Theorem on the statistics of identical particles

A. B. Govorkov
Joint Institute for Nuclear Research, Dubna

Fiz. Elem. Chastits At. Yadra **24**, 1341–1413 (September–October 1993)

Possible quantum statistics of identical particles are investigated in a general form and the constraints imposed on the corresponding quantization schemes of (free) fields are determined on the basis of the general requirements of the theory. It is shown that a small violation of ordinary statistics is impossible, but ordinary statistics can be extended to para-Fermi and para-Bose statistics of finite orders. It is also shown that in the limiting case of infinite statistics the very existence of antiparticles forbids a small violation of ordinary statistics. It is established that the only possible generalizations are: 1) Green's well-known paraquantization scheme and 2) a new charge-asymmetric scheme. For the latter scheme the limit of infinite-order parastatistics yields a nonlocal theory, describing the classical Maxwell–Boltzmann statistics.

1. INTRODUCTION

Our experience teaches us that all particles which at one or another stage of our knowledge we consider to be elementary obey either Bose or Fermi statistics. We assume, often without any special grounds for doing so, that this *theorem on Bose and Fermi statistics* is applicable not only to particles which are directly observable in our laboratories (electrons, photons, nucleons, mesons, etc.) but also to imagined objects, such as quarks and gluons, which, apparently, in principle cannot be observed in a free state outside hadrons.

We expect that the reason for this general law of nature is hidden in deep properties of matter itself—its identity and repeatability. But there immediately arise two questions: First, how accurately can we formulate a physically acceptable concept of particle identity? Second, even if we are able to do so, will we be able to derive the above-indicated *Bose-Fermi theorem* from this definition and other general tenets of quantum theory? Which generalizations of ordinary statistics will then be admissible and what meaning can be attached to these generalizations? For example, are small violations of ordinary statistics, say, outside the limits set by modern experimental checks (see Ref. 1 for a discussion of a check of Pauli's principle for electrons and protons) which are consistent with the more general assumptions that form the foundation of modern quantum field theory? This review attempts to answer these questions.

The first part of the review, Secs. 2–11, is devoted to defining particle identity and its consequences within the framework of nonrelativistic theory. The analysis is based not on the definition of some symmetry properties of the wave functions of identical particles but rather on the natural *symmetry of the density matrix* for the particles.

While studying the general properties of the density matrix for Fermi and Bose statistics N. N. Bogolyubov noted[2] this general property for these statistics and termed it *classical symmetry*, underscoring thereby its correspondence to the symmetry of the classical distribution function for *identical* but distinguishable classical particles. In contrast to this, he proposed the term *quantum symmetries* for the symmetry properties of the wave functions of fermions and bosons.

It seems to me to be more consistent to adopt the invariance of the density matrix with respect to permutations of all variables of identical particles as the starting definition of indistinguishability of the particles and to study all consequences following from this definition. It is this definition of indistinguishability of identical particles that was adopted in Ref. 3 (see also Ref. 4), where generalizations likewise admitted by this definition for the statistics of identical particles were investigated. The sense of such a definition will be explained in greater detail in Sec. 2.

It is well known that systems of identical particles are best described by the method of second quantization, in which all information about the permutation properties of the wave functions of the particles is contained in the commutation properties of the particle creation and annihilation operators. Since, however, in general the wave functions are not known in advance, constraints are imposed on the commutation properties of the operators only on the basis of the indicated general symmetry property of the density matrix and for this reason they are of a most general character. For this reason the density matrix itself must be second-quantized. This method was developed in Refs. 3 and 5, and it will be briefly repeated in Sec. 2. We note that N. N. Bogolyubov extensively employed the method of direct quantization of the density matrix in application to Bose and Fermi statistics.[2]

Such a general formulation of identity, however, still does not give more concrete relations for particle creation and annihilation operators that would express the special properties of *quantum* statistics. In order to obtain such relations additional constraints based on the concept of the *elementarity* of identical particles must be imposed. This

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concept is formulated and discussed in Sec. 3, where the question of the statistics of composite particles is also briefly discussed.

As a result of combining the conclusions which we draw from our definitions of particle identity and elementarity we still arrive at quite general *trilinear commutation relations* for particle creation and annihilation operators. Next we must elucidate the symmetry properties which such relations give to the wave functions of identical particles.

In Sec. 5 we construct the Fock representation of the obtained relations and in Sec. 6 we formulate the principal result in the form of two theorems: The *positive-definiteness* of the norm of the particle state vectors in Fock space imposes very stringent constraints on possible generalizations of the statistics. It is proved that only generalizations in the form of so-called parastatistics of finite orders and parastatistics of infinite order are consistent with this condition. Finite-order *para-Bose* and *para-Fermi* statistics permit placing a definite number of particles in an *antisymmetric* state and a *symmetric* state, respectively, not exceeding a fixed number called the *order* of the parastatistics. Evidently, ordinary Bose and Fermi statistics fit within this definition, if they are taken to be of unit order (i.e., they do *not* admit two and more particles in an antisymmetric state for bosons and in a symmetric state for fermions). *Infinite statistics* does not impose any constraints on the number of particles in either symmetric or antisymmetric states. It is found, however, that finite-order parastatistics corresponds to not one but two second-quantization schemes based on *different* trilinear relations. Green's well-known scheme,[6] which has been investigated in detail in the literature,[7–9] is one such scheme. Another scheme was discovered very recently.[10,11] In Secs. 7 and 8 the characteristic features of these schemes will be investigated briefly. The impossibility of formulating intermediate schemes, which fall between the two schemes indicated above, leads to the impossibility of formulating within the framework of finite-order parastatistics a *small* violation, proposed by Ignat'ev and Kuz'min[12] and Greenberg and Mohapatra,[13] of the ordinary statistics. We note that the previously published proof[14,15] of the impossibility of such violation was based on the simplest particular case of second-order para-Fermi statistics. The first general proof for both para-Fermi and para-Bose statistics of arbitrary finite order is given in the present review. We also note that the proof of this theorem does not require the existence of antiparticles.

In Sec. 9 infinite statistics are studied as parastatistics in the limit of infinite order. Since, however, the previously proved theorems no longer hold in this limit, infinite statistics must be regarded as limiting statistics for the initial trilinear relations. In the limit of infinite order the initial trilinear relations are found to be equivalent to the so-called q-deformed bilinear relations, which in the cases $q=\pm 1$ become the ordinary commutation or anticommutation relations. There arises a continuum of possible infinite statistics corresponding to continuous variation of the parameter q within the admissible limits $-1\leqslant q\leqslant 1$, and all statistics are consistent with the positive-definiteness of the norm of vectors in the corresponding Fock space. On the basis of this fact Greenberg[16] proposed discussing once again ordinary statistics but now within the framework of infinite statistics.

In order to understand the physical significance of parastatistics of both finite and infinite orders, in Sec. 10 an interpretation of parastatistics as ordinary statistics with degeneracy with respect to some additional internal degree of freedom, the number of states of which is equal to the order of the parastatistics, is examined. It is found that *unitary* symmetry is associated with this internal degree of freedom. The values of the Casimir operators of this symmetry are observables representing some *generalized nonabelian charges* and completely describe paraparticle states. On the basis of this interpretation it is possible to formulate, even within the nonrelativistic theory, the concept of conjugate generalized charges and the corresponding *antiparticles*; Sec. 11 is devoted to a discussion of this concept. It turns out, however, that in the case of infinite statistics it is impossible to introduce the conjugate sector and the corresponding antiparticle. This completes the discussion of the nonrelativistic formulation of quantization schemes for identical particles. In Part 2 of the review, consisting of Secs. 12–15, we examine a generalized quantization of fields that corresponds to the previously discussed second quantization of parastatistics. In so doing, we make the assumption that the particles and previously defined antiparticles must appear in the *same* free field, which depends locally on the coordinates and time. In Sec. 12 general trilinear commutation relations which are consistent with the previously obtained relations for particles are postulated for such a field. We show that the field theory based on such general commutation relations will be local (observables separated by space-like intervals commute) and *canonically self-consistent* (the Hamiltonian corresponding to the equation of motion of this field is also the generator of time translations in the Heisenberg equation). With the exception of Green's paraquantization, however, the theory is in general noninvariant under charge conjugation (C conjugation). At the same time the theory remains CPT invariant.

In Sec. 13 commutation relations for particles and antiparticles are obtained and their combined Fock representation is constructed in Sec. 14. Our theorems on the limited choice of schemes of generalized quantization of fields remain valid for the case of finite parastatistics. Moreover, it can be shown that particles and antiparticles described by the same field have the same parastatistics and it is possible to prove on this basis a generalized Pauli theorem on the connection of spin to *parastatistics*.

Finally, in Sec. 15 we return to the question of the existence of infinite statistics, but now within the framework of field theory. Once again we consider such statistics as a limiting case of parastatistics as its order approaches infinity and we find that the trilinear relations are replaced by q-deformed bilinear relations. Now, however, it turns out that the *existence of antiparticles* itself imposes strict constraints on the possible values of the deformation parameter q, limiting it to the same three values: ± 1 and 0.

Thus even in the case of infinite statistics this restriction forbids a small violation of ordinary statistics, but now this requires the existence of antiparticles in the same field on an equal footing with particles. We note that the bilinear relations obtained in the limit of infinite-order parastatistics become nonlocal. We also note that in Refs. 17 and 18 the same result was obtained on the basis of the starting assumption that the field satisfies at the outset q-deformed *bilinear* relations. In principle, as Greenberg suggested,[16] a single field incorporating particles and antiparticles may not satisfy any commutation relations, and in this case a small violation of ordinary statistics becomes possible (*CPT* invariance is still preserved). Such unification of particles and antiparticles into a single field seems, however, very artificial.

The admissible values $q=\pm 1$ evidently correspond to the ordinary method of quantization describing fermions and bosons. The case $q=0$ is of greatest interest. It corresponds to the relations directly proposed by Greenberg[19] for describing infinite statistics. As we can see, when antiparticles are included in the analysis, this case is the only suitable one for such a description. How do antiparticles behave in this case? It turns out that they can exist only in a *pair* with particles, and such pairs themselves *commute* with particle operators and behave like independent bosons. The theory is thus nonlocal. In this case infinite statistics is itself equivalent to classical Maxwell–Boltzmann statistics. Physically this equivalence points to an interpretation of parastatistics as ordinary statistics with degeneracy with respect to a unitary internal degree of freedom with this degeneracy approaching infinity.[10,19]

The review is concluded with a section where all assertions which we have proved are summarized in a condensed form.

We now briefly enumerate the works on this subject and compare the results obtained there to our results.

It is well known that the general spin–statistics theorem in the case of a choice between Fermi and Bose statistics was established by Pauli.[10] A modern proof of this theorem, on the basis of Wightman axiomatics, based on the general tenets of quantum field theory, has been given by Burgoyne[21] and Luders and Zumino[22] (see also Ref. 23). All these proofs, however, are of a negative character and assert that integer-spin fields cannot be quantized with the help of anticommutators and half-integer-spin fields cannot be quantized with the help of commutators. Other possibilities were not proposed.

