Linear and nonlinear excitations of nuclear density

V. G. Kartavenko
Joint Institute for Nuclear Research, Dubna

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A review is made of the development of methods for describing nonlinear phenomena and relaxation processes in complex nuclear systems formed by the interaction of heavy ions with nuclei at energies 10–100 MeV/nucleon. A form of nuclear hydrodynamics based on current and density algebra is developed to analyze linear and nonlinear excitations of the nuclear density. It is shown that for irrotational motion in the semiclassical limit the equations of nuclear hydrodynamics can be reduced to the nonlinear Schrödinger equation. It is shown that the spherically symmetric single-soliton solutions describe well the properties of the nuclear density in the ground state. The possible existence of spherically symmetric states of the nuclear density with nodes is pointed out. Analytic solutions for one-dimensional nonlinear excitations are obtained in the form of "cnoidal" waves, "kinks," and "holes" in nuclear matter. The basic equations for describing vortical "disks" of nuclear matter are obtained. The evolution of the "disk" boundary is analogous to the propagation of a nonlinear dispersion wave in a plane. The basic properties of the "disk" states are analyzed qualitatively. The inverse scattering method for the mean field of the nucleus is used to analyze the static (description of the density profiles and single-particle potentials) and dynamic (simple model for analyzing the evolution of a nucleus compressed at the initial time) properties of nuclear systems. It is shown that the giant monopole isoscalar resonances can be regarded as linearized soliton vibrations of the nuclear density. A self-consistent analytic solution is constructed for the vibrations of a double nuclear system. An investigation is made of the influence of the channel of coherent excitation of giant resonances on the emission of nonequilibrium light particles and the dissipation of the kinetic energy.

1. INTRODUCTION

In recent years, there has been a great growth of interest in the study of nonlinear dynamical phenomena in practically all branches of modern physics. To a large degree, this is due to the progress achieved in the methods of exact solution of nonlinear partial differential equations, for which the fundamental concept is the soliton—a nonlinear solitary wave capable of propagating without change of shape and energy. This remarkable phenomenon was discovered by Russell more than 150 years ago[1] in an analysis of waves traveling on the surface of water. The modern development of soliton theory was initiated by investigations into energy thermalization in nonlinear oscillatory systems.[2] The absence of thermalization in numerical computer experiments remained for a long time a paradox. The explanation of the phenomenon was found in 1965 by Zabusky and Kruskal.[3] They showed that long waves traveling in one direction along a string with quadratic interaction between the particles can be described by the Korteweg–de Vries (KdV) equation, which was first derived in an investigation of long waves in a channel with water.[4] It was shown that the absence of thermalization in the considered systems is explained by the presence of stable objects, which were called solitons. The discovery in 1967 by Gardner, Green, Kruskal, and Miura[5] of the fact that for the KdV equation $u_t-6uu_x+u_{xxx}=0$ (where the indices denote differentiation with respect to the corresponding variable) there exists an analytic method of solving the Cauchy problem and the subsequent investigations, which showed that similar methods can be applied to many other evolution equations, gave rise to a revolution in nonlinear physics.[6–8]

Investigations that use the soliton concept to analyze hadronic systems can be divided nominally into two main fields, relativistic and nonrelativistic nuclear physics, and the employed solitonlike solutions can themselves be divided into two main types.

Topological solitons. The necessary requirement here is that the boundary condition at infinity for the soliton state must differ topologically from the corresponding boundary condition for the physical vacuum. Some of the first examples of topological solitons were obtained by Polyakov[9] and 't Hooft,[10] who showed how in systems with spontaneous symmetry breaking one can have not only the ordinary sector of mesonic excitations but also the realization of heavy bosons, which arise as excitations over the nontrivial vacuum. From the quantum-field point of view, such monopoles and solitons are bunches of energy that are localized in space and can be described by classical fields. A second example is the description of baryons as topological solitons of mesonic and chiral fields.[11,12]

Nontopological solitons. The boundary conditions at infinity for the solitonlike solutions and the physical vacuum are the same. Degeneracy of the vacuum state is not required. A necessary condition for the existence of nontopological soliton solutions is the presence of additive conservation laws. In addition, for the existence of soliton

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solutions in a coordinate space of two or more dimensions it is necessary to have either an additional gauge field with nonvanishing spin or to consider solutions that depend explicitly on the energy but are nondispersive.[13]

To a large degree, this review is devoted to investigations that use the concept of a nontopological soliton in nonrelativistic nuclear physics. Besides the studies already cited, references to topological and nontopological solitons in relativistic nuclear physics can be found in the monographs and reviews of Refs. 14–17.

Nonlinear collective excitations in nonrelativistic nuclear systems are of interest for the following main reasons.

First, the development of experimental heavy-ion physics has opened up a real possibility of studying collective excitations of large amplitude in nuclear systems: giant resonances, high-spin states, superdeformed nuclei, and highly excited states of various kinds. Of particular interest are the investigations aimed at the search for new unusual modes of motion that arise when heavy ions of intermediate energy ($E\sim 10$–100 MeV/nucleon) collide with nuclei.

Second, large-amplitude collective motions constitute one of the main problems of modern theoretical nuclear physics, requiring the development of new theoretical methods.

Third, it is always of interest to find new unusual solutions of known equations that cannot be obtained by the methods of perturbation theory near some equilibrium state. In addition, the majority of nonlinear evolution equations can be reduced by means of scale transformations to a universal form, and therefore the solutions obtained, for example, for nuclear systems can be used to analyze analogous systems in plasma theory, in radiophysical media, etc.

Let us expand on this.

One of the main reasons for the intensive investigation of collisions of heavy ions with nuclei at intermediate energies is the hope of finding the unusual effects known in plasma physics[18,19] at a collision energy near the speed of "sound" ($E\sim 30$–50 MeV/nucleon) in nuclear matter.

The first indications of the possible existence of shock waves in nuclear matter appeared more than 30 years ago.[20] The classical picture of a supersonic hadronic particle passing through a heavy target nucleus is very transparent but not adequate for finite quantum nuclear systems. A microscopic description of processes associated with the passage of a fast light particle through a large nucleus was considered in Refs. 21–27. In the linear approximation, it was shown that in the "supersonic" regime collective density excitations (giant resonances) are generated in the target nucleus, forming a sharply delineated wave packet—a Cherenkov cone. The "splashing" of nucleons from the target by the front of the Cherenkov cone is very similar to the action of a classical shock wave. Despite this, the microscopic method of investigation does not require the establishment of local thermodynamic equilibrium, which is important for phenomenological classical hydrodynamics.

About 15 years ago, it was suggested that there exist new types of nonlinear waves of nuclear density in cold nuclear matter: quantum shock waves and solitons.[28–30] In Refs. 26, 28, 30, and 31, the soliton corresponded to the propagation in nuclear matter of not only isoscalar solitons but also solitons of other types: spin, isospin, and spin–isospin.[32] The use of the soliton concept gave a new impulse to investigations aimed at the search for states of nuclear matter possessing an unusual topological structure.

The traditional example is provided by "bubble" (or "hole") states of nuclear matter. The first indications of the existence of such states were found about 50 years ago,[33] and a partial investigation of these states was made in Refs. 34 and 35. Such states could be formed, in principle, by the bombardment of heavy nuclei by the lightest antiparticles (antiprotons, antideuterons,...), by annihilation with protons in $1s$, $1p$,... orbits, or in the interaction of two sufficiently heavy nuclear systems.[35] In principle, the development of experimental heavy-ion physics makes it possible to observe such states. Soliton theory introduced new nuances into this problem. Hole states of nuclear matter moving with a velocity near the speed of "sound" and leading to fragmation of the nuclear system when the soliton reaches the surface were considered in Refs. 25, 26, 31, 36, and 37. An original mechanism of formation of bubble states in nuclear systems of a new type—localized kaon states—was proposed in Refs. 38 and 39, in which an analogy between the behavior of the K^+ meson in nuclear matter and an electron in liquid helium[40] was noted. The physical picture of bubble formation is that for a sufficiently strong interaction of the impurity (kaon) with the particles of the medium (nucleons) it is energetically advantageous to create a local lowering of the density, this being disadvantageous with regard to the energy of the medium but advantageous from the point of view of the energy of the impurities, which are localized in the bubble. The properties of such excitations were investigated in detail. It was found that the typical values of the bubble parameters are for the energy $E_b\sim 100$ MeV, the mass $m\sim 5$ GeV, the radius $a\sim 2$ fm, and the minimum number of kaons in the bubble $N>4$. An increase of the K^+-meson lifetime as a result of the localization and broadening of the momentum distributions of μ^+ mesons in $K\mu$ decay was predicted.

We may add that recently theoretical predictions were again made of the possible formation in the collision region of unusual objects such as nuclear "disks"[41,42] and "blisters" and "rings."[43]

With regard to the methods used to describe collective excitations of large amplitude in complex nuclear systems, we should mention the following. The basic Schrödinger equation is linear in the many-particle wave function, but as yet it is not possible to find the complete solution of the many-particle problem. Any reduced description in a restricted space of collective degrees of freedom leads to a nonlinear problem that, in the majority of cases, can be linearized near some equilibrium state. A quantum description of the collision of two complex nuclear systems has not yet been constructed, at least for processes accompanied by large energy and matter transfer and significant rearrangement of the structure of the colliding nuclei. Collisions of two complex nuclei are mainly described by means of classical or semiclassical methods [various forms of hydrodynamics,[44,45] the time-dependent Hartree–Fock (TDHF) method,[46,47] dynamical Thomas–Fermi theory (DTFT),[48,49] and the approaches of transport theory based on the equations of kinetic type (including a self-consistent mean field and collision terms) of Boltzmann–Nordheim–Vlasov,[50] Boltzmann–Uhling–Uhlenbeck,[51] etc.]. All these approaches require appreciable computational power and resemble a black box, at the output of which the basic quantities are the density profiles and the velocity fields of the particles, the interpretation of which always contains some ambiguity. Therefore, there is always the desire to have simpler approaches, similar as regards the physics but enabling one to obtain at least some results in analytic form and make more definite predictions about the evolution of the nonlinear system. Soliton theory can also give some assistance in this question.

The review is arranged as follows. In Sec. 2 we give the basic relations for the hydrodynamic representation of the Hamiltonian. Sections 3 and 4 are devoted to vibrational and rotational excitations of nuclear density, respectively. Section 5 gives examples of the use of the inverse scattering method. In Sec. 6 there is a brief review of dissipative effects.

2. HAMILTONIAN

The use of hydrodynamic concepts in nuclear physics goes back to early studies of Bohr[52,53] and Frenkel', who proposed the liquid-drop model to describe fission. For the description of the vibrations of the drop, the two-fluid hydrodynamic model[54] was developed, and later the dynamic collective model,[55] which led to an understanding of many important properties of isoscalar giant resonances.[56] The use of hydrodynamic concepts in the description of high-energy collisions of hadrons[45] and nuclei with nuclei was associated with early studies of Landau.[57] The successful use of hydrodynamic concepts in the analysis of varied phenomena in nuclear systems initiated investigations aimed at the construction of equations of hydrodynamic type on the basis of general properties of systems of interacting nucleons (Refs. 48, 49, and 58–60). We must mention here the pioneering studies of Bogolyubov,[61,62] in which hydrodynamic equations were obtained for the first time in general form for a system of interacting particles. In recent years, various forms of this approach have been used in theoretical nuclear physics, and the approach has become known as fluid dynamics (see, for example, the reviews of Refs. 45 and 63).

We recall briefly the scheme of the quantum-hydrodynamic approach that we developed to describe relaxation phenomena in heavy-ion reactions (Refs. 24, 26, and 64–66).

The term "hydrodynamics" means that we shall describe the dynamical behavior of the nuclear system in a restricted space of collective variables representing the density and nucleon current of the system. As was pointed out in Ref. 67, the use of current operators as collective coordinates of a nonrelativistic system is by no means new and goes back to the early hydrodynamic studies of Landau.[57] Many of the ideas used in the present review have long been known in the theory of liquid helium.[40] Currents can be regarded as fundamental dynamical variables just like the corresponding canonical field operators. Such an approach is based on the algebra of current operators. The Hamiltonian of the system can be rewritten formally in terms of the current operators. The use of equal-time commutation relations between the field and current operators gives a complete dynamical description of the system. We realized these ideas for nuclear hydrodynamics.[26,65]

The logical construction of the approach corresponds to the logic of modern theories of collective nuclear motion, namely, one first selects a space of certain collective variables and then seeks an expression for the collective Hamiltonian in terms of the chosen collective variables that reproduces the commutation relations of the original many-particle Hamiltonian and the collective operators. As an example, one could take any form of theory of collective motion based on the method of boson representation of fermion operators.

