Quantization of Yang–Mills fields in linear gauges

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Different methods of quantization of Yang–Mills fields in linear gauges are considered. Canonical quantization in the $A_0=0$ gauge and calculation of the propagator of the Yang–Mills field in the light-cone gauge by transforming from the $A_0=0$ gauge are discussed. BRST quantization in arbitrary linear gauges is examined in detail.

INTRODUCTION

In this review we examine different methods of quantization of Yang–Mills fields in linear gauges. Linear gauges have been employed in physics for many years. They have found applications in QCD,[1] the supersymmetric Yang–Mills theory,[2,3] and string theory.[4] For example, the light-cone gauge is convenient for analyzing ultraviolet divergences in supersymmetric theories. One variant of the proof of the finiteness of $N=4$ supersymmetric Yang–Mills theory employs the light-cone gauge.[2,3] There are a number of reasons for the interest in linear gauges. First, Faddeev–Popov ghosts do not occur in these gauges, and this simplifies the diagram technique, the generalized Ward identities,[5,6] and the Schwinger–Dyson equations for the Green's functions. Gribov "copies" do not occur in these gauges,[7] and for this reason there is hope that linear gauges can also be employed outside the framework of perturbation theory (in this case, however, nonperturbative solutions of the equation $\varphi|\Psi\rangle=0$ must be sought).

We consider briefly different methods of quantization of gauge-invariant systems. The most general gauge-invariant Lagrangian can be represented, after second-class constraints are eliminated, in the form

$$\mathscr{L}=p_i\dot{q}^i-H+\lambda^a\varphi_a(p,q). \tag{1}$$

Here (p_i,q^i) is a pair of canonically conjugate variables, $\varphi_a(p,q)$ is a first-class constraint, and λ^a are Lagrange multipliers. The system described by the Lagrangian (1) is called a generalized Hamiltonian system (the concepts of first- and second-class constraints and a generalized Hamiltonian system were introduced by Dirac[8,9]).

In order to quantize such systems the gauge condition must be imposed. There are three basic classes of gauge conditions.

1. Unitary gauges $\chi^a(p,q)=0$. In such gauges unphysical degrees of freedom are eliminated with the help of the conditions $\chi^a(p,q)=0$ and the constraints $\varphi_a(p,q)=0$. Then only the physical degrees of freedom are considered.[10] Examples of such gauges are the Coulomb gauge in the Yang–Mills theory and the light-cone gauge in the theory of first-quantized strings.

2. Relativistic gauges $\lambda^a=f^a(p,q,\lambda)$. The problem of constructing an effective action was solved for the case of independent first-class constraints, forming a Lie algebra, in 1967 by L. D. Faddeev and V. N. Popov[11] and by B. De Witt.[12] The standard Faddeev–Popov procedure is, however, inapplicable to the case of dependent constraints and it must be extended. For some theories, which are similar to the nonabelian antisymmetric tensor field, such an extension was constructed in Ref. 13. Nonetheless, in the general case the problem remains open.

The BRST (Becchi–Rouet–Stora–Tyutin) approach enables quantization of an arbitrary gauge system in the relativistic gauge.[14–21] Lagrangian and Hamiltonian BRST quantization are distinguished. A Lagrangian BRST approach was proposed in Ref. 16 (an alternative and simpler approach is described in Ref. 17). Hamiltonian BRST quantization[18,19] is the most highly developed method of BRST quantization (see the reviews Refs. 20 and 21).

3. Hamiltonian gauges $\lambda^a=f^a(p,q)$. These gauges are a generalization of the Hamiltonian (or temporal) gauge $\lambda^a=0$. The Hamiltonian gauge, in contrast to the unitary and relativistic gauges, does not completely fix gauge arbitrariness. Longitudinal excitations are present in the theory and they must be quantized. An arbitrary Hamiltonian gauge is reduced to a purely Hamiltonian gauge with the help of the substitution

$$\lambda^a-f^a(p,q)\to\lambda^a.$$

The following methods can be employed in order to quantize a system in the Hamiltonian gauge. First, the system can be quantized in an extended phase space, including all pairs of variables (p_i,q^i), after the which the physical space is singled out by the condition $\varphi|\Psi\rangle=0$. As a rule, however, this condition results in unnormalizable state vectors, which is unsatisfactory from the standpoint of quantum mechanics. Second, the Faddeev–Popov trick can be used to transform from a known gauge (in which the system can be quantized) to the Hamiltonian gauge. Finally, the system can be BRST-quantized in the Hamiltonian gauge. In this case, however, the question of the norm of the physical subspace must be considered.

In the present paper we apply these methods in order to quantize Yang–Mills fields in linear gauges $n_\mu A_\mu=0$.

Linear gauges are divided into three types depending on the vector n_μ:

1) Hamiltonian or temporal gauge $A_0=0$, $n_\mu=(1,0,0,0)$. Any gauge $n_\mu A_\mu=0$ with $n^2>0$ can be transformed to a Hamiltonian gauge by a Lorentz rotation.

2) Light-cone gauge $n_\mu A_\mu=0$, $n^2=0$.

3) Axial gauge $A_3=0$, $n_\mu=(0,0,0,1)$. Any gauge $n_\mu A_\mu=0$ with $n^2<0$ can be transformed to an axial gauge.

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All linear gauges are Hamiltonian gauges $A_0=\mathbf{n}\cdot\mathbf{A}/n_0$ (the axial gauge is obtained in the limit $n_0\to 0$).

In constructing a perturbation theory in linear gauges there arises the problem of determining correctly the propagator of the gauge field. This problem is associated with the existence of an additional singularity of the Green's function. The propagator of Yang–Mills fields in the gauge $n_\mu A_\mu=0$ contains, besides a pole at $k^2=0$, a singularity at $n\cdot k=0$. The rule for encircling the pole at $k^2=0$ is determined by the Feynman boundary conditions for transverse quanta and reduces to the standard substitution $k^2\to k^2+i0$. In order to determine the rule for encircling the pole at $n\cdot k=0$ the boundary conditions for the longitudinal components must be specified, and this rule will depend on the chosen boundary conditions. Several prescriptions have been proposed for encircling the pole at $n\cdot k=0$ (see Ref. 22 for a recent review and a detailed list of references); these prescriptions usually depend on the vector n_μ.

In the present paper we discuss correct methods of quantization which enable determining the rule for encircling this pole.

Section 1 is devoted to canonical (Dirac) quantization of Yang–Mills fields in the Hamiltonian gauge $A_0=0$. Historically, the first prescription for encircling the pole at $k_0=0$ (and $n\cdot k=0$) was the principle-value prescription.[23–25] Calculations of Wilson's loop in the lowest orders of perturbation theory, as done in Ref. 26, however, did not agree with the results obtained in the Coulomb gauge. In order to achieve agreement with the Coulomb gauge Karacciolo *et al.*[26] postulated a longitudinal propagator in the form

$$D^L(x-y)=\tfrac{1}{2}\Delta^{-1}[\,|t-s|\pm\tfrac{1}{2}(t+s)+\gamma].\tag{2}$$

Here the first term in brackets corresponds to encircling the pole at $k_0^2=0$ in the principle-value sense. The second term engenders a translationally noninvariant (with a time translation) contribution to the Green's function. The third term γ is an arbitrary constant.

We shall show that the propagator (2) can be derived within the framework of canonical quantization with a suitable definition of the physical states.

In the case of quantum electrodynamics it has been proved that the translationally noninvariant terms do not contribute to the S matrix.

The results obtained in Ref. 27 are presented in Sec. 1.

Section 2 is devoted to quantization of Yang–Mills fields in the light-cone gauge $A_0=A_3$. Encircling the pole at $n\cdot k=0$ in the principle-value sense is likewise unsuitable for the light-cone gauge. The reason is that this prescription results in an infrared divergence of the single-loop Green's function in calculations employing dimensional regularization. Mandelstam[3] and Leibbrandt[28] proposed a prescription that corresponds to replacing the term $(n\cdot k)^{-1}$ in the propagator by $n^*k/(nk{*}n^*k\,+\,i\varepsilon)$, where $n=(n_0,\mathbf{n})$ and $n^*=(n_0,-\mathbf{n})$. Leibbrandt showed that this prescription does not lead to infrared divergences of the single-loop Green's function. Moreover, the Mandelstam–Leibbrandt prescription enables Wick rotation, which is important for analyzing ultraviolet divergences in supersymmetric theories.[3]

At first glance it appears natural to study the light-cone gauge by analogy to the $A_0=0$ gauge, using the cone variables $x^\pm=x^0\pm x^3$ (cone variables were employed in QED back in Refs. 29 and 30). In these variables the field $A_-=A_0-A_3$ plays the role of a Lagrange multiplier. It turns out, however, that the precedure employed in the Hamiltonian gauge (see Sec. 1) cannot be transferred directly. In particular, in this case the subspace of physical states cannot be separated in an invariant manner. If, on the other hand, an attempt is made to quantize in the variables (t,x), then the obtained propagator is translationally noninvariant under time translations, just as in the Hamiltonian gauge (this is connected to the choice of the momentum representation for longitudinal quanta and the unnormalizability of vectors from the physical subspace).

A possible method for circumventing these difficulties was proposed in Ref. 31, where the Yang–Mills theory with a modified Lagrangian, including a term of the form $\lambda(n_\mu A_\mu)$, ensuring that the gauge condition $n_\mu A_\mu=0$ is satisfied in the strong sense, was considered. The method employed in Ref. 31 is the analog of the well-known Gupta–Bleuler method (see, for example, Ref. 32): An extended Hilbert state space with an indefinite metric is introduced and the physical subspace is separated by imposing an additional condition on the state vectors. It is then proved that Gauss's law and Poincaré invariance hold in the physical subspace. The propagator obtained in Ref. 31 is identical to the Mandelstam–Leibbrandt propagator. It should be noted, however, that the procedure proposed in Ref. 31 is specially adapted for obtaining the Mandelstam–Leibbrandt propagator and cannot be used to obtain the complete set of admissable propagators.

In order to solve this problem, in Sec. 2 the Faddeev–Popov device is used, taking into account the boundary conditions, in order to switch from the Hamiltonian gauge to the light-cone gauge (a similar procedure for switching from the Coulomb into the Lorentz gauge is described, for example, in Ref. 33).

The starting point is the expression obtained in Sec. 1 for the S matrix in the $A_0=0$ gauge. Without loss of generality it can be assumed that $n_\mu=(1,0,0,1)$. We introduce the notation $A_-=A_0-A_3$ and $\partial_-=\partial_0-\partial_3$. In order to switch to the $A_-=0$ gauge, following the standard procedure, the expression for the S matrix in the Hamiltonian gauge must be multiplied by "one":

$$\Delta(A_-)\int d\omega(\mathbf{x},t)\delta(A_-^\omega)=1,\quad \omega(\mathbf{x},t')=\omega(\mathbf{x},t'')=1.\tag{3}$$

Unit boundary conditions must be imposed on the gauge-transformation parameter $\omega(\mathbf{x},t)$, so that the substitution of variables $A_\mu\to A_\mu^\omega$ would not change the boundary conditions in the functional integral. It is easy to see, however, that the condition (3) cannot be satisfied for arbitrary fields A_- [the equation $A_-^\omega=0$ is a linear equation, and its solution for arbitrary A_- does not satisfy two boundary conditions on $\omega(\mathbf{x},t)$]. This obstacle can be circumvented by regularizing the $A_-=0$ gauge, i.e., by switching to the gauge $F_\kappa(A_-)=0$, where $F_\kappa(A_-)\to A_-$ as $\kappa\to 0$, and the equation $F_\kappa(A_-^\omega)=0$, $\omega(\mathbf{x},t')=\omega(\mathbf{x},t'')=1$ has a solution for any field $A_-(\mathbf{x},t)$. The gauge $A_-=0$ does not completely fix the gauge arbitrariness: Transformations with functions $\omega(t)$ which do not depend on x_- remain. In order that the limit $F_\kappa(A_-)\to A_-$ as $\kappa\to 0$ be nonsingular, the gauge $F_\kappa(A_-)=0$ must have this property. In Sec. 2 we employ two different regularizations of the $A_-=0$ gauge which have this property, and we obtain two different sets of propagators, one of which contains the Mandelstam–Leibbrandt propagator.

The proposed method of gauge regularization is suitable for finding the propagator of a Yang–Mills fields in any linear gauge $n_\mu A_\mu=0$ [as well as in any Hamiltonian gauge $A_0=f(A_i,E_i)$]. Other rules for encircling the pole at $n\cdot k=0$ can be found by using other regularizations of the gauge $n_\mu A_\mu=0$. In particular, there exists a regularization for which the pole is encircled as follows: $\mathbf{n}\cdot\mathbf{k}\to\mathbf{k}\cdot\mathbf{n}+i\delta\varepsilon(\mathbf{k}\cdot\mathbf{m})$, where $\mathbf{m}$ is an arbitrary three-dimensional vector (for $\mathbf{m}=\mathbf{n}$ this prescription is identical to the Mandelstam–Leibbrandt prescription).

The exposition in Sec. 2 follows Ref. 34.

In Sec. 3 the method of Hamiltonian BRST quantization in a Hamiltonian gauges $\lambda^a=f^a(p,q)$ is developed and applied to quantization of Yang–Mills fields in linear gauges.

As mentioned above, in the case of canonical quantization of systems with first-class constraints in Hamiltonian gauges the physical subspace is separated by the condition $\varphi|\Psi\rangle=0$, and this condition leads, as a rule, to unnormalizable physical vectors. It is the unnormalizability of the state vectors that leads to the translationally noninvariant propagator (2) of Yang–Mills fields in the gauge $A_0=0$. The method of BRST quantization enables separating a physical subspace with positive and finite norm. In Sec. 3.1 the basic components of the BRST approach are examined and it is explained why the method of Hamiltonian BRST quantization developed in Refs. 18 and 19 is directly applicable only to relativistic gauges. Next it is shown that the results obtained with the help of the method of Hamiltonian BRST quantization can be used to construct an effective action in the Hamiltonian gauge, and it is demonstrated how any Hamiltonian gauge can be reduced to a purely Hamiltonian gauge. In order to complete the quantization it must be proved, however, that the physical subspace separated by the condition $Q|\Psi\rangle=0$, where Q is a BRST operator, has a positive and finite norm. This question is examined in Sec. 3.2, where the fact that an even number of constraints is required in order for the vectors from the physical subspace to be regular is also explained. A representation of the commutation relations guaranteeing finiteness of the norm, at least in perturbation theory, is found for the case of first-class constraints, which form a Lie algebra. The search for the correct representation of the Heisenberg algebra is a nontrivial problem, since in this case not all representations are equivalent. It is shown that Dirac's condition $\varphi_a|\Psi\rangle=0$, where $a=1,\ldots,2L$, is replaced in the case of BRST quantization by the condition $\varphi_i|\Psi\rangle=0$, where $i=1,\ldots,L$ and the constraints φ_i are complex and form a subalgebra of the initial Lie algebra (more accurately, its complexification). In subsequent sections BRST quantization of Yang–Mills fields in linear gauges is examined. In Sec. 3.3 a holomorphic representation of the commutation relations is constructed and an expression is found for the evolution operator and the S matrix in terms of a path integral. This analysis is largely a repetition of the corresponding analysis for the standard holomorphic representation of creation and annihilation operators (see, for example, Ref. 33). In Secs. 3.4 and 3.5 Yang–Mills fields are BRST-quantized in linear gauges. The method of Hamiltonian BRST quantization enables a universal analysis of all linear gauges, and for this reason the rules for encircling the pole at $\mathbf{n}\cdot\mathbf{k}=0$ are identical for any vectors n_μ (in contrast to the situation occurring in Secs. 1 and 2 of this review). Two prescriptions for the propagator of Yang–Mills fields are found with the help of two different representations of the commutation relations. The first prescription corresponds to encircling the pole in the principle-value sense. This result shows that there must exist a regularization enabling calculations with such a propagator. The second prescription is a generalized Mandelstam–Leibbrandt prescription: $\mathbf{n}\cdot\mathbf{k}\to\mathbf{k}\cdot\mathbf{n}+i\delta\varepsilon(\mathbf{k}\cdot\mathbf{m})$. For $\mathbf{m}=\mathbf{n}$ this prescription is identical to the one proposed in Ref. 35. In the case of a Hamiltonian gauge, the choice $\mathbf{m}=(1,0,0)$ yields the prescription proposed in Ref. 36.

Section 3 is based on the results obtained in Ref. 37.