An example of generalized quantization of fields by a method different from the ordinary method but nonetheless satisfying all reguirements of local quantum field theory was first proposed by Green[6] and by Volkov (for the particular case of second-order para-Fermi statistics).[24] The theory of such quantization was then studied by many authors (see Refs. 8 and 9).

On the other hand, the founders of modern quantum theory—Pauli,[25] Dirac,[26] and others—have always underscored the possibility of describing the statistics of identical particles not only by means of symmetric and antisymmetric representations but also by multidimensional representations of the group of permutations of the coordinates (and spin variables) of particles.

The first attempt to construct a statistics intermediate between Fermi and Bose statistics was apparently made by Gentile.[27] Later, several authors studied the properties of a paragas.[28] Okayama attempted to establish commutation relations for operators corresponding to multidimensional representations of permutation groups.[29] But, as proved in Ref. 30, this attempt was unsuccessful.

Hartle, Stolt, and Taylor[31] proved, on the basis of the cluster properties of systems of identical particles, a theorem about the fact that the statistics of identical particles can be para-Fermi or para-Bose statistics of finite order or infinite statistics.

The most systematic investigation of the statistics of identical particles was made by Doplicher, Haag, and Roberts[32] on the basis of the axiomatic approach based on the assumptions of a *local* algebra of observables (Haag has given an excellent review of these works[33]). In an unrestricted exposition the concept of particles in such an approach is replaced by a system that is localized in a finite region of space and time and characterized by its own algebra of observables (as proved in Ref. 34, such space-time regions can also be open, but they are space-like cones). The principle of localization consists of the fact that the existence of such a system is not correlated with the rest of the universe in the sense that its removal or addition to the universe does not influence measurements performed in the space-like complement to the region of localization of the system. Such particle-like objects can be characterized by *generalized charge (not necessarily abelian) quantum numbers*, separating all of their states into superselection sectors. Further, the principle of localization makes it possible to introduce composition of states. Then the unitary representation of the group S_n of permutations of n elements, which act just as permutations of an n-particle wave function in quantum mechanics, can be associated to an n-fold product of states of this sector. It is thus not surprising that the classification established in such an approach[1)] for admissible particle statistics is identical to the classification given in Ref. 31 on the basis of nonrelativistic quantum mechanics.

Thus far, however, a complete correspondence between such parastatistics and generalized schemes for quantizing fields has not been established. Moreover, doubts (which, as we shall see below are well-founded) have been raised about the correspondence of Green's paraquantization to parastatistics defined strictly as the statistics of identical particles.[36,37] Chernikov[38] first proved that Green–Volkov second-order para-Fermi quantization[6,24] does indeed describe two types of particles which can be distinguished with the help of an interaction that is formulated within the same parafield theory (see also Ref. 39). An extension of this result to arbitrary order was proposed in Ref. 3 on the basis of so-called unitary quantization. Admittedly, we can always talk about two or several types of different particles as identical particles occupying different internal (isospin, unitary, etc.) states.

The main goal of the approach expounded in the present review is to establish the most general schemes of second quantization and generalized quantization of fields that correspond to possible statistics of identical particles. We shall see that besides Green's quantization there exists a new paraquantization based on different trilinear relations. Such quantization no longer distinguishes the internal states of particles and for this reason is more suitable for describing parastatistics with no additional constraints. We shall also see that such a quantization scheme can be regarded as an implementation of the abstract scheme of Haag *et al.*[32,33] Casimir's operators of unitary symmetry, which arises in our approach, correspond to generalized nonabelian charge quantum numbers, and our definition of antiparticles as particles transforming according to additional representations of this group corresponds to antiparticles associated with the conjugate sector in the algebra of observables. Finally, the absence of such representations and, correspondingly, antiparticles for infinite statistics is in complete agreement with Fredenhagen's theorem[40] on the absence of a conjugate sector in the local algebra of observables. Moreover, by regarding infinite statistics as the infinite-order limit of parastatistics we shall be able to trace how a local theory becomes nonlocal.

Our working scheme, initially formulated within a nonrelativistic approach (though later extended to relativistic field theory), naturally, is not as general as an axiomatic approach. But this could be an advantage. In particular, the problems encountered in the axiomatic approach due to its generality do not arise here. These problems are, in a certain sense, technical but they significantly impede application to theories which now form the foundation of our knowledge about elementary particles—quantum electrodynamics and chromodynamics. The difficulty arises because the axiomatic approach is still limited to only massive objects, and adding massless photons presents a problem (see Refs. 34 and 41 for a discussion of infraparticles). Another limitation is that this approach considers only particle-like objects which appear in an asymptotically free state after a collision. This restriction makes the theory inapplicable to quantum chromodynamics, whose objects are quarks and gluons confined inside hadrons (the problem of introducing local gauge groups into the axiomatic scheme is discussed in Refs. 33 and 42).

In our approach the paracommutation relations can be formulated for any, including massless, fields. When interactions are included these fields must be regarded as Heisenberg fields and the commutation relations must be regarded as single-time relations. The next step in the investigation of such schemes is to associate them to physical symmetries. The formulating gauge symmetries, such as color symmetry, within the parafield approach is, however, only just beginning,[11,43] and is not discussed in the present review.

We note, finally, that objects (so-called anyons) which can exhibit arbitrary (neither integer nor half-integer) spin and satisfy fractional statistics (see, for example, the review Ref. 44) were discovered comparatively recently in two-dimensional models of the fractional Hall effect. The spin arbitrariness is connected to the fact that the two-dimensional rotation group $O(2)$ [isomorphic to $U(1)$] is abelian. Fractional statistics, however, are due to the nontrivial nature of the topology of the two-dimensional space itself, while the topology of the three-dimensional space is assumed to be trivial. Nonetheless the possibility of the existence of string models with nontrivial topology, within which unusual statistics resembling fractional statistics could arise in three-dimensional space, is not precluded. However, we shall not study objects of this type, since they do not satisfy the principle of elementarity, which we propose below and they are not described by a *local* field theory.[44]

2. INDISTINGUISHABLITY OF IDENTICAL PARTICLES AND THE DENSITY MATRIX IN THE SPACE OF DOUBLE OCCUPATION NUMBERS

It is well known (see, for example, Ref. 45) that any quantum-mechanical system can be described with the help of a density matrix $\rho(x;x';t)$ which satisfies a bilateral Schroedinger equation—the quantum analog of Liouville's equation. A system of N nonrelativistic particles is described by the density matrix $\rho(x_1,x_2,\dots,x_N;x_1',x_2',\dots,x_N';t)$—a complex function of time t and two sets of single-particle coordinates: $x_1,x_2,\dots,x_N$ and $x_1',x_2',\dots,x_N'$, which we term primary and secondary, respectively. Thus this function is defined in a $3N$-dimensional configuration space [if spin is present, the spin variables can be included in the arguments x_i, and then the dimension of the space is $(2S+1)^N$ times larger]. The average value of any operator $\hat{Q}$ is defined as

$$\langle Q\rangle=\int dx_1\dots\int dx_N\int dx_1'\dots\int dx_N'\hat{Q}(x_1,\dots,x_N)\times\rho(x_1,\dots,x_N;\ x_1',\dots,x_N';\ t)\times\delta(x_1-x_1')\dots\delta(x_N-x_N'),\tag{1}$$

the integration of the δ functions being performed after the action of the operator on the density matrix is calculated. In order that the average values of the hermitian[2)] operators representing observables be real, the density matrix must be hermitian (* denotes complex conjugation):

$$\rho^*(x_1,\dots,x_N;\ x_1',\dots,x_N';\ t)=\rho(x_1',\dots,x_N';\ x_1,\dots,x_N;\ t).\tag{2}$$

Definition. *Particles for which all hermitian operators of observables are symmetric are said to be identical*:

$$\hat{Q}(x_{\mathscr{P}1},\dots,x_{\mathscr{P}N})=\hat{Q}(x_1,\dots,x_N),\tag{3}$$

where $\mathscr{P}$ is any transposition of the indices $1,2,\dots,N$ to some indices $\mathscr{P}1=i_1$, $\mathscr{P}2=i_2,\dots$, $\mathscr{P}N=i_N$ from the same set. This definition implies that the density matrix itself must be symmetric under any such simultaneous transposition of primary and secondary indices:[3)]

$$\rho(x_1,\dots,x_N;\ x_1',\dots,x_N';\ t)=\rho(x_{\mathscr{P}1},\dots,x_{\mathscr{P}N};\ x_{\mathscr{P}1}',\dots,x_{\mathscr{P}N}';\ t).\tag{4}$$

In order to prove this it is sufficient to make the substitution of variables $x_1 \to x_{\mathcal{P}1}$, $x_2 \to x_{\mathcal{P}2},...,$ $x_N \to X_{\mathcal{P}N}$, and apply the invariance (3) and the condition that the average value of each hermitian operator of an observable is unique. The requirement that the primary and secondary indices must be transposed simultaneously arises here due to the invariance of the δ functions under such transpositions.

The diagonal elements of such a symmetric density matrix

$$\rho(x_1,...,x_N;\ x_1,...,x_N;\ t)$$

determine the probability density of finding *identical* particles at the points $x_1,x_2,...,x_N$ at time t. Obviously, they must be positive.

Our definition incorporates the condition that we know *all* hermitian operators of observables and they are all symmetric. It can happen, however, that only some characteristics of the state of the system are observable while averaging is performed over the other characteristics. This can happen, for example, when averaging over spin variables, and the density matrix, which depends only on the coordinates of (nonrelativistic) particles with nonzero spin, is also symmetric. In such a situation one sometimes talks about particles that are the *same* with respect to external manifestations but are not identical. However, we can never be sure that we know the *complete* set of observables and that the symmetry of the density matrix is not a result of averaging over degenerate internal variables. (Such a situation can be imagined in the form of a system of neutrons and protons in nuclei, regarding the electromagnetic and weak interaction as switched off, i.e., unimportant, and neglecting the mass difference. A better example are quarks and gluons, which are degenerate with respect to the color degree of freedom, which, apparently, is fundamentally unobservable.) Our main result will consist of a proof of the fact that *no other situation is possible and that any generalized statistics of identical particles can always be interpreted as ordinary Fermi and Bose statistics in the presence of degeneracy with respect to some internal coordinate.*[4)] Before reaching this conclusion, however, we must develop a theory of second quantization for theories in which this general symmetry property of the density matrix (4) holds.

