The dispersion-relation method (dedicated to the memory of N. N. Bogolyubov)

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The physical and mathematical foundations of the dispersion-relation method are considered. A decisive contribution to the creation and development of the dispersion-relation method (DRM) was made by Bogolyubov. The paper reviews the contribution of scientists at the Laboratory of Theoretical Physics at Dubna to the development and application of the DRM during the period 1956–1965, when Academician N. N. Bogolyubov was Director of the Laboratory. A review is given of double dispersion relations, Reggeology, the bootstrap method, and superconvergent (finite-energy) sum rules that follow from the assumptions of analyticity of the scattering amplitudes. An understanding of the depth of the physical and mathematical foundations of these methods enables us to evaluate the possibility of using them in quantum chromodynamics.

INTRODUCTION

From the point of view of "common sense," the mathematical definition of causality is relatively simple. Let us consider the case in which by striking the key of a piano we produce an acoustic signal, to which a tuning fork then reacts. Is it possible for the tuning fork to sound earlier, predicting that the piano key will soon be struck? It is obvious that from the point of view of common sense (or our habitual experience) this cannot be! Let $g(t)$ be the response of the tuning fork to our striking the key, and t_0 be the instant of time at which the acoustic signal reaches the tuning fork. It is obvious that $g(t)=0$ for $t<t_0$. Let the function $F(t,t')$ describe the coupling between the tuning fork and the signal $f(t)$ [$f(t)$ represents the blow on the piano key]. We assume that $f(t')=\delta(t'-t)$. We assume that the coupling function $F(t,t')$ is a linear coupling that depends on the time difference $t-t'$; this appears obvious. Then the coupling between the signal $f(t)$ and the response $g(t)$ of the tuning fork can be expressed by

$$g(t)=\int_{-\infty}^{+\infty} F(t-t')f(t')dt' \tag{1}$$

or

$$g(t)=\int_{-\infty}^{+\infty} F(t-t')\delta(t'-t_0)dt'=F(t-t_0),$$

and since $g(t)\equiv 0$ for $t<t_0$, it follows that $F(t-t_0)\equiv 0$ for $t<t_0$.

We apply a Fourier transformation to the expression (1).

Let

$$f(\omega)=\int_{-\infty}^{+\infty} f(t)e^{i\omega t}dt. \tag{2}$$

Substituting (2) in (1), we obtain $g(\omega)=F(\omega)f(\omega)$, where $F(\omega)=\int_{-\infty}^{+\infty}F(\tau)e^{i\omega\tau}d\tau$.

We have found that the tuning fork responds only to the acoustic frequency that is given out by the piano key. However, this is not all. It is obvious that the function $F(\omega)$ is an analytic function of the variable ω in the upper half-plane of ω:

$$F(\omega)=\int_{-\infty}^{+\infty} F(\tau)e^{i(\omega_r+i\omega_i)\tau}. \tag{3}$$

If $\tau>0$, then in the integrand of (3) the cutoff factor $e^{-\omega_i\tau}$ arises.

If $\tau<0$, then $F(\tau)\equiv 0$.

If $F(\omega)$ is an analytic function of the variable ω in the upper half-plane of the complex variable ω, then for this function we can use Cauchy's theorem:

$$F(\omega)=\frac{1}{2\pi i}\int_C \frac{F(\omega')d\omega'}{\omega'-\omega} \quad (\mathrm{Im}\,\omega>0).$$

Lowering the variable ω to the real axis of ω, and using the symbolic identity

$$\lim_{\substack{\varepsilon>0\\ \varepsilon\to 0}} \frac{1}{\omega'-(\omega+i\varepsilon)}=\mathscr{P}\frac{1}{\omega'-\omega}\, i\pi\delta(\omega'-\omega),$$

we obtain

$$F(\omega)=\frac{\mathscr{P}}{\pi i}\int \frac{F(\omega')d\omega'}{\omega'-\omega}, \tag{4}$$

$$F(\omega)=\mathrm{Re}\,F(\omega)+i\,\mathrm{Im}\,F(\omega). \tag{5}$$

Substituting (5) in (4), we obtain

$$\begin{aligned}\mathrm{Re}\,F(\omega)&=\frac{\mathscr{P}}{\pi}\int_{-\infty}^{+\infty}\frac{\mathrm{Im}\,F(\omega')d\omega'}{\omega'-\omega},\\ \mathrm{Im}\,F(\omega)&=-\frac{\mathscr{P}}{\pi}\int_{-\infty}^{+\infty}\frac{\mathrm{Re}\,F(\omega')d\omega'}{\omega'-\omega}.\end{aligned} \tag{6}$$

Equations (6) are called dispersion relations. As we have shown, they follow from the causality principle: $F(t)\equiv 0$ for $t<0$. In Eqs. (6), $F(\omega)$ is the Fourier transform of the function $F(t)$. For functions that possess at infinity poles of nth order (polynomially increasing functions), we can also write down dispersion relations, but in this case it is necessary to make n subtractions:

$$\lim_{\substack{\varepsilon\to 0\\ \varepsilon>0}} \frac{f(E)}{(E-E_0+i\varepsilon)^{n+1}} = \mathscr{P}\left\{\frac{1}{(E-E_0)^{n+1}} - \frac{i\pi(-1)^n}{n!}\delta^n(E-E_0)\right\};$$

then

$$F(E)=\frac{(E-E_0)^{n+1}}{i\pi}\mathscr{P}\int_{-\infty}^{+\infty}\frac{F(E')dE'}{(E'-E)(E'-E_0)^{n+1}} + F(E_0)+\ldots\frac{F^n(E_0)}{n!}(E-E_0)^n.$$

In this case, the dispersion relations are determined up to a polynomial of degree n. This is a consequence of the causality principle, which we have introduced from a "common sense" point of view.

1. ONE-DIMENSIONAL DISPERSION RELATIONS

Does there exist a connection between dispersion relations and quantum field theory? To what extent can one generalize field theory without violating the validity of dispersion relations? Bogolyubov formulated fundamental propositions of field theory under which dispersion relations can be derived, proving in this manner their existence in the framework of quantum field theory.[1] The propositions included requirements of a general nature such as the following: 1) invariance of the amplitudes of states with respect to the transformations L of some group G (besides Lorentz transformations, the group G may include isotopic, gauge, etc., transformations); 2) the existence of asymptotic states of the system of particles such that when the particles are infinitely far apart there is no interaction between them, and also quantities such as the energy, momentum, etc., are additive; 3) the existence of a translation operator U_{L_a} of the single-particle state that, acting on the state vector $|p\rangle$ with definite value of the energy–momentum vector p, can be expressed in the form

$$U_{L_a}|p\rangle = e^{-ipa}|p\rangle;$$

4) the existence of a vacuum state $|0\rangle$ such that

$$U_{L_a}|0\rangle = 0;$$

5) the existence of a unitary operator S describing the transition between the initial and final asymptotic states with the property

$$SS^+ = 1,$$

and some other properties.

Besides these general properties of field theory, there were formulated certain local properties, among which the following causality condition in variational form was decisive:

$$\frac{\delta}{\delta\varphi(x)}\left(\frac{\delta S}{\delta\varphi(y)}S^+\right)=0 \quad \text{for } x\lesssim y,$$

where $x\lesssim y$ means that the point x lies earlier than the point y or is separated from it by a spacelike interval.

To pass from the purely mathematical consequences of the causality condition to dispersion relations possessing physical meaning, we must relate the real and imaginary parts of the function $F(\omega)$ to physically measurable quantities.

Let the function $F(\omega)$ [see Eqs. (5) and (6)] be the scattering amplitude of a physical process. For definiteness, we shall for the moment assume that this is the amplitude for scattering of a pion by a nucleon, and that ω is the energy of the incident pion ($\omega\equiv E$). We write the relation (6) in a different form:

$$\mathrm{Re}\,F(E)=\frac{\mathscr{P}}{\pi}\int_0^{\infty}\frac{\mathrm{Im}\,F(E')dE'}{E'-E} + \frac{\mathscr{P}}{\pi}\int_{-\infty}^{0}\frac{\mathrm{Im}\,F(E')dE'}{E'-E}. \tag{7}$$

The region of negative values of E can be expressed in terms of the complex-conjugate value of the amplitude $F(E)$.

Indeed, the Fourier transform of the function $F(E)$:

$$F(E)=\int e^{iEt}F(t)dt$$

possesses the property

$$F^*(E)=F(-E),$$

from which it follows that

$$\mathrm{Re}\,F(E)=\mathrm{Re}\,F(-E);\quad \mathrm{Im}\,F(E)=-\mathrm{Im}\,F(-E). \tag{8}$$

Thus, the relation (7) can be written in the form

$$\mathrm{Re}\,F(E)=\frac{\mathscr{P}}{\pi}\int_0^{\infty}\frac{\mathrm{Im}\,F(E')dE'}{E'-E} + \frac{\mathscr{P}}{\pi}\int_0^{\infty}\frac{\mathrm{Im}\,F(E')dE'}{E'+E} \tag{9}$$

(in the second integral, the plus sign is replaced by the minus sign four times).

As a result, the expression (9) takes the form

$$\mathrm{Re}\,F(E)=\frac{2\mathscr{P}}{\pi}\int_0^{\infty}\frac{E'\,\mathrm{Im}\,F(E')dE'}{E'^2-E^2}. \tag{10}$$

For charged particles, the properties of charge symmetry and the parity relation for the real and imaginary parts of the amplitude $F(E)$ (8) have the consequence that the region of negative values of the energies for the scattering process $\pi^++p\to\pi^++p$ can be expressed in terms of the positive values of the energies of the physical scattering process $\pi^-+p\to\pi^-+p$. Of course, to write down a dispersion relation for $\pi+p\to\pi+p$ processes, it would be necessary to prove that the scattering amplitude in quantum field theory, in which, in particular, there exist particle creation and annihilation processes, is an analytic function of the complex energy variable E.

The first derivation of dispersion relations in the framework of quantum field theory was proposed by Gell-Mann, Goldberger, and Thirring.[2] They considered the analytic properties of the amplitude for forward scattering (at 0°) of pions by nucleons and studied the problem of analytic continuation with respect to the energy into the upper half-plane under the assumption that the degree of polynomial divergence at infinity is not more than the first. It is convenient to consider the forward scattering amplitude because its imaginary part is expressed in accordance with the optical theorem in terms of the total cross section:

$$\mathrm{Im}\, F(E)=\frac{q\sigma(E)}{4\pi}, \tag{11}$$

$$\begin{aligned}|F(E)|^2&=|\mathrm{Re}\, F(E)|^2+|\mathrm{Im}\, F(E)|^2\\&=|\mathrm{Re}\, F(E)|^2+\left|\frac{q\sigma(E)}{4\pi}\right|^2\\&=\left.\frac{d\sigma}{d\Omega}\right|_{\theta=0^\circ}.\end{aligned} \tag{12}$$

Here, q is the wave vector of the incident pion, and θ is the pion scattering angle in the laboratory system. The only unknown quantity in the relation (12) is $\mathrm{Re}\, F(E)$. It is possible to measure $q\sigma(E)/4\pi$ and $d\sigma/d\Omega|_{\theta=0^\circ}$ experimentally. On the other hand, $\mathrm{Re}\, F(\omega)$ can be calculated by the dispersion-relation method [see (10)]. Thus, comparing $\mathrm{Re}\, F(E)_{\mathrm{exp}}$ with $\mathrm{Re}\, F(E)_{\mathrm{theor}}$, one can verify the dispersion relation and, therefore, verify the basic principles of quantum field theory and, in the first place, the causality principle.

However, in Ref. 2 the proof of analyticity for scattering amplitudes for massive particles was not convincing. In addition, in the dispersion relation (10) the integration with respect to the energy is in the interval $(-\infty,+\infty)$, whereas in a real physical experiment the physical energies are measured in the range of positive values greater than the masses of the particles participating in the reaction. Therefore, in the dispersion relation it is necessary to exclude the unobservable region $0<E<M+\mu$, where μ is the pion rest mass, M is the mass of the nucleon at rest, and E is the energy of the πN system of particles. In this region, bound states may exist. If the bound states form a discrete spectrum, the corresponding integrals can be readily calculated, but if their spectrum is continuous, certain complications arise.

At an international conference of theoretical physicists at Seattle (USA) in 1956, Bogolyubov presented a mathematically rigorous proof of the existence of dispersion relations for the amplitude of pion scattering by nucleons through nonzero angle (up to momentum transfers $q^2=-8\mu^2$).[3] The proof was based on the "edge-of-the-wedge" theorem, which carries Bogolyubov's name. The content of this theorem is that generalized functions (distributions) of several complex variables satisfying definite conditions of growth at infinity can be analytically continued into the upper half-plane of one complex variable (for example, the energy), and for such functions one can write down dispersion relations of physical interest.[1)]

Bogolyubov's studies in the middle of the fifties in quantum field theory, his consistent axiomatic scheme for constructing quantum field theory, and his proof of the existence of dispersion relations had a huge influence on the development of international theoretical physics and brought about a deep shift in priorities, toward rigorous mathematical methods of studying physical phenomena. Bogolyubov's studies raised theoretical elementary-particle physics to a new level of high mathematical culture and rigor of theoretical thinking.

