Energy levels of hydrogen-like atoms and fundamental constants

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Fiz. Elem. Chastits At. Yadra **25**, 144–228 (January–February 1994)

This review contains a description of the theoretical methods of studying the spectra of hydrogen-like systems. Different versions of the quasipotential approach and the method of the effective Dirac equation are considered, including relatively new methods which have appeared in the 1980s. These include the method of studying spectra using the quasipotential equation with the relativistic reduced mass and the method of selecting logarithmic corrections by means of the renormalization-group equation. Special attention is paid to the construction of perturbation theory and the selection of graphs which make it possible to calculate the contributions of various orders in the fine-structure constant α to the energy of the fine and hyperfine splitting in hydrogen-like atoms (positronium, muonium, and the hydrogen atom). Then the experimental and theoretical results are compared for a wide range of problems such as the fine and hyperfine splitting in hydrogen-like systems, the Lamb shift, and the anomalous magnetic moments of the electron and muon. The related problem of the precision determination of the numerical value of the fine-structure constant is also discussed.

1. INTRODUCTION

In recent years, like earlier, much attention has been paid to the study of the energy levels of hydrogen-like (HL) systems—positronium, muonium, hydrogen, and so on, which form the basis for verifying the predictions of quantum electrodynamics (QED). An earlier review[1] has been devoted to this topic. The current status of research in this area is analyzed in the present article.

As is well known, the Dirac equation allows the inclusion of the spin–orbit and spin–spin interactions in HL systems and, accordingly, the prediction of the fine and hyperfine structure of the energy levels with accuracy $O(\alpha^4)$. However, the deviations in the fine structure of the levels, corrections for hyperfine splitting (HFS) induced by radiation effects, and the existence of the Lamb shift (splitting of the $2S_{1/2}$ and $2P_{1/2}$ levels, which coincide in the Dirac theory) can be described only by using the quantum-field methods of QED.

In nonrelativistic quantum mechanics the two-body problem reduces to two simpler problems: the uniform motion of the center of mass and the motion of a particle with reduced mass in a potential field. In the relativistic case of QED it is impossible to separate explicitly the center-of-mass motion and introduce the ordinary potential concept. The distinguishing feature of most quantum methods of describing bound states is the use of the technique of two-particle Green functions and the determination of the spectrum from the locations of its poles.

The equation for the Green function of two fermions can be written in the Schwinger form:[2]

$$\{(\gamma\pi-M)_1(\gamma\pi-M)_2-I_{12}\}G=I, \tag{1}$$

where $\pi=p_\alpha-eA_\alpha$, where p^i_α is the 4-momentum of the ith particle, A^i_α is the external field acting on the ith particle, e is the electron charge, M_i is the mass operator of the ith particle, I_{12} is the interaction kernel of particles 1 and 2, and I is the unit operator. Finally,

$$G(x_1,x_2,x_3,x_4)=\frac{\langle 0|T\{\psi_a(x_1)\psi_b(x_2)\bar{\psi}_a(x_3)\bar{\psi}_b(x_4)S\}|0\rangle}{\langle 0|S|0\rangle} \tag{2}$$

is the complete two-particle Green function in the interaction picture and $\psi(x_i)$ are the field operators of the composite particles.

Equation (1) can also be rewritten in Bethe–Salpeter (BS) form (Ref. 3):[1)]

$$G(x_1,x_2;x_3,x_4)=G_0(x_1,x_2;x_3,x_4)+G_0(x_1,x_2;x_1',x_2')\times K_{BS}(x_1',x_2';x_3',x_4')G(x_3',x_4';x_3,x_4), \tag{3}$$

$$G_0(x_1,x_2;x_3,x_4)=iG_a(x_1,x_3)G_b(x_2,x_4), \tag{4}$$

where $G_{a,b}$ is the free-fermion Green function, and K_{BS} is the kernel of the BS equation, which is related to the particle interaction operator I_{12}, represented as the sum of two-particle-irreducible Feynman graphs. The state of the two-particle system is determined by the two-time wave function ψ, the solution of the homogeneous equation corresponding to (3):

$$(G_0^{-1}-K_{BS})\Psi_{\mathscr{P}}(x_1,x_2)=0, \tag{5}$$

$$\Psi_{\mathscr{P}}(x_1,x_2)=\langle 0|T\{\psi_a(x_1)\psi_b(x_2)\}|\mathscr{P},\nu\rangle. \tag{6}$$

The vector $|\mathscr{P},\nu\rangle$ characterizes the bound system with 4-momentum $\mathscr{P}$ and the set of additional quantum numbers ν as a whole.

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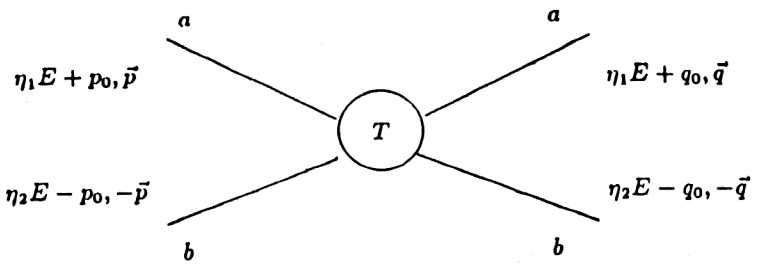


FIG. 1. Parametrization of the off-shell two-particle scattering amplitude T in the c.m. frame ($\eta_1=(E^2+m_1^2-m_2^2)/2E^2$ and $\eta_2=(E^2+m_2^2-m_1^2)/2E^2$).

Using the translational invariance and choosing the c.m. frame $\mathscr{P}_\mu=(E,\vec{0})$, we can obtain the wave function corresponding to a state with definite energy E:

$$\Psi_{\mathscr{P}}(x_1,x_2)=e^{-iEX_0}\phi_E(x) \tag{7}$$

(X_0 is the time coordinate of the center of mass, and x is the relative coordinate).

The bound-state problem in relativistic quantum mechanics can be solved only approximately, using perturbation theory. For the basic approximation one usually uses something corresponding to the instantaneous (Coulomb) interaction. The energy spectrum corresponds to Coulomb levels determined from the wave functions, and the corrections to them are obtained from higher orders of perturbation theory:[4]

$$\Delta E=-i\bar{\phi}_{K_C}(x)(\tilde{K}+\tilde{K}G_C\tilde{K}+\text{etc.})\phi_{K_C}(x'), \tag{8}$$

where $\tilde{K}=K_{BS}-K_C$, K_C is the Coulomb part of the kernel of the BS equation, G_C is the solution of Eq. (3) with the kernel K_C, and $\phi_{K_C}(x)$ is the solution of Eq. (5) with the kernel K_C. However, the state $\phi_{K_C}(x)$ is not stationary, and the relation between the function ϕ_{K_C} and the solution of the Schrödinger (or Dirac) equation with the Coulomb potential is rather complicated. Difficulties also arise in the normalization and formulation of the boundary conditions on the wave function depending on the relative time. All this in the final analysis affects the accuracy of the calculations.

The formalism of three-dimensional equations in relativistic bound-state theory arose even before the appearance of the covariant field-theoretic formalism.[5–7] Of special importance were the development of the quasipotential method[8,9] and the Gross approach.[10] The quantum-field equations in these approaches are free of the defects of the BS equation and are very close in form to the nonrelativistic Schrödinger equation with potential V. They can be viewed as a direct generalization to the relativistic case of two-body potential theory. The basic idea behind all three-dimensional approaches is that of choosing the "bare" two-particle propagator which gives a good description of the physics of the problem. In the quasipotential approach this choice is made by transforming to the two-time Green function in which the relative-time parameter t_a-t_b is eliminated. In the Gross approach the bare two-particle propagator is chosen in the form of a projector on the mass shell of the heavy particle and the free electron propagator.

2. THE QUASIPOTENTIAL APPROACH IN QUANTUM FIELD THEORY

As already noted in the Introduction, the quantum field equations for a two-particle system in three-dimensional approaches reduce to Schrödinger-type equations with quasipotential determined by two-time Green functions. In spite of the absence of explicit relativistic covariance, the quasipotential method retains its information about the properties of the scattering amplitude, which can be obtained by starting from the fundamental principles of quantum field theory. Therefore, the quasipotential equation can be used to study both the analyticity properties and the asymptotic behavior of the scattering amplitude, and also some of the regularities of potential scattering, in particular, at high energies.[11] The renormalization of the quasipotential equation reduces, as in ordinary S-matrix theory, to mass and charge renormalization.[12]

The quasipotential method is very effective also for determining the relativistic and radiative corrections to the spectra of hydrogen-like atoms. It is often convenient to introduce instead of the Green function (2) the off-shell two-particle scattering amplitude:

$$G=G_0+G_0TG_0, \tag{9}$$

which is related to the kernel of the Bethe–Salpeter equation as

$$T=K_{BS}+K_{BS}G_0T \tag{10}$$

or

$$T=K_{BS}+K_{BS}GK_{BS}. \tag{11}$$

On the mass shell ($p_0=q_0=0$, $\sqrt{\vec{p}^2+m_1^2}+\sqrt{\vec{p}^2+m_2^2}=\sqrt{\vec{q}^2+m_1^2}+\sqrt{\vec{q}^2+m_2^2}=E$) the amplitude T coincides with the physical scattering amplitude (see Fig. 1).[2)] Henceforth, we shall take $\sqrt{\vec{p}^2+m_{1,2}^2}\equiv\varepsilon_{1,2p}$.

The quasipotential approach is universal and symmetric in describing the two particles. Because of this it is applicable for studying any system of particles with arbitrary masses. However, the operation of equating the times does not lead to the Dirac equation for one of the particles

when the mass of the other tends to infinity. In studying HL atoms, in which the nucleus has a large charge and mass, it is convenient to use as the initial approximation the exact solutions of the Dirac equation with Coulomb potential. This method was suggested in Ref. 10.

The authors of Ref. 13 have constructed a self-consistent three-dimensional formalism leading to an effective (modified) Dirac equation based on the ideas of the quasipotential approach. In contrast to the usual procedure, when the time arguments of the operators $\psi_a(x_1)$ and $\psi_b(x_2)$ in Eq. (6) are equated, in that study the limit $x_2^0 \to \infty$ is taken.[3)] This leads to the following representation for the wave function (WF) in momentum space:

$$\Psi_{\mathscr{P}}(p_1^0;\vec{p}_1,\vec{p}_2)=\frac{1}{\sqrt{2\varepsilon_{2p}}}\langle \vec{p}_2,\sigma_2|\psi_a(0)|\mathscr{P},\nu\rangle, \qquad (12)$$

for which it is easy to transform to the variables corresponding to the total and relative momentum. A similar procedure performed also for the free Green function G_0 [see Eqs. (2.5), (3.4), and (3.5) of Ref. 13] leads directly to the equation

$$(\eta_1\hat{\mathscr{P}}+\hat{p}-m_1)\Psi_{\mathscr{P}}(\vec{p})=\frac{1}{(2\pi)^3}\int d\vec{q}V(\vec{p},\vec{q})\Psi_{\mathscr{P}}(\vec{q}), \qquad (13)$$

which in the limit $m_2 \to \infty$ becomes the Dirac equation for particle 1 in an external field. This is a characteristic feature of the method of the effective Dirac equation (EDE).

Although the Logunov–Tavkhelidze quasipotential approach and the Gross approach (the EDE method) are based on different physical assumptions, the expressions which can be used to calculate the corrections to the energy levels of composite systems are very similar in these two approaches (the momentum notation corresponds to Fig. 1).

QUASIPOTENTIAL APPROACH AND THE EDE METHOD

Bare propagator for the two-fermion system

$$\hat{G}_0(\vec{p},\vec{q};E)=\frac{1}{(2\pi)^2}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}dp_0dq_0G_0(\vec{p},\vec{q};p_0,q_0;E);$$

$$G_0(\vec{p},\vec{q};p_0,q_0;E)=i(2\pi)^4\frac{1}{\hat{p}_1-m_1}\frac{1}{\hat{p}_2-m_2}\delta^{(4)}(p-q).$$

$$g(p,q;E)=\Lambda_2S(p,q;E)$$

$$=2\pi i\theta(p_2^0)\delta(p_2^2-m_2^2)(2\pi)^4\frac{\hat{p}_2+m_2}{\hat{p}_1-m_1}\delta^{(4)}(p-q).$$

The relative-time parameter is set equal to zero in the first approach. In momentum space this operation corresponds to integration over the relative energies;

$$\hat{G}_0^+=F=(2\pi)^3\delta(\vec{p}-\vec{q})(E-\varepsilon_{1p}-\varepsilon_{2p})^{-1}$$

is the free Green function projected on states with positive energy.

Λ_2 denotes the projection operator on the mass shell of the second particle. The momenta of the constituents are usually written as $p_1=\mathscr{P}-p_2$. In the case of the c.m. system $\mathscr{P}=(E,\vec{0})$ we have $p_1^0=E-\sqrt{\vec{p}^2+m_2^2}$.

The complete Green function

The two-time complete Green function is related to the off-shell scattering amplitude as

$$\hat{G}^+=\hat{G}_0^++[G_0\widehat{TG}_0]^+,$$

which can be rewritten using the quasipotential

$$\hat{V}=(\hat{G}_0^+)^{-1}-(\hat{G}^+)^{-1}$$

$$=(\hat{G}_0^+)^{-1}[G_0\widehat{TG}_0]^+(\hat{G}^+)^{-1}=\tau[1+\hat{G}_0^+\tau]^{-1};$$

$$\hat{G}^+=\hat{G}_0^++\hat{G}_0^+\hat{V}\hat{G}^+.$$

The effective Dirac equation is constructed by using the new two-particle Green function G:

$$G=g+gTg,$$

which satisfies the equation

$$G=g+g\hat{V}_{\text{EDE}}G.$$

It can be used to find the three-dimensional Green function:

$$G(p,q;E)=i(2\pi)^2\delta(p_2^2-m_2^2)\hat{G}(\vec{p},\vec{q};E)\delta(q_2^2-m_2^2),$$

obeying an analogous equation.

Kernel of the equation

It is the quasipotential expressed in terms of the amplitude

$$\tau=(\hat{G}_0^+)^{-1}[G_0\widehat{TG}_0]^+(\hat{G}_0^+)^{-1}$$

as

$$\tau=\hat{V}+\hat{V}\hat{G}_0^+\tau.$$

It is introduced in the second method by the relation

$$T=\hat{V}_{\text{EDE}}+\hat{V}_{\text{EDE}}gT.$$

Equation for the wave function

The single-time wave function obeys the corresponding homogeneous equation:

$$(\hat{G}_0^+)^{-1}\Psi=\hat{V}\Psi.$$

The homogeneous equation for the wave function in symbolic form in the second method is

$$S^{-1}\phi=\Lambda_2\hat{V}_{\text{EDE}}\phi.$$

In expanded form for the two-fermion system these are written as

$$(E-\varepsilon_{1p}-\varepsilon_{2p})\Psi(\vec{p})=\int\frac{d\vec{q}}{(2\pi)^3}\hat{V}(\vec{p},\vec{q};E)\Psi(\vec{q});$$

$$(\hat{p}_1-m_1)\phi(\vec{p};E)=(\hat{p}_2+m_2)\int\frac{d\vec{q}}{(2\pi)^32\varepsilon_{2q}}i$$

$$\times\hat{V}_{\text{EDE}}(\vec{p},\vec{q};E)\phi(\vec{q};E).$$

Since the WF in the muon index satisfies the Dirac equation, the above equation can be rewritten for the WF Ψ containing only the electron spin factor:

$$(\hat{p}_1-m_1)\Psi(\vec{p};E)=\int\frac{d\vec{q}}{(2\pi)^3}\,i\hat{V}_{\rm EDE}(\vec{p},\vec{q};E)\Psi(\vec{q};E).$$

The energy spectrum

Solving the resulting equations by perturbation theory, we obtain the expression for the corrections to the energy levels of an HL atom (Ref. 14):

$$E_n=E_n^0+\langle n|\Delta\hat{V}^{(2)}+\hat{V}^4+\ldots|n\rangle\times E_n$$
$$=E_n^0+\langle n|i\delta\hat{V}_{\rm EDE}|n\rangle$$

multiplied by

$$\left(1+\left\langle n\left|\frac{\partial\Delta\hat{V}^{(2)}}{\partial E}\right|n\right\rangle\right)$$
$$+\left\langle n\left|\sum_{m\neq n}\Delta\hat{V}^{(2)}\frac{|m\rangle\langle m|}{E_n-E_m}\Delta\hat{V}^{(2)}\right|n\right\rangle+\ldots,$$

or

$$\left(1+\left\langle n\left|i\frac{\partial\delta\hat{V}_{\rm EDE}}{\partial E}\right|n\right\rangle\right)+\langle n|i\delta\hat{V}_{\rm EDE}G_{n0}i\delta\hat{V}_{\rm EDE}|n\rangle$$
$$\times\left(1+\left\langle n\left|i\frac{\partial\delta\hat{V}_{\rm EDE}}{\partial E}\right|n\right\rangle\right)+\ldots,$$

where

$$\Delta\hat{V}^{(2)}=\hat{V}^{(2)}-v_C,$$

and v_C is the Coulomb potential, or where

$$\hat{V}_{\rm EDE}=\hat{V}_0+\delta\hat{V}_{\rm EDE},$$

and V_0 is the bare potential. The derivative with respect to E is taken at the point E_n^0.

It should be noted that there is a different method of constructing the quasipotential, using the physical on-shell (but off-shell with respect to the system as a whole) scattering amplitude. However, for precision calculations of the eigenvalues of HL atoms (for example, for calculating the corrections to the Fermi energy of the HFS of the ground state with accuracy better than α^5) it is quite important to include the effects due to the particle binding and the relativistic nature of the particle interaction. The analysis of these effects is much more efficient in the first method. This is because in higher orders of perturbation theory the on-shell scattering amplitude can have poles in the infrared region. If graphs up to fourth order in the charge are considered, the infrared divergences of the quasipotential cancel in the sum of the graphs. In the analysis of higher orders in α the singularities in the behavior of the quasipotential lead to the appearance of contributions logarithmic in the fine-structure constant. The possibilities of including such effects by means of the on-shell scattering amplitude are limited. In addition, the presence of poles in the virtual-particle propagators considerably complicates the integration over the relative momenta (see Sec. 3 and Ref. 15).

In the construction of the quasipotential using the two-particle Green functions the total energy of the system explicitly enters into the quasipotential. As a result, the conditions for normalization and orthogonality of the eigenfunctions[16] along with perturbation theory acquire special features.

Let the eigenvalue equation have the form

$$[F^{-1}(E)-V(\vec{p},\vec{q};E)]\Psi_E(\vec{q})=0, \tag{14}$$

where[4)]

$$F(E)=\hat{G}_0^+=(2\pi)^3\delta^{(3)}(\vec{p}-\vec{q})(E-\varepsilon_{1p}-\varepsilon_{2p})^{-1}, \tag{15}$$

and the quasipotential is defined above. Then for the eigenfunction Ψ_n corresponding to the state with eigenvalue E_n we have

$$\hat{G}^+(E)W(E,E_n)\Psi_n=\frac{\Psi_n}{E-E_n}, \tag{16}$$

with

$$W(E,E_n)=\frac{F^{-1}(E)-F^{-1}(E_n)}{E-E_n}-\frac{V(E)-V(E_n)}{E-E_n}. \tag{17}$$

Since near a pole ($E\simeq E_n$)

$$\hat{G}^+(E)\simeq\frac{\Psi_n\Psi_n^*}{E-E_n}, \tag{18}$$

the orthogonality and normalization conditions take the form

$$\Psi_m^*W(E_m,E_n)\Psi_n=\delta_{mn} \tag{19}$$

and

$$\Psi_m^*\Psi_n=\delta_{mn}+\Psi_m^*\left.\frac{\partial V(E)}{\partial E}\right|_{E=E_n}\Psi_n\quad(E_m=E_n). \tag{20}$$

Since the energy levels of a composite particle are poles of the exact scattering amplitude, and this pole is absent in each individual term of the expansion of this amplitude in powers of the charge, there must exist an infinite sequence of graphs for bound states leading to contributions to the energy of the bound system of the same order in α. The graphs making up this sequence are reducible Feynman diagrams in the sense of two-particle cross sections. However, such an effect of the binding can occur also for irreducible diagrams (for example, in the calculation of the Lamb shift of the energy levels).

As was shown in Refs. 3 and 17, diagrams with successive exchange of Coulomb photons in the infrared region give a contribution of the same order in α in the calculation of the hyperfine splitting of an HL system. Therefore, it is necessary to make the corresponding modification of the procedure of constructing the quasipotential owing to the need to selectively sum an infinite series of diagrams. Let us introduce the Coulomb Green function G_C satisfying the equation

$$(G_0^{-1}-K_C)G_C=I, \tag{21}$$

where

$$K_C(\vec{p},\vec{q})=-\frac{e(Ze)\Gamma_0}{(\vec{p}-\vec{q})^2}=\nu_C\Gamma_0 \tag{22}$$

is the Coulomb kernel ($\Gamma_0=\gamma_{10}\gamma_{20}$). Equation (21) can be rewritten as

$$G_C=G_0+G_0K_CG_C=G_0+G_CK_CG_0, \tag{23}$$

and the complete Green function can be written as in (9)–(11):

$$G=G_C+G_C\widetilde{T}G_C, \tag{24}$$

$$\widetilde{T}=\widetilde{K}+\widetilde{K}G_C\widetilde{T}, \tag{25}$$

$$\widetilde{T}=\widetilde{K}+\widetilde{K}G\widetilde{K} \tag{26}$$

(here $\widetilde{K}=K_{BS}-K_C$).