To this end, as usual, we switch to a discrete basis and expand the density matrix in terms of a complete set of discrete single-particle states r_i ($i=1,2,...,\infty$) for both primary and secondary arguments. The expansion coefficients will determine the density matrix in a new representation, and by virtue of Eqs. (2) and (4) they will be hermitian and symmetric

$$c^*(r'_1,...,r'_N;\ r_1,...,r_N;\ t)=c(r_1,...,r_N;\ r'_1,...,r'_N;\ t), \tag{5}$$

$$c(r_{\mathcal{P}1},...,r_{\mathcal{P}N};\ r'_{\mathcal{P}1'},...,r'_{\mathcal{P}N'};\ t) =c(r_1,...,r_N;\ r'_1,...,r'_N;\ t), \tag{6}$$

and their diagonal elements will be positive:

$$c(r_1,...,r_N;\ r_1,...,r_N;\ t)\geqslant 0. \tag{7}$$

In general, however, in our scheme the symmetry properties of the density matrix and its elements under permutation of the primary arguments only or the secondary arguments only are not known in advance, and we cannot talk about the occupation numbers of only primary or only secondary states occupied by identical particles. But since the density matrix is symmetric with respect to both sets of states, we can introduce the concept of combined or *double occupation numbers.*[3] We define a double occupation number n_{ij} as the number of identical particles occupying a definite state $r^{(i)}$ among all primary states $r_1,...,r_N$ and in a definite state $r^{(j)}$ among all secondary states $r'_1,...,r'_N$. Evidently,

$$\sum_{i,j=1}^{\infty} n_{ij}=N. \tag{8}$$

The total numbers of particles in the primary state i or the secondary state j are

$$N_i=\sum_{j=1}^{\infty} n_{ij},\quad N'_j=\sum_{i=1}^{\infty} n_{ij}\,. \tag{9}$$

The state of a system of identical particles is determined completely by prescribing the values of all double occupation numbers. We write this briefly as $\{n_{ij}\}$. We term this definition the density matrix in the *space of double occupation numbers* and we define it by the coefficients of the matrix

$$C(\{n_{ij}\},\quad i,j=1,...,\infty;t).$$

The diagonal elements (i.e., the elements for which only the double numbers with identical state numbers $n_{ii}\equiv n_i$ are different from zero) must be positive:

$$C(\{n_i\};\ t)\geqslant 0, \tag{10}$$

since they determine the probability of finding n_i identical particles in the state $r^{(i)}$ ($i=1,2,...,\infty$), normalized by the condition

$$\sum_{n_1,n_2,...=1}^{\infty} C(\{n_i\};\ t)=1. \tag{11}$$

According to Eq. (5) the hermiticity condition must also hold:

$$C(\{n_{ij}\};\ t)=C^*(\{n_{ji}\};\ t). \tag{12}$$

We can introduce in the space of double occupation numbers orthonormalized basis vectors with fixed values of these numbers:

$$|\{n^0_{ij}\}\rangle=\prod_{i,j=1}^{\infty}\delta_{n_{ij}n^0_{ij}}, \tag{13}$$

$$\langle\{n^0_{ij}\}|\{m^0_{ij}\}\rangle=\sum_{\{n_{ij}\}=0}^{\infty}\ \prod_{i,j=1}^{\infty}\delta_{n_{ij}n^0_{ij}}\prod_{k,l=1}^{\infty}\delta_{n_{ij}m^0_{kl}} =\prod_{i,j=1}^{\infty}\delta_{n^0_{ij}m^0_{ij}}. \tag{14}$$

Next, we can decompose any matrix in this space in terms of the basis:

$$C(\{n_{ij}\};t)=\sum_{\{n_{ij}^0\}=0}^{\infty} f(\{n_{ij}^0\};\ t)|\{n_{ij}^0\}\rangle, \tag{15}$$

where the coefficients are given by the projections

$$f(\{n_{ij}^0\};\ t)=\sum_{\{n_{ij}\}=0}^{\infty}\langle\{n_{ij}^0\}|C(\{n_{ij}\};\ t)\rangle. \tag{16}$$

We note that vectors in the space of double occupation numbers are the density matrices themselves (and not wave functions).

We can now determine the operator of the transformation from one primary state r into another primary state s without a change in the secondary states by means of the following action on the basis vectors (13):[3]

$$N_{sr}|\{n_{ij}^0\}\rangle=\sum_{q=1}^{\infty}[(n_{sq}^0+1-\delta_{rs})n_{rq}^0]^{1/2}|\{...,n_{rq}^0-1,...,n_{sq}^0+1,...\}\rangle. \tag{17}$$

The normalization was chosen so that for $s=r$ we obtain the operator (9) for the total number of particles in the primary state s:

$$N_s|\{n_{ij}^0\}\rangle=\left(\sum_{q=1}^{\infty} n_{sq}^0\right)|\{n_{ij}^0\}\rangle. \tag{18}$$

An identical operator N'_{sr} can be defined for transitions of secondary states. The Schroedinger equation in the space of double occupation numbers can be written in terms of the operators N_{sr} and N'_{sr}.[3] With the help of the definition (17) it can be verified directly that the transformation operators satisfy the relations

$$[N_{lm},N_{sr}]_-=\delta_{lr}N_{sm}-\delta_{ms}N_{lr}, \tag{19}$$

where the brackets with the suffix "-" denote a commutator. The operators N'_{sr} also satisfy similar relations, and the primed and unprimed operators commute with one another.

We define the hermitian-conjugate operator in the ordinary manner:

$$N_{rs}^{+}=N_{sr}^{*}. \tag{20}$$

The conditions for the operators (19) and (20) must hold in any representation. Any theory of identical particles must satisfy these *necessary* conditions. For Fermi and Bose statistics these relations were given by N. N. Bogolyubov (Ref. 2, p. 333).

3. PARTICLE CREATION AND ANNIHILATION OPERATORS AND DEFINITION OF PARTICLE ELEMENTARITY

The transition operators satisfying Eqs. (19) and (20) are necessary but not sufficient for a complete definition of the statistical properties of identical particles. For such a definition we must now introduce particle creation and annihilation operators a_r^+ and a_r for some state r and formulate commutation relations for them, so that we can introduce a statistical operator of the following form for a system of N identical particles

$$\hat{\rho}=\sum_{r_1,...,r_N;\ r'_1,...,r'_N}\rho(r_1,...,r_N;\ r'_1,...,r'_N)\times a_{r1}^+...a_{rN}^+|0\rangle\langle 0|a_{r'_N}...a_{r'_1} \tag{21}$$

and calculate the average values of observables over states of a system of such particles.

We now formulate the following proposition.

Proposition 1. *There exist creation operators a_r^+ and annihilation operators a_r of particles in a primary single-particle state r such that the relations*

$$(a_r^+)^+=a_r,\quad (a_ra_s)^+=a_s^+a_r^+ \tag{22}$$

and

$$[N_{sr},a_t]_-=-\delta_{st}a_r,\quad [N_{sr},a_t^+]_-=\delta_{rt}a_s^+ \tag{23}$$

are satisfied.

For $r=t$ we have for the last relation

$$N_{sr}a_r^+=a_r^+N_{sr}+a_s^+$$

and we can interpret it as replacing, under the action of the transition operator N_{sr}, the operator creating a particle in the state r by an operator creating a particle in the state s. It is also obvious that for $s=r$ we have the particle number operator

$$N_sa_t^+=a_t^+(N_s+\delta_{ts}). \tag{24}$$

We now state a more decisive proposition.

Proposition 2. *The transition operators N_{sr} are bilinear in the particle creation and annihilation operators*

$$N_{sr}=-\rho^{-1}[a_r,a_s^+]_{-q}+c_{sr}, \tag{25}$$

where ρ, q, and c_{sr} are constants, and the brackets are defined as

$$[a_r,a_s^+]_{-q}\equiv a_ra_s^+-qa_s^+a_r. \tag{26}$$

The minus sign in front of ρ^{-1} and q is used for convenience in the exposition given below.

The condition (20) leads to the fact that the parameters ρ and q must be real

$$\rho^*=\rho,\quad q^*=q, \tag{27}$$

and the constant c_{sr} must satisfy the relation

$$c_{rs}^*=c_{sr}. \tag{28}$$

We call the proposition (25) the *hypothesis of elementarity* of particles or, in other words, the hypothesis of linearity of the theory. Indeed, when we switch to the field theory we shall see that this proposition corresponds to the fact that the Hamiltonian and other observables have the bilinear form (26) and describe *free particles* whose equations are linear. It is only at the next step that we can introduce interactions and switch to the Heisenberg fields, postulating for them single-time relations as generalizations of such relations for free fields. In other words, our proposition reduces to the fact that we can quantize free linear (elementary) fields *first* and *then* include the interaction. The particles themselves are then regarded as the quanta of such a free field, which is why the particles are identical, they did not appear at some enormous distances from one another.

If, however, it is found that the theory cannot be constructed by this method and the interaction or self-interaction of the fields must be taken into account already when the quantization rules for the fields are formulated, then the corresponding excitations of the fields must be regarded as being "collective." Such "fields" usually arise in an average description of groups of strongly-coupled "elementary" particles. I do not know of any systematic implementation of such a quantization procedure, at least within the framework of a relativistic local field theory (in the Wightman axiomatic formulation of fields which interact at the outset the main problem in constructing a nontrivial theory satisfying all axioms on which such a formulation is based has still not been solved; see, for example, Ref. 46).

It is well known that in quantum electrodynamics the possibility of quantizing free fields first and then introducing their interaction is associated with the smallness of the dimensionless coupling constant and with the possibility of implementing the relativistic renormalization procedure. In quantum chromodynamics the coupling constant of quarks and gluons is by no means small, and the gluon fields themselves are inherently strongly nonlinear. But, luckily, it turned out that it is this nonlinearity that makes the theory asymptotically free at large momentum transfers (or at small distances). Once again we have obtained a range of energies at which quarks and gluons can be regarded as asymptotically free and we can start to construct a theory from this starting point. This is also true for any nonabelian theory.

On the other hand, there often arises the question of the quantization of composite particles regarded as a single entity, as happened in its time for nuclei consisting of nucleons and arises now for the nucleons themselves, which consist of quarks. Such a picture is possible only if, according to the Ehrenfest–Oppenheimer theorem,[47] the effect of interactions of composite particles on the internal motion of their constituents can be *completely* neglected. If the internal state of such clusters is the same, then their statistics are determined by the statistics of the constituents; if, however, the internal state is different, then the statistics of the composite particles is not determined, and the particles can be regarded as *different* particles, even though the particles consist of the same components. Evidently, such a description cannot be violated "a little" and corrected by taking into account the direct interaction between the components; it is more of an approximation than an approach to a limit. The motion of the constituents in a complicated system, taking into account their interaction, cannot be decomposed with respect to their internal states inside separate clusters, since states which do not refer to separate clusters arise in the spectrum of the states of the complete system. For this reason it is meaningless to talk about a small violation of the statistics of clusters caused by the interaction of the constituents of the clusters, as proposed recently as a model of an "apparent" violation of Pauli's principle.[48]

A good example of this situation is the nucleon model of the nucleus, which describes perfectly well the low-energy nuclear states taking into account the Fermi statistics of the neutrons and protons. When the fact that the nucleons themselves are colorless clusters, consisting of three "differently colored" quarks, is taken into account, however, one obtains not violation of the Fermi statistics of the nucleons, which is a consequence of the Fermi statistics of quarks taking into account their color, but rather the possibility of the appearance of fundamentally different, nonnucleonic bound states. Such states are multiquark (say, six-quark) states, which are colorless on the whole, but do not have any smaller colorless substructures. This phenomenon has been termed hidden color. Thus the nucleonic model is an approximation of some states (in particular the ground state) of a complicated many-quark system. The state space of the latter system is richer and cannot be decomposed only into states of the nucleonic model. Moreover, in many nucleonic models the indicated states with hidden color must be admixed to the nucleonic states, which are colorless clusters of the entire system as a whole. The Pauli principle, however, is now satisfied not for nucleons, regarded as a whole, but rather for the quarks themselves, taking into account their color state.