The system of A nucleons with two-particle interaction is described by the Hamiltonian

$$\hat{H}=\frac{\hbar^2}{2m}\int d^3x\nabla\Psi^+(\mathbf{x})\nabla\Psi(\mathbf{x}) + \int d^3x d^3y \Psi^+(\mathbf{x})\Psi(\mathbf{x})U(\mathbf{x}-\mathbf{y})\Psi^+(\mathbf{y})\Psi(\mathbf{y}), \tag{1}$$

where the nucleon-field operators $\Psi^+(\mathbf{x})$, $\Psi(\mathbf{x})$ satisfy the anticommutation relations

$$\{\Psi^+(\mathbf{x}),\Psi(\mathbf{y})\}_+=\delta(\mathbf{x}-\mathbf{y}).$$

The collective operators of the density and current of the nucleons in the second-quantization representation are expressed in the usual manner:

$$\hat{\rho}(\mathbf{x})\equiv\Psi^+(\mathbf{x})\Psi(\mathbf{x}),$$

$$\hat{j}_k(\mathbf{x})=\frac{\hbar}{2mi}\left(\Psi^+(\mathbf{x})\frac{\partial}{\partial x_k}\Psi(\mathbf{x})-\frac{\partial}{\partial x_k}\Psi^+(\mathbf{x})\Psi(\mathbf{x})\right).$$

The operators $\hat{\rho}(\mathbf{x})$ and $\hat{j}_k(\mathbf{x})$ satisfy the commutation relations

$$\begin{aligned}
&[\hat{\rho}(\mathbf{x}),\hat{j}_k(\mathbf{y})]=-i\frac{\hbar}{m}\frac{\partial}{\partial \mathbf{x}_k}(\delta(\mathbf{x}-\mathbf{y})\hat{\rho}(\mathbf{x})),\\
&[\hat{\rho}(\mathbf{x}),\hat{\rho}(\mathbf{y})]=0,\\
&[\hat{j}_k(\mathbf{x}),\hat{j}_l(\mathbf{y})]=-i\frac{\hbar}{m}\frac{\partial}{\partial \mathbf{x}_l}(\delta(\mathbf{x}-\mathbf{y})\hat{j}_k(\mathbf{x}))\\
&\qquad+i\frac{\hbar}{m}\frac{\partial}{\partial \mathbf{x}_k}(\delta(\mathbf{x}-\mathbf{y})\hat{j}_l(\mathbf{y})).
\end{aligned} \tag{2}$$

We consider the equations of motion for the operators $\hat{\rho}(\mathbf{x})$ and $\hat{j}_k(\mathbf{x})$:

$$\frac{\partial\hat{\rho}(\mathbf{x})}{\partial t}=\frac{1}{i\hbar}[\hat{\rho}(\mathbf{x}),\ \hat{H}]=-\sum_{n=1}^{3}\frac{\partial}{\partial x_k}\hat{j}_k(\mathbf{x}), \tag{3}$$

$$\frac{\partial \hat{j}_k(\mathbf{x})}{\partial t}=\frac{1}{i\hbar}[\hat{j}_k(\mathbf{x}), \quad \hat{H}]$$

$$=-\frac{\hbar^2}{2m^2}\sum_{n=1}^{3}\frac{\partial}{\partial x_n}\left(\hat{T}_{nk}(\mathbf{x})-\frac{1}{2}\frac{\partial^2\hat{\rho}(\mathbf{x})}{\partial x_n\partial x_k}\right)$$

$$-\frac{2}{m}\hat{\rho}(\mathbf{x})\frac{\partial}{\partial x_k}\int d^3yU(\mathbf{x}-\mathbf{y})\hat{\rho}(\mathbf{y}), \qquad (4)$$

where the tensor of the kinetic-energy density

$$\hat{T}_{nk}(\mathbf{x})=\frac{\partial}{\partial x_n}\Psi^+(\mathbf{x})\frac{\partial}{\partial x_k}\Psi(\mathbf{x})$$

$$+\frac{\partial}{\partial x_k}\Psi^+(\mathbf{x})\frac{\partial}{\partial x_n}\Psi(\mathbf{x})$$

satisfies the commutation relations

$$\left[\int \hat{T}_{kl}(\mathbf{x}')d^3x',\hat{\rho}(\mathbf{x})\right]$$

$$=-\frac{2m}{i\hbar}\left(\frac{\partial}{\partial \mathbf{x}_k}\hat{j}_l(\mathbf{x})+\frac{\partial}{\partial \mathbf{x}_l}\hat{j}_k(\mathbf{y})\right),$$

$$\left[\int \hat{T}_{kl}(\mathbf{x}')d^3x',\hat{j}_n(\mathbf{x})\right] \qquad (5)$$

$$=\frac{i\hbar}{m}\left(\frac{\partial}{\partial \mathbf{x}_k}\hat{T}_{ln}(\mathbf{x})+\frac{\partial}{\partial \mathbf{x}_l}\hat{T}_{kn}(\mathbf{x})-2\frac{\partial}{\partial \mathbf{x}_k}\frac{\partial}{\partial \mathbf{x}_l}\frac{\partial}{\partial \mathbf{x}_n}\hat{\rho}(\mathbf{x})\right).$$

It can be seen from this that if we find an expression for $\hat{T}_{kl}$ in terms of operators $\hat{\rho}(\mathbf{x})$ and $\hat{j}_k(\mathbf{x})$ that satisfy the commutation relations (5), we should have a closed system of equations of motion containing only the operators $\hat{\rho}(\mathbf{x})$ and $\hat{j}_k(\mathbf{x})$.

There exists[67] the following expression for $\hat{T}_{kl}$ in terms of operators $\hat{\rho}(\mathbf{x})$ and $\hat{j}_k(\mathbf{x})$ satisfying the commutation relations (5) and, from this point of view, equivalent to the original expression for $\hat{T}_{kl}$ in terms of the nucleon-field operators $\Psi^+(\mathbf{x})$, $\Psi(\mathbf{x})$:

$$\hat{T}_{kl}=\frac{m^2}{\hbar^2}(\hat{j}_k\hat{\rho}^{-1}\hat{j}_l+\hat{j}_l\hat{\rho}^{-1}\hat{j}_k)+\frac{1}{2}\hat{\rho}^{-1}\frac{\partial}{\partial \mathbf{x}_k}\hat{\rho}\frac{\partial}{\partial \mathbf{x}_l}\hat{\rho}$$

$$+\text{const}\,\delta_{kl}.$$

Finally, we can write down the following approximate hydrodynamic expression for the Hamiltonian of the system that is equivalent to the original one (1) from the point of view of the equations of motion (2)–(4) for the operators $\hat{\rho}(\mathbf{x})$ and $\hat{j}_k(\mathbf{x})$:

$$\hat{H}_h=\frac{m}{2}\sum_{k=1}^{3}\int \hat{j}_k\frac{1}{\hat{\rho}}\hat{j}_kd^3x+\frac{\hbar^2}{8m}\int\frac{|\nabla\hat{\rho}|^2}{\hat{\rho}}d^3x$$

$$+\int d^3xd^3y\hat{\rho}(\mathbf{x})U(\mathbf{x}-\mathbf{y})\hat{\rho}(\mathbf{y}). \qquad (6)$$

The equations of motion (3) and (4) for the operators $\hat{\rho}(\mathbf{x},t)$ and $\hat{j}_k(\mathbf{x},t)$ have formally the form of an operator continuity equation and an operator equation of Euler type:

$$\frac{\partial\hat{\rho}}{\partial t}+\sum_{k=1}^{3}\frac{\partial\hat{j}_k}{\partial x_k}=0, \qquad (7)$$

$$\frac{\partial\hat{j}_k}{\partial t}=-\sum_{n=1}^{3}\frac{\partial}{\partial x_k}(\hat{j}_k\hat{\rho}^{-1}\hat{j}_n-\hat{P}_{kn}), \qquad (8)$$

where $\hat{P}_{kn}(\mathbf{x},t)$ is the operator of the "pressure" tensor.

We give the expressions for the integrals of the motion—the total particle number A, the total momentum $\mathbf{P}$, and the angular momentum $\mathbf{L}$ of the system:

$$A=\int\rho d^3x, \quad \mathbf{P}=m\int\mathbf{j}(\mathbf{x})d^3x,$$

$$\mathbf{L}=m\int(\mathbf{x}\times\mathbf{j}(\mathbf{x}))d^3x.$$

In classical hydrodynamics,[69] the fundamental variables are the local density $\rho(\mathbf{x})$ and the local velocity field $\mathbf{v}(\mathbf{x})$, by means of which the flow function is defined as follows:

$$\hat{\mathbf{j}}(\mathbf{x})=\tfrac{1}{2}[\hat{\rho}(\mathbf{x})\hat{\mathbf{v}}(\mathbf{x})+\hat{\mathbf{v}}(\mathbf{x})\hat{\rho}(\mathbf{x})]. \qquad (9)$$

We use a symmetrized form of expression for the transition to quantum mechanics. However, the concept of velocity is not so well defined in quantum mechanics as that of the flux. The velocity is not a canonical variable. In addition, $\mathbf{j}(\mathbf{x})$ and $\rho(\mathbf{x})$ are related by the particle conservation law in the form of the continuity equation. Therefore, there exist problems in the construction of a velocity operator that plays the corresponding role in classical hydrodynamics. Landau[57] defined the velocity operator as follows:

$$\hat{\mathbf{v}}(\mathbf{x})=\frac{1}{2}\left[\frac{1}{\hat{\rho}(\mathbf{x})}\hat{\mathbf{j}}(\mathbf{x})+\hat{\mathbf{j}}(\mathbf{x})\frac{1}{\hat{\rho}(\mathbf{x})}\right].$$

To determine the commutation relations of the velocity operator, we shall use (9). We regard this choice as preferable; it enables us to defer the use of the operator of the reciprocal density to the final stage.

By analogy with classical hydrodynamics, we can separate in operator form the irrotational and purely vortical components of the flux or velocity operators.[68]

The irrotational condition for $\hat{\mathbf{v}}(\mathbf{x})$ can be ensured if we introduce in the usual manner an operator $\hat{\phi}(\mathbf{x})$ of the velocity potential by means of the relation

$$\hat{\mathbf{v}}(\mathbf{x})=\nabla\hat{\phi}(\mathbf{x}), \qquad (10)$$

which reduces the commutation relations (2) to the canonical form

$$[\hat{\rho}(\mathbf{x}),\hat{\phi}(\mathbf{y})]=\frac{i\hbar}{m}\delta(\mathbf{x}-\mathbf{y}), \quad [\hat{\rho}(\mathbf{x}),\hat{\rho}(\mathbf{y})]=0,$$

$$[\hat{\phi}(\mathbf{x}),\hat{\phi}(\mathbf{y})]=0. \qquad (11)$$

A nonlinear quantum-mechanical description of nuclear systems has not yet been constructed (although certain efforts have been made in this direction[70]), and we cannot construct a general solution of the nonlinear operator equations (2), (3) or (7), (8). In the following sections, we shall give solutions of the nonlinear equations obtained in the semiclassical approximation and linearized solutions for quantum vibrations.

3. VIBRATIONAL EXCITATIONS

General remarks

Vibrations of the nuclear density are the subject of constant interest in nuclear physics. They represent the simplest type of excitation (nonrotational motion). Currently, the best studied are the giant monopole resonances, which are linear vibrations of the equilibrium density of an isolated nucleus. The study of heavy-ion reactions posed new problems in the theoretical description of vibrations of the nuclear density.

To understand the mechanism of deep inelastic interaction of heavy ions with nuclei,[71] it is helpful to use the concept of doorway states, which are formed in the initial stage of the collision and determine the further relaxation of the system. Among the numerous degrees of freedom of colliding nuclei, some of the most probable candidates for doorway states are vibrations of the nuclear density (giant resonances) in the colliding nuclei or in the double nuclear system formed as a result of the interaction. Such states may be important for understanding the dissipation of kinetic energy[22,64,72] and the mechanism of formation of fast light particles[22–24,27] in deep inelastic collisions.

The method used to describe the vibrations of the nuclear density in such calculations must satisfy several requirements simultaneously, namely, it must admit direct generalization to the case of a double nuclear system, be sufficiently consistent (the properties of the excitations must be consistent with the original many-particle Hamiltonian of the system, and the fundamental conservation laws must hold), adequate (reproduce as many of the existing experimental data on the properties of the excitations as possible), and simple, since information on the properties of such excitations is used as input data for subsequent rather complicated calculations. Among the existing methods for describing vibrations of the nuclear density,[56,73–76] semimicroscopic nuclear hydrodynamics matches these requirements best.

In Ref. 65, we began the description of vibrations of the nuclear density for relaxation processes in collisions of heavy ions with nuclei. We obtained a very simple analytic solution that models the description of density vibrations in a double nuclear system.[66] Analysis of the vibrations of an isolated nucleus possessing a diffuse boundary revealed[26] a connection between this problem and a traditional problem of soliton theory—the stability of nonlinear waves.[77–79] The nucleus is regarded as a nuclear-density soliton. The existence of the diffuse layer is a consequence of the choice of the effective interaction and the symmetry of the problem. The fundamental conservation laws are satisfied. The scheme used to describe the vibrations is completely self-consistent. After scale transformations, the nucleus is completely equivalent to other nonlinear dispersive media that evolve in accordance with a nonlinear Schrödinger equation of the ψ^3–ψ^5 type. We briefly present below the basic relations of the approach.

Nonlinear Schrödinger equation

In the semiclassical limit, the equations of motion (7), (8) in the case of irrotational flow (10) can be reduced to the nonlinear Schrödinger equation[26]

$$i\hbar\frac{\partial u}{\partial t}=-\frac{\hbar^2}{2m}\Delta u+\frac{\delta\mathscr{E}[|u|^2]}{\delta|u|^2}u, \tag{12}$$

with local density and velocity potential

$$\rho(\mathbf{x},t)=|u(\mathbf{x},t)|^2,\quad \phi(\mathbf{x},t)=(\hbar/m)\arg u(\mathbf{x},t). \tag{13}$$

The functional derivative of the effective interaction with respect to the density, $\delta\mathscr{E}[\rho]/\delta\rho$, completely determines the type of nonlinearity of Eqs. (12) and (13).

We choose the interaction functional in the form of Skyrme forces[80] and renormalize the parameters of the Hamiltonian:

$$\begin{gathered}\int d^3x d^3y\rho(\mathbf{x})U(\mathbf{x}-\mathbf{y})\rho(y)\Rightarrow\int d^3x\mathscr{E}[\rho(\mathbf{x})],\\ m\Rightarrow m^*\equiv(m^{-1}+(3t_1+5t_2)\rho_N/8\hbar^2)^{-1},\\ \frac{\hbar^2}{8m}\Rightarrow\frac{\hbar^2\eta^2}{8m}\equiv\frac{\hbar^2}{8m}+\frac{\rho_N}{64}(9t_1-5t_2),\\ \mathscr{E}[\rho]\Rightarrow\frac{3}{8}t_0\rho^2+\frac{1}{16}t_3\rho^3,\end{gathered} \tag{14}$$

where ρ_N is the density of the nuclear matter, and t_0, t_1, t_2, t_3 are the parameters of the Skyrme forces.