The dispersion relations for the pion–nucleon scattering amplitudes are written not for scalar functions but for matrices in the ordinary spin space and the isotopic spin space. In the isotopic space, the matrix has the form

$$T_{\alpha\beta}(E)=\delta_{\alpha\beta}T^{(1)}(E)+\tfrac{1}{2}[\tau_\alpha,\tau_\beta]T^{(2)}(E),$$

where E is the total energy of the pion in the laboratory coordinate system $[E=(k^2+\mu^2)^{1/2}]$. The amplitudes $T^{(i)}$ have real and imaginary parts:

$$T^{(i)}(E)=\mathrm{Re}\ T^i(E)+\mathrm{i}\ \mathrm{Im}\ T^i(E).$$

Isotopically pure states are related to the amplitudes $T^{(i)}$ by

$$\left.\begin{aligned}T^{3/2}(E)&=T^{(1)}(E)-T^{(2)}(E)\\T^{1/2}(E)&=T^{(1)}(E)+2T^{(2)}(E).\end{aligned}\right\} \tag{13}$$

It follows from (13) that

$$\begin{aligned}T^{(1)}(E)&=\tfrac{2}{3}T^{3/2}+\tfrac{1}{3}T^{1/2},\\T^{(2)}(E)&=\tfrac{1}{3}T^{1/2}-\tfrac{1}{3}T^{3/2}.\end{aligned} \tag{14}$$

The $\pi^++p\to\pi^++p$ scattering amplitude $T^{(+)}$ corresponds to the pure $T^{3/2}$ state; the $\pi^-+p\to\pi^-+p$ scattering amplitude $T^{(-)}$ corresponds to the mixture of states $T^{(-)}=\frac{2}{3}T^{1/2}+\frac{1}{3}T^{3/2}$. By means of the $T^{(+)}$ and $T^{(-)}$ states, we can express the relations (14) in the different form

$$\begin{aligned}T^{(1)}(E)&=\tfrac{1}{2}[T^{(-)}(E)+T^{(+)}(E)],\\T^{(2)}(E)&=\tfrac{1}{2}[T^{(-)}(E)-T^{(+)}(E)].\end{aligned} \tag{15}$$

By the optical theorem

$$\mathrm{Im}\ T^{(\pm)}(E)=\frac{k}{4\pi}\sigma_\pm(E), \tag{16}$$

where $k=(E^2-\mu^2)^{1/2}$, and $\sigma_\pm(E)$ are the total cross sections for scattering of $\pi^\pm$ mesons by protons.

In the ordinary spin space, the structure of the matrix is

$$T_{ss'}(E)=\sigma_{ss'}T_{(1)}+i(\sigma\cdot[\mathrm{p}\times\lambda\mathrm{e}])_{ss'}T_{(2)}. \tag{17}$$

Omitting the details of the proof (which the reader can find in the monograph of Ref. 4), we list the symmetries of the imaginary and real parts of the amplitudes $T_{(i)}^{(k)}$ (i, $k=1$, 2):

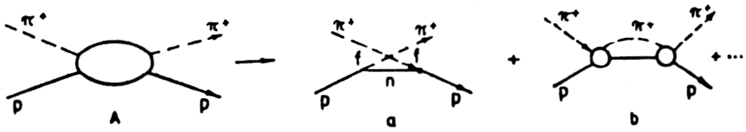

FIG. 1. Graphical representation of the expansion of the $\pi^+ + p \to \pi^+ + p$ scattering amplitude (A) in a series with respect to intermediate states of the $\pi^+ p$ system. Diagram a corresponds to the single-nucleon contribution (the pole term); f is the coupling constant of the πN interaction in perturbation theory. Diagram b describes the two-particle intermediate state and is responsible for the cut with respect to the pion energy: $\mu \leqslant E \leqslant \infty$.

$$\begin{aligned}
&\mathrm{Re}\, T(E)_{\mathrm{even}} = \{\mathrm{Re}\, T^{(1)}_{(1)}(E),\ \mathrm{Re}\, T^{(2)}_{(2)}(E)\},\\
&\mathrm{Re}\, T(E)_{\mathrm{odd}} = \{\mathrm{Re}\, T^{(2)}_{(1)}(E),\ \mathrm{Re}\, T^{(1)}_{(2)}(E)\},\\
&\mathrm{Im}\, T(E)_{\mathrm{even}} = \{\mathrm{Im}\, T^{(2)}_{(1)}(E),\ \mathrm{Im}\, T^{(1)}_{(2)}(E)\},\\
&\mathrm{Im}\, T(E)_{\mathrm{odd}} = \{\mathrm{Im}\, T^{(1)}_{(1)}(E),\ \mathrm{Im}\, T^{(2)}_{(2)}(E)\}.
\end{aligned} \tag{18}$$

Combining the relations (12)–(17), we can write down dispersion relations for $\pi^\pm + p \to \pi^\pm + p$ scattering at zero angle in the form[1]

$$\begin{aligned}
\mathrm{Re}\, T^{(\pm)}(E) = {}& \frac{1}{2}\left(1+\frac{E}{\mu}\right)\mathrm{Re}\, T^{(\pm)}(\mu) + \frac{1}{2}\left(1-\frac{E}{\mu}\right)\\
&\times \mathrm{Re}\, T^{(\mp)}(\mu)\\
&+ \frac{k^2}{4\pi^2}\int_\mu^\infty \frac{dE'}{k'}\left[\frac{\sigma_\pm(E')}{E'-E} + \frac{\sigma_\mp(E')}{E'+E}\right]\\
&+ \frac{2f^2}{\mu^2}\,\frac{k^2}{E \mp \dfrac{\mu^2}{2M}}.
\end{aligned} \tag{19}$$

In the dispersion relation (19), there is no longer any integration over the unphysical region of energies. The final term in (19) is the contribution of the single-nucleon intermediate state to the real part of the amplitudes $\mathrm{Re}\, T^{(\pm)}(E)$ (see Fig. 1). From the relation (19), the theoretical value $\mathrm{Re}\, T^{(\pm)}(E)$ can be determined. The values of $\mathrm{Re}\, T^{(\pm)}(\mu)$ can be calculated in terms of the $\pi^+ + p \to \pi^+ + p$ and $\pi^- p \to \pi^- p$ scattering lengths. The constant f can be determined from the condition of best agreement with the experimental data. The values of $\sigma_\pm(E)$ are taken from the experimental data in the experimentally accessible energy ranges (and within the experimental errors). The high-energy region can be approximated by reasonable theoretical estimates. The theoretical value of $\mathrm{Re}\, T^\pm(E)$ found from (19) can be compared with $\mathrm{Re}\, T^\pm(E)$ calculated using the experimental data in accordance with the expression (12).

The main uncertainty in the dispersion relation is in the behavior of the total cross sections $\sigma_\pm(E)$ at infinitely large values of the energy. It was usually assumed that the total cross sections of πN processes tend to a constant at infinity. As a rule, it was found that the uncertainty in the integrated contribution from this part was 5% of the contribution from the low-energy region.

Tests of the dispersion relation showed that in a wide range of energies good agreement between the theoretical $\mathrm{Re}\, T(E)$ curves and the experimental data can be obtained for $f^2 = 0.08$ [$f^2 = (1/4\pi)(g\mu/M)^2$, where g is the renormalized coupling constant of the meson–nucleon interaction]; see, for example, Ref. 5.

An important role in the testing of dispersion relations was played by Pomeranchuk's theorem, which states that the total cross sections for interaction of particles and antiparticles are equal at infinitely high energies.[6] In accordance with this theorem, we have

$$\begin{aligned}
&\sigma_-(\infty) = \sigma_+(\infty)\\
&(\text{for example, } \sigma(\pi^- p \to \pi^- p) = \sigma(\pi^+ p \to \pi^+ p),\\
&E \to \infty;\\
&\sigma(p\bar{p}) = \sigma(pp),\quad E \to \infty;\ \text{etc.}).
\end{aligned} \tag{20}$$

Whenever a new elementary-particle accelerator was commissioned, the theorem was tested at the highest energies accessible through it. Tests of the theorem using the 76-GeV accelerator at the Institute of High Energy Physics (Protvino) led to the discovery of the so-called Serpukhov effect. We shall return to this phenomenon later.

At the present time, the dispersion-relation method and Pomeranchuk's theorem will be tested on new more powerful accelerators such as the LHC at CERN and the SSC (Texas, United States).

Soon after the derivation of the dispersion relations for πN scattering, dispersion relations for nucleon–nucleon scattering were derived.[7] The dispersion relations obtained in Ref. 7 could not be compared with experiment, since they contained unobservable quantities or unknown quantities associated with antinucleon–nucleon scattering. In Ref. 8, approximate dispersion relations that did permit comparison with experimental data were obtained. Later, dispersion relations for nucleon–nucleon scattering were obtained by some other authors. The problem of the scattering of protons by nucleons was analyzed in detail from the experimental point of view in the review of Ref. 9, and experimental data were compared with the predictions of the theory, with the predictions of the dispersion relations.

The process $\gamma + N \to \pi + N$ of pion photoproduction on nucleons was investigated in very great detail. In Refs. 10–13, dispersion relations were obtained for the first time for pion photoproduction, and the most general properties of the $\gamma + N \to \pi + N$ process were investigated. Dispersion relations were expressed in a form convenient for experimental tests. This confirmed the behavior due to the pole term of the dispersion relations found earlier in Refs. 14 and 15 in the weak-coupling approximation. It may be added that a pseudoscalar form of meson theory was established in Refs. 14 and 15. Later, pion photoproduction on nucleons was described by means of double dispersion relations,[16,17] which we shall discuss in the following section of the review. The series of studies on pion photoproduction made by the JINR theoreticians A. M. Baldin, A. A. Logunov, L. D. Solov'ev, and A. N. Tavkhelidze was awarded a State Prize of the USSR in 1973. Evaluating the scientific significance of this series, Academicians N. N. Bogolyubov and B. M. Pontecorvo wrote: "... The authors have, for the first time, formulated and proved, on the basis of fundamental principles of quantum field theory, dispersion relations for meson photoproduction. On this basis, it has been possible to relate physical properties of pion photoproduction to the properties of the strong interaction of pions with nucleons and to obtain reliable quantitative results for photoproduction processes in a fairly wide range of energies. By virtue of this, the foundations of the theoretical description of photoproduction processes have been laid. Careful experimental testing of the dispersion relations has confirmed the validity of the fundamental physical principles of the theory for the given range of energies ..."

Dispersion relations for photon scattering by nucleons were also derived by a number of authors.[18] A rigorous proof of the existence of a dispersion relation for the Compton effect on nucleons, valid for photon scattering through any angle and obtained by Bogolyubov's method, was given in Ref. 19. The lowest approximation in e^2 was considered, and only strong interactions were taken into account. Dispersion relations for electron bremsstrahlung on nucleons and pair production by photons on nucleons were used in order to test quantum electrodynamics at short distances.[20] Dispersion relations for weak interactions (for neutrino scattering by nucleons) were postulated in Ref. 21.

2. DOUBLE DISPERSION REPRESENTATIONS

To compare the dispersion relations with experimental data, very limited experimental material was needed—the total interaction cross sections and the differential cross sections for scattering at zero angle. This was not satisfactory for physicists.

Mandelstam's double dispersion representations[22] removed the limitations, but the existence of the representations was not proved in the framework of field theory. Mandelstam's double representations postulate analytic properties of the scattering amplitudes with respect to two complex variables—the energy and the momentum transfer. In principle, for fixed momentum transfer they go over into ordinary one-dimensional dispersion relations. Essentially, the Mandelstam representations arise, as it were, from the assumption that the imaginary parts of the scattering amplitude that occur in the integrand in ordinary dispersion relations have, in their turn, additional cuts along the real values of the energies and momenta. Later, it was shown in the framework of perturbation theory that in some restricted ranges of variation of these variables such analyticity does exist.

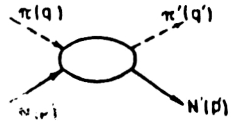

FIG. 2. Generalized Feynman diagram representing scattering of the π meson by the nucleon.

We consider the scattering of a pion by a nucleon. We represent the generalized Feynman diagram for this process in the form shown in Fig. 2. All the 4-momenta q, q', p, p' lie on the mass shell. The Green's function, which depends on the variables q, q', p, p', can describe three processes simultaneously:

$$\begin{aligned} &\mathrm{I}\quad \pi+N\rightarrow\pi'+N' \\ &\mathrm{II}\quad \bar{\pi}'+N\rightarrow\bar{\pi}+N' \\ &\mathrm{III}\quad \pi+\bar{\pi}'\rightarrow\bar{N}+N'. \end{aligned} \tag{21}$$

One introduces three invariant variables:

$$\begin{aligned} &s=(q+p)^2,\quad u=(q+p')^2,\quad t=(q+q')^2, \\ &s+u+t=2M^2+2\mu^2, \end{aligned} \tag{22}$$

where M is the nucleon mass, and μ is the pion mass.

For process I [see (21)], s is the energy variable, and t is the 4-dimensional momentum transfer. In the center-of-mass system of process I, the variables s, u, t have the form

$$s=M^2+\mu^2+2\mathbf{q}^2+2\sqrt{(q^2+\mu^2)(q^2+M^2)},$$

$$u=M^2+\mu^2-2\mathbf{q}^2\cos\theta_1-2\sqrt{(q^2+\mu^2)(q^2+M^2)},$$

$$t=-2\mathbf{q}^2(1-\cos\theta_1).$$

Here, $\mathbf{q}$ is the 3-dimensional momentum of the pion, and θ_1 is the pion scattering angle in the first channel (in process I).

For the second channel (process II), the variables s and u are interchanged—this is the usual property of crossing symmetry. The third channel (reaction III) is annihilation of pions into a nucleon–antinucleon pair; in it, t [see the definition (22)] is an energy variable, while s plays the role of a momentum transfer. Here, the substitution $s\rightleftarrows t$ is not the ordinary property of crossing symmetry but an allowed substitution rule. The only justification for the operation is the fact that one and the same Green's function describes all three channels. In the case of equal masses of all four particles, the substitution rule is transformed into exact crossing symmetry.

The crossing-symmetry and substitution properties and the physical and unphysical regions of the processes I–III are shown in transparent form in Fig. 3.