1. QUANTIZATION OF YANG–MILLS FIELDS IN THE GAUGE $A_0=0$

This section is devoted to quantization of Yang–Mills fields in the Hamiltonian gauge $A_0=0$. We employ the coordinate and momentum representation for the longitudinal components of the fields and we obtain, taking into account Gauss' law, by the method of functional integration the translationally noninvariant propagator proposed in Ref. 26 for Yang–Mills fields.

1.1. *S* matrix

The Yang–Mills Lagrangian has the form

$$\mathscr{L}=-\tfrac{1}{4}F^a_{\mu\nu}F^a_{\mu\nu}, \tag{4}$$

where $F^a_{\mu\nu}=\partial_\mu A^a_\nu-\partial_\nu A^a_\mu-gt^{abc}A^b_\mu A^c_\nu$ is the tensor of the intensity of the Yang–Mills field and t^{abc} are the structure constants of the Lie algebra corresponding to the gauge group under consideration.

In order to construct the Hamiltonian formalism it is convenient to rewrite the Lagrangian (14) (to within a divergence) in the form

$$\mathscr{L}=E^a_k\partial_0A^a_k-\tfrac{1}{2}[(E^a_k)^2+\tfrac{1}{2}(F^a_{ik})^2]+A^a_0G^a, \tag{5}$$

where $E^a_k=F^a_k$, $aG^a\equiv\partial_kE^a_k+gt^{abc}E^b_kA^c_k$ and $i,j,k=1,2,3$.

It is obvious from this formula that the pairs (E^a_k,A^a_k) are conjugate canonical variables,

$$h=\tfrac{1}{2}(E^a_k)^2+\tfrac{1}{4}(F^a_{ik})^2 \tag{6}$$

is the Hamiltonian, A_0^a are Lagrange multipliers, and G^a are constraints on the canonical variables.

In the gauge $A_0=0$ the Lagrangian (5) describes a Hamiltonian system and the constraints $G^a=0$ must be taken into account separately.

The gauge $A_0=0$ does not completely fix gauge arbitrariness; transformations with a time-independent gauge function are still possible. We show that the constraints $G^a(\mathbf{x})$ are generators of the corresponding gauge transformations. To an arbitrary matrix $\alpha(\mathbf{x})$ in an adjoint representation of the Lie algebra we can associate the functional $G(\alpha)$ defined as

$$G(\alpha)=\int d\mathbf{x}G^a(\mathbf{x})\alpha^a(\mathbf{x})=-\tfrac{1}{2}\int d\mathbf{x}\,\mathrm{tr}[G(\mathbf{x})\alpha(\mathbf{x})], \tag{7}$$

where $G(\mathbf{x})=G^a(\mathbf{x})A^a$ and $\alpha(\mathbf{x})=\alpha^a(\mathbf{x})T^a$, and T^a are generators of the Lie algebra in the adjoint representation.

The standard Poisson bracket for the variables E_k^a and A_k^a generates the following commutation relations for the functionals $G(\alpha)$:

$$\{G(\alpha),G(\beta)\}=gG([\alpha,\beta]). \tag{8}$$

This shows that $G(\alpha)$ gives a representation of the Lie algebra of the gauge-transformation group consisting of the matrices $\alpha(\mathbf{x})$. The action of this representation on the variables E_k^a and A_k^a is given by the formula

$$\delta A_k^a(\mathbf{x})=\{G(\alpha),A_k^a(\mathbf{x})\}=\partial_k\alpha^a(\mathbf{x})-gt^{abc}A_k^b(\mathbf{x})\alpha^c(\mathbf{x}),$$
$$\delta E_k^a(\mathbf{x})=\{G(\alpha),E_k^a(\mathbf{x})\}=-gt^{abc}E_k^b(\mathbf{x})\alpha^c(\mathbf{x}). \tag{9}$$

The Poisson bracket of the constraints G^a with the Hamiltonian h is zero, so that $G(\mathbf{x},t)$ generates an infinite set of integrals of motion.

The observables $O(A_i,E_i)$ are gauge-invariant and therefore they must commute with $G(\alpha)$. This condition enables expressing one of the functions A_i or E_i, on which $O(A_i,E_i)$ depends, in terms of the other functions. Together with the constraint $G^a=0$ this reduces the number of independent functions to four.

In the process of quantization the Poisson brackets between the variables A_i and E_i are replaced by commutators. Since, however, the operators A_i and E_i are considered to be independent, we cannot require that the constraint operator G^a vanish. Instead, the constraint equation is imposed on the admissable states:

$$G|\Phi\rangle=0. \tag{10}$$

Our problem is to construct the scattering matrix describing a transition from the asymptotic state $|\Phi'\rangle$ into the state $|\Phi''\rangle$:

$$\langle\Phi''|S|\Phi'\rangle=\lim_{\substack{t''\to+\infty\\ t'\to-\infty}}\langle\Phi''|\exp\{iH_0t''\}\exp\{-iH(t''-t')\}\times\exp\{-iH_0t'\}|\Phi'\rangle. \tag{11}$$

The space of asymptotic states can be realized as a tensor product of the space of transverse photons and the space of longitudinal photons: $H=H^T\otimes H^L$. For H^T we choose the standard Fock space. In contrast to the case of transverse photons, the free Lagrangian for longitudinal photons, as is obvious from Eq. (5), contains only a kinetic term. For this reason, it is natural to choose for H^L the coordinate or momentum representation. In what follows we used the notation $|\Psi^L\rangle\equiv|E^L\rangle$, if H^L is realized in the momentum representation, and $|\Psi^L\rangle\equiv|A^L\rangle$, if H^L is realized in the coordinate representation. The physical states are separated by the condition (10). It is usually assumed that Gauss' law linearizes for the asymptotic states:

$$\lim_{|t|\to\infty}e^{iH_0t}Ge^{-iH_0t}=G_0,\quad G_0=\partial_kE_k. \tag{12}$$

Here the fact that the coefficient functions of the operator

$$e^{iH_0t}(G-G_0)e^{-iH_0t} \tag{13}$$

are quadratic in the fields and therefore approach zero in the limit $|t|\to\infty$ is employed. Since, however, the solutions of the free equation for A^L grow asymptotically, this assertion may not be valid. Nonetheless, the terms quadratic in the transverse components A^T and in the matter fields vanish asymptotically, and it can be asserted that

$$\lim_{|t|\to\infty}e^{iH_0t}Ge^{-iH_0t}=G_{as},\quad G_{\mathrm{as}}^a\equiv\partial_kE_k^{L,a}+gt^{abc}E_k^{L,b}A_k^{L,c}, \tag{14}$$

where G_{as}^a commutes with the free Hamiltonian H_0.[38]

It is obvious (at least, within the framework of perturbation theory) that the conditions

$$G_0|\Phi\rangle=0,\quad G_{\mathrm{as}}|\Phi\rangle=0 \tag{15}$$

determine the same set of physical states which can be represented in the form

$$|\Phi\rangle=|\Psi^T\rangle\otimes|0\rangle\equiv|\Phi^T,0\rangle, \tag{16}$$

where $|\Psi^T\rangle$ is an arbitrary Fock vector in the transverse-state space.

Repeating word for word the arguments presented in Ref. 33, it is easy to show that the S matrix commutes with G_{as} and is therefore unitary in the physical-state space. Hence it follows, in particular, that

$$\langle E^L,\Psi^T|S|\tilde{\Psi}^T,0\rangle\sim\delta(E^L). \tag{17}$$

We shall make extensive use of this observation below.

The matrix element of the S matrix between arbitrary states can be represented in the form

$$\langle E^L,\Psi^T|S|\tilde{\Psi}^T,\tilde{E}^L\rangle=\exp\left\{-i\int H_{\mathrm{int}}\left(\frac{1}{i}\frac{\delta}{\delta J}\right)dx\right\}\times\langle E^L,\Psi^T|S(J)|\tilde{\Psi}^T,\tilde{E}^L\rangle, \tag{18}$$

where

$$\langle E^L,\Psi^T|S(J)|\tilde{\Psi}^T,\tilde{E}^L\rangle$$
$$= \lim_{\substack{t''\to+\infty\\ t'\to-\infty}} \left\langle E^L,\Psi^T\right|\exp\{iH_0t''\}$$
$$\times\exp\left\{-iH_0(t''-t')+\int_{t'}^{t''} J_i^a(\mathbf{x})A_i^a(\mathbf{x})d\mathbf{x}dt\right\}$$
$$\times\exp\{-iH_0t'\}\left|\tilde{\Psi}^T,\tilde{E}^L\right\rangle. \tag{19}$$

This matrix element determines the propagator in the Feynman diagrammatic technique. Knowing this propagator it is possible to calculate the S matrix of interest from Eq. (18). In the matrix element (19) the dependence on the longitudinal (transverse) components of the field factorizes, and for this reason the problem reduces to calculating independently $\langle\Psi^T|S(J^T)|\tilde{\Psi}^T\rangle$ and $\langle E^L|S(J^L)|\tilde{E}^L\rangle$, where $J^{T(L)}$ is the transverse (longitudinal) component of the source.

The answer for the transverse part is well known (see Ref. 33). For the normal symbol $S^T(a,a^*)$ we have

$$S^T(a,a^*)=\exp\left\{i\int J_i^{a,T}(x)A_{i(0)}^{a,T}(x)dx\right.$$
$$\left.+\frac{i}{2}\int J_i^a(x)\,D_{ij}^{ab,T}(x-y)J_i^b(y)dxdy\right\}, \tag{20}$$

where

$$D_{ij}^{ab,T}(x)=-\frac{\delta^{ab}}{(2\pi)^4}\int\frac{dke^{-ikx}}{k^2+i\varepsilon}\left(\delta_{ij}-\frac{k_ik_j}{\mathbf{k}^2}\right), \tag{21}$$

$A_{i(0)}^{a,T}$ is the solution of the free equation

$$\partial_i(\partial_iA_j-\partial_jA_i)=0$$

in the holomorphic representation.

We now examine the matrix element $\langle E^L|S(J)|\tilde{E}^L\rangle$. Using the completeness of the system of vectors $|A^L\rangle$ this matrix element can be represented in the form

$$\langle E^L,|S(J)|\tilde{E}^L\rangle$$
$$= \lim_{\substack{t''\to+\infty\\ t'\to-\infty}}\int \exp\left\{i\int d\mathbf{x}(\tilde{E}^L\tilde{A}^L-E^LA^L)\right\}$$
$$\times\langle A^L|\exp\{iH_0t''\}|A_2^L\rangle\left\langle A_2^L\right|\exp\left\{-iH_0(t''-t')\right.$$
$$\left.+\int J^L(x)A^L(x)dx\right\}\left|A_1^L\right\rangle$$
$$\times\langle A_1^L|\exp\{-iH_0t'\}|A^L\rangle\mathscr{D}\tilde{A}^L\mathscr{D}A^L\mathscr{D}A_1^L\mathscr{D}A_2^L. \tag{22}$$

Each matrix element appearing on the right-hand side is expressed in terms of a functional integral:

$$\langle A_i^L|\exp\{iH(t_2-t_1)\}A_j^L\rangle$$
$$=\int\mathscr{D}A^L\exp\left\{i\int_{t_1}^{t_2}\mathscr{L}(\mathbf{x},t)d\mathbf{x}dt\right\} \tag{23}$$

with the boundary conditions $A^L(\mathbf{x},t_2)=A_i^L(\mathbf{x})$; $A^L(\mathbf{x},t_1)=A_i^L(\mathbf{x})$ and

$$\mathscr{L}(\mathbf{x},t)=\tfrac{1}{2}(\partial_0A_i^{a,L})^2+J_i^{a,L}(x)A_i^{a,L}(x) \tag{24}$$

($J^L=0$ for the first and third matrix elements). The integral (23) is Gaussian, so that it is equal to the integrand calculated at the extremum of the integrand. The extremal values of A^L are determined by solving the corresponding classical equations and are given by the formulas

$$A_k^L(\mathbf{x},t)=\frac{1}{2}\int_{t_1}^{t_2}|t-s|J_k^{a,L}(\mathbf{x},s)ds+C_k^a(\mathbf{x})t+D_k^a(\mathbf{x}),$$

$$C_k^a(\mathbf{x})=\frac{1}{t_2-t_1}\left(A_{k,i}^{a,L}-A_{k,j}^{a,L}+\int_{t_1}^{t_2}sJ_k^{a,L}(\mathbf{x},s)ds\right.$$
$$\left.-\frac{1}{2}(t_2+t_1)\int_{t_1}^{t_2}J_k^{a,L}(\mathbf{x},s)ds\right),$$

$$D_k^a(\mathbf{x})=A_{k,j}^{a,L}-\frac{1}{2}\int_{t_1}^{t_2}(s-t_1)J_k^{a,L}(\mathbf{x},s)ds-C_k^a(\mathbf{x})t_1. \tag{25}$$

At the extremum the action in the exponent in the exponential in Eq. (23) has the form

$$I=\int_{t_1}^{t_2}\mathscr{L}(\mathbf{x},t)d\mathbf{x}dt$$
$$=\int d\mathbf{x}\left[\frac{1}{2}\left(\frac{1}{2}\int_{t_1}^{t_2}J_k^{a,L}(\mathbf{x},s)ds+C_k^a(\mathbf{x})\right)A_{k,i}^{a,L}\right.$$
$$-\frac{1}{2}\left[-\frac{1}{2}\int_{t_1}^{t_2}J_k^{a,L}(\mathbf{x},s)+C_k^a(\mathbf{x})\right]A_{k,j}^{a,L}$$
$$\left.+\frac{1}{2}\int_{t_1}^{t_2}J_k^{a,L}(\mathbf{x},s)A_k^{a,L}(\mathbf{x},t)ds\right]. \tag{26}$$

Substituting the corresponding expressions into the right-hand side of Eq. (22) and integrating over A^L we obtain

$$\langle E^L|S(J)|\tilde{E}^L\rangle$$

$$= \lim_{\substack{t''\to+\infty \\ t'\to-\infty}} \int \exp\left\{\frac{1}{4}\int_{t'}^{t''} J_k^{a,L}(\mathbf{x},t)\Big|t\right.$$

$$-s\Big|J_k^{a,L}(\mathbf{x},s)dtdsd\mathbf{x}$$

$$-\frac{i}{2}\int_{t'}^{t''} J_k^{a,L}(\mathbf{x},t)J_k^{a,L}(\mathbf{x},s)dtdsd\mathbf{x}$$

$$\left.+i\int_{t'}^{t''} tJ_k^{a,L}(\mathbf{x},t)E_k^{a,L}(\mathbf{x},)d\mathbf{x}dt\right\}$$

$$\times\delta\left(E_k^{a,L}(\mathbf{x})-\tilde{E}_k^{a,L}(\mathbf{x})-\int_{t'}^{t''} J_k^{a,L}(\mathbf{x},t)dt\right). \tag{27}$$

For the physical matrix elements of interest to us $E^L=\tilde{E}^L=0$, and as one can see from Eq. (27) the element $\langle 0|S(J)|0\rangle$ is different from zero only if

$$\int_{-\infty}^{\infty} J_k^{a,L}(\mathbf{x},t)dt=0. \tag{28}$$

The origin of this constraint is easy to understand. For the theory with the Hamiltonian

$$H_0+\int J_k^a(x)A_k^a(x)d\mathbf{x} \tag{29}$$

Gauss' law is an integral of motion only if the condition (28) holds. Otherwise the S matrix converts a physical vector into an unphysical vector with $E^L\neq 0$.

