;

•study of the geometry of the singularities of the formal expansion;

•identification of the sets of propagators (*IR* subgraphs) whose singularities overlap at different points of the region of integration (*completeness condition*);

•study of the analytic structure of the singularities (counting of the powers);

•construction of the counterterms. *The formal expansion* has the form

$$\frac{1}{p_1^2+m^2} \times \frac{1}{p_2^2+m^2} \times \frac{1}{(p_1-p_2)^2+m^2} \approx \frac{1}{p_1^2} \times \frac{1}{p_2^2}$$
$$\times \frac{1}{(p_1-p_2)^2} + O(m^2). \tag{22}$$

The geometry of the singularities on the right-hand side of (22) is shown in Fig. 8. Each denominator that can vanish generates a manifold on which the singularity is localized. In our case, there are three different two-dimensional subspaces corresponding to each of the three factors. Such singular manifolds can intersect. Each intersection must be regarded as a new singular manifold, because the nature of the singularity at the corresponding points can be nontrivial. In this context, "nontrivial" means that the singularity does not factorize. In our case, such an intersection is the point $p_1 = p_2 = 0$; here, the singularity does not factorize. If any of the three factors is absent, the singularity becomes factorizable.

The listing of all the intersections does not present difficulties—it is sufficient to list all the subsets of the singular propagators with allowance for the restrictions imposed by momentum conservation. This is known as the *complete-*

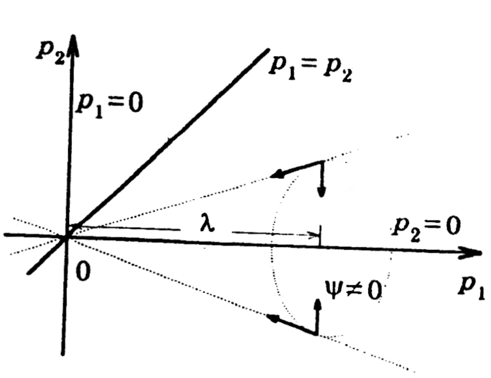

FIG. 8.

ness condition.[49,50,106] Indeed, if we set equal to zero the momenta of a certain chosen set of propagators (this corresponds to intersection of the corresponding manifolds), then there may be propagators whose momenta are automatically annihilated because of momentum conservation. This means that their singularities are superimposed on the singularities of the set, and such propagators must be included in the set. The sets of propagators that cannot be completed in such a manner are precisely the *IR subgraphs*. In the present case, there are four *IR* subgraphs: $\gamma_1=\{$the propagator that depends on $p_1\}$; $\gamma_2=\{$the propagator that depends on $p_2\}$; $\gamma_3=\{$the propagator that depends on $p_1-p_2\}$; $\gamma_{123}=\{$the set of all three propagators$\}$. None of the three pairs of propagators possesses the completeness property, and therefore is not an *IR* subgraph.

It is then necessary to study the analytic nature of the singularities. It is at this point that the localization condition is introduced. Indeed, as was explained above, an expansion valid on test functions ψ that are not equal to zero only in the region shown in Fig. 8 is given by formal expansion of factors regular in this region multiplied by the expansion in the sense of distribution theory for the propagator that is singular in this region. The expansion of the isolated propagator is already known [see (18)]; in the two-dimensional case, a counterterm proportional to the δ function must be included already for the first term of the formal expansion, which in this case will have a logarithmic singularity at zero momentum. Therefore, the expansion in the region has the form (we here use dimensional regularization to give meaning to the intermediate expressions)

$$\frac{1}{p_1^2}\times\left(\frac{1}{p_2^2}+c_0(m)\delta(p_2)\right)\times\frac{1}{(p_1-p_2)^2}+O(m^2)$$
$$=\frac{1}{p_1^2}\times\frac{1}{p_2^2}\times\frac{1}{(p_1-p_2)^2}+\frac{1}{p_1^2}\times c_0(m)\delta(p_2)\times\frac{1}{(p_1-p_2)^2}+O(m^2).$$