Mandelstam's dispersion representation has the rather cumbersome form

$$F(s,u,t)=\frac{1}{\pi}\int_{s_0}^{\infty}\frac{\mathrm{Im}\,F_s(s')ds'}{s'-s}$$

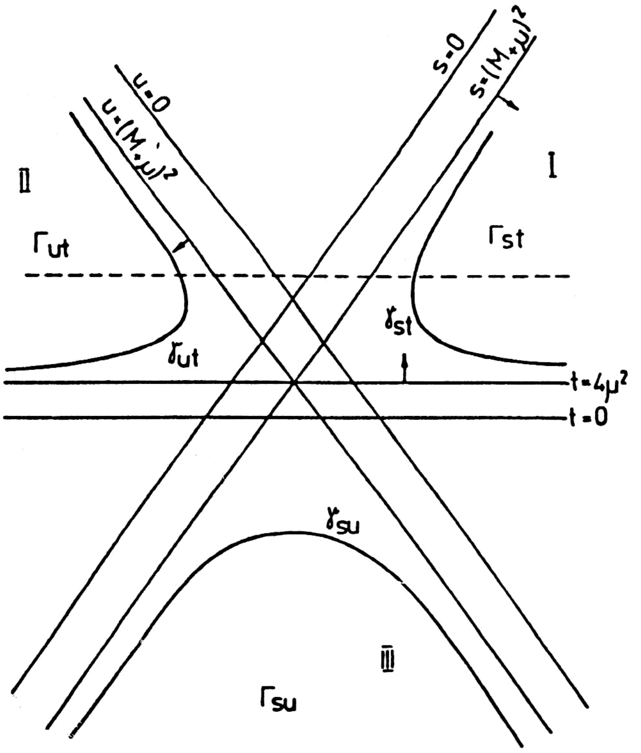

FIG. 3. The physical region of channel I corresponds to $S\geqslant(M+\mu)^2$, $u<0$, $t<0$, of channel II to $u\geqslant(M+\mu)^2$, $s<0$, $t<0$, and of channel III to $t\geqslant 4M^2$, $s<0$, $u<0$ (an arbitrary scale has been adopted in the figure).

$$+\frac{1}{\pi}\int_{u_0}^{\infty}\frac{\operatorname{Im}F_u(u')du'}{u'-u}$$

$$+\frac{1}{\pi}\int_{t_0}^{\infty}\frac{\operatorname{Im}F_t(t')dt'}{t'-t}$$

$$+\frac{1}{\pi^2}\int_{\gamma_{su}}\frac{\Gamma_{su}(s',u')ds'du'}{(s'-s)(u'-u)}$$

$$+\frac{1}{\pi^2}\int_{\gamma_{st}}\frac{\Gamma_{st}(s',t')ds'dt'}{(s'-s)(t'-t)}$$

$$+\frac{1}{\pi^2}\int_{\gamma_{ut}}\frac{\Gamma_{ut}(u',t')du'dt'}{(u'-u)(t'-t)}. \qquad (23)$$

The limits of integration γ_{su}, γ_{st}, γ_{ut} are shown schematically in Fig. 3; $\operatorname{Im}F_s$, $\operatorname{Im}F_u$, $\operatorname{Im}F_t$ are the imaginary parts of the amplitude $F(s,u,t)$ corresponding to the imaginary parts of the real physical processes in the first, second, and third channels of the reaction. The functions Γ_{st}, Γ_{su}, Γ_{ut} are called spectral functions. They are real and describe the analytic properties of the scattering amplitude with respect to the two complex variables. An example of a double spectral representation is discussed in the monograph of Ref. 23 for a superposition of Yukawa potentials:

$$V=V_0\int_a^{\infty}\frac{f(z)e^{-zr}}{r}dz.$$

On the transition to the one-dimensional case, the Mandelstam representations go over exactly into ordinary dispersion relations.

We choose the energy variable s [see (22)]. In this case, there remain two integrals along the cuts in the s and u channels and a certain subtraction constant:

$$F(s,u,t)=F_0(u,t)+\frac{1}{\pi}\int\frac{\operatorname{Im}\tilde{F}_s(s')ds'}{s'-s}+\frac{1}{\pi}\int\frac{\operatorname{Im}\tilde{F}_u(u')du'}{u'-u},$$

where $\operatorname{Im}\tilde{F}_s(s')$ will contain, besides the integral of $\operatorname{Im}F_s(s')$, a combination of double integrals of Γ_{su} and Γ_{st}, and $\operatorname{Im}\tilde{F}_u(u')$ will contain, besides the integral of $\operatorname{Im}\tilde{F}_u(u')$, a combination of double integrals of Γ_{us} and Γ_{ut}.

The imaginary parts of the amplitude $F(s,u,t)$ can be determined from the unitarity condition for the S matrix:

$$SS^+=1, \qquad (24)$$

$$\langle a|S|b\rangle=\delta_{ab}+i\langle a|T|b\rangle, \qquad (25)$$

where $\langle a|$ and $|b\rangle$ are the final and initial states, respectively, and $\langle a|T|b\rangle$ is the matrix element of the transition from the state $|b\rangle$ to the state $|a\rangle$.

Using the relations (24) and (25), we obtain

$$\operatorname{Im}\langle a|T|b\rangle=\tfrac{1}{2}\sum_n\langle a|T^+|n\rangle\langle n|T|b\rangle,$$

where Σ_n is the sum over all the intermediate states that are allowed by the conservation laws and the other requirements of field theory. In the language of Feynman diagrams, the unitarity condition takes a transparent form (see Figs. 4 and 5).

In the s channel, the cut begins at $s=(M+\mu)^2$, the next cut begins at $s=(M+2\mu)^2$, etc. In the t channel, the physical cut of the $\pi\pi\to N\bar{N}$ reaction must begin at $t=4M^2$. However, it begins earlier, at $t=4\mu^2$; the next cut begins at $t=16\mu^2$. There is then a cut from the $\pi\pi\to K\bar{K}$ reaction at $t=4M_K^2$, etc. The range of values $4\mu^2\leqslant t\leqslant 4M^2$ is unphysical (for the $\pi\pi+N\bar{N}$ process) and is regarded as the analytic continuation of the physical region $t\geqslant 4M^2$ to the unphysical region $4\mu^2\leqslant t<4M^2$. Accordingly, the unitarity condition in the region $4\mu^2\leqslant t<4M^2$ is also regarded as the analytic continuation of the unitarity condition from the region $t\geqslant 4M^2$.

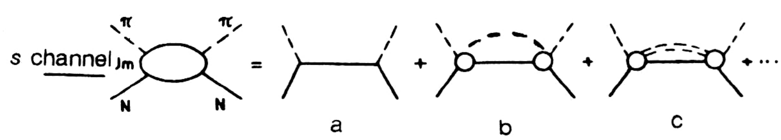

FIG. 4. Graphical representation of the unitarity condition for $\pi N\to\pi N$ scattering in the s channel. Diagram a represents the single-nucleon state (the pole term), diagram b represents the two-particle intermediate state, and diagram c the three-particle intermediate state.

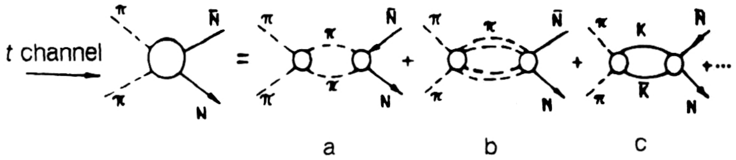

FIG. 5. Unitarity condition in the t channel (i.e., for the $\pi\pi\to N\bar{N}$ process): a) 2π-meson intermediate state; b) 4π-meson intermediate state; c) $K\bar{K}$-intermediate state. The 3π-meson intermediate state is forbidden (because of the different parities of the 2π- and 3π-meson states in the initial and intermediate states, respectively; this is the analog of the Furry theorem in QED).

A distinctive feature of the double Mandelstam representation is that under certain assumptions it leads to a system of equations that includes different physical processes and can be written down not only for the amplitudes themselves but also for the partial waves of the considered processes, since as variables we now have both the energy of the incident particle (or the total energy of the system of particles) and the scattering angle.

At the end of the fifties and the beginning of the sixties, the energies of particle accelerators were not yet very high, and it was natural to assume that the main role in the scattering processes would be played by the lowest partial waves: *s, p*, and, perhaps, *d* waves. Using the unitarity condition for the partial waves, one could write down a system of nonlinear singular integral equations that included the partial waves of different processes. For example, for the process of πN scattering, the system contains not only the partial waves of this process but also the lowest partial waves of the process of $\pi\pi\to\pi\pi$ scattering and the $\pi\pi\to K\bar{K}$ and $\pi\pi\to N\bar{N}$ annihilation processes. The only possible exception is the system of equations for the partial waves of $\pi\pi$ scattering, which may be approximately closed, since the contribution of the partial waves from the $\pi\pi\to K\bar{K}$ or $\pi\pi\to N\bar{N}$ annihilation processes may be negligibly small because the physical cuts from these processes begin very far away ($t\geqslant 4M_K^2$ or $t\geqslant 4M_N^2$) compared with the beginning of the physical threshold of the $\pi\pi\to\pi\pi$ scattering reaction ($t\geqslant 4\mu^2$).

The scheme of successive inclusion of strong-interaction processes into the Mandelstam representations has a kind of hierarchy, which is shown in Fig. 6. If we stop at this level of understanding, we may note that the Mandelstam representations in the low-energy region are based on a number of assumptions: 1) the postulate of analyticity; 2) the crossing substitution in the third channel is included in the concept of crossing symmetry; 3) a restriction to two-particle unitarity is introduced; 4) a restriction on the number of partial waves is introduced.

It can be shown[23] that the first, second, and third conditions are incompatible, and one must exercise a certain mathematical care to ensure that the approximate system of equations is consistent.

At the Laboratory of Theoretical Physics at the JINR, under the leadership of Academician Bogolyubov and his students, Academician A. A. Logunov and D. V. Shirkov (corresponding member of the Russian Academy of Sciences), much work was done on the theoretical foundations of one-dimensional dispersion relations and the double representations of Mandelstam. The analytic properties of the Feynman diagrams of the processes was studied in order to obtain dispersion relations and spectral representations for the amplitudes of the considered processes by means of the method of majorizing Feynman diagrams.[24] Since convergence of the complete perturbation-theory series was not proven, the domain of analyticity of the complete perturbation-theory series remained unknown. One could only speak of analyticity in any finite order of perturbation theory. A semiphenomenological theory of strong interactions was developed in the low-energy region on the basis of the idea of short-wavelength repulsion.[25] Many studies were devoted to application of the Mandelstam representations to the description of observed physical processes such as pion scattering by nucleons, or pion photoproduction on nucleons, and also to the study of pion and nucleon form factors, etc. In those years, the Laboratory of Theoretical Physics at the JINR was one of the leading theoretical centers in the world.

$\pi\pi\to\pi\pi$ scattering. In 1959, Chew and Low[26] pointed out the possibility of using unstable particles as targets. Let us consider pion production in πN collisions:

$$\pi+N\to\pi+\pi+N. \tag{26}$$

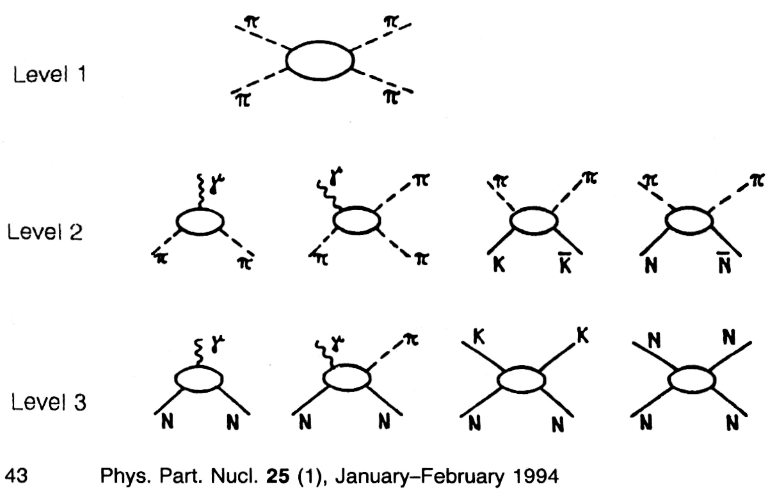

FIG. 6. Hierarchy of interactions. The first level is the process of $\pi\pi$ scattering, which by virtue of the unitarity condition occurs in all the processes of the second and third levels. The second level also occurs by virtue of the unitarity condition in the processes of the third level.

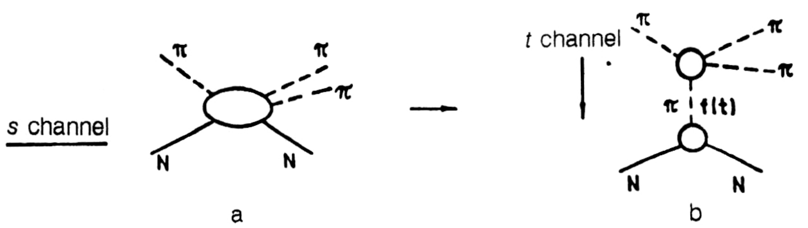

FIG. 7. Pion production in πN collisions: a) in the general case; b) in the case of a peripheral collision.

At small momentum transfers, it can be assumed that the incident pion interacts with the nucleon peripherally, i.e., with the pion cloud of the nucleon (I do not see how in the quark–gluon model of a nucleon one can distinguish peripheral collisions from central ones!). Under the assumption of a peripheral interaction, the process (26) can be represented by the Feynman diagram shown in Fig. 7. The amplitude of the process shown in Fig. 7 can be expressed in the form

$$F(s,t) \sim \frac{f(t)T(\pi\pi \to \pi\pi)}{t-\mu^2}.$$

Here, the pole $t=\mu^2$ corresponds to pion exchange; μ is the pion mass; $f(t)$ is the π-meson form factor of the nucleon; $\lim f(t) = g_{\pi\pi N}$ is the pion–nucleon coupling constant; and $T(\pi\pi \to \pi\pi)$ is the amplitude for π-meson scattering by a virtual π meson in which we are interested. For the partial waves of the $\pi\pi \to \pi\pi$ amplitude, we write down Mandelstam representations and find the energy dependence of the $\pi\pi$ scattering phase shifts.