At the same time, in order to construct a perturbation theory on the basis of Eq. (18) it is necessary to know the matrix element $\langle 0,\Psi^T|S(J)|\tilde{\Psi}^T,0\rangle$ for arbitrary J. The way out of this position is as follows. As mentioned, the complete S matrix is unitary in the space of physical states. For this reason, in accordance with Eq. (17) the matrix element of interest to us can be written in the form

$$\langle 0,\Psi^T|S(J)|\tilde{\Psi}^T,0\rangle$$

$$=\int \mathscr{D}E^L(\mathbf{x})\exp\left\{\tfrac{1}{2}\gamma\int E_k^L(\mathbf{x})E_k^L(\mathbf{x})d\mathbf{x}\right\}$$

$$\times\langle E^L,\Psi^T|S|\tilde{\Psi}^T,0\rangle, \tag{30}$$

where γ is an arbitrary constant. Replacing in all equations the matrix element $\langle 0,\Psi^T|S(J)|\tilde{\Psi}^T,0\rangle$ on the right-hand side of Eq. (30), we obtain

$$\langle 0|S(J^L)|0\rangle=\int \mathscr{D}E^L(\mathbf{x})\exp\left\{\frac{i}{2}\gamma\int E_k^L(\mathbf{x})E_k^L(\mathbf{x})d\mathbf{x}\right\}$$

$$\times\langle E^L|S(J^L)|0\rangle$$

$$=\exp\left\{\frac{i}{2}\int J_i^{a,L}(x)D_{ik}^{ab,L}(x,y)J_k^{b,L}(y)dxdy\right\}, \tag{31}$$

where

$$D_{ij}^{ab,L}(x,y)=\bar{D}_{ij}^{ab,L}(\mathbf{x}-\mathbf{y})D(x_0,y_0),$$

$$\bar{D}_{ij}^{ab,L}(\mathbf{x}-\mathbf{y})=\frac{\delta^{ab}}{(2\pi)^3}\int d\mathbf{k}e^{i\mathbf{k}(\mathbf{x}-\mathbf{y})}\frac{k_ik_j}{\mathbf{k}^2}, \tag{32}$$

$$D(x_0,y_0)=\tfrac{1}{2}|x_0-y_0|-\tfrac{1}{2}(x_0+y_0)+\gamma.$$

If we had regularized the initial state instead of the final state, then we would have obtained for the function $D(x_0,y_0)$ the expression

$$D(x_0,y_0)=\tfrac{1}{2}|x_0-y_0|+\tfrac{1}{2}(x_0+y_0)+\gamma. \tag{33}$$

Combining these results with Eq. (21) we obtain the following representation for the Green's function of the Yang–Mills field in the gauge $A_0=0$:

$$D_{ij}^{ab}(x,y)=-\frac{\delta^{ab}}{(2\pi)^4}\int\frac{dke^{-ik(x-y)}}{k^2+i\varepsilon}\left(\delta_{ij}-\frac{k_ik_j}{\mathbf{k}^2}\right)$$

$$+[\tfrac{1}{2}|x_0-y_0|\pm\tfrac{1}{2}(x_0+y_0)+\gamma]$$

$$\times\frac{\delta^{ab}}{(2\pi)^3}\int d\mathbf{k}e^{i\mathbf{k}(\mathbf{x}-\mathbf{y})}\frac{k_ik_j}{\mathbf{k}^2}. \tag{34}$$

The propagator (34) depends not only on the difference $(x-y)$, which, at first glance, destroys the translational invariance of the theory. It is easy to show, however, that the gauge-invariant quantities do exhibit translational invariance. For this it is sufficient to switch, with the help of a standard procedure into for example the Coulomb gauge, in which translationally noninvariant terms are known not to occur.

1.2. Quantum electrodynamics

In the abelian case it is easy to show that the translationally noninvariant terms do not contribute directly to the gauge $A_0=0$. Consider, for example, the generating functional for the scattering matrix in spinor electrodynamics. It can be written in the form of a functional integral

$$S(J^T)=\int \mathscr{D}\bar{\psi}\mathscr{D}\psi\mathscr{D}A\delta(A_0)$$

$$\times\exp\left\{i\int(\mathscr{L}(\bar{\psi},\psi,A)+J_i^T(x)A_i^T(x))dx\right\}, \tag{35}$$

where the field propagator A_i is determined by Eq. (34). Integrating explicitly over A^L we obtain

$$S(J^T)=\int \mathscr{D}\bar{\psi}\mathscr{D}\psi\mathscr{D}A^T$$

$$\times\exp\left\{\frac{i}{2}\int j_i(x)\right.$$

$$\left.\times D^L_{ij}(x,y)j_j(y)dxdy\right\}$$

$$\times\exp\left\{i\int[\mathscr{L}(\bar{\psi},\psi,A^T)+J^T_i(x)A^T_i(x)]dx\right\},\tag{36}$$

where

$$j_i=e\bar{\psi}(x)\gamma_i\psi(x).\tag{37}$$

The argument of the exponential contains the translationally noninvariant term

$$\int \partial_k j_k(x)\partial_n j_n(y)x_0\bar{D}(\mathbf{x}-\mathbf{y})dxdy.\tag{38}$$

We now show that

$$\int\left[\int \partial_k j_k(\mathbf{x},t)dt\int s\partial_n j_n(\mathbf{y},s)\bar{D}(\mathbf{x}-\mathbf{y})dsd\mathbf{x}d\mathbf{y}\right]^n$$

$$\times\exp\left\{i\int\mathscr{L}(\bar{\psi},\psi,A^T)dx\right\}\mathscr{D}\bar{\psi}\mathscr{D}\psi\mathscr{D}A^T=0.\tag{39}$$

To this end, we make in the integral

$$I=\int\exp\left\{i\int(\mathscr{L}(A^T)+i\bar{\psi}(\hat{\partial}-ie\hat{A}^T)\psi+J^T_iA^T_i)dx\right\}$$

$$\times\left[\int s\partial_n j_n(\mathbf{y},s)\bar{D}(\mathbf{y}-\mathbf{z})dsd\mathbf{y}\right]^n\mathscr{D}\bar{\psi}\mathscr{D}\psi\tag{40}$$

the substitution of variables

$$\psi(x)\to e^{i\lambda(x)}\psi(x);\quad \bar{\psi}(x)\to e^{-i\lambda(x)}\bar{\psi}(x).\tag{41}$$

Since such a substitution does not change the integral, we can equate to zero the derivative with respect to $\lambda(x)$ of the transformed integral. This gives

$$\int\left[\int\partial_k j_k(\mathbf{z},t)s\partial_n j_n(\mathbf{y},s)\bar{D}(\mathbf{z}-\mathbf{y})dsdtd\mathbf{x}d\mathbf{y}\right]$$

$$\times\exp\left\{i\int\mathscr{L}(\bar{\psi},\psi,A^T)dx\right\}\mathscr{D}\bar{\psi}\mathscr{D}\psi\mathscr{D}A^T=0.\tag{42}$$

Vanishing of the integral of any power of the expression in brackets is proved completely analogously. This proves that the noninvariant term (38) does not contribute to the physical matrix elements and the S matrix in the gauge $A_0=0$ is explicitly translationally invariant.

In the nonabelian case the situation is more complicated. The question of the contribution of the translationally noninvariant term has thus far not been studied in detail. The correct solution of this question requires a more detailed specification of the regularization scheme.

2. PROPAGATOR OF YANG–MILLS FIELDS IN THE LIGHT-CONE GAUGE

This section is devoted to deriving the propagator of Yang–Mills fields in the light-cone gauge $A_0=A_3$. For this we switch from the Hamiltonian gauge into the light-cone gauge. We obtain two classes of propagators, one of which contains the Mandelstam–Leibbrandt propagator.[3,28]

2.1. *S* matrix

The S matrix in the gauge $A_0=0$ can be written in the form

$$S=\int\mathscr{D}\tilde{E}^L\mathscr{D}\tilde{A}^L\mathscr{D}A^L_1\mathscr{D}A^L_2\mathscr{D}A^L\mathscr{D}a^*_1\mathscr{D}a_1\mathscr{D}b^*_1\mathscr{D}b_1$$

$$\times\exp\left\{\frac{i}{2}\int\tilde{E}^L_k(\mathbf{x})\gamma(\mathbf{x}-\mathbf{y})\tilde{E}^L_k(\mathbf{y})d\mathbf{x}d\mathbf{y}\right\}$$

$$\times\exp\left\{-i\int d\mathbf{x}\tilde{E}^L(\mathbf{x})\tilde{A}^L(\mathbf{x})\right\}\langle\tilde{A}^L,b|e^{iH_0t''}|A^L_2,b_1\rangle$$

$$\times\langle A^L_2,b_1|e^{-iH(t''-t')}|A^L_1,a_1\rangle\langle A^L_1,a_1|e^{-iH_0t'}|A^L,a\rangle.\tag{43}$$

The equation (43) describes the transition matrix element $\langle\tilde{E}^L,b|S|0,a\rangle$, integrated with the function

$$\exp\left\{\frac{i}{2}\int\tilde{E}^L_k(\mathbf{x})\gamma(\mathbf{x}-\mathbf{y})\tilde{E}^L_k(\mathbf{y})d\mathbf{x}d\mathbf{y}\right\},$$

where $\gamma(\mathbf{x})$ is an arbitrary function of $\mathbf{x}$. We employed here the completeness of the system of intermediate states, using, in accordance with the analysis in the preceding section, a holomorphic representation for the transverse states and a coordinate representation for the longitudinal states.

The matrix element of the evolution operator can be represented in the form

$$\langle A^L_2;b_1|e^{-iH(t''-t')}|A^L_1,a_1\rangle$$

$$=\langle b_1|a_1\rangle\exp\left\{-\int d\mathbf{k}b^*_{1r}(\mathbf{k})a_{1r}(\mathbf{k})\right\}\int\mathscr{D}a^*_r(\mathbf{k},t)\mathscr{D}a_r(\mathbf{k},t)\mathscr{D}E^L_i(\mathbf{x},t)\mathscr{D}A^L_i(\mathbf{x},t)\mathscr{D}A_0(\mathbf{x},t)\delta(A_0)$$

$$\times\exp\left\{\int d\mathbf{k}\left(a^*_r(\mathbf{k},t'')a_r(\mathbf{k},t'')+\int_{t'}^{t''}dt(-a^*_r\dot{a}_r-i\omega a^*_ra_r)\right)+i\int_{t'}^{t''}dtd\mathbf{x}(E^{a,L}_i\partial_0A^{a,L}_i-\tfrac{1}{2}(E^{a,L}_i)^2+A^a_0\partial_iE^a_i-V_{\rm int})\right\}.\tag{44}$$

The following boundary conditions are presumed in the integral (44):

$$a_r^*(\mathbf{k},t'')=b_{1r}^*(\mathbf{k}),\quad a_r(\mathbf{k},t')=a_{1r}(\mathbf{k}),\quad A_i^L(\mathbf{x},t'')=A_{i2}^L(\mathbf{x}),\quad A_i^L(\mathbf{x},t')=A_{il}^L(\mathbf{x}).$$

We recall briefly the origin of this equation. The matrix element of an arbitrary operator $\mathbf{V}(\mathbf{a}^*,\mathbf{a})$ has the form

$$\langle b_1|\mathbf{V}(\mathbf{a}^*,\mathbf{a})|a_1\rangle=\langle b_1|\mathbf{V}_{kl}(\mathbf{a}^*)^k(\mathbf{a})^l|a_1\rangle.$$

Since in the holomorphic representation $\mathbf{a}|a_1\rangle=a_1|a_1\rangle$, this expression can be rewritten in the form

$$\langle b_1|a_1\rangle\mathbf{V}_{kl}(b_1^*)^k(a_1)^l=\langle b_1|a_1\rangle\mathbf{V}(b_1^*,a_1),$$

where $\mathbf{V}(b_1^*,a_1)$ is the normal symbol of the operator $\mathbf{V}(\mathbf{a}^*,\mathbf{a})$. Using the representation of a normal symbol as a path integral (see Ref. 33), we obtain Eq. (44).

It can be assumed, without loss of generality, that $n_\mu=(1,0,0,1)$. We introduce the notation $A_-=A_0-A_3$, $\partial_-=\partial_0-\partial_3$. In order to switch to the gauge $A_-=0$, following the standard procedure, the expression for the S matrix in the Hamiltonian gauge must be multiplied by "unity":

$$\Delta(A_-)\int d\omega(\mathbf{x},t)\delta(A_-^\omega)=1,\quad \omega(\mathbf{x},t')=\omega(\mathbf{x},t'')=1. \tag{45}$$

Unit boundary conditions must be imposed on the parameter $\omega(\mathbf{x},t)$ of gauge transformations in order that the substitution of variables $A_\mu\to A_\mu^\omega$ not change the boundary conditions in the path integral. It is easy to see, however, that the condition (45) cannot be satisfied for arbitrary fields A_- [the equation $A_-^\omega=0$ is a first-order equation, and its solution for arbitrary A_- does not satisfy two boundary conditions on $\omega(\mathbf{x},t)$]. In order to circumvent this obstacle the gauge $A_-=0$ can be regularized, i.e., we can switch to the gauge $F_\kappa(A_-)=0$, where $F_\kappa(A_-)\to A_-$ in the limit $\kappa\to0$, and the equation $F_\kappa(A_-^\omega)=0$, $\omega(\mathbf{x},t')=\omega(\mathbf{x},t'')=1$ has a solution for any field $A_-(\mathbf{x},t)$. The gauge $A_-=0$ does not completely fix the gauge arbitrariness, since transformations with functions $\omega(x)$ which do not depend on x_- remain. In order that the limit $F_\kappa(A_-)\to a_-$ as $\kappa\to0$ not be singular, the $F_\kappa(A_-)=0$ gauge must have this property. We employ two different regularizations of the gauge $A_-=0$ that have this property, and we obtain two different sets of propagators, one of which contains the Mandelstam–Leibbrandt propagator.

2.2. Regularization $A_-+\kappa\partial_-A_-=0$

The first regularization of the gauge $A_-=0$ is

$$A_-+\kappa\partial_-A_-=0. \tag{46}$$

In this case Eq. (45) becomes

$$\Delta_\kappa(A_-)\int d\omega(\mathbf{x},t)\delta(A_-^\omega+\kappa\partial_-A_-^\omega)=1,$$

$$\omega(\mathbf{x},t')=\omega(\mathbf{x},t'')=1. \tag{47}$$

We now prove that $\Delta_\kappa(A_-)\to1$ as $\kappa\to0$. It is easy to see that on the surface $A_-+\kappa\partial_-A_-=0$

$$\Delta_\kappa(A_-)=\det M,$$

where $M=\partial_-+\kappa\partial_-^2-\kappa[A_-,\partial_-]$. Using the formula

$$\det M=\int d\bar{c}dc\exp\left\{i\int dx\bar{c}Mc\right\},$$

we obtain

$$\Delta_\kappa(A_-)=\int d\bar{c}dc\exp\left\{i\int dx(-\tfrac{1}{2}\operatorname{tr}(\bar{c}(\partial_-+\kappa\partial_-^2-\kappa[A_-,\partial_-])c))\right\}.$$

The propagator of Faddeev–Popov ghosts is determined from the equation

$$\partial_-D^{ab}(t,s,\mathbf{x}-\mathbf{y})+\kappa\partial_-^2D^{ab}(t,x,\mathbf{x}-\mathbf{y})=\delta^{ab}\delta(x-y), \tag{48}$$

where the boundary conditions formulated above are presumed. It is equal to

$$D^{ab}(t,s,\mathbf{x}-\mathbf{y})=\frac{\delta^{ab}}{(2\pi)^3}\int d\mathbf{k}e^{-i\mathbf{kx}}D(t,s,\mathbf{k}), \tag{49}$$

$$D(t,s,\mathbf{k})=-\frac{\exp\{ik_3(t-s)\}\exp\left\{\frac{1}{\kappa}(s-t')\right\}}{1-\exp\left\{\frac{1}{\kappa}(t''-t')\right\}}\times\left(\left(1-\exp\left\{\frac{1}{\kappa}(s-t'')\right\}\right)\times\left(1-\exp\left\{\frac{1}{\kappa}(t-t')\right\}\right)\vartheta(s-t)+(t\leftrightarrow s)\right).$$

The limit of the propagator $D(t,s,\mathbf{k})$ as $\kappa\to$exists. We shall see below that the limit of the propagator of the Yang–Mills field as $\kappa\to0$ exists, so that the contribution of ghost fields vanishes in the limit $\kappa\to0$, and the Faddeev–Popov determinant is equal to 1.