The wave function characterizing the state of a two-fermion system with total energy E must then satisfy an equation of the quasipotential type:

$$[(\hat{G}_C^+(\vec{p},\vec{q};E))^{-1}-\widetilde{V}(\vec{p},\vec{q};E)]\Psi_E(\vec{q})=0, \tag{27}$$

where again there is understood to be an integration over repeated momenta. The corresponding quasipotential has the form

$$\begin{aligned}\widetilde{V}(\vec{p},\vec{q};E)&=[\hat{G}_C^+(\vec{p},\vec{q};E)]^{-1}-[\hat{G}^+(\vec{p},\vec{q};E)]^{-1}\\&=(\hat{G}_C^+)^{-1}[G_C\widehat{\widetilde{T}G_C}]^+(\hat{G}^+)^{-1},\end{aligned} \tag{28}$$

and $\hat{G}^+$ is the two-time Green function projected on positive-frequency states (see above).

The technique developed in Ref. 18 using the Coulomb Green function makes it possible to include successive multiple exchange of Coulomb photons and to write down expressions for the energy-level shift ΔE relative to the Coulomb energy.

We see from Eq. (24) that the complete two-particle Green function (projected onto positive-frequency states) has the inverse

$$\begin{aligned}&(\hat{G}^+)^{-1}=(\hat{G}_C^+)^{-1}-(\hat{G}_C^+)^{-1}\widetilde{T}_C^+(\hat{G}^+)^{-1},\\&\widetilde{T}_C^+=[G_C\widetilde{T}G_C]^+\end{aligned} \tag{29}$$

and the quasipotential can be written as

$$\widetilde{V}(\vec{p},\vec{q};E)=(\hat{G}_C^+)^{-1}\widetilde{T}_C^+(\hat{G}^+)^{-1}=\widetilde{\tau}_C-\widetilde{\tau}_C\hat{G}_C^+\widetilde{\tau}_C+..., \tag{30}$$

$$\widetilde{\tau}_C=(\hat{G}_C^+)^{-1}\widetilde{T}_C^+(\hat{G}_C^+)^{-1}. \tag{31}$$

Projecting the two-time Coulomb Green function onto positive-frequency states, we find

$$\hat{G}_C^+=F+F(\nu_C\hat{G}_C)^+=F+(\hat{G}_C\nu_C)^+F. \tag{32}$$

Here we have used the expression

$$\hat{G}_0=(\Lambda^{++}F-\Lambda^{--}F')\Gamma_0, \tag{33}$$

where F is given by (15) and

$$F'=(2\pi)^3\delta(\vec{p}-\vec{q})(E+\varepsilon_{1p}+\varepsilon_{2p})^{-1}. \tag{34}$$

$\Lambda^{++}=\Lambda_1^+(\vec{p})\Lambda_2^+(-\vec{p})$ and $\Lambda^{--}=\Lambda_1^-(\vec{p})\Lambda_2^-(-\vec{p})$, $\Lambda^{\pm}(\vec{p})$ are projection operators.

Transforming Eq. (32), using the relations

$$[QG_C]^+=[Q\Sigma\Gamma_0]^+\hat{G}_C^+, \tag{35}$$

$$[G_CQ]^+=\hat{G}_C^+[\Sigma'\Gamma_0Q]^+ \tag{36}$$

(where $\Sigma=(1+\Lambda^{--}F'\nu_C)^{-1}$ and $\Sigma'=(1+\nu_C\Lambda^{--}F')^{-1}$, and Q has an arbitrary matrix structure), we obtain a closed equation for the function $\hat{G}_C^+$:

$$\hat{G}_C^+=F+FK_\Sigma\hat{G}_C^+=F+\hat{G}_C^+K_\Sigma F \tag{37}$$

with the kernel $K_\Sigma=u_1^*u_2^*\widetilde{\Sigma}u_1u_2$ and $\widetilde{\Sigma}=\nu_C\Sigma=\Sigma'\nu_C$. The inverse Green function $(\hat{G}_C^+)^{-1}$ in this case is

$$(\hat{C}_C^+)^{-1}=F^{-1}-K_\Sigma, \tag{38}$$

and the kernel K_Σ contains projections of the Coulomb interaction on negative-frequency states:

$$K_\Sigma=K_C^+ +\delta K_\Sigma=K_C^+-u_1^*u_2^*\nu_C\Lambda^{--}F'\nu_Cu_1u_2. \tag{39}$$

It is important to note that F', in contrast to F, does not have a pole for $\vec{p},\vec{q}\to 0$, $E\to m_1+m_2$. In the approximation of the large components ($u_i=\binom{w_i}{0}$, where w_i is a normalized Pauli spinor) the kernel K_C^+ coincides with the Coulomb potential, and the term containing the projection operator Λ^{--} vanishes. Therefore, the principal part of the kernel K_Σ is equal to K_C^+, and the function $(\hat{G}_C^+)^{-1}$ coincides with the inverse Green function of the nonrelativistic Schrödinger equation with Coulomb potential.

In many cases the quasipotential is conveniently chosen in terms of the amplitude $\widetilde{T}_0^+=[G_0\widehat{\widetilde{T}G_0}]^+$. This can easily be accomplished by using the auxiliary expressions (35) and (36). Since

$$\begin{aligned}\widetilde{T}_C^+=\widetilde{T}_0^+&+(\widetilde{T}_0K_C\hat{G}_C)^+ +(\hat{G}_CK_C\widetilde{T}_0)^+\\&+(\hat{G}_CK_C\widetilde{T}_0K_C\hat{G}_C)^+,\end{aligned} \tag{40}$$

after using the transformations (35) and (36) we have

$$\begin{aligned}\widetilde{T}_C^+=\widetilde{T}_0^+&+\hat{G}_C^+(\Sigma'\nu_C\widetilde{T}_0)^+ +(\widetilde{T}_0\nu_C\Gamma_0\Sigma\Gamma_0)^+\hat{G}_C^+\\&+\hat{G}_C^+(\Sigma'\nu_C\widetilde{T}_0K_C\Sigma\Gamma_0)^+\hat{G}_C^+.\end{aligned} \tag{41}$$

Equation (41) can more conveniently be written as

$$\widetilde{T}_C^+=(\hat{G}_C^+u_1^*u_2^*\widetilde{\Sigma}I+Iu_1^*u_2^*)\widetilde{T}_0(\Gamma_0u_1u_2I+\Gamma_0I\widetilde{\Sigma}u_1u_2\hat{G}_C^+), \tag{42}$$

or, using the fact that

$$\begin{aligned}&\lambda^+ +\lambda^-=I,\\&\lambda--=\Lambda1+(\vec{p})\Lambda2-(-\vec{p})+\Lambda1-(\vec{p})\Lambda2+(-\vec{p})+\Lambda--,\end{aligned} \tag{43}$$

in the form

$$\begin{aligned}\widetilde{T}_C^+=\hat{G}_C^+[&F^{-1}\widetilde{T}_0^+F^{-1}+F^{-1}(\widetilde{T}_0\Gamma_0\lambda^-\widetilde{\Sigma}\Gamma_0)^+\\&+(\widetilde{\Sigma}\lambda^-\widetilde{T}_0)^+F^{-1}+(\widetilde{\Sigma}\lambda^-\widetilde{T}_0\Gamma_0\lambda^-\widetilde{\Sigma}\Gamma_0)^+]\hat{G}_C^+.\end{aligned} \tag{44}$$

Therefore, the amplitude $\tilde{\tau}_C$ can be written as the sum $\tilde{\tau}_C=\tilde{\tau}_0+\rho$, where $\tilde{\tau}_0$ is the scattering amplitude containing the matrix $\tilde{T}$ and the other terms correspond to the higher-order interaction with subtraction of the corresponding iterations.

The graphical interpretation of the amplitude $\tilde{\tau}_0$ is obvious: it represents the set of all irreducible graphs, excluding the graph with exchange of one Coulomb photon, and also reducible graphs with exchanges of Coulomb photons only in intermediate states. The corresponding quasipotential can also be split into two parts:

$$\tilde{V}(\vec{p},\vec{q};E)=\tilde{V}_{\tilde{\tau}_0}+\tilde{V}_{\rho}+\text{higher orders}, \tag{45}$$

where

$$\tilde{V}_{\tilde{\tau}_0}=\tilde{\tau}_0-\tilde{\tau}_0\hat{G}_C^{+}\tilde{\tau}_0, \tag{46}$$

$$\tilde{V}_{\rho}=\rho-\rho\hat{G}_C^{+}\rho-\tilde{\tau}_0\hat{G}_C^{+}\rho-\rho\hat{G}_C^{+}\tilde{\tau}_0, \tag{47}$$

with

$$\rho=F^{-1}\Delta RF^{-1}, \tag{48}$$

$$\Delta R=F(\tilde{\Sigma}\lambda^{-}\tilde{T}_0)^{+}+(\tilde{T}_0\Gamma_0\lambda^{-}\tilde{\Sigma}\Gamma_0)^{+}F$$
$$+F(\tilde{\Sigma}\lambda^{-}\tilde{T}_0\Gamma_0\lambda^{-}\tilde{\Sigma}\Gamma_0)^{+}F. \tag{49}$$

After substituting the amplitude $\tilde{T}$ from (26) ($\tilde{T}\simeq\tilde{K}+\tilde{K}G_C\tilde{K}$) into (46), we have

$$\tilde{V}_{\tilde{\tau}_0}=F^{-1}[\tilde{K}_0^{+}+(\tilde{K}G_C\tilde{K})_0^{+}-\tilde{K}_0^{+}F^{-1}\hat{G}_C^{+}F^{-1}\tilde{K}_0^{+}$$
$$+\text{etc.}]F^{-1}, \tag{50}$$

in which we have used the notation

$$\tilde{K}_0^{+}=[G_0\widehat{\tilde{K}G_0}]^{+}, \tag{51}$$

$$(\tilde{K}G_C\tilde{K})_0^{+}=[G_0\widehat{\tilde{K}G_C\tilde{K}}G_0]^{+}. \tag{52}$$

To calculate the shift of the energy levels of the two-fermion system up to order $\alpha^6\ln\alpha$ it is sufficient to include

$$(\tilde{K}G_C\tilde{K})_0^{+}=(\tilde{K}G_0\tilde{K})_0^{+}+(\tilde{K}_0K_C\tilde{K}_0)^{+}$$
$$+(\tilde{K}_0\Gamma_0\nu_C\hat{G}_C\nu_C\Gamma_0\tilde{K}_0)^{+}. \tag{53}$$

Using the representation of the unit operator in terms of the projection operators, we find

$$(\tilde{K}_0\Gamma_0I\nu_C\hat{G}_C\nu_C\Gamma_0I\tilde{K}_0)^{+}$$
$$=\tilde{K}_0^{+}(\nu_C\hat{G}_C\nu_C)^{+}\tilde{K}_0^{+}+\tilde{K}_0^{+}(\nu_C\hat{G}_C\nu_C\Gamma_0\lambda^{-}\tilde{K}_0)^{+}$$
$$+(\tilde{K}_0\Gamma_0\lambda^{-}\nu_C\hat{G}_C\nu_C)^{+}\tilde{K}_0^{+}$$
$$+(\tilde{K}_0\Gamma_0\lambda^{-}\nu_C\hat{G}_C\nu_C\Gamma_0\lambda^{-}\tilde{K}_0)^{+}. \tag{54}$$

The first term in this expression gives the largest contribution in the calculation of the energy-level shifts.

Using the definition of the operator Σ, we can make the transformation

$$(\nu_C\hat{G}_C\nu_C)^{+}=[\nu_C\Sigma(I+\Lambda^{--}F'\nu_C)\hat{G}_C\nu_C]^{+}$$
$$=(\tilde{\Sigma}\lambda^{+}\hat{G}_C\nu_C)^{+}+(\tilde{\Sigma}\lambda^{-}\hat{G}_C\nu_C)^{+}$$
$$+(\tilde{\Sigma}\Lambda^{--}F'\nu_C\hat{G}_C\nu_C)^{+}. \tag{55}$$

Starting from the definition of the kernel K_Σ (39), we have

$$(\tilde{\Sigma}\lambda^{+}\hat{G}_C\nu_C)^{+}=K_\Sigma(\hat{G}_C\nu_C)^{+}=K_\Sigma\hat{G}_C^{+}K_\Sigma. \tag{56}$$

Since we have the equation $\lambda^{-}\hat{G}_C=-\Lambda^{--}F'(\Gamma_0+\nu_C\hat{G}_C)$, the sum of the last two terms in (55) can be transformed to

$$(\tilde{\Sigma}\lambda^{-}\hat{G}_C\nu_C)^{+}+(\tilde{\Sigma}\Lambda^{--}F'\nu_C\hat{G}_C\nu_C)^{+}$$
$$=-(\tilde{\Sigma}\Lambda^{--}F'\Gamma_0\nu_C)^{+}=K_\Sigma-K_C^{+}. \tag{57}$$

Therefore,

$$(\nu_C\hat{G}_C\nu_C)^{+}=K_\Sigma\hat{G}_C^{+}K_\Sigma+K_\Sigma-K_C^{+}. \tag{58}$$

The iteration term in the quasipotential expression (50) including (37) is written as

$$\tilde{K}_0^{+}F^{-1}\hat{G}_C^{+}F^{-1}\tilde{K}_0^{+}=\tilde{K}_0^{+}(F^{-1}+K_\Sigma+K_\Sigma\hat{G}_C^{+}K_\Sigma)\tilde{K}_0^{+}. \tag{59}$$

Substituting the last two equations into the expression for the quasipotential, we convince ourselves that the projection of the block $\nu_C\hat{G}_C\nu_C$ on positive-frequency states drops out of the expression for the quasipotential (50). Here

$$\tilde{V}_{\tilde{\tau}_0}=F^{-1}\{\tilde{K}_0^{+}+(\tilde{K}G_0\tilde{K})_0^{+}-\tilde{K}_0^{+}F^{-1}\tilde{K}_0^{+}+(\tilde{K}_0K_C\tilde{K}_0)^{+}$$
$$-\tilde{K}_0^{+}K_C^{+}\tilde{K}_0^{+}+\tilde{K}_0^{+}(\nu_C\hat{G}_C\nu_C\Gamma_0\lambda^{-}\tilde{K}_0)^{+}$$
$$+(\tilde{K}_0\Gamma_0\lambda^{-}\nu_C\hat{G}_C\nu_C)^{+}\tilde{K}_0^{+}$$
$$+(\tilde{K}_0\Gamma_0\lambda^{-}\nu_C\hat{G}_C\nu_C\Gamma_0\lambda^{-}\tilde{K}_0)^{+}\}F^{-1}. \tag{60}$$

Information about the corrections to the Coulomb energy levels can be obtained by constructing perturbation theory on the basis of the above arguments and Eq. (27) written as

$$(F^{-1}(E_C)+\Delta E-K_C^{+}-\tilde{\tilde{V}}(E))\Psi_E=0, \tag{61}$$

where

$$\tilde{\tilde{V}}=\tilde{V}+K_\Sigma-K_C^{+}, \tag{62}$$

and $\Delta E=E-E_C$ is the correction to the ground state of the nonrelativistic Schrödinger equation with Coulomb potential.

If E' is the eigenvalue of the wave function of the equation with kernel K_C^{+},

$$(\hat{G}_C^{+}(E'))^{-1}\Psi_{E'}=(F^{-1}(E')-K_C^{+})\Psi_{E'}=0, \tag{63}$$

the initial equation (61) takes the form

$$((\hat{G}_C^{+}(E'))^{-1}+\Delta E'-\tilde{\tilde{V}}(E))\Psi_E=0, \tag{64}$$

with

$$\Delta E'=E-E' \quad\text{and}\quad \Delta E=\Delta E'+\Delta E_C=\Delta E'+E'-E_C. \tag{65}$$

Let us assume that the eigenfunction $\Psi_{E'}$ and energy eigenvalues are known. According to first-order perturbation theory, we find ($\Psi_E=\Psi_E+\Psi_I$)

$$((\hat{G}_C^{+}(E'))^{-1}\Psi_I+\Delta E_I\Psi_{E'}-\tilde{\tilde{V}}(E')\Psi_{E'})=0. \tag{66}$$

Multiplying both sides of this equation on the left by $\Psi^*_{E'}$ and using the normalization condition, we obtain

$$\Delta E'_{\rm I}=\langle\Psi_{E'}|\,\widetilde{\widetilde{V}}(E')\,|\Psi_{E'}\rangle, \tag{67}$$

$$\Psi_{\rm I}=\left(G^+_C(E')-\frac{\Psi_{E'}\Psi^*_{E'}}{E-E'}\right)\widetilde{\widetilde{V}}(E')\Psi_{E'},\quad E\to E'. \tag{68}$$

Similarly, for the correction to the energy level in second-order perturbation theory we have

$$\Delta E'_{\rm II}=\langle\Psi_{E'}|\,\widetilde{\widetilde{V}}(E')(1+G'^+_C(E')\widetilde{\widetilde{V}}(E'))\,|\Psi_{E'}\rangle, \tag{69}$$

where

$$G'^+_C(E')=G^+_C(E')-\frac{\Psi_{E'}\Psi^*_{E'}}{E-E'}. \tag{70}$$

The term containing the energy derivative has dropped out of the final result, since $\Psi_{E'}$ are the eigenfunctions of the equation with kernel K^+_C, which is independent of the energy.

To determine $\Psi_{E'}$ from Eq. (64) we introduce the auxiliary function $\Phi_{E'}$ determined by the equality

$$\Psi_{E'}(\vec{q})=\frac{(\varepsilon_{1q}+\eta_1E)(\varepsilon_{2q}+\eta_2E)}{2\mu(E+\varepsilon_{1q}+\varepsilon_{2q})}\Phi_{E'}(\vec{q})$$

$$=\frac{(2E'\varepsilon_{1q}+E'^2+m_1^2-m_2^2)(2E'\varepsilon_{2q}+E'^2+m_2^2-m_1^2)}{8\mu E'^2(E'+\varepsilon_{1q}+\varepsilon_{2q})}$$

$$\times\Phi_{E'}(\vec{q}), \tag{71}$$

satisfying a nonrelativistic Schrödinger type of equation:

$$(g_C^{-1}-\Delta\varepsilon-\delta K_C)\Phi_{E'}=0, \tag{72}$$

where

$$g_C^{-1}=W_C-\frac{p^2}{2\mu}-\nu_C,\quad W_C=E_C-m_1-m_2,\quad \mu=\frac{m_1m_2}{m_1+m_2}, \tag{73}$$

$$\Delta\varepsilon=W_C+\frac{((m_1+m_2)^2-E'^2)(E'^2-(m_1-m_2)^2)}{8\mu E'^2}$$

$$\simeq-\Delta E_C \tag{74}$$

and

$$\delta K_C=K^+_C(\vec{p},\vec{q})\frac{(\varepsilon_{1q}+\eta_1E)(\varepsilon_{2q}+\eta_2E)}{2\mu(E+\varepsilon_{1q}+\varepsilon_{2q})}-\nu_C$$

$$\simeq K^+_C\frac{W_C-\frac{\vec{q}^2}{2\mu}}{E'-\varepsilon_{1q}-\varepsilon_{2q}}-\nu_C. \tag{75}$$

The Coulomb WF of the $1S$ state

$$\phi_C(\vec{p})=\frac{8\pi Z\alpha\mu}{(\vec{p}^2+Z^2\alpha^2\mu^2)^2}\,|\phi_C(r=0)|, \tag{76}$$

$$|\phi_C(r=0)|^2=\frac{(\alpha\mu)^3}{\pi}, \tag{77}$$

obviously satisfies the equation

$$g_C^{-1}\phi_C=0. \tag{78}$$

The correction to this Coulomb energy level ΔE_C and the function $\Phi_{E'}$ are found (like $\Delta E'$ earlier) by using quantum-mechanical perturbation theory:

$$\Delta E_C=\langle\phi_C|\,\delta K_C(1+g'_C\delta K_C)\,|\phi_C\rangle, \tag{79}$$

$$g'_C=g_C-\frac{\phi_C\phi^*_C}{E-E_C},\quad \text{for } E\to E_C. \tag{80}$$

In second-order perturbation theory the eigenfunction of Eq. (63) has the form

$$\Psi_{E'}(\vec{p})=\frac{(\varepsilon_{1p}+m_1)(\varepsilon_{2p}+m_2)-\vec{p}^2}{4m_1m_2}$$

$$\times[\phi_C(\vec{p})+g'_C(\vec{p},\vec{k})\delta K_C(\vec{k},\vec{q})\phi_C(\vec{q})] \tag{81}$$

(there is understood to be an integration over the 3-vectors $\vec{k}$ and $\vec{q}$).

The final expression for the total shift of the energy level E relative to the Coulomb ground state is the sum of the correction ΔE_C and the corrections from the quasipotentials $\delta K_\Sigma=K_\Sigma-K^+_C$ and $\widetilde{V}(E)$. The expressions for these corrections are given in Refs. 18 and 19:

$$\Delta E=\Delta E_{KK}+\Delta E_{KV}+\Delta E_{VV}, \tag{82}$$

$$\Delta E_{KK}=\langle\phi_C|\,\delta K_C(1+g'_C\delta K_C)\,|\phi_C\rangle$$

$$+\langle\Psi_{E'}|\,\delta K_\Sigma(1+G'_C(E')\delta K_\Sigma)\,|\Psi_{E'}\rangle, \tag{83}$$

$$\Delta E_{KV}=\langle\Psi_{E'}|\,\widetilde{V}(E')G'_C(E')\delta K_\Sigma$$

$$+\delta K_\Sigma G'_C(E')\widetilde{V}(E')\,|\Psi_{E'}\rangle, \tag{84}$$

$$\Delta E_{VV}=\langle\Psi_{E'}|\,\widetilde{V}(E')(1+G'_C(E')\widetilde{V}(E'))\,|\Psi_{E'}\rangle. \tag{85}$$

The question of constructing the kernel of the equation has been studied in Refs. 18–20. The kernel $\widetilde{K}$ can be expanded in a perturbation series:

$$\widetilde{K}=K_T+K^{(2)}, \tag{86}$$

where the subscript T denotes a transverse photon in the Coulomb gauge, and $K^{(2)}$ is constructed from the graphs of second order in the fine-structure constant.