The invariant amplitude of $\pi\pi$ scattering in the isotopic space can be represented as a combination of amplitudes corresponding to the values $I=0$, 1, 2 of the total isotopic spin: A_0, A_1, A_2. Expansion of these amplitudes in partial waves has the consequence that the amplitudes with even value of the isotopic spin (A_0,A_2) contain only even partial waves, and the amplitude A_1 contains only odd partial waves. The condition of crossing symmetry (the substitutions $s \rightleftarrows u$, $s \rightleftarrows t$, $u \rightleftarrows t$) is here satisfied exactly (because of the equality of the masses of the particles participating in the reaction), and it mixes the isotopic amplitudes of the different channels. Without going into the mathematical details (which the reader can find in the monograph of Ref. 23), we note that for the partial waves of the $\pi\pi$ system it is possible to write down a system of nonlinear singular integral equations. The system is nonlinear because in the unitarity condition

$$\operatorname{Im} A_i(E) = k(E)\,|A_i(E)|^2 \tag{27}$$

[here, $k(E)$ is some kinematic factor] the imaginary part of the partial-wave amplitude is proportional to the square of the amplitude, and in the integrand there are squares of the corresponding partial-wave amplitudes (or squares of the amplitudes themselves if the Mandelstam representations are written down for the amplitudes themselves). The system of equations for the partial waves of $\pi\pi$ scattering was investigated in detail in Refs. 26–30. On the other hand, the experimentalists had developed a method for separating the partial waves of the $\pi\pi$ scattering process from experimental data[31] on the process (26). Comparison of the theoretical calculations and the experimental data on the energy dependence of the $\pi\pi$ scattering phase shifts led to remarkably good agreement. The general behavior of the $\pi\pi$ phase shifts in the region of low energies could be characterized as follows:

a) There was a large s wave in the state with isotopic spin $I=0$.

b) There were strong indications of the presence of a resonance in the p wave of $\pi\pi$ scattering in the state with isotopic spin $I=1$. This was necessary to give a satisfactory explanation of the behavior of the electromagnetic form factor as a function of the momentum transfer q^2 (this resonance was subsequently called the ρ meson).

c) In the region of energies $\lesssim 1$ GeV of the incident pion, the contribution of the partial waves higher than the d wave was small. The attempt to take into account an infinite number of partial waves in order to obtain exact equations was meaningless in view of the fact that we did not know how to take into account the contributions from the inelastic processes, and in any case was hardly possible from the mathematical point of view in view of the presence of the spectral functions, leading to the summation of series of a definite kind.[32] This general picture of the $\pi\pi$ interaction in the region of energies $\lesssim 1$ GeV has remained unchanged to the present day.

A similar study was made of a more complicated type of interaction—the $K\pi \to K\pi$ scattering process.[33–35] This included the $K\bar{K} \to \pi\pi$ annihilation channel and the $\pi\pi \to \pi\pi$ scattering process. The solution of the equations for the partial waves of the $K\bar{K} \to \pi\pi$ annihilation process was reduced to the solution of an inhomogeneous Riemann boundary-value problem:[36]

$$f_l(E+\mathrm{i}0) = G_l(E) f_l(E-\mathrm{i}0) + 2\mathrm{i}g_l(E).$$

The coefficient $G_l(E)$ of the Riemann problem is determined by the choice of the $\pi\pi$ phase shifts, and the inhomogeneous term $g_l(E)$ of the Riemann problem is found from the $K\pi$ scattering process.

$\pi N \to \pi N$ scattering. The process of elastic πN scattering (Fig. 8) was subjected to a complete comprehensive investigation by means of Mandelstam representations. The presence of different masses of the particles that participate in the process (as in $K\pi \to K\pi$ scattering) and the existence of the nucleon spin lead to a more complicated kinematics of the process, a more complicated structure of the amplitude of the process, and a more complicated system of singular integral equations than for the processes of $\pi\pi \to \pi\pi$ and $K\pi \to K\pi$ scattering.

The amplitude $T(\pi N \to \pi N)$ in the isotopic space has the form

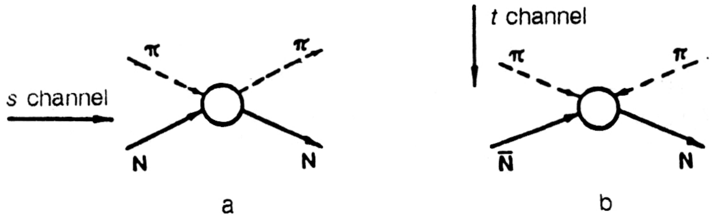

FIG. 8. The process of elastic πN scattering in the s channel corresponds to the process of $\pi\pi\to N\bar{N}$ annihilation in the t channel. This last process includes, in particular, the amplitude of $\pi\pi\to\pi\pi$ scattering.

$$T_{\alpha_1 n_1,\alpha_2 n_2}=\delta_{\alpha_1\alpha_2}\delta_{n_1 n_2}T^{(+)}+\tfrac{1}{2}[\tau_{\alpha_1},\tau_{\alpha_2}]T^{(-)}, \tag{28}$$

where α_1 and α_2 are the isotopic indices of the pion, n_1 and n_2 are the isotopic indices of the nucleon, and $T^{(\pm)}$ are scalar functions in the isotopic space and matrices in the spin space [in Eqs. (12)–(15), the spin structure is not considered].

We introduce notation for the physically observable processes:

$$\begin{aligned}
&(\mathrm{a})\quad T^{++}_{++}\to\pi^+ +p\to\pi^+ +p,\\
&(\mathrm{b})\quad T^{--}_{++}\to\pi^- +p\to\pi^- +p,\\
&(\mathrm{c})\quad T^{-0}_{+0}\to\pi^- +p\to\pi^0 +n,\\
&(\mathrm{d})\quad T^{+0}_{0+}\to\pi^+ +n\to\pi^0 +p,\\
&(\mathrm{e})\quad T^{--}_{00}\to\pi^- +n\to\pi^- +n.
\end{aligned} \tag{29}$$

Processes in which the neutral π^0 meson is the incident pion are physically unobservable. They include

$$\begin{aligned}
&(\mathrm{f})\quad \pi^0+p\to\pi^+ +n,\\
&(\mathrm{g})\quad \pi^0+p\to\pi^0 +p,\\
&(\mathrm{h})\quad \pi^0+n\to\pi^0 +n,\\
&(\mathrm{i})\quad \pi^0+n\to\pi^- +p.
\end{aligned}$$

The physically observable processes (a), (b), and (c) can be expressed in terms of amplitudes with a given value of the isotopic spin of the πN system ($I=3/2$ or $1/2$) as follows:

$$T^{++}_{++}=T^{3/2},$$

$$T^{--}_{++}=\frac{1}{3}\,T^{3/2}+\frac{2}{3}\,T^{1/2},$$

$$T^{-0}_{+0}=\frac{\sqrt{2}}{3}\,(T^{3/2}-T^{1/2}),\ \text{etc.}$$

In their turn, the amplitudes $T^{(+)}$ and $T^{(-)}$ can be expressed in terms of the processes (a)–(i).

In the ordinary spin space, each of the matrices $T^{(\pm)}$ can be expressed in the form

$$T^{(\pm)}=A^{(\pm)}+\frac{\hat{q}_1+\hat{q}_2}{2}\,B^{(\pm)}, \tag{30}$$

where $\hat{q}_1=q_1^0\gamma^0-\mathbf{q}_1\gamma$, $\hat{q}_2=q_2^0\gamma^0-\mathbf{q}_2\gamma$, q_μ is the 4-vector of the incident pion, and γ^μ are the Dirac matrices. The πN scattering amplitudes (17) and (30) are expressed in different forms, but each of them is convenient for the respective purposes.

The kinematic variables s, u, t for the s channel of πN scattering in the center-of-mass system of the πN system have the form

$$s=M^2+\mu^2+2\mathbf{q}^2+2p^0q^0,$$

$$u=M^2+\mu^2-2\mathbf{q}^2\cos\theta-2p^0q^0,$$

$$t=-2q^2(1-\cos\theta),$$

where $\mathbf{p}$ and p^0 are the components of the nucleon 4-vector, $\mathbf{q}$ and q^0 are the components of the pion 4-vector, and $\cos\theta$ is the pion scattering angle [see also Eqs. (22) and below].

The original amplitude $T_{\alpha_1 n_1,\alpha_2 n_2}$ satisfies the condition of crossing symmetry

$$T_{\alpha_1 n_1,\alpha_2 n_2}(q_1p_1;q_2p_2)=T_{\alpha_2 n_1,\alpha_1 n_2}(p_1,-q_2\,;p_2,-q_1),$$

which corresponds to interchange of the π mesons (or the substitution $s\rightleftarrows u$). The spin structure functions $A^{(\pm)}$ and $B^{(\pm)}$ behave under the substitution $s\rightleftarrows u$ as follows:

$$A^{(\pm)}(s,u)=\pm A^{(\pm)}(u,s);\quad B^{(\pm)}(s,u)=\mp B^{(\pm)}(u,s).$$

The transition to dispersion relations for the partial waves is made by means of the chain of relations given below. The expression for the differential cross section is

$$\frac{d\sigma}{d\Omega}=\sum\left|\left\langle f\left|f_1+\frac{(\sigma_1\mathbf{q}_1)(\sigma_2\mathbf{q}_2)}{q_1q_2}\,f_2\right|i\right\rangle\right|^2,$$

where i and f are the initial and final states for two-row Pauli spinors, Σ denotes a summation over the final spin variables and an average over the initial states, and f_1 and f_2 are the amplitudes with given helicity introduced by Jacob and Wick. They can be expanded in a partial-wave series with the appropriate value of the total angular momentum $J=l\pm 1/2$ as follows:

$$f_1(s,\theta)=\sum_l\,[f_{l_+}p'_{l+1}(\cos\theta)-f_{l\_}p'_{l-1}(\cos\theta)],$$

$$f_2(s,\theta)=\sum_l\,[f_{l\_}-f_{l_+}]p'_l(\cos\theta),$$

where

$$f_{l\pm}=\frac{1}{2iq}\,(e^{2i\delta_{l\pm}}-1);$$

and the indices $\pm$ correspond to the sign of the total angular momentum $J=l\pm 1/2$. The isotopic index $I=3/2$,

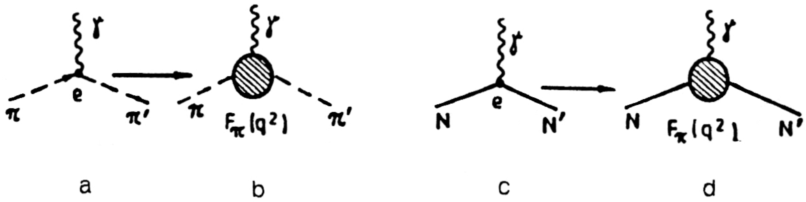

FIG. 9. Electromagnetic interaction of a photon with a pion in the lowest order of perturbation theory (a). b) The same as in case (a) but with allowance for the strong interactions of the pion. c) Electromagnetic interaction of a photon with a nucleon in the lowest order of perturbation theory. d) The same as in case (c) but with allowance for the strong interactions of the nucleon.

1/2 is omitted (to simplify the expression of the idea of the calculations). Finally, we establish the connection

$$\frac{1}{4\pi}A=\frac{W+M}{p^0+M}f_1-\frac{W-M}{p^0-M}f_2,$$

$$\frac{1}{4\pi}B=\frac{1}{p^0+M}f_1-\frac{1}{p^0-M}f_2,$$

where $W=p^0+q^0$ is the total energy of the πN system. The Mandelstam representations are postulated for the amplitudes $A^{(\pm)}$ and $B^{(\pm)}$, and from them we go over, in accordance with the chain of relations given above, to nonlinear singular integral equations for the partial waves. Usually, a restriction is made to the lowest partial waves—the s and p waves, and, at the most, d waves. The kernels of the equations contain the phase shifts of the lowest partial waves of $\pi\pi$ scattering. The relativistic system of equations for πN scattering has a very cumbersome form even for two partial (s and p) waves. It does admit numerical solution, but because it is so cumbersome there is no point in giving either the system itself or its solution. As an illustration, we give the system of equations in the static limit, for which in the relativistic system of equations one makes an expansion in powers of $1/M$, where M is the nucleon mass, and in the integrands one retains only one large p wave describing the 33 resonance of the πN system.[37] In this case, the system of equations for the s and p waves has the form

$$\mathrm{Re}\, f_s^{(-)}(\nu)=a^-\omega F_\pi(\nu),$$

$$\mathrm{Re}(f_{p_{1/2}}^{(-)}(\nu)-f_{p_{3/2}}^{(-)}(\nu))$$
$$=\frac{\nu(a_1^- - a_3^-)F_\pi(\nu)}{2}-\frac{\nu}{2\pi}\int_0^\infty \frac{\nu'+\nu}{\nu'-\nu}\frac{\mathrm{Im}\, f_{p_{3/2}}^{(-)}(\nu')d\nu'}{\nu'^2}$$
$$-\frac{\nu^2}{2\pi}\int_0^\infty \frac{\mathrm{Im}\, f_{p_{3/2}}^{(-)}(\nu')}{\nu'^2(\nu'-\nu)}\left[\frac{F_\pi(\nu)}{F_\pi(\nu')}-1\right]d\nu', \tag{31}$$

$$\mathrm{Re}(f_{p_{1/2}}^{(-)}(\nu)+2f_{p_{3/2}}^{(-)}(\nu))=-2\frac{\nu}{\omega}f^2+a^-\omega[1-F_\pi(\nu)]$$
$$+\frac{2\nu\omega}{\pi}\int_0^\infty \frac{\mathrm{Im}\, f_{p_{3/2}}^{(-)}(\nu')d\nu'}{\nu'(\nu'-\nu)\omega}.$$

In deriving Eqs. (31), we have made one subtraction at the point $\nu=0$. In the expressions (31), $\nu=q^2$ is the square of the momentum in the πN center-of-mass system, $F_\pi(\nu)$ is the pion electromagnetic form factor, $\omega^2=\nu+\mu^2$ is the pion mass, a^-, a_1^-, a_3^- are the scattering lengths in the s, $p_{1/2}$, and $p_{3/2}$ waves, respectively, and $f^2=(1/4\pi)(q\mu/M)^2=0.08$ is the coupling constant of the πN interaction. Comparison of the theoretically calculated s and p waves with the experimental data gives reasonable agreement for the energy behavior of the phase shifts.