Thus, the analysis of the singularity of the singular manifold $p_2=0$ has shown the need for the introduction of a counterterm on the right-hand side. One can easily consider similarly the two singular manifolds $p_1=0$ and $p_1=p_2$. We obtain the expression

$$\frac{1}{p_1^2}\times\frac{1}{p_2^2}\times\frac{1}{(p_1-p_2)^2}+\frac{1}{p_1^2}\times c_0(m)\delta(p_2)\times\frac{1}{(p_1-p_2)^2}+\frac{1}{p_1^2}\times\frac{1}{p_2^2}\times c_0(m)\delta(p_1-p_2)+c_0(m)\delta(p_1)\times\frac{1}{p_2^2}\times\frac{1}{(p_1-p_2)^2}+O(m^2). \quad (23)$$

This expansion is valid in the sense of distribution theory on test functions that can be different from zero everywhere except a small neighborhood of the point $p_1=p_2=0$.

One can also say that after the removal of the point $p_1=p_2=0$ the three manifolds cease to intersect (the singularities "decay," so that the problem effectively breaks up into simpler subproblems that correspond to singular manifolds of higher dimension and with smaller number of propagators contributing to the corresponding singularities.

It remains to transform the expansion (23) into an expansion valid on the complete space of integration. This can be done in two logical steps. *First*, we note that (as in the case of the *R* operation in the coordinate representation) the expansion (23) (*together* with the already added counterterms) is integrable with test functions belonging to a larger set that do not necessarily vanish in the neighborhood of the origin. The assertion, essentially equivalent to the Bogolyubov–Parasiuk theorem, is that such test functions must have only a zero of second order (a detailed discussion of the Bogolyubov–Parasiuk theorem in the coordinate representation from the point of view of distribution theory is given in Refs. 56 and 97). It is natural to assume that the approximation properties of the expansion (23) are also preserved on such test functions. This is the key analytic step in the proof. Assuming that this has been shown, it is easy to make the *second* step, which consists of constructing the counterterm for the singularity at the point $p_1=p_2=0$ in exactly the same way as in the case of an individual propagator. Namely, the counterterm must be a linear combination of the δ function and its first-order derivatives, and the coefficients have a form analogous to (19). For example, for the coefficient of the δ function without derivatives we have

$$c_0^{\{\gamma_3\}}(m)=\int dp_1 dp_2\,\frac{1}{p_1^2+m^2}\times\frac{1}{p_2^2+m^2}\times\frac{1}{(p_1-p_2)^2+m^2}\,.$$

(A discussion of such expressions can be found in Ref. 106).

We now concentrate on the key analytic point of the above argument—the counting of the powers in (23) at the point $p_1=p_2=0$ in the presence of three counterterms. We again consider a test function localized in the region shown in Fig. 8. We introduce the parameter λ as shown in the figure in order to parametrize the test function $\psi_\lambda(p_1,p_2)\equiv\psi(p_1/\lambda,p_2/\lambda)$ (*without assuming* anything about the zeros of ψ). Clearly, study of the singularity of (23) at $p_1=p_2=0$ is equivalent to studying the dependence of the value of the distribution (23) on ψ_λ as $\lambda\to 0$. As we have already said, the proposition, essentially equivalent to the

Bogolyubov—Parasiuk theorem, is that for fixed m the leading power-law behavior is λ^{-2}, i.e., the same as is obtained by the naive counting of the powers. On the other hand, for fixed λ we know with regard to the remainder term of the expansion [i.e., the difference between the initial expression, i.e., the left-hand side of (22), and (23)] that it has the order $O(m^2)$. In order to verify now that the approximation property holds, we must obtain an estimate of the remainder term that should combine both types of dependence in factorized form, i.e., $O(\lambda^{-2})\times O(m^2)$.