We note that in the case of a two-particle system consisting of a particle and an antiparticle, it is necessary to include the annihilation channel of the interaction in constructing the interaction kernel.[19,21]

A method based on the use of the local quasipotential equation with relativistic reduced mass in the c.m. frame has been suggested for studying the relativistic energy spectrum of a two-particle bound system.[22–24] The following equation was obtained by "rationalization" of the Logunov–Tavkhelidze equation:[25]

$$\left(\frac{b^2(E)}{2\mu_R}-\frac{\vec{p}^2}{2\mu_R}\right)\Psi_E(\vec{p})=I(E,\vec{p})\int\frac{d\vec{q}}{(2\pi)^3}V(\vec{p},\vec{q};E)\Psi_E(\vec{q}), \tag{87}$$

where

$$I(E,\vec{p})=\frac{(E+\varepsilon_{1p}+\varepsilon_{2p})(E^2-(\varepsilon_{1p}-\varepsilon_{2p})^2)}{8E_1E_2E}. \tag{88}$$

In accordance with the notation of Fig. 1,

$$E_1=\eta_1E=\frac{E^2-m_2^2+m_1^2}{2E}, \tag{89}$$

$$E_2=\eta_2E=\frac{E^2-m_1^2+m_2^2}{2E}, \tag{90}$$

$E_1+E_2=E$ is the bound-state energy, and

$$b^2(E)=\frac{1}{4E^2}[E^2-(m_1+m_2)^2][E^2-(m_1-m_2)^2] \tag{91}$$

is the squared relative on-shell momentum. Finally,

$$\mu_R=\frac{E_1E_2}{E}=\frac{E_1E_2}{E_1+E_2}=\frac{E^4-(m_1^2-m_2^2)^2}{4E^3} \tag{92}$$

is the relativistic reduced mass, determined in accordance with the relativistic expression for the center-of-mass coordinate. It should be noted that there are also other definitions of the relativistic reduced mass, like $\tilde{\mu}_R=[E^2-(m_1-m_2)^2]/4E$ (Ref. 25) and $\mu_R'=m_1m_2/E$ (Ref. 26). The quantity $m'=\sqrt{m_1m_2}$ introduced in Ref. 27 (the effective mass) is closely related to the concept of the relativistic reduced mass. This definition allowed the authors of Ref. 27 to reduce the relativistic two-body problem to the case of the motion of a particle of mass m' in a quasipotential field. In the nonrelativistic limit $E_{1,2}\to m_{1,2}$ the relativistic reduced mass μ_R becomes the ordinary reduced mass $\mu=m_1m_2/(m_1+m_2)$.

As shown earlier, the quasipotential can be considered to be defined from an equation of the Lippmann–Schwinger type with scattering amplitude at zero relativity of the particle energy: $p_0=q_0=0$, $p_2^0=E_1$, $p_2^0=E_2$. On the energy shell $E=\varepsilon_{1p}+\varepsilon_{2p}=\varepsilon_{1q}+\varepsilon_{2q}$, $\varepsilon_{ip}=\sqrt{\vec{p}^2+m_i^2}$, and $\varepsilon_{iq}=\sqrt{\vec{q}^2+m_i^2}$. In this case $I(E,\vec{p})=1$ and Eq. (87) can be solved exactly for the Coulomb interaction.

In the case of the interaction of two spinor particles with masses m_1 and m_2 and charges $(-e)$ and Ze, as shown in Ref. 22, the main contribution to the particle binding energy comes from the modified Coulomb potential:

$$V_C(\vec{p},\vec{q};E)=-\frac{Ze^2}{(\vec{p}-\vec{q})^2}\left(1+\frac{b^2(E)}{E_1E_2}\right). \tag{93}$$

The quantization of the energy levels is determined by a relation analogous to that obtained in Ref. 28:

$$\frac{b^2E^2}{(b^2+E_1E_2)^2}=\frac{Z^2\alpha^2}{n^2} \tag{94}$$

($n=1,2,...$ is the principal quantum number), which leads to a version of the relativistic Balmer formula:[29]

$$E_n^2=m_1^2+m_2^2+2m_1m_2\left(1+\frac{(Z\alpha)^2}{n^2}\right)^{-1/2}. \tag{95}$$

This equation can be rewritten in the form of an expansion of the binding energy B in powers of α^2:

$$B=E-m_1-m_2=-\frac{\mu}{2}\frac{Z^2\alpha^2}{n^2}+\frac{\mu}{8}\frac{Z^4\alpha^4}{n^4}\left(3-\frac{\mu^2}{m_1m_2}\right), \tag{96}$$

$$\mu=\frac{m_1m_2}{m_1+m_2}.$$

The relativistic Balmer formula includes recoil effects, but does not describe the fine and hyperfine structure related to the spin–orbit and spin–spin interactions. Corrections of this type have been studied in Ref. 24 in the formalism of the local quasipotential equation with relativistic reduced mass. In first-order perturbation theory

$$\Delta E_{\rm I}=\langle\Psi_C'|\Delta\hat{V}_\gamma+\hat{V}_{2\gamma}+\text{etc.}|\Psi_C'\rangle, \tag{97}$$

where $\Delta\hat{V}_\gamma=\hat{V}_\gamma-V_C$ describes the difference of the quasipotential of one-photon exchange and the modified Coulomb potential (93), and

$$V_{2\gamma}=T_{2\gamma}^+(\vec{p},\vec{q},p_0=0,q_0=0)$$
$$-\int\frac{d\vec{k}}{(2\pi)^3}\frac{\hat{V}_\gamma(\vec{p},\vec{k};E)\hat{V}_\gamma(\vec{k},\vec{q};E)}{\frac{b^2(E)}{2\mu_R}-\frac{\vec{k}^2}{2\mu_R}}. \tag{98}$$

In accordance with perturbation theory the second-order corrections are given by

$$\Delta E_{\rm II}=\langle\Psi_C'|\Delta\hat{V}_\gamma|\Psi_C'\rangle\left\langle\Psi_C'\left|\frac{\partial\Delta\hat{V}_\gamma}{\partial E}\right|\Psi_C'\right\rangle$$
$$+\sum_{n=2}^{\infty}\frac{\langle\Psi_C'|\Delta\hat{V}_\gamma|\Psi_{nC}'\rangle\langle\Psi_{nC}'|\Delta\hat{V}_\gamma|\Psi_C'\rangle}{E_1^C-E_n^C} \tag{99}$$

with eigenfunctions of the Pauli type which are the solution of a local quasipotential equation with modified Coulomb potential:

$$\Psi_C'(\vec{p})=\frac{8\pi Z\alpha\mu_{\rm eff}}{(\vec{p}^2+Z^2\alpha^2\mu_{\rm eff}^2)^2}|\Psi_C(0)|$$
$$\times\left[1-\frac{1}{2}(Z\alpha)^2\left(1-\frac{\mu^2}{m_1m_2}\right)\right]\chi_1\chi_2, \tag{100}$$

$$|\Psi_C(0)|=\sqrt{\frac{Z^3\alpha^3\mu_{\rm eff}^3}{\pi}} \tag{101}$$

($\chi_{1,2}$ are two-component Pauli spinors),

$$\mu_{\rm eff}=\frac{b^2(E)+E_1E_2}{E_n}=\mu_R+\frac{b^2(E)}{E_n}=\frac{m_1m_2}{E_n}\left(1+\frac{(Z\alpha)^2}{n^2}\right)^{-1/2}. \tag{102}$$

The corrections to the hyperfine structure of muonium of order $(Z\alpha)^2E_F$ and $(m_1/m_2)(Z\alpha)^2E_F$ obtained in this approach are given in Sec. 4 of the present review.

3. METHODS OF CALCULATING THE ENERGY SPECTRUM IN DIFFERENT ORDERS IN α

As can be seen from the preceding section, the calculation of the fine and hyperfine splitting of the energy levels reduces to finding the matrix elements of the quasipotential V. In the nonrelativistic limit the WF of a weakly coupled system can be approximated by a Dirac δ function. The use of the Coulomb wave function of the $1S$ state allows us to consider the matrix elements at nonzero values of the momenta, $\vec{p}$, $\vec{q}\neq 0$. However, the relativistic corrections are more accurately included in the description of the bound state using the WF in the form (81).

In the perturbation theory developed in Sec. 2 the quasipotential and the interaction kernel incorporate the Coulomb Green function in an essential way. The methods of using this function to study the problem of the fine and hyperfine structure of HL atoms are different. In the first case one is particularly interested in the low-frequency region of virtual momenta, where the interaction is nonrelativistic in nature. Here to include bound-state effects in virtual states of the interaction kernel it is important to consider the block of Coulomb interactions as a whole, for example, using the explicit expression for the Green function of the nonrelativistic Schrödinger equation with Coulomb potential.

When analyzing the HFS energy levels it is necessary to be able to consider contributions from exchanges of one or two transverse photons and an arbitrary number of Coulomb photons. Here it is sufficient to use the ordinary expansion of the Coulomb Green function, keeping only the required number of terms in the expansion.[5)]

If we restrict ourselves to accuracy $O(\alpha^5)$, then in the calculations of the $1S$ level based on the quasipotential V constructed from the graphs of order α^2 and higher, it is sufficient to use the approximate WFs (Ref. 30):

$$\phi_C^{\text{approx}}(\vec{p})=(2\pi)^3\delta(\vec{p})\,|\phi_C(r=0)|, \quad E\simeq m_1+m_2. \tag{103}$$

The above statement is based on the fact that the squared modulus of the WF (which is present in the matrix element) in coordinate space for $r=0$ is of order α^3:

$$|\phi_C(r=0)|^2=\frac{(\alpha\mu)^3}{\pi n^3}\sim O(\alpha^3). \tag{104}$$

Using the known representation of the δ function

$$\delta(x)=\lim_{\alpha\to 0}\frac{1}{\pi}\frac{\alpha}{\alpha^2+x^2}, \tag{105}$$

we find

$$\frac{\pi\delta(x)}{2x^2}=\lim_{\alpha\to 0}\frac{\alpha}{(\alpha^2+x^2)^2} \tag{106}$$

and, therefore,

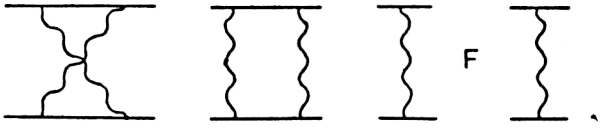
$$\lim_{\alpha\to 0}\phi_C(\vec{p})=\lim_{\alpha\to 0}\frac{8\pi\alpha\mu\,|\phi_C(0)|}{(\vec{p}^2+\alpha^2\mu^2)^2}=8\pi|\phi_C(0)|\,\frac{\pi\delta(p)}{2p^2}. \tag{107}$$

Using the expression $\delta(\vec{p})=\delta(p)/2\pi p^2$, which is valid for spherical symmetry, we finally obtain

$$\lim_{\alpha\to 0}\phi_C(\vec{p})=(2\pi)^3|\phi_C(0)|\,\delta(\vec{p}). \tag{108}$$

It is clear from the form of the Coulomb WF (76) that the main contribution to the splitting of the energy levels comes from momenta $\vec{p}^2\sim Z^2\alpha^2\mu^2$, as a result of which the expansion of the integrand in p/m is equivalent to the expansion of the entire integral in α (with the condition that the integral converges).

This property has proved to be very useful in calculating the contributions of order α^5 to the HFS of the ground state of positronium from the graphs[30]



In the calculation of the matrix elements only the upper "large" components of the bispinors have survived and it has been possible to set $E^2=m^2$ and $\vec{p}, \vec{q}=0$ in the interaction amplitude corresponding to these diagrams.

The situation is changed in the calculation of higher orders, owing to the singular behavior of the integrand at small momenta.

As an example of a graph with exchange of a single transverse photon, let us consider the isolation of the order-$\alpha^2\ln\alpha$ contributions to the Fermi energy of the hyperfine splitting, $E_F=\frac{2}{3}(\alpha^4\mu^3/m_1m_2)\langle\vec{\sigma}_1\vec{\sigma}_2\rangle$. The expression for the correction from this graph has the form

$$\Delta E_{1T}=\langle\Phi_C'|F^{-1}[G_0K_TG_0]^+F^{-1}|\Phi_C'\rangle. \tag{109}$$

$$K_T=-\frac{4\pi\alpha\Gamma_{12}(\vec{k})}{k_0^2-\vec{k}^2+i\varepsilon} \tag{110}$$

is the kernel corresponding to the graph with exchange of a single transverse photon,

$$\Gamma_{12}(\vec{k})=\vec{\gamma}_1\vec{\gamma}_2-\frac{(\vec{\gamma}_1\vec{k})(\vec{\gamma}_2\vec{k})}{\vec{k}^2}, \tag{111}$$

and Φ_C', the WF in second-order perturbation theory, can (in the calculation with the accuracy that we need) be replaced by $\phi_C(\vec{p})$, determined by Eq. (76) and corresponding to the exact solution for the $1S$ state of the nonrelativistic Schrödinger equation with the Coulomb potential.

An analytic expression for (109) is given by

$$\Delta E_{1T}=-\frac{4\alpha^2\mu^2|\phi_C(0)|^2}{(2\pi)^6}\int\frac{d\vec{p}d\vec{q}dp_0dq_0}{(\vec{p}^2+\alpha^2\mu^2)^2(\vec{q}^2+\alpha^2\mu^2)^2}$$
$$\times F^{-1}(p)F^{-1}(q)\int dk_0dk_0'$$
$$\times[S_1(p_1)S_2(p_2)K_T(k_0,k_0';\vec{p},\vec{q})S_1(q_1)S_2(q_2)]^+$$

$$\times\delta(p_0-k_0)\delta(k_0'-q_0). \tag{112}$$

Using the Fourier representation of the δ function and residue theory, after integration over the variables p_0, q_0, k_0, and k_0' we find

$$\Delta E_{1T}=\frac{4i\alpha^3\mu^2|\phi_C(0)|^2}{(2\pi)^3}\times\int\frac{d\vec{p}d\vec{q}}{|\vec{p}-\vec{q}|}\frac{u_1^*(\vec{p})u_2^*(-\vec{p})\vec{\alpha}_1\vec{\alpha}_2u_1(\vec{q})u_2(-\vec{q})}{(\vec{p}^2+\alpha^2\mu^2)^2(\vec{q}^2+\alpha^2\mu^2)^2}\times\int dte^{-i|\vec{p}-\vec{q}||t|}\{\vartheta(t)e^{-i|t|(\varepsilon_{1q}+\varepsilon_{2p}-E-i\varepsilon)}+\vartheta(-t)e^{-i|t|(\varepsilon_{1p}+\varepsilon_{2q}-E-i\varepsilon)}\}. \tag{113}$$

After separating the spin–spin interaction from the spin structures [see the numerator of Eq. (113)] and using the symmetries of the expressions for the spin structure under the exchange $p_i\Leftrightarrow q_j$, for example,

$$\int d\vec{p}d\vec{q}(\vec{\sigma}_1\vec{p})(\vec{\sigma}_2\vec{q})f(\vec{p}^2,\vec{q}^2,(\vec{p}-\vec{q})^2)=\tfrac{1}{3}(\vec{\sigma}_1\vec{\sigma}_2)\int d\vec{p}d\vec{q}(\vec{p}\vec{q})f(\vec{p}^2,\vec{q}^2,(\vec{p}-\vec{q})^2), \tag{114}$$

we can write down an integral expression for $\Delta E_T^{\rm str}$ (the case of different masses $m_1\neq m_2$):

$$\Delta E_{1T}=\frac{\alpha^3\mu^2}{3\pi^3}|\phi_C(0)|^2\langle\vec{\sigma}_1\vec{\sigma}_2\rangle\int d\vec{p}d\vec{q}\frac{1}{(\vec{p}^2+\alpha^2\mu^2)^2(\vec{q}^2+\alpha^2\mu^2)^2}\frac{1}{|\vec{p}-\vec{q}|+(\varepsilon_{1p}+\varepsilon_{2q}-E-i\varepsilon)}\Xi\left\{\vec{p}^2(\vec{q}^2-\vec{p}^2)\left[\frac{M_{1q}^+}{\varepsilon_{2p}+\varepsilon_{2q}}+\frac{M_{2q}^+}{\varepsilon_{1p}+\varepsilon_{1q}}\right]+\vec{q}^2(\vec{p}^2-\vec{q}^2)\left[\frac{M_{1p}^+}{\varepsilon_{2p}+\varepsilon_{2q}}+\frac{M_{2p}^+}{\varepsilon_{1p}+\varepsilon_{1q}}\right]+(\vec{p}-\vec{q})^2[M_{2p}^+M_{1q}^++M_{1p}^+M_{2q}^+]-\frac{(\vec{p}\cdot\vec{q})^2}{(\vec{p}-\vec{q})^2}\frac{(\vec{p}^2-\vec{q}^2)^2}{(\varepsilon_{1p}+\varepsilon_{1q})(\varepsilon_{2p}+\varepsilon_{2q})}\right\}. \tag{115}$$

Above, we have used the notation

$$\Xi=\frac{1}{\sqrt{\varepsilon_{1p}\varepsilon_{2p}\varepsilon_{1q}\varepsilon_{2q}M_{1p}^+M_{2p}^+M_{1q}^+M_{2q}^+}}, \tag{116}$$

$M_{ip}^+=\varepsilon_{ip}+m_i$, and $M_{iq}^+=\varepsilon_{iq}+m_i$.

As a rule, even before the integration it is possible to estimate the order in α of the contribution of each of the integrals to the total value of the hyperfine splitting. In Ref. 19 it was shown that the integral conventionally referred to as the standard one always gives a logarithmic correction of order $\alpha^2\ln\alpha$ to the Fermi energy:

$$I_{\rm st}(\ln\alpha)=\frac{1}{8\pi^2}\int\frac{d\vec{p}}{\varepsilon_{1p}\varepsilon_{2p}(\vec{p}^2+\alpha^2\mu^2)}\times\int\frac{d\vec{q}}{(\vec{q}^2+\alpha^2\mu^2)(\vec{p}-\vec{q})^2}=\int_0^\infty\frac{pdp}{\varepsilon_{1p}\varepsilon_{2p}(\vec{p}^2+\alpha^2\mu^2)}\int_0^\infty\frac{qdq}{\vec{q}^2+\alpha^2\mu^2}\times\ln\frac{p+q}{|p-q|}=\frac{\pi^2}{2m_1m_2}\ln\alpha^{-1}+O(\alpha),$$

$$\alpha^6I_{\rm st}\sim\alpha^6\ln\alpha. \tag{117}$$

The additional powers of p or q in the numerator and of the factors $(\vec{p}^2+\alpha^2\mu^2)$ or $(\vec{q}^2+\alpha^2\mu^2)$ in the denominator of the integrand lead to contributions of order α^6 or α^4 and α^5, respectively.

The terms in the curly brackets containing the product $\vec{p}^2\vec{q}^2$ lead to the standard integral in (115). In addition, here the difference between the factor $(|\vec{p}-\vec{q}|+\varepsilon_{1p}+\varepsilon_{2q}-E)$ and the factor $|\vec{p}-\vec{q}|$ in the denominator of (115) turns out to be important. The main contribution of this term is of order α^4, and the higher terms of the expansion of the denominator lead to the standard integral.

TABLE I. Contributions to the HFS of muonium from the graphs of Fig. 2. $\Delta E_{\rm Mu}^{\rm HFS}=(\mu^2\alpha^2/m_em_\mu)E_F\ln\alpha K_i$, $\mathscr{M}=m_e/m_\mu+m_\mu/m_e$.

Graph	Contribution K_i to $\Delta E_{\rm Mu}^{\rm HFS}$
a	1/4
b	$\mathscr{M}+2$
c	9/2
d	$-(\mathscr{M}+2)$
e	$3\mathscr{M}$
f	$-2(\mathscr{M}+2)$
g	5/4
h	$-\mathscr{M}$
sum	2

The other graphs also give contributions $\sim\alpha^6\ln\alpha$ to the HFS of muonium, which is calculated in a similar manner. The results of the calculations[31] are given in Table I.

In the method using the amplitude T (on-shell constituent particles, $p_1^0=\varepsilon_{1p}$, $p_2^0=\varepsilon_{2p}$, $q_1^0=\varepsilon_{1q}$, $q_2^0=\varepsilon_{2q}$), the problem of the correct inclusion of retardation effects arises already at the stage of calculating the contributions from graphs with the exchange of one transverse photon. Depending on the method of writing ω^2 in the denominator of the photon propagator

$$D_{il}=-\frac{4\pi}{\omega^2-\vec{k}^2}\left(\delta_{il}-\frac{k_ik_l}{\vec{k}^2}\right), \tag{118}$$

TABLE II. Contribution to the HFS of a two-fermion system from the one-photon exchange diagram (ω^2 is the zeroth component of the photon 4-momentum).

ω^2	$\Delta E_T^{\mathrm{HFS}}$ $(\alpha^6 \ln \alpha)$, $m_1 \neq m_2$	$\Delta E_T^{\mathrm{HFS}}$ $(\alpha^6 \ln \alpha)$, $m_1 = m_2$
0	$E_F \frac{\mu^2\alpha^2}{m_1 m_2} \mathfrak{M} \ln \alpha^{-1}$	$\frac{1}{2} E_F \alpha^2 \ln \alpha^{-1}$
$[\varepsilon_{1p} - \varepsilon_{1q}]^2$	$E_F \frac{\mu^2\alpha^2}{m_1 m_2} (\mathfrak{M} - 2\frac{m_2}{m_1}) \ln \alpha^{-1}$	0
$[\varepsilon_{2p} - \varepsilon_{2q}]^2$	$E_F \frac{\mu^2\alpha^2}{m_1 m_2} (\mathfrak{M} - 2\frac{m_1}{m_2}) \ln \alpha^{-1}$	0
$[(\varepsilon_{1p} - \varepsilon_{1q})(\varepsilon_{2q} - \varepsilon_{2p})]$	$E_F \frac{\mu^2\alpha^2}{m_1 m_2} (\mathfrak{M} + 2) \ln \alpha^{-1}$	$E_F \alpha^2 \ln \alpha^{-1}$

the contribution $\sim\alpha^6 \ln \alpha$ is different (see Table II).