The contribution of the $\pi\pi$ interaction to the partial waves of πN scattering in the static limit enters additively and possesses certain symmetry properties.[38] We denote by $G_{lJ}^{(\pm)}$ the partial contribution of the $\pi\pi$ terms to the πN scattering. For $G_{lJ}^{(\pm)}$ there exist a number of symmetry relations [the symbols ($\pm$) of $G_{lJ}^{(\pm)}$ correspond to the symbols ($\pm$) of $A^{(\pm)}$ and $B^{(\pm)}$; see (30)]:

$$G_s^{(\pm)}(W)=-3G_{p_{3/2}}^{(\pm)}(W),$$

$$G_{p_{3/2}}^{(+)}(W)-G_{p_{1/2}}^{(+)}(W)=\frac{p^0-M}{p^0+M}G_s^{(+)}(W),$$

$$W[G_{p_{1/2}}^{(+)}(W)-G_{p_{3/2}}^{(+)}(W)]=W'[G_{p_{1/2}}^{(+)}(W')$$
$$-G_{p_{3/2}}^{(+)}(W')],\ \text{etc.},$$

where $W=p^0+\omega$ is the total energy of the πN system, and $W'=p^0-\omega$. Similar symmetry properties of the contributions of the $\pi\pi$ interactions also exist for the πK scattering process.[38]

In the process of analysis of the energy behavior of the πN phase shifts, it was established that the double dispersion representations give a good description of the experimental situation in the region of low energies (up to $\lesssim 500$ MeV) when the two lowest s and p partial waves are used. At higher energies (up to 1 GeV), it is necessary to take into account higher partial waves (d, and possibly, the f wave). However, in absolute magnitude they remain small, though the errors in the calculations begin to increase. Allowance for the higher d and f partial waves is associated with an excessive growth of the calculations not commensurate with the quality of the obtained scientific results.

Electromagnetic form factors of pions, kaons, and nucleons.[39] Electromagnetic form factors of elementary particles arise in the calculation of vertex functions of the type shown in Fig. 9. The analytic properties of the vertex functions $F_\pi(q^2)$, $F_N^2(q^2)$, etc., which depend on one variable—the 4-dimensional momentum transfer q^2—can be established, for example, in the framework of perturbation theory. As a function of the 4-dimensional variable q^2 (or t in the Mandelstam notation), the form-factor function $F_\pi(t)$ is an analytic function in the t plane with a cut $4\mu^2 \leqslant t \leqslant \infty$. For example, for the annihilation process $e^+ + e^- \to \pi^+ + \pi^-$ (it is in this process that a vertex of the type of Fig. 9b arises) the form factor $F_\pi(t)$ is complex, whereas in the region $t<0$ (i.e., in the region of the phys-

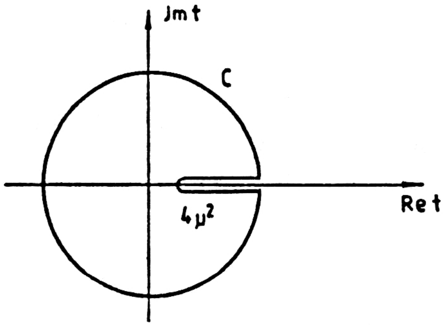

FIG. 10. Contour of integration in the complex plane of the variable t for the pion form factor. The cut begins at the point $t=4\mu^2$.

ically observable $e+\pi\to e+\pi$ process) $\mathrm{Im}\,F_\pi(t)=0$ and the form factor $F_\pi(t)$ is a real function. Thus, the dispersion relation for $F_\pi(t)$ has the form

$$F_\pi(t)=\frac{1}{\pi}\int_C \frac{\mathrm{Im}\,F_\pi(t')dt'}{t'-(t+i\varepsilon)}. \tag{32}$$

The contour of integration C is shown in Fig. 10.

In (32), we go to the limit $\varepsilon\to 0$: $\lim_{\substack{\varepsilon>0\\ \varepsilon\to 0}} 1/[t' - (t + i\varepsilon)] = \mathscr{P}[1/(t'-t)] + i\pi\delta(t'-t)$, and we obtain

$$\mathrm{Re}\,F_\pi(t)=\frac{\mathscr{P}}{\pi}\int_C \frac{\mathrm{Im}\,F_\pi(t')dt'}{t'-t}. \tag{33}$$

If we assume that $\mathrm{Im}\,F_\pi(\infty)=\text{const}$, then we must make one subtraction. It is made at the point $t=0$, at which the form factor $F_\pi(0)$ is normalized to the value of the electric charge: $F_\pi(0)=e$. Then the expression (33) becomes

$$\mathrm{Re}\,F_\pi(t)=e+\mathscr{P}\,\frac{t}{\pi}\int_{4\mu^2}^{\infty} \frac{\mathrm{Im}\,F_\pi(t')dt'}{t'(t'-t)}. \tag{34}$$

Using the unitarity condition for $\mathrm{Im}\,F_\pi(t)$,

$$\mathrm{Im}\langle 0|j_v|\pi^+\pi^-\rangle=\sum_\alpha \langle 0|j_v|\alpha\rangle\langle\alpha|T^+|\pi^+\pi^-\rangle,$$

and restricting ourselves to the lowest 2π-meson state, we obtain

$$\mathrm{Im}\langle 0|j_v|\pi^+\pi^-\rangle=\langle 0|j_v|\pi\pi\rangle\langle\pi\pi|T^+|\pi^+\pi^-\rangle.$$

If we assume that the contribution to the $\pi\pi\to\pi\pi$ interaction in the region of low energies is sufficiently fully described by a single p wave (the ρ meson), then the vertex $\langle 0|j_v|\pi\pi\rangle$ will also be described by a single p wave. In this case

$$\mathrm{Im}\,F_\pi(t)=F_\pi(t)e^{-i\delta_1(t)}\sin\delta_1(t)$$

or

$$\mathrm{Im}\,F_\pi(t)=\mathrm{Re}\,F_\pi(t)\tan\delta_1(t),$$

where $\delta_1(t)$ is the phase shift of the $\pi\pi$ interaction with total angular momentum $J=1$ and isospin $I=1$. Substituting $\mathrm{Im}\,F_\pi(t)$ in (34), we obtain

$$\mathrm{Re}\,F_\pi(t)=e+\frac{\mathscr{P}}{\pi}\int_{4\mu^2}^{\infty}\frac{t\mathrm{Re}\,F_\pi(t')\tan\delta_1(t')}{t'(t'-t)}dt'. \tag{35}$$

Thus, the form-factor function in the considered approximations satisfies a linear singular integral equation, which can be solved by the Muskhelishvili–Omnès method.[36] The general solution of Eq. (35) has a cumbersome form. We write down the solution of the integral equation (35) without a subtraction:

$$F_\pi(t)=e\cdot\exp\left\{\frac{\mathscr{P}}{\pi}\int_{4\mu^2}^{\infty}\frac{t\delta_1(t')dt'}{t'(t'-t)}\right\}. \tag{36}$$

Making different assumptions about the behavior of the $\pi\pi$ scattering phase shift δ_1, we can obtain different expressions for the form factor $F_\pi(t)$. In Refs. 40 and 41, the expression for $\delta_1(t)$ was chosen in the form of the ratio of polynomials $P(t)$ and $Q(t)$:

$$k^3\cot\delta_1(t)=\frac{P(t)}{Q(t)},$$

where $k^2=t/4-1$. In this case, the integration in (36) can be carried through to the end. The expression obtained for $F_\pi(t)$ in the pole approximation reduces to the usual expression for the pion form factor:

$$F_\pi(t)=\frac{1}{1-t/t_\rho}.$$

In this expression, we have taken the charge $e=1$, and t_ρ is the square of the ρ-meson mass ($m_\rho\approx 750$ MeV). The form-factor function of the π meson was studied experimentally in π-meson scattering by electrons of nuclei in the region of low energies of the incident π mesons.[42] Analogous experiments were then made using the 76-GeV accelerator at IHEP (Protvino)[43] and the FNAL accelerator at Batavia.[44] The IHEP measurements were made using a pion beam with momentum 50 GeV/c in the range of momentum transfers $0.014\leqslant q^2\leqslant 0.035$ (GeV/c)^2. The difficulties of this experiment were in determining the coplanarity of the process, allowance for the radiative corrections, etc. (allowance was made for 14 corrections that depend on q^2 and 12 corrections that do not depend on q^2). The electromagnetic radius of the pion was determined from the expression (see Ref. 39)

$$F_\pi(q^2)=1+\tfrac{1}{6}q^2\langle r_\pi^2\rangle$$

and was found to be $\langle r_\pi^2\rangle=(0.61\pm 0.15)$ fm^2. This experiment was continued at FNAL, using pion beams with momentum 100 GeV/c in the range of momentum transfers $0.03\leqslant q^2\leqslant 0.07$ (GeV/c)^2.

It was found that $\langle r_\pi^2\rangle=(0.31\pm 0.04)$ fm^2.

Accelerators with colliding electron–positron beams (Orsay, Frascati, Novosibirsk, Hamburg) were used to obtain experimental data on the dependence of the pion form factor in the timelike region of variation of the variable $t\geqslant 4\mu^2$. In this region, it is well described by the function

$$F_\pi(t)=\frac{m_\rho^2}{m_\rho^2-t-im_\rho\Gamma}.$$

An analytic expression taking into account all the fundamental properties of the pion form factor and describing the experimental data in both the spacelike and timelike regions of q^2 was obtained in Ref. 45, using the method of conformal mappings.

Similar theoretical considerations were used to describe the kaon form factors. As in the case of π mesons, the cross section for scattering of K mesons by the electrons of nuclei was expressed in the form

$$\frac{d\sigma}{dq^2}=\left(\frac{d\sigma}{dq^2}\right)_{\text{point}}|F_K(q^2)|^2,$$

where $(d\sigma/dq^2)_{\text{point}}$ is the cross section for scattering by a point K meson. The expression for the form factor $F_K(q^2)$ in the region of small momentum transfers q^2 is chosen in the form

$$|F_K(q^2)|^2=[1-\tfrac{1}{6}q^2\langle r_K^2\rangle]^{-2}.$$

A group of Soviet and American scientists measured $F_K(q^2)$ at FNAL using K-meson beams with q^2 in the range of momentum transfers $0.037\leqslant q^2\leqslant 0.119$ $(\text{GeV}/c)^2$. The electromagnetic radius of the K meson was found to be $\langle r_K^2\rangle=(0.28\pm 0.05)$ fm^2.

The electromagnetic radius of the neutral K^0 meson, measured by the method of K^0 regeneration, was found to be negative: $\langle r_{K^0}^2\rangle = -0.054 \pm 0.026$ fm^2, and this can be justified theoretically in the framework of the quark $SU(3)$ model if it is assumed that the mass of the strange quark is greater than the masses of the u and d quarks.[46] The problem of describing the nucleon form factors is more complicated.[47] All conclusions about the behavior of the form factors as functions of the momentum transfer are based on the Rosenbluth expression for the differential cross section for electrons on protons. In the lowest order of perturbation theory (single-photon exchange), it has the form

$$\frac{d\sigma}{d\Omega}=\sigma_M\left\{F_{1p}(q^2)-\frac{q^2}{4M^2}2[F_{1p}(q^2)+\kappa F_{2p}(q^2)]\right.$$
$$\left.\times\tan^2\frac{\theta}{2}-\frac{q^2}{4M^2}\kappa^2F_{2p}^2(q^2)\right\}, \qquad (37)$$

where σ_M is the differential cross section for electron scattering by the Coulomb field (the Mott cross section), $q^2\approx -4EE'\sin^2(\theta/2)$, E and E' are the energies of the electron before and after the collision, respectively, θ is the angle of the scattering of the electron by the proton in the laboratory coordinate system, and κ is the anomalous magnetic moment of the proton. As can be seen from (37), there are two proton form factors: F_{1p} and F_{2p}. Similarly, there are two neutron form factors F_{1n} and F_{2n}. Each of the form factors must have isotopically scalar and isotopically vector parts, since the nucleon form factors arise because of the strong interaction with the pions. Their isotopic structure has the form

$$F_{1p}(q^2)=F_{1p}^s(q^2)+\tau_3F_{1p}^v(q^2),$$
$$F_{2p}(q^2)=F_{2p}^s(q^2)+\tau_3F_{2p}^v(q^2).$$

Here, F^s and F^v are, respectively, the isotopically scalar and vector parts of the form factor, and $\tau_3=\left(\begin{smallmatrix}1 & 0\\ 0 & -1\end{smallmatrix}\right)$ is a two-row matrix. Thus,

$$F_{(1,2)p}=F_{(1,2)p}^s+F_{(1,2)p}^v,$$
$$F_{(1,2)n}=F_{(1,2)n}^s-F_{(1,2)n}^v.$$

The form-factor functions of the protons and neutrons are normalized at the point $q^2=0$ as follows:

$$F_{1p}(0)=e;\quad F_{1n}(0)=0;$$

$$F_{2p}(0)=\mu_p=1.79\frac{e}{2M};\quad F_{2n}(0)=\mu_n=-1.91\frac{e}{2M}.$$

Dispersion relations for the nucleon form factors can be written down for their isovector and isoscalar parts in complete analogy with the dispersion relations for the π-meson and K-meson form factors:

$$F_{1,2}^v(t)=F_{1,2}^v(0)+\frac{t}{\pi}\int_{4\mu^2}^{\infty}\frac{\operatorname{Im}F_{1,2}^v(t')dt'}{t'(t'-t)},$$

$$F_{1,2}^s(t)=F_{1,2}^s(0)+\frac{t}{\pi}\int_{9\mu^2}^{\infty}\frac{\operatorname{Im}F_{1,2}^s(t')dt'}{t'(t'-t)}.$$

The beginnings of the cuts for the isovector and isoscalar form factors are determined from the unitarity condition. The isovector imaginary parts $\operatorname{Im}F_{1,2}^v(t)$ are nonvanishing in the region $t\geqslant 4\mu^2$, since contributions to them are made only by the states even with respect to the number of π mesons, while the isoscalar parts $\operatorname{Im}F_{1,2}^s$ are nonzero in the region $t\geqslant 9\mu^2$, since only states odd with respect to the number of π mesons contribute to them. Naturally, the intermediate states may also include $K\bar{K}$ and $N\bar{N}$ pairs and other more complicated many-particle states—their cuts begin at correspondingly rather high values of t. The double Mandelstam representations make it possible to use analytic continuations of the unitary conditions from the physical region $t\geqslant 4M^2$ (where M is the nucleon mass) to the unphysical region $4\mu^2\leqslant t<4M^2$. Therefore, there are no fundamental difficulties in performing the theoretical calculations. At small values of t, it is sufficient to make a restriction in the unitarity condition for the isovector form factors to the lowest two-particle (2π-meson) intermediate state:

$$\operatorname{Im}F_{1,2}^v(t)=F_\pi(t)\langle\pi\pi|T_{1,2}^+(t)|N\bar{N}\rangle,$$

where $F_\pi(t)$ is the pion form factor, and $\langle\pi\pi|T_{1,2}^+|N\bar{N}\rangle$ are the parts of the amplitude of the $\pi\pi\to N\bar{N}$ annihilation process that contribute to the Dirac or Pauli form factors, respectively.