Note that this means that the type of the dependence on λ and m is here *the same as in the first neglected term*—a property that is well known from simpler cases of asymptotic expansions. Bearing in mind that the analytic nature of the integrals with which we must work (integrals of rational functions) is rather simple (despite being superficially cumbersome, which has a purely combinatorial origin), it is hard to suppose that this could be otherwise. In any case, such estimates are undoubtedly valid for the individual propagators, and it is only necessary to make a formal calculation in order to derive such a property for the product as a whole. The existence of the recursion is a great help, since it enables one to use (the inductive assumption, which, in essence, must also be verified, consists of this) the factorized estimate of the power-law behavior for an individual propagator (more generally, for the *IR* subgraph corresponding to the singular manifold with which one must work).

It remains to note that a convenient formal language for the exact description of such estimates and implementation of calculations of such type—essentially, counting powers in the presence of logarithmic modifications—was developed in Refs. 56 and 97 (so-called *d inequalities*).

Finally, we make a remark on the combinatorial recovery of the expansions in global operator form. It is clear that the final expressions for the expansions in the formalism of the asymptotic operation are sums over *IR* subgraphs that have a simple characterization (see the completeness condition above). Moreover, it is readily seen from the expressions for the counterterms that they admit direct interpretation as integrated Feynman diagrams corresponding to certain matrix elements. This leads to *radical simplifications of the combinatorix of exponentiation* compared with the usual arguments (for the details, see Refs. 87, 98, and 107).

Working with ultraviolet-divergent diagrams

Let us consider an unrenormalized diagram. Let $G(p,M_{\text{tot}})$ be its integrand, with p denoting the set of momenta of integration and M_{tot} the set of all of its external momenta and all masses on which its propagators depend. We introduce a smooth cutoff at large $p\sim\Lambda$ using a test function $\Phi(p/\Lambda)$ such that $\Phi(0)=1$. Then going to the limit $\Lambda\to\infty$ corresponds to lifting the cutoff and restoring the integration over the complete momentum space. We represent the cutoff function as a sum over spherical layers (cf. Fig. 9):

$$\Phi\left(\frac{p}{\Lambda}\right)=\int_0^\Lambda \frac{d\lambda}{\lambda}\,\phi\left(\frac{p}{\lambda}\right).$$

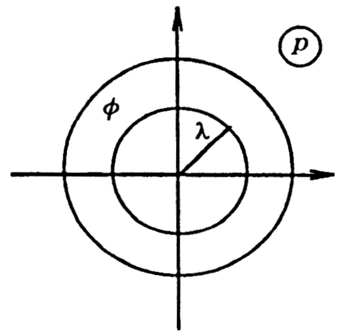

FIG. 9.

Changing the order of the integrations with respect to p and λ and making the substitution $p\to\lambda p$, we see that study of the integral $\int \mathbf{d}p\Phi(p/\Lambda)G(p,M_{\text{tot}})$ as $\Lambda\to\infty$ is equivalent to study of the expression $\int \mathbf{d}p\,\phi(p)G(p,M_{\text{tot}}/\lambda)$ as $\lambda\to\infty$. Now this last is exactly the same as study of the asymptotic expansion of the integrand as $M_{\text{tot}}\to 0$ in the sense of distribution theory for $p\neq 0$. We can immediately use the formalism of the As operation and obtain *exact exhaustive information about the ultraviolet divergences of the diagram.* It turns out that by subtracting from the integrand those terms, and only those terms, of the expansion that are responsible for the ultraviolet divergence one can not only ensure automatic convergence in the ultraviolet region but also obtain the correct ultraviolet renormalization equivalent to the Bogolyubov R operation.[75,98]

It can be shown that after this the study of the expansions of the renormalized diagrams reduces to study of a double asymptotic expansion (the As operation with respect to a hierarchy of parameters that consists, on the one hand, of the light expansion parameters and, on the other, of the set of all dimensional parameters of the diagram, as shown above).[75,98] As before, all that one must do is obtain factorized estimates for the remainder term of the double expansion. Such a problem does not contain significant new difficulties.