These very simple density-dependent effective forces have been successfully used in Hartree–Fock calculations to describe the properties of nuclei in the ground state[81,82] and in various dynamical calculations.[47,48]

In the case of the Skyrme-type interaction (14), making the scale transformation[26]

$$\mathbf{x}\equiv(\hbar\eta/m^*c_s)\mathbf{x}',\quad t\equiv(m^*c_s^2/2\hbar)\tau, \tag{15}$$

$$\rho(\mathbf{x},t)\equiv\rho_N\psi^2(\mathbf{x}',\tau), \tag{16}$$

where c_s is the speed of sound in nucler matter,

$$c_s\equiv\left(\frac{1}{m^*}\rho^2\frac{\delta^2(\mathscr{E}/\rho)}{\delta\rho^2}\bigg|_{\rho=\rho_N}\right)^{1/2}, \tag{17}$$

we obtain the dimensionless form of Eq. (12):

$$i\frac{\partial\psi}{\partial\tau}=-\Delta'\psi-4|\psi|^2\psi+3|\psi|^4\psi. \tag{18}$$

Note the very fact that the equations of irrotational nuclear hydrodynamics can be expressed in the form of a nonlinear Schrödinger equation. Such equations with different types of nonlinearity have been successfully used to analyze various nonlinear phenomena.[7,8] The most popular and best studied is the cubic, ψ^3, nonlinear Schrödinger equation.[83,84] The most important reasons for this are that this equation is completely integrable, for the given type of equation there exists a well-developed scheme of integration by means of the inverse scattering method, expressions for calculating many-soliton solutions, etc.

The nonlinear Schrödinger equation (18) of the $\psi^3-\psi^5$ type is not, unfortunately, a completely integrable system, although some results can be obtained in analytic form.[26,31,36] The $\psi^3-\psi^5$ equation has already been encountered in the theory of plasma waves.[85–87] An equation of this type has not yet been well studied.

It is interesting to note that field-theory equations can be associated with the equations of relativistic hydrodynamics. For example, the relativistic form of Eq. (18) corresponds to the nonlinear $\Phi^4-\Phi^6$ field theory ($\Box\Phi=\kappa^2\Phi-\mu^2|\Phi|^2\Phi+|\Phi|^4\Phi$), which was studied in Refs. 88 and 89.

The scale transformation (15), (16) gives a convenient set of dimensional factors: the effective wavelength $\hbar\eta/m^*c_s$, the energy $m^*c_s^2/2$, and the density ρ_N. This makes it possible to separate general properties of the system determined by the type of nonlinearity (interaction polynomial in the density and competition between attraction and effective repulsion) from the effects associated with the choice of the interaction parameters, which will determine the scale of the quantities.

For appropriate choice of initial and boundary conditions, the nonlinear Schrödinger equation (18) can be used, in conjunction with the already mentioned other semiclassical methods, to analyze various dynamical phenomena in complex nuclear systems. We list some problems.

- It can be shown that the nucleus itself is a nonlinear density wave, and one can consider the collision of two nonlinear waves (nuclei) or the process of interaction of a wave with an external field.
- It is interesting to find solutions of the type of various stationary and moving nonlinear waves in nuclear systems: localized excitations (solitons), nonlinear vibrations of the system, nonlinear vortices, blisters, rings, etc.
- One can analyze the evolution of excited nuclear systems and the relaxation of various excitations. Particularly interesting is the problem of the breakup of excited nuclear systems into individual subsystems (analogy with multifragmentation).
- One can show that the equations we obtained earlier to describe vibrations of nuclear density[65,66] are directly related to the conditions for stability of the single-soliton solutions of the nonlinear Schrödinger equation (18). Giant monopole resonances can be regarded as linearized solutions of Eq. (18), as vibrations of a nonlinear wave.

Solutions of some of these problems are given in this review.

Properties of soliton solutions

We consider the properties of soliton solutions of Eqs. (12) and (18). Such solitary waves are usually sought in the form

$$\psi_0(\mathbf{x}',\tau)\equiv\psi_0(\mathbf{r})\exp(i(\lambda-v'^2/2)\tau+i\mathbf{r}\mathbf{v}'+\sigma_0),$$

where $\mathbf{r}=\mathbf{x}'-\mathbf{v}'\tau-\mathbf{r}_0$, and the parameters $\mathbf{r}_0$, σ_0, $\mathbf{v}'$, λ are determined, respectively, by the initial position $\mathbf{r}_0$ of the center of the wave, the initial phase shift σ_0, and the momentum and energy of the wave. The parameter λ is determined from the condition of conservation of the number of nucleons of the system:

$$A=\rho_N\left(\frac{m^*c_s}{\hbar\eta}\right)^3\int d^3r\psi_0(\mathbf{r})^2. \tag{19}$$

The function $\psi_0(\mathbf{r})$ is the solution of the equation

$$\Delta_r\psi_0-\lambda\psi_0+4\psi_0^3-3\psi_0^5=0 \tag{20}$$

that satisfies the boundary conditions

$$\psi_0(\infty)=\frac{\partial\psi_0}{\partial r}(\infty)=\frac{\partial\psi_0}{\partial r}(0)=0. \tag{21}$$

This equation realizes an extremum of the action with Lagrangian

$$\mathscr{L}=\mathscr{T}+\lambda\mathscr{I}_2/2-\mathscr{I}_4+\mathscr{I}_6/2,$$

$$\mathscr{T}\equiv\frac{1}{2}\int_0^\infty drr^{M-1}\left|\frac{\partial}{\partial r}\psi_0\right|^2,\quad \mathscr{I}_n\equiv\int_0^\infty drr^{M-1}\psi_0^n,$$

where M is the dimension of space. The functionals $\mathscr{T}(\lambda)$, $\mathscr{I}_n(\lambda)$ are related by two constraints. We multiply Eq. (20) by $r^{M-1}\psi_0$ and integrate over r from zero to infinity. With allowance for the boundary conditions (21),

$$2\mathscr{T}+\lambda\mathscr{I}_2-4\mathscr{I}_4+3\mathscr{I}_6=0. \tag{22}$$

We obtain a second relation by means of the scale transformation $r'\equiv\alpha r$, $\psi_0^\alpha\equiv\psi_0(\alpha,r)$, under which $\mathscr{T}^\alpha=\alpha^{M-2}\mathscr{T}$, $\mathscr{I}_n^\alpha=\alpha^M\mathscr{I}_n$, and the variational principle $(d\mathscr{L}^\alpha/d\alpha)|_{\alpha=1}=0$.

As a result, we have the virial theorem

$$(M-2)\mathscr{T}+M(\lambda\mathscr{I}_2/2-\mathscr{I}_4+\mathscr{I}_6/2)=0. \tag{23}$$

Eliminating from (22) and (23) the functional $\mathscr{T}$, we obtain the exact relations

$$\mathscr{I}_4=\lambda\mathscr{I}_2,\quad M=3,\quad \lambda\mathscr{I}_2-3\mathscr{I}_2+2\mathscr{I}_6=0,\quad M=1. \tag{24}$$

The relations (22)–(24) facilitate the calculation of physical observables. In particular, calculating the mean energy of the system per nucleon (with allowance for conservation of the total number of nucleons),

$$\frac{E-\Lambda A}{A}=\frac{mv^2}{2}-\Lambda+\frac{m^*c_s^2}{\mathscr{I}_2}\left(\mathscr{T}-\mathscr{I}_4+\frac{\mathscr{I}_6}{2}\right),$$

and using the relation (22), we can eliminate the gradient terms:

$$\frac{E-\Lambda A}{A}=\frac{mv^2}{2}-\Lambda+\frac{m^*c_s^2}{2}(-\lambda+2\mathscr{I}_4-2\mathscr{I}_6/\mathscr{I}_2).$$

A further simplification is achieved by means of the relations (24):

$$\frac{E-\Lambda A}{A}=\frac{mv^2}{2}-\Lambda+\frac{m^*c_s^2}{2}(\lambda-2\mathscr{I}_6/\mathscr{I}_2),\quad M=3,$$

$$\frac{E-\Lambda A}{A}=\frac{mv^2}{2}-\Lambda-\frac{m^*c_s^2}{2}(\mathscr{I}_4/\mathscr{I}_2),\quad M=1.$$

The density of the nuclear system in the ground state is $\rho_0(\mathbf{x})=\rho_N\psi_0(\mathbf{r})^2$. The expression for the mean momentum of the system per nucleon is $\mathbf{P}/A=m\mathbf{v}$.

One-dimensional solutions. In the one-dimensional case, Eqs. (20) have the analytic single-soliton solutions

$$\psi_0(r)=(\lambda/(1+(1-\lambda)^{1/2}\cosh(\lambda^{1/2}r))^{1/2}. \quad (25)$$

The analytic form of the solution (25) makes it possible to obtain an explicit expression for the "particle number"

$$\mathscr{A}\equiv\int_{-\infty}^{\infty}dr\rho(r)=\rho_0\frac{\hbar\eta}{m^*c_s}\operatorname{arctanh}(\sqrt{\lambda}),$$

and the energy per particle:

$$\frac{E-\Lambda\mathscr{A}}{\mathscr{A}}=\frac{m^*v^2}{2}-\Lambda-\frac{m^*c_s^2}{2}\left(1-\frac{\sqrt{\lambda}}{\operatorname{arctanh}(\sqrt{\lambda})}\right).$$

The single-soliton solution (25) describes the so-called layer used in one-dimensional TDHF[46,47] and DTTF[48] calculations when the three-dimensional collision of two nuclei is modeled by the collision of two one-dimensional "layers." The number of nucleons of the system is related as follows to the "number of particles": $A\approx\mathscr{A}^3\pi/6\rho_N^2$. The existence of the analytic solution (25) simplifies dynamical calculations, since the initial conditions can be specified analytically, and one need not solve numerically equations of the Thomas–Fermi type. The distribution of the "layer" density can be expressed in the form of the symmetrized Fermi distribution

$$\rho_0(x)=\rho_N\frac{\sinh(\mathscr{R}/\mathscr{D})}{\cosh(\mathscr{R}/\mathscr{D})+\cosh(x/\mathscr{D})}, \quad (26)$$

with "layer radius"

$$\mathscr{R}=\frac{\hbar\eta}{m^*c_s}\frac{1}{2\sqrt{\lambda}}\operatorname{arctanh}(\sqrt{\lambda}),$$

and "diffuseness"

$$\mathscr{D}=\frac{\hbar\eta}{m^*c_s}\frac{1}{2\sqrt{\lambda}}.$$

Spherically symmetric solutions. To analyze spherically symmetric solutions of Eq. (20), we rewrite it in the form

$$\psi'_{r'r'}+2\psi'_{r'}=\psi'-\psi'^3+B\psi'^5, \quad (27)$$

making the auxiliary scale transformation

$$\psi'\equiv2\lambda^{-1/2}\psi,\quad r'\equiv\lambda^{1/2}r,\quad B\equiv\lambda B_0,\quad B_0\equiv3/16. \quad (28)$$

In this section, the parametrization $\lambda\to B$ will be more convenient for analyzing the transition to the well-studied cubic Schrödinger equation with $B\to0$.

We consider Eq. (27) in the phase plane $(\psi',\psi'_{r'})$. For this, it is convenient to identify r' with a "time," and regard (27) as the equation of motion of an effective "particle" under the influence of conservative "forces" $F(\psi')\equiv-dV/d\psi'$ and "friction forces" $-2\psi'_{r'}/r'$. The "potential" $V\equiv\psi'^2/2+\psi'^4/4-B\psi'^6/6$ has minima at the points

$$\psi'=\pm X_1\equiv\pm\left(\frac{1-\sqrt{1-4B}}{2B}\right)^{1/2}$$

and maxima at

$$\psi'=0,\quad \psi'=\pm X_2\equiv\pm\left(\frac{1+\sqrt{1-4B}}{2B}\right)^{1/2}.$$

The "potential" V vanishes at the points

$$\psi'=0,\quad \psi'=\pm X_3\equiv\pm\left(\frac{1-\sqrt{1-16B/3}}{(4B/3)}\right)^{1/2},$$

$$\psi'=\pm X_4\equiv\pm\left(\frac{1+\sqrt{1-16B/3}}{(4B/3)}\right)^{1/2}.$$

The picture of the phase trajectories for Eq. (27) is qualitatively the same as in the case of cubic nonlinearity, $B=0$,[86] or in the case with saturation of the nonlinearity ("force" $F(\psi')=\psi'^3/(1+\psi'^2)$, Ref. 87). For $B<B_0$, any trajectory that begins on the axis $\psi'_{r'}=0$ ends as $r'\to\infty$ at one of three points: (0,0) or in the "wells" $(\pm X_1,0)$. By a gradual change of the initial amplitude $\psi'(0)$ one can obtain a countable set of soliton trajectories that begin on the $\psi'(0)$ axis and end as $r'\to\infty$ at the origin. To each such trajectory there corresponds a strictly definite initial value of the amplitude $\psi'_c(0)$. An arbitrarily small deviation of $\psi'(0)$ from $\psi'_c(0)$ has the consequence that as $r'\to\infty$ the trajectories will end, not at the origin, but in one of the "wells." The region of existence of soliton solutions is $B<B_0=3/16$, and $X_3(B)<\psi'_c(0)<X_2(B)$. Examples of phase trajectories for an equation of the type (27) can be found in Ref. 88.