Comparing the results given in Table II with the result calculated by the first method (two-time Green functions)

$$\Delta E_T^{\mathrm{HFS}}(\alpha^6 \ln \alpha) = E_F \frac{\mu^2\alpha^2}{m_1 m_2} \left(\frac{m_1}{m_2} + \frac{m_2}{m_1} + 2\right) \ln \alpha^{-1}, \tag{119}$$

we convince ourselves that it is preferable to use the symmetric form of writing ω^2 in the quasipotential continued off the energy shell ($|\vec{p}|^2 \neq |\vec{q}|^2$).

The situation is more complicated in the higher-order graphs when the above-described method of going to the mass shell is used. First, the method of symmetrization is not clear (i.e., how to go off the mass shell). Second, the problem of the presence of additional poles significantly complicates the calculations. The integration over the loop momentum, for example, is already impossible to understand in the sense of the principal value. In this situation the use of various expansions of the integrand and interchange of the order of integration become problematic.

Therefore, retardation effects are included most accurately when the first method of constructing the quasipotential is used. However, in this case already when the one-photon exchange graphs are included the problem arises of anomalous contributions $\sim\alpha^5 \ln \alpha$ in the HFS of the ground-state level of the two-fermion system, which do not arise in the second method. This problem is not a specific feature of the quasipotential approach, but is of general nature for the theory of bound states.[4]

The graphs which can give a contribution $\sim\alpha^5 \ln \alpha$ to the hyperfine splitting of the ground state of positronium in the direct channel are shown in Fig. 3. The corresponding quasipotential is given in Refs. 15 and 32.

As already noted, these contributions arise from the infrared behavior of the matrix elements. The presence of iteration terms for each reducible graph improves its behavior in the infrared region and makes it possible in a given order in α to avoid the summation of ladder graphs. In calculations with exact Coulomb WFs (76) in the first

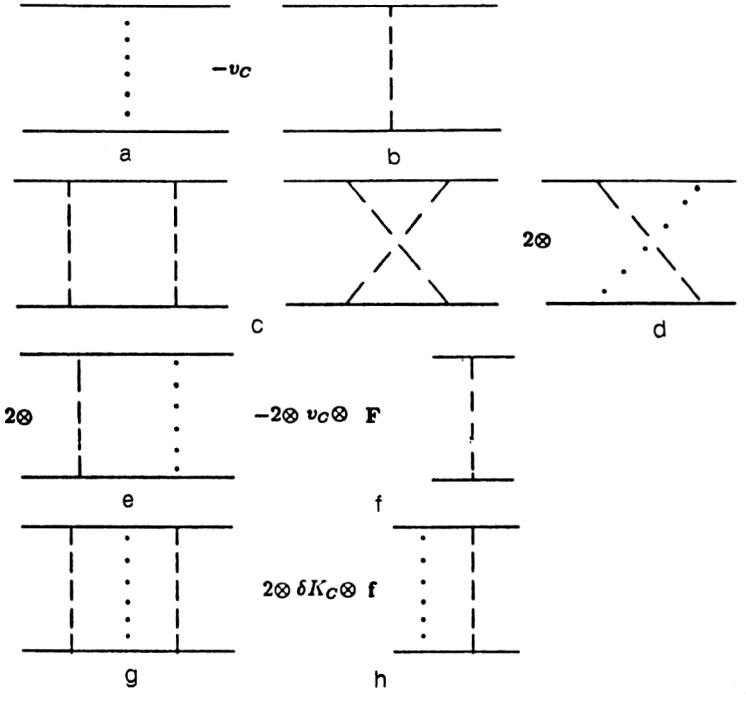


FIG. 2. Graphs giving contributions of order $\sim\alpha^6 \ln \alpha$ to the HFS of muonium.

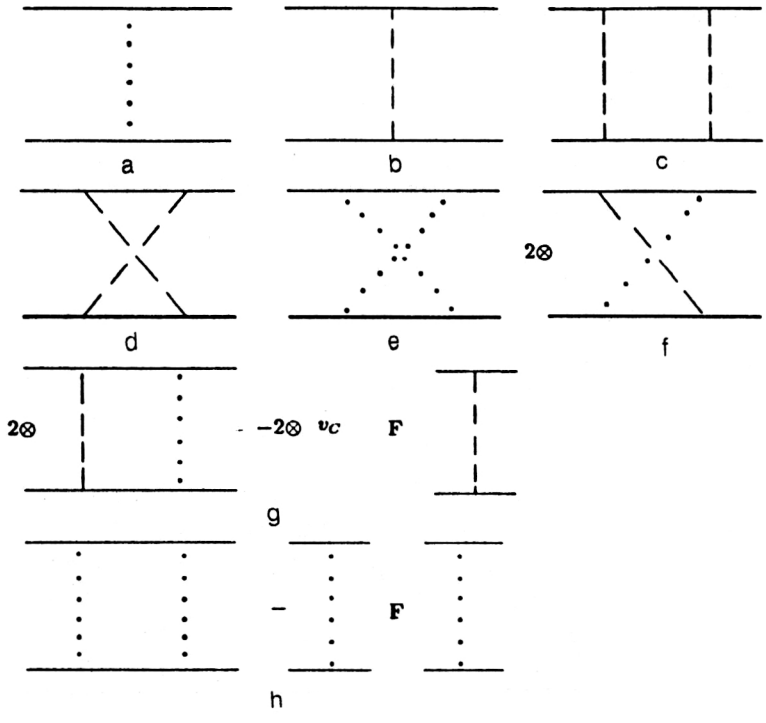


FIG. 3. Graphs used to analyze the anomalous contributions of order $\alpha^5 \ln \alpha$.

version of the quasipotential approach it has been established that the parameter $(\alpha\mu)$ plays the role of a cutoff factor for the infrared singularities. The cancellation of these anomalous terms[15] is demonstrated in Table III.

A series of new corrections to the hyperfine splitting of muonium energy levels has been obtained by Eides *et al.*[33–35] In particular, in these studies the corrections $\sim\alpha(Z\alpha)(m_e/m_\mu)$ and $\sim Z^2\alpha(Z\alpha)(m_e/m_\mu)$ to the Fermi energy have been calculated using the method of the effective Dirac equation (EDE).[6)] Let us consider the isolation of these contributions from graphs with radiative phonons. The distinguishing feature of these studies is the use of the Fried–Yennie gauge[38–40] for the photon propagator:

$$D_{\mu\nu}=\frac{1}{q^2+i\varepsilon}\left(g_{\mu\nu}+2\frac{q_\mu q_\nu}{q^2+i\varepsilon}\right). \tag{120}$$

The infrared divergences are softened in this gauge. Any graph with a radiative correction possesses softer behavior near the mass shell than the corresponding skeleton block. The attractive feature of the Fried–Yennie gauge is the possibility of carrying out the renormalization procedure on the mass shell without the introduction of an unphysical photon mass λ. This feature simplifies the estimates of the integrals which arise in problems concerning the energy levels of HL atoms.

In the Fried–Yennie gauge the WF renormalization constant Z_2 is infrared-finite, and the renormalized self-energy operator possesses soft behavior on the mass shell:

$$\Sigma_{FY}^{(R)}(p)=(\hat{p}-m)^2\left(-\frac{3\alpha\hat{p}}{4\pi m^2}\right)(1+O(\rho)), \tag{121}$$

$$\rho=\frac{m^2-p^2}{m^2}\ll 1,$$

which distinguishes this gauge from, for example, the Feynman gauge:

TABLE III. Cancellation of anomalous contributions of order $\alpha^5 \ln \alpha$ to the HFS of positronium.

Graph in Fig. 3	$\Delta E(\alpha^5 \ln \alpha)$
a	0
b	$\frac{2\alpha}{\pi} E_F \ln \alpha$
c	$-\frac{\alpha}{2\pi} E_F \ln \alpha$
d	$\frac{\alpha}{2\pi} E_F \ln \alpha$
e	0
f	$-\frac{2\alpha}{\pi} E_F \ln \alpha$
g	0
h	0
Sum	0

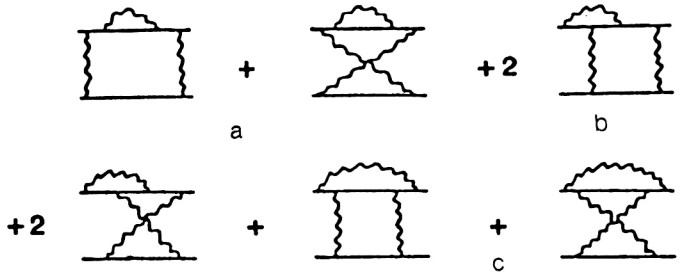


FIG. 4. Complete gauge-invariant set of graphs for calculating the recoil corrections to the Fermi energy of the HFS of muonium of order $\alpha(Z\alpha)E_F$ and $Z^2\alpha(Z\alpha)E_F$.

$$\Sigma_F^{(R)}(p)=(\hat{p}-m)\frac{\alpha}{\pi}\left[\ln\frac{\lambda}{m}-\ln\rho+1\right], \qquad (122)$$

$$\frac{\lambda}{m}\ll\rho\ll 1.$$

As far as the vertex function is concerned, the hardest behavior is that of the expression corresponding to the fermion anomalous magnetic moment:

$$-\frac{\alpha}{2\pi}\sigma_{\mu\nu}\frac{k_\nu}{2m}. \qquad (123)$$

However, redefining the renormalized vertex operator,

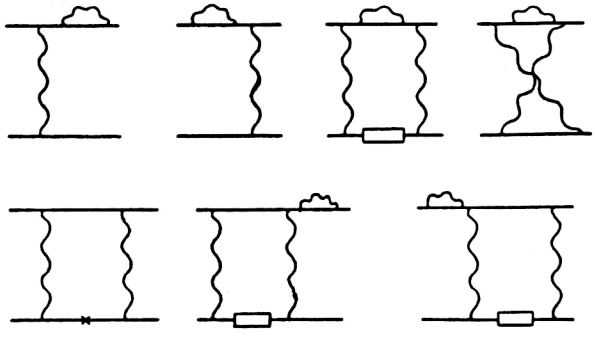
$$\Lambda_\mu(p_1,p_2)=\gamma_\mu\Lambda(0,0)-\frac{\alpha}{2\pi}\sigma_{\mu\nu}\frac{k^\nu}{2m}+\Lambda_\mu^{(R)}(p_1,p_2), \qquad (124)$$

we obtain at zero momentum transfer and $\rho\ll 1$

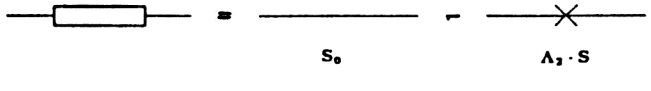
$$\Lambda_{\mu,FY}^{(R)}=-\gamma_\mu\frac{3\alpha}{4\pi}\frac{(\hat{p}-m)\hat{p}}{m^2}, \quad \rho\ll 1, \qquad (125)$$

which according to the Ward identity is consistent with the asymptote of the self-energy operator. The contributions from the anomalous magnetic-moment term are analyzed separately.

Let us trace the selection of graphs for calculating the corrections $\alpha(Z\alpha)(m_e/m_\mu)E_F$. Seven graphs exhaust the contributions to the EDE kernel related to the mass operator:

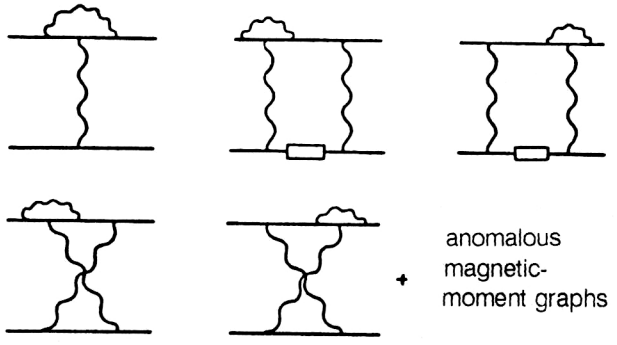


Here



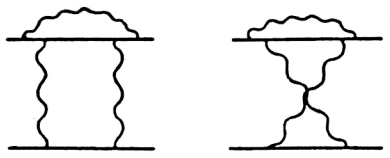
and S_0 is the free two-particle propagator; $g=\Lambda_2 S$ is the projection operator onto the muon mass shell, multiplied by the electron propagator.

The simplest graphs with the vertex correction

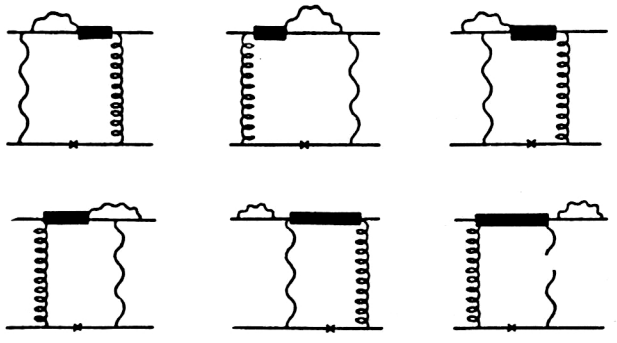


also give a contribution of order $\alpha(Z\alpha)(m_e/m_\mu)$.

In addition, there are graphs with more than one photon:



and graphs of second-order perturbation theory:



In analyzing the graphs entering into the EDE kernel on the subject of the presence of corrections $\alpha(Z\alpha)(m_e/m_\mu)$ to the Fermi energy, it became clear that

contributions of this order can come only from graphs from a gauge-invariant set (see Fig. 4).[7)] Here it proved possible in the claddings of the matrix element to use the approximate WFs (103), i.e., the matrix elements must be calculated between the large components of the electron and muon spinors, neglecting the momenta of the wave functions inside the graphs. These conditions are referred to as standard by the authors of Refs. 33–35.

The summation of the complete gauge-invariant set shown in Fig. 4 leads to an infrared- and ultraviolet-finite matrix element. Therefore, the further calculations can be carried out using any convenient gauge, both for the exchanged photons and for the radiative photons. By explicit calculation it is possible to show that the anomalous magnetic moment does not lead to corrections of the required order, and it is assumed to be subtracted from the vertex operator. The iteration graphs in the EDE method (with a $\times$ on the muon line) are in some sense analogous to the iteration graphs of the quasipotential approach. As in the quasipotential approach, there exist two iteration graphs for the graphs of two-photon exchange with a radiative insertion ($\tau^{(4)}F\tau^{(2)}$ and $\tau^{(2)}F\tau^{(4)}$), and "uncancelled" iteration graphs arise in the EDE method. They are regenerated by the kernel of the BS equation containing the inverse muon propagator. It has been shown[34] that the iteration graphs in this order cancel with the one-loop graph with vertex correction after construction of the EDE kernel and perturbation theory for the energy levels.

As a result, we obtain the following expression for the contribution from the graphs of Fig. 4a:

$$\delta E_\Sigma=\frac{\alpha(Z\alpha)}{\pi^2}\frac{m_e}{m_\mu}E_F\frac{3i}{8\pi^2\hat\mu^2}\int_0^1 dx\int_0^x dy \times\int\frac{d^4k}{k^4}\left(\frac{1}{k^2+\hat\mu^{-1}k_0+i\varepsilon}+\frac{1}{k^2-\hat\mu^{-1}k_0+i\varepsilon}\right) \times\frac{1}{-k^2+2k_0+a_1^2(x,y)-i\varepsilon}\left[h_1(x,y)k_0-h_2(x,y) \times\left(k_0^2-\frac{2}{3}\vec k^2\right)\right]\equiv\delta E_{\Sigma1}+\delta E_{\Sigma2}, \tag{126}$$

where x and y are the Feynman parameters, $\hat\mu=m_e/2m_\mu$,

$$h_1(x,y)=\frac{1+x}{y},\quad h_2(x,y)=\frac{1-x}{y}\left[1-\frac{2(1+x)}{x^2+\lambda^2}y\right], \tag{127}$$
$$a_1^2(x,y)=\frac{x^2+\lambda^2}{(1-x)y},$$

and λ is the dimensionless infrared mass of the radiative photon in units of the electron mass.

The dominant contribution to the integral of order $1/\hat\mu$ comes from the residue at the muon pole corresponding to the motion of the muon on the mass shell. This residue is also related to the leading infrared divergence of the integral proportional to $\lambda^{-1/2}$, and the other contributions diverge only logarithmically. It is therefore convenient to separate the calculations of the mass-shell contributions and the contributions of the remaining difference:

$$\delta E_\Sigma(\text{m.s.})=\frac{1}{2\hat\mu}\left(-2I_\lambda+\frac{11\pi^2}{6}\right)+\left(I_\lambda+\frac{23\pi^2}{24}\right), \tag{128}$$

where

$$I_\lambda=\frac{4\pi}{3}\int_0^1 dx\left(\frac{x}{x^2+\lambda^2}\right)^{3/2}(1-x)^{1/2}\sim\frac{1}{\lambda^{-1/2}} \tag{129}$$

is an infrared-divergent integral, which cancels in the sum of the pole contributions from the graphs of Fig. 4.

Using a number of mathematical tricks (see Refs. 34 and 35) such as division of the range of integration into two parts (small and large momenta) after integration over the angular variables, subtraction of the pole contribution in the integrand, and the use of various identities, we obtained the expressions for the contributions from the sum of all the graphs of the gauge-invariant set of Fig. 4:

$$\delta E_\Sigma=\alpha(Z\alpha)E_F\left[\ln 2-\frac{13}{4}\right]+\frac{\alpha(Z\alpha)}{\pi^2}\frac{m_e}{m_\mu}E_F \times\left[\frac{15}{4}\ln\frac{m_\mu}{m_e}+6\zeta(3)+3\pi^2\ln 2+\frac{\pi^2}{2}+\frac{17}{8}\right]. \tag{130}$$

There is another method which allows the calculation of the logarithmic contribution of order $\alpha^3(m_e/m_\mu)\ln^3(m_e/m_\mu)$ to the hyperfine splitting of muonium: the renormalization-group method (Ref. 41).[8)]

The contribution of radiative corrections to the recoil is $\Delta E=-E_F R_\mu$, where R_μ in lowest order is calculated from the two-photon exchange graphs. The authors of Ref. 43 also considered graphs with radiative insertions in the electron and photon lines and at the electron vertex. There they give the logarithmic contributions

$$R_\mu^{(2)}=-\frac{3\alpha}{\pi}\frac{m_e}{m_\mu}\ln\frac{m_e}{m_\mu}+\left(\frac{\alpha}{\pi}\right)^2\frac{m_e}{m_\mu}\left[2\ln^2\frac{m_e}{m_\mu}+\frac{31}{12}\ln\frac{m_e}{m_\mu}\right], \tag{131}$$

which arise as a consequence of integration over the momentum variable in the region $m_e^2\ll k^2<m_\mu^2$, i.e., in the asymptotic region for the contribution of the electron vacuum polarization to the photon propagator. The muon loop does not contribute in the integration over the momentum variable in the region $k^2<m_\mu^2$. An estimate of the next term in the series in perturbation theory is given in Ref. 41.

If we consider the physical quantity R calculated using perturbation theory, it obviously must satisfy the condition

$$\frac{\partial R}{\partial\tau}=0, \tag{132}$$

where the variable $\tau=-\beta_0\ln(m/\Lambda)$ characterizes the renormalization scheme (RS) which is used, μ is an arbitrary parameter with the dimension of a mass, Λ is a scale parameter,[44] and $\beta_0=2/3$ is the first coefficient in the renormalization-group equation with running coupling constant g:

$$\mu\frac{\partial g}{\partial\mu}=\beta(g)=\beta_0 g^2+\beta_1 g^3+\dots. \tag{133}$$

Accordingly, for the quantity R in second-order perturbation theory, chosen in the form

$$R^{(2)}=r_0 g(1+r_1 g), \tag{134}$$

we have

$$\frac{\partial R^{(2)}}{\partial \tau}=O(g^3), \tag{135}$$

and from this it follows that

$$\frac{\partial r_0}{\partial \tau}=0, \quad \frac{\partial r_1}{\partial \tau}=1. \tag{136}$$

We therefore see that r_0 is independent of the choice of RS, and $r_1=\tau+\rho_1$, where the constant ρ_1 can be calculated if the value of r_1 is known for any RS.

The τ dependence of the next (the third) term of the perturbation series should be constructed in such a way that the dependence of $R^{(2)}$ on τ is canceled up to order g^4:

$$\frac{\partial (R^{(2)}+\Omega^{(2)} g^3)}{\partial \tau}=O(g^4). \tag{137}$$

From this it follows that

$$\frac{\partial \Omega^{(2)}}{\partial \tau}=r_0\left(2r_1+\frac{\beta_1}{\beta_0}\right). \tag{138}$$

After integration using Eqs. (136) we obtain

$$\Omega^{(2)}=r_0 r_1\left(r_1+\frac{\beta_1}{\beta_0}\right)+\text{const}. \tag{139}$$

Furthermore, to determine the arbitrary constant of integration it is necessary to know the "optimal" RS for which the quantity $R^{(2)}$ is closest to R. In Ref. 41 it is specified by the condition

$$\Omega^{(2)}(r_1,\tau)\,|_{\tau=\tau_{\text{opt}}}=0, \quad \tau_{\text{opt}}=-\beta_0 \ln\left(\frac{\mu_{\text{opt}}}{\Lambda}\right). \tag{140}$$

If as the initial scheme we choose the RS on the mass shell ($r_1=K_1$), then

$$\Omega^{(2)}(K_1)=r_0 K_1\left(K_1+\frac{\beta_1}{\beta_0}\right)-r_0 r_1^{\text{opt}}\left(r_1^{\text{opt}}+\frac{\beta_1}{\beta_0}\right),$$

$$r_1^{\text{opt}}=r_1(\tau_{\text{opt}}). \tag{141}$$

For the physical quantity ΔE, the energy of the hyperfine splitting in muonium, from (131) we obtain

$$r_0=-3\frac{m_e}{m_\mu}\ln\frac{m_e}{m_\mu}, \quad K_1=-\tfrac{2}{3}\ln\frac{m_e}{m_\mu}-\tfrac{31}{36} \tag{142}$$

and from (141)

$$\Omega^{(2)}=-\frac{4}{3}\frac{m_e}{m_\mu}\ln^3\frac{m_e}{m_\mu}-\frac{35}{18}\frac{m_e}{m_\mu}\ln^2\frac{m_e}{m_\mu}+A\frac{m_e}{m_\mu}\ln\frac{m_e}{m_\mu}, \tag{143}$$

where

$$A=-\beta_0\ln\frac{m_\mu}{\mu_{\text{opt}}}\left(\frac{35}{36}-\beta_0\ln\frac{m_\mu}{\mu_{\text{opt}}}\right). \tag{144}$$

As a result,[9)]

$$\Delta E=E_F\left(\frac{\alpha}{\pi}\right)^3\frac{m_e}{m_\mu}\left[\frac{4}{3}\ln^3\frac{m_e}{m_\mu}+\frac{35}{18}\ln^2\frac{m_e}{m_\mu}\right]\simeq -0.04\ \text{kHz}. \tag{145}$$

The first term in (145) coincides with that obtained in Ref. 33 by direct calculation. The value of the coeficient A in (143) depends on the choice of the optimal scheme, and its contribution to the HFS can be estimated only as $0<-\Delta E<1$ kHz.