The Euclidean As operation

A more or less complete theory of the As operation has by now been developed for the case of Euclidean asymptotic regimes, although the philosophy and a large proportion of the techniques are completely general. The extension principle was found in Refs. 33 and 47 (see also Ref. 106 and the references given there). The original motivations were as follows:

•analysis of the derivation of Bogolyubov's R operation[1,6] in connection with multiloop calculations of theoretical groups at the JINR (Dubna)[22] and the Institute for Nuclear Research of the Russian Academy of Sciences (Moscow);[36]

•the method of renormalization-group calculations developed in Refs. 15, 20, 22, and 28; this method is based on the idea of using the simple dependence of the ultraviolet counterterms in the MS scheme on the masses and momenta[52] in order to simplify the calculations; essentially, the method included Taylor expansions with respect to masses and momenta, and this sometimes led to difficulties on account of infrared divergences of the type discussed above;

•analysis of the operator expansion at short distances with the aim of applying the computational algorithms for three-loop massless integrals of propagator type (Refs. 29, 32, 38, and 84) to the calculation of coefficient functions; the standard rules proposed by BPHZ theory were found to be of no use for these purposes.

The motivations had a clearly applied nature. Accordingly, the first results were also associated with practical calculations.

The R^* operation[35] generalizes the method of renormalization-group calculations mentioned above. The heuristics behind the prescription of the R^* operation[35,108] is as follows. We take a diagram whose counterterm must be calculated in renormalized form, i.e., together with all counterterms. We now make an expansion of such an expression with respect to some parameter of the diagram (for example, with respect to one of the masses) in accordance with the prescriptions of the theory of the As operation. (Such an expansion certainly simplifies the diagram from the computational point of view.) At the same time, we take into account the fact that the ultraviolet counterterms in the MS scheme (including the one that must be calculated) are polynomials in dimensional parameters,[52] and therefore they are expanded trivially and enter the final expression in a known manner; without loss of generality, it can be assumed that the diagram diverges logarithmically in the ultraviolet region and that the calculated counterterm occurs in the expression as an additive correction. We now note that it is not necessary to calculate the entire diagram but only this last counterterm. It is then unimportant whether or not the obtained expression is an approximation of the original diagram. But then the approximation properties of the counterterms introduced by the As operation are also unimportant; putting it simply, we can ignore the finite parts of the infrared counterterms and choose them solely on the basis of the requirement of infrared convergence. Schematically,

$$\mathbf{R}_{\mathrm{MS}}\circ G \rightarrow \mathrm{As}\circ \mathbf{R}_{\mathrm{MS}}\circ G \rightarrow \tilde{\mathbf{R}}\circ \mathbf{R}_{\mathrm{MS}}\circ G \equiv \mathbf{R}^*\circ G,$$

where $\tilde{\mathbf{R}}$ denotes an operation that differs from the As operation by the absence of finite parts in the counterterms. The resulting expressions admit independent verification by direct calculations, since even in one and the same diagram one can have variations associated with different choices of the expansion parameters; the structures of the infrared divergences that are then obtained may be completely different.

The definition of the *IR* subgraph in the original Ref. 35 was not fully general, since an attempt was made to formulate it in the language of the coordinate representation, the motivation for which came from earlier computational algorithms.[36] A general analytic characterization of the infrared counterterms was proposed in Refs. 49 and 50 (see the example of the completeness condition above and the detailed exposition in Ref. 106).

If the α representation is used, then the analytic description of the *IR* subgraphs in terms of the completeness condition is no longer sufficient, and for work with the α representation a special graph-theoretical version of the definition of *IR* subgraphs was specially worked out.[54] Although unsuitable for practical work on account of its extreme complexity, this version is necessary for the rigorous proof of already known results on multiloop diagrams. The reason for this is that the structure of the integrand in the representation is characterized indirectly in terms of secondary graph-theoretical concepts.[17)] Further on the transition to the parametric representations the simple multiplicative structure of the integrands in the momentum representation is lost, and the structure of the integrand in the α representation is described by secondary graph-theoretical concepts (ditrees, cocycles, etc.). Accordingly, the completeness condition, which can be formulated in a very simple and natural manner in the language of the momentum representation,[106] becomes completely rigorous when rewritten in terms of cocycles.[54]

From the analytic point of view, the essence of the mechanism of the R^*-operation method reduces to the property of commutativity of the double As operation discussed above (complete proofs of this property in a form that does not depend on the regularization are given in Refs. 87 and 97).