We multiply Eq. (43) by Eq. (47), make the substitution of variables $A_\mu\to A_\mu^\omega$, and integrate over $\omega(\mathbf{x},t)$. Adding to the action the source $J_\mu A_\mu$ and setting $b_r^*(\mathbf{k})=a_r(\mathbf{k})=0$ we obtain Green's generating functional in the gauge $A_-+\kappa\partial_-A_-=0$:

$$Z(J)=\int \mathscr{D}\tilde{E}^L\mathscr{D}\tilde{A}^L\mathscr{D}A_1^L\mathscr{D}A_2^L\mathscr{D}A^L\mathscr{D}a^*\mathscr{D}a\mathscr{D}b''\mathscr{D}b$$

$$\times\exp\left\{\frac{i}{2}\int \tilde{E}_k^L(\mathbf{x})\gamma(\mathbf{x}-\mathbf{y})\tilde{E}_k^L(\mathbf{y})d\mathbf{x}d\mathbf{y}\right\}$$

$$\times\exp\left\{-i\int d\mathbf{x}\tilde{E}^L(\mathbf{x})\tilde{A}^L(\mathbf{x})\right\}$$

$$\times\langle\tilde{A}^L,0|e^{iH_0t''}|A_2^L,b\rangle\langle A_2^L,b|e^{-iH(t''-t')}|A_1^L,a\rangle$$

$$\times\langle A_1^L,a|e^{-iH_0t'}|A^L,0\rangle, \tag{50}$$

where

$$\langle A_2^L,b|e^{-iH(t''-t')}|A_1^L,a\rangle=\langle b|a\rangle\exp\left\{-\int d\mathbf{k}b_r^*(\mathbf{k})a_r(\mathbf{k})\right\}\int \mathscr{D}a^*\mathscr{D}a\mathscr{D}E^L\mathscr{D}A^L\mathscr{D}A_0\delta\left(\int_{t'}^{t''}A_0(\mathbf{x},t)dt\right)\delta(A_-$$
$$+\kappa\partial_-A_-)\Delta_\kappa(A_-)\exp\left\{\int d\mathbf{k}\left(a_r^*(\mathbf{k},t'')a_r(\mathbf{k},t'')+\int_{t'}^{t''}dt(-a_r^*\dot{a}_r-i\omega a_r^*a_r-i\gamma_r^*a_r-i\gamma_ra_r^*)\right)\right.$$
$$\left.+i\int_{t'}^{t''}dtd\mathbf{x}(E_i^L\partial_0A_i^L-\tfrac{1}{2}(E_i^L)^2+A_0\partial_iE_i^L-J_0A_0+J_i^LA_i^L-V_{\text{int}})\right\}, \tag{51}$$

$$\gamma_r^*(\mathbf{k},t)=\frac{1}{\sqrt{2\omega}}\frac{1}{(2\pi)^{3/2}}\int d\mathbf{x}e^{i\mathbf{kx}}J_l(\mathbf{x},t)e_l^r(\mathbf{k}),$$

$$\gamma_r(\mathbf{k},t)=-\frac{1}{\sqrt{2\omega}}\frac{1}{(2\pi)^{3/2}}\int d\mathbf{x}e^{-i\mathbf{kx}}J_l(\mathbf{x},t)e_l^r(\mathbf{k}).$$

Here $e_l^r(\mathbf{k})$ are unit vectors orthogonal to the vector $\mathbf{k}$ and to one another. The following boundary conditions are presumed in the integral (51):

$$a_r^*(\mathbf{k},t'')=b_r^*(\mathbf{k}),\quad a_r(\mathbf{k},t')=a_r(\mathbf{k}),$$

$$A^L(\mathbf{x},t'')=A_2^L(\mathbf{x}),\quad A^L(\mathbf{x},t')=A_1^L(\mathbf{x}).$$

The δ function from $\int_{t'}^{t''}A_0(\mathbf{x},t)dt$ was used due to the unit boundary conditions on $\omega(\mathbf{x},t)$ in the integral $\int d\omega\delta(A_0^\omega)$, $\omega(\mathbf{x},t')=\omega(\mathbf{x},t'')=1$. In the perturbation theory $Z(J)$ is represented in the form

$$Z(J)=\exp\left\{-i\int dxV_{\text{int}}\left(-\frac{1}{i}\frac{\delta}{\delta J(x)}\right)\right\}Z_0(J), \tag{52}$$

where $Z_0(J)$ is the generating functional of the free theory and is given by

$$Z_0(J)=\int \mathscr{D}\tilde{E}^L\exp\left\{\frac{i}{2}\int \tilde{E}_k^L(\mathbf{x})\gamma(\mathbf{x}-\mathbf{y})\tilde{E}_k^L(\mathbf{y})d\mathbf{x}d\mathbf{y}\right\}$$
$$\times\langle\tilde{E}^L,0|S_0(J)|0,0\rangle$$
$$=\exp\left\{\frac{i}{2}\int J_\mu^a(x)D_{\mu\nu}^{ab}(x,y)J_\nu^b(y)dxdy\right\}. \tag{53}$$

Here $D_{\mu\nu}^{ab}(x,y)$ is the propagator of the Yang–Mills field. Since, as mentioned above, the limit $\kappa\to0$ is nonsingular, we set in this formula $\Delta_\kappa(A_-)=1$. The functional $Z_0(J)$ is determined by a Gaussian integral, and for this reason the functional is equal to the value of the integrand at the extremum. In order to find the extremum it is convenient to transform into the k representation according to the formulas

$$E_i^L(\mathbf{x},t)=\frac{1}{(2\pi)^{2/3}}\int d\mathbf{k}e^{-i\mathbf{kx}}\frac{k_i}{\omega}E(\mathbf{k},t),$$

$$A_0(\mathbf{x},t)=\frac{1}{(2\pi)^{3/2}}\int d\mathbf{k}e^{-i\mathbf{kx}}A_0(\mathbf{k},t).$$

After integrating over the fields $\mathscr{D}a^*\mathscr{D}a\ \mathscr{D}b^*\mathscr{D}b$ we obtain

$$Z_0(J)=\int \mathscr{D}\tilde{E}\mathscr{D}\tilde{A}\mathscr{D}A_1\mathscr{D}A_2\mathscr{D}A\exp\left\{\frac{i}{2}\int d\mathbf{k}(-\tilde{E}(\mathbf{k})\gamma(\mathbf{k})\tilde{E}(-\mathbf{k})+2\tilde{E}(\mathbf{k})\tilde{A}(-\mathbf{k})\right\}$$
$$\times\langle\tilde{A}^L,0|e^{iH_0t''}|A_2^L,0\rangle\langle A_2^L,0|e^{-iH(t''-t')}|A_1^L,0\rangle\langle A_1^L,0|e^{-iH_0t'}|A^L,0\rangle, \tag{54}$$

$$\langle A_2^L,0|e^{-iH(t''-t')}|A_1^L,0\rangle=\int \mathscr{D}a^*\mathscr{D}a\mathscr{D}E\mathscr{D}A\mathscr{D}A_0\mathscr{D}\lambda_1(\mathbf{k},t)\mathscr{D}\lambda_2(\mathbf{k})\exp\left\{\int_{t'}^{t''}d\mathbf{k}dt\left(-a_r^*\dot{a}_r-i\omega a_r^*a_r-i\gamma_r^*a_r\right.\right.$$

$$-i\gamma_r a_r^*-iE(\mathbf{k},t)\partial_0A(-\mathbf{k},t)+\frac{i}{2}E(\mathbf{k},t)E(-\mathbf{k},t)-\omega A_0(\mathbf{k},t)E(-\mathbf{k},t)-iJ_0(\mathbf{k},t)A_0$$

$$(-\mathbf{k},t)-iJ(\mathbf{k},t)A(-\mathbf{k},t)+i\lambda_2(\mathbf{k})A_0(-\mathbf{k},t)+i\lambda_1(-\mathbf{k},t)\left(\left[A_0-\frac{k_3}{\omega}A-\frac{e_3^r(\mathbf{k})}{\sqrt{2\omega}}(a_r^*(\mathbf{k},t)\right.\right.$$

$$\left.\left.\left.\left.+a_r(-\mathbf{k},t))\right](1-i\kappa k_3)+\kappa\left[\partial_0A_0-\frac{k_3}{\omega}\partial_0A-\frac{e_3^r(\mathbf{k})}{\sqrt{2\omega}}(\dot{a}_r^*(\mathbf{k},t)+\dot{a}_r(-\mathbf{k},t))\right]\right)\right)\right\}. \tag{55}$$

The integral (55) is calculated with the following boundary conditions: $a_r^*(\mathbf{k},t'')=a_r(\mathbf{k},t')=0$, $A(\mathbf{k},t'')=A_2(\mathbf{k})$, $A(\mathbf{k},t')=A_1(\mathbf{k})$. Varying the argument of the exponential in Eq. (55) we obtain the system of equations

$$\dot{a}_r(\mathbf{k},t)+i\omega a_r(\mathbf{k},t)+i\gamma_r(\mathbf{k},t)+i\frac{e_3^r(\mathbf{k})}{\sqrt{2\omega}}\lambda(-\mathbf{k},t)=0,$$

$$a_r(\mathbf{k},t')=0,$$

$$\dot{a}_r^*(\mathbf{k},t)-i\omega a_r^*(\mathbf{k},t)-i\gamma_r^*(\mathbf{k},t)-i\frac{e_3^r(\mathbf{k})}{\sqrt{2\omega}}\lambda(\mathbf{k},t)=0,$$

$$a_r^*(\mathbf{k},t'')=0,$$

$$\partial_0E(\mathbf{k},t)-J(\mathbf{k},t)+\frac{k_3}{\omega}\lambda(\mathbf{k},t)=0,\quad A(\mathbf{k},t')=A_1(\mathbf{k}),$$

$$\partial_0A(\mathbf{k},t)-E(\mathbf{k},t)-i\omega A_0(\mathbf{k},t)=0,\quad A(\mathbf{k},t'')=A_2(\mathbf{k}),$$

$$-\omega E(\mathbf{k},t)-iJ_0(\mathbf{k},t)+i\lambda_2(\mathbf{k})+i\lambda(\mathbf{k},t)=0,$$

$$\int_{t'}^{t''}A_0(\mathbf{k},t)dt=0,$$

$$\left[A_0(\mathbf{k},t)-\frac{k_3}{\omega}A-\frac{e_3^r(\mathbf{k})}{\sqrt{2\omega}}(a_r^*(\mathbf{k},t)\right.$$

$$\left.+a_r(-\mathbf{k},t))\right](1-i\kappa k_3)+\kappa\left[\partial_0A_0-\frac{k_3}{\omega}\partial_0A\right.$$

$$\left.-\frac{e_3^r(\mathbf{k})}{\sqrt{2\omega}}(\dot{a}_r^*(\mathbf{k},t)+\dot{a}_r(-\mathbf{k},t))\right]=0,$$

$$\lambda(\mathbf{k},t)=\lambda_1(\mathbf{k},t)(1+i\kappa k_3)-\kappa\partial_0\lambda_1(\mathbf{k},t),$$

$$\lambda_1(\mathbf{k},t')=\lambda_1(\mathbf{k},t'')=0. \tag{56}$$

Solving the system of equations (56), we find the matrix element (55). Next, we substitute this matrix element into Eq. (54) and integrate over $\widetilde{E}$, $\widetilde{A}$, A_1, A_2, and A. Passing to the limits $\kappa\to0$, $t'\to-\infty$, $t''\to+\infty$, we obtain (the calculations are presented in the Appendix to Sec. 2)

$$D_{\mu\nu}(k)=\frac{1}{k^2+i\varepsilon}\left(g_{\mu\nu}-\frac{k_\mu n_\nu}{kn\pm i0}\right.$$

$$\left.-\frac{k_\nu n_\mu}{kn\pm i}\right)+k_\mu k_\nu\widetilde{\gamma}(\mathbf{k})\delta(kn),$$

$$\widetilde{\gamma}(\mathbf{k})=\frac{2\pi}{\omega^2}\left(\gamma(\mathbf{k})+t''-\frac{\omega^2+(\mathbf{kn})^2}{2i\omega(\omega^2-(\mathbf{kn})^2)}\right), \tag{57}$$

where $\gamma(\mathbf{k})$ is an arbitrary function of $\mathbf{k}$. In particular, it can be chosen so that $\gamma(\mathbf{k})=0$. The lower sign in Eq. (57) corresponds to the limit $\kappa\to-0$ ($\kappa<0$) and the upper sign corresponds to the limit $\kappa\to+0$ ($\kappa>0$).

2.3. Mandelstam–Leibbrandt propagator

We obtained above the family of Green's function for the Yang–Mills field in the light-cone gauge. This family does not contain, however, the Mandelstam–Leibbrandt propagator. From the derivation of Eq. (57) it is evident that in order to encircle the pole in the manner proposed by Mandelstam and Leibbrandt the following regularization must be employed (a more detailed discussion is given in the Appendix):

$$A_-+\frac{i\kappa}{\pi}\int dy^3\mathscr{P}\frac{1}{x^3-y^3}\partial_-A_-(x^1,x^2,y^3,t)=0. \tag{58}$$

This formula acquires a pellucid form in Fourier variables:

$$A_-(\mathbf{k},t)+\kappa\varepsilon(k_3)(\partial_-A_-)(\mathbf{k},t)=0. \tag{59}$$

The condition (58) is complex, so that in order for it to be satisfied the fields A_μ must be complex. The fields A_μ in the initial integral (43) are assumed to be real, and for this reason in order to switch to the gauge (58) it is necessary to fields must be complexified.

As usual, we introduce the functional $\Delta_\kappa(A_-)$:

$$\Delta_\kappa(A_-)\int d\omega(\mathbf{x},t)$$

$$\times\delta\left(A_-^\omega+\frac{i\kappa}{\pi}\int dy^3\mathscr{P}\frac{1}{x^3-y^3}\partial_-A_-^\omega(x^1,x^2,y^3,t)\right)=1,$$

$$\omega(\mathbf{x},t')=\omega(\mathbf{x},t'')=1. \tag{60}$$

In this equation the fields A_- are real, since the fields in Eqs. (43) and (44) are real, and the integration extends over the complex variables $\omega(\mathbf{x},t)$. It can be shown, as in the preceding case, that $\Delta_\kappa(A_-)\to 1$ as $\kappa\to 0$. Multiplying Eq. (44) by Eq. (60) and making the substitution of variables $A_\mu\to A_\mu^\omega$, we switch to integration over the complex fields $A_\mu(\mathbf{x},t)$. In order to understand the point of this integration over complex fields we examine a finite-dimensional approximation of Eq. (44) and the linear substitution of variables $A_\mu\to A_\mu+\partial_\mu\alpha$, where $\partial_\mu\alpha$ is a complex number. This substitution transforms the integral over $A_\mu(x_n,t_m)$, which can be regarded as an integral along the real axis in the complex plane, transforms into an integral over a complex variable along a straight line in the complex plane. A similar situation obtains with the substitution $A_\mu\to A_\mu^\omega$, only in this case we switch to integration along a curve in the complex plane. In spite of the fact that the fields A_μ are complex, the action remains real, since it is invariant under the substitutions $A_\mu\to A_\mu^\omega$. For this reason, exponentially diverging terms do not arise in the integral, and standard formulas can be used to calculate the Gaussian integral. Calculations similar to those performed above yield the propagator

$$D_{\mu\nu}(k)=\frac{1}{k^2+i\varepsilon}\left(g_{\mu\nu}-\frac{(k_\mu n_\nu+k_\nu n_\mu)(k_0n_0+\mathbf{kn})}{(k_0n_0)^2-(\mathbf{kn})^2\pm i0}\right)$$
$$+k_\mu k_\nu\tilde{\gamma}(\mathbf{k})\delta(kn),$$

$$\gamma(\mathbf{k})=\frac{2\pi}{\omega^2}(\gamma+t'')\exp\{-ik_3(t''-t')\}. \qquad (61)$$

In Eq. (61) the lower and upper signs correspond to the limits $\kappa\to-0$ ($\kappa<0$) and $\kappa\to+0$ ($\kappa>0$), respectively. Thus if $\gamma=-t''$ and the upper sign are chosen, then the propagator (61) is identical to the Mandelstam–Leibbrandt propagator.

In conclusion we note the following.

1. It is possible to switch not from the Hamiltonian gauge but rather from the Coulomb gauge. In this case we obtain the propagator corresponding to the choice $\gamma=-t''$ in Eqs. (57) and (61).

2. As noted previously, encircling the pole at $k\cdot n=0$ in the principle-value sense leads to infrared divergences associated with the fact that $n^2=0$ (see, for example, Ref. 28). It is easy to see that the class of propagators (57) also leads to infrared divergences. For this reason, in spite of the fact that the propagators (57) are formally admissable, constructing a perturbation theory with these propagators is problematic. The Mandelstam–Leibbrandt propagator, contained in the class (61), does not lead to infrared divergences and it makes it possible to construct a renormalized perturbation theory at least in the single-loop approximation.

3. In order to obtain the propagators (61) we had to switch to integration over complex fields. It should be noted that the light-cone gauge is not the only gauge for which such a switch is required. For example, in order to obtain the propagator in the Lorentz gauge it is necessary to make the substitution $\mathbf{k}^2\to\mathbf{k}^2-i\varepsilon$, which actually means regularization of the Lorentz gauge: We switch from the gauge $\partial_kA_k=0$ to the complex gauge $\partial_0A_0-(1-i\varepsilon)\partial_kA_k=0$, requiring introduction of complex fields A_μ.