For the imaginary parts of the isoscalar form factors, there remains the lowest three-meson intermediate state:

$$\operatorname{Im}F_{1,2}^s(t)=\langle\gamma|T_{1,2}|\pi\pi\pi\rangle\langle\pi\pi\pi|T_{1,2}^+|N\bar{N}\rangle,$$

where $\langle\gamma|T|\pi\pi\pi\rangle$ is the $\gamma\to 3\pi$ vertex, and $\langle\pi\pi\pi|T_{1,2}^+|N\bar{N}\rangle$ are the parts of the $N\bar{N}\to 3\pi$ annihilation

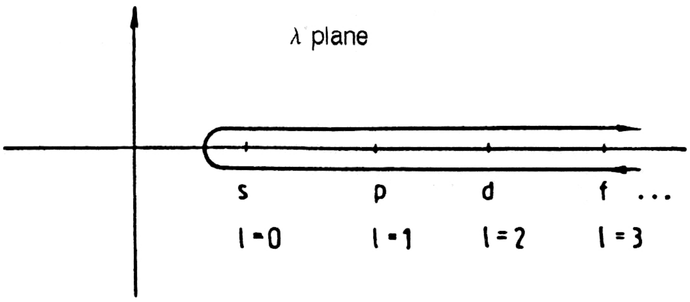

FIG. 11. Contour of integration C of the Watson integral (39).

process that, like the vector parts of the form factors, contribute to the Dirac or Pauli form factors. The subsequent calculations of the electromagnetic form factors of the nucleons have much in common with the manner in which the pion or kaon form factors are calculated. The functional dependence of the form factors on the four-dimensional spacelike momentum transfers q^2 predicted by the dispersion-relation method reproduces well the experimental data on the proton and neutron form factors that were obtained at different accelerators around the world (USA, France, German Federal Republic, Great Britian) in the wide range $0 \leqslant |q^2| \leqslant 20$ $(\text{GeV}/c)^2$.

Data on the nucleon form factors in the timelike region of momentum transfers are scarce. The vector-dominance model with allowance for the contributions from the ρ, ω, φ, ρ', ρ'' mesons and the contributions of the family of J/ψ particles satisfactorily describes this region $q^2>0$ too. Despite the listed successes, one cannot regard the problem of describing the form factors as solved. "... The kinematics of the process of elastic scattering by particles whose form factors are studied in elementary-particle physics is remarkably simple; the expressions that describe elastic scattering do not contain any other unknown functions apart from the form factors; the form-factor functions of elementary particles have been studied since the middle of the fifties. However, despite the apparent simplicity, the problem of investigating the form-factor functions of elementary particles, which is intimately related to the study of the structure of the elementary particles, has not yet been solved."[47]

The definitions and expressions obtained on the basis of the dispersion-relation method and the basic theoretical results have retained their significance up to the present day, when we now consider the behavior of the form factors in the range of momentum transfers up to $q^2 \leqslant 100$ $(\text{GeV}/c)^2$. The problem of investigating the form-factor functions is still topical today in the period of quark notions of the structure of matter.

Hitherto, in considering the double Mandelstam representations, we have avoided the question of the number of subtractions that must be made when the integrals are calculated in order to ensure their convergence. The subtraction problem is one of the unresolved problems of dispersion relations. It was the stimulus to further development of theoretical thought in several very important directions: a) the extension of analyticity of the S matrix through the assumption of analytic continuation with respect to the angular momentum into the complex plane of this variable; b) study of the asymptotic behavior of scattering amplitudes; c) dispersion sum rules; d) the bootstrap idea, and some others.

3. REGGEOLOGY

Regge[48] considered the analytic properties of the scattering amplitude $f(E,\theta)$ in quantum mechanics and showed that in the complex plane of the angular momentum one can write down Mandelstam representations for the amplitude $f(E,\theta)$. We write down the Schrödinger equation

$$\Delta\psi(\mathbf{r})+E\varphi(\mathbf{r})=V\psi(\mathbf{r}),$$

where $\hbar=c=1$; $2M=1$; $\mathbf{r}=\mathbf{r}(x,y,z)$ is the vector giving the position of the scattered particle; and $V(\mathbf{r})$ is a local potential that depends only on r. The asymptotic behavior of $\psi(\mathbf{r})$ is

$$\psi(r)\sim e^{ikr}+f(E,\theta)\frac{e^{ikr}}{r},$$

where $\mathbf{k}$ is the momentum of the scattered particle, and

$$f(E,\theta)=\frac{1}{2ik}\sum_{l=0}^{\infty}(e^{2i\delta_l(k)}-1)(2l+1)P_l(\cos\theta). \quad (38)$$

Here, l is the angular momentum; $P_l(\cos\theta)$ are Legendre polynomials; and $\delta_l(k)$ is the phase shift corresponding to the angular momentum l. The total cross section is given by

$$\sigma(E)=\int d\Omega\,|f(E,\theta)|^2=\frac{4\pi}{E}\sum_{l=0}^{\infty}(2l+1)\sin^2\delta_l.$$

The function $f(E,\theta)$ is the quantum-mechanical analog of the scattering matrix $T(E)$ [see (25)]. One can show that there exists an analytic continuation of the function $S_l(k)=e^{2i\delta_l(k)}$ to the region of complex variables λ such that when λ takes the value $\lambda=l+1/2$ the function $S(\lambda,k)$ goes over into the function $S_l(k)$ (Watson transformation):[2)]

$$f(E,\theta)=-\frac{1}{2k}\int_C\frac{\lambda d\lambda}{\cos\pi\lambda}P_{\lambda-1/2}(-\cos\theta)\times[S(\lambda,k)-1]. \quad (39)$$

The contour of integration of the Watson integral is shown in Fig. 11. So far as is known, the only singularities of the amplitude (39) in the l plane are poles and cuts.

The integral (39) determines the properties of the function $S(\lambda,k)$ for arbitrary values of λ and k; in particular, the asymptotic behavior of $S(\lambda,k)$ for large values of λ is determined. The integral (39) converges for a specific class of Yukawa potentials that satisfy the following conditions:[49]

1) The potential $V(z)$ has the representation

$$V(z)=\int_{\mu>0}\sigma(\mu)\,\frac{e^{-\mu z}}{z}\,d\mu$$

with a suitably chosen weight function $\sigma(\mu)$.

2) The potential function $V(z)$ can be continued into the half-plane $\operatorname{Re} z>0$.

3) Along any ray $\arg z=\sigma$, $|\sigma|<\pi/2$, we have

$$\int|V(z)z|ds<M<\infty,\quad ds=|dz|.$$

In this restricted case, the Mandelstam representation mentioned above can be written down for the total amplitude $f(E,\theta)$.

Theoreticians noted that there exists a connection between the asymptotic behavior of the scattering amplitude and the nature of the poles of the function $f(E,\theta)$ [see (39)].

Suppose that in the s channel there is a resonance at the value M_r^2 with width Γ. The amplitude at the pole can be written in the form

$$f(s,t)\approx\frac{g(s)P_l(\cos\theta_s)}{s-M_r^2+i\Gamma M_r}. \tag{40}$$

Here, $f(s,t)$ is equivalent to the function $f(E,\theta)$. The symbol s of $\cos\theta_s$ means $\cos\theta$ in the s channel. If Γ is small compared with M_R, then we can write

$$\operatorname{Im} f(s,t)\approx g(s)P_l(\cos\theta)\delta(s-M_r^2).$$

The asymptotic behavior of the Legendre polynomials for infinitely large values of the argument is

$$P_l(\cos\theta)\underset{\cos\theta_s\to\infty}{\sim}\cos\theta_s^l,\quad l>-\tfrac{1}{2}.$$

From the definition of the invariants $s=(p_1+p_2)^2$, $u=(p_1+p_3)^2$, $t=(p_1+p_4)^2$ [see (22)], we find that $\cos\theta_s\sim t$, whence

$$\operatorname{Im} f(s,t)\sim t^l.$$

Thus, if the asymptotic behavior with respect to t for all s is bounded by $\sim t^l$, then we obtain a bound on the largest value of the spin of the corresponding pole in the s channel. Naturally, however, any number of particles with $\lesssim l$ may exist. If we speak of a Regge representation of the scattering amplitude, in place of l we must understand some analytic function $\alpha(s)$ that depends on the square of the energy s. The function $\alpha(s)$ at the pole $s=s_0=M_r^2$ can be expanded in the series

$$\alpha(s_0)=l+\varepsilon(s_0)+i\eta(s_0)+(s+s_0)\left.\frac{d\alpha}{ds}\right|_{s=s_0}+\dots,$$

where $\varepsilon\ll l$, $\eta\ll l$, and at the point s_0 we have $\lambda=l+1/2$. The function $\alpha(s)$ is called a Regge trajectory. The trajectory with value $\alpha(0)=1$ is called the vacuum trajectory, or the Pomeranchuk trajectory.

Using the explicit expression for the function $f(s,t)$ [or $f(E,\theta)$; see (38)], we can show that for forward scattering ($\cos\theta_s=1$) the amplitude behaves as follows:[50]

$$\underset{\substack{\cos\theta_s=1\\ s\to\infty}}{|f(s,t)|}<\text{const}\cdot s\ln^2 s, \tag{41}$$

and for nonvanishing angles

$$\underset{s\to\infty}{|f(s,t)|}<\text{const}\cdot s^{3/4}\ln^{3/2}(s). \tag{42}$$

The bounds (41) and (42) on the $s\to\infty$ behavior of the amplitude are called Froissart bounds.[51] The appearance of the logarithmic factors in the relations (41) and (42) is not entirely understood and can be explained only by the approximations used in the derivation of these expressions. It is usually very difficult to separate logarithmic behavior in an experiment—$\ln s$ is a too slowly varying function of the energy s. Therefore, the logarithmic dependence is usually omitted, and the Froissart bounds take the form

$$\lim_{s\to\infty} f(s,t)=0(s^{N(t)}),$$

where $N(t)\leqslant 1$, $t<0$.

In the crossed t channel, this bound has the symmetric form

$$\lim_{t\to\infty} f(s,t)=0(t^{N(s)}),$$

where $N(s)\leqslant 1$, $s<0$.

If we recall the expression for the optical theorem [see (11)], then we obtain

$$\sigma(s)\sim s^{N(0)-1}\leqslant\text{const}.$$

Thus, on the one hand, the scattering amplitude in the Mandelstam representation has been expressed in the form

$$f(s,t)\sim\frac{1}{\pi^2}\int_{s_0}^{\infty}ds'\int_{t_0}^{\infty}dt'\,\frac{\Gamma_{st}(s',t')}{(s'-s)(t'-t)}+\dots$$

[see (23)]. On the other hand, in the Regge-pole representation it has the form

$$f(s,t)\sim-\sum 16\pi^2(2\alpha_i(s)+1)\beta_i(s)\,\frac{P_{\alpha_i}(s)(-z_s)}{\sin\pi\alpha_i(s)}$$
$$-\frac{16}{\pi}\int_{-1/2-i\infty}^{-1/2+i\infty}(2l+1)A(s,l)\,\frac{P_l(-z)}{\sin\pi l}\,dl, \tag{43}$$

where $\alpha_i(s)$ is the trajectory of the Regge pole, $\beta_i(s)$ is the residue at the pole s_{ir}, $\alpha_i(s_r)=l$, and $P_{\alpha_i(s)}(-z)$ is a Legendre polynomial.

The expression (43) is derived from (39) and indicates that the amplitude $f(s,t)$, represented as a sum of Regge poles, has, in addition to these, cuts from these poles. Much work was done in the attempt to prove the equivalence of the two representations (23) and (43), and for a

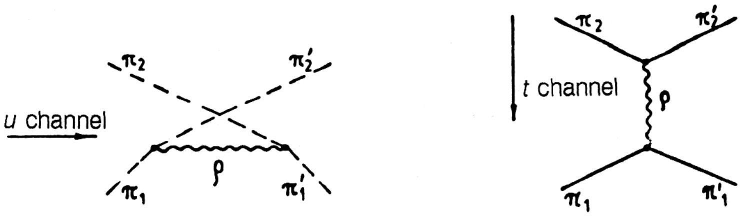

FIG. 12. The ρ-exchange diagrams in the u and t channels.

number of artificial cases this succeeded. In quantum field theory, it has not proved possible to find a rigorous justification of the Regge representation (43) of the scattering amplitude.

There is no need to go deeper into the theory of Regge poles and, exhibit the remarkable successes of this theory in the description of the systematics of particles and the asymptotic behavior of scattering amplitudes. This can be found in a number of reviews.[52,53] It was important for us to point out that the proof of analyticity of the scattering amplitude with respect to one variable in the theory of strong interactions generated the double representations of Mandelstam and then the theory of Regge poles, the basis of which is the assumption of a further extension of the analyticity of the interaction amplitudes of elementary particles—in the case of Reggeology, by the assumption of analyticity with respect to the angular momentum l.