An interesting method of calculating the corrections to the QED spectrum of a system which are logarithmic in α was suggested in Ref. 46. As was shown in those articles, the corrections logarithmic in α arise from integrals which diverge logarithmically, with the contribution coming from the momentum range

$$\mu\alpha\leqslant q\leqslant\mu \tag{146}$$

(μ is the reduced mass, $\mu=mM/(m+M)$). The lower limit is the characteristic momentum for QED bound states [see the earlier discussion of the WF (76)], and the upper limit corresponds to the limit of applicability of the nonrelativistic approximation.

The shift of the level with quantum numbers n and l is calculated in Ref. 46 as the matrix element of the operator $\hat{V}(q)$:

$$\hat{V}^{(N)}(q)=[A+B(\vec{\sigma}_1\vec{\sigma}_2)]\frac{\pi\alpha^N}{mM}\ln\frac{\mu}{q} \tag{147}$$

between WFs of the type (103).

The calculation of the operator $\hat{V}(q)$ in this approach reduces to the logarithmic approximation for calculating the on-shell scattering amplitude from Feynman diagrams of order α^N.

The authors of Ref. 46 use ordinary quantum-mechanical perturbation theory with the insertions

$$\sum_i\frac{|i\rangle\langle i|}{E-E_i} \tag{148}$$

in intermediate states of the interaction operator.

For example, for graphs of order $O(\alpha^3)$ with a purely Coulomb interaction the result for the operator $\hat{V}$ is

$$\hat{V}=-\int\frac{d\vec{k}}{(2\pi^3)}\int\frac{d\vec{k}'}{(2\pi^3)}\frac{(4\pi\alpha)^3}{k^2k'^2(\vec{k}-\vec{k}')^2}$$

$$\times\frac{\Lambda_1^+(\vec{k}')\Lambda_1^+(\vec{k})\otimes\Lambda_2^+(-\vec{k}')\Lambda_2^+(-\vec{k})}{(E-\varepsilon_{1k}-\varepsilon_{2k})(E-\varepsilon_{1k'}-\varepsilon_{2k'})}, \tag{149}$$

which leads (in conjunction with the results from other graphs with exchange of Coulomb photons) to the level shift

$$\delta E_C(n,l)=\frac{\mu^5}{m^2M^2}\alpha^6\ln\frac{1}{\alpha}\left(-\frac{3}{2}+\frac{\vec{\sigma}_1\vec{\sigma}_2}{6}\right)\frac{\delta_{l0}}{n^3}. \tag{150}$$

I. V. Khriplovich *et al.* have obtained theoretical values for the decay widths *o*–Ps and *p*–Ps for the fine and hyperfine structure of muonium levels, which are discussed in the next section and are compared with the results of

Fell[47] obtained on the basis of the relativistic two-particle equations. See Ref. 348 for a correction to Ref. 46.

4. COMPARISON OF THE THEORETICAL AND EXPERIMENTAL RESULTS

4.1. The positronium decay width

Quantum electrodynamical systems consisting of a particle and an antiparticle are distinguished by special features. In addition to the scattering channel, an annihilation channel appears. The positronium atom is unstable as such a system. Its lifetime (decay width) is the subject of precision experimental and theoretical studies. The charge parity of positronium $C=(-1)^{L+S}$ (where L is the orbital angular-momentum eigenvalue and S is the eigenvalue of the total spin of the system) is an integral of the motion. Therefore, all its states are split into charge-even ($C=1$) and charge-odd ($C=-1$) states. The total spin of positronium is also conserved, so that its energy levels can be split into singlet ($S=0$, parapositronium) and triplet ($S=1$, orthopositronium) states. In the S state ($L=0$) parapositronium is charge-even, and orthopositronium is charge-odd. Owing to conservation of charge parity in electromagnetic interactions, parapositronium decays into an even and orthopositronium into an odd number of photons.

At present there is a considerable discrepancy between the theoretical and experimental values of the orthopositronium decay width. Theory predicts the value[48–51]

$$\Gamma_3^{\text{theor}}(o-\text{Ps})=\frac{\alpha^6 mc^2}{h}\frac{2(\pi^2-9)}{9\pi}\left[1-A_3\frac{\alpha}{\pi}-\frac{1}{3}\alpha^2\ln\alpha^{-1}+B_3\left(\frac{\alpha}{\pi}\right)^2+\ldots\right]$$
$$=\Gamma_0+\frac{m\alpha^7}{\pi^2}\left\{\begin{pmatrix}-1.984(2)\\-1.9869(6)\end{pmatrix}\right\}+\frac{m\alpha^8}{\pi}\ln\alpha^{-1}\left[-\frac{4}{9}\zeta(2)+\frac{2}{3}\right]+\frac{m\alpha^8}{\pi^3}\chi+\ldots$$
$$=7.03831(5)\,\mu\text{sec}^{-1}, \tag{151}$$

where

$$A_3=-10.266\pm0.011 \text{ (Ref. 50)}, \tag{152}$$

$$A_3=-10.282\pm0.003 \text{ (Ref. 51)}. \tag{153}$$

The most recent experimental values are[10)]

$$\Gamma^{\text{exp}}(o-\text{Ps})=7.0514(14)\ \mu\text{sec}^{-1} \text{ (Ref. 52)}, \tag{154}$$

$$\Gamma^{\text{exp}}(o-\text{Ps})=7.0482(16)\ \mu\text{sec}^{-1} \text{ (Ref. 53)}. \tag{155}$$

The result in Ref. 52 gives 9.4 standard deviations, and that of Ref. 53 gives 6.2 standard deviations from the theoretically predicted value. The coefficient $B_3=1$ of the term $O(\alpha^8)$ can contribute only 3.5×10^{-5} μsec^{-1} (or 5 ppm of the contribution from Γ_3). In order to eliminate this disagreement with experiment, the coefficient B_3 must be $\cong 250\pm40$, which is extremely improbable. However, as was shown in Ref. 53, this case cannot be excluded *a priori*. Therefore, the calculation of the coefficient B_3 is very important at the present time.

The dominant contribution to the decay width of orthopositronium

$$\Gamma_0(o-\text{Ps})=-2\,\text{Im}(\Delta E_{3\gamma})$$
$$=\frac{2}{9\pi}(\pi^2-9)m\alpha^6=7.21117\ \mu\text{sec}^{-1} \tag{156}$$

was first calculated in Ref. 48. The $O(\alpha)$ corrections to the width of orthopositronium decay into three photons were first calculated numerically (Refs. 49, 51, 54, and 55), and later some corrections were calculated analytically (Refs. 46, 59, and 56–58) in the Feynman gauge. For example, Adkins[57,58] calculated the corrections from graphs with self-energy and vertex insertions:

$$\Gamma_{0V}=\Gamma_0\frac{\alpha}{\pi}\left\{D+\frac{3}{4(\pi^2-9)}\left[-26-\frac{115}{3}\ln2+\frac{91}{18}\zeta(2)\right.\right.$$
$$\left.\left.+\frac{443}{54}\zeta(3)+\frac{3419}{108}\zeta(2)\ln2-R\right]\right\}$$
$$=\Gamma_0\frac{\alpha}{\pi}[D+2.9711385(4)], \tag{157}$$

$$\Gamma_{SE}=\Gamma_0\frac{\alpha}{\pi}\left\{-D-4+\frac{3}{4(\pi^2-9)}\left[-7+\frac{67}{3}\ln2\right.\right.$$
$$\left.\left.+\frac{805}{36}\zeta(2)-\frac{1049}{54}\zeta(3)-\frac{775}{54}\zeta(2)\ln2\right]\right\}$$
$$=\Gamma_0\frac{\alpha}{\pi}[-D+0.78498], \tag{158}$$

$$\Gamma_{1V}=\Gamma_0\frac{\alpha}{\pi}\left\{\frac{1}{2}D+\frac{3}{4(\pi^2-9)}\left[-4-\frac{34}{2}\ln2-\frac{841}{36}\zeta(2)\right.\right.$$
$$+\frac{1253}{36}\zeta(2)\ln2+\frac{1589}{54}\zeta(3)+\frac{17}{40}\zeta^2(2)$$
$$\left.\left.-\frac{7}{8}\zeta(3)\ln2+\frac{5}{2}\zeta(2)\ln^2 2-\frac{1}{24}\ln^4 2-\alpha_4\right]\right\}$$
$$=\Gamma_0\frac{\alpha}{\pi}\left[\frac{1}{2}D+0.160677\right], \tag{159}$$

where

$$R=\int_0^1 dx\,\frac{\ln(1-x)}{2-x}\,[\zeta(2)-Li_2(1-2x)]$$
$$=-1.7430338337(3), \tag{160}$$

$$a_4=Li_4\left(\frac{1}{2}\right)=\sum_{n=1}^{\infty}\frac{1}{n^4 2^n}=0.517479061674 \tag{161}$$

$$\zeta(2)=\frac{\pi^2}{6},\quad \zeta(3)=1.2020569032 \tag{162}$$

and

$$D=\frac{1}{2-w}-\gamma_E+\ln(4\pi) \tag{163}$$

is the standard expression arising in dimensional regularization ($2w$ is the space dimension). This result is consistent with that obtained by Stroscio[56] if to the latter we add

$$\Gamma_0\frac{\alpha}{\pi}\,[-D-4-2\ln(\lambda^2/m^2)], \tag{164}$$

which is necessary owing to the different regularization procedures used in Refs. 56 and 57, 58.

A program for calculating these corrections in the Fried–Yennie gauge has recently been completed:[59]

$$\Gamma_{SE}=\frac{m\alpha^7}{\pi^2}\left[-\frac{13}{54}\zeta(3)+\frac{461}{108}\zeta(2)\ln 2-\frac{251}{72}\zeta(2)\right.$$
$$\left.-\frac{29}{6}\ln 2+\frac{9}{2}\right]$$
$$=\frac{m\alpha^7}{\pi^2}(-0.007132904)=\Gamma_0\frac{\alpha}{\pi}(-0.036911113), \tag{165}$$

$$\Gamma_{0V}=\frac{m\alpha^7}{\pi^2}\left[-\frac{88}{54}\zeta(3)-\frac{299}{216}\zeta(2)\ln 2+\frac{49}{18}\zeta(2)\right.$$
$$\left.+\frac{13}{6}\ln 2-2-\frac{1}{6}R\right]$$
$$=\frac{m\alpha^7}{\pi^2}(0.732986380)=\Gamma_0\frac{\alpha}{\pi}(3.793033599). \tag{166}$$

The contributions from the remaining graphs (with a radiative insertion in the vertex of an internal photon, with a radiative photon spanning two vertices, graphs including binding effects, and the annihilation graph—see Fig. 1 in Ref. 59b) have been calculated numerically. These $O(\alpha)$ corrections when summed give

$$\frac{m\alpha^7}{\pi^2}\,[-1.98784(11)]=\Gamma_0\frac{\alpha}{\pi}\,[-10.2866(6)]. \tag{167}$$

Then[11)]

$$\Gamma_3^{\text{theor}}(o\text{–Ps})=7.038236(10)\ \mu\text{sec}^{-1}\ \text{(Ref. 59)}, \tag{168}$$

which at present is the most accurate theoretical result.

In order to resolve the existing contradiction in the results of the theoretical and experimental studies, the authors of Refs. 60 and 61 studied the question of the 5-photon decay mode of o–Ps and the 4-photon decay mode of p–Ps.[12)] The following theoretical estimates were obtained:

$$\frac{\Gamma_5(o\text{–Ps})}{\Gamma_3(o\text{–Ps})}=0.177\left(\frac{\alpha}{\pi}\right)^2\cong 0.96\cdot 10^{-6}\ \text{(Ref. 60,}$$

$$\frac{\Gamma_4(p\text{–Ps})}{\Gamma_2(p\text{–Ps})}=0.274\left(\frac{\alpha}{\pi}\right)^2\cong 1.48\cdot 10^{-6}\ \text{(Ref. 60}$$

and

$$\Gamma_5(o\text{–Ps})=0.0189(11)\alpha^2\Gamma_0\ \text{(Ref. 61)},$$

$$\Gamma_4(p\text{–Ps})=0.01389(6)m\alpha^7\ \text{(Ref. 61)},$$

which agree with each other and with the results of earlier studies (Ref. 62):[13)]

$$\Gamma_4(p\text{–Ps})=0.01352m\alpha^7$$
$$=11.57\cdot 10^{-3}\text{sec}^{-1}\ \text{(Ref. 62)}. \tag{169}$$

In connection with the present situation regarding the orthopositronium decay width, the studies devoted to seeking alternative decay modes of this system (for example, o–Ps$\to\gamma+a$, where a is an axion, a pseudoscalar particle with mass $m_a<2m_e$) are also important.[64–69] The following experimental limit on the ratio of the decay widths was obtained in Ref. 67:

$$\text{Br}=\frac{\Gamma(o\text{–Ps}\to\gamma+a)}{\Gamma(o\text{–Ps}\to 3\gamma)}<5\cdot 10^{-6}-1\cdot 10^{-6}\ (30\ \text{ppm}), \tag{170}$$

if it is located in the range 100–900 keV. If the axion mass is less than 100 keV (as envisioned in the hypothesis of Samuel,[68] according to which[14)] $m_a<5.7$ keV and $g_{a^+e^-}\sim 2\cdot 10^{-8}$), the limits on the Br are[69]

$$\text{Br}=7.6\cdot 10^{-6},\quad \text{if } m_a\sim 100\ \text{keV},$$

$$\text{Br}=6.4\cdot 10^{-5},\quad \text{if } m_a<30\ \text{keV}.$$

These limits are almost two orders of magnitude smaller than the value needed to eliminate the discrepancy.

Finally, the decay o–Ps$\to$nothing (i.e., weakly interacting, undetected particles) has been studied in Ref. 71. The result

$$\frac{\Gamma(o\text{–Ps}\to\text{nothing})}{\Gamma(o\text{–Ps}\to 3\gamma)}<5.8\cdot 10^{-4}\ (350\ \text{ppm}) \tag{171}$$

includes the possibility that this decay mode is responsible for the discrepancy between theory and experiment.

The decay of o–Ps into two photons, which violates CP invariance (as already noted in Refs. 72 and 73) was experimentally refuted in Ref. 74.[16)]

It should also be noted that the contribution of the weak interaction has been studied in Ref. 76. However,

TABLE IV.

Year	Reference	$\Gamma_3(o-\text{Ps})$, μsec	Error ppm	Method
1968	[82]	7,262(15)	2070	gas
1973	[83]	7,262(15)	2070	gas
1973	[84]	7,275(15)	2060	gas
1976	[85]	7,104(6)	840	SiO_2 powder
1976	[86]	7,09(2)	2820	vacuum
1978	[87]	7,056(7)	990	gas
1978	[88]	7,045(6)	850	gas
1978	[89]	7,050(13)	1840	vacuum
1978	[90]	7,122(12)	1680	vacuum
1982	[91]	7,051(5)	710	gas
1987	[92]	7,031(7)	1000	vacuum
1987	[93]	7,0516(13)	180	gas
1989	[52]	7,0514(14)	200	gas
1990	[53]	7,0482(16)	230	vacuum

since it is proportional to m_e^2/M_W^2, it cannot significantly affect the results. In that article the weak decay modes were estimated to be

$$\frac{\Gamma(p-\text{Ps}\to 3\gamma)}{\Gamma(p-\text{Ps}\to 2\gamma)}\cong\frac{\Gamma(o-\text{Ps}\to 4\gamma)}{\Gamma(o-\text{Ps}\to 3\gamma)}\cong\alpha(G_F m_e^2 g_V)^2\cong 10^{-27}, \tag{172}$$

where G_F is the Fermi weak-interaction constant,

$$g_V=1-4\sin^2\Theta_W\cong 0.08, \tag{173}$$

and Θ_F is the Weinberg angle. The existing experimental limits are[77,78]

$$\frac{\Gamma(p-\text{Ps}\to 3\gamma)}{\Gamma(p-\text{Ps}\to 2\gamma)}\leqslant 2.8\cdot 10^{-6},$$

$$\frac{\Gamma(o-\text{Ps}\to 4\gamma)}{\Gamma(o-\text{Ps}\to 3\gamma)}\leqslant 8\cdot 10^{-6}.$$

In Table IV we give all the experimental results known to us on the decay width of orthopositronium.[17)]

As far as the results for the decay width of parapositronium are concerned, until recently the situation has been much better. The theoretical value determined in the 1950s (Refs. 94 and 95)

$$\Gamma_2^{\text{theor}}(p-\text{Ps})=-2\,\text{Im}(\Delta E_{2\gamma})=\frac{1}{2}\frac{\alpha^5 mc^2}{h}\left[1-\frac{\alpha}{\pi}\left(5-\frac{\pi^2}{4}\right)\right]$$

$$=7.9852\ \text{nsec}^{-1}, \tag{174}$$

and confirmed in Refs. 96 and 97, accurately coincides with the result of the direct experimental measurement:

$$\Gamma^{\text{exp}}(p-\text{Ps})=7.994\pm 0.011\ \text{nsec}^{-1}\ \text{(Ref. 91)}. \tag{175}$$

The experimental values of the parapositronium decay width are summarized in Table V.[18)]

It has been shown in Refs. 50 and 97 that a correction logarithmic in α must be added to the result of Harris and Brown. The logarithmic corrections to $\Gamma_3(o\text{-Ps})$ and $\Gamma_2(p\text{-Ps})$ were calculated again in Ref. 46a, and the result for the parapositronium decay width differs from that found earlier in Refs. 50 and 97:

$$\Gamma_2(p-\text{Ps},\alpha^2\ln\alpha)=\frac{m\alpha^5}{2}2\alpha^2\ln\alpha^{-1}\ \text{(Ref. 46)},$$

$$\Gamma_2(p-\text{Ps},\alpha^2\ln\alpha)=\frac{m\alpha^5}{2}\frac{2}{3}\alpha^2\ln\alpha^{-1}\ \text{(Refs. 50,97)}.$$

Finally, we should mention the completely unexpected (and undesirable from the viewpoint of experiment) result given in the report of Remiddi *et al.*[101] Their calculations indicate the presence of a correction

$$\Gamma_2(p-\text{Ps},\alpha\ln\alpha)=\frac{m\alpha^5}{2}\left(\frac{\alpha}{\pi}\right)2\ln\alpha\ \text{(Ref. 101)}, \tag{176}$$

TABLE V.

Year	Reference	$\Gamma_3(o-\text{Ps})$, μsec	Error, %	Method
1952	[98]	7,63(1,02)	13	gas
1954	[99]	9,45(1,41)	15	gas
1970	[100]	7,99(11)	1,38	gas
1982	[91]	7,994(11)	0,14	gas

TABLE VI.

Year	Reference	ΔE, GHz	Error, ppm
1952	[104]	203,2(3)	1500
1954	[105]	203,350(50)	250
1955	[106]	203,380(40)	200
1957	[107]	203,330(40)	200
1970	[100]	203,403(12)	60
1972	[108]	203,396(5)	24
1975	[109]	203,3870(16)	8
1975	[110]	203,3849(12)	6
1977	[111]	203,384(4)	20
1983	[112]	203,3875(16)	8
1984	[103]	203,38910(74)	3,6

which, as they explain, arises when the dependence of the nuclear interaction on the relative momenta is included (see Fig. 1).

All this indicates the importance of continuing calculations of the decay widths of both ortho- and parapositronium using more accurate methods.

4.2. The hyperfine splitting

Positronium. Comparison of the experimental and theoretical results for the hyperfine splitting of the ground state in positronium and muonium over many years is viewed as an important check of our understanding of the bound-state problem. The value of this splitting for positronium was first estimated in 1951 (Ref. 102) as $(9.4\pm1.4)\times10^{-4}$ eV, and the most recent measured result was obtained in Ref. 103:

$$\Delta E_{\rm HFS}^{\rm exp}({\rm Ps})=203389.10\pm0.74\ {\rm MHz}\ \ (3.6\ {\rm ppm}). \tag{177}$$

In Table VI we summarize all the published values of precision experimental measurements of the HFS of the ground state of positronium.

All the experiments use the method based on observation of Zeeman transitions in Ps and then substitution of the results into the well known Breit–Rabi equation, from which the value of ΔE is determined.

The theoretical result (firmly established) at present is (Refs. 4, 31, 50, and 113–117)

$$\Delta E_{\rm HFS}^{\rm theor}({\rm Ps})=m\alpha^4\left[\frac{7}{12}-\frac{\alpha}{\pi}\left(\frac{8}{9}+\frac{\ln 2}{2}\right)+\frac{5}{24}\alpha^2\ln\alpha^{-1}+O(\alpha^2)\right]\cong 203400.3\ {\rm MHz}. \tag{178}$$

The coefficient of 1 in front of the term $\sim\alpha^6$ can contribute ≈ 18.7 MHz to the HFS energy, i.e., the estimated uncertainty is almost 50 ppm, which is an order of magnitude larger than the experimental value. As a result of the calculation of errors $O(\alpha^2)$ and $O(\alpha^3\ln\alpha)$ the theoretical error would be decreased to 1 ppm. Work in this area has been going on since the 1970s (Refs. 43 and 118–127).