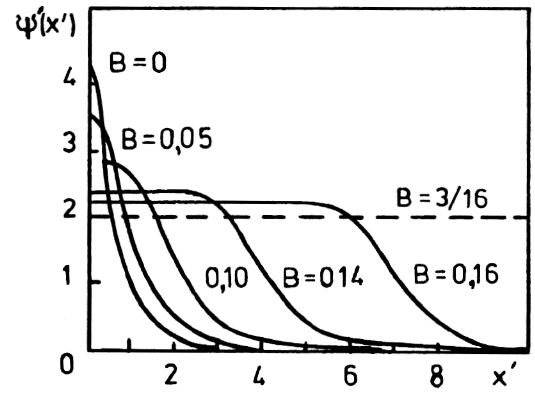

FIG. 1. Nodeless spherically symmetric single-soliton solutions of Eq. (27).

Figure 1 gives the results of numerical integration of Eq. (27) for nodeless single-soliton solutions. It can be seen that in this case too the amplitudes $\psi'_c(0)$ lie in the fairly narrow range $2\leqslant\psi'_c(0)\leqslant4.337$. As in the one-dimensional case, the transition $B\to B_0$ corresponds to transition to nuclear matter.

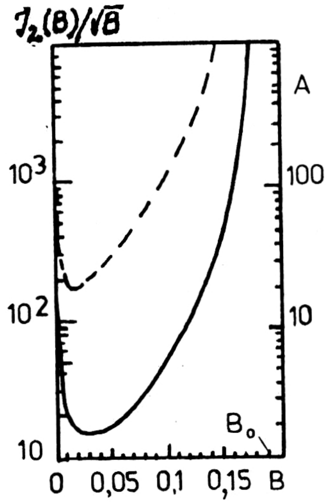

FIG. 2. Dependence of the dimensionless normalization $\mathscr{I}_2/\sqrt{B}$ on the parameter B. The ordinate A is the corresponding number of nucleons of the system. The continuous curve represents the nodeless solutions, and the broken curve represents the solutions with one node.

Figure 2 gives the results of calculating the dimensionless renormalization function $\mathscr{I}_2' \equiv \int_0^\infty dr' r'^2 \psi'^2$, which is related to the number of nucleons of the system by a relation analogous to (19):

$$A = \rho_N \left(\frac{m^* c_s}{\hbar \eta} \right)^3 \pi \left(\frac{B_0}{B} \right)^{1/2} \mathscr{I}_2'. \tag{29}$$

It can be seen that the dependence $A(B)$ is very pronounced. Moreover, the region from light ions, $A > 0$, to all existing nuclei lies in the range $0.17 > B > 0.11$ or $1 > \lambda > 0.02$.

For $B > 0.1$, one can construct an approximate analytic description of the exact single-soliton solutions of Eq. (27) or (20). For this, we choose a trial function in the form

$$\psi_{SF}(r') = \psi_c(0) \left[\frac{\sinh(\mathscr{R}'/\mathscr{D}')}{\cosh(\mathscr{R}'/\mathscr{D}') + \cosh(r'/\mathscr{D}')} \right]^{1/2}. \tag{30}$$

This trial dependence is chosen by analogy with the single-soliton solution (26) of the one-dimensional problem and satisfies the necessary boundary conditions. In addition, since

$$\rho(r) = \rho_N \left(\frac{B}{4B_0} \right) (\psi'(r'))^2, \tag{31}$$

the function $\rho(r)$ in this case has a form that is close to the spherically symmetric Fermi distribution $\rho_{SF}(r)$:[90]

$$\rho_{SF}(r) \equiv \rho_N \frac{\sinh(R/d)}{\cosh(R/d) + \cosh(r/d)}. \tag{32}$$

In all the main properties, ρ_{SF} is close to the ordinary Fermi distribution ρ_F, namely, R determines the half-decay of the density, and d is the width of the surface layer. At large distances, ρ_{SF} decreases exponentially. In addition, ρ_{SF} possesses at least two advantages compared with ρ_F: 1) the most important integrals of ρ_{SF} (normalization, rms radius, Fourier transform) can be calculated analytically; 2) at the origin, ρ_{SF} has a derivative equal to zero. This last circumstance is necessary if the approximate solution (31)–(32) is to satisfy the requirements of translational invariance. Thus, the popular phenomenological expression for the density ρ_{SF} can be regarded as an approximate solution of the ψ^3–ψ^5 nonlinear Schrödinger equation.

The obtained solutions of Eqs. (20) and (27) match the results of Hartree–Fock calculations and reproduce well the properties of the nuclear density in the ground state. The $\sim A^{1/3}$ dependence of the rms radius on the mass number is correctly reproduced. It can be seen in Fig. 1 that the density at the center of the nucleus gradually decreases on the transition from light to heavier nuclei. For medium and heavy nuclei there exists a region of constant density and a surface layer. The layer diffuseness $d = (\hbar\eta/m^* c_s)\sqrt{B_0/B}\mathscr{D}'(B)$ is approximately constant.

Note that the constructed description of the properties of the nuclear density on the basis of the solutions of Eqs. (20) and (27) is significantly better than the description based on the KdV equation or the cubic Schrödinger equation,[83,91] for which $\rho \sim \mathrm{sech}(r')$ (see Fig. 1, $B=0$).

Solutions with nodes. Besides the nodeless solutions considered above, Eqs. (20) and (27) can have solutions with nodes.

We consider the one-dimensional case. We multiply both sides of Eq. (18) by $\psi'_{r'}$ and integrate once:

$$\psi'_{r'} = \pm \sqrt{2(E' - V'(\psi'))}$$

$$= \pm (2E' + \psi'^2 - \psi'^4/2 + B\psi'^6/3)^{1/2},$$

in which the constant of integration E' is an "energy," and V' is a "potential." The substitution $Z \equiv \psi'^2$ reduces this equation to the form

$$Z_{r'} = \pm 2(Z(2E' + Z - Z^2/2 + BZ^3/3))^{1/2}.$$

Since the radicand is a polynomial of fourth degree, this expression can be integrated in elliptic functions. The explicit form of the solution is determined by the constants of integration and boundary conditions. The single-soliton solution (25) corresponds to the separatrix $E'=0$, $Z_{r'}(\infty) = Z(\infty) = Z_{r'}(0) = 0$. Among the other solutions there are solutions that are analogs of the well-known cnoidal waves of the KdV equation. For the nucleus, they will have the meaning of nonlinear periodic vibrations of the nuclear surface. Choosing nonzero boundary values, we can obtain solutions of kink type (motion of a wave front) and "bubble" solutions (a localized rarefaction pulse moving through nuclear matter). A detailed analysis of these solutions can be found in Refs. 26 and 36.

For spherical symmetry, the spectrum of the nodes with solutions is much less rich. There are no periodic solutions, but there are soliton solutions possessing nodes.

Figure 3 shows an example of a solution with one node. In Fig. 2 we give the graph of the dependence $A(B)$ for the solutions with one node. It should be noted that the solutions with one node can appear at a certain minimum number of nucleons. For the choice that we have made for

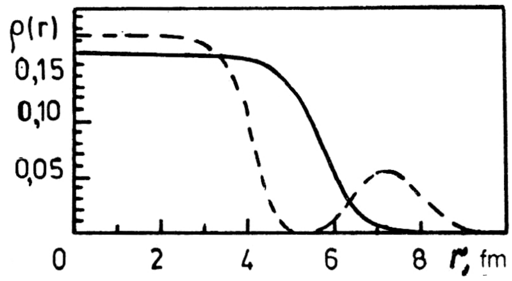

FIG. 3. Example of spherically symmetric nuclear density in the ground state (continuous curve) and isomer state with one node (broken curve); $A \sim 130$.

the parameters of the Skyrme forces, the number is $A > 19$ (Fig. 2). In Fig. 3, for an $A \sim 130$ nucleus, we show the lowest nodeless solution (the ground state of the nucleus) and the first solution with a node (density isomer). This state is metastable and has the high excitation energy $E^* \sim 10$ MeV/nucleon.

Stability of soliton solutions. To analyze the stability of the single-soliton solutions, we use the Q theorem,[15,85,89] which for the given nonlinear Schrödinger equation is formulated as follows:[26] the nodeless soliton solutions are stable when $dA/d\lambda > 0$. In the one-dimensional case, the derivative can be calculated analytically and $d\mathscr{A}/d\lambda > 0$ for $\lambda < 1$ (or $B < B_0$), i.e., the condition of stability in the one-dimensional case is identical to the condition for the existence of the soliton solution. In the three-dimensional case, $dA/d\lambda > 0$ for $d\mathscr{I}_2/d\lambda > 0$ or $d(\mathscr{I}_2'/\sqrt{B})/dB > 0$. It can be seen from Fig. 2 that the condition of stability for spherically symmetric single-soliton solutions is satisfied when $B > 0.025$ (or $A > 1.7$), i.e., for all nuclei. Note that for $B=0$ the solution is unstable. In plasma theory, this corresponds to the collapse of the Langmuir plasma waves[87] or instability of the particlelike solutions in a nonlinear field theory of the Φ^4 type.[88,89] For nuclear systems, $B=0$ means the absence of repulsion, $t_3=0$, and therefore the remaining attraction leads to collapse of the nuclear system. One can also show that the stability of the single-soliton solution corresponds to a real spectrum of vibration frequencies of a spherical nucleus with a diffuse boundary.[77–79]

Solitons in nuclear matter. We obtain approximate solutions for a density fluctuation moving in nuclear matter with speed differing little from the speed of "sound."[26,30,31]

In the semiclassical limit, Eqs. (3), (4) or (7), (8) for the Skyrme-type forces (14) can be written as follows:

$$\frac{\partial \rho}{\partial t} + \nabla(\rho \mathbf{v}) = 0,$$
$$\frac{\partial \mathbf{v}}{\partial t} + \mathbf{v}\nabla\mathbf{v} = \frac{\hbar^2\eta^2}{4m^{*2}}\left(\frac{\Delta\rho}{\rho} - \frac{|\nabla\rho|^2}{2\rho^2}\right) - \frac{c_s^2}{2}\nabla\left(3\left(\frac{\rho}{\rho_N}\right)^2 - 4\frac{\rho}{\rho_N}\right). \quad (33)$$

We consider the motion of a density fluctuation in nuclear matter with speed close to the speed of "sound." For this we use the scale transformation

$$\xi \equiv \alpha^{1/2}(\mathbf{x}_1 - c_s t)/c_s^2, \quad \tau \equiv \alpha^{3/2} t/c_s, \quad x_2 = x_2, \quad x_3 = x_3, \quad (34)$$

where the x_1 axis is taken along the ion beam that is incident on the nuclear matter and creates the fluctuation of the nuclear density. The transition $(x_1, t) \to (\xi, \tau)$ corresponds to transition to a coordinate system moving with speed close to the speed of "sound." The parameter $\alpha \ll 1$ characterizes the deviation of this speed of "sound" c_s. Solutions for (33) are sought in the form of expansions with respect to the parameter α:

$$\rho(\mathbf{x}, t) = \rho_N(1 + \alpha\rho_{(1)} + \alpha^2\rho_{(2)} + \ldots), \quad \mathbf{v} = c_s(\alpha\mathbf{v}_{(1)} + \alpha^2\mathbf{v}_{(2)} + \ldots). \quad (35)$$

Substituting (34) and (35) in the equations of motion (33) and collecting together the terms having equal powers in α, we find that the motion along the x_2 and x_3 axes separates in the first orders in α from the motion along the ξ and τ axes. Therefore, in what follows we omit the vector symbol of the variables $\mathbf{v}_{(1)}$ and $\mathbf{v}_{(2)}$, understanding by this their projections onto the x_1 axis. Separating in Eqs. (33) the terms linear in α, we obtain

$$\frac{\partial \rho_{(1)}}{\partial \xi} = \frac{\partial v_{(1)}}{\partial \xi}. \quad (36)$$

The terms quadratic in α give a system of two coupled equations for $\rho_{(1)}, v_{(1)}, \rho_{(2)}, v_{(2)}$:

$$\frac{\partial \rho_{(1)}}{\partial \tau} + \frac{\partial}{\partial \xi}(v_2 + \rho_{(2)} + v_{(1)}\rho_{(1)}) = 0, \quad \mathscr{C} \equiv \frac{\hbar^2\eta^2}{4m^* c_s^6}, \quad (37)$$

$$\frac{\partial v_{(1)}}{\partial \tau} + \frac{\partial}{\partial \xi}(\rho_{(2)} - v_{(2)} + v_{(1)}^2/2 + 3\rho_{(1)}^2/2) - \mathscr{C}\frac{\partial^3 \rho_{(1)}}{\partial \xi^3} = 0. \quad (38)$$

Integrating (36) and combining Eqs. (37) and (38), we obtain an equation of KdV type for $\rho_{(1)}(\xi, \tau)$:

$$\frac{\partial \rho_{(1)}}{\partial \tau} + 3\rho_{(1)}\frac{\partial \rho_{(1)}}{\partial \xi} - \frac{\mathscr{C}}{2}\frac{\partial^3 \rho_{(1)}}{\partial \xi^3} = 0,$$

which has the single-soliton solutions

$$\rho_{(1)}(\xi, \tau) = -\frac{\mathscr{B}}{\cosh^2(\sqrt{\mathscr{B}/2\mathscr{C}}(\xi + \mathscr{B}\tau))}, \quad \mathscr{B} > 0.$$

In the ordinary coordinate space, this solution has the form

$$\rho(x_1, t) = \rho_N\left(1 - \frac{\alpha\mathscr{B}}{\cosh^2\left(\frac{m^* c_s}{\hbar\eta}\sqrt{2\alpha\mathscr{B}}(x_1 - c_s(1 - \alpha\mathscr{B})t)\right)}\right). \quad (39)$$

It can be seen that the solution (39) describes the propagation through nuclear matter of a pulse disturbance with speed $c_s(1-\alpha\mathscr{B})$. The pulse width $(\hbar\eta/m^*c_s)\sqrt{2\alpha\mathscr{B}}$ and the amplitude of the deviation $\alpha\mathscr{B}$ from the density of nuclear matter are determined by the deviation of the pulse speed from the speed of "sound."

Our conclusion that there exist "subsonic" rarefaction solitons, rather than "supersonic" compression solitons, is indirectly confirmed by the results of one-dimensional TDFH[46,47] or DTTF[48] calculations. These calculations reveal rather clearly bubble rarefaction solitons leading to fragmentation of the layer that models the nucleus when the soliton reaches the surface. Moreover, there is no simple example in which a compression propagates through the layer in the form of a pulse.

Linear vibrations

We shall describe vibrational excitations of the nuclear density in terms of the operators of the density and the velocity potential of the nucleon field, which satisfy the canonical commutation relations (11).

From the density operator $\hat{\rho}(\mathbf{x})$ we separate a certain mean density $\rho_0(\mathbf{x})$ and an operator of the density fluctuation: $\hat{\rho}(\mathbf{x})\equiv\rho_0(\mathbf{x})+\delta\hat{\rho}(\mathbf{x})$.

Requiring that $\rho_0(\mathbf{x})$ be the equilibrium density, and linearizing the equations of motion for $\delta\hat{\rho}(\mathbf{x})$ and $\hat{\phi}(\mathbf{x})$, we obtain an equation of Thomas–Fermi type for $\rho_0(\mathbf{x})$. The dimensionless analog of this equation is Eq. (20), and the nucleus itself in the ground state can be regarded as a nuclear-density soliton (Refs. 26, 65, 77, and 78): $\rho_0(\mathbf{x})=\rho_N\psi_0(\mathbf{r})^2$. To analyze steady vibrations, the operators $\delta\hat{\rho}$ and $\hat{\phi}$ can be expanded in terms of bosonic creation, b_j^+, and annihilation, b_j operators:

$$[b_j,b_l^+]=\delta_{jl},\quad [b_j,b_l]=0,\quad [b_j^+,b_l^+]=0, \tag{40}$$

$$\delta\hat{\rho}(\mathbf{x})=\rho_N^{1/2}\left(\frac{m^*c_s}{\hbar\eta}\right)^{3/2}\psi_0(\mathbf{r})\sum_j g_j(\mathbf{r})(b_j^++b_j), \tag{41}$$

$$\hat{\phi}(\mathbf{x})=\frac{i\hbar}{2m}\rho_N^{-1/2}\left(\frac{m^*c_s}{\hbar\eta}\right)^{3/2}\psi_0^{-1}(\mathbf{r})\sum_j f_j(\mathbf{r})(b_j^+-b_j). \tag{42}$$

The amplitudes $g_j(\mathbf{r})$ and $f_j(\mathbf{r})$ satisfy the orthogonality relations

$$\sum_j g_j(\mathbf{r})f_j(\mathbf{r}')=\delta(\mathbf{r}-\mathbf{r}'),\qquad \int d^3r g_j(\mathbf{r})f_l(\mathbf{r})=\delta_{jl}, \tag{43}$$
$$\delta(\mathbf{x}-\mathbf{y})=\left(\frac{m^*c_s}{\hbar\eta}\right)^{3/2}\delta(\mathbf{x}'-\mathbf{y}'),$$

which follow from the definition (41)–(42), and from the commutation relations (11) and (14) between the operators $\delta\hat{\rho}$, $\hat{\phi}$, b_j^+, b_j.

The frequencies of the vibrational excitations, $\hat{H}_h\to\Sigma_j\hbar\omega_j b_j^+b_j+\text{const}$,

$$\hbar\omega_j\equiv(m^*c_s^2/2)\Omega_j, \tag{44}$$

and the amplitudes $g_j(\mathbf{r})$ and $f_j(\mathbf{r})$ are found by solving the system of equations

$$\hat{\mathscr{L}}_0 f_j(\mathbf{r})=\Omega_j g_j(\mathbf{r}),\quad \hat{\mathscr{L}}_0\equiv-\Delta_r+\lambda-4\psi_0^2+3\psi_0^4, \tag{45}$$

$$\hat{\mathscr{L}}_1 g_j(\mathbf{r})=\Omega_j f_j(\mathbf{r}),\quad \hat{\mathscr{L}}_1\equiv-\Delta_r+\lambda-12\psi_0^2+15\psi_0^4. \tag{46}$$

Thus, we have a self-consistent quantum scheme analogous to the popular scheme of the random-phase approximation for the description of monopole vibrations of the nuclear density.

It was found to be not at all a simple problem to solve Eqs. (45)–(46) to find the amplitudes and frequencies of the vibrations at the same time as solving Eqs. (20)–(21) for the equilibrium density and the orthogonality relations (43). The type of dimensionless equations that are obtained is common for many nonlinear dynamical systems that have been studied in recent years (Refs. 7, 15, and 85–89). Therefore, the development of numerical methods for their solution is a topical problem. We proposed two different approaches to find bound states of the quantum-mechanical systems of the two coupled second-order differential equations (45)–(46). The first is based on the method of bar splines.[77] The second approach uses a separation procedure[78] for the effective potentials that occur in Eqs. (45) and (46). For all the considered physical systems, only states in the continuous spectrum describing decay modes with particle emission were obtained.[79]

Density vibrations and emission process

We show that a coherent state of vibrational excitations of nuclear density can imitate a source in the process of emission of high-energy light particles in reactions with heavy ions.[24]

The emission of light particles accompanying fusion or deep inelastic collisions of heavy ions with nuclei carries information about the initial far-from-equilibrium stage of the collision, which determines the subsequent development of the process. It is this that explains the great interest shown in study of the emission of light particles in heavy-ion reactions. Phenomenological analysis of the experimental emission spectra shows that they can be represented as the sum of two components, the first of which is due to the equilibrium emission of particles from a compound nucleus and describes the low-energy part of the emission spectrum, while the second includes harder particles emitted predominantly forward. To explain the origin of the second component, several emission models have been proposed (see, for example, the references in Ref. 24). Most models use a source concept to interpret the data on the emission of fast light particles in nucleus–nucleus collisions. The concept of a radiating thermalized heated region (nuclear fireball model[92]), introduced to interpret the inclusive spectra of light particles at high energies >100 MeV/nucleon of the incident ion, was extended to lower energies ~ 10 MeV/nucleon.[24,93] Although such a phenomenological description is transparent and convenient for analysis of the experimental data, one cannot speak of a real radiating piece of the nucleus at energies ~ 10 MeV/nucleon. However, it was possible to construct, in the linear approximation, a microscopic description for which the basic properties are very close to those of the source model.[24]

We shall assume the following picture of the process of collision of nuclei. Beginning at the instant of contact, there is a rearrangement of the mean fields of the colliding nuclei and the formation of a mean field common to the composite system. As a result of this process, some of the energy of the relative motion goes over into internal excitation. The other part of it is concentrated on the collective mode described by the variable $\mathbf{R}_{\rm rel}$—the relative distance between the centers of the colliding nuclei.

From the density operator ρ we separate the averaged part ρ_0 and the fluctuation $\delta\rho\equiv\rho-\rho_0$. The current $\mathbf{j}$ is equal to the sum of the current associated with the relative motion of the parts of the composite system, $\mathbf{j}_{\rm rel}$, and the single-particle current $\mathbf{j}_{\rm sp}$:

$$\mathbf{j}=\mathbf{j}_{\rm rel}+\mathbf{j}_{\rm sp},\quad \mathbf{j}_{\rm rel}(\mathbf{x})\equiv\rho(\mathbf{x})\mathbf{p}_{\rm rel}(\mathbf{x})/m,$$

where $\mathbf{p}_{\rm rel}(\mathbf{x})$ is the momentum of the relative motion per nucleon. The momentum changes direction on the passage through the neck that connects the two parts of the composite system. We make our investigation in the center-of-mass system. We obtain

$$H_h=\frac{1}{2m}\int d^3x\rho(\mathbf{x})\mathbf{p}_{\rm rel}^2(\mathbf{x})+\mathscr{E}[\rho_0]+\frac{m}{2}\int d^3x\mathbf{j}_{\rm sp}(\mathbf{x})\rho^{-1}\mathbf{j}_{\rm sp}(\mathbf{x})+(\mathscr{E}[\rho_0+\delta\rho]-\mathscr{E}[\rho_0])+\int d^3x\mathbf{p}_{\rm rel}(\mathbf{x})\mathbf{j}_{\rm sp}(\mathbf{x}).$$

The first two terms are the kinetic and potential energies of the relative motion. For simplicity, we have separated all the gradient terms in the effective interaction functional $\mathscr{E}[\rho]$. After the formation of the mean field of the composite system, $\mathscr{E}[\rho_0]$ becomes the ground-state energy, calculated in the hydrodynamic approximation, and the kinetic energy becomes the energy of the collective motion of the composite system associated with the relative motion of its parts. The following two terms represent the Hamiltonian of the internal motion. We take it in the single-particle approximation. The last term describes the coupling between the relative motion and the internal motion. In what follows, we shall assume that we know the single-particle potential $U(\mathbf{x})$, the single-particle energies, and the eigenfunctions of the single-particle Hamiltonian.

Thus, the internal state of the composite system is described by the time-dependent Hamiltonian

$$H_h=\frac{1}{2m}\int d^3x\rho(\mathbf{x})\mathbf{p}_{\rm rel}^2(\mathbf{x})+\mathscr{E}[\rho_0]+\sum_s\varepsilon_s a_s^+a_s+\int d^3x\mathbf{p}_{\rm rel}(\mathbf{x})\mathbf{j}_{\rm sp}(\mathbf{x}),$$

$$\mathbf{j}_{\rm sp}(\mathbf{x})=\frac{\hbar}{2mi}(\Psi^+(\mathbf{x})\nabla\Psi(\mathbf{x})-\nabla\Psi^+(\mathbf{x})\Psi(\mathbf{x})),$$

$$\Psi^+(\mathbf{x})=\sum_s\varphi_s(\mathbf{x})a_s^+,\quad [a_s,a_{s'}^+]=\delta_{ss'}.$$

Since in the collision process a large number of different particle–hole states are excited and each particular state has a small excitation probability, we shall follow the scheme of the random-phase approximation. In the framework of the quasiboson approximation for the binary fermion operators, we can readily find the eigenstate

$$|\Psi\rangle\sim\exp\left(\sum_{ss'}\gamma_{ss'}(t)\theta(\varepsilon_s-\varepsilon_{\rm F})\theta(\varepsilon_{\rm F}-\varepsilon_{s'})a_s^+a_{s'}\right)|0\rangle,$$

$$\gamma_{ss'}=-\hbar^{-1}\int_0^t dt'\exp(-\hbar^{-1}(\varepsilon_s-\varepsilon_{s'})\times(t-t'))\chi_{ss'}(t'),\qquad (47)$$

$$\chi_{ss'}=\frac{\hbar}{2mi}\int d^3x\mathbf{p}_{\rm rel}(\mathbf{x},t)(\varphi_s^*(\mathbf{x},t)\nabla\varphi_{s'}(\mathbf{x},t)-\nabla\varphi_s^*(\mathbf{x},t)\varphi_{s'}(\mathbf{x},t)),$$

where $|0\rangle$ is the ground state of the system, and calculate the density of the distribution over the nuclear system of the internal excitation energy and the nucleon current. These results can be obtained without particularizing the form of the single-particle functions $\varphi_s(\mathbf{x})$ by using only the condition of their completeness and the fact that the sum $\Sigma_{\varepsilon_s<\varepsilon_{\rm F}}\varphi_s^*(\mathbf{x})\varphi_s(\mathbf{y})$ is equal to $\rho_0(\mathbf{x})$ at $\mathbf{x}=\mathbf{y}$ and decreases rapidly with increasing $|\mathbf{x}-\mathbf{y}|$ We assumed that

$$\sum_{\varepsilon_s<\varepsilon_{\rm F}}\varphi_s^*(\mathbf{x})\varphi_s(\mathbf{y})=\rho_0\left(\frac{\mathbf{x}+\mathbf{y}}{2}\right)\exp\left(\frac{(\mathbf{x}-\mathbf{y})}{2r_0^2}\right),$$

where r_0 is in order of magnitude equal to the internucleon separation. It can be shown that the excitation energy and current are concentrated either where there is an abrupt change of the momentum $\mathbf{p}_{\rm rel}$ (i.e., in the region of contact of the nucleus) or where the gradient of the density ρ_0 is large (i.e., in the surface layer). If we ignore surface effects, then for the density of the excitation energy we obtain (for small t)

$$E^*(\mathbf{x},t)=-\frac{5\hbar^2}{2mr_0^2}\rho_0(\mathbf{x})S\times\int_0^t dt'\dot{\mathbf{R}}_{\rm rel}(t')\mathbf{R}_{\rm rel}(t')/|\mathbf{R}_{\rm rel}(t')|.$$

The excitation energy concentrated in the region of contact is

$$E^*(t)=\frac{5\hbar^*}{2mr_0^2}\rho_N S\int_0^t dt'\dot{\mathbf{R}}_{\rm rel}(t')\mathbf{R}_{\rm rel}(t')/|\mathbf{R}_{\rm rel}(t')|,$$

where $\dot{\mathbf{R}}_{\rm rel}$ is the velocity of the relative motion of the nuclei, and S is the cross-sectional area of the neck. The value of $\rho_N S\int_0^t dt'\mathbf{R}_{\rm rel}(t')\mathbf{R}_{\rm rel}(t')/|\mathbf{R}_{\rm rel}(t')|$ is approximately equal to the number of nucleons in the region of contact, and $5\hbar^2/2mr_0^2\approx 100$ MeV. Thus, there are concentrated in the region of contact both the momentum introduced by the incident ion and a high excitation energy.

Using the wave function (47), we can calculate the momentum distribution $N(\mathbf{p},t)$ of the nucleons in the excited composite system. In this case, it is necessary to particularize the form of the functions φ_s. For qualitative estimates, we replace φ_s by plane waves. This means that we ignore surface effects. If for simplicity we assume that $\dot{\mathbf{R}}(t)=\mathbf{v}\theta(\tau-t)$, then for $N(\mathbf{p},\tau)$ we obtain the result

$$N(\mathbf{p},\tau)=2(2\pi)^{-5}\left(\frac{A_pA_t}{2\rho_N(A_p+A_t)}\right)\frac{(\mathbf{p}\mathbf{y})^2}{p}\int_0^{p_F}dqq(1-q/p)^2\frac{1-\cos(\hbar^{-1}((\varepsilon_p-\varepsilon_q)-\mathbf{p}\mathbf{v}(1-q/p))\tau)}{(\hbar^{-1}(\varepsilon_p-\varepsilon_q)-\mathbf{p}\mathbf{v}(1-q/p))^2}\times\left(R_p^{-2}\exp\left(-\frac{1}{2}p^2R_p^2(1-q/p)^2\right)+R_t^{-2}\exp\left(-\frac{1}{2}p^2R_t^2(1-q/p)^2\right)\right).$$

This distribution is characterized by the following basic features.

The factors $(\mathbf{p}\cdot\mathbf{v})^2$ and

$$\frac{1-\cos(\hbar^{-1}((\varepsilon_p-\varepsilon_q)-\mathbf{p}\mathbf{v}(1-q/p))\tau)}{(\hbar^{-1}(\varepsilon_p-\varepsilon_q)-\mathbf{p}\mathbf{v}(1-q/p))^2}$$

determine a preferred excitation of particles with momenta directed along the velocity of the incident ion, and the angular distribution of the emitted nucleons has a forward peaking.

The slope of the high-energy part of the spectrum does not depend directly on the velocity (energy per nucleon) of the incident ion. However, the contribution of the emission from the initial state of the system to the total emission spectrum will increase with increasing v in proportion to v^2 and simultaneously there will be an increase in the effective slope of the energy spectrum (envelope of the evaporation and pre-equilibrium components), i.e., the effective temperature. The total number of nucleons with $p>p_F$ increases with increasing number of nucleons in the incident ion. It is these features that characterize the momentum distribution of the nucleons in the moving-source model.[24]

We should also mention the result obtained in Ref. 25, namely, that the interaction of the soliton with the particles of the medium can lead to their having an almost Maxwellian spectrum. Thus, the nuclear system, although remaining cold, can radiate as a hot body with an effective temperature roughly proportional to the inverse scale of the collective excitation propagating through the nucleus.[22–27]

4. ROTATIONAL DENSITY EXCITATIONS

General remarks

Rotational states were always at the center of interest of theoretical and experimental nuclear physics. High-spin states[94,95] are the most popular type of rotational motion in nuclear systems but are by no means the only possible ones. Numerous attempts have been made to find nontrivial vortical states. In Ref. 96 the suggestion was made that there exist vortical isomer nuclei (superconducting component of a nuclear fluid with quantized vortex along the drop axis). A few years ago a suggestion having a similar physical significance was made.[97]

In Refs. 98 and 99, the liquid-drop model was used to obtain approximate solutions describing stable vortical formations. In the first case, they corresponded to a hot spot produced in peripheral collisions.[98] In Ref. 99, solitons on the nuclear surface were associated with cluster configurations.

By analogy with impurity electrons and positrons in dense gases, one can conjecture the existence in nuclear systems of analogs of vortex rings at impurity hadrons.[40,100]

Recently, there have been theoretical predictions of the possible formation, in the region of collision of two complex nuclei, of exotic objects such as disks[41,42] and rings.[97]

In this section, we consider one of the possible types of vortical states of incompressible nuclear matter—plane vortex disks.[42] These are regions of constant vorticity bounded by a uniformly rotating boundary. These states can be regarded as a generalization of the elliptic vortices of Kirchhoff.[69] In the framework of semimicroscopic nuclear hydrodynamics,[26] a nonlinear integrodifferential equation was obtained for the description of the disk boundary. The method can be used to analyze disks of any type without the use of additional assumptions of an ellipsoidal nature of the boundary or assumptions of small deviations of the boundary from circular.[69]

Basic equations

For the description of purely vortical states of incompressible ($\rho\equiv\rho_N$) nuclear matter, the nonlinear Euler-type equation (4), (8) can be rewritten in the kinematic form

$$\frac{\partial}{\partial t}\operatorname{curl}\mathbf{v}=\operatorname{curl}[\mathbf{v}\times\operatorname{curl}\mathbf{v}],\quad \mathbf{j}\equiv\rho_N\mathbf{v} \tag{48}$$

for the velocity field $\mathbf{v}(\mathbf{x},t)$.

It is convenient to give the further treatment in terms of the vorticity ζ and the vector potential $\mathbf{A}$:

$$\begin{gathered}\mathbf{v}\equiv\operatorname{curl}\mathbf{A},\quad \operatorname{div}\mathbf{A}=0,\\ \zeta\equiv\operatorname{curl}\mathbf{v}=\operatorname{curl}\operatorname{curl}\mathbf{A}=\operatorname{grad}\operatorname{div}\mathbf{A}-\Delta\mathbf{A}=-\Delta\mathbf{A},\\ \frac{\partial}{\partial t}\zeta+(\mathbf{v}\nabla)\zeta=0.\end{gathered} \tag{49}$$

In what follows, we restrict ourselves to the simplest vortex motion, motion on a plane:

$$\begin{gathered}\mathbf{A}=A\mathbf{e}_z,\quad \zeta=\zeta\mathbf{e}_z,\quad \mathbf{v}(r,\phi)=v_r\mathbf{e}_r+v_\phi\mathbf{e}_\phi,\\ \frac{\partial}{\partial t}\zeta+v_r\frac{\partial}{\partial r}\zeta+v_\phi\frac{1}{r}\frac{\partial}{\partial\phi}\zeta=0,\quad \zeta=-\Delta A,\\ \mathbf{v}(r,\phi)=v_r\mathbf{e}_r+v_\phi\mathbf{e}_\phi,\quad v_r(r,\phi)=\frac{1}{r}\frac{\partial A}{\partial\phi},\\ v_\phi(r,\phi)=-\frac{\partial A}{\partial r},\end{gathered} \tag{50}$$

where (r,ϕ) are the polar coordinates of the considered point.

The flow function can be obtained from the Poisson equation by means of the two-dimensional Green's function for the Laplacian:

$$A(r,\phi)=\frac{1}{2\pi}\int d\phi' dr' r' \ln(|\delta\mathbf{r}|)\zeta(r',\phi'),$$

$$|\delta\mathbf{r}|=(r^2+r'^2-2rr'\cos(\phi-\phi'))^{1/2}.$$

If the region of constant vorticity, $\zeta(r',\phi',t)\equiv\zeta_0$, is bounded by the contour

$$\Gamma(r,\phi)\equiv r-R(\phi)=0, \tag{51}$$

then the expressions simplify, and the integration is performed over the region bounded by this contour:

$$A(r,\phi)=\frac{\zeta_0}{2\pi}\int_\Gamma d\phi' dr' r' \ln(|\delta\mathbf{r}|). \tag{52}$$

The velocity projections v_r and v_ϕ are determined in accordance with the expressions (50) by differentiating $A(r,\phi)$ with respect to r and ϕ. Therefore, in their calculation there remains only a one-dimensional integration around the contour.

Up to now, our treatment has been purely kinematic. It must be augmented by a dynamical condition on the contour, $(\mathbf{n}\cdot\mathbf{v})=(\mathbf{n}\cdot\mathbf{v}_{\text{contour}})$, which is determined by means of (51). Here, $\mathbf{n}$ is the vector of the unit normal to the boundary:

$$\mathbf{n}=\frac{\sigma\nabla\Gamma}{|\nabla\Gamma|}=\sigma(\mathbf{e}_r-S(\phi)\mathbf{e}_\phi)(1+S(\phi)^2)^{-1/2}, \tag{53}$$

$$\Omega\frac{dR}{d\phi}+v_r-v_\phi S(\phi)=0,\quad S(\phi)\equiv\frac{1}{R}\frac{dR}{d\phi}, \tag{54}$$

where Ω is the angular velocity of uniform rotation of the contour, and $\sigma=\pm1$ determines the orientation of the contour.

By means of the expressions (50)–(53), Eq. (54) for the disk boundary can be rewritten as a nonlinear integrodifferential equation:

$$\begin{aligned}\frac{2\pi\Omega}{\zeta_0}\frac{dR}{d\phi}=&\int_0^{2\pi}d\phi' R(\phi')\ln(|\delta\mathbf{R}|)\\&\times[(1+S(\phi)S(\phi')\sin(\phi'-\phi)+(S(\phi)\\&-S(\phi'))\cos(\phi'-\phi)],\\|\delta\mathbf{R}|=&(R(\phi)^2+R(\phi')^2-2R(\phi)R(\phi')\\&\times\cos(\phi-\phi'))^{1/2}.\end{aligned} \tag{55}$$

Note that the equations of motion for the single-particle density and the density of the velocity field are nonlinear, and Eqs. (54) and (55) were obtained without the usually employed procedure of linearization of the equations of motion with respect to powers of $\mathbf{v}$ and the deviation of the single-particle density from the equilibrium value. Before we analyze these equations, we recall the basic properties of already known solutions.

PROPERTIES OF THE ELLIPTIC VORTICES OF KIRCHHOFF

At the present time, we know well the elliptic vortices of Kirchhoff and small perturbations of a circular vortex (we refer to the fundamental monograph of Lamb,[69] where one can find references to the original studies, which are no longer readily accessible).

The simplest case of a circular section was investigated by Kelvin.[69] We consider small perturbations of a circular vortex: $\Gamma(r,\phi)\equiv r-R_0$, where R_0 is the radius of the circle. Making a direct calculation of the flow function in accordance with (52), we obtain for $r\leqslant R_0$

$$A_0(r,\phi)=\tfrac{1}{4}\zeta_0(R_0^2-r^2).$$

A small irrotational perturbation

$$\delta A(r,\phi)=\alpha\frac{\zeta_0}{2}R_0^2\left(\frac{r}{R_0}\right)^l\cos(l\phi-\omega t), \tag{56}$$

where l is an integer, corresponds to the following equation of the contour (for small $\alpha\ll R_0$) that is usually adopted for nuclear physics:

$$R(\phi)=R_0(1+\alpha\cos(l\phi-\omega t)). \tag{57}$$

Thus, the small perturbation (56), expressed by the trigonometric functions (57), is a corrugation that is displaced around the circle of the vortex with angular velocity $\Omega=\zeta_0\omega/l=(l-1)/2l$. For example, for $l=2$ the perturbed cross section is an ellipse that rotates around its center with angular velocity $\zeta_0/4$, i.e., twice as slow as the fluid within the contour. Perturbations of higher symmetry $l\geqslant3$ rotate even more slowly. The special case of an elliptic perturbation was solved exactly by Kirchhoff.[69]

For the ellipse $x^2/a^2+y^2/b^2=1$, it is possible to obtain the following relation between the rotation frequency of the contour and the vorticity within the contour: $\Omega=\zeta_0 ab/(a+b)^2$.

From the special cases considered above, we deduce the following:

1) although the inner part of the disk rotates with constant angular velocity, the motion is not that of a rigid body, since the boundary rotates with a different angular velocity, more slowly;

2) the velocity of the boundary depends on the symmetry of the perturbation, the rotation being slower, the higher the symmetry of the state;

3) a fixed value of the ratio Ω/ζ_0 and the symmetry of the states completely determine the shape of the contour (for example, for the elliptical vortex its eccentricity).

The stability of these states needs a separate investigation. It was shown that the above special cases of disks are stable. However, if the contour shape is not a stationary solution of Eq. (10) for fixed Ω/ζ_0, the disk will be unstable. Therefore, the ratio Ω/ζ_0 will be a bifurcation parameter.

Analysis of disk states of general form

For the quantitative analysis of Eq. (55), one must construct a discretized analog of it. We are currently making such investigations. However, in this paper we restrict ourselves to a qualitative analysis, which can be made by analogy with the results of the previous section.

It is to be expected that the exact states of the nonlinear equation (55) can also be approximately classified by means of the parameters l (l=2, 3, 4...), i.e., by means of the symmetry of the boundary of the disk with respect to rotations through angle $2\pi/l$. Then, if a Fourier analysis of the contour is made, the expansion amplitudes of the corresponding $\cos(l\phi)$ must be greatest.

It is also to be expected that the parameter Ω/ζ_0 will be a bifurcation parameter and characterize the stability of the state.

Equation (55) in conjunction with the definition of the projections of the velocity field (50) describes the motion of the contour as the propagation of a nonlinear dispersive wave in a plane. As it moves, the contour will of necessity begin to be distorted. However, if the state is stable, the competition of the nonlinearity and the dispersion leads to a return of the initial shape of the contour. If it can be shown that there exist such states, then vortical disks of nuclear matter will be analogous to solitons in a plane.

If the motion of the contour is unstable, it will be very interesting to follow its evolution. Conserved quantities are the area of the contour—the two-dimensional analog of the particle number—and the circulation, which is determined by means of ζ_0. Therefore, when the contour decays one can expect the appearance of filaments and, perhaps, decay of the disk into individual disks or filaments.

It would be very interesting to analyze finite three-dimensional axisymmetric disks. Here, we should mention the recent preprint of Ref. 41, in which calculations were made to simulate the collision of two almost symmetric heavy ions using the Boltzmann–Nordheim–Vlasov equation. The formation of states of disk type in the collision process was established. The disk thickness decreased, while the diameter increased monotonically with increasing collision energy. Sufficiently thin disks began to break up into pieces having diameters comparable to the disk thickness. Thus, the problem of the stability of disks of nuclear matter is related to the process of multifragmentation.

We have attempted to consider purely vortical motion without the usual approximations of a small excitation amplitude or additional assumptions about the shape of the nuclear system. When the vortical component of the velocity field is separated from the equations of motion, the gradient components of the "pressure" terms drop out, and the equations of motion for curl **v** formally have a purely kinematic form, at least for forces of Skyrme type. The use of other forms of interaction containing a velocity dependence will lead to more complicated equations. It would also be interesting to analyze analogous equations obtained in the framework of a kinetic approach.

Finally, in this section we have obtained the basic equations for describing vortical disks of nuclear matter. The evolution of the disk boundary is analogous to the propagation of a nonlinear dispersive wave in a plane. We have made a qualitative analysis of the main properties of disk states.

5. USE OF THE INVERSE SCATTERING METHOD

General remarks

One of the most important parts of soliton theory is the inverse scattering method and its application to the integration of nonlinear partial differential equations.[5–8] Some definite experience of the use of this method in nuclear physics too has already been accumulated. In this section, we shall follow the inverse mean-field method (Imefim).[101] This method makes it possible to obtain useful information about the radii of nuclei[102] and optical potentials.[103] It was proposed to use the method to describe the dynamics,[104] relaxation,[105] and breakup of nuclear systems.[106,107] A specific feature of the use of the method to integrate nonlinear equations of motion is the fact that data on the structure of the initial state (the number of bound states and their scheme, data on the reflection coefficients) make it possible to predict the evolution of the initial state (Refs. 5, 8, 106, and 107). This makes it possible to reexamine the spectroscopy of nuclear states.

Below, we give two examples of the use of the inverse scattering method for static (description of density profiles and single-particle potentials of nuclear systems[108]) and dynamic (simple model for analysis of the evolution of a nucleus that is compressed at the initial time[106,107]) problems of nuclear physics.

Basic equations

The types of system that we shall consider are uncharged layers of nuclear matter.[46,47] The layers are finite in the direction of the z axis but infinite and uniform in the transverse directions.

The basic equations that describe a layer in the mean-field approximation are

$$\begin{aligned}
&\psi_{\mathbf{k}_\perp n}(\mathbf{x})=\frac{1}{\Omega}\,\psi_n(z)\exp(i\mathbf{k}_\perp\mathbf{r}),\quad \varepsilon_{\mathbf{k}_\perp n}=\frac{\hbar^2k_\perp^2}{2m}+e_n,\\
&-\frac{\hbar^2}{2m}\frac{d^2}{dz^2}\psi_n(z)+U(z)\psi_n(z)=e_n\psi_n(z),\\
&\rho(\mathbf{x})\Rightarrow\rho(z)=\sum_{n=1}^{N_0}a_n\psi_n^2(z),\\
&A\Rightarrow\mathscr{A}=(6A\rho_N^2/\pi)^{1/3}=\sum_{n=1}^{N_0}a_n,\\
&a_n=\frac{2m}{\pi\hbar^2}(e_{\rm F}-e_n),\\
&\frac{E}{A}\Rightarrow\frac{\hbar^2}{2m\mathscr{A}}\left(\sum_{n=1}^{N_0}a_n\int_{-\infty}^{\infty}\left(\frac{d\psi_n}{dz}\right)^2dz+\frac{\pi}{2}\sum_{n=1}^{N_0}a_n^2\right)\\
&\qquad+\frac{1}{\mathscr{A}}\int_{-\infty}^{\infty}\mathscr{E}[\rho(z)]dz,
\end{aligned}\tag{58}$$

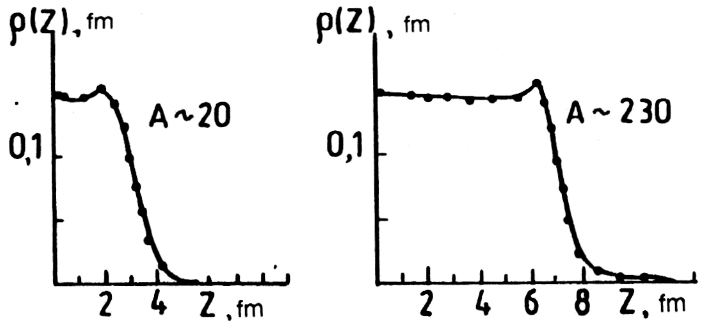

FIG. 4. Density profiles (continuous curve for HF, dots for INV).

where N is the total number of bound states, N_0 is the number of those that are occupied, $\mathscr{A}$ is the layer thickness, and $\mathscr{E}$ is the interaction functional.

A direct method of solving the problem is to specify the interaction functional and the number of particles (or the layer thickness) and solve the Hartree–Fock equations for the spectrum of single-particle states and wave functions $\psi_n(z)$.

Following the inverse scattering method, we must find a function $K(x,y)$ that solves the integral Gel'fand–Levitan–Marchenko equation:[109–111]

$$K(x,y)+B(x+y)+\int_x^\infty B(y+z)K(x,z)dz=0. \quad (59)$$

The kernel B is determined by means of the reflection coefficients $R(k)$ $(e_k=\hbar^2k^2/2m)$ and the set of N energy eigenvalues of the bound states $(e_n=-\hbar^2\kappa_n^2/2m)$:

$$B(z)=\sum_{n=1}^N C_n^2(\kappa_n)+\frac{1}{\pi}\int_{-\infty}^\infty R(k)\exp(ikz)dk. \quad (60)$$

The coefficients C_n are determined by the boundary conditions:

$$C_n(\kappa_n)=\lim_{z\to\infty}\psi_n(z)\exp(\kappa_n z). \quad (61)$$

The required single-particle potential is

$$U(z)=-\frac{\hbar^2}{m}\frac{\partial}{\partial z}K(z,z). \quad (62)$$

By means of the given phase shifts and the spectrum of bound states of the system, the integral equation (59) can be solved only numerically. However, in the case of reflectionless potentials, $R(k)=0$, one can construct the following solution:

$$U(z)=-\frac{\hbar^2}{m}\frac{\partial^2}{\partial z^2}\ln(\det\|M\|)=-\frac{2\hbar^2}{m^2}\sum_{n=1}^N \kappa_n\psi_n^2(z),$$

$$\psi_n(z)=\sum_{n=1}^N (M^{-1})_{nl}\lambda_l(z),$$

$$\lambda_n(z)=C_n(\kappa_n)\exp(-k_n z), \quad (63)$$

$$M_{nl}(z)=\delta_{nl}+\frac{\lambda_n(z)\lambda_l(z)}{\kappa_n+\kappa_l},$$

$$C_n(\kappa_n)=\left(2\kappa_n\prod_{l\neq n}^N\frac{\kappa_n+\kappa_l}{\kappa_n-\kappa_l}\right)^{1/2}.$$

Thus, the wave functions, potential, and density profile are completely determined by the spectrum of bound states.

Profiles of densities and potentials

In Ref. 108, we made a series of calculations of layers not only in the ground states but also in various external fields. For the direct part of the calculation by the Hartree–Fock method, the interaction functional was chosen in the form of effective Skyrme forces. The calculated spectrum of bound states was fed into the scheme of the inverse scattering method, and the relations (63) were used to recover the wave functions of the states, the single-particle potentials, and the densities.

Figure 4 gives the results of calculations of the density profiles for layers that simulate light ($\mathscr{A}=1.0$, $A\approx20$) and heavy ($\mathscr{A}=2.15$, $A\approx230$) nuclei. The continuous curve shows the results of calculation by the Hartree–Fock (HF)

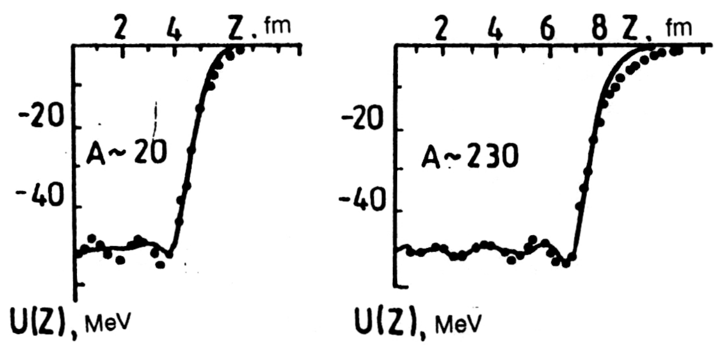

FIG. 5. Single-particle potentials.

method, and the dotted curve is the result obtained by the inverse mean-field method (INV). It can be seen that the density profiles are reproduced very well.

Figure 5 gives the results of calculation of the corresponding single-particle potentials. The notation is as before. It can be seen that although the mean properties of the potentials (depths and effective radius) are reproduced well, there are weak oscillations in the inner region of the potential.

Figure 6 shows on a logarithmic scale the dimensionless profiles of the density and potential in the region of the diffuse layer and in the asymptotic region. It can be seen that the density "tails" are reproduced very well but the potential "tails" less well. It should be mentioned that the calculation of the potential by the Hartree–Fock method was made using the first variation of the interaction functional with respect to the density, and therefore for Skyrme forces the slopes of the density and the potential are equal in the logarithmic scale (see Fig. 6). In the framework of the inverse scattering method, all bound states are taken into account in the calculation of the potentials, but in the calculation of the density only the occupied states are taken into account [see Eqs. (58)]. Therefore, the slope of the "tails" of the potentials and the densities will be different. Since the scheme of the inverse scattering method is realized in the approximation of reflectionless potentials, the following conclusions can be drawn: 1) allowance for the reflection terms in the calculation of the density is unimportant; 2) allowance for the reflection terms in the calculation of the potentials will lead to smoothing of the oscillations in the inner part and to correction of the "tail" of the potential.

All that we have said above related to calculation of layers in the ground state. For a further testing of the method, we calculated layers in various external fields. In this way one usually forms initial nonequilibrium states in dynamical calculations in order to analyze the evolution of a system. In all the cases that we considered, the density profiles were practically identical to the results of calculations by the Hartree–Fock method. This makes it possible to simulate with very good accuracy excited states of nuclear systems by means of the inverse scattering method.

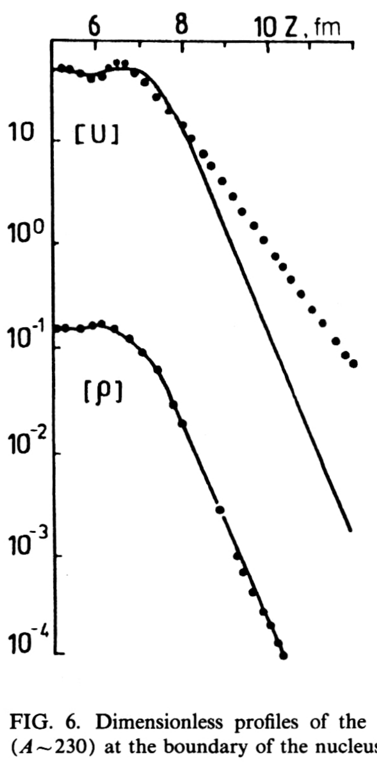

FIG. 6. Dimensionless profiles of the density [ρ] and potentials [U] ($A \sim 230$) at the boundary of the nucleus and in the asymptotic region.

Figure 7 shows examples of the construction of various states of a nuclear system for fixed particle number by means of the inverse scattering method.

Finally, we may draw the conclusion that the inverse mean-field method (in the approximation of reflectionless single-particle potentials) gives a simple analytic description of the density and potential profiles of nuclear systems.

Evolution of a cold compressed nucleus

In this section, we present a simple analytic model for describing the evolution of cold nuclear systems that are compressed at the initial time.[106,107,112]

To analyze the evolution of the nuclear systems, one should, in principle, solve the many-particle Schrödinger equation, augmenting it by the necessary initial and boundary conditions. However, the exact solution of this problem is as yet impossible. A well-known method for overcoming many difficulties in the solution of the many-particle problem is to use the mean-field approximation. Then the many-particle Schrödinger equation reduces to a system of single-particle Schrödinger equations (the state index is omitted for simplicity):

$$i\hbar \frac{\partial \psi(\mathbf{x},t)}{\partial t} = -\frac{\hbar^2}{2m} \Delta\psi(\mathbf{x},t) + U(\mathbf{x},t)\psi(\mathbf{x},t), \tag{64}$$

which must be augmented by the condition of self-consistency for the single-particle potential.

The basic idea of the inverse mean-field method when applied to the dynamics of nuclear systems was to use instead of (64) an equation of KdV type for the single-particle potential

$$\sum_{n=1}^{N} \frac{\partial U}{\partial (S_n t)} = 6U \frac{\partial U}{\partial z} - \frac{\hbar^2}{2m} \frac{\partial^3 U}{\partial z^3} \tag{65}$$

and the well-known scheme of integrating the KdV equation by means of the inverse scattering method. The scheme is based on the existence of the nonlinear transformation $U = V^2 + V_z$, $V \equiv \psi_z/\psi$ (up to scale transformations), after which the nonlinear equation (65) is associated with a linear equation of Schrödinger type (all the notation is as in the previous section):

$$-\frac{\hbar^2}{2m} \frac{\partial^2}{\partial z^2} \psi_n(z,t) + U(z,t)\psi_n(z,t) = e_n \psi_n(z,t), \tag{66}$$

where the time plays the role of a parameter. The above scheme of integration of the reflectionless potentials (63) is fully valid if one makes the single change of variables $z \to (z,t)$ in the calculation of $\lambda_n(z)$:

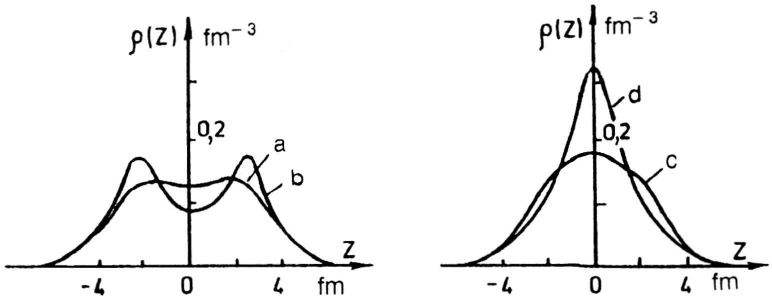

FIG. 7. Examples of density profiles calculated by the inverse scattering method ($A \sim 16$): a) ground state; b) nucleus with two centers; c) and d) weakly and strongly compressed nucleus, respectively.

$$\lambda_n(z,t) \equiv C_n(\kappa_n)\exp(-\kappa_n z + 2\hbar^2\kappa_n^3 S_n t/m).$$

Despite definite progress in the application of the inverse scattering methods in nuclear physics, the given method is as yet unusual. As an illustration, we give the basic expressions for the wave functions of a one-dimensional three-level system.[106] The simplest one- and two-dimensional systems were considered in detail earlier in Ref. 113. The three-level system is helpful for modeling light nuclei, for example, oxygen (Ref. 112) with $\kappa_3 > \kappa_2 > \kappa_1$:

$$\psi_1(z,t) = \left(2\kappa_1\left(\frac{\kappa_2+\kappa_1}{\kappa_2-\kappa_1}\right)\left(\frac{\kappa_3+\kappa_1}{\kappa_3-\kappa_1}\right)\right)^{1/2} D^{-1}(z,t) \times\left(\cosh(\xi_2+\xi_3) - \left(\frac{\kappa_3+\kappa_2}{\kappa_3-\kappa_2}\right) \times\cosh(\xi_3-\xi_2)\right),$$

$$\psi_2(z,t) = \left(2\kappa_2\left(\frac{\kappa_2+\kappa_1}{\kappa_2-\kappa_1}\right)\left(\frac{\kappa_3+\kappa_2}{\kappa_3-\kappa_2}\right)\right)^{1/2} D^{-1}(z,t) \times\left(\sinh(\xi_3+\xi_1) - \left(\frac{\kappa_3+\kappa_1}{\kappa_3-\kappa_1}\right)\cosh(\xi_3-\xi_1),\right.$$

$$\psi_3(z,t) = \left(2\kappa_3\left(\frac{\kappa_3+\kappa_2}{\kappa_3-\kappa_2}\right)\left(\frac{\kappa_3+\kappa_1}{\kappa_3-\kappa_1}\right)\right)^{1/2} D^{-1}(z,t) \times\left(\cosh(\xi_1+\xi_2) - \left(\frac{\kappa_2+\kappa_1}{\kappa_2-\kappa_1}\right) \times\cosh(\xi_2-\xi_1)\right),$$

$$D(z,t) = \cosh(\xi_1+\xi_2+\xi_3) + \left(\frac{\kappa_2+\kappa_1}{\kappa_2-\kappa_1}\right) \times\left(\frac{\kappa_3+\kappa_1}{\kappa_3-\kappa_1}\right)\cosh(\xi_3+\xi_2-\xi_1) + \left(\frac{\kappa_2+\kappa_1}{\kappa_2-\kappa_1}\right) \times\left(\frac{\kappa_3+\kappa_2}{\kappa_3-\kappa_2}\right)\cosh(\xi_3+\xi_1-\xi_2) + \left(\frac{\kappa_3+\kappa_1}{\kappa_3-\kappa_1}\right) \times\left(\frac{\kappa_3+\kappa_2}{\kappa_3-\kappa_2}\right)\cosh(\xi_2+\xi_1-\xi_3),$$

$$\xi_n(z,t) \equiv \kappa_n z - 2\hbar^2\kappa_n^3 S_n t/m, \quad n=1,2,3.$$

The asymptotic behavior of the wave functions and the single-particle potential for fixed ξ_n are

$$\lim_{t\to\infty} \psi_k = \delta_{kn}\sqrt{\kappa_n}\,\mathrm{sech}(\xi_n - \xi_n^0),$$

$$\lim_{t\to\infty} U(z,t) = -\frac{\hbar^2\kappa_n^2}{m}\mathrm{sech}^2(\xi_n - \xi_n^0),$$

$$\xi_n^0 = \frac{1}{2}\ln\left(\prod_{k=1}^{n-1}\left(\frac{\kappa_n-\kappa_k}{\kappa_n+\kappa_k}\right)^2 \prod_{m\neq n}^{n}\left|\frac{\kappa_n+\kappa_m}{\kappa_n-\kappa_m}\right|\right).$$

Thus, for large z and t the time-dependent single-particle potential and the corresponding density distribution are described by a set of solitary waves. The spectrum of the initial compressed nonequilibrium system completely determines the widths, velocities, and phase shifts of the solitons. Allowance for the reflection terms has the consequence that in the exit channel there will be small-amplitude waves ("ripples"), which are identified with the emission of light particles. The nonequilibrium system compressed at the initial time expands, and in the exit channel one can observe stable solitary waves. Such a picture is in qualitative agreement with the results of numerical simulation of the evolution of the ^{16}O nucleus in the framework of the TDHF method.[112] The decay leads to the presence of a collective flow and clustering.

The model discussed above is too simplified to describe the evolution of nonequilibrium nuclear systems. However, it can be used for illustrative purposes and to demonstrate the general scheme of the inverse mean-field method, the nonlinear superposition principle, and, most importantly, the idea that the formation of stable objects (clustering) during the breakup of nuclear systems can be due to the presence of nonlinear and dispersive effects.

6. DISSIPATIVE EFFECTS

The dissipative effects associated with excitations of the nuclear density can be nominally divided into two groups:

1) damping due to the effect of the medium on the moving collective density excitation;

2) the influence of excitations of the nuclear density, treated as internal degrees of freedom, on a certain collective motion. For example, one can regard the excitation of giant resonances in deep inelastic[71] collisions of heavy ions with nuclei as a mechanism of dissipation of kinetic energy.[64]

We recall that solitons exist because of the presence of two factors—nonlinearity and dispersion. From the point of view of spectral analysis, nonlinearity enriches the spectrum with high-frequency harmonics, while dispersion, i.e., dependence of the velocity on the wavelength, leads to spreading of the wave packet that describes the analyzed excitation. A balance of the two factors can be achieved only for certain types of nonlinearity, this ensuring the existence of solitonlike solutions. However, even in these cases the soliton will be stable as long as the system can be regarded as weakly dissipative. Nevertheless, the existence of stable waves is possible if the necessary supply of energy is ensured.[19,25] These waves are largely similar to hydrodynamic shock waves, but they arise in a regime in which there are no collisions between the particles of the medium. Therefore, in plasma physics they were called collisionless shock waves.[19] For a small but finite nonlinearity, the equation that describes such a wave differs only by the presence of a dissipative term (KdV–Burgers equation).[6–19]

Dissipative effects of the given type for nuclear systems were investigated in Ref. 25. In the framework of the theory of finite Fermi systems,[76] a kinetic equation with collision integral was obtained in the linear and dispersionless approximation. The collision integral arises when allowance is made for the nonzero damping of the quasiparticles in the lowest order in the parameter γ/ε_F.

This equation, with allowance for the contribution of nonlinearity and dispersion, was used in the limit of small amplitudes to obtain an equation for nonlinear dispersive waves in a plasma. Its solutions—quantum shock waves—are analogous to collisionless shock waves in a plasma, which differ appreciably from the shock waves of classical hydrodynamics. First, the existence of quantum shock waves does not require a regime of thermalization of the medium. Second, the state of the matter behind the front of such a wave is not one of thermodynamic equilibrium because of the presence of strong collective vibrations. Compression is achieved, not by heating of the medium, but by the coherent addition of vibrational excitations. Third, such waves have a complicated (oscillatory or turbulent) structure in the transition region. Fourth, the quantum shock waves propagate with a speed greater than the speed of zero sound, c_0, and not first (hydrodynamic) sound c_s. Since $c_0 > c_s$, the quantum wave travels faster than a hydrodynamic wave.

The propagation of the wave leads to "splashing"[21] of nucleons from the nucleus. The energy spectrum of the nucleons is of Maxwellian type. The effective "temperature" is $T \sim 10$ MeV. The Mach cones have not yet been observed experimentally or, at least, clearly. However, from the theoretical point of view too one cannot expect clearly expressed Mach cones. The reasons for this are, first, the finiteness of the nucleus (distortion of the waves by reflection from the boundary of the nucleus), second, the decay instability of the vibrations, leading to turbulence of the nonlinear density wave, and, third, the interaction of the wave with nucleons, which distorts the wave through the potential created by it. The most important thing is, as was noted in Sec. 3, the fact that all approaches give a quasi-Maxwellian emission spectrum of nonequilibrium nucleons. On the one hand, this justifies the phenomenological method of analysis of the nucleon energy spectra by means of the model of an effective source, but, on the other hand, it introduces a large degree of uncertainty in the attempt to elucidate the mechanism of formation of such particles.

The investigation of the process of dissipation of kinetic energy in deep inelastic collisions of ions with nuclei is one of the most interesting problems of the theoretical physics of heavy ions. In this review, we consider only the small part of this problem that relates to the role played in this phenomenon by the channel corresponding to excitation of density vibrations.

The losses of the energy of the relative motion on internal excitation of the nuclei (the inelasticity of the process) determine the main characteristics of deep inelastic collisions of heavy ions[71]—the angle and mass distributions of the products.

The amount of dissipation of the kinetic energy characterizes the duration of the reaction, since the energy of the relative motion always goes over in part into the energy of internal excitation of the nuclei, and the process of establishment of equilibrium does not come to an end during the time of the deep inelastic part of the collision. This circumstance makes the amount of kinetic-energy dissipation a kind of "clock" that makes it possible to obtain information about the development in time of the interaction process. The scale of the kinetic-energy loss, 100–300 MeV, indicates that the excited nuclear states can have a complicated structure. However, since the energy of the relative motion is "spilled" during a very short time ($\sim 10^{-22}$ sec), the existence of a coherent mechanism of energy loss is possible. The large energy loss indicates that the "doorway" states for the reaction must be the excitation modes with the highest frequencies. Among them, the most probable candidates are multipole vibrations of the density of the nuclear matter (giant resonances).[22,64,72]

Using these considerations, we obtained in Ref. 64 equations that describe, in the framework of linear response theory,[114] the relative motion of the colliding nuclei in the classical approximation. These equations have a number of features that distinguish them from the equations that are usually employed in phenomenological models and include "friction" forces.[115] First, because the time during which the energy is transferred to the internal degrees of freedom and the relaxation time have the same order of magnitude, an important role in the equations is played by the explicit time dependence of the coefficients of friction and the nucleus–nucleus potential and mass parameter, which are both renormalized by the excitation of the internal degrees of freedom. Second, the asymptotic value of the coefficient of radial friction is significantly less than the value that is used in phenomenological models.[115] These two facts served as the basis for the assumption that not only radial "friction" but also an explicitly time-dependent interaction potential acts as a source of the irreversible loss of kinetic energy. It was shown that in the framework of these assumptions it is possible to explain the loss of kinetic energy and the correlation between the mean energy and the deflection angle of the reaction products.

Important here is simultaneous allowance for the time-dependent potential and the variable (in time) "friction." Details can be found in the studies of Ref. 64. It should be mentioned that most conclusions of these studies hold for all systems whose Hamiltonian consists of three parts: a collective Hamiltonian, a Hamiltonian of the "internal" motion (described by a set of noninteracting oscillators), and an interaction term (in the linear approximation in the amplitudes of the "internal" oscillators). From the physical point of view, the dissipation mechanism—coherent excitation of density vibrations—corresponds to the mechanism described in the previous sections of the review.

Investigation of the role of viscosity and heat conduction in vortical motion (the coupling of the collective hydrodynamic motion to the "internal" degrees of freedom) is particularly interesting. However, the information on the tangential viscosity leading to dissipation of the angular momentum is as yet very uncertain even for the currently best investigated process of dissipation of kinetic energy in deep inelastic heavy-ion collisions.

7. CONCLUSIONS

In this review, we have developed methods for describing nonlinear phenomena and relaxation processes in the complex nuclear systems formed by the interaction of heavy ions with nuclei at energies 10–100 MeV/nucleon.

To analyze linear and nonlinear density excitations of various systems, we have developed a form of nuclear hydrodynamics based on current and density algebra. We have shown that for irrotational motion in the semiclassical limit the equations of nuclear hydrodynamics with effective Skyrme interaction can be reduced to the ψ^3–ψ^5 nonlinear Schrödinger equation.

We have investigated the basic properties of one-dimensional and spherically symmetric single-soliton solutions and shown that they describe well the properties of the nuclear density in the ground state. We have pointed out the possible existence of spherically symmetric states of the nuclear density with nodes. We have obtained analytic solutions for one-dimensional nonlinear excitations: cnoidal waves, kinks, and "bubbles" in nuclear matter.

We have obtained the basic equations for the description of vortical disks of nuclear matter. The evolution of the disk boundary is analogous to the propagation of a nonlinear dispersive wave in a plane. We have made a qualitative analysis of the basic properties of disk states.

We have proposed a simple model for analyzing the evolution of a cold nuclear system that is compressed at the initial instant and have shown that the inverse scattering method makes it possible to predict that in the exit channel there will be stable density waves ("clusters"—solitons) and small-amplitude density waves ("radiation").

In the framework of the inverse mean-field method we have constructed an analytic description of the density profiles and single-particle potentials of nuclear layers. By this method one can not only reproduce well the results of Hartree–Fock calculations but also model various excitations of nuclear systems (compressed nuclei, nuclei with two centers, etc.).

Giant monopole isoscalar resonances can be treated as linearized vibrations of a nuclear-density soliton. We have obtained the basic equations for the description of such states and developed numerical methods for their solution. We have constructed an analytic self-consistent solution for the vibrations of a double nuclear system.

A coherent state of vibrational excitations of the nuclear density can simulate a source in the process of emission of high-energy light particles in heavy-ion reactions. We have investigated the influence of the channel in which the nuclear density is excited on the process of kinetic-energy dissipation in deep inelastic collisions of heavy ions. A significant fraction of the kinetic-energy loss can be explained by the excitation of monopole vibrations of the nuclear density. The main contribution to the energy dissipation is made not by friction forces proportional to the velocity but by the explicitly time-dependent interaction potential of the heavy ions, renormalized with allowance for the excitation of the nuclei.

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