4. The proposed method is obviously applicable not only to the light-cone gauge, but also to any linear gauge of the form $A_0-\mathbf{n}\cdot\mathbf{A}=0$. One possible regularization is similar to Eqs. (58) and (59):

$$A_0-\mathbf{nA}+\kappa\varepsilon(\mathbf{mk})[\partial_\mathbf{n}(A_0-\mathbf{nA})](\mathbf{k},t)=0,$$

where $\mathbf{m}$ is an arbitrary vector and $\partial_\mathbf{n}=\partial_0-\mathbf{n}\partial$. This regularization leads to the generalized Mandelstam–Leibbrandt prescription, which will be described in greater detail in the next section.

Appendix

The solution of the system (56) has the form

$$a_r(\mathbf{k},t)=-\int_{t'}^{t}ds\,e^{-i\omega(t-s)}\left(i\gamma_r(\mathbf{k},s)+i\frac{e_3^r(\mathbf{k})}{\sqrt{2\omega}}\lambda(-\mathbf{k},s)\right),$$

$$a_r^*(\mathbf{k},t)=-\int_{t}^{t''}ds\,e^{i\omega(t-s)}\left(i\gamma_r^*(\mathbf{k},s)+i\frac{e_3^r(\mathbf{k})}{\sqrt{2\omega}}\lambda(\mathbf{k},s)\right),$$

$$E(\mathbf{k},t)=E(\mathbf{k},t')+\int_{t'}^{t}ds\left(J(\mathbf{k},s)-\frac{k_3}{\omega}\lambda(\mathbf{k},s)\right),$$

$$E(\mathbf{k},t')=\frac{1}{t''-t'}\left(A_2(\mathbf{k})-A_1(\mathbf{k})-\int_{t'}^{t''}dt(t''-t)\left(J(\mathbf{k},t)-\frac{k_3}{\omega}\lambda(\mathbf{k},t)\right)\right),$$

$$A(\mathbf{k},t)=\tilde{A}(\mathbf{k},t)+\int_{t'}^{t''}ds\,G(t,s,\mathbf{k})f(\mathbf{k},s),$$

$$f(\mathbf{k},t)=\left(E(\mathbf{k},t)+i\sqrt{\frac{\omega}{2}}e_3^r(\mathbf{k})(a_r^*(\mathbf{k},t)+a_r(-\mathbf{k},t))\right)(1-i\kappa k_3)+\kappa\left(\partial_0E(\mathbf{k},t)+i\sqrt{\frac{\omega}{2}}e_3^r(\mathbf{k})(\dot{a}_r^*(\mathbf{k},t)+\dot{a}_r(-\mathbf{k},t))\right),$$

$$\tilde{A}(\mathbf{k},t)=\frac{A_1(\mathbf{k})-A_2(\mathbf{k})\exp\left\{i\left(k_3+\frac{i}{\kappa}\right)(t'-t'')\right\}}{1-\exp\left\{\frac{1}{\kappa}(t''-t')\right\}}e^{ik_3(t-t')}$$
$$+\frac{A_2(\mathbf{k})-A_1(\mathbf{k})\exp\{ik_3(t''-t')\}}{1-\exp\left\{\frac{1}{\kappa}(t''-t')\right\}}\times\exp\left\{i\left(k_3+\frac{i}{\kappa}\right)(t-t'')\right\},$$

$$G(t,s,\mathbf{k})=\frac{\exp\{ik_3(t-s)\}\exp\left\{\frac{1}{\kappa}(s-t')\right\}}{1-\exp\left\{\frac{1}{\kappa}(t''-t')\right\}}$$

$$\times\left(\left(1-\exp\left\{-\frac{1}{\kappa}(s-t'')\right\}\right)\right.$$

$$\left.\times\left(1-\exp\left\{-\frac{1}{\kappa}(t-t')\right\}\right)\vartheta(s-t)+(t\leftrightarrow s)\right),$$

$$A_0(\mathbf{k},t)=\frac{i}{\omega}E(\mathbf{k},t)-\frac{i}{\omega}\partial_0A(\mathbf{k},t),$$

$$\lambda(\mathbf{k},t)=\frac{\exp\{ik_3t\}}{1-\exp\left\{\frac{1}{\kappa}(t''-t')\right\}}\int_{t'}^{t''}ds\partial_\mu J_\mu(\mathbf{k},s$$

$$\times)\left(1-\exp\left\{\frac{1}{\kappa}(t''-t')\right\}\right)\exp\{-ik_3s\}$$

$$+\int_{t'}^{t}ds\partial_\mu J_\mu(\mathbf{k},s)e^{ik_3(t-s)}. \qquad (62)$$

Substituting these solutions into the integrand in Eq. (55) gives

$$I=\langle A_2^L,0|e^{-iH(t''-t')}|A_1^L,0\rangle$$

$$=\exp\left\{\int_{t'}^{t''}d\mathbf{k}dt\left(i\frac{e_3^r(\mathbf{k})}{\sqrt{2\omega}}\lambda(-\mathbf{k},t)a_r^*(\mathbf{k},t)\right.\right.$$

$$-i\gamma_r^*(\mathbf{k},t)a_r(\mathbf{k},t)-\frac{i}{2}E(\mathbf{k},t)E(-\mathbf{k},t)$$

$$\left.\left.+\frac{1}{\omega}J_0(\mathbf{k},t)E(-\mathbf{k},t)+\frac{1}{\omega}\partial_\mu J_\mu(\mathbf{k},t)A(-\mathbf{k},t)\right)\right\}, \qquad (63)$$

where λ, a_r, a_r^*, γ_r, A, and E are determined by Eqs. (62). For $J_\mu=0$ this expression reduces to the following:

$$I_0=\exp\left\{\frac{i}{2}\int\frac{(A_2(\mathbf{k})-A_1(\mathbf{k}))^2}{t''-t'}d\mathbf{k}\right\}$$

$$=\exp\left\{\frac{i}{2}\int d\mathbf{k}[E(\mathbf{k},t')]^2(t''-t')\right\}. \qquad (64)$$

Substituting Eqs. (62) and (63) into Eq. (54) for the generating functional gives

$$Z_0(J)=\int\mathscr{D}\widetilde{E}\mathscr{D}\widetilde{A}\mathscr{D}A_1\mathscr{D}A_2\mathscr{D}A\mathscr{D}E_1(\mathbf{k},t'')\mathscr{D}E_2(\mathbf{k},t')\mathscr{D}E(\mathbf{k},t')\delta\left(E_1(\mathbf{k},t'')+\frac{\widetilde{A}(\mathbf{k})-A_2(\mathbf{k})}{t''}\right)$$

$$\times\delta\left(E_2(\mathbf{k},t')-\frac{A_1(\mathbf{k})-A(\mathbf{k})}{t'}\right)\delta\left(E(\mathbf{k},t')-\frac{1}{t''-t'}\left(A_2(\mathbf{k})-A_1(\mathbf{k})-\int_{t'}^{t''}dt(t''-t)\left(J(\mathbf{k},t)-\frac{k_3}{\omega}\lambda(\mathbf{k},t)\right)\right)\right)$$

$$\times\exp\left\{\int d\mathbf{k}\left(-\frac{i}{2}\widetilde{E}(\mathbf{k})\gamma(\mathbf{k})\widetilde{E}(-\mathbf{k})i\widetilde{E}(\mathbf{k})\widetilde{A}(-\mathbf{k})+\frac{i}{2}t''E_1(\mathbf{k},t'')E_1(-\mathbf{k},t'')-\frac{i}{2}t'E_2(\mathbf{k},t')E_2(-\mathbf{k},t')\right.\right.$$

$$+\int_{t'}^{t''}dt\left(i\frac{e_3^r(\mathbf{k})}{\sqrt{2\omega}}\lambda(-\mathbf{k},t)a_r^*(\mathbf{k},t)-i\gamma_r^*(\mathbf{k},t)a_r(\mathbf{k},t)-\frac{i}{2}E(\mathbf{k},t)E(-\mathbf{k},t)+\frac{1}{\omega}J_0(\mathbf{k},t)E(-\mathbf{k},t)\right.$$

$$\left.\left.\left.+\frac{1}{\omega}\partial_\mu J_\mu(\mathbf{k},t)A(-\mathbf{k},t)\right)\right)\right\}. \qquad (65)$$

We are interested in the propagator in the limit $\kappa\to0$. Passing in Eqs. (62) to the limit $\kappa\to+0$ we obtain

$$\lambda(\mathbf{k},t)=\int_{t'}^{t}ds\partial_\mu J_\mu(\mathbf{k},s)e^{ik_3(t-s)},$$

$$G(t,s,\mathbf{k})=-\vartheta(s-t)e^{ik_3(t-s)},$$

$$f(\mathbf{k},t)=E(\mathbf{k},t)+i\sqrt{\frac{\omega}{2}}e_3^r(\mathbf{k})[a_r^*(\mathbf{k},t)+a_r(-\mathbf{k},t)],$$

$$\widetilde{A}(\mathbf{k},t)=A_2(\mathbf{k})e^{ik_3(t-t'')}. \qquad (66)$$

Substituting Eqs. (66) into Eq. (65) and passing to the limits $t'\to-\infty$ and $t''\to+\infty$ we obtain after simple but long calculations the expression (57) for the propagator of the Yang–Mills field.

In order to understand the origin of the complex gauge (58) we consider now the propagator proposed by Mandelstam and Leibbrandt:

$$D_{ij}(x-y)=\frac{1}{(2\pi)^4}\int\frac{dke^{ik(x-y)}}{k^2+i0}\times\left(-\delta_{ij}-\frac{(k_i\delta_{3j}+k_j\delta_{3i})(k_0+k_3)}{k_0^2-k_3^2+i0}\right)$$

$$=\frac{1}{(2\pi)^3}\int dke^{-i\mathbf{kx}}\left[\frac{\vartheta(s-t)e^{i\omega(t-s)}}{2\omega i}\times\left(-\delta_{ij}-\frac{k_i\delta_{3j}+k_j\delta_{3i}}{\omega-k_3}\right)+\frac{\vartheta(s-t)e^{-i\omega(t-s)}}{2\omega i}\times\left(-\delta_{ij}+\frac{k_i\delta_{3j}+k_j\delta_{3i}}{\omega+k_3}\right)-\frac{i(k_i\delta_{3j}+k_j\delta_{3i})}{\omega^2-k_3^2}\times e^{ik_3(t-s)}[\vartheta(k_3)\vartheta(s-t)-\vartheta(-k_3)\vartheta\times(t-s)]\right].$$

We can see that in order to obtain such a propagator $\lambda(\mathbf{k},t)$ and $G(t,s,\mathbf{k})$ in the limit $\kappa\to 0$ must depend on $\vartheta(k_3)$. The simplest choice corresponds to the substitution $\kappa\to\kappa\varepsilon(k_3)$, where $\varepsilon(k_3)$ is the sign function. In this case we obtain in the limit $\kappa\to+0$

$$G(t,s,\mathbf{k})=-(\vartheta(k_3)\vartheta(s-t)-\vartheta(-k_3)\vartheta(t-s)e^{ik_3(t-s)}.$$

The substitution $\kappa\to\kappa\varepsilon(\kappa_3)$ changes the gauge (46) into the gauge (48). The system of equations in this case is obtained from the system (56) by making the substitution

$$\kappa\to\kappa\varepsilon(k_3),\quad \lambda(\mathbf{k},t)\to\lambda'(\mathbf{k},t),$$

$$\lambda'(\mathbf{k},t)=\lambda_1(\mathbf{k},t)(1-i\kappa\varepsilon(k_3)k_3)+\kappa\varepsilon(k_3)\partial_0\lambda_1(\mathbf{k},t).$$

The solution of the system is obtained from Eqs. (62) by the substitution

$$G(t,s,\mathbf{k},\kappa)\to G[t,s,\mathbf{k},\kappa\varepsilon(k_3)],$$

$$\tilde{A}(t,s,\mathbf{k},\kappa)\to\tilde{A}[t,s,\mathbf{k},\kappa\varepsilon(k_3)],$$

$$\lambda(t,s,\mathbf{k},\kappa)\to\lambda[t,s,\mathbf{k},-\kappa\varepsilon(k_3)].$$

3. BRST QUANTIZATION OF GAUGE THEORIES IN HAMILTONIAN GAUGES

In this section we examine the general scheme of BRST quantization of systems with first-class constraints in gauges of the form $\lambda^a=f^a(p,q)$. As an example, BRST quantization of Yang–Mills fields in linear gauges $n_\mu A_\mu=0$ is performed.

3.1. Hamiltonian BRST quantization

The BRST approach to quantization of systems with constraints includes the following steps. First, the initial phase space is enlarged by including in it nonphysical degrees of freedom (Lagrange multipliers, Faddeev–Popov ghosts, and so on). Next, the nilpotent BRST generator Q is found, the BRST-invariant effect of action or Hamiltonian is constructed, and finally the positiveness and finiteness of the norm of the physical subspace singled out by the condition $Q|\Psi\rangle=0$ is proved. The composition of the enlarged space depends on the gauge in which the theory is to be quantized. For the case of relativistic gauges, these problems have been solved on the basis of the Hamiltonian BRST approach, but the positiveness and finiteness of the norm of the physical subspace has been proved only within perturbation theory.[39] BRST quantization of systems with constraints in a Hamiltonian gauge includes many steps of the Hamiltonian BRST approach in relativistic gauges. For this reason, we shall briefly describe this approach. Hamiltonian BRST quantization can be applied to systems with arbitrary first-class constraints (in particular, the constraints can form a Lie algebra and be dependent).[19] However, we shall confine our attention only to independent constraints.

Consider a system with constraints which is describable by the Lagrangian (1). The enlarged phase space includes, besides the initial variables (p_i,q_i), the following canonically conjugate pairs:[18,19]

a) Lagrange multipliers (π_a,λ^a);

b) Faddeev–Popov ghosts:

$$(\bar{C}_a,P^a),(\bar{P}_a,C^a).$$

These variables have the following ghost numbers:

$$\mathrm{gh}(p_i)=-\mathrm{gh}(q^i)=0,$$

$$\mathrm{gh}(\pi_a)=-\mathrm{gh}(\lambda^a)=0,$$

$$\mathrm{gh}(\bar{C}_a)=-\mathrm{gh}(P^a)=\mathrm{gh}(\bar{P}_a)=-\mathrm{gh}(C^a)=-1. \quad (67)$$

For simplicity, we assumed that p and q are boson variables. In this case the canonical variables satisfy the following statistics:

$$\varepsilon(P)=\varepsilon(X)\equiv\mathrm{gh}(P)\equiv\mathrm{gh}(X)(\mathrm{mod}\ 2), \quad (68)$$

where P and X are a canonically conjugate pair $(|X,P|_\pm=i)$.

We adopt the following conjugation condition for the operators introduced above:

$$(\bar{C}_a)^+=-\bar{C}_{a'}(P^a)^+=P^a,$$

$$(\bar{P}_a)^+=-\bar{P}_{a'}(C^a)^+=C^a,$$

$$(\pi_a)^+=\pi_{a'}(\lambda^a)^+=\lambda^a. \quad (69)$$

These conditions are compatible with the canonical commutation relations. By virtue of the conditions (69), the constraints φ_a are also hermitian (in order that Lagrangian be hermitian).

Thus far we have not said anything about the gauge in which we shall perform the quantization, but we assumed implicitly that quantization is performed in a relativistic gauge. This is associated with the composition of the enlarged phase space. First, we introduce the Lagrange multipliers with the corresponding momenta. The effective action is constructed according to the rule

$$\mathscr{L}_{\mathrm{eff}}=P\dot{X}-H_{\mathrm{eff}}, \quad (70)$$

where P and X are all pairs of canonically conjugate variables from the enlarged phase space. For this reason, the effective action will contain a term of the form $\pi_a\dot{\lambda}^a$. The term fixing the gauge arises after integration over the momenta π_a, and therefore the gauge condition will have the form $\dot{\lambda}^a = f^a(p,q,\lambda)$. Second, we introduce two pairs of canonically conjugate Faddeev–Popov ghosts. This indicates that the corresponding Faddeev–Popov determinant will be quadratic in the time derivative. Hence we also conclude that a relativistic gauge is employed.

The nilpotent BRST operator and BRST-invariant effective Hamiltonian are constructed as follows.