The R^* operation has been used in a considerable number of calculations (see, for example, Refs. 46, 51, 79, 89, and 90). Arguments of a similar nature were used in Ref. 80.

Algorithms for the coefficient functions of operator expansions[39,45] in the framework of the MS scheme have proved to be very successful (see, for example, the recent three-loop calculations of Refs. 46, 51, 61, 82, 83, 85, and 89). The theoretical and practical importance of the requirement of complete factorization in operator expansions was proved in Refs. 39 and 40.

General expressions for asymptotic expansions in Euclidean regimes in dimensionally-regularized form and satisfying the criterion of complete factorization were given in Refs. 50, 57, and 58 (see also Ref. 107) and were discussed in the literature from the traditional point of view in Refs. 63, 68, and 74. The method of combinatorial exponentiation of expansions that was developed in the theory of the As operation (in particular, the method of the inverse R operation[62]) stimulated a review of the combinatorial aspects in BPHZ theory too.[81]

A systematic description of the theory of the Euclidean As operation with emphasis on applications in the framework of dimensional regularization was given in Refs. 106, 107, and 108.

An analysis of the theory of the Euclidean As operation independent of the employed regularization was undertaken in Refs. 56, 64, 75, 77, 78, 87, 97, and 98. For a detailed comparison of the study of multiloop diagrams in the theory of the As operation and the BPHZ formalism, the representation for ultraviolet-renormalized diagrams developed in the framework of the theory of the As operation[64,75,98] is very important. This representation gives a convenient method for formulating computational schemes of the type of the MS scheme[14] without using dimensional regularization. The de-

TABLE IV.

Graeco–Roman arithmetic notation in Europe up to the ninth century		Decimal-place system introduced by the Arabs from India
enables one to start work immediately: **I+I=II**		first one must learn new rules: **1+1=2**
	HOWEVER	
In complicated problems difficulties soon arise: **MCMXCIII**		after the new rules have been mastered, there are no problems: **1993**
Low-level languages:	←→	High-level languages:
BPHZ		*As* operation

velopment of purely four-dimensional analytical formalisms is very important in view of the fact that the powerful computational methods based on the use of helicity amplitudes (see, for example, Ref. 65) are occupying a central position in the practice of perturbative QCD.

Besides the already mentioned studies of Refs. 63 and 68, the expressions of the Euclidean asymptotic expansions obtained in the framework of the theory of the As operation (Refs. 33, 39, 47, 48, 50, 57, 58, 106, and 107) were the subject of verification in a rather unusual series of studies[71,74] published by representatives of the Moscow school of supporters of the α representation (see the basic text Ref. 21 of this school). These studies are unusual in that although they do not contain any significant new results they do give formalized descriptions of already known results which are very helpful for the rigorous theory of the α representation.

An interesting mathematical problem of a general nature is to analyze and elucidate the use of dimensional regularization in applications directly in the momentum representation (see the original definition of Ref. 11 and any paper with calculations in perturbative QCD). Formal translation of the heuristically obtained results into the language of parametric representations, as was done, for example, in Refs. 19 and 81, remains unsatisfactory, since it does not give anything at all at the heuristic level. There are partial results in the right direction (see Ref. 37 and the references given there), but they are evidently insufficient. In this connection, we should mention the recently discovered effect of the violation of dimensional regularization in non-Euclidean asymptotic regimes.[101] This result emphasizes the necessity for a more substantial mathematical study of dimensional regularization.

Toward the non-Euclidean As operation

When the method of the As operation is extended to non-Euclidean asymptotic regimes, we encounter technical problems due to the fact that the quadratic forms in the denominators of the Feynman propagators are not, in contrast to the Euclidean case, positive definite. Therefore, the singularities of the individual factors are localized on second-order manifolds that can be singular in the sense of differential geometry (singularity at apex of the light cone). The following difficulties arise.