4. BASIC COMMUTATION RELATIONS

Combining the relations (23) and (25) we obtain the *basic commutation relations* for identical elementary particles

$$[[a_r,a_s^+]_{-q},a_t]_-=\rho\delta_{ts}a_r \tag{29}$$

and the hermitian-conjugate relations

$$[[a_r,a_s^+]_{-q},a_t^+]_-=-\rho\delta_{rt}a_s^+, \tag{30}$$

which we shall investigate below. The parameters q and ρ can assume any real values, including zero.[5)] We note that for $\rho\neq0$ this arbitrary parameter can always be eliminated by a simple renormalization: $a\to\rho^{1/2}a$, $a^+\to\rho^{1/2}a^+$. For our purposes, however, it is more convenient to keep this parameter arbitrary and later assign to it suitable values.

The relations (29) and (30) were regarded in Ref. 5 as the most general commutation relations corresponding to parastatistics.[6)] Later they were "rediscovered" for a single-level system by Ignat'ev and Kuz'min[12] and for the corresponding field theory by Greenberg and Mohapatra[13], who proposed using these relations for formulating a small violation of Pauli's principle.

A consequence of Eqs. (29) and (30) is the relation[7)]

$$[[a_r,a_s^+]_{-q},[a_k,a_l^+]_{-q}]_- =\rho\delta_{ks}[a_r,a_l^+]_{-q}-\rho\delta_{rl}[a_k,a_s^+]_{-q}. \tag{31}$$

Taking into account Eq. (25) (with $\rho\neq0$), we can put the relation (31) into the form (19) by setting

$$c_{sr}=c\delta_{sr}, \tag{32}$$

where c is a real constant.

We note that the relations (29) and (30) are invariant under a unitary transformation

$$a_r'=\sum_j u_{rj}a_j, \quad (a_r')^+=\sum_j u_{rj}^* a_j^+ \tag{33}$$

with

$$\sum_j u_{sj}^* u_{rj}=\delta_{sr}. \tag{34}$$

This property of the basic relations is very important, because it ensures that they are independent of the initial basis of single-particle states.[8)] Bialynicki-Birula indicated the importance of such a transformation within the framework of Green's quantization.[50] We note that no relations besides (29) and (30) are postulated. For example, we do not postulate any relations for three annihilation operators (or three creation operators). We shall show below that the relations (29) and (30) completely determine the Fock space of the particle states.

5. FOCK REPRESENTATION OF THE BASIC COMMUTATION RELATIONS

We now consider the Fock representation of the relations (29) and (30), basing this representation, as usual, on the following postulate:

Postulate. *There exists a unique vacuum state* $|0\rangle$ *such that*

$$a_r|0\rangle=0 \quad \text{and} \quad \langle 0|a_r^+=0 \tag{35}$$

for all states r.

Corollary. *If* $r\neq 0$, *then*

$$a_r a_s^+|0\rangle=p\delta_{rs}|0\rangle, \tag{36}$$

where p is an arbitrary constant (not depending on the state) number.

Proof. We operate with the left- and right-hand sides of Eq. (29) on the vacuum vector. Using Eq. (35) we obtain

$$a_t a_r a_s^+|0\rangle=0,$$

whence, by virtue of the assumed *uniqueness* of the vacuum vector, the relation

$$a_r a_s^+|0\rangle=f_{rs}|0\rangle,$$

where f_{rs} is a number which, generally speaking, depends on the states r and s, must hold. Next, we operate on the vacuum vector by the left- and right-hand sides of Eq. (31). We obtain

$$\rho(\delta_{ks}f_{rl}-\delta_{rl}f_{ks})=0.$$

By definition $\rho\neq 0$ and therefore the expression in parentheses vanishes, whence it follows that $f_{rl}=p\delta_{rl}$, where p is a number that does not depend on r and l, and we arrive at Eq. (36).

The case $\rho=0$, excluded from this analysis, corresponds to the fact that the expression (26) commutes with all creation and annihilation operators and therefore it is a c number. We shall examine this case below.

The relations (35) and (36) completely determine the representation of the trilinear relations (29) and (30). Indeed, in the Fock representation the basis vectors are obtained by operating with monomials of creation operators a^+ on the vacuum vector. The action of annihilation operators on such vectors can be determined by using the relation (30) to advance these operators to the right up to the vacuum vector and then the relations (35) and (36). The result is the general formula

$$\begin{aligned} a_r a_{r_1}^+ a_{r_2}^+ \dots a_{r_n}^+|0\rangle = & \, p\delta_{sr_1}a_{r_2}^+\dots a_{r_N}^+|0\rangle+\sum_{k=2}^{n}\delta_{rr_k}\Big[q^{k-2}(qp-\rho)a_{r_1}^+\dots a_{r_{k-1}}^+ \\ & -\rho\sum_{l=1}^{k-2}q^{k-l-2}a_{r_1}^+\dots a_{r_{k-l-2}}^+ a_{r_{k-l}}^+\dots a_{r_{k-l}}^+ a_{r_{k-l-1}}^+\Big]a_{r_{k+1}}^+\dots a_{r_n}^+|0\rangle. \end{aligned} \tag{37}$$

An arbitrary vector in the Fock representation can be written in the form of an expansion in the basis vectors

$$|\Psi\rangle=\Psi_0|0\rangle+\sum_{n=1}^{\infty}\sum_{r_1,\dots,r_n}\Psi^{(n)}(r_1,\dots,r_n)a_{r_1}^+\dots a_{r_n}^+|0\rangle, \tag{38}$$

in which the amplitudes $\Psi^{(n)}(r_1,\dots,r_n)$ do not have in advance any symmetry with respect to permutations of their arguments. Using Eq. (37) we obtain for the projections of this vector on the basis vectors

$$\langle 0|\Psi\rangle=\Psi_0, \tag{39a}$$

$$\langle 0|a_{r_1}|\Psi\rangle=p\Psi^{(1)}(r_1), \tag{39b}$$

$$\langle 0|a_{r_2}a_{r_1}|\Psi\rangle=p^2\Psi^{(2)}(r_1,r_2)+p(qp-\rho)\Psi^{(2)}(r_2,r_1), \tag{39c}$$

$$\begin{aligned} \langle 0|a_{r_3}a_{r_2}a_{r_1}|\Psi\rangle = & \, p^3\Psi^{(3)}(r_1,r_2,r_3)+p^2(qp-\rho)\Psi^{(3)} \\ & \times(r_1,r_3,r_2)+p^2(qp-\rho)\Psi^{(3)} \\ & \times(r_2,r_1,r_3)+p(qp-\rho)^2\Psi^{(3)} \\ & \times(r_3,r_1,r_2)+p[q(qp-\rho)^2 \\ & -p\rho]\Psi^{(3)}(r_3,r_2,r_1)+p(qp \\ & -\rho)^2\Psi^{(3)}(r_2,r_3,r_1) \end{aligned} \tag{39d}$$

and so on. We can see that in the general case the relations between the projections and amplitudes are not as simple as for the usual fermion and boson operators. In order to have one-to-one correspondence between the projections and amplitudes it is necessary to construct orthogonal combinations from the basis vectors. Such orthogonalization is performed by forming combinations with definite symmetry properties with respect to permutations of the arguments. The symmetrized amplitudes acquire the corresponding meaning as wavefunctions of the system in one or another symmetrized state. We shall perform below such orthogonalization explicitly for the particular cases of two- and three-particle systems.

Thus far we have not imposed any constraints on the parameters p, q, and ρ (one of the parameters $p\neq 0$ or $\rho\neq 0$ can be eliminated, as indicated above, by renormalizing the operators). We now consider the constraints arising from the positive-definiteness of the norms of the state vectors in Fock space.

Consider an arbitrary vector of a single-particle system

$$|\Psi^{(1)}\rangle=\sum_r \Psi^{(1)}(r)a_r^+|0\rangle. \tag{40}$$

We obtain for the norm of such a vector with the help of Eq. (39b)

$$\begin{aligned}\|\Psi^{(1)}\|^2&\equiv\langle\Psi^{(1)}|\Psi^{(1)}\rangle\\ &=\left\langle 0\middle|\sum_r \Psi^{(1)*}(r)a_r\middle|\Psi^{(1)}\right\rangle\\ &=p\sum_r|\Psi^{(1)}(r)|^2.\end{aligned} \tag{41}$$

The condition that the norm be positive definite means that $p^*=p\geqslant 0$. The case $p=0$ is trivial and for this reason we assume everywhere below

$$p>0. \tag{42}$$

There now arise two possibilities: The value of the real parameter p can be assumed beforehand to be either *finite* or *infinite* (i.e., larger than any prescribed number). In the rest of this section and up to Sec. 9 we shall consider the case of finite values of the parameter p.

We now consider symmetrized two-particle states determined by the vectors

$$|\Psi_\lambda^{(2)}\rangle=\sum_{r_1,r_2}\Psi_\lambda(r_1,r_2)a_{r_1}^+a_{r_2}^+|0\rangle, \tag{43}$$

where the symmetric and antisymmetric wave functions are constructed from an arbitrary two-particle function in the form of the known combinations[9]

$$\Psi_\lambda^{(2)}(r_1,r_2)=\frac{1}{\sqrt{2}}[\Psi(r_1,r_2)+\lambda\Psi(r_2,r_1)],\quad \lambda=\pm 1. \tag{44}$$

The norms of the vectors (43) can be calculated using Eq. (39c):

$$\begin{aligned}|\Psi_\lambda^{(2)}|^2&=\sum_{r_1,r_2}\Psi_\lambda^*(r_1,r_2)\langle 0|a_{r_2}a_{r_1}|\Psi_\lambda^{(2)}\rangle\\ &=p[p+\lambda(qp-\rho)]\sum_{r_1,r_2}|\Psi_\lambda(r_1,r_2)|^2.\end{aligned} \tag{45}$$

The condition of positive-definiteness of these norms means that

$$p\pm(qp-\rho)\geqslant 0. \tag{46}$$

This can be rewritten in the form

$$-1+\frac{\rho}{p}\leqslant q\leqslant 1+\frac{\rho}{p}. \tag{47}$$

We consider next symmetrized three-particle states, but we choose in advance only three-particle wave functions which exhibit λ symmetry under transposition of the two last arguments:

$$\Psi(r_1,[r_2,r_3]_\lambda)=\lambda\Psi(r_1,[r_3,r_2]_\lambda), \tag{48}$$

which we indicated by enclosing these arguments in brackets with a suffix λ. From these functions we can construct only three orthogonal combinations: one function that is completely λ-symmetric with respect to all arguments:

$$\begin{aligned}\Psi_\lambda(r_1,r_2,r_3)=\frac{1}{\sqrt{3}}\{&\Psi(r_1,[r_2,r_3]_\lambda)+\lambda\Psi(r_2,[r_1,r_3]_\lambda)\\ &+\Psi(r_3,[r_1,r_2]_\lambda)\}\end{aligned} \tag{49}$$

and two combinations forming the basis of a mixed representation:[10]

$$\Psi_m'(r_1,r_2,r_3)=\frac{1}{\sqrt{2}}\{\Psi(r_1,[r_2,r_3]_\lambda)-\Psi(r_3,[r_1,r_2]_\lambda\}, \tag{50}$$

$$\begin{aligned}\Psi_m''(r_1,r_2,r_3)=\frac{1}{\sqrt{6}}\{&\Psi(r_1,[r_2,r_3]_\lambda)-2\Psi(r_2,[r_1,r_3]_\lambda)\\ &+\Psi(r_3,[r_1,r_2]_\lambda)\}.\end{aligned} \tag{51}$$