The first contributions of this order were calculated from the one-photon annihilation graph with fourth-order polarization insertion (Ref. 119):[19)]

$$\Delta E_{\rm v.p.}^{\rm HFS}=\frac{1}{2}\alpha^2R_\infty\left(\frac{\alpha}{\pi}\right)^2\left[\frac{13}{324}+\frac{21}{8}\zeta(3)+\frac{\pi^2}{4}\ln 2-\frac{35\pi^2}{32}\right]=-2.78\ {\rm MHz}. \tag{179}$$

The contributions from three-photon annihilation,[120] also calculated analytically, were recently improved:[121]

$$\Delta E_{3\gamma}^{\rm HFS}=\frac{\alpha^4R_\infty}{\pi^2}\left\{\frac{3}{4}\zeta(3)-\frac{1}{3}\zeta(2)\ln 2-\frac{1}{6}\zeta(2)-4\ln 2+\frac{3}{2}-i\pi\left[\frac{4}{3}\zeta(2)-2\right]\right\}, \tag{180}$$

which numerically is small, ${\rm Re}(\Delta E_{3\gamma})=-0.969$ MHz, and has been confirmed by the authors of Ref. 120 in Ref. 122.

The two-photon annihilation contributions[123]

$$\Delta E_{2\gamma}^{\rm HFS}=-\frac{\alpha^4R_\infty}{2\pi^2}\left[1+\frac{35}{9}\pi^2+\left(\frac{41}{4}+\pi^2\right)\ln 2-\frac{85}{4}\zeta(3)-i\pi\left(5-\frac{\pi^2}{4}\right)\right] \tag{181}$$

at present give the contribution ${\rm Re}(\Delta E_{2\gamma})=-13.13$ MHz, which numerically is the largest of the corrections of this order.

It should be noted that various approximation methods have been used for the calculation of the $\sim\alpha^6$ corrections, which in many respects complicates the comparison of the results of certain authors with those of others. For example, the authors of Ref. 125 studied the case of a static interaction kernel corresponding to the fourth component of a vector potential independent of the relative times (the effective-Hamiltonian method). Up to third-order perturbation theory they obtained the result

$$\Delta E_{\rm III}=\frac{1}{12}\alpha^4R_\infty\left[-\frac{1}{2}\left(1+\ln\frac{1}{2}\alpha\right)+\frac{2}{\pi}G+\frac{1}{\pi}(1-4F)\right]=\alpha^4R_\infty\left[\frac{1}{24}\ln\alpha^{-1}+0.031\right]$$

$$=(1.9+0.3)\ \text{MHz}, \tag{182}$$

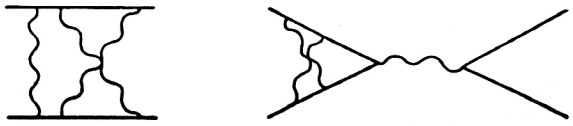
where

$$G=\int_0^1 \frac{\tan^{-1}p}{p}\,dp=0.91596... \tag{183}$$

is the Catalan constant and

$$F=\int_0^1 \frac{(\tan^{-1}p)^2}{p}\,dp=0.3897....$$

The contributions from the various graphs were calculated numerically in Refs. 43 and 127. The importance of isolating the contributions $\sim\alpha^6$ from the other uncalculated graphs (Fig. 7 in Ref. 43) was noted there:



Muonium. Earlier reviews (Refs. 35, 43, and 128–130) give the following theoretical result for the HFS of the ground state of muonium:

$$\Delta E_{\text{HFS}}^{\text{theor}}(\text{Mu})=E_F(1+a_\mu)\left[1+a_e+\frac{3}{2}(Z\alpha)^2+\varepsilon+\frac{\delta_\mu}{1+a_\mu}\right], \tag{184}$$

$$\varepsilon=\alpha(Z\alpha)\left(\ln 2-\frac{5}{2}\right)-\frac{8\alpha(Z\alpha)^2}{3\pi}\ln(Z\alpha)\left[\ln(Z\alpha)-\ln 4+\frac{281}{480}\right]+\frac{\alpha(Z\alpha)^2}{\pi}(15.38(29))+\frac{\alpha^2(Z\alpha)}{\pi}D, \tag{185}$$

$$\delta_\mu=-\frac{3(Z\alpha)}{\pi}\frac{m_e m_\mu}{m_\mu^2-m_e^2}\ln\frac{m_\mu}{m_e}+(Z\alpha)^2\frac{m_e m_\mu}{(m_e+m_\mu)^2}\times\left[-2\ln(Z\alpha)-8\ln 2+3\frac{11}{18}\right]. \tag{186}$$

The early, classical studies of Refs. 131–133 were devoted to the calculation of the corrections neglecting the finiteness of the mass of the heavy particle (recoilless corrections). The result for the corrections was confirmed, and the contribution of order $\alpha(Z\alpha)^2E_F$ was found numerically in Refs. 43 and 127. The leading corrections for recoil have been calculated by several authors (Refs. 25, 116, 117, 128 and 134–136). The contributions from graphs with radiative insertions into electron and muon lines depending on the mass ratio m_e/m_μ were calculated analytically in Refs. 34 and 35:[20)]

$$\delta_\mu\left(\frac{m_e}{m_\mu};\text{el. line}\right)=\frac{\alpha(Z\alpha)}{\pi^2}\frac{m_e}{m_\mu}\left[\frac{15}{4}\ln\frac{m_\mu}{m_e}+6\zeta(3)+3\pi^2\ln 2+\frac{\pi^2}{2}+\frac{17}{8}\right], \tag{187}$$

$$\delta_\mu\left(\frac{m_e}{m_\mu};\text{mu. line}\right)=\frac{Z^2\alpha(Z\alpha)}{\pi^2}\frac{m_e}{m_\mu}\left[\frac{9}{2}\zeta(3)-3\pi^2\ln 2+\frac{39}{8}\right]. \tag{188}$$

The above-mentioned result for the contributions from the electron line was confirmed in Ref. 137. The calculations were carried out in the Fried–Yennie gauge for the radiative photons.

The contributions from the vacuum-polarization graphs have been calculated earlier:[138,139]

$$\delta_\mu\left(\frac{m_e}{m_\mu};\text{v.p.}\right)=\left(\frac{\alpha}{\pi}\right)^2\frac{m_e}{m_\mu}\left[-2\ln^2\frac{m_\mu}{m_e}-\frac{8}{3}\ln\frac{m_\mu}{m_e}-\frac{28}{9}-\frac{\pi^2}{3}\right]; \tag{189}$$

and the leading logarithmic corrections have been calculated in Refs. 41 and 45:[21)]

$$\delta_\mu\left(\frac{m_e}{m_\mu}\ln\frac{m_\mu}{m_e}\right)=-\frac{\alpha^2(Z\alpha)}{\pi^3}\frac{m_e}{m_\mu}\left[\frac{4}{3}\ln^3\frac{m_\mu}{m_e}-\frac{4}{3}\ln^2\frac{m_\mu}{m_e}\right]. \tag{190}$$

In addition, a program of calculating the purely radiative corrections of order $\alpha^2(Z\alpha)E_F$ [i.e., the coefficient D in (185); Refs. 36 and 37] is being concluded.[21*] The $\alpha^2(Z\alpha)E_F$ corrections induced by graphs with vacuum-polarization insertions in external photons (see Figs. 1a–1c in Ref. 36a) have been calculated in Ref. 36a:

$$\Delta E(\alpha^2(Z\alpha))=\frac{\alpha^2(Z\alpha)}{\pi}E_F\left[-\frac{4}{3}\ln^2\frac{1+\sqrt{5}}{2}-\frac{20}{9}\sqrt{5}\ln\frac{1+\sqrt{5}}{2}+\frac{608}{45}\ln 2+\frac{\pi^2}{9}-\frac{38}{15}\pi+\frac{91639}{37800}\right]$$
$$\cong -2.23\frac{\alpha^2(Z\alpha)}{\pi}E_F\cong\left\{\begin{pmatrix}-1.2\ \text{kHz} & \text{for Mu}\\ -0.34\ \text{kHz} & \text{for H}\end{pmatrix}\right\}; \tag{191}$$

and those induced by insertions of the vacuum polarization into the photon lines of radiative photons (see Fig. 1 in the cited article) have been calculated in Ref. 36b:

$$\Delta E(\alpha^2(Z\alpha))=\frac{\alpha^2(Z\alpha)}{\pi}E_F\left[-\frac{149}{270}+\frac{2}{9\pi}\int_0^1 dq\,D(q)\times\left[\frac{3}{1+q}\arctan\sqrt{\frac{2q}{1-q}}+\sqrt{\frac{2q}{1-q}}\times\left(-\frac{5}{4}\frac{1}{1+q}-\frac{2927}{2400}+\frac{10169}{3600}q\right)\right]\right]$$
$$\cong -0.310742...\frac{\alpha^2(Z\alpha)}{\pi}E_F$$

TABLE VII.

Year	Reference	ΔE, kHz	Error, ppm
1962	[148]	4461300 (2200)	493
1964	[149]	4463330 (190)	43
1964	[150]	4463150 (60)	13
1969	[151]	4463260 (40)	9,0
1969	[152]	4463317 (21)	4,7
1970	[153]	4463302,2 (8,9)	2,0
1971	[154]	4463311 (12)	2,7
1971	[155]	4463301,17 (2,3)	0,5
1972	[156]	4463240 (120)	26,9
1973	[157]	4463304,0 (1,8)	0,4
1975	[158]	4463302,2 (1,4)	0,3
1977	[159]	4463302,35 (52)	0,12
1980	[160]	4463302,90 (27)	0,06
1982	[145]	4463302,88 (16)	0,036

$$\cong\left\{\begin{pmatrix}-0.17\ \text{kHz} & \text{for Mu}\\ -0.054\ \text{kHz} & \text{for H}\end{pmatrix}\right\} \quad (192)$$

[$D(q)$ is a complete elliptic integral]; finally, those induced by graphs with insertions of light-on-light scattering subgraphs (see Fig. 1 in the cited article) have been calculated in Ref. 36c:

$$\Delta E(\alpha^2(Z\alpha))\cong -0.48213...\frac{\alpha^2(Z\alpha)}{\pi}E_F$$

$$\cong\left\{\begin{pmatrix}-0.26\ \text{kHz} & \text{for Mu}\\ -0.084\ \text{kHz} & \text{for H}\end{pmatrix}\right\}. \quad (193)$$

Let us also note the corrections of second order in the recoil, first calculated in Ref. 140; the corrections of order $(Z\alpha)^2E_F$ and $(m_e/m_\mu)(Z\alpha)^2E_F$ calculated in the quasipotential approach (Ref. 24):[22)]

$$\Delta E=E_F\left\{1+(Z\alpha)^2\left[\frac{3}{2}+\frac{m_em_\mu}{(m_e+m_\mu)^2}\left(\frac{19}{2}-\frac{1}{72}-\frac{2}{3}\pi^2\right)\right]\right\}; \quad (194)$$

the contributions of the hadron vacuum polarization (Refs. 43 and 141):

$$\delta_\mu(\text{hadrons})=3.7520\pm 0.2373\left(\frac{\alpha}{\pi}\right)^2\frac{m_em_\mu}{m_\pi^2}$$

$$\cong 0.250\pm 0.0016\ \text{kHz (Ref. 141)}; \quad (195)$$

and the estimated contribution of the weak interaction (Refs. 142 and 143):

$$\Delta E(\text{weak int.})=\frac{3}{4\sqrt{2}\pi}(Z\alpha)^{-1}G_Fm_em_\mu E_F$$

$$\cong 0.065\ \text{kHz}. \quad (196)$$

As a result, the theoretical prediction of the HFS of the ground state of muonium is

$$\Delta E_{\text{HFS}}^{\text{theor}}(\text{Mu})=4463303.0(0.2)(0.7)(0.6)\ \text{kHz}. \quad (197)$$

Here the first uncertainty arises from the experimental error in the value of α (Ref. 144), the second from the experimental error in determining the mass ratio of the muon and the electron,[145] and the third from the incompletely calculated coefficient D (see footnote 21*).

Regarding the experimental determination of $\Delta E^{\text{HFS}}(\text{Mu})$, in Table VII we summarize the published values of the measurements of the HFS of the ground state of muonium from the time of its discovery in 1960 (Ref. 146).[23)]

The agreement between the theoretical and experimental results is excellent. Starting from this fact, by comparing them [the expressions (184) and the last value in Table VII] we can extract the value of the fine-structure constant [see (244)].

Hydrogen. The result of the analytic calculation of the quantum-electrodynamical corrections to the HFS of hydrogen neglecting the proton recoil is obviously the same as for muonium: (184), (185).[24)] Numerically, the value is[161]

$$\Delta E(\text{QED})=1420.45195(14)\ \text{MHz}. \quad (198)$$

The corrections for recoil and the dynamical corrections due to the presence of the nuclear structure have been calculated in the classical studies of Refs. 134 and 162–164, and in addition there are the results of Refs. 165–170 and 161:[25)]

$$\Delta E(\text{structure})=E_F[\delta_p(\text{Zemach})+\delta_p(\text{recoil})+\delta_p(\text{polar.})],$$

$$\delta_p(\text{Zemach})=-2\mu\alpha\langle r_p\rangle\cong -38.72(56)\ \text{ppm},$$

$$\delta_p(\text{recoil})\cong 5.68\ \text{ppm},$$

$$|\delta_p(\text{polar.})|<4\ \text{ppm},$$

where $\langle r_p\rangle$ is the mean proton radius, related to the charge distribution.

TABLE VIII.

Year	Reference	Isotope	ΔE, kHz	Error, ppm
1948	[173]	H	1420410 (6)	4224
1948	[173]	D	327384 (3)	9164
1952	[174]	H	1420405,1 (2)	141
1952	[174]	D	327384,24 (8)	244
1955	[175]	H	1420405,73 (5)	35
1955	[175]	D	327384,302 (30)	92
1956	[176]	H	1420405,80 (6)	42
1960	[177]	H	1420405,726 (30)	21
1960	[177]	D	327384,349 (5)	15
1960	[177]	T	1516701,396 (30)	20
1962	[178]	H	1420405,762 (4)	2,8
1962	[179]	H	1420405,7491 (60)	4,2
1962	[179]	T	1516701,4768 (60)	4,0
1963	[180]	H	1420405,751827 (20)	0,014
1963	[181]	H	1420405,751800 (28)	0,019
1964	[182]	H	1420405,751827 (20)	0,014
1965	[183]	H	1420405,751778 (16)	0,011
1965	[184]	H	1420405,751785 (16)	0,011
1966	[185]	H	1420405,751781 (16)	0,011
1966	[186]	H	1420405,7517860 (46)	0,003
1966	[187]	H	1420405,7517864 (17)	0,001
1970	[188]	H	1420405,7517667 (9)	0,0006

A discussion of the contributions from the proton polarization obtained using the data on the inelastic scattering of polarized electrons on nucleons can be found also in Ref. 171.

In Table VIII we summarize the published experimental values of the measurements of the HFS of the ground state of hydrogen and deuterium.[26)]

The difference between the theoretical and experimental values can then be written as

$$\frac{\Delta E^{\text{theor}}-\Delta E^{\text{exp}}}{E_F}$$

$$=(-0.48\pm0.56\pm\text{uncalculated contribs.})\ \text{ppm}. \tag{199}$$

The uncertainty of 0.56 ppm arises from the error in the value of the fine-structure constant and, mainly, from the inaccuracy of the data on the proton elastic form factor. The as yet uncalculated contributions may be of order 1 ppm (Refs. 161 and 189).

The muonic helium atom. The muonic helium atom, $^4He^{++}\mu^-e^-$, was discovered experimentally in 1975 (Refs. 190 and 191). From the viewpoint of the electron structure, this system can be treated as a heavy isotope of hydrogen with the "pseudonucleus" $(^4He^{++}\mu^-)^+$, which has size intermediate between the typical nuclear and atomic dimensions ($\sim$130 F). The first measurements of the HFS carried out in 1980 (Refs. 192 and 194) gave

$\Delta E^{\text{exp}}=4464.95(6)$ MHz (13 ppm) (Ref. 192),

$\Delta E^{\text{exp}}=4464.02(10)$ MHz (22 ppm) (Ref. 193),

$\Delta E^{\text{exp}}=4465.004(29)$ MHz (6.5 ppm) (Ref. 194).

The last experiment also made it possible to determine the magnetic moment of the negative muon:

$$\frac{\mu_{\mu^-}}{\mu_p}=3.18328(15)\quad(47\ \text{ppm}), \tag{200}$$

which makes it possible to check the predictions of *CPT* invariance, according to which the magnetic moments of a particle and its antiparticle must be the same. This quantity for a positively charged muon, measured considerably more accurately in experiments on the HFS of muonium, is[145]

$$\frac{\mu_{\mu^+}}{\mu_p}=3.1833461(11)\quad(0.36\ \text{ppm}), \tag{201}$$

and also from observations of the muon spin rotation in a liquid (Larmor precession):[195]

$$\frac{\mu_{\mu^+}}{\mu_p}=3.1833441(17)\quad(0.53\ \text{ppm}), \tag{202}$$

and agrees with (200) at the level of several tens of ppm. For comparison, the agreement between the electron and positron magnetic moments is at the level of accuracy 0.13 ppm (Ref. 196), and that between the magnetic moments of the proton and antiproton is at the level 7500 ppm (Ref. 197).

The theoretical description of the muonic helium atom is in many respects related to the methods used to describe muonium; see Refs. 198–201. There is a significant contribution from relativistic and radiative corrections to the Fermi energy for this system (M is the "pseudonucleus" mass):

$$E_F=\frac{16}{3}\alpha^2 R_\infty c\frac{m_e}{m_\mu}\left(1+\frac{m_e}{M}\right)^{-3}=4516.96\ \text{MHz}. \tag{203}$$

The theoretical values are[27)]

$\Delta E^{\text{theor}}=4465.1(1.0)$ MHz (Ref. 198),

$\Delta E^{\text{theor}}=4462.6(3.0)$ MHz (Ref. 199),

$\Delta E^{\text{theor}}=4464.8(5)$ MHz (Ref. 200),

$\Delta E^{\text{theor}}=4460$ MHz (Ref. 201).

The first result was obtained using variational methods, the second and third were obtained including the graphs of perturbation theory, and the fourth was obtained using the Born–Oppenheimer theory.

4.3. The fine structure

Positronium and muonium. The fine-structure interval in positronium, which in contrast to the hydrogen atom is nondegenerate already at order $\alpha^2 R_\infty$, was first studied experimentally in Ref. 204:

$$\Delta E(2^3S_1-2^3P_2)=(8628.4\pm 2.8)\ \text{MHz}. \tag{204}$$

Experiments of this type make it possible to verify QED for excited states of purely leptonic systems.

Recent experiments[205,349] have attained higher accuracy (in MHz):[28)]

$$\Delta E(2^3S_1-2^3P_2)=\begin{cases}8619.6(2.7)(0.9)\\ 8624.38(0.54)(1.40)\end{cases},$$

$$\Delta E(2^3S_1-2^3P_1)=\begin{cases}13001.3(3.9)(0.9)\\ 13012.42(0.67)(1.54)\end{cases},$$

$$\Delta E(2^3S_1-2^3P_0)=\begin{cases}18504.1(10.0)(1.7)\\ 18499.65(1.20)(4.00)\end{cases}.$$

The theoretical calculations based on the BS equation predict the following for the first excited states (Refs. 46, 47, 113, 114, and 206; see also Refs. 207 and 208):

$$E(1^3S_1)=R_\infty\left\{-\frac{1}{2}+\frac{49}{96}\alpha^2+\frac{3}{2\pi}\alpha^3\ln\alpha^{-1}+\frac{\alpha^3}{\pi}\left[-\frac{1}{15}+\frac{4}{3}\ln 2-\frac{4}{3}\ln R(1,0)\right]+A_{1S}\alpha^4\ln\alpha^{-1}+B_{1S}\alpha^4+\ldots\right\},$$

$$E(2^3S_1)=\frac{1}{8}R_\infty\left\{-1+\frac{65}{192}\alpha^2+\frac{3}{2\pi}\alpha^3\ln\alpha^{-1}+\frac{\alpha^3}{\pi}\left[\frac{97}{120}+\frac{1}{6}\ln 2-\frac{4}{3}\ln R(2,0)\right]+A_{2S}\alpha^4\ln\alpha^{-1}+B_{2S}\alpha^4+\ldots\right\},$$

$$E(2^3P_2)=\frac{1}{8}R_\infty\left\{-1-\frac{43}{960}\alpha^2-\frac{\alpha^3}{\pi}\left[\frac{1}{45}+\frac{4}{3}\ln R(2,1)\right]+A_{P2}\alpha^4\ln\alpha^{-1}+B_{P2}\alpha^4+\ldots\right\},$$

$$E(2^3P_1)=\frac{1}{8}R_\infty\left\{-1-\frac{47}{192}\alpha^2-\frac{\alpha^3}{\pi}\left[\frac{5}{36}+\frac{4}{3}\ln R(2,1)\right]+A_{P1}\alpha^4\ln\alpha^{-1}+B_{P1}\alpha^4+\ldots\right\},$$

$$E(2^3P_0)=\frac{1}{8}R_\infty\left\{-1-\frac{95}{192}\alpha^2-\frac{\alpha^3}{\pi}\left[\frac{25}{72}+\frac{4}{3}\ln R(2,1)\right]+A_{P0}\alpha^4\ln\alpha^{-1}+B_{P0}\alpha^4+\ldots\right\},$$

where $R(n,l)$ is the Bethe logarithm:[209,210]

$\ln R(1,0)\cong 2.9841285$,

$\ln R(2,0)\cong 2.8117699$,

$\ln R(2,1)\cong -0.0300167$,

and the coefficients A_{1S} and A_{2S} have been recently calculated, but the results of Refs. 46 and 47 differ from each other:[29)]

$$\Delta E(\alpha^6\ln\alpha)=\tfrac{5}{24}m\alpha^6\ln\alpha^{-1}\frac{\delta_{l0}\delta_{s1}}{n^3}\quad\text{(Ref. 46)},$$

$$\Delta E(\alpha^6\ln\alpha)=\tfrac{1}{12}m\alpha^6\ln\alpha^{-1}\frac{\delta_{l0}\delta_{s1}}{n^3}\quad\text{(Ref. 47)},$$

and the coefficients B have not yet been calculated. The coefficient 1 of the term $\alpha^4\ln\alpha^{-1}R/8$ gives a contribution of 5.7 MHz, and that in front of $\alpha^4 R/8$ gives 1.2 MHz.