Osculating ("sticking") singularities (nongeneric intersections of singular manifolds). To analyze the nature of such singularities, it is necessary to make rectifying deformations of the coordinates. Such deformations must be found in explicit form, and this is not always trivial. Another consequence is that the rescalings of the variables that must be made in order to count the powers (cf. above) are asymmetric, i.e., the degrees of dilitation along different directions are different.

There can be problems due to *algebraic dependence* between scalar invariants in the denominators. We recall that there always exists linear dependence between vectors whose number exceeds the dimension of the space. This can make a simple characterization of *IR* subgraphs impossible. However, in a number of specific cases a complete analysis does appear to be possible.

Inhomogeneous expressions for counterterms, which arise from self-consistency conditions. In this case, it is necessary to make a secondary expansion (so-called homogenization). As a result, there arise factors that no longer correspond to the standard propagators and have unusual ultraviolet behavior. Homogenization in the general case can be a very delicate procedure with unusual interaction between ultraviolet and infrared divergences.

Complicated two-level recursion, which is associated with the secondary expansion in the counterterms (homogenization) and in no way admits explicit solution of the type of the three-point products or forest formula that arise in BPHZ theory. This is probably an insuperable obstacle for ordinary methods of proof.

Inapplicability of dimensional regularization[101] is, by all appearances, a very general feature of non-Euclidean expansions. The consequences of this fact for computational applications, in which the role of dimensional regularization is very great, may be dramatic. At the least, it seems evident that we must bid farewell to the simplicity (often deceptive) of the expressions to which we have become accustomed in the framework of dimensional regularization.

CONCLUSIONS

It is convenient to complete this review by means of the following analogy from the history of mathematics, which illustrates rather accurately some psychological aspects of the manner in which new paradigms are received in the scientific community (see Table IV).

It is helpful to recall that in Florence already in 1299 the use of the decimal-place system in bank account ledgers was forbidden by law.[7] It seems improbable today, but at that time the decimal system appeared complicated and incomprehensible. We now know that ultimately the decimal system laid the way to the discovery of decimal fractions, the binary system, and to arithmetic with floating point, which is built into any personal computer. It is here appropriate to recall the reactions to the early studies on the theory of the As operation received from experts on BPHZ theory (see the earlier quotations).[18)] This analogy may help the reader to understand why, for example, it is difficult to satisfy the usual request—to explain the basic ideas of the theory of As operation in a simple single-loop example; the analysis of such examples consists mainly of new definitions, the meaning and significance of which become clear only in an analysis of the rather cumbersome integrals that it is very difficult to treat by direct methods. At the same time, in simple examples one can easily get by without distribution theory (cf. I+I=II↔1+1+2).

Nevertheless it is perfectly obvious that the theory of the As operation has quite definitely proved its potential—both at the heuristic level, indicating unique computational algorithms, as well as at the formal level, giving new and very compact methods of formal proofs. Moreover, at the present time it appears quite definite that the problems listed above in the discussion of non-Euclidean asymptotic regimes admit quite definite solutions in the framework of the theory of As operation. I hope to have the possibility of discussing these solutions in the near future.

[1)]Based on lectures given at Symposium on SSC Physics (Madison, April 1992), Seventh Symposium on High Energy Physics (Sochi, October 1992), the Symposium on New Computational Methods in Perturbative QCD (Zurich, December 1992), at the Joint Institute for Nuclear Research (Dubna), the Institute for Theoretical and Experimental Physics (Moscow), the Research Institute for Nuclear Physics at the Moscow State University (Moscow), at the Institute for High Energy Physics (Protvino), at national laboratories in the United States (FERMILAB, SEVAF, Argonne) and at state universities in Oklahoma, Pennsylvania, and New York (Stony Brook), and also McGill (Canada). The work was done with financial support of the collaboration CTEQ/TNRLC.