We are now interested in the last two combinations. Using Eq. (39d), we obtain for the norms of the corresponding vectors

$$\begin{aligned}|\Psi_m^{\prime,\prime\prime}|^2&=p(1\pm x)[p^2-(pq-\rho)^2]\\ &\times\sum_{r_1,r_2,r_3}|\Psi_m^{\prime,\prime\prime}(r_1,r_2,r_3)|^2,\end{aligned} \tag{52}$$

where we have introduced the notation

$$x=\lambda q, \tag{53}$$

and the upper sign in front of x corresponds to the combination (50) and the lower sign corresponds to the combination (51). Taking into account the condition (46), we conclude that the positive-definiteness of the norm (52) means that

$$-1\leqslant x\leqslant 1. \tag{54}$$

Setting $\lambda=\pm 1$ we obtain the restrictions on the possible values of the initial parameter q:

$$-1 \leqslant q \leqslant 1. \tag{55}$$

We note that the different expressions (52) for the norms of the two combinations (50) and (51) mean that these states can be physically distinguished (for example, for $x=\pm 1$ one of these combinations vanishes identically). For this reason, by permutations of their arguments we mean not permutations of states of *different* enumerated particles (which is meaningless for identical particles) but rather *permutations of the states* occupied by *some* identical particles. The classification of the functions according to such irreducible representations with respect to permutations of the states is required simply for constructing orthogonal combinations.

Now we consider n particles with respect to whose states complete symmetrization or complete antisymmetrization is performed. We once again designate this as a λ-symmetric state, making the association

$$\lambda=\{1 \text{ for a symmetric vector}, \; -1 \text{ for an asymmetric vector}\}, \tag{56}$$

$$|\Psi_\lambda^{(n)}\rangle=\sum_{r_1,\dots,r_n}\Psi_\lambda(r_1,\dots,r_n)a_{r_1}^+\dots a_{r_n}^+|0\rangle, \tag{57}$$

where $\Psi_\lambda(r_1,\dots,r_n)$ is a completely λ-symmetric function of its arguments.

We shall always choose the λ-symmetry of the vector (57) so that the quantity

$$\omega=\lambda\rho \tag{58}$$

is positive

$$\omega\geqslant 0. \tag{59}$$

This means that if the initial parameter $\rho\leqslant 0$, then we choose $\lambda=-1$ and, therefore, we consider antisymmetric vectors; if, however, $\rho\geqslant 0$, then we choose $\lambda=1$ and we consider symmetric vectors.

Using Eq. (37) we obtain for such vectors

$$a_r|\Psi_\lambda^{(n)}\rangle=R(n,x)\sum_{r_2,\dots,r_n}\Psi_\lambda(r,r_2,\dots,r_n)a_{r_2}^+\dots a_{r_n}^+|0\rangle, \tag{60}$$

where

$$R(n,x)=\sum_{k=0}^{n-1}[p-\omega(n-k-1)]x^k \tag{61}$$

is a numerical function of x, defined by the relation (53). For example,

$$R(1,x)=p, \tag{62a}$$

$$R(2,x)=p(1+x)-\omega, \tag{62b}$$

$$R(3,x)=p(1+x+x^2)-\omega(2+x), \tag{62c}$$

$$R(4,x)=p(1+x+x^2+x^3)-\omega(3+2x+x^2) \tag{62d}$$

and so on.

The calculation of the norms of the vectors (57) simply reduces to repeated application of Eq. (60) and yields the expression

$$\begin{aligned}|\Psi_\lambda^{(n)}|^2&=\sum_{r_1,\dots,r_n}\Psi_\lambda^*(r_1,\dots,r_n)\langle 0|a_{r_n}\dots a_{r_1}|\Psi_\lambda^{(n)}\rangle\\&=R(1,x)R(2,x)\dots R(n,x)\\&\quad\times\sum_{r_1,\dots,r_n}|\Psi_\lambda(r_1,\dots,r_n)|^2.\end{aligned} \tag{63}$$

We are now prepared to prove the basic theorems on the statistics of identical particles.

6. THEOREMS ON FINITE STATISTICS OF IDENTICAL PARTICLES

Theorem I. *If the parameter p, appearing in the condition (36), assumes finite values, then the number of particles in an symmetric or antisymmetric state cannot exceed some finite number M.*

Proof. Setting successively $n=1,2,\dots$, we obtain from the condition of positive-definiteness of the norms of the corresponding λ-symmetric vectors (57) the requirements

$$R(1,x)\geqslant 0, \quad R(2,x)\geqslant 0,\dots . \tag{64}$$

Therefore, for arbitrary n we have the condition

$$p\sum_{k=0}^{n-1}x^k\geqslant\omega\sum_{k=0}^{n-1}(n-k-1)x^k. \tag{65}$$

The condition (42) implies that p is positive. Further, according to our choice of the λ-symmetric vector depending on the sign of ρ, Eq. (59) is always satisfied, and we have for x, according to Eq. (44), the restriction $|x|\leqslant 1$. Therefore, the left- and right-hand sides of the inequality (65) are positive. But, as n increases, the right-hand side increases more rapidly than the left-hand side. For this reason, for some n and prescribed finite values of p and ω the right-hand side exceeds the left-hand side and the condition that the norm of the vectors is positive-definite breaks down. In order that this not happen, for some $n=M+1$ the inequality (65) must become an equality. Then the norm of the corresponding λ-symmetric vector and, likewise, according to Eq. (63), the norms of all subsequent such vectors vanish and therefore the vectors themselves also vanish. Thus the following relation must be satisfied:

$$R(M+1,x)=0. \tag{66}$$

Hence we can express the parameter ω in terms of the other parameters:

$$\omega=p\left(\sum_{k=0}^{M}x^k\right)\Big/\left[\sum_{k=0}^{M}(M-k)x^k\right]. \tag{67}$$

Thus, for a given finite value of the parameter ρ the number of particles in the λ-symmetric state, chosen depending on the sign of ρ [see Eq. (58) and (59)], indeed cannot exceed the finite number M.

We find that in the case $\rho\leqslant 0$ the number of possible particles in the antisymmetric state ($\lambda=-1$) is limited. If, however, $\rho \geqslant 0$, then the possible number of particles in the symmetric state is limited. Thus we have the correspondence

$$\rho \leqslant 0, \quad \lambda = -1 \quad \text{para-Bose statistics,} \tag{68a}$$

$$\rho \geqslant 0, \quad \lambda = 1 \quad \text{para-Fermi statistics} \tag{68b}$$

of order M.

In the particular case of two particles in a λ-symmetric state we have for the norm of the vector, according to Eqs. (45), (53), and (58), after substituting Eq. (67),

$$\|\Psi_\lambda^{(2)}\|^2 = p^2\beta^2 \sum_{r_1,r_2} |\Psi_\lambda(r_1,r_2)|^2, \tag{69}$$

where we have introduced the notation

$$\beta^2 = \left[M - 1 + 2\sum_{k=1}^{M} (M-k)x^k\right] \Big/ \left[\sum_{k=0}^{M} (M-k)x^k\right]. \tag{70}$$

The coefficient β^2 is a measure of the violation of the ordinary statistics. We now consider systematically the simplest cases.

For $M=1$ we obtain $\beta^2=0$, as should be for ordinary Bose and Fermi statistics, which do not admit any deviations from symmetric or antisymmetric states, respectively. In this case, according to Eq. (62b), we also have

$$p(1+x) - \omega = 0. \tag{71}$$

It can be shown (see Appendix 1) that this condition is not only necessary but also sufficient for describing ordinary statistics.

For $M=2$ we have

$$\beta^2 = (1+2x)/(2+x). \tag{72}$$

As $x \to -1/2$ the quantity $\beta^2 \to 0$; this means that the deviations of such parastatistics from the ordinary (Bose or Fermi) statistics are small. For example, in the case $\lambda=1$ such para-Fermi statistics permits placing two (but not more!) particles in the same state with low probability, proportional to β^2 (in the limit $q \to -1/2$). This is the possibility considered by Ignat'ev and Kuz'min[12] in formulating the question of a small violation of Pauli's principle, for example, for electrons in a single atom. Greenberg and Mohapatra proposed a field-theoretic analysis of this possibility on the basis of the trilinear relations (29).[13] Inverting Eq. (72) (with $\lambda=1$ and $x=q$) we obtain

$$q = (1-2\beta^2)/(-2+\beta^2). \tag{73}$$

In Refs. 12 and 13 the parameter p was arbitrarily set equal to 1, and then, according to Eq. (67), the parameter ρ can also be expressed in terms of the small parameter β^2:

$$\rho = (1-\beta^2+\beta^4)/(2-\beta^2). \tag{74}$$

However, we shall soon show that in this case, as in all subsequent cases, the condition (66) is found to be necessary but insufficient for describing the corresponding parastatistics, as was established in general form in Ref. 5. We shall see that additional restrictions, following *only from the general requirement that the norm of the state vectors be positive-definite*, lead to very stringent restrictions on the possible values of the parameter q, and on the basis of this theory a small violation of the ordinary statistics becomes impossible. In the case $M=2$ this was proved directly in Refs. 14 and 15 for the relations (29) and (30) with the parameterization (73) and (74). The proof given here is the first general proof of this assertion for arbitrary finite M.

We also note that in this case for $M=2$ and $x=-1$ we have $\beta^2=-1$, which violates the condition that the norm of the vector (69) be positive-definite. The restriction $\beta^2 \geqslant 0$ leads, in this case, to the restriction $-1/2 \leqslant x$, and instead of Eq. (54) we now have

$$-1/2 \leqslant x \leqslant 1. \tag{75}$$

We examine also the particular case $M=3$. According to Eq. (70)

$$\beta^2 = 2(1+x)^2/[2+(1+x)^2]. \tag{76}$$

The quantity β^2 becomes small as $x \to -1$. Inverting the relation (76) gives

$$x = -1 + \sqrt{2\beta^2(2-\beta^2)}, \tag{77}$$

where we have also used the restriction (54) and dropped the solution with the minus sign in front of the square root. Setting $p=1$ we obtain for the expression (67)

$$\omega = \sqrt{2\beta^2/(2-\beta^2)}\,[1 - \sqrt{\beta^2(2-\beta^2/2}]. \tag{78}$$

We note that $\beta^2=0$ for $x=-1$; this corresponds to ordinary statistics. As one can see from Eq. (78), $\omega=0$, and the right-hand side in the initial relation (29) vanishes. This means that, as expected, the expression (26), which in this case is a commutator or anticommutator, becomes a c number.