There is a simple expression for the contributions of order $\alpha^2 R_\infty$ to the S levels of the electron–positron system:[207]

$$E(\alpha^2 R_\infty)=\frac{m\alpha^4}{n^3}\left[\frac{11}{64}\frac{1}{n}-\frac{1}{2}+\frac{7}{12}\delta_{1S}\right]. \tag{205}$$

The contributions beginning at order $\alpha^3 R$ arise from the inclusion of radiative corrections (of the vertex function, the vacuum polarization, the self-energy of the electron–positron field, and the annihilation channel of the interaction). Apparently, the expression for the S states of the electron–positron system given in Ref. 207 is incorrect, as stated in, for example, Refs. 47 and 208. From the result given in the latter article it is clear that these corrections are no longer proportional to $1/n^3$:

$$\Delta E(\alpha^3 R_\infty)=\frac{m\alpha^5}{8\pi n^3}\left\{\frac{14}{3}\left[\frac{7}{15}+\ln\frac{2}{n}+\frac{n-1}{2n}+\sum_{k=1}^{n}\frac{1}{k}\right]+\frac{14}{3}\ln 2-6\ln\alpha-\frac{16}{3}\ln R(n,0)-4\left(\frac{16}{9}+\ln 2\right)\delta_{1S}\right\}. \tag{206}$$

TABLE IX.

Year	Reference	Isotope	$\Delta E(1S-2S)$	$E(LS,1S)$
1975	[223]	D	—	8300(300)
1975	[223]	H	—	8600(800)
1975	[224]	D	—	8250(110)
1975	[224]	H	—	8200(100)
1980	[225]	D	—	8177(30)
1980	[225]	H	—	8151(30)
1986	[226][34]	H	2466061395.6(4.8)	8184.8(5.4)
1986	[227]	H	2466061397(25)	8182(25)
1987	[228]	H	2466061413.8(1.5)	8173.3(1.7)
1989	[229]	H	2466061413.19(1.75)	8173.9(1.9)
1989	[230]	D	2466732408.5(7)	8183.7(6)
1989	[230]	H	2466061414.1(8)	8172.6(7)
1990	[231]	H	2466061413.182(45)	8172.804(83)

The numerical values of the theoretical predictions (including logarithmic corrections) are (Refs. 46 and 47; see also Ref. 348):[30)]

$$\Delta E(2^3S_1-1^3S_1)$$

$$=1233607211.7 \text{ MHz}; \quad 1233607221.69 \text{ MHz};$$

$$\Delta E(2^3S_1-2^3P_2)=8627.7 \text{ MHz}; \quad 8626.21 \text{ MHz};$$

$$\Delta E(2^3S_1-2^3P_1)=13013.3 \text{ MHz}; \quad 13011.86 \text{ MHz};$$

$$\Delta E(2^3S_1-2^3P_0)=18498.5 \text{ MHz}; \quad 18497.10 \text{ MHz}.$$

In connection with the development of experimental methods based on Doppler-less two-photon spectroscopy, it has become possible also to measure the intervals of the "large structure" ($1S$–$2S$) with an accuracy of several MHz. These experiments make it possible to determine the value of a fundamental constant like the Rydberg constant with record accuracy (see below).

References 211 and 351 give the following results for the measurement of the interval $\Delta E(1^3S_1-2^3S_1)$ in positronium and muonium:[31)]

$$\Delta E_{\text{Ps}}^{\text{exp}}(2^3S_1-1^3S_1)=\begin{cases}(1233607218.9\pm10.7) \text{ MHz}\\ (1233607216.4\pm3.2) \text{ MHz}\end{cases}. \quad (206')$$

$$\Delta E_{\text{Mu}}^{\text{exp}}(2^3S_1-1^3S_1)=(2455527936\pm120\pm140) \text{ MHz}.$$

The result for muonium agrees with the theory:[217]

$$\Delta E_{\text{Mu}}^{\text{theor}}(2^3S_1-1^3S_1)=2455527959.6(3.6) \text{ MHz}.$$

For positronium the result is 16.6 (10.7) MHz higher than the theoretical value of Fulton. However, since many authors consider the result of Ref. 207 for the $1S$ state to be inaccurate, it is preferable to compare the experimental value (206′) with the result of Ref. 208, in which the fine structure was calculated using the simple potential approach:[218]

$$\Delta E_{\text{Ps}}^{\text{theor}}(2^3S_1-1^3S_1)=1233607221.69 \text{ MHz},$$

$$\Delta E_{\text{Mu}}^{\text{theor}}(2^3S_1-1^3S_1)=2455528055 \text{ MHz},$$

and with the recent results of Fell, Khriplovich *et al.*[47,348]

The fine structure of an excited state of muonium was studied in Refs. 219–221 (see below).

We also make note of Refs. 76 and 222, in which the question of verifying CP conservation in the leptonic sector was studied. Experimental and theoretical limits on transitions of the type $2^3S_1\to2^1P_1$ were obtained in those studies.

Hydrogen and deuterium. For hydrogen the results of measurements of the $2S$–$1S$ interval are given in Refs. 223–231.[32)] The importance of these studies is clear, since they provide information on the Lamb shift of the $1S$ level,[33)] which cannot be obtained by the methods of radio-frequency spectroscopy used in experiments on the $2S$ Lamb shift, and they allow the numerical value of the Rydberg constant to be determined with record accuracy.

In Table IX we give the results of measurements of the interval and of the Lamb shift of the $1S$ level. All energies are measured in MHz.

The result of Ref. 230 for the Lamb shift of the $1S$ level was obtained using the following value of the Rydberg constant (the average of the results of Refs. 252 and 253):

$$R_\infty=109737.315714(19) \text{ cm}^{-1}. \quad (207)$$

There are several values for the isotopic shift $\Delta E(H-D)$ determined using this method:

$$\Delta E(H-D)=670993(56) \text{ MHz (Ref. 224)},$$

$$\Delta E(H-D)=670992.3(6.3) \text{ MHz (Ref. 225)},$$

$$\Delta E(H-D)=670994.33(64) \text{ MHz (Ref. 230)},$$

$$\Delta E(H-D)=670994.337(22) \text{ MHz (Ref. 347)}.$$

The numerical values of the Lamb shift of the $1S$ level determined using the theoretical calculations of Refs. 236–242 are (see Refs. 229 and 230):

$$E^{\text{theor}}(H,LS,1S)=8172.89(9) \text{ MHz (Ref. 229)};$$

$$E^{\text{theor}}(H,LS,1S)=8173.03(9) \text{ MHz (Ref. 230)};$$

$$E^{\text{theor}}(D,LS,1S)=8184.08(12) \text{ MHz (Ref. 230)}.$$

TABLE X.

Article	Year	Value of the constant (cm^{-1})	Interval
[246]	1974	109737.3143(10)	$2P - 3D$
[247]	1978	109737.31476(32)	$2S - 3P$
[248]	1980	109737.31513(85)	$2S - 3P$, $2P - 3D$
[249]	1981	109737.31521(11)	$2S - 3P$
[226]	1986	109737.31492(22)	$1S - 2S$
[227]	1986	109737.3150(11)	$1S - 2S$
[250]	1986	109737.31569(7)	$2S - 3P$
[251]	1986	109737.31569(6)	$2S - 8D$, $10D$
[252]	1987	109737.31573(3)	$2S - 4P$
[228]	1987	109737.31571(7)	$1S - 2S$
[253]	1989	109737.315709(18)	$2S - 8D$, $10D$, $12D$
[229]	1989	109737.31569(8)	$1S - 2S$
[230]	1989	109737.31573(3)	$1S - 2S$
[231]	1992	109737.3156841(42)	$1S - 2S$
[254]	1992	109737.3156830(31)	$2S - 8S$, $8D$

The proton charge radius in Ref. 230 is set equal to 0.862(12) F (Ref. 244),[35] and that of the deuteron is 2.116(12) F (Ref. 245).

The theoretical value of the isotopic splitting

$$\Delta E^{\text{theor}}(H-D)=670994.39(12)\ \text{MHz}\ \text{(Ref. 230)} \quad (208)$$

agrees with the results of recent experiments within the error.

If we take the value of the Lamb shift to be given theoretically, by comparing the calculated value of the $1S$–$2S$ interval with this experiment we can obtain information on the Rydberg constant (see the following section).

4.4. The Rydberg constant

The recent measurements of the Rydberg constants are given in Table X.

4.5. The Lamb shift

Hydrogen. The results of optical measurements of the Lamb shift of the $1S$ level of the hydrogen atom are reported in Ref. 255:

$$E(LS,1S,H)=8172.82(11)\ \text{MHz}\ (13\ \text{ppm}). \quad (209)$$

The technique is based on comparing the frequencies of two-photon $1S$–$2S$ and $2S$–$4S,4D$ transitions and is fundamentally different from experiments on the indirect determination of this quantity (see the preceding section) by the method of Doppler-less two-photon spectroscopy of the $1S$–$2S$ transition. That study[255] also gives the first optical measurement of the Lamb shift of the $4S$ level of hydrogen:

$$E(LS,4S,H)=131.66(4)\ \text{MHz}\ (300\ \text{ppm}), \quad (210)$$

which can be compared with the theoretical predictions:[233]

$$E^{\text{theor}}(LS,4S,H)=133.084(1)\ \text{MHz},$$

$$E^{\text{theor}}(LS,4S,D)=133.254(3)\ \text{MHz},$$

and with the radio-frequency measurements (Refs. 256–258):

$$E^{\text{exp}}(LS,4S,D)=133(10)\ \text{MHz}\ \text{(Ref. 256)},$$

$$E^{\text{exp}}(LS,4S,H)=133.18(59)\ \text{MHz}\ \text{(Ref. 257)},$$

$$E^{\text{exp}}(LS,4S,H)=133.53^{+0.58}_{-0.78}\ \text{MHz}\ \text{(Ref. 258)}.$$

See Ref. 259 for a discussion of the Lamb shift for $n=3$ in the hydrogen atom. Here we reproduce the results of that study:

$$E^{\text{exp}}(LS,3S,D)=314.93(40)\ \text{MHz}\ \text{(Ref. 256)},$$

$$E^{\text{exp}}(LS,3S,H)=313.6(2.9)\ \text{MHz}\ \text{(Ref. 260)},$$

$$E^{\text{exp}}(LS,3S,D)=315.3(8)\ \text{MHz}\ \text{(Ref. 260)},$$

TABLE XI.

Year	Reference	ΔE, MHz	Error, ppm
1953	[262]	1057.774(100)	94.5
1969	[263]	1057.772(63)	59.6
1970	[264]	1057.90(6)	56.7
1975	[265]	1057.892(20)	18.9
1979	[266]	1057.862(20)	18.9
1981	[267]	1057.845(9)	8.5
1982	[268]	1057.8594(19)	1.8
1983	[269]	1057.851(2)	1.9

$$E^{\text{exp}}(LS,3S,H)=315.11(89)\ \text{MHz}\ \text{(Ref. 261)},$$

$$E^{\text{exp}}(LS,3S,H)=314.819(48)\ \text{MHz}\ \text{(Ref. 259)},$$

$$E^{\text{theor}}(LS,3S,H)=314.898(3)\ \text{MHz}\ \text{(Ref. 237)}.$$

The accuracy of these experiments approaches that of the measurements of the Lamb shift of the $2S$ level, which were extremely important for checking the QED predictions. In Table XI we summarize the results of all the measurements for the $n=2$ level.

The results depend significantly on the parameter τ, the lifetime of the $2P$ state. Since there are no experimental data on this decay constant, its value has been calculated theoretically.[268,269] Including relativistic corrections,

$$\gamma_{\text{rel}}=4\pi c\left(\frac{2}{3}\right)^8 R_H\alpha^3\left(1+\alpha^2\ln\frac{9}{8}\right)$$

$$=\left(\frac{2}{3}\right)^8\frac{me^4}{h^3}\frac{\alpha^3\left(1+\alpha^2\ln\frac{9}{8}\right)}{1+\frac{m}{\mu_p}} \tag{211}$$

(R_H is the Rydberg constant including the finite proton mass).

The leading radiative corrections (the self-energy and the vacuum polarization) give an additional contribution

$$\delta\gamma_{\text{rad}}=4\pi c\left(\frac{2}{3}\right)^8 R_H\alpha^3\left[\frac{R(2,1)}{8}-R(1,0)-\ln\frac{1}{\alpha^2}-\frac{1}{64}-\frac{19}{30}\right] \tag{212}$$

[$R(n,l)$ is the Bethe logarithm].

Numerically, including these corrections, we have

$$\gamma=\frac{1}{\tau}=6.2648812(20)\cdot 10^8\ \text{sec}^{-1},$$

$$\tau=1.59619946(48)\cdot 10^{-9}\ \text{sec}.$$

For deuterium ($n=2$) we know the following experimental results:

$$E^{\text{exp}}(LS,D)=1059.00(10)\ \text{MHz}\ \text{(Ref. 262)},$$

$$E^{\text{exp}}(LS,D)=1059.24(6)\ \text{MHz}\ \text{(Ref. 270)}.$$

We note that further improvement of the accuracy of the experimental results for the Lamb shift of the $n=2$ level of hydrogen will be very difficult, because the natural width of the $2P$ state is of order ~ 100 MHz.

The complete theoretical expression for the Lamb shift ($n=2$) in hydrogen is (Ref. 27; see also the improvements in Ref. 348)

$$\Delta E_{LS}=\Delta E_{2S_{1/2}}-\Delta E_{2P_{1/2}}$$

$$=\frac{\alpha(Z\alpha)^4m}{6\pi}\left(\frac{\mu}{m}\right)^3\left\{\frac{1}{8}\frac{m}{\mu}+\ln(Z\alpha)^{-2}-2.207909\right.$$

$$+\pi Z\alpha\left(\frac{427}{128}-\frac{3}{2}\ln 2\right)+(Z\alpha)^2$$

$$\times\left[-\frac{3}{4}\ln^2(Z\alpha)^{-2}+\left(4\ln 2+\frac{55}{48}\right)\ln(Z\alpha)^{-2}\right]$$

$$\left.+(Z\alpha)^2[G_{\text{s.e.}}(Z\alpha)+G_{\text{v.p.}}(Z\alpha)]+\alpha\left(\frac{0.323}{\pi}\right)\right\}$$

$$+\frac{(Z\alpha)^5m^2}{6\pi M}\left\{\frac{1}{4}\ln(Z\alpha)^{-2}+2.39977+\frac{3}{4}\pi Z\alpha\left[\frac{5}{2}\right.\right.$$

$$\left.\left.+\ln(2Z\alpha)^{-1}-4.25\right]\right\}+\frac{1}{12}(Z\alpha)^4m^3\langle r_p^2\rangle$$

$$-\frac{1}{48}\frac{(Z\alpha)^4m^3}{M^2}+\frac{\alpha(Z\alpha)^5m^2}{8M}$$

$$\times\left[\left(\frac{35}{4}\ln 2-\frac{39}{5}-\frac{31}{192}\right)+(-0.415\pm 0.004)\right], \tag{2.13}$$

where the self-energy and vacuum-polarization contributions ($G_{\text{s.e.}}$ and $G_{\text{v.p.}}$) can be represented in the form of the well known Wichman–Kroll expansion (Refs. 272 and 236c):

$$G_{\text{v.p.}}=-\tfrac{1199}{2100}+\tfrac{5}{128}\pi(Z\alpha)\ln(Z\alpha)^{-2}+0.5(Z\alpha)+\ldots, \tag{214}$$

$$G_{\text{s.e.}}=-24.1+7.5(Z\alpha)\ln(Z\alpha)^{-2}+12.3(Z\alpha)\pm 1.2 \tag{215}$$

and give a contribution -24.0 ± 1.2 (Ref. 271) to the sum for the case of the hydrogen atom.[36)]

The value of the Lamb shift of the $n=2$ level of hydrogen, corrected by taking into account the new calculated corrections, is given in Ref. 271:

$$E^{\text{theor}}(LS,H)=(1057.855\pm 0.011)\ \text{MHz}$$

$$(\text{for}\ \langle r_p\rangle$$

$$=0.805(11)),$$

$$E^{\text{theor}}(LS,H)=(1057.873\pm 0.011)\ \text{MHz}$$

$$(\text{for}\ \langle r_p\rangle$$

$$=0.862(12)).$$

The earliest theoretical studies in which the recoil corrections $(Z\alpha)^5(m^2/M)$ were calculated are Refs. 114c, 169, and 273–275. The corrections obtained by including graphs with radiative exchanges, $\alpha(Z\alpha)^4(m^2/M)$ and $\alpha(Z\alpha)^5(m^2/M)$, have been calculated in the external-field approximation in Ref. 241. The corrections of order $(Z\alpha)^4(m^2/M^2)$ are found in Refs. 274 and 221. The contributions arising from the inclusion of the finite size of the proton are discussed in Ref. 271, where the corrections $(Z\alpha)^6m^2/M$ are calculated.

The correction of order $\alpha^2(Z\alpha)^5m$, the binding correction to two-loop radiative effects, has not yet been calculated.[37)] It would be good to calculate it in order to raise the accuracy of the theoretical calculations to ~ 1 kHz.

TABLE XII.

Year	Reference	$E(LS)$, MHz	Error, ppm
1950	[282]	14020(100)	7130
1952	[283]	14021(60)	4280
1955	[244]	14043(13)	930
1957	[285]	14040,2(1,8)	128
1971	[286]	14046,2(1,2)	85
1979	[287]	14040,9(2,9)	207
1987	[280]	14041,9(1,5)	107
1988	[281]	14042,22(35)	25

Muonium. At present there are two experimental results for the Lamb shift $(2S_{1/2}-2P_{1/2}, J=1)$ in muonium:

$$E^{\exp}(LS,\mathrm{Mu})=(1070^{+12}_{-15}\pm 2)\ \text{MHz (Ref. 278)},$$

$$E^{\exp}(LS,\mathrm{Mu})=(1054\pm 22)\ \text{MHz (Ref. 279)}.$$

The theoretical value was given by Owen (Refs. 219, 220, and 221):

$$E^{\text{theor}}(LS,\mathrm{Mu})=1047.03\ \text{MHz (Ref. 219)},$$

like the fine-structure interval $2P_{3/2}-2P_{1/2}$:

$$E^{\text{theor}}(FS,\mathrm{Mu})=10921.50\ \text{MHz (Ref. 219)}.$$

In contrast to the hydrogen atom, as in the case of calculations of the HFS in muonium no problems with the structure arise.

Helium. The accuracy of the experimental measurements[280,281] of the frequencies of transitions between Rydberg states of $^4He^+$ by the methods of beam foil spectroscopy has now reached a level where it has become possible to check the $(2S_{1/2}-2P_{1/2})$ Lamb shift in helium at the level of corrections of order $\sim\alpha(Z\alpha)^6mc^2$. In Table XII we give all the experimental results for this quantity, including earlier ones based on the technique of Lamb and Rutherford.[262]

The first theoretical studies of the Lamb shift in the helium atom were carried out in Ref. 289. Modern calculations (Refs. 232, 237, 276, and 281; see also Ref. 238) give widely differing results:[38)]

$$E(LS,{}^4He^+)^{\text{theor}}=14044.5(5.2)\ \text{MHz (Ref. 232)},$$

$$E(LS,{}^4He^+)^{\text{theor}}=14045.12(55)\ \text{MHz (Ref. 237)},$$

$$E(LS,{}^4He^+)^{\text{theor}}=14042.36(55)\ \text{MHz (Ref. 276a)},$$

$$E(LS,{}^4He^+)^{\text{theor}}=14042.26(50)\ \text{MHz (Ref. 281)}.$$

Recent calculations of $G_{\text{s.e.}}(Z=2)$ give

$$G_{\text{s.e.}}(Z=2)=-22.8\pm 2.0\ \text{(Ref. 290)}, \tag{216}$$

$$G_{\text{s.e.}}(Z=2)=-22.0\pm 0.3\ \text{(Ref. 291)}. \tag{217}$$

In addition, we should note the recent studies devoted to high-lying states of the helium atom (Refs. 292–296 and 291) and the table of recent results for ions of other atoms in Ref. 281. Muonic helium was studied in Ref. 297.

4.6. The anomalous magnetic moment

The electron.