First, the minimal sector including the initial variables p and q together with the pair of Faddeev–Popov ghosts $\bar{P}_a$ and C^a is considered. Using variables from this sector, the minimal BRST operator $Q_{\min}$ and the minimal Hamiltonian $H_{\min}$ satisfying the following conditions are found:

$$[Q_{\min}, Q_{\min}] = 0, \quad \mathrm{gh}(Q_{\min}) = 1, \quad Q^{+}_{\min} = Q_{\min}, \tag{71}$$

$$[H_{\min}, Q_{\min}] = 0, \quad \mathrm{gh}(H_{\min}) = 0, \quad H^{+}_{\min} = H_{\min} \tag{72}$$

If the constraints form a Lie algebra

$$[\varphi_a, \varphi_b] = i f_{abc} \varphi_c, \tag{73}$$

where f_{abc} do not depend on the variables p and q, then the operator $Q_{\min}$ equals

$$Q_{\min} = C^a \varphi_a - \frac{1}{2} f_{abc} \bar{P}_a C^b C^c. \tag{74}$$

The complete BRST operator Q and the effective Hamiltonian H_{eff} depend on all variables from the enlarged phase space:

$$Q = Q_{\min} + P^a \pi_a, \tag{75}$$

$$H_{\text{eff}} = H_{\min} + i[Q, \Psi], \tag{76}$$

Ψ is the so-called gauge fermion operator,[18,19] fixing a gauge whose properties are not important for our purposes.

As mentioned above, in order to complete BRST quantization it is necessary to prove that the norm of the physical subspace separated by the condition

$$Q|\Psi\rangle = 0.$$

is positive and finite. For this we must choose a representation of the commutation relations of the Heisenberg algebra for variables from the enlarged phase space. This problem is nontrivial, since not all representations of the Heisenberg algebra are equivalent. For example, the coordinate representation of the commutation relations leads to unnormalizable vectors of the physical subspace and for this reason is unsuitable for BRST quantization. A suitable representation of the Heisenberg algebra was found in Ref. 39 for a theory with arbitrary linearizable first-class constraints. In the case of independent constraints the free BRST operator Q_0 has the form

$$Q_0 = C^a p_a + P^a \pi_a. \tag{77}$$

Here, with the help of a canonical transformation we made the constraints $\varphi_a^{(0)}$ equal to the momenta p_a.

The representation of the commutation relations that guarantees that the norm of the physical subspace will be positive and finite can be written as follows:[39]

$$a_a^{\pm} = p_a \pm i\pi_a; \quad \bar{a}_a^{\pm} = \tfrac{1}{2}(\pm iq^a + \lambda^a), \tag{78}$$

$$c_a^{\pm} = \frac{1}{2}(C^a \pm iP^a); \quad \bar{c}_a^{\pm} = -i\bar{P}_a \pm \bar{C}_a, \tag{79}$$

$$[a_a^-, \bar{a}_b^+] = [\bar{a}_a^-, a_b^+] = [c_a^-, \bar{c}_b^-] = [\bar{c}_a^-, c_b^+] = \delta_{ab}, \tag{80}$$

and all other commutators are zero.

This representation entangles the pair of initial variables (p,q) with the pair of Lagrange multipliers (π,λ). Using the operators $a^{\pm}$ and $c^{\pm}$ the operator Q_0 can be rewritten in the form

$$Q_0 = c_a^+ a_a^- + c_a^- a_a^+. \tag{81}$$

The representation (81) of the operator Q_0 makes it possible to investigate easily the structure of physical states. We introduce the operator of the number of unphysical particles

$$R = c_a^+ \bar{c}_a^- + a_a^+ \bar{a}_a + \text{h.c.}$$

This operator commutes with Q_0 and admits the representation

$$R = [K, Q_0]_+,$$

where

$$K = \bar{a}_a^+ \bar{c}_a^- + \bar{c}_a^+ \bar{a}_a^-.$$

Any vector from the subspace of states can be expanded in the eigenvectors of the operator R:

$$R|\Psi\rangle = r|\Psi\rangle.$$

For $r \neq 0$ the vector $|\Psi\rangle$ can be represented in the form

$$|\Psi\rangle = \frac{1}{r} R|\Psi\rangle = \frac{1}{r} KQ_0|\Psi\rangle + \frac{1}{r} Q_0 K|\Psi\rangle.$$

For physical vectors the first term on the right-hand side is zero. Therefore any physical vector with $r \neq 0$ has the form $|\Psi\rangle_{\text{ph}} = Q_0|\chi\rangle$ and, by virtue of the nilpotence of Q_0 its norm is zero. The state with $r=0$ does not contain any unphysical particles. For this reason, any physical vector can be represented as

$$|\Psi\rangle_{\text{ph}} = |\Psi\rangle_0 + Q_0|\chi\rangle,$$

where $|\Psi\rangle_0$ is constructed only with the help of physical operators.

Thus have proved that the space of physical states has a nonnegative norm. The vector $Q_0|\chi\rangle$ is orthogonal to all vectors in this space and for this reason makes a zero contribution to all matrix elements. This enables identifying all vectors differing by $Q_0|\chi\rangle$. Factorizing the physical space with respect to the zero vectors, we obtain a space with strictly positive norm.

We are now ready to consider BRST quantization of systems with constraints in Hamiltonian gauges. As mentioned above, it is first necessary to enlarge the phase space. It turns out that the enlarged phase space in the gauge $\lambda^a = f^a(p,q)$ coincides with the minimum spectrum of the expanded space in the relativistic gauge. The absence of pairs (π_a, λ^a) and $(\bar{C}_a, P^a)$ is easy to understand. As explained above, the introduction of a canonically conjugate pair (π_a,λ^a) always presumes quantization in a relativistic gauge. The pair $(\bar{C}_a,P^a)$ is absent, since in the gauge $\lambda^a=f^a(p,q)$ the Faddeev–Popov determinant is linear in the time derivative, and for this reason the Faddeen–Popov antighost is the Faddeev–Popov momentum ghost.

The BRST operator in the Hamiltonian gauge is identical to the minimal BRST operator, and the effective Hamiltonian is determined with the help of the minimal effective Hamiltonian:

$$Q=Q_{\min},$$

$$H_{\text{eff}}=H_{\min}+i[Q,\Psi], \tag{82}$$

where the fermion gauge operator Ψ for the gauge $\lambda^a=f^a(p,q)$ is

$$\Psi=\bar{P}_a f^a(p,q).$$

The effective Lagrangian in the gauge $\lambda^a=f^a(p,q)$ is determined as follows:

$$\mathscr{L}_{\text{eff}}=p_i\dot{q}^i+\bar{P}_a\dot{C}^a-H_{\text{eff}}. \tag{83}$$

For most physical theories ghosts will not contribute to the S matrix. In the general case, however, ghosts can interact nontrivially with the initial fields.

We note also that quantization of systems with constraints in Hamiltonian gauges $\lambda^a=f^a(p,q)$ is easily reduced to quantization in a purely Hamiltonian gauge. For this it is sufficient to redefine the field λ^a:

$$\lambda^a-f^a(p,q)\rightarrow\lambda^a. \tag{84}$$

Then the Lagrangian (1) assumes the form

$$\mathscr{L}=p_i\dot{q}^i-H+f^a(p,q)\varphi_a(p,q)+\lambda^a\varphi_a(p,q) \tag{85}$$

with the new Hamiltonian $H'=H-f^a(p,q)\varphi_a(p,q)$.

Now, in order to quantize the system (1) in the gauge $\lambda^a=f^a(p,q)$ it is sufficient to quantize the system (85) in the gauge $\lambda^a=0$. In order to complete the BRST quantization, however, it is necessary to examine the question of the norm of the physical subspace.

3.2. Physical subspace

We consider here the simplest case of independent first-class constraints, which form a semisimple Lie algebra L with the BRST operator Q:

$$Q=C^a\varphi_a-\tfrac{1}{2}f_{abc}\bar{P}_aC^bC^c. \tag{86}$$

In the perturbation theory, however, our discussions will be valid for arbitrary first-class constraints.

The physical subspace is singled out by the conditions

$$Q|\Psi\rangle=0, \tag{87}$$

$$N|\Psi\rangle=0. \tag{88}$$

Here $N=\frac{1}{2}(\bar{P}_aC^a-C^a\bar{P}_a)$ is the ghost number operator. The condition (88) can be imposed on vectors of the physical subspace, since the operator N commutes with the Hamiltonian. In the perturbation theory the condition (88) is not required, since any vector satisfying the condition (87) can be represented in the form

$$|\Psi'\rangle=|\Psi\rangle_{\text{ph}}+Q|\chi\rangle, \tag{89}$$

where the vector $|\Psi\rangle_{\text{ph}}$ satisfies the condition (88).

As explained in Sec. 3.1, in order for the norm of the physical subspace to be positive and finite it is necessary to find the correct representation of the commutation relations. We shall strive to represent the operator Q in a form as close as possible to the form of the free BRST operator Q_0 (81). It is helpful to note that the operator Q_0 can be written as follows:

$$Q_0=\Omega+\Omega^+, \tag{90}$$

$$\Omega=c_a^+a_a^-;\quad \Omega^+=c_a^-a_a^+. \tag{91}$$

The operators Ω and Ω^+ are nilpotent, hermitian conjugates of one another and they anticommute with one another. Moreover, both operators Ω and Ω^+ annihilate the vacuum state. We shall try to represent the operator (86) in the form (90).

It is obvious that such a representation is possible only in the case when the number of constraints φ_a is even. For this reason, we first consider this case. Let the number of constraints φ_a be $2L$.

By virtue of Gauss' decomposition, for any complex Lie algebra the following representation is valid:

$$L=L^++H+L^-. \tag{92}$$

Here H is a Cartan subalgebra, and L^+ and L^- are nilpotent subalgebras. In the algebras L^+ and L^- there exist bases which satisfy the conjugation condition

$$(e_\alpha^+)^*=e_\alpha^-, \tag{93}$$

where $e_\alpha^+(e_\alpha^-)$ is the basis in the subalgebra $L^+(L^-)$; the operator * is the conjugation of the algebra L (if the matrix realization of the algebra L is being considered, then * is the standard hermitian conjugation of matrices). Choosing in the Cartan subalgebra H an analogous basis, we obtain the following decomposition for the algebra L:

$$L=L_1^++L_1^-, \tag{94}$$

where $L_1^+(L_1^-)$ is the subalgebra of algebra L with the basis $\{e_\alpha^+,e_\mu^+\}(e_\alpha^-,e_\mu^-)$ (e_μ^+ and e_μ^- is the complex basis of the Cartan subalgebra).

The constraints φ_a form a complex Lie algebra (which can be the complexification of a real Lie algebra). For this reason we can switch from the constraints φ_a to the constraints φ_i^+ and φ_i^-, satisfying the conditions

$$(\varphi_i^+)^+=\varphi_i^-,\quad i=1,...,L, \tag{95}$$

$$[\varphi_i^+,\varphi_j^+]=\varphi_k^+\bar{U}_{ij}^k,\quad [\varphi_i^-,\varphi_j^-]=\varphi_k^-U_{ij}^k, \tag{96}$$

where the overbar indicates complex conjugation. This transformation is made with the help of the complex matrix A:

$$\varphi_a=A_a^i\varphi_i^++A_a^{L+i}\varphi_i^-;\quad \bar{A}_a^i=A_a^{L+i}, \tag{97}$$

$$\varphi_i^+=(A^{-1})_i^a\varphi_a;\quad \varphi_i^-=(\bar{A}^{-1})_i^a\varphi_a. \tag{98}$$

It is easy to obtain from Eqs. (96)–(98) the following relations between the structure constants $f_{abc}, \bar{U}^k_{ij}$ and the matrix A:

$$\bar{U}^k_{ij} = (A^{-1})^a_i (A^{-1})^a_j i f_{abc} A^k_c,$$

$$(A^{-1})^a_i (A^{-1})^a_j f_{abc} \bar{A}^k_c = 0. \quad (99)$$

We now introduce the ghost creation and annihilation operators:

$$c^+_i = \bar{A}^i_a C^a; \quad c^-_i = A^i_a C^a, \quad (100)$$

$$\bar{c}^-_i = -i(\bar{A}^{-1})^a_i \bar{P}_a; \quad \bar{c}^+_i = -i(A^{-1})^a_i P_a. \quad (101)$$

This representation of the commutation relation is analogous to the representation (79) employed in Sec. 1.

Using Eqs. (96)–(100) it is not difficult to show that the operator Q has the following form (see, for comparison, Ref. 40):

$$Q_0 = \Omega + \Omega^+; \quad (\Omega^+)^+ = \Omega; \quad \Omega^2 = \Omega\Omega^+ + \Omega^+\Omega = 0,$$

$$\Omega = c^+_i \varphi^-_i - 1/2 c^+_k c^+_j \bar{c}^-_i U^i_{jk} - \bar{c}^+_k c^+_j c^-_i U^i_{jk}. \quad (102)$$

Indeed, the operators (86) and (102) differ by terms arising due to normal ordering.

We now define the vacuum with respect to the ghost creation and annihilation operators:

$$c^- |0\rangle = \bar{c}^- |0\rangle = 0; \quad \langle 0|0\rangle = 1. \quad (103)$$

Consider the state

$$|\Psi\rangle = |\Psi(p,q)\rangle \otimes |0\rangle, \quad (104)$$

where the state $|\Psi(p,q)\rangle$ does not depend on the ghost fields. The operator Q operating on such a state gives

$$Q|\Psi\rangle = \varphi^-_i |\Psi(p,q)\rangle \otimes c^+_i |0\rangle.$$

Thus in order for the operator Q to annihilate the state $|\Psi\rangle$ the following condition must be satisfied:

$$\varphi^-_i |\Psi(p,q)\rangle = 0. \quad (105)$$

For theories such as the Yang–Mills theory, this condition exhausts all physical states and replaces the usually employed condition

$$\varphi_a |\Psi(p,q)\rangle = 0. \quad (106)$$

We note that there are half as many conditions (105) as conditions (10), but they are complex.

Thus far we have said nothing about the representation of the commutation relations for the initial variables p and q. In the perturbation theory the parts of the constraints φ^+_i and φ^-_i which are linear in the variables p and q can be taken as the creation and annihilation operators. Outside the framework of perturbation theory the representation must be chosen so that the physical vectors have a positive and finite norm.

However, we have examined only the case of an even number of constraints. If the number of constraints is odd, then it is easy to see that there does not exist a suitable representation of the Heisenberg algebra: Any representation will lead to irregular vectors. This difficulty can be circumvented in a natural and simple manner. We consider the auxiliary Lagrangian

$$\mathcal{L}' = p_{2L}\dot{q}_{2L} - H(p_{2L}, q_{2L}) + \lambda_{2L} p_{2L} \quad (107)$$

and add it to the Lagrangian (1) [here $2L-1$ is the number of constraints in the Lagrangian (1)]. Adding the Lagrangian (107) does not change the dynamics because of the constraint $p_{2L}=0$. After this we have a system with an even number of constraints, which can be studied just as above. The Hamiltonian $H(p_{2L}, q_{2L})$ must satisfy the condition $[H(p_{2L}, q_{2L}), p_{2L}] \sim p_{2L}$ and it must be chosen so that the complete free Hamiltonian $\tilde{H}_0 = H_0(p,q) + H_0(p_{2L}, q_{2L})$ annihilates the vacuum of perturbation theory. In concluding this section we note that the representation $Q_0 = \Omega + \Omega^+$ probably exists for a theory with any number of first-class constraints.

3.3. Holomorphic representation

In what follows we shall make extensive use of a holomorphic representation of the commutation relations (78) that is somewhat different from the one usually employed. For this reason we shall describe it in greater detail. We shall consider only the case of boson variables, since in the Yang–Mills theory ghosts in linear gauges do not contribute to the S matrix. As usual, we construct the holomorphic representation for the minimum number of creation and annihilation operators; the extension to the case of an arbitrary number of operators is trivial.

Thus we consider four creation and annihilation operators, satisfying the commutation relations and conjugation rules:

$$[\mathbf{a}^-, \bar{\mathbf{a}}^+] = [\bar{\mathbf{a}}^-, \mathbf{a}^+] = 1, \quad (108)$$

$$(\mathbf{a}^-)^+ = \mathbf{a}^+, \quad (\bar{\mathbf{a}}^-)^+ = \bar{\mathbf{a}}^+. \quad (109)$$

All other commutators are zero. Here boldface symbols are operators.