The examples considered above show that the value $x=-1$ is either forbidden completely for $M=2$ or it corresponds to ordinary quantization with $M=3$. This result can be generalized, and it can be shown with the help of Eq. (70) that for even $M=2m$ $(m=1,2,...)$ the value $x=-1$ is forbidden [it contradicts the condition $\beta^2 \geqslant 0$ for the vector (69)],[11)] and for any odd $M=2m+1$ $(m=1,2,...)$ the value $x=-1$ corresponds to $\beta^2=-1$, i.e., ordinary statistics. In considering the nontrivial generalization of ordinary statistics we can eliminate this value and instead of Eq. (54) we can always assume that

$$-1 < x \leqslant 1. \tag{79}$$

Theorem II. *For finite $M \geqslant 2$, defined by theorem I, the condition that the norms of state vectors in the Fock representation of the initial trilinear relations be positive-definite admits only two values of x: 0 and 1.*[12)]

Proof. Consider a vector in which n particles are in the λ-symmetric state [defined in accordance with the sign of ρ according to the rules (68)], but, in addition, there exists an "extra" particle which is not in a λ-symmetric state with the other particles. Let the vector of such a state be

$$|\psi_{[n]_\lambda+1}\rangle = \sum_{\mathscr{P} \in S_n} \lambda^{\eta(\mathscr{P})} a^+_{r_{\mathscr{P}1}} a^+_r a^+_{r_{\mathscr{P}2}} \ldots a^+_{r_{\mathscr{P}n}} |0\rangle, \tag{80}$$

where $r \neq r_1,...,r_n$, and $\mathscr{P}$ is any permutation of the indices $1,2,...,n$ and $\eta(\mathscr{P})$ is the parity of such a permutation. The norm of such a vector can be calculated by repeatedly applying (37). The norm will be

$$\|\psi_{[n]_\lambda+1}\|^2 = R(1,x)...R(n-1,x)\{(px-2\omega)xR(n,x) + p(1-x^2)[p-\omega(n-1)] + \omega x[2p-\omega(n-1)]\}, \quad (81)$$

where we have employed Eqs. (53) and (58). We set $n=M+1$. Using Eqs. (66) and (67) we obtain the following expression for the norm (81):

$$\|\psi_{[M+1]_\lambda+1}\|^2 = R(1,x)...R(M,x) \times\left\{-(M+1)p^2x^2(1-x^2) \times\frac{[\sum_{k=0}^{[M/2]-1}x^{2k}(\sum_{l=0}^{M-2(k+1)}x^l)^2]}{[\sum_{k=0}^{M}(M-k)x^k]^2}\right\}, \quad (82)$$

where the sum over k extends over the integer part $[M/2]-1$. According to the condition of positive-definiteness of λ-symmetric vectors, for $n=1,2,...,M$ all factors $R(1,x),...,R(M,x)$ must be positive. All other factors cannot be negative, since they are all positive sums of squares and the factor $1-x^2$ is nonnegative by virtue of Eq. (79). But there is a minus sign in front of the entire product, and for this reason the entire expression as a whole is negative! This can be avoided only by setting the sum equal to zero. There are three ways of doing this: set $x=0$ or ± 1 or equating the sum over k to zero. But the latter sum is a sum of squares and can vanish only if all of its terms vanish. In particular, the first term for $k=0$ must vanish:

$$\sum_{l=0}^{M-2} x^l = 0. \quad (83)$$

In the case of even $M \geqslant 2$ this equation has no real roots. For odd $M \geqslant 3$ the equation has a single root $x=-1$, which, however, is eliminated by the condition (79). Thus there remain only two possibilities for the expression (82) to vanish: $x=0$ and $x=1$, which is what we were required to prove.

Now it remains for us only to prove that no additional restrictions arise in these two admissable cases.

7. GREEN'S RELATIONS

The value $x=1$ corresponds to Green's relations. In this case the relation (67) becomes

$$\omega = 2p/M. \quad (84)$$

It is convenient to set the arbitrary parameter p equal to the order of the parastatistics

$$p=M \quad \text{and} \quad \omega=2. \quad (85)$$

It is easy to see that then all conditions (64) acquire the simple form

$$n(M-n+1) \geqslant 0 \quad (86)$$

and are satisfied asymptotically for all $n \leqslant M$.

Recalling Eqs. (53) and (58), we obtain in this case for the initial parameters

$$q=\lambda \quad \text{and} \quad \rho=2\lambda. \quad (87)$$

Thus the initial relations (29) assume the form

$$[[a^s,a_r^+]_{-\lambda},a_t]_- = 2\lambda\delta_{rt}a_s, \quad (88)$$

and, according to the rule (68), together with the condition (36) with $p=M$, they will define *para-Fermi statistics* of order p with $\lambda=1$ and *para-Bose statistics* of order p with $\lambda=-1$. These relations were postulated directly by Green,[6] and Greenberg and Messiah[7] proved, using the condition (36), that they indeed describe parastatistics of order p.

In order to illustrate the characteristic features of Green's quantization we shall consider only the simplest case of parastatistics of *second* order ($p=M=2$). According to Eq. (87), in this case we have $qp-\rho=0$, and the general relation (37) becomes

$$a_r a_{r_1}^+ a_{r_2}^+ ... a_{r_n}^+|0\rangle = 2\delta_{rr_1}a_{r_2}^+...a_{r_n}^+|0\rangle + \sum_{k=1}^{[(n-1)/2]} 2\delta_{rr_{2k+1}} \times(-\lambda)^k a_{r_2}^+ a_{r_1}^+ a_{r_4}^+ a_{r_3}^+ ... a_{r_{2k}}^+ a_{r_{2k-1}}^+ \times a_{r_{2k+2}}^+ a_{r_{2k+3}}^+ ... a_{r_n}^+|0\rangle. \quad (89)$$

With the help of this relation it can be shown that, besides the general trilinear relations (88), in this case the additional relations

$$a_{r_1}a_{r_2}a_{r_3}^+ + \lambda a_{r_3}^+ a_{r_2} a_{r_1} = 2\delta_{r_2r_3}a_{r_1}, \quad (90)$$

$$a_{r_1}a_{r_2}a_{r_3}^+ + \lambda a_{r_3}a_{r_2}a_{r_1} = 0 \quad (91)$$

and the hermitian-conjugates of these expressions are also satisfied. It is found that the relations (90) and (91) replace the general relations (88) with $p=2$.

The relation (91) indicates that the wave functions appearing in the general vector (38) must, in this case, exhibit $(-\lambda)$-symmetry (antisymmetry for $\lambda=1$ and simple symmetry for $\lambda=-1$) with respect to the arguments occupying odd locations and separately with respect to arguments occupying even locations. We note this property by writing such arguments in two rows and marking the brackets with the subscript $-\lambda$:

$$\Psi\begin{bmatrix} r_1 & & r_3 & & \cdots \\ & r_2 & & r_4 & \cdots \end{bmatrix}_{-\lambda}. \quad (92)$$

Evidently, such symmetry of the wave functions makes it impossible for more than two particles to occupy a λ-symmetric state: in the symmetric state for parafermions ($\lambda=1$) and in the antisymmetric state for parabosons ($\lambda=-1$).

The second-quantization scheme based on the relations (90) and (91) was formulated initially by Green[6] and independently by D. V. Volkov[24] and it was later subjected to numerous investigations.[36-39] One can guess, already according to the form of the wave functions (92) on which the scheme is defined, that actually this scheme describes a set of ordinary fermions or bosons of *two* types. It has been proved that such a scheme indeed can be interpreted as a scheme with ordinary fermions or bosons, exhibiting some hidden degree of freedom, such as isospin.[39] A more detailed discussion of the possibility of describing internal symmetries on the basis of Green's quantization can be found in the reviews Refs. 8 and 9 and in the literature cited there.

We now wish to point out that in this scheme the additional relations (90) and (91), leading to the definite symmetry property (92) of the wave functions, arose automatically (within the Fock representation). For example, in the case of para-Fermi statistics, if the states r_1 and r_3 are identical, then it follows from the "hermitian-conjugate" relation (91) that

$$a_r^+ a_{r_2}^+ a_r^+ = 0, \qquad (93)$$

which means that the corresponding three-particle vector also vanishes.[36,39] In the language of hidden internal symmetry this means that particles occupying odd states in the function (92) have the same isospin projections and for this reason, now as simple fermions, they cannot occupy one and the same quantum-mechanical state (see Ref. 39). Particles occupying even locations in the function (92) also have a similar property.

If particles occupying odd and even locations are assigned isospin states $i_3 = 1/2$ and $-1/2$, respectively, then the functions (92) will correspond to the projection of the total isospin I_3 of either 1/2 (for odd number of particles) or zero (for even number of particles). Thus isospin states with only these two projections of the isospin are realized in the Fock space.[13)]

A similar separation of some internal states also happens in the cases of parastatistics of third and higher orders, when the states of the paraparticles are interpreted as states of ordinary fermions or bosons with internal states of the type "strangeness" and so on.[39] This is the essence of the perculiarity of Green's quantization, based on the relations (88).

8. NEW PARAQUANTIZATION

We now consider the second possibility $x=0$. In this case the relation (67) also acquires the simple form

$$\omega = p/M, \qquad (94)$$

and once again it is convenient to set $p=M$. Then $\omega=1$ and all conditions (64) acquire the form

$$M-n-1 \geqslant 0 \qquad (95)$$

and are likewise automatically satisfied for all $n \leqslant m$. Once again recalling Eqs. (53) and (58), we have in this case for the initial parameters

$$q=0 \quad \text{and} \quad \rho=\lambda, \qquad (96)$$

and thus the starting relations (29) can be written as

$$[a_s a_r^+, a_t]_- = \lambda \delta_{rt} a_s. \qquad (97)$$

Just as in the preceding case, these relations, together with the condition (36) with $p=M$, will determine para-Fermi statistics of order M with $\lambda=1$ and para-Bose statistics of order M with $\lambda=-1$. Surprisingly, two quantization schemes, based on the relations (88) and (97), respectively, can be used to describe the same parastatistics of finite orders. We cannot give preference, as yet, to one of these schemes over the other, but we can elucidate the characteristic features.

According to Eq. (96), in this case $qp-\rho=-\lambda$, and the general relation (37) is now

$$a_r a_{r_1}^+ a_{r_2}^+ \dots a_{r_n}^+ |0\rangle = p\delta_{rr_1} a_{r_2}^+ \dots a_{r_n}^+ |0\rangle - \lambda \sum_{k=2}^{n} \delta_{rr_k} a_{r_2}^+ a_{r_3}^+ \dots a_{r_{k-1}}^+ \times a_{r_1}^+ a_{r_{k+1}}^+ \dots a_{r_n}^+ |0\rangle, \qquad (98)$$

where $p=M$. The projections (39) in this case have the form

$$\langle 0 | a_{r_2} a_{r_1} | \Psi \rangle = p^2 \Psi^{(2)}(r_1, r_2) - \lambda p \Psi^2(r_2, r_1), \qquad (99a)$$

$$\langle 0 | a_{r_3} a_{r_2} a_{r_1} | \Psi \rangle = p^3 \Psi^{(3)}(r_1, r_2, r_3) - \lambda p^2 \Psi^{(3)}(r_1, r_3, r_2) - \lambda p^2 \Psi^{(3)}(r_2, r_1, r_3) - \lambda p^2 \Psi^{(3)} \times (r_2, r_1, r_3) - \lambda p^2 \Psi^{(3)}(r_3, r_2, r_1) + p\Psi^{(3)}(r_3, r_1, r_2) + p\Psi^{(3)} \times (r_2, r_3, r_1). \qquad (99b)$$

In general, the projection for particles can be written in the following general form:

$$\langle 0 | a_{r_n} a_{r_{n-1}} \dots a_{r_2} a_{r_1} | \Psi \rangle = \sum_{\mathscr{P} \in S_n} (-\lambda^{\mathscr{N}(\mathscr{P})} p^{n-\mathscr{N}(\mathscr{P})} \Psi^{(n)}(r\mathscr{P}_1, r\mathscr{P}_2, \dots, r\mathscr{P}_n), \qquad (100)$$

where $\mathscr{P}$ is any permutation of the indices $1,2,\dots,n$ and $\mathscr{N}(\mathscr{P})$ is the minimum number of transpositions required in order to restore the initial natural order.