The theoretical analysis of Regge trajectories led to the conclusion that they have a nonlinear dependence on the variable s (or t). Comparision with experimental data led, nevertheless, to the conclusion that the Regge trajectories are very nearly linear. If, besides linearity, one assumes that the slopes of all the trajectories are the same, then we pass from the representation (43) of the scattering amplitude to the Veneziano model for the scattering amplitude:

$$\Phi_b(\alpha_i(s),\alpha_j(t))=\frac{\Gamma[1-\alpha_i(s)]\Gamma[1-\alpha_j(t)]}{\Gamma[N-\alpha_i(s)-\alpha_j(t)]}$$
$$+\text{terms with the substitutions}$$
$$s\to u,\ t\to u, \qquad (44)$$

where Φ_b are generalized Euler β functions, $\alpha_{i,j}$ are linear Regge trajectories, Γ is the gamma function, and N is a positive integer. The function (44) has poles at the points $\alpha_i(s)$, $\alpha_j(s)=l$ and possesses crossing symmetry ($s\rightleftarrows t\rightleftarrows u$) and Regge asymptotic behavior. However, the discussion of the Veneziano model would take us too far from the subject of our review.

4. BOOTSTRAP METHOD

At the beginning of the sixties, the "bootstrap" hypothesis was very popular. It followed naturally from all the preceding development of the theoretical and experimental elementary-particle physics. The essence of the conjecture is that there are no fundamental constituents (like quarks) that underlie matter, but that all particles are on an equal footing and consist of each other. In collisions of a pair of any particles, any other particles can be produced (naturally, all the conservation laws must be respected). As a mathematical formalism adequate for the bootstrap idea, the method of double Mandelstam representations was used. The mathematical method of solving the equations was called the bootstrap method.[54]

Let us consider the simplest example of the interaction of two pions. We assume that their interaction occurs through the exchange of only the ρ meson (spin $I=1$, isotopic spin$=1$, negative parity). This means that the π mesons interact in the p state. The dispersion relations for the p wave have the form

$$A_{l=1}(\nu)=\frac{1}{\pi}\int_0^{\infty}\frac{\operatorname{Im}A_l(\nu')d\nu'}{\nu'-\nu}+\frac{1}{\pi}\int_{-\infty}^{-1}\frac{\operatorname{Im}A_l(\nu')d\nu'}{\nu'-\nu}. \qquad (45)$$

Here, ν is the c.m.s. momentum of the pions, and the mass of the pion is $\mu=1$; the square of the total energy is $s=4(\nu+1)$. The second integral gives the contribution from ρ-meson exchange in the u and t channels (see Fig. 12). To solve Eq. (45), we represent the amplitude $A_l(\nu)$ in the form of the ratio

$$\frac{1}{\nu}A_l(\nu)=\frac{N_l(\nu)}{D_l(\nu)}. \qquad (46)$$

[In the expression (46) and below, the index $l=1$ is omitted]. Suppose that $N(\nu)$ has only a left-hand cut, and $D(\nu)$ has only a right-hand cut. We write down the Cauchy theorem for $N(\nu)$ and $D(\nu)$:

$$\operatorname{Im}N(\nu)=\frac{1}{\nu}D(\nu)\operatorname{Im}A(\nu),$$

$$\operatorname{Im}D(\nu)=\nu N(\nu)\operatorname{Im}\left[\frac{1}{A(\nu)}\right]=-\nu N(\nu)\frac{\operatorname{Im}A^*(\nu)}{|A(\nu)|^2}.$$

We adopt the normalization $N(\nu)\underset{\nu\to\infty}{=}0$; $D(\nu)\underset{\nu\to\infty}{=}1$,

$$\operatorname{Im}A^*(\nu)=\sqrt{\frac{\nu}{\nu+1}}|A(\nu)|^2.$$

We can now write the Cauchy theorem in the form

$$N(\nu)=\frac{1}{\pi}\int_{-\infty}^{-1}\frac{\operatorname{Im}A(\nu')D(\nu')}{\nu'(\nu'-\nu)}d\nu',$$
$$D(\nu)=1-\frac{1}{\pi}\int_0^{\infty}\sqrt{\frac{\nu'}{\nu'+1}}\frac{N(\nu')d\nu'}{\nu'-\nu}. \qquad (47)$$

We have assumed that the function $D(\nu)$ has no zeros, i.e., there are no poles of the function $A(\nu)$. In general, zeros of the function $D(\nu)$ must be taken into account; for the poles of the amplitude $A(\nu)$ correspond to resonances on

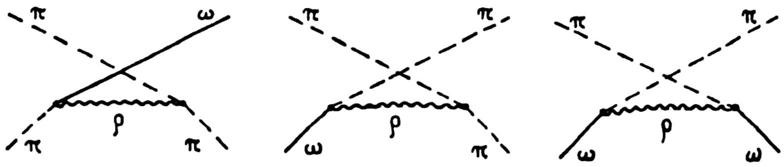

FIG. 13. Additional "forces" in the 2-channel problem for determination of the ρ-meson parameters.

the unphysical sheet or bound states on the physical sheet. In this case, the $\nu \to \infty$ behavior of the function $D(\nu)$ has the form

$$D_l(\nu) \underset{\nu \to \infty}{\sim} \nu^{-M_l+P_l+\frac{1}{\pi}[\delta_l(\infty)-\delta_l(0)]},$$

and since we have chosen the normalization $D_l(\infty)=1$, the exponent of $D_l(\nu)$ must be zero:

$$\delta_l(\infty)-\delta_l(0)=\pi[-M_l+P_l],$$

where P_l is the number of poles in the phase shift δ_l, and M_l is the number of bound states. This result agrees with Levinson's theorem. One can set, for example, $\delta_l(0)=\pi M_l$, and then $\delta_l(\infty)=\pi P_l$. Returning to Eqs. (47), we see that instead of the singular integral equation (45) we have obtained the system of equations (47). After substitution of $D(\nu)$ in the equation for $N(\nu)$, the linear equation can be solved if we know the form of the function $\operatorname{Im} A(\nu)$. We assume that $\operatorname{Im} A(\nu)$ is determined by the sum of the contributions of the diagrams from the second and third channels (see Fig. 12). Then the first iteration in the N/D method leads to the solution

$$D(\nu)=1, \quad \operatorname{Im} N(\nu)=\frac{1}{\nu}\operatorname{Im} A(\nu), \quad N(\nu)=\frac{1}{\nu}A(\nu), \tag{48}$$

$$\frac{1}{\nu}A(\nu)=\frac{N(\nu)}{1-\frac{1}{\pi}\int_0^\infty \sqrt{\frac{\nu'}{\nu'+1}}\frac{N(\nu')}{\nu'-\nu}d\nu'}.$$

If necessary, the iterative procedure can be repeated the necessary number of times. If the mathematical scheme were adequate for the bootstrap philosophy, then in (48) the real part of the denominator must vanish at the resonance point corresponding to the ρ-meson mass, $\nu_\rho=m_\rho^2/4-1$, and the theoretical width of the resonance, expressed in terms of $N(\rho_\nu)$, must be equal to the experimental width of the ρ meson. Such a solution would represent the bootstrap method in its pure form. However, the calculation gave a self-consistent solution at the value $m_\rho \approx 350$ MeV (instead of $m_\rho \approx 750$ MeV). Since the model was very primitive, the calculations were regarded as promising and it was resolved to improve them by introducing additional exchange forces, i.e., it was decided to go over from a single- to a two-channel problem. To this end, besides the $\pi\pi$ scattering amplitude, the $\pi\pi \to \pi\omega$, $\pi\omega \to \pi\pi$, and $\pi\omega \to \pi\omega$ processes (see Fig. 13) were included in the problem. The generalization of the N/D method to the two-channel (multichannel) case was considered in Ref. 55. The scattering amplitude is now represented by the matrix

$$T=\begin{Vmatrix} t_{11} & t_{12} \\ t_{21} & t_{22} \end{Vmatrix},$$

where t_{11} corresponds to the $\pi\pi \to \pi\pi$ scattering process, t_{12} to $\pi\pi \to \pi\omega$ scattering, t_{21} to $\pi\omega \to \pi\pi$ scattering, and t_{22} to $\pi\omega \to \pi\omega$ scattering.

Allowance for the second channel led to a significant improvement of the bootstrap solution. The value obtained was $m_\rho \approx 660$ MeV. The technique of the calculation in the two-channel bootstrap method is a trivial generalization of the single-channel case. The amplitude T is again expressed in the form N/D, where N and D are square second-rank matrices:

$$N=\begin{Vmatrix} n_{11} & n_{12} \\ n_{21} & n_{22} \end{Vmatrix}, \quad D=\begin{Vmatrix} d_{11} & d_{12} \\ d_{21} & d_{22} \end{Vmatrix}.$$

All the elements n_{ik} have only left-hand cuts, while the elements d_{ik} have only right-hand cuts. For each element t_{ik} of the matrix T, we have

$$t_{ij}=\sum_k \frac{n_{ik}D_{jk}}{\det D},$$

where D_{jk} is the cofactor of the determinant D. The unitarity condition is expressed in the form $\operatorname{Im} t_{ij} = \Sigma_k \rho_i t_{ik} t_{kj}^*$. One then writes down the Cauchy theorem for n_{ij} and d_{ij}. For example, for d_{ij} it has the form

$$d_{ij}=\delta_{ij}-\frac{1}{\pi}\int_{s_{\rm thr}}^\infty \frac{ds'\rho(s')n_{ij}(s')}{s'-s},$$

where the phase factor $\rho(s')$ is chosen in such a way as to ensure the correct threshold behavior of the amplitude and the behavior of the amplitude at infinity, and undesirable kinematic singularities are eliminated. For a number of simple problems, this can be done, and at the same time the unitarity condition can be satisfied.

A deeper study of the bootstrap solutions led to the conclusion that they are not invariant with respect to time reversal and in a number of cases change the original asymptotic behavior. In order to eliminate the last shortcoming, it was resolved to Reggeize the bootstrap, i.e., to introduce a Regge nature of the behavior of the amplitudes. Then after bootstrapping the calculated amplitude should also have Regge behavior, i.e., must "recover" the parameters of the original trajectory $\alpha(s)$—this was the idea of the Reggeization of the bootstrap. This also could not always be obtained. More complicated cases of bootstrapping the parameters of particles were considered. Thus, attempts were made to use the bootstrap method to obtain $SU(3)$ symmetry (or some symmetry of another type).

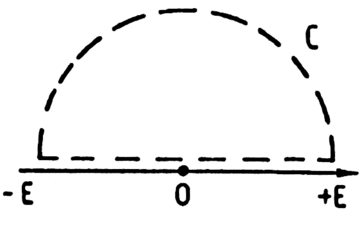

FIG. 14. Contour of integration for the function $F(E)$.

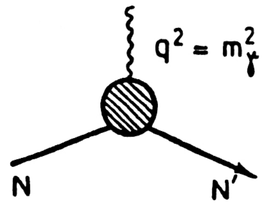

FIG. 15. Vertex of the interaction of a virtual photon with a nucleon.

For example, attempts were made to obtain a bootstrap octet of vector and pseudovector mesons, i.e., a definite set of mesons, and their quantum numbers, was assumed in advance, and the octet was sought in the form of the ratios of the coupling constants:

$$g^2_{\pi\rho\rho}:g^2_{\rho K\bar{K}}:g^2_{\omega K\bar{K}}:g^2_{K^*\pi K}:g^2_{\eta K^* K}=\frac{4}{3}:\frac{2}{3}:2:1:1.$$

The resulting solution did satisfy this relation predicted by $SU(3)$ symmetry.

The pretensions of the bootstrap method to be the philosophy of the universal dynamical system of equations describing the entire physical world of elementary particles proved to be excessive. It was more natural to speak of bootstrap solutions that gave reasonable agreement with the experimental data and of solutions capable of predicting certain parameters of new unstable particles (their masses, decay widths, coupling constants, etc.).

5. SUPERCONVERGENT (DISPERSION) SUM RULES[56]

We have previously discussed the problem of the behavior of the amplitudes of physical processes at infinitely high energies. It was already clear when dispersion relations first arose that if the amplitude $F(E)$ is analytic in the E plane (E is the energy) with a cut along the real axis does not have singularities in the upper half-plane, and decreases sufficiently rapidly at infinity, then by the trivial result of the Cauchy theorem for such a function (the contour of integration has the form shown in Fig. 14) we must have

$$\int_{-E}^{+E} F(E')dE' + \int_C F(E')dE' = 0. \tag{49}$$

If for large values of E the amplitude $F(E)$ decreases sufficiently rapidly, for example, as $E^{-1}\ln^a E$, $a<-1$ (or more rapidly), the integral around the contour C tends to zero as $E\to\infty$, and we obtain the superconvergent sum rule

$$\int_{-\infty}^{+\infty} \operatorname{Im} F(E')dE' = 0.$$

What can be obtained from such sum rules? Let us consider, for example, the invariant amplitude that describes the scattering of a virtual isovector photon with mass $m_\gamma^2 = q^2$ (see Fig. 15) and assume that the conditions of superconvergence hold for the amplitude difference $F(E,q^2)-F(E,0)$. In this case, we obtain for the form factors the Cabibbo–Radicatti relations

$$\int_{-\infty}^{+\infty} \operatorname{Im}[F(E,q^2)-F(E,0)]dE=0.$$

If as the amplitude $F(E,q^2)$ we consider the real Compton effect on a meson (with isotopic spin 2 in the t channel), then we obtain sum rules in the form[57]

$$\frac{1}{m_\pi^2}=\frac{1}{2\pi^2\alpha}\int_{E_0}^{\infty}[\sigma(\gamma\pi^0)-\sigma(\gamma\pi^+)]dE.$$

In the summer of 1965, when he was analyzing calculations of magnetic moments obtained by means of current algebra, Bogolyubov noted that the same results could be obtained from ordinary one-dimensional dispersion relations. In particular, he noted the possibility of using for such purposes sum rules for Compton scattering. These sum rules were used in conjunction with $SU(6)$ symmetry to obtain relations between the total magnetic moments of the proton, μ_p, and neutron, μ_n:[58]

$$\mu_p-\frac{e}{2M}+\mu_n=0,$$

where M is the nucleon mass (the same result was obtained in a somewhat different way in Ref. 59). If the amplitude $F(E)$ does not converge sufficiently rapidly, it is necessary to calculate the integral around the contour C [see Fig. 14, Eq. (49)]. This can be done if the behavior of $F(E)$ at large E is known.