[2)]Such a transition is called factorization or exponentiation; a better word would evidently be *globalization*.

[3)]There is here a direct analogy with the problem in classical geometry of measuring the diagonal of a square, the solution of which is also expressed in terms of certain "generalized" objects—irrational numbers. Note that the discovery of such generalized solutions is, as a rule, accompanied by confusion, rejection, and other emotions on the part of supporters of the old methods. The theory of the asymptotic operation is here no exception.

[4)]In fact, since any expression with generalized functions can be rewritten in terms of ordinary integrals, the old methods can be used to verify results—*provided these results have already been found by some other more adequate method*. Actually, such verification is truly possibly only in simple cases, since it is exceptionally cumbersome. Even in the case of Euclidean asymptotic regimes it has proved to be practically impossible to implement it fully in explicit form (see the discussion below). In any case, the practical scientific value of such verifications is extremely small.

[5)]We may mention here that phenomenologically significant problems are always formulated in the language of the momentum representation, so that a literal understanding of the words "short distances," etc., without concrete analysis of actual computational practice leads merely to the proof of useless theorems.

[6)]I am grateful to L. Frankfurt for a discussion of this question.

[7)]I am grateful to A. V. Radyushkin for a discussion of this question.

[8)]I am grateful to R. K. Ellis and W. T. Giele for explaining the double-box problem.

[9)]The term was introduced by A. V. Radyushkin.

[10)]The analytic idea of all such proofs is rather simple—multiplications by rational functions and one-dimensional integrations does not take us out of the class of power-logarithmic functions. The main difficulty is to carry out a more or less explicit reduction of the problem to a succession of one-dimensional integrals and construct the argument for an arbitrary diagram of some fairly large class. The most clearly defined method of organizing such arguments (leading to significantly stronger results) is given by the theory of the As operation and is based on the use of recursion in the problem (see below).

[11)]We recall that the perturbation-theory series can be generated by iterations of the Dyson–Schwinger equations.

[12)]It must be emphasized here that in the original integral the factor that we replaced by a test function is not itself a test function. In the first place, we are speaking here about heuristics that makes it possible to concentrate on the most important part of the problem. Note that the choice of the class of test functions is here to a large degree arbitrary and is dictated exclusively by convenience.

[13)]One should distinguish between Bogolyubov's heuristic derivation of the *R* operation and its "proof" in BPHZ theory—they are based on two completely opposite ways of resolving the dilemma that is discussed below.

[14)]Incidentally, they did not do this systematically and thereby merely aggrevated the technical difficulties of the proof.

[15)]We may mention in passing that this circumstance explains psychologically why the modern supporters of BPHZ theory are so zealous in insisting on formal rigor, moreover in their form of it, in the study of multiloop diagrams; for when working with the forest formula, one attempts to solve the problem in its entirety, at one stroke, without structuring it in a hierarchy of simpler subproblems. In such an approach, the primitive counting of powers that is the basis of the arguments is hidden under a layer of formalism that is needed to trace the calculations between terms of a rather complicated sum; as a result, one can work with the forest formula only in a scrupulously formal manner. Therefore, it is not surprising that among those who began to study the theory of multiloop diagrams with the forest formula there is an ineradicable suspicion of new approaches that give better results with fewer efforts.

[16)]In order to see how the recursion will continue, it is here necessary to consider some properties of two-loop diagrams of products that are not needed for the expansion of strictly two-loop integrals. This is still less cumbersome than study of the complete three-loop case. More complicated examples do not exhibit new phenomena. The complications associated with non-Euclidean regimes already make single-loop diagrams more difficult, from the point of view of the construction of expansions, than Euclidean two-loop diagrams. Nevertheless, the general principles still work in this case.

[17)]It is sometimes asserted that an indirect (and, we note, extremely cumbersome) description of this kind is more "rigorous" than a simple analytic description. This is a clear example of how rigor is sometimes confused with the empty clutter of badly formalized "proofs."

[18)]It is well known that the main method by which human societies master new methods of thought is by the succession of generations.

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