It is evident from Eqs. (99a,b) that if $p(=M)=2$, then two particles can be in a λ-symmetric state, while three particles cannot be in such a state [the sum of the coefficients on the right-hand side of Eq. (99b) p^3-3p^2+2p vanishes in this case]. Thus we once again are dealing with second-order para-Fermi ($\lambda=1$) or para-Bose ($\lambda=-1$) statistics. However, no other relations arise for the three-particle state. This is what distinguishes this case from Green's quantization.

In general, if $p=M$, then the λ-symmetric projection (100) vanishes for $n=M+1$. Therefore the parastatistics of order M is defined, without any additional constraints, on the functions obtained from any function in the form of a combination standing on the right-hand side of Eq. (100). Such functions can be regarded as generalizations of symmetric and antisymmetric wave functions for Bose and Fermi statistics. The λ-symmetric combination automatically vanishes on these functions:

$$\sum_{\mathscr{P}\in S_{M+1}} \lambda^{\eta(\mathscr{P})} a_{r_{\mathscr{P}1}} \ldots a_{r_{\mathscr{P}(M+1)}} = 0, \tag{101}$$

where $\eta(\mathscr{P})$ is the parity of the permutation $\mathscr{P}$. But no additional relation between the creation and annihilation operators arise.

It will be shown in Sec. 10 that for such a parastatistics scheme the states of the paraparticles can be interpreted as states of ordinary fermions or bosons which are degenerate with respect to some internal degree of freedom. But, then, no internal states are singled out, as happened in the case of Green's quantization.

9. INFINITE STATISTICS

In the proof of theorems I and II we assumed that the parameter p appearing in the relation (36) is bounded. We now let the value of this parameter pass to infinity. In this case our theorems no longer work and this case of infinite p must be studied specially.

We now consider the limiting expression of the relation (37), dropping in it all terms on the right-hand side except terms proportional to $p\to\infty$. We obtain the following simple result for the action of the annihilation operator on the basis vectors of the Fock representation:

$$a_s a^+_{r_1} \ldots a^+_{r_n}|0\rangle = p \sum_{k=1}^{n} \delta_{sr_k} q^{k-1} a^+_{r_1} \ldots a^+_{r_{k-1}} a^+_{r_{k+1}} \ldots a^+_{r_n}|0\rangle. \tag{102}$$

It follows from this relation that in this representation the creation and annihilation operators must satisfy in this limiting case *bilinear* relations

$$a_s a^+_r - q a^+_r a_s = p\delta_{sr}. \tag{103}$$

In order to exclude the infinitely growing factor p, we renormalize the operators

$$a_r,\ a^+_r \to \sqrt{p}a_r,\ \sqrt{p}a^+_r, \tag{104}$$

and then the relations (103) assume the form

$$a_s a^+_r - q a^+_r a_s = \delta_{sr}. \tag{105}$$

The algebra of operators satisfying these relations was recently examined by Greenberg[16] in order to use it for formulating a small violation of ordinary statistics (in particular, the Pauli principle), but now within a nonlocal theory.[14)]

As one can see by comparing Eqs. (26) and (105), the initial relations (29) and (30) contain a $(-q)$-mutator, now transforming into the c-number Kronecker δ function, which commutes with all operators. The left-hand sides of Eqs. (29) and (30) likewise vanish, since after the renormalization (104) we have

$$\rho/p \to 0 \tag{106}$$

for finite ρ.

We note that in the general case, if $q^2\neq 1$, no bilinear relations between two operators a or two operators a^+ arise in this limiting case. For the projections we have

$$\langle 0|a_{r_2}a_{r_1}|\Psi\rangle = \Psi^{(2)}(r_1,r_2) + q\Psi^{(2)}(r_2,r_1), \tag{107}$$

$$\langle 0|a_{r_3}a_{r_2}a_{r_1}|\Psi\rangle = \Psi^{(3)}(r_1,r_2,r_3) + q\Psi^{(3)}(r_1,r_3,r_2) + q\Psi^{(3)}(r_2,r_1,r_3) + q^2\Psi^{(3)} \times (r_3,r_1,r_2) + q^2\Psi^{(3)}(r_2,r_3,r_1) + q^2\Psi^{(3)}(r_3,r_2,r_1). \tag{108}$$

We can see that the functions on the right-hand sides of Eqs. (107) and (108), generally speaking, do not have any symmetry, if only $q^2\neq 1$ (i.e., if we are not dealing with ordinary statistics). It is easy to formulate a general rule for the projection in terms of a combination of amplitudes:

$$\langle 0|a_{r_n}a_{r_{n-1}} \ldots a_{r_2}a_{r_1}|\Psi\rangle = \sum_{\mathscr{P}\in S_n} q^{\mathscr{M}(\mathscr{P})}\Psi^{(n)} \times (r_{\mathscr{P}1}, r_{\mathscr{P}2}, \ldots, r_{\mathscr{P}n}), \tag{109}$$

where $\mathscr{M}(\mathscr{P})$ is the minimum number of successive (neighboring) transpositions [compare Eq. (100), where we dealt with the minimum number of simple transpositions—permutations of two numbers occupying any locations].

By virtue of the fact that the projections (109) do not have in advance any symmetry, any combination, belonging to all irreducible representations of the permutation group S_n, can be constructed from them. In particular, there are no restrictions on the possible number of particles in symmetric or antisymmetric states. Such statistics without any restrictions on admissible Young's diagrams have been termed infinite statistics. We can see that such statistics can be continuous with the parameter q, characterizing a given infinite statistics, varying continuously.

The conditions (64) of positive-definiteness of the norms of symmetric ($\lambda=1$) or antisymmetric ($\lambda=-1$) vectors now assume the form [taking into account Eqs. (61) and (106)]

$$\sum_{k=0}^{n-1} x^k \geqslant 0 \tag{110}$$

for any $n=2,3,\ldots$. For even $n=2m$ $(m=1,2,\ldots)$ this condition means that

$$(1+x)\sum_{k=0}^{m-1} x^{2k} \geqslant 0 \tag{111}$$

and it always holds, if

$$-1\leqslant x. \tag{112}$$

For odd $n=2m+1$ $(m=1,2,...)$ the condition (110) always holds, since in this case it has the form

$$(1+x)\sum_{k=0}^{m-1} x^{2k}+x^{2m}\geqslant 0. \tag{113}$$

Thus the condition (112) is the only constraint for the parameter x. Recalling that $x=\lambda q$, and requiring that the condition (112) hold for both symmetric ($\lambda=1$) and antisymmetric ($\lambda=-1$) vectors, we obtain finally the admissible range of values of the parameter q:

$$-1\leqslant q\leqslant 1. \tag{114}$$

It can be shown that the condition of positive-definiteness of norms of any vectors does not impose any other restrictions on the parameter q.[16]

As one can see from the expressions (107) and (108), as q approaches its lower and upper limits the weight of the antisymmetric and symmetric projections increases correspondingly, and we obtain a continuous approach to the Fermi and Bose statistics. Recently, Greenberg[16] employed this fact in order to develop, on the basis of bilinear relations (105), the scheme of violation of ordinary statistics. As we can see, this turns out to be entirely possible, in contrast to the preceding scheme of finite parastatistics, on the basis of a scheme with only particles satisfying infinite statistics. Here, however, it is pertinent to recall the theorem, proved within the local algebra of observables by Fredenhagen,[40] on the fact that within this algebra it is impossible to formulate for infinite statistics the conjugate sector of antiparticles, and such statistics thereby contradicts locality. Moreover, in Sec. 15, where we switch to formulation of local relativistic fields, including both particles and antiparticles, we shall see that in the limit $p\to\infty$ the theory does indeed become nonlocal, but then even in this nonlocal theory the parameter q is found to be limited to only three values $q=\pm 1$, 0 as a result of the very presence of antiparticles. Here we merely note the definite behavior of the particle statistics for these values of q in the limit $p\to\infty$.

In the case of Green's quantization for $q=\pm 1$ it follows from Eq. (103) that the para-Fermi statistics ($\lambda=1$) pass in this limit into Bose statistics, and para-Bose statistics ($\lambda=-1$) pass into Fermi statistics! This change in the very character of the statistics in the limit for Green's quantization was discovered by Greenberg and Messiah.[54] Since for finite statistics the generalized Pauli theorem on the correct connection of spin to parastatistics holds, in this limiting case such a correct connection evidently breaks down. The reason for this breakdown lies precisely in the fact that the theory becomes nonlocal in this limit.

The limit $p\to\infty$ in the case $q=0$ has a different form. In this case, after the renormalization (104) and the substitution $q=0$ in Eq. (105), we obtain the simple bilinear relation

$$a_s a_r^+=\delta_{sr}. \tag{115}$$

This relation was proposed by Greenberg[19] directly as an example of infinite statistics. The relation (115) was derived in Ref. 10 as the limiting case of the trilinear relations (97) in the limit $p\to\infty$. According to Eqs. (107)–(109) in the case $q=0$ we obtain a one-to-one correspondence between the projections and the amplitudes:

$$\langle 0|a_{r_n}...a_{r_2}a_{r_1}|\Psi\rangle=\Psi^{(n)}(r_1,r_2,...,r_n). \tag{116}$$

The functions on the right-hand side of Eq. (116) do not exhibit any *a priori* symmetry. For this reason all states symmetrized according to the corresponding Young tableaux have equal weights and as a result we obtain the classical Maxwell–Boltzmann statistics![10,19] This fact can be interpreted as follows. In the next section it will be shown that any parastatistics of finite order can be interpreted as ordinary Bose or Fermi statistics in the presence of degeneracy with respect to some additional internal degree of freedom, whose number is equal to the order of the parastatistics. Passage of parastatistics to infinity corresponds to degeneracy with respect to an internal degree of freedom, which takes on an infinite number of values. It is these hidden degrees of freedom that can distinguish (in principle) particle states which are equally probable in the statistical sense. As a result, there arises the classical statistics for identical but distinguishable (according to the internal state) particles.

As we shall see in Sec. 15, quantum field theory, corresponding to the case $q=0$, likewise becomes nonlocal in the limit $q\to\infty$: In this theory the particles and antiparticles are quantized differently. Thus Fredenhagen's theorem remains valid in this case also.