The renormalizability of QED guarantees the possibility of calculating the electron anomalous magnetic moment (AMM) by expansion in a perturbation series in α/π with finite coefficients a_i:

$$\frac{g-2}{2}=a_e(\mathrm{QED})$$
$$=a_{\mathrm{II}}\frac{\alpha}{\pi}+a_{\mathrm{IV}}\left(\frac{\alpha}{\pi}\right)^2+a_{\mathrm{VI}}\left(\frac{\alpha}{\pi}\right)^3+a_{\mathrm{VIII}}\left(\frac{\alpha}{\pi}\right)^4+\dots . \tag{218}$$

At present the value of the electron AMM is known through 8 significant figures. The calculation of this contribution has been completed in Refs. 298–302. The result for the eighth-order contributions to the electron AMM is[39)]

$$a_{e,\mathrm{VIII}}^{\text{theor}}=-1.434(138). \tag{219}$$

If the most accurate value of the fine-structure constant α (Ref. 304), determined by using the quantum Hall effect, is used,

$$\alpha^{-1}=137.0359979(32)\ (0.024\ \text{ppm}), \tag{220}$$

the most accurate value of the electron AMM will be[40)]

$$a_e^{\text{theor}}=1159652140(27.1)(5.3)(4.1)\cdot 10^{-12}, \tag{221}$$

which agrees[41)] to within 1.7 standard deviations with the experimental values for the electron and the positron (Ref. 144):[42)]

$$a_{e^-}^{\exp}=1159652188.4(4.3)\cdot 10^{-12}, \tag{222}$$

$$a_{e^+}^{\exp}=1159652187.9(4.3)\cdot 10^{-12}. \tag{223}$$

The theoretical result includes the analytically calculated second- and fourth-order contributions:

$$a_{e,\mathrm{II}}=0.5\ \text{(Ref. 307)}, \tag{224}$$

$$a_{e,\mathrm{IV}}=\left[\frac{197}{144}+\frac{\pi^2}{12}-\frac{\pi^2\ln 2}{2}+\frac{2\zeta(3)}{4}\right]$$
$$=-0.328478965\ \text{(Ref. 308)},^{43)} \tag{225}$$

$$a_{e,\mathrm{IV}}\left(\frac{m_e}{m_\mu}\right)=\frac{1}{45}\left(\frac{m_e}{m_\mu}\right)^2+O\left(\left(\frac{m_e}{m_\mu}\right)^4\ln\frac{m_e}{m_\mu}\right)$$
$$=5.198\cdot 10^{-7}\ \text{(Ref. 313)}^{44)} \qquad (226)$$

and the sixth-order contribution, consisting of 72 graphs (not all of them have been calculated analytically),[314–317] corrected in Refs. 302 and 350:

$$a_{e,\mathrm{VI}}=1.17613(42),^{45)} \qquad (227)$$

and also the small correction terms which appear as a result of the inclusion of the muon, τ-lepton, and hadronic vacuum-polarization loops, and also the contribution of the weak interaction:

$$\Delta a_e(\text{muon})=2.804\cdot 10^{-12},$$

$$\Delta a_e(\tau\ \text{lepton})=0.010\cdot 10^{-12},$$

$$\Delta a_e(\text{hadrons})=1.6(2)\cdot 10^{-12},$$

$$\Delta a_e(\text{weak int.})=0.05\cdot 10^{-12}.$$

The contribution of light-on-light scattering subdiagrams to the sixth-order AMM a_e recently calculated analytically is also interesting:[319]

$$a_{e,\mathrm{IV}}(\gamma\gamma)=\tfrac{5}{6}\zeta(5)-\tfrac{5}{18}\pi^2\zeta(3)-\tfrac{41}{540}\pi^4-\tfrac{2}{3}\pi^2\ln^2 2+\tfrac{2}{3}\ln^4 2$$
$$+15a_4-\tfrac{4}{2}\zeta(3)-24\pi^2\ln 2+\tfrac{931}{54}\pi^2+\tfrac{5}{9}$$
$$\cong 0.3710052921 \qquad (228)$$
$$\left(a_4=\sum_{n=1}^{\infty}\frac{1}{2^n n^4}\cong 0.517479061...\right),$$

which agrees with the numerical estimates:

$$a_{e,\mathrm{VI}}(\gamma\gamma)=0.36(4)\ \text{(Ref. 320)},$$

$$a_{e,\mathrm{VI}}(\gamma\gamma)=0.37112(8)\ \text{(Ref. 321)},$$

$$a_{e,\mathrm{VI}}(\gamma\gamma)=0.370986(20)\ \text{(Ref. 315)},$$

but disagrees with $a_{e,\mathrm{VI}}(\gamma\gamma)=0.398(5)$ (Ref. 318).

The muon. The AMM of a lepton of mass m_1 can be expressed in the following most general form:

$$a_l=a_1+a_2\left(\frac{m_1}{m_2}\right)+a_2\left(\frac{m_1}{m_3}\right)+a_3\left(\frac{m_1}{m_2},\frac{m_1}{m_3}\right), \qquad (229)$$

where m_2 and m_3 are the masses of the other leptons.

As for the electron AMM,

$$a_i=a_{i,\mathrm{II}}\left(\frac{\alpha}{\pi}\right)+a_{i,\mathrm{IV}}\left(\frac{\alpha}{\pi}\right)^2+a_{i,\mathrm{VI}}\left(\frac{\alpha}{\pi}\right)^3+a_{i,\mathrm{VIII}}\left(\frac{\alpha}{\pi}\right)^4+\dots. \qquad (230)$$

Clearly, $a_{2,\mathrm{II}}=a_{3,\mathrm{II}}=a_{3,\mathrm{IV}}=0$, since the Feynman graphs which can give such contributions are absent.

Let us consider the current status of studies of the muon AMM. The value is known from the calculation of the electron AMM. The value of $a_{2,\mathrm{IV}}(m_\mu/m_e)$, in contrast to the electron AMM, is not so small, owing to the large value of the ratio m_μ/m_e. The result is known analytically:[322]

$$a_{2,\mathrm{IV}}\left(\frac{m_\mu}{m_e}\right)=\frac{1}{3}\ln\frac{m_\mu}{m_e}-\frac{25}{36}+\frac{\pi^2}{4}\frac{m_\mu}{m_e}$$
$$=\left(\frac{m_\mu}{m_e}\right)^2\ln\frac{m_\mu}{m_e}+3\left(\frac{m_\mu}{m_e}\right)^3+O\left(\frac{m_\mu}{m_e}\right)^3$$
$$=1.0942596.... \qquad (231)$$

From this expression we can also extract the contribution

$$a_{2,\mathrm{IV}}\left(\frac{m_\mu}{m_\tau}\right)\cong 7.794(32)\cdot 10^{-5}. \qquad (232)$$

The sixth-order contribution $a_{2,\mathrm{VI}}(m_\mu/m_e)$ arising from the 18 Feynman diagrams containing vacuum-polarization loops has been obtained analytically (Refs. 313, 323, and 324):

$$a_2^{\mu,\mathrm{VI}}\left(\frac{m_e}{m_{\mu;\mathrm{v.p.}}}\right)=a_e^{\mathrm{VI}}(\mathrm{v.p.})+\left(\frac{\alpha}{\pi}\right)^3\left[\frac{1075}{216}-\frac{25}{3}\zeta(2)\right.$$
$$+10\zeta(2)\ln 2-2\zeta(3)+3c_4$$
$$+\left(\frac{31}{27}+\frac{2}{3}\zeta(2)-4\zeta(2)\ln 2+\zeta(3)\right)$$
$$\left.\times\ln\frac{m_\mu}{m_e}+\frac{2}{9}\ln^2\frac{m_\mu}{m_e}\right]$$
$$=1.94404\left(\frac{\alpha}{\pi}\right)^3, \qquad (233)$$

and the contribution of the six graphs with light-on-light scattering subdiagrams has been calculated numerically (Refs. 321, 325, and 326). The most recent calculations of this order of the muon AMM give the following values (Refs. 321 and 350):

$$a_{2,\mathrm{VI}}\left(\frac{m_\mu}{m_e};\text{vac.pol.}\right)=1.9204550(2),$$

$$a_{2,\mathrm{VI}}\left(\frac{m_\mu}{m_e};\text{light-on-light}\right)=20.9479242(9),$$

$$a_{2,\mathrm{VI}}\left(\frac{m_\mu}{m_e};\text{sum}\right)=22.8683792(9).$$

The eighth-order contribution to the muon AMM is calculated from 469 Feynman diagrams, each of which contains electron loops of the vacuum-polarization type or light-on-light scattering subdiagrams. The result of the numerical calculations for eighth order is given in Refs. 327 and 350:[46)]

$$a_2^{\mu,\mathrm{VIII}}\left(\frac{m_\mu}{m_e}\right)=127.47. \qquad (234)$$

The asymptotic (for $m_e/m_\mu\to 0$) contributions to the muon AMM[47)] arising from eighth-order diagrams containing one electron vacuum-polarization loop have been obtained analytically in Refs. 328–330, using the renormalization-group method:[330]

$$a_{\mu,\mathrm{VIII}}^{\infty,1}(\mathrm{v.p.})=-\tfrac{1}{32}\ln\frac{m_\mu}{m_e}+\tfrac{17}{48}+\tfrac{5}{8}\zeta(2)-\zeta(2)\ln 2$$

$$+\tfrac{99}{128}\zeta(3)-\tfrac{5}{4}\zeta(5)=-0.290987\ldots, \quad (235)$$

and that from graphs containing two electron loops is (Refs. 328 and 329):[48)]

$$a_{\mu,\mathrm{VIII}}^{\infty,2}(\mathrm{v.p.})=-\frac{1}{12}\ln^2\frac{m_\mu}{m_e}+\left[\frac{1}{3}\zeta(3)-\frac{2}{3}\right]\ln\frac{m_\mu}{m_e}+\frac{1531}{1728}+\frac{5}{12}\zeta(2)-\frac{1025}{1152}\zeta(3)=-1.452570\ldots. \quad (236)$$

The estimated tenth-order contribution is[327]

$$a_{2,\mathrm{X}}\left(\frac{m_\mu}{m_e}\right)=570(140). \quad (237)$$

The contributions $a_{3,\mathrm{VI}}(m_\mu/m_e,m_\mu/m_\tau)=5.24(1)\cdot10^{-4}$ and $a_{3,\mathrm{VIII}}(m_\mu/m_e,m_\mu/m_\tau)=0.079(3)$ were calculated in the same study. Combining the results with the contribution a_1 known earlier,[302] we obtain the purely quantum-electrodynamical contribution (Ref. 350):[49)]

$$a_\mu(\mathrm{QED})=1165846984(17)(28)\cdot10^{-12}, \quad (238)$$

and after adding it to the hadronic contribution (Refs. 333–336):[50)]

$$a_\mu(\mathrm{hadrons})=7.03(19)\cdot10^{-8} \quad (239)$$

and the weak-interaction contribution (Refs. 337, 338, 350):[51)]

$$a_\mu(\text{weak int.})=195(10)\cdot10^{-11}, \quad (240)$$

we arrive at the following result for the muon AMM (Refs. 327 and 350):

$$a_\mu^{\mathrm{theor}}=116591920(176)\cdot10^{-11}, \quad (241)$$

which agrees well with the known experimental values:[339]

$$a_{\mu^-}^{\mathrm{exp}}=1165937(12)\cdot10^{-9},$$

$$a_{\mu^+}^{\mathrm{exp}}=1165911(11)\cdot10^{-9}.$$

However, we note that the experiments planned at Brookhaven will raise the experimental accuracy by a factor of 20 (up to $\pm0.05\times10^{-8}$; Ref. 340).

The τ lepton. Theoretical calculations of the AMM of the τ lepton were performed in Refs. 341 and 342. The following numerical estimates were obtained:

$$a_\tau=11773(3)\cdot10^{-7}.$$

The experimental value of a_τ has not been studied as well as the theoretical one. No direct experiment in this area is even being planned, owing to the extremely small lifetime of the τ lepton,[343] $T\sim3\times10^{-13}$ sec.

4.7. The fine-structure constant

The experimental values of the fine-structure constant are the following:

(a) determined using the quantum Hall effect:[304]

$$\alpha^{-1}(\mathrm{QHE})=137.0359979(32) \quad (0.024\ \text{ppm}); \quad (242)$$

(b) determined by calculating the eighth-order electron AMM and comparing it with the experimental value (Refs. 302 and 114):[52)]

$$\alpha^{-1}(a_e)=137.03599222(51)(63)(48) \quad (0.0069\ \text{ppm}); \quad (243)$$

(c) determined using the Josephson effect:[344]

$$\alpha^{-1}(\mathrm{JE})=137.0359770(77) \quad (0.056\ \text{ppm});$$

(d) determined by comparing the experimental and theoretical values of the HFS in muonium:[130]

$$\alpha^{-1}(\mu-hfs)=137.035992(22) \quad (0.16\ \text{ppm}). \quad (244)$$

The first two values are consistent up to 0.05 ppm, thus indicating that QED is valid at that level of accuracy. It is necessary to further increase the accuracy of measurements of α^{-1}(QHE) by using other methods. A possible source of uncertainty between (242) and (243) is some unknown interaction between particles more massive than the $W^\pm$ and Z^0 bosons,[53)] and also a possible subquark structure of the electron.[345]

In conclusion, the authors are deeply grateful to V. G. Kadyshevskiĭ, V. I. Savrin, B. A. Arbuzov, É. É. Boos, D. Broadhurst, N. B. Skachkov, R. Fell, I. B. Khriplovich, O. A. Khrustalev, and M. I. Éĭdes for useful discussions, to P. S. Isaev for his interest in the study and for the suggestion to write this review, to the Center for the Science and Technology of the Control and Use of Physical Fields and Radiation of the Russian Academy of Sciences for pleasant working conditions, and also to I. V. Musatov and I. E. Monakhova for their help in preparing the manuscript.

[1)] In (3) and below, repeated variables are understood to be integrated over.

[2)] The physical meaning of the quantities p_0, q_0, $\vec{p}$, $\vec{q}$ is clear from Fig. 1; m_1 and m_2 are the constituent-particle masses.

[3)] For the Green function (2) it is also necessary to take $x_4^0\to\infty$.

[4)] In view of the fact that the free Green function of the two-fermion system has no inverse, in the quasipotential approach it is necessary to project $\hat{G}_0$ onto positive-energy states.

[5)] In calculating the HFS of the ground state of positronium and muonium with accuracy $\alpha^6\ln\alpha$ it is possible to confine ourselves to graphs with three-photon exchange.

[6)] The same authors recently calculated the corrections of order $\alpha^2(Z\alpha)E_F$. See Sec. 4 and Refs. 36 and 37 for more details.

[7)] The graphs in which a radiative photon spans more than two exchanged photons do not lead to contributions of order $\alpha(Z\alpha)E_F$ in the Fried–Yennie gauge.

[8)] This method has also been used to calculate the contribution of terms depending on the logarithm of the mass ratio to the value of the muon anomalous magnetic moment.[42]

[9)] The value of the coefficient of $\ln^2(m_e/m_\mu)$ was improved in Ref. 45.

[10)] See Table I for the previous experimental results.

[11)] Neglecting the as yet uncalculated corrections $O(\alpha^8)$.

[12)] Owing to angular-momentum conservation and the isotropy of space, orthopositronium must decay into an odd number of photons, and parapositronium into an even number; see above.

[13)] The result of Ref. 63 is incorrect: it is 4 times smaller than the values quoted. The explanation of this is given in Ref. 61.

[14)] The suggested values do not give rise to contradictions in comparing the theoretical and experimental results for the electron anomalous magnetic moment.

[15)]For example, Glashow[70] has suggested decay into an invisible particle belonging to a "mirror" world.

[16)]The physical basis of these speculations is the possible existence of an anisotropic vector field with nonzero vacuum expectation value,[75] with which the electron and positron can interact: $\mathscr{L}=g\bar{\psi}O_{\alpha\beta}\psi A^{\alpha}\Omega^{\beta}$, where $\mathscr{L}$ is the interaction Lagrangian.

[17)]The results of Refs. 79–81 can be viewed only as rough estimates.

[18)]In the 1952 and 1954 experiments the ratio of the decay widths of para- and ortho-positronium $\Gamma_2(p-\mathrm{Ps})/\Gamma_3(o-\mathrm{Ps})$ was measured. The results given in Table V were calculated using the first experimental value:[82] $\Gamma_3(o-\mathrm{Ps})=7.262(15)\mu\mathrm{sec}^{-1}$.

[19)]For positronium only the contribution of electron–positron loops is important, owing to the smallness of m_e.

[20)]These corrections were first found numerically.[127]

[21)]The result of Ref. 41 calculated using the method of the preceding section has been added to that of Ref. 45.

[21*)]Recently,[346] Kinoshita presented a preliminary result of calculations of the last remaining graph of order $\alpha^2(Z\alpha)$ with crossed virtual photons: $\Delta E(\alpha^2(Z\alpha))\cong(-0.64\pm0.06)(\alpha^2(Z\alpha)/\pi)E_F=-0.353(33)$ kHz for Mu. This makes it possible to decrease the theoretical error in (197) to 0.17 kHz.

[22)]Three-photon exchange graphs were not included in this calculation.

[23)]The result of 1961 (Ref. 147) is only an estimate: $\Delta E=2250$–9000 MHz.

[24)]Of course, with the corresponding substitutions $m_{\mu}\to m_p$ and $a_{\mu}\to a_p$.

[25)]The results are given in units relative to E_F.

[26)]The only experimental result known to us on the HFS of the hydrogen level with principal quantum number $n=2$, that of Ref. 172: $\Delta E^{\mathrm{HFS}}(2S,H)=177556.6(3)$ kHz, accurately satisfies the equation obtained by Breit:[131] $\Delta E(2S)=\Delta E(1S)/(8-5\alpha^2)$.

[27)]The calculation of some contributions can also be found in Refs. 200 and 203, but the numerical results given there are very different from the experimental results.

[28)]The first error is statistical, and the second is systematic.

[29)]Experiments with accuracy ~ 10 ppm are needed to check the result of the studies of Fell and Khriplovich *et al.* The result of Ref. 47 was confirmed in Ref. 348.

[30)]The result of Khriplovich *et al.* is given in the first column, and that of Fell in the second.

[31)]The previous experiments gave the following results for positronium:
$\frac{1}{2}\Delta E^{\mathrm{exp}}=\frac{3}{16}cR_{\infty}=41.4(5)$ GHz (Ref. 212),

$\Delta E^{\mathrm{exp}}=1233607185(15)$ MHz$=\frac{3}{8}cR_{\infty}-83545(15)$ MHz (Ref. 213).

$\Delta E^{\mathrm{exp}}=1233607142.9(10.7)$ MHz (Ref. 214),
and for muonium (see the discussion in Ref. 215):
$\frac{1}{4}\Delta E^{\mathrm{exp}}=(613881924\pm30\pm35)$ MHz (Ref. 216).

[32)]The early results on the determination of $2^2S_{1/2}-2^2P_{3/2}$ are reviewed in Refs. 232 and 233.

[33)]The first successful attempt to measure this quantity was made in Ref. 234: $E^{\mathrm{exp}}(LS,1S)=(7.9\pm1.1)$ GHz. See also Ref. 235, where the value of the isotopic splitting of hydrogen was first discussed on the basis of observations of the Lyman series for hydrogen and deuterium.

[34)]The result from Ref. 226 given in this table was obtained using R_{∞} from Ref. 249. Better agreement with the theoretical result is obtained when R_{∞} from Ref. 247 is used: $E(LS,1S)=8174.8(8.7)$ MHz.

[35)]The previous measurement of the proton radius[243] cannot be ignored, because for some experiments it leads to better agreement with the theoretical predictions.

[36)]For other ions ($Z\neq1$) (Refs. 236 and 276).

[37)]See the studies in this area,[277] where the corrections of this order from graphs with polarization insertions in external Coulomb lines and from graphs with radiative insertions in an electron line and with one polarization insertion into a Coulomb line have been calculated.

[38)]The calculation of Ref. 281 was done using a new value of the nuclear radius, 1.673(1) F (Ref. 288), and is in excellent agreement with experiment.

[39)]This result is considerably more accurate than the previous one: $a_{e,\mathrm{VIII}}=(-0.8\pm2.5)$ (Ref. 303).

[40)]The most significant uncertainty (27.1) arises from the uncertainty in the value of the fine-structure constant; the second one arises from the error in the calculations of $a_{e,\mathrm{VI}}^{\mathrm{theor}}$, and the third arises from the error in the calculations of the eighth order $a_{e,\mathrm{VIII}}^{\mathrm{theor}}$.

[41)]The suggestion[305] about a connection between this difference (1.7σ) and the existence of a scalar partner of the electron (and not only with the error in measuring α and the inaccurate numerical integration) is not yet taken seriously.

[42)]The history of the experimental results is given in Ref. 306.

[43)]The contribution $a_{e,\mathrm{IV}}$ was recently calculated again in the Fried–Yennie gauge.[309] The result agrees with the previous one (Refs. 308 and 310–312).

[44)]The fourth-order term $a_{e,\mathrm{IV}}(m_e/m_{\tau})$ due to the existence of graphs with radiative insertion of a τ lepton at the vertex is $(m_{\mu}/m_{\tau})^2$ times smaller than the value (226). The term $a_{e,\mathrm{VI}}(m_e/m_{\tau},m_e/m_{\tau})$ of order $(\alpha/\pi)^3(m_e/m_{\mu})^2(m_e/m_{\tau})^2$ is negligible at the present level of experimental accuracy.

[45)]The result of Samuel,[318] $a_{e,\mathrm{VI}}=1.184(5)$, is presently considered to be too high.

[46)]A preliminary result was given in Ref. 325: $a_{2,\mathrm{VIII}}=140(6)$.

[47)]Upon completing this review, we learned of the remarkable results of Ref. 331, where the $(n+1)$-loop contributions to the muon anomaly were obtained analytically by substitution into the one-loop graphs of all graphs with an n-loop photon propagator containing $(n-1)$ electron loops.

[48)]See also Ref. 332, in which the same technique is used to calculate the eighth-order Callan–Symanzik β function, which is used for the above-mentioned analytic calculations.

[49)]The first uncertainty is the estimate of the theoretical uncertainty mainly from $a_{2,\mathrm{VI}}(m_{\mu}/m_e)$, and the second is the uncertainty due to the error in measuring α by the quantum Hall effect.

[50)]The recently published result is more accurate than that given in the study by Kinoshita: $a_{\mu}(\mathrm{hadrons})=-(6.986\pm0.042\pm0.016)\times10^{-8}$ (Ref. 336). The values of the experimental errors and the model errors are given.

[51)]The quoted uncertainty is due to the absence of information about the Higgs mass.

[52)]The first uncertainty is from the measurement error, the second is from the uncertainty in the sixth-order calculations, and the third is from the uncertainty in the eighth-order calculations. The result in Ref. 302 is three times more accurate than that in Ref. 304, and even more accurate that the results of other methods.

[53)]However, the contribution of this interaction cannot be large, owing to the effect of the factor m/M. The value of a_e^{theor} is more sensitive to the case where there exists an as yet unknown light particle interacting with the electron.[68]

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Translated by Patricia A. Millard