The holomorphic representation is a representation in the space of holomorphic functions of two variables $f(a^+, \bar{a}^+)$ with the following scalar product (we follow Ref. 33 in constructing the holomorphic representation):

$$(f_1, f_2) = \int d^2a\, d^2\bar{a} [f_1(\bar{a}^+, a^+)]^* f_2(a^+, \bar{a}^+) \times e^{-a^+a^- - \bar{a}^+\bar{a}^-}. \quad (110)$$

Here $d^2a = da^+ da^-/2\pi i$, $d^2\bar{a} = d\bar{a}^+ d\bar{a}^-/2\pi i$; the variables a^- and $\bar{a}^-$ are the complex conjugates of the variables a^+ and $\bar{a}^+$. The equation (110) differs from the usual equation for the scalar product by a transposition of the variables in the function $f_1(a^+, \bar{a}^+)$. The operators a^+, $\bar{a}^+$, a^-, and $\bar{a}^-$ operate as follows:

$$\mathbf{a}^+ f(a^+,\bar{a}^+)=a^+ f(a^+,\bar{a}^+);$$

$$\bar{\mathbf{a}}^+ f(a^+,\bar{a}^+)=\bar{a}^+ f(a^+,\bar{a}^+),$$

$$\bar{\mathbf{a}}^- f(a^+,\bar{a}^+)=\partial f(a^+,\bar{a}^+)/\partial a^+;$$

$$\mathbf{a}^- f(a^+,\bar{a}^+)=\partial f(a^+,\bar{a}^+)/\partial \bar{a}^+. \qquad (111)$$

The scalar product (110) introduced above and the realization (111) of the creation and annihilation operators satisfy the commutation relations (108) and the conjugation rules (109).

The monomials

$$\Psi_{nm}(a^+,\bar{a}^+)=(a^+)^n/\sqrt{n!}(\bar{a}^+)^m/\sqrt{m!} \qquad (112)$$

form a basis in the space of holomorphic functions. This basis is normalized as follows (it is not orthonormalized, and for this reason the scalar product (110) is not positive-definite):

$$(\Psi_{n_1m_1},\Psi_{n_2m_2})=\delta_{n_1m_2}\delta_{n_2m_1}. \qquad (113)$$

Two methods are employed to describe operators in this representation. The first method is the representation of an arbitrary operator **A** in the form of an integral operator with the kernel $A(a^+,\bar{a}^+;a^-,\bar{a}^-)$:

$$(Af)(a^+,\bar{a}^+;a^-,\bar{a}^-)=\int d^2bd^2\bar{b}e^{-b^+b^- -\bar{b}^+\bar{b}_*} \times A(a^+,\bar{a}^+;b^-,\bar{b}^-)f(\bar{b}^+,b^+). \qquad (114)$$

The product of operators $\mathbf{A}_1$ and $\mathbf{A}_2$ corresponds to convolution of the kernels:

$$(\mathbf{A}_1\mathbf{A}_2)(a^+,\bar{a}^+;a^-,\bar{a}^-)=\int d^2bd^2\bar{b}e^{-b^+b^- -\bar{b}^+\bar{b}_*} \times A_1(a^+,\bar{a}^+;b^-,\bar{b}^-) \times A_2(\bar{b}^+,b^+;a^-,\bar{a}^-). \qquad (115)$$

The normal symbol of an operator is employed as the second representation for operators. If the operator **A** is given as a sum of normal products

$$\mathbf{A}=\sum K_{n_1m_1,n_2m_2}(\mathbf{a}^+)^{n_1}(\bar{\mathbf{a}}^+)^{m_1}(\mathbf{a}^-)^{n_2}(\bar{\mathbf{a}}^-)^{m_2}, \qquad (116)$$

then such an operator is associated to the function $K(a^+,\bar{a}^+;a^-,\bar{a}^-)$, called the normal symbol of the operator **A**:

$$K(a^+,\bar{a}^+;a^-,\bar{a}^-)=\sum K_{n_1m_1,n_2m_2} \times(a^+)^{n_1}(\bar{a}^+)^{m_1}(a^-)^{n_2}(\bar{a}^-)^{m_2}. \qquad (117)$$

Using Eqs. (111), (114), and (117) it is easy to show that the kernel of the operator **A** is related to the normal symbol as follows:

$$A(a^+,\bar{a}^+;b^-,\bar{b}^-)=K(a^+,\bar{a}^+;b^-,\bar{b}^-)e^{a^+b^- +\bar{a}^+\bar{b}}. \qquad (118)$$

The equations (115) and (118) enable constructing easily the kernel of the evolution operator $\mathbf{U}(t''-t')=\exp\{-i\mathbf{H}(t''-t')\}$ in the form of a path integral over the functions $a^+(t),\bar{a}^+(t),a^-(t),\bar{a}^-(t)$. The corresponding construction is identical to the arguments given in Ref. 33, so that we present the answer without derivation:

$$U(a^+,\bar{a}^+;a^-,\bar{a}^-;t''-t') = \int d^2\alpha(t)d^2\bar{\alpha}(t)\exp\{\alpha^+(t'')\bar{\alpha}^-(t'') +\bar{\alpha}^+(t'')\alpha^-(t'')+i\int_{t'}^{t''}dt[i\alpha^+(t)\dot{\bar{\alpha}}^-(t) +i\bar{\alpha}^+(t)\dot{\alpha}^-(t)-H(\alpha^+,\bar{\alpha}^+,\alpha^-,\bar{\alpha}^-)]\}. \qquad (119)$$

The following boundary conditions are presupposed in Eq. (119):

$$\alpha^+(t'')=a^+,\ \bar{\alpha}^+(t'')=\bar{a}^+,\ \alpha^-(t')=a^-,\ \bar{\alpha}^-(t')=\bar{a}^-. \qquad (120)$$

The function $H(\alpha^+,\bar{\alpha}^+,\alpha^-,\bar{\alpha}^-)$ is the normal symbol of the Hamiltonian. The S matrix is determined with the help of the evolution operator

$$\mathbf{S}=\lim_{t',t''\to\infty}\exp(i\mathbf{H}_0(t'')\exp[-i\mathbf{H}(t''-t')]\exp(-i\mathbf{H}_0(t'). \qquad (121)$$

Using Eqs. (119), (115), and (118) it is easy to find the kernel and the normal symbol of the S matrix. The perturbation theory is usually constructed with the help of the generating functional $Z(J)$. It is defined as the vacuum average of the S matrix in the presence of an external current J and can be calculated with the help of the kernel of the S matrix according to the formula

$$Z(J)=S(0,0,0,0). \qquad (122)$$

3.4. Quantization of Yang–Mills fields in linear gauges

We shall apply the above-described scheme for quantizing Yang–Mills fields in linear gauges. As noted in the introduction, these gauges are all Hamiltonian gauges: $A_0=\mathbf{n}\cdot\mathbf{A}/n_0$ (with the exception, strictly speaking, of the purely axial gauge). For this reason, from the standpoint of the BRST approach, quantization in these gauges is performed universally and therefore we shall have to obtain a universal prescription for any linear gauge. Our problem is to find the propagator of the Yang–Mills fields. For this reason, it is sufficient to examine the free Yang–Mills theory, i.e., electrodynamics.

The Lagrangian of electrodynamics in the first-order formalism is (see, for example, Ref. 33):

$$\mathscr{L}=E_i\partial_0A_i-\tfrac{1}{2}E_i^2-\tfrac{1}{4}(\partial_iA_j-\partial_jA_i)^2+A_0\partial_iE_i. \qquad (123)$$

The BRST operator in the Hamiltonian gauge is

$$Q=\int d\mathbf{x}c(\mathbf{x})\partial_i\dot{E}_i(\mathbf{x}). \qquad (124)$$

We have one constraint $\partial_i E_i$, so that in order to perform the quantization correctly we must examine the auxiliary Lagrangian $\mathscr{L}$:

$$\mathscr{L}'=E'\partial_0 A'-H'(E',A')+A_0'E'. \quad (125)$$

The operator Q of the complete system $\mathscr{L}+\mathscr{L}'$ is

$$Q=\int d\mathbf{x}[c(\mathbf{x})\partial_i E_i(\mathbf{x})+c'(\mathbf{x})E'(\mathbf{x})]. \quad (126)$$

We separate the fields E_i and A_i into longitudinal and transverse components:

$$A_i(\mathbf{x})=A_i^T(\mathbf{x})+A_i^L(\mathbf{x});\quad A_i^L(\mathbf{x})=-\partial_i A(\mathbf{x}),$$

$$E_i(\mathbf{x})=E_i^T(\mathbf{x})+E_i^L(\mathbf{x});\quad E_i^L(\mathbf{x})=\Delta^{-1}\partial E(\mathbf{x}). \quad (127)$$

Using this separation we rewrite the complete Lagrangian $\tilde{\mathscr{L}}=\mathscr{L}+\mathscr{L}'$ and the complete BRST operator Q in the form

$$\tilde{\mathscr{L}}=E_i\partial_0 A_i+E\partial_0 A+E'\partial_0 A'-\tfrac{1}{2}(E_i^T)^2-\tfrac{1}{4}(\partial_i A_j^T-\partial_j A_i^T)^2$$
$$+\tfrac{1}{2}\Delta^{-1}E^2-H'(E',A')+A_0E+A_0'E', \quad (128)$$

$$Q=\int d\mathbf{x}[c(\mathbf{x})E(\mathbf{x})+c'(\mathbf{x})E'(\mathbf{x})]. \quad (129)$$

We introduce the creation and annihilation operators, just as in Eqs. (78)–(80):

$$c^{\pm}=\tfrac{1}{2}(c\pm ic'),\quad \bar{c}^{\pm}=-i\bar{P}\pm\bar{P}', \quad (130)$$

$$a^{\pm}=E\pm iE',\quad \bar{a}^{\pm}=\tfrac{1}{2}(\pm iA+A'). \quad (131)$$

Then the operator Q will assume the form (81) and it will separate the subspace with a positive norm.

The next step in BRST quantization is choosing the Hamiltonian $H'(E',A')$. This choice depends on the gauge in which quantization is performed. We shall consider the simplest case of the Hamiltonian gauge $A_0=0$. In this gauge the Hamiltonian $H'(E',A')$ must be chosen as $-\frac{1}{2}\Delta^{-1}E'^2$ (in this case the vacuum will be an eigenvector of the Hamiltonian with zero eigenvalue). In the gauge $A_0=0$ the longitudinal and transverse components of the fields do not interact with one another. For this reason, the Lagrangian can be studied separately for the longitudinal and transverse components. Quantization of the transverse components is performed as usual and yields a propagator that is identical to the propagator in the Coulomb gauge (see Ref. 33). Thus it is sufficient to consider quantization of only the longitudinal components. The Hamiltonian for the longitudinal components in the gauge $A_0=0$ is

$$H^L=-\tfrac{1}{2}\Delta^{-1}E^2-\tfrac{1}{2}\Delta^{-1}E_1^2+AJ, \quad (132)$$

where we introduced the external current J.

Using the operators $a^{\pm},\bar{a}^{\pm}$ we rewrite the Hamiltonian H^L in the form

$$H^L=-\tfrac{1}{2}\Delta^{-1}a^+a^--i\bar{a}^+J+i\bar{a}^-J. \quad (133)$$

We must now calculate the kernel of the evolution operator $U(t''-t')$. Then, using Eqs. (121) and (122) we shall be able to find the longitudinal propagator. In accordance with Eq. (119) the kernel of the evolution operator is

$$U(a^+,\bar{a}^+;a^-,\bar{a}^-;t''-t')$$
$$=\int d^2\alpha(\mathbf{x},t)d^2\bar{\alpha}(\mathbf{x},t)\exp\Big\{\int d\mathbf{x}[\alpha^+(\mathbf{x},t'')\bar{\alpha}^-(\mathbf{x},t'')$$
$$+\bar{\alpha}^+(\mathbf{x},t'')\alpha^-(\mathbf{x},t'')+i\int_{t'}^{t''}dt(i\alpha^+(x,t)\dot{\bar{\alpha}}^-(\mathbf{x},t)$$
$$+i\bar{\alpha}^+(\mathbf{x},t)\dot{\alpha}^-(\mathbf{x},t)+\tfrac{1}{2}\Delta^{-1}\alpha^+(\mathbf{x},t)+i\bar{\alpha}^+(\mathbf{x},t)J(\mathbf{x},t)$$
$$-i\bar{\alpha}^-(\mathbf{x},t)J(\mathbf{x},t)]\Big\}, \quad (134)$$

where the boundary conditions (120) are presumed.

The integral in Eq. (134) is a Gaussian integral. Simple computations yield

$$U(a^+,\bar{a}^+;a^-,\bar{a}^-;t''-t')$$
$$=\exp\Big\{\int d\mathbf{x}\Big[a^+\Big(\bar{a}^-+\frac{i}{2}\Delta^{-1}a^-(t''-t')\Big)+\bar{a}^+a^-$$
$$+i\int_{t'}^{t''}dtJ(t)\Big(-\frac{i}{2}\Delta^{-1}a^+(t''-t)$$
$$+\frac{i}{2}\Delta^{-1}a^-(t-t')-\bar{a}^++\bar{a}^-\Big)$$
$$-\frac{i}{4}\Delta^{-1}\int_{t'}^{t''}dtdsJ(t)J(s)\Big|t-s\Big|\Big]\Big\}. \quad (135)$$

With the help of Eqs. (115), (121), (122), and (135) it is easy to calculate the generating functional:

$$Z(J)=S(0,0,0,0)$$
$$=\exp\Big\{-\frac{i}{4}\Delta^{-1}\int_{t'}^{t''}dtdsJ(t)J(s)\Big|t-s\Big|\Big\}. \quad (136)$$

Thus the longitudinal propagator

$$D^L(x-y)=\tfrac{1}{2}\Delta^{-1}|t-s| \quad (137)$$

corresponds to encircling the pole at $k_0=0$ in the principle-value sense.

In Ref. 26 the Wilson loop was calculated with the propagator (137) on the basis of dimensional regularization. As asserted in Ref. 26, the result is different from that obtained in the Coulomb gauge. In these calculations, however, a number of nonobvious assumptions were employed and the question of the possibility of using the operator (137) requires, in our opinion, further investigation. The main difficulty here lies in choosing an adequate invariant regularization.

We also note that the propagator (137) can be obtained using a different representation of the commutation relations. Instead of introducing the auxiliary field A', the creation and annihilation operators of the longitudinal quanta can be constructed using the operators $A^L(\mathbf{x})$ and $E^L(\mathbf{x})$ at different spatial points:

$$E^{\pm}(\mathbf{x})=E(\mathbf{x})\pm iE(-\mathbf{x});$$

$$A^{\pm}(\mathbf{x})=\tfrac{1}{2}[\pm iA(\mathbf{x})+A(-\mathbf{x})],$$

where the operators $E^{\pm}(\mathbf{x})$ and $A^{\pm}(\mathbf{x})$ are determined for $\mathbf{m}\cdot\mathbf{x}>0$, where $\mathbf{m}$ is an arbitrary vector.

Quantization in an arbitrary linear gauge $A_0=\mathbf{n}\cdot\mathbf{A}$ is performed by the same method. In the general case, however, the longitudinal and transverse modes will interact, and this greatly complicates the calculations. Nonetheless, the answer will be the same as in the Hamiltonian gauge—the pole at $n\cdot k=0$ is encircled in the principle-value sense (indeed, the rule obtained in Ref. 34 for encircling the pole for the light-cone gauge is possible; this prescription is identical to encircling the pole in the principle-value sense in the Hamiltonian gauge).

3.5. General Mandelstam–Leibbrandt prescription

In the discussion above we introduced the auxiliary variables A' and E' in order to quantize correctly Yang–Mills fields in linear gauges. We assumed that electrodynamics is a system with one constraint. However, this is not entirely correct. In reality, electrodynamics is a system with an infinite number of constraints (there is one constraint at each point $\mathbf{x}$ of space). For this reason, if we were able to separate this infinity into two infinities, then we could construct a representation of the commutation relations that is analogous to the representation (78)–(88) and guarantees that the norm of the physical subspace would be positive. In this section we shall find such a representation and we shall show that it leads to the general Mandelstam–Leibbrandt prescription.

The constraints $E(\mathbf{x})$ $(E=\partial_i E_i)$ are most easily separated for the Fourier transform of the constraint $E(\mathbf{x})$. The corresponding representation is

$$E(\mathbf{x})=\frac{i}{(2\pi)^{3/2}}\int\sqrt{2\omega}dk[-e^{i\mathbf{kx}}\vartheta[f(\mathbf{k})]\bar{A}^-(\mathbf{k})+e^{-i\mathbf{kx}}\vartheta[f(\mathbf{k})]\bar{A}^+(\mathbf{k})], \tag{138}$$

$$A(\mathbf{x})=\frac{1}{(2\pi)^{3/2}}\int\frac{d\mathbf{k}}{\sqrt{2\omega}}[e^{i\mathbf{kx}}\vartheta[f(\mathbf{k})]A^-(\mathbf{k})+e^{-i\mathbf{kx}}\vartheta[f(\mathbf{k})]A^+(\mathbf{k})]. \tag{139}$$

Here $f(\mathbf{k})$ is an arbitrary odd function: $f(-\mathbf{k})=-f(\mathbf{k})$; the operator $A^+(\bar{A}^+)$ is conjugate to the operator $A^-(\bar{A}^-)$. All operators $A^+,\bar{A}^+,A^-,\bar{A}^-$ introduced are defined in the upper "half-plane" $f(\mathbf{k})>0$ and satisfy in this half-plane the following commutation relations:

$$[A^-(\mathbf{k}),\bar{A}^+(\mathbf{k}')]=[\bar{A}^-(\mathbf{k}),A^+(\mathbf{k}')]=\delta(\mathbf{k}-\mathbf{k}'), \tag{140}$$

other commutators being equal to zero.