We can now understand the significance of the contradiction of the limiting statistics with spin for Green's quantization. For such quantization only definite states are selected (in the Fock space) from all internal states. The antisymmetric function with respect to internal states of fermions (or bosons) corresponds to symmetric states in the case of parafermions (or antisymmetric states in the case of parabosons). As $p\to\infty$, the weight of these antisymmetric internal states with a fixed total internal state of the fermions (bosons) increases compared to other internal states, and as a result, for such distinguished internal states there arises "apparent Bose statistics" for fermions and "apparent Fermi statistics" for bosons, degenerate with respect to the infinite internal degree of freedom for its fixed total value. In the case $q=0$, however, such fixation does not occur and all possible values of statistics are possible.

In the latter case, for infinite statistics, determined by the relation (115), the particle number operator can no longer be written in a bilinear form. However, Greenberg[19] has proposed in this case for the particle number operator the following expression in the form of an infinite series:

$$N_r=a_r^+a_r+\sum_{r_1}a_{r_1}^+a_r^+a_ra_{r_1}+\sum_{r_1,r_2}a_{r_2}^+a_{r_1}^+a_r^+a_ra_{r_1}a_{r_2}+\dots . \tag{117}$$

It is easy to show that this operator indeed satisfies the properties which the particle number operator must satisfy:

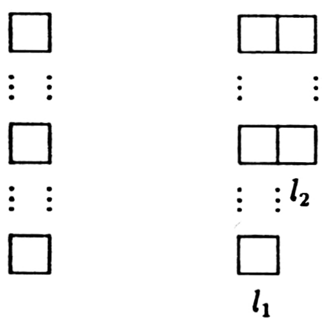
$$[N_r,a_s]_-=-\delta_{rs}a_s,\quad [N_r,a_s^+]_-=\delta_{rs}a_s^+ . \tag{118}$$

The question of particle statistics for intermediate values of the parameter q in permissible intervals $-q<q<0$ and $0<q<1$ have still not been investigated completely.[15)] It can be expected that in the limit $q\to\pm 1$ the weight of antisymmetric or symmetric states will increase, and we shall have a continuous passage from classical Maxwell–Boltzmann statistics (for $q=0$) to Bose and Fermi statistics. The particle number operator was recently constructed for such cases in Ref. 55.

10. INTERPRETATION OF PARASTATISTICS AS ORDINARY STATISTICS WITH EXACT DEGENERACY

We consider first the simplest example of second-order para-Fermi statistics, and then we extend this result to parastatistics of arbitrary order.

In this case more than two particles cannot occupy a symmetric state. The functions satisfying this condition are constructed from arbitrary functions according to the rule (100) with $p=2$. Such functions decompose into irreducible representations, described by Young tableaux with not more than two columns:



In this case each irreducible representation (l) of the group S_n of permutations of the states of n identical particles is determined uniquely by two integers: the lengths of these columns

$$(l_1,l_2),\quad l_1\geqslant l_2\geqslant 0,\quad l_1+l_2=n, \tag{119}$$

and its dimension is

$$N_l=n!(l_1-l_2+1)/(l_1+1)!l_2!$$

For a fixed number n of particles this irreducible representation is uniquely characterized by only one parameter: the difference of the column lengths. For what follows it will be convenient for us to introduce the half-difference

$$I=(l_1-l_2)/2. \tag{120}$$

This parameter can be regarded as an observable characterizing the state of the particles which is determined by the symmetry of the given irreducible representation. We call it the "isospin."

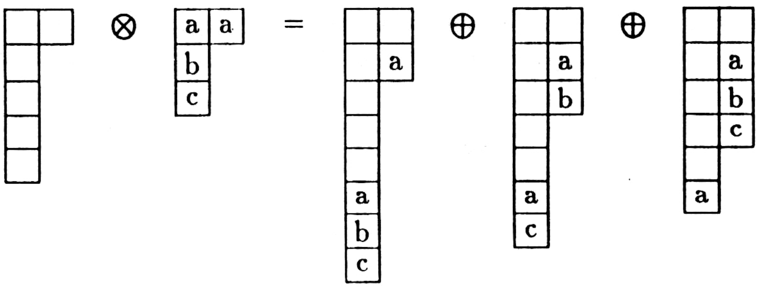
We now show that when two systems of n_1 and n_2 particles, characterized by the isospins I_1 and I_2, are combined the total isospins of these systems indeed add as isospins. Let these two systems be characterized by the Young tableaux with column lengths $(l_1^{(i)},l_2^{(i)})$, $i=1,2$. Then the corresponding isospins will be

$$I_i=(l_1^{(i)}-l_2^{(i)})/2,\quad i=1,2. \tag{121}$$

Combining these two systems yields a system consisting of $n=n_1+n_2$ particles. Generally speaking, to this total system there corresponds an irreducible representation of the group S_n that is constructed as a direct (exterior) product of the two initial irreducible representations. Its basis vectors are obtained as all possible products of the basis vectors of the systems being multiplied. But, since the particles are identical, it is necessary to sort through all possible partitions of n particles, enumerated as $1,2,\dots,n$, into n_1 particles with any numbers and n_2 particles with the remaining numbers.[16)] Further, each of these products must be symmetrized according to the rule (100) (with $p=2$) in accordance with the admissible class of functions for the given parastatistics.

The space consisting of such products of basis vectors must now be partitioned into irreducible representations of S_n, and we must thereby find which isospins arise when the isospins I_1 and I_2 are added. But in so doing it should be kept in mind that for such a partitioning, for the given class of functions, only Young tableaux with not more than two columns arise.

The indicated partitioning can be performed with the help of the following graphical method (see Ref. 56, p. 297). Assume that we are multiplying two Young tableaux, corresponding to two irreducible representations S_{n_1} and S_{n_2}, as shown in the diagram below for the case $n_1=6$ and $n_2=4$. In the second tableau all cells in the first row must be filled with the same letter a, all cells in the second row must be filled with the letter b, and so on.



$I_1=2 \qquad I_2=1 \qquad I=3 \qquad I=2 \qquad I=1$

We now add to the first Young tableau the cells of the second tableau according to the following rules: 1) two identical letters must never appear in the same column and 2) with each addition of these letters the correct *lattice schemes* must be obtained. This means that when at each step a "word" consisting of the added letters is constructed reading according to the "eastern" method from right to left and top to bottom the number of letters a must not be less than the number of letters b, the number of letters b must be less than the number of letters c, and so on (in our example such "eastern words" are *cbaa*, *caba*, *acba* and, for example, the "word" *baca*, which we would have obtained if we had interchanged b and c on the right-hand side in the second diagram, is forbidden).

Obviously, such multiplication is commutative and the multiplied schemes can be interchanged. They can always be chosen so that

$$l_1^{(1)}-l_2^{(1)}\geqslant l_1^{(2)}-l_2^{(2)}, \text{ i.e. } I_1\geqslant I_2. \tag{122}$$

With the help of the rule presented above it is easy to prove that the following irreducible representations of S_n are obtained:

$$\begin{aligned}&(l_1^{(1)}+l_1^{(2)}, l_2^{(1)}+l_2^{(2)}),\\ &(l_1^{(1)}+l_1^{(2)}-1, l_2^{(1)}+l_2^{(2)}+1),\\ &\ldots\\ &(l_1^{(1)}+l_2^{(2)}, l_2^{(1)}+l_1^{(2)}).\end{aligned} \tag{123}$$

The corresponding isospins are

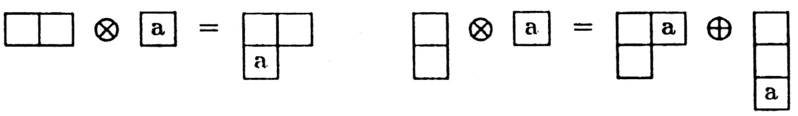
$$I=I_1+I_2,\quad I_1+I_2-1,\ldots,\quad I_1-I_2.$$

If the case when $I_1\leqslant I_2$ is also included, then the addition rule has the form

$$I=I_1+I_2,\quad I_1+I_2-1,\ldots,\quad |I_1-I_2|. \tag{124}$$

Thus this rule indeed corresponds to the rule for adding ordinary isospins.

We can introduce isospins $\iota=1/2$ for each separate particle (cell) and construct the total isospin of a system of n particles by successively adding the isospins. For example, adding the isospins of two particles gives isospins $I=0$ and $I=1$, corresponding to symmetric and antisymmetric wave functions. If a third particle is now added, then for second-order parastatistics the following tableaux can arise:



$I=0 \quad \iota=\frac{1}{2} \quad I=\frac{1}{2} \qquad I=1 \quad \iota=\frac{1}{2} \quad I=\frac{1}{2} \quad I=\frac{3}{2}$

Here the same final isospins (in our example $I=1/2$) can arise with different methods for adding the isospins of separate particles. Such identical final states can be characterized with the help of the symmetry of the subsystem obtained by removing a single particle from the system. It should be noted that we are talking not about removing some enumerated (say, the third) particle, but rather physical removal of one of the *identical* particles.[31]

We now consider, as an example, the case of three parafermions (of second order), whose state belongs to a mixed representation. It is determined by a set of two functions:

$$\Psi'_m(x_1,x_2,x_3)=[\Psi(x_1,x_2,x_3)+\Psi(x_2,x_1,x_3)-\Psi(x_3,x_2,x_1)-\Psi(x_3,x_1,x_2)]/2, \quad (125)$$

$$\Psi''_m(x_1,x_2,x_3)=[\Psi(x_1,x_2,x_3)+2\Psi(x_1,x_3,x_2)-\Psi(x_2,x_1,x_3)-2\Psi(x_2,x_3,x_1)-\Psi(x_3,x_2,x_1)+\Psi(x_3,x_1,x_2)]/(2\sqrt{3}). \quad (126)$$

We now remove *one of the identical particles*, i.e., we assume that the wave function of the three particles can be represented as a product

$$\Psi(x_1,x_2,x_3)=f(x_1,x_2)\psi(x_3), \quad (127)$$

under the condition that the regions where the functions f and ψ are different from zero overlap. For the probability of finding identical particles at the points x_1, x_2, and x_3, we must construct an expression that is *invariant under permutations of the particle coordinates*:

$$W(x_1,x_2,x_3)=[|\Psi'_m(x_1,x_2,x_3)|^2+|\Psi''_m(x_1,x_2,x_3)|^2]/2=\{[|f_s(x_1,x_2)|^2+(1/3)|f_a(x_1,x_2)|^2]|\psi(x_3)|^2+\text{two analogous terms obtained by cyclic transposition of the indices: (2,3,1) and (3,1,2)}\}/4, \quad (128)$$

where

$$f_s(x_1,x_2)=[f(x_1,x_2)+f(x_2,x_1)]/\sqrt{2}, \quad (129)$$

$$f_a(x_1,x_2)=[f(x_1,x_2)-f(x_2,x_1)]/\sqrt{2}. \quad (130)$$

We now prepare a definite state of two close identical particles (say, by measuring their total relative momentum), for example, a symmetric state, such that

$$f(x_1,x_2)=f(x_2,x_1). \quad (131)$$

Next we include the third, removed particle. The vectors (125) and (126) will become expressions similar to Eqs. (50) and (51):

$$\Psi'_m(x_1,x_2,x_3)=[2f_s(x_1,x_2)\psi(x_3)-f_s(x_3,x_2)\psi(x_1)-f_s(x_3,x_1)\psi(x_2)]/\sqrt{6}, \quad (132)$