Representing the amplitude $F(E)$ in the region of large E as a sum of Regge poles,

$$F(E)=\sum_n \alpha_n E^{\alpha n},$$

we obtain

$$\int_{-E}^{+E} \operatorname{Im} F(x)dx = \sum_n \frac{E^{\alpha n+1}}{\alpha n+1}\operatorname{Im}\alpha_n(1+e^{i\pi\alpha n}), \tag{50}$$

where E is a sufficiently high limit with respect to the energy.

In the considered case, the dispersion sum rules are obtained using the hypothesis of Regge behavior of the scattering amplitude at high energies. It follows from the relation (50) that if we have information about the behavior of Im $F(E)$ in the region of lower energies ($\lesssim E$) then we can draw certain conclusions about the behavior of the Regge trajectories $\alpha_n(E)$ that govern the behavior of the cross section at high energies.

Some important results that are still of significance in the present day were obtained for electromagnetic processes. We give one example. The amplitude of Compton scattering (at zero scattering angle) can be expressed in the form

$$F(\omega)=f_1(\omega)\mathbf{e}'^*\mathbf{e}+f_2(\omega)i\sigma[\mathbf{e}'^*\times\mathbf{e}],$$

where $\mathbf{e}$ and $\mathbf{e}'$ are the vectors of the transverse polarization of the incident and forward-scattered photons, respectively, ω is the photon energy in the laboratory coordinate system, and $f_1(\omega)$ and $f_2(\omega)$ are analytic scalar functions that describe photon scattering by a certain target. They are related to the total cross sections for absorption of a photon by the target as follows:

$$\mathrm{Im}\, f_1(\omega)=\frac{\omega}{8\pi}(\sigma_A+\sigma_p),$$

$$\mathrm{Im}\, f_2(\omega)=\frac{\omega}{8\pi}(\sigma_A-\sigma_p),$$

where σ_p is the total cross section for interaction of a circularly polarized photon with helicity $+1$ (i.e., a polarized photon) on a target completely polarized along the z axis, which is directed along the momentum of the incident photon, and σ_A is the total cross section for a left-polarized photon (helicity -1) for the same polarization of the target. The system of units is such that $\hbar=e=1$.

In the low-energy limit, the functions $f_1(0)$ and $f_2(0)$ are related to the fine-structure constant α, the charge e, the mass M, and the anomalous magnetic moment κ of the particle on which the Compton scattering of the photon occurs:

$$f_1(0)=-\frac{\alpha}{M}e^2,$$
$$f_2(0)=-\frac{\alpha}{2M^2}\kappa^2 \qquad (51)$$

(Low–Gell-Mann–Goldberger theorem[60]).

It is well known that the function $f_1(\omega)$ satisfies a dispersion relation with one subtraction:

$$f_1(\omega)=-\frac{\alpha}{M}e^2+\frac{\omega^2}{2\pi^2}\int_0^\infty \frac{[\sigma_p(\omega')+\sigma_A(\omega')]d\omega'}{\omega'^2-\omega^2}.$$

The behavior of the function $f_2(\omega)$ at infinity is unknown. We assume that the function $f_2(\omega)$ satisfies an unsubtracted dispersion relation:

$$\mathrm{Re}\, f_2(\omega)=\frac{2\omega}{\pi}\mathscr{P}\int_0^\infty \frac{\mathrm{Im}\, f_2(\omega')d\omega'}{\omega'^2-\omega^2}. \qquad (52)$$

Differentiating (52) with respect to ω and going to the limit $\omega\to 0$, we obtain

$$\mathrm{Re}\, f_2'(\omega)=\frac{2}{\pi}\mathscr{P}\int_0^\infty \frac{\mathrm{Im}\, f_2(\omega')d\omega'}{\omega'^2}=f_2(0).$$

Using now the Low–Gell-Mann–Goldberger theorem [see (51)], we obtain

$$\frac{2\pi^2}{M^2}\kappa^2=\int_0^\infty \frac{[\sigma_p(\omega)-\sigma_A(\omega)]}{\omega}d\omega. \qquad (53)$$

The possible existence of a sum rule like (53) was first pointed out in Ref. 61. It was obtained in explicit form by Gerasimov[62] and was generalized by him to the case of a target with arbitrary spin. Later, the relation (53) was obtained by Drell and Hearn[63] and by a number of other authors and entered the literature under the name of the Gerasimov–Drell–Hearn sum rules. It is presently used, in particular, to analyze polarization phenomena.

6. CONSEQUENCES OF THE HYPOTHESIS OF ANALYTICITY OF THE SCATTERING AMPLITUDE

The notion of the scattering amplitude as a single analytic function of the energy and momentum variables proved to be very fruitful for the subsequent development of the theory of the strong interactions of elementary particles. Already the assumption of analyticity of the scattering amplitude in a certain part of the complex plane of one of the variables makes it possible to obtain definite physical conclusions.

Let s be the square of the total energy of a system of two particles with equal masses, and t be the momentum transfer: $t=-2q^2(1-z)$, where q is the three-dimensional c.m.s. momentum, and $z=\cos\theta$, where θ is the c.m.s. scattering angle. Let the amplitude $F(q,z)$, as a function of the variable z, be analytic within an ellipse with foci at the points $z=\pm 1$. The lengths of the semiaxes of the ellipse are determined by the energies and masses of the particles in the intermediate states. In accordance with the analyticity assumption, the series

$$F(q,z)=\frac{1}{q}\sum_{l=0}^{\infty}(2l+1)F_l(q)P_l(z)$$

converges within the ellipse, and the partial-wave amplitudes $F_l(q)$ for sufficiently large $l>l_0$ decrease rapidly with increasing l. Then

$$F(q,z)\approx\frac{1}{q}\sum_{l=0}^{l_0}(2l+1)F_l(q)P_l(z)=\tilde{F}(q,z).$$

By virtue of the unitarity condition $|F_l(q)|\lesssim 1$, so that for all $|z|\leqslant 1$

$$\tilde{F}(q,z)\leqslant\frac{1}{q}\sum_{l=0}^{l_0}(2l+1)=\frac{(l_0+1)^2}{q}.$$

If we introduce the concept of the effective interaction range $R\approx l_0/q$ and recall the optical theorem,

$$\sigma_{\mathrm{tot}}=\frac{4\pi}{q}\mathrm{Im}\, F(q,1),$$

then we obtain the estimate

$$\sigma_{\mathrm{tot}}\leqslant 4\pi R^2.$$

(Generally speaking, the radius R must depend on the energy.) The behavior of the cross sections of elastic and inelastic processes at high energies were considered in studies by Logunov and collaborators[64] on the basis of the analyticity properties of the scattering amplitude. They obtained important conclusions about the behavior of the

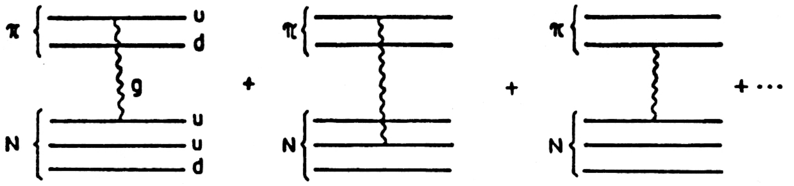

FIG. 16. The set of diagrams in QCD that describe the $\pi^+p\to\pi^+p$ scattering process.

differential cross section for multiparticle production with increasing energy in the region of large scattering angles for processes of the type

$$a+b\to a+b, \tag{I}$$

$$a+b\to A_j, \tag{II}$$

$$a+b\to c+A_j, \tag{III}$$

$$a+b\to c+d+A_j, \tag{IV}$$

where A_j is a collection of hadrons, and a, b, c, d are individual particles. The attentive reader will readily note that the concept of the extensive class of processes that today are called deep inelastic processes was first introduced, and the processes themselves studied with reasonable generality by Logunov and his collaborators already in 1967 (Ref. 65) (see the process III). On the basis of assumptions about the nature of the analyticity of the amplitudes of elastic and inelastic processes as functions of the angle variables and the unitarity condition, bounds on the behavior of the cross sections of elastic and inelastic processes were established. In particular, for strong interactions in which pions with mass m_π play the role of exchange particles the following relation was obtained:

$$\sigma_{el}(s)\geqslant\frac{m_\pi^2}{4\pi}\frac{\sigma_{\rm tot}^2(s)}{\ln^2\left(\frac{s}{s_0}\right)},$$

from which it follows that if the total cross section $\sigma_{\rm tot}(s)$ tends with increasing energy s to a constant limit, then the elastic cross section of the same process cannot decrease more rapidly than $\ln^2(s/s_0)$, where s_0 is some constant.[66] For the two-particle inelastic scattering $a+b\to c+d$ the following bound was obtained:

$$\frac{d\sigma_{\rm inel}}{dt}\leqslant\frac{\pi}{4m_\pi^4}\ln^4\left(\frac{s}{s_0}\right),$$

where $d\sigma_{\rm inel}/dt$ is the differential cross section of the considered inelastic scattering, and t is the invariant momentum transfer between particles a and c.

In 1971, a group of experimentalists at IHEP and a collaboration of IHEP and CERN physicists completed a series of experiments in which they measured the total cross sections in the region of Serpukhov energies, 30–70 GeV, which at that time were inaccessible to other accelerators in the world. The results were completely unexpected: The total cross sections of the $\pi^\pm p$, pp, and $p\bar{p}$ processes ceased to decrease with increasing energy and tended approximately to constants, while the K^+p cross section, which at lower energies was approximately constant, began to increase. This change in the behavior of the cross sections with increasing energy was called the "Serpukhov effect." Theoretical investigations based on analyticity of the scattering amplitudes, which indicated a growth of the total cross sections with increasing energy, found confirmation already at the energies of the Serpukhov accelerator. The experimental data obtained later using the FNAL (USA) and CERN accelerators at higher energies also agree with the conclusions of the theoreticians. The confrontation of the theoretical predictions given above with experimental data is valuable because the experiments convince us of the validity of the most general principles of quantum field theory, from which there is no need to depart in our time either. These principles include the precise mathematical formulation of the causality principle given by Bogolyubov when creating the axiomatic approach to quantum field theory. A mathematical formalism that straddles the interface between the theory of generalized functions and the theory of analytic functions of many complex variables was used by Bogolyubov to establish the dispersion relations—to prove analyticity of the scattering amplitudes in the theory of strong interactions. On the basis of the principles of analyticity of the physical amplitudes it is possible to obtain relationships between different physical processes, and these were found to be particularly simple in the region of asymptotically high energies.

7. CONCLUSIONS

The approaches to the solution of the problems of the theory of strong interactions that we have considered here were the main approaches for 10–15 years, beginning from 1955. The transition to a quark structure of elementary particles displaced from the awareness of theoreticians the problems associated with the concept of analyticity of scattering amplitudes and, with it, the unresolved problems of the dispersion-relation method, Reggeology, and the physics of phenomena in the region of very high energies based on ideas about the analyticity of scattering amplitudes.

It is obvious that one cannot automatically extend the propositions of quantum field theory that Bogolyubov used to prove analyticity of the scattering amplitudes in the theory of strong interactions to the analogous propositions in QCD in order to prove analyticity of the corresponding scattering amplitude in QCD. For example, to the process of π^+-meson scattering by a proton in QCD in the first approximation there will correspond a series of diagrams (Fig. 16) whose analytic properties are manifestly not the same as those of the $\pi^+p\to\pi^+p$ scattering amplitude in the old quantum field theory, in which the π^+ and p particles were regarded as point particles. It is entirely unclear how a pole term in QCD corresponding to the term in quantum field theory can arise. Quite generally, it appears simply impossible to make any comparison of the analytic properties of the considered process of π^+p scattering in the framework of the two approaches: the old one (in quantum field theory) and the new one (in the framework of QCD). One cannot simultaneously argue in the language of QCD and use in the framework of that theory mathematical relations obtained from analyticity and proved in the framework of classical quantum field theory such as dispersion relations, dispersion sum rules, the relations of Reggeology, and some others, which have not been proved in QCD. This would manifestly contradict the tradition of rigorous theoretical thinking that was cultivated by Bogolyubov during his entire creative life.

Does this mean that we must give up attempting to introduce the idea of analyticity into quantum chromodynamics? I believe the answer is no. We know that bound states can be described by local interpolating fields for which causality conditions exist. Thus, dispersion relations can also be derived in the framework of QCD. In perturbative QCD without confinement, the cuts that arise on account of the unitarity condition begin at the origin, since gluons have zero rest mass and there is no "hole" needed to obtain retarded and advanced functions. It follows from this that the problem of proving dispersion relations in QCD is intimately related to the confinement problem in QCD. If the virus of analyticity penetrates into QCD, then the philosophy of the sixties and the mathematical approaches may be reborn but, naturally, in a modified form. The dispersion relations in quantum field theory established relationships between the differential and total cross sections, between elastic and inelastic cross sections, and between physical constants and partial-wave amplitudes of different processes; in QCD, the analyticity properties of the scattering amplitudes could be helpful in establishing relations between the distributions of quarks and gluons in particles, in the proof of confinement of elementary particles, and in some asymptotic relations in processes involving the interaction of elementary particles.

[1)]See supplement A in the book of Ref. 1.

[2)]All the following arguments must be made for amplitudes $f(E,\theta)$ with definite signature: f^+ or f^- (for the definition of the signature, see, for example, Ref. 50). However, the conceptual aspect of the exposition is not distorted if the signature symbol is omitted in the following exposition.

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