If $f(\mathbf{k})=\mathbf{k}\cdot\mathbf{m}$, then we obtain the representation employed in Ref. 31 for canonical quantization of Yang–Mills fields in the light-cone gauge and in Refs. 41 and 42 for all other cases.

Using the representation (138) it is easy to rewrite the operator Q in the form

$$Q=\int d\mathbf{k}\vartheta[f(\mathbf{k})][c^+(\mathbf{k})\bar{A}^-(\mathbf{k})+c^-(\mathbf{k})\bar{A}^+(\mathbf{k})], \tag{141}$$

where the operators c^+ and c^- are introduced analogously to the operators $\bar{A}^+$ and $\bar{A}^-$. Thus we have written the operator Q in the form (81), and therefore the norm of the physical subspace will be positive and finite.

We now calculate the propagator of Yang–Mills fields in linear gauges $n_\mu A_\mu=0$. For this we must calculate the generating functionals $Z_0(J)$ of free Yang–Mills fields. This functional can be formally represented as follows:

$$Z_0(J)=\int dA_0 dA_i dE_i\delta(n_\mu A_\mu)\times\exp\left\{i\int dx(\mathscr{L}(x)-A_\mu(x)J_\mu(x))\right\}, \tag{142}$$

where $\mathscr{L}(x)$ is defined by Eq. (123).

In order to determine $Z_0(J)$ correctly it is necessary to use the holomorphic representation introduced in Sec. 3.4 and the definitions (121) and (122). This will be done later. Let us redefine the field A_0 as

$$A_0-n_iA_i\to A_0 \tag{143}$$

and integrate over the new field A_0. Then we obtain the following expression for $Z_0(J)$ (with no loss of generality we can set $n_0=1$):

$$Z_0(J)=\int dA_i dE_i\exp\left\{i\int dx(\mathscr{L}'(x)-n_iA_iJ_0+A_iJ_i)\right\}, \tag{144}$$

where the Lagrangian $\mathscr{L}'(x)$ is

$$\mathscr{L}'(x)=E_i\partial_0A_i-\tfrac{1}{2}E_i^2-\tfrac{1}{4}(\partial_iA_j-\partial_jA_i)^2+n_iA_i\partial_jE_j. \tag{145}$$

Redefining the current J_i as

$$J_i-n_iJ_0\to J_i, \tag{146}$$

we obtain finally

$$Z_0(J)=\int dA_i dE_i\exp\left\{i\int dx(\mathscr{L}'(x)+A_iJ_i)\right\}. \tag{147}$$

Thus in order to calculate the propagator it is sufficient to calculate the integral (147) and then make the substitution inverse to (146):

$$J_i\to J_i-n_iJ_0. \tag{148}$$

As mentioned above, the holomorphic representation described in Sec. (3.3) and Eq. (119) for the evolution operator must be employed in order to calculate the path integral (147). Since, however, for nonzero n the Lagrangian does not split into a sum of Lagrangians for the longitudinal and transverse components, the contribution of the transverse components must be taken into account. For this, some representation must be prescribed for the transverse components and a path integral must be constructed for the evolution operator. We choose the standard holomorphic representation for the transverse components (see, for example, Ref. 33):

$$A_i^T(\mathbf{x}) = \frac{1}{(2\pi)^{3/2}} \int \frac{d\mathbf{k}}{\sqrt{2\omega}} [e^{i\mathbf{kx}} a_r^-(\mathbf{k}) u_i^r(\mathbf{k}) + e^{-i\mathbf{kx}} a_r^+(\mathbf{k}) u_i^r(\mathbf{k})],$$

$$E_i^T(\mathbf{x}) = \frac{i}{(2\pi)^{3/2}} \int \sqrt{\omega/2}\, d\mathbf{k} [-e^{i\mathbf{kx}} a_r^-(\mathbf{k}) u_i^r(\mathbf{k}) + e^{-i\mathbf{kx}} a_r^+(\mathbf{k}) u_i^r(\mathbf{k})]. \qquad (149)$$

Here $a_r^-(\mathbf{k})$ and $a_r^+(\mathbf{k})$ are the usual creation and annihilation operators with the commutation relations $[a_r^-(\mathbf{k}), a_s^+(\mathbf{k}')] = \delta_{rs}\delta(\mathbf{k}-\mathbf{k}')$ and $u_i^r(\mathbf{k})$, $i=1$ and 2, are two orthonormal polarization vectors which are orthogonal to the vector $\mathbf{k}$.

Using Eqs. (138), (139), and (149) it is not difficult to rewrite the Hamiltonian H of the system (145) in the form

$$\begin{aligned} H = \int & d\mathbf{k}\, \vartheta[f(\mathbf{k})][\omega a_r^+(\mathbf{k}) a_r^+(-\mathbf{k}) + \omega a_r^+ \\ & \times(-\mathbf{k}) a_r^-(\mathbf{k}) + i a_r^-(-\mathbf{k}) \\ & \times n_r(-\mathbf{k}) \bar{A}^-(\mathbf{k}) - i a_r^+ \\ & \times(-\mathbf{k}) n_r(-\mathbf{k}) \bar{A}^+(\mathbf{k}) + i a_r^+(\mathbf{k}) n_r(\mathbf{k}) \bar{A}^-(\mathbf{k}) \\ & - i a_r^-(\mathbf{k}) n_r(\mathbf{k}) \bar{A}^+(\mathbf{k}) + \bar{A}^+(\mathbf{k}) \bar{A}^-(\mathbf{k}) \\ & - \mathbf{kn} \bar{A}^+(\mathbf{k}) A^-(\mathbf{k}) - \mathbf{kn} A^+(\mathbf{k}) \bar{A}^-(\mathbf{k}) \\ & - A^-(\mathbf{k}) J^+(\mathbf{k}) - A^+(\mathbf{k}) J^-(\mathbf{k}) \\ & - a_r^-(\mathbf{k}) \eta_r^*(\mathbf{k}) \\ & - a_r^+(\mathbf{k}) \eta_r(\mathbf{k}) - a_r^-(-\mathbf{k}) \eta_r^*(-\mathbf{k}) \\ & - a_r^+(-\mathbf{k}) \eta_r(-\mathbf{k})]. \end{aligned} \qquad (150)$$

Here the currents $J^+(\mathbf{k})$ and $\eta_r^*(\mathbf{k})$ are the complex conjugates of the currents $J^-(\mathbf{k})$ and $\eta_r(\mathbf{k})$ and are defined as

$$J^+(\mathbf{k}) = \frac{\vartheta[f(\mathbf{k})]}{(2\pi)^{3/2}} \int \frac{d\mathbf{x}}{\sqrt{2\omega}} e^{i\mathbf{kx}} \partial_i J_i(\mathbf{x}),$$

$$\eta_r^*(\mathbf{k}) = \frac{u_i^r(\mathbf{k})}{(2\pi)^{3/2}} \int \frac{d\mathbf{x}}{\sqrt{2\omega}} e^{i\mathbf{kx}} J_i(\mathbf{x}), \qquad (151)$$

and $n_r(\mathbf{k}) = n_i u_i^r(\mathbf{k})$.

The path integral for the kernel of the evolution operator, taking into account the transverse components, is

$$\begin{aligned} U(a_r^+, a_r^-, A^+, \bar{A}^+, A^-, \bar{A}^-; t'' - t') = \int & d^2 a_r d^2\alpha d^2\bar{\alpha} \exp\Big| a_r^+(t'') \alpha_r^-(t'') \\ & + \alpha^+(t'') \bar{\alpha}^-(t'') + \bar{\alpha}^+(t'') \alpha^-(t'') \\ & + i \int_{t'}^{t''} dt (i\alpha_r^+ \dot{a}_r^- + i\alpha^+ \dot{\bar{\alpha}}^- + i\bar{\alpha}^+ \dot{\alpha}^- \\ & - H(a_r^+, a_r^-, \alpha^+, \bar{\alpha}^+, \alpha^-, \bar{\alpha}^-))\Big|, \end{aligned} \qquad (152)$$

where we have used the boundary conditions (120) for the fields α and $\bar{\alpha}$ together with the standard Feynman boundary conditions for the longitudinal components. The propagator can be calculated with the help of the equations obtained above. The intermediate equations are very complicated. We therefore present only the final answer for the propagator of Yang–Mills fields in the gauge $n_\mu A_\mu = 0$:

$$D_{\mu\nu}(k) = -\frac{1}{k^2 + i\varepsilon} \left(g_{\mu\nu} - \frac{k_\mu n_\nu + k_\nu n_\mu}{kn - i\delta\varepsilon[f(\mathbf{k})]} + \frac{n^2 k_\mu k_\nu}{(kn - i\delta\varepsilon[f(\mathbf{k})])^2} \right). \qquad (153)$$

Here $\varepsilon[f(\mathbf{k})]$ is the sign function.

If $f(\mathbf{k}) = -\mathbf{k}\cdot\mathbf{n}$ ($\mathbf{n} \neq 0$), then we obtain the Mandelshtam–Leibbrandt prescription for any linear gauge[35] (in the case of an axial gauge there is the requirement $n_i \neq 0$). For a purely Hamiltonian gauge $A_0 = 0$ we can choose $f(\mathbf{k}) = -k_1$, which gives the prescription proposed in Refs. 36, 43, and 44. Quantization in the purely temporal gauge was considered in Refs. 44 and 45 within the framework of canonical quantization (and in Refs. 41 and 42 it was extended to all other gauges). In these works the physical space was separated with the help of the weak Dirac condition $\langle \Psi | \varphi | \Psi \rangle = 0$. The prescription obtained in Ref. 44 is identical to the prescription (143) for the particular case $\mathbf{n} = 0$ and $f(\mathbf{k}) = -k_1$. The regularized Hamiltonian gauge was employed in Ref. 45, so that the propagator obtained in this work is somewhat different from the propagator (153).

CONCLUSIONS

We have described different methods for quantizing Yang–Mills fields in linear gauges. These methods make it possible to determine correctly the propagator of a vector field. It was shown that the incomplete fixing, inherent in these models, of the gauge arbitrariness leads to arbitrariness in the choice of propagator, and an admissible class of propagators was described. The proposed methods solve the problem of quantization in linear gauges, but the problem of renormalization has not been investigated. This problem is far from trivial. For example, investigation of renormalization in the light-cone gauge shows that in this case nonlocal counterterms must be introduced.[46] The question of renormalization in the Hamiltonian gauge $A_0 = 0$ remains absolutely unclear. In particular, the question of Wilson's loop in the gauge $A_0 = 0$ with different choices of the propagator requires additional investigation.

It is also of great interest to investigate gauge fields in linear gauges outside the framework of perturbation theory. These gauges enable circumventing the problem of Gribov nonuniqueness, arising in the case of quantization in other gauges. In this case the main problem is to find nonperturbative solutions of the constraint equations.

[1] Y. L. Dokshitzer, D. I. Dyakonov, and S. I. Troyan, Phys. Rep. **58**, 271 (1980).
[2] I. Brink, O. Lindgren, and B. F. Nilson, Nucl. Phys. B **212**, 401 (1983).
[3] S. Mandelstam, Nucl. Phys. B. **213**, 149 (1983).

[4] J. H. Schwarz, Phys. Rep. **89**, 223 (1982).
[5] A. A. Slavnov, Teor. Matem. Fiz. **10**, 99 (1972).
[6] J. C. Taylor, Nucl. Phys. B **33**, 436 (1971).
[7] V. N. Gribov, Nucl. Phys. B **139**, 1 (1978).
[8] P. A. M. Dirac, Can. J. Math. **2**, 129 (1950); Proc. Roy. Soc. London A **246**, 326 (1958).
[9] P. A. M. Dirac, *Lectures on Quantum Mechanics* [Russian translation], Mir, Moscow, 1968.
[10] L. D. Faddeev, Teor. Matem. Fiz. **1**, 3 (1969).
[11] L. D. Faddeev and V. N. Popov, Phys. Lett. B **25**, 30 (1967).
[12] B. de Witt, Phys. Rev. 160, **119**, 1195 (1967).
[13] A. A. Slavnov and S. A. Frolov, Teor. Matem. Fiz. **75**, 201 (1988).
[14] G. Curci and R. Ferrary, Nuovo Cimento A **35**, 273 (1976).
[15] T. Kugo and I. Ojima, Suppl. Prog. Theor. Phys. **66**, 1 (1979).
[16] I. A. Batalin and G. A. Vilkovisky, Phys. Rev. D **28**, 2567 (1983).
[17] S. A. Frolov and A. A. Slavnov, Nucl. Phys. B **347**, 333 (1990).
[18] E. S. Fradkin and G. A. Vilkovisky, Phys. Lett. B **55**, 224 (1975); I. A. Batalin and G. A. Vilkovsky, Phys. Lett. B **69**, 309 (1977).
[19] I. A. Batalin and E. S. Fradkin, Phys. Lett. B **122**, 157 (1983).
[20] M. Henneaux, Phys. Rep. **129**, 1 (1985).
[21] I. A. Batalin and E. S. Fradkin, Nuovo Cimento **9**, 1 (1986).
[22] G. Leibbrandt, Rev. Mod. Phys. **59**, 1067 (1987).
[23] W. Kummer, Acta Phys. Austr. **41**, 315 (1975).
[24] W. Konetshny and W. Kummer, Nucl. Phys. B **100**, 106 (1975).
[25] J. F. Willemsen, Phys. Rev. **17**, 574 (1978).
[26] S. Caracciolo, G. Curci, and P. Menotti, Phys. Lett. B **113**, 311 (1982).
[27] A. A. Slavnov and S. A. Frolov, Teor. Matem. Fiz. **68**, 360 (1986).
[28] G. Leibbrandt, Phys. Rev. D **29**, 1699 (1984).
[29] J. B. Kogut and D. E. Soper, Phys. Rev. D **1**, 2901 (1970).
[30] J. H. Ten Eyck and F. Rohrlich, Phys. Rev. D **9**, 2237 (1974).
[31] A. Bassetto, M. Dalbosco, I. Lazzizzera, and R. Soldati, Phys. Rev. D **31**, 2012 (1985).
[32] N. N. Bogolyubov and D. V. Shirkov, *Introduction to Quantum Field Theory* [in Russian], Nauka, Moscow, 1984.
[33] A. A. Slavnov and L. D. Faddeev, *Introduction to the Quantum Theory of Gauge Fields* [in Russian], Nauka, Moscow, 1988.
[34] A. A. Slavnov and S. A. Frolov, Teor. Matem. Fiz. **73**, 199 (1987).
[35] G. Leibbrandt, Nucl. Phys. B **310**, 405 (1988).
[36] H. Huffel, P. V. Landshoff, and J. C. Taylor, Phys. Lett. B **217**, 147 (1989).
[37] S. A. Frolov, Teor. Matem. Fiz. **87**, 188 (1991).
[38] F. M. Saradzhev and V. Ya. Fainberg, Trudy FIAN **165**, 4 (1989).
[39] S. A. Frolov and A. A. Slavnov, Phys. Lett. B **218**, 461 (1989).
[40] S. Hwang and R. Marnelius, Preprint CERN-TH.5235/88, 1988.
[41] I. Lazzizzera, Nuovo Cimento A **102**, 1385 (1989).
[42] A. Burnel, Phys. Rev. D **40**, 1221 (1989).
[43] P. Gaigg and M. Kreuzer, Phys. Lett. B **205**, 530 (1988).
[44] I. Lazzizzera, Phys. Lett. B **210**, 188 (1988).
[45] P. V. Landshoff, Phys. Lett. B **227**, 431 (1989).
[46] A. Bassetto, M. Dalbosco, and R. Soldati, Phys. Rev. D **36**, 3138 (1987); H. Skarke and P. Gaigg, Phys. Rev. D **38**, 3205 (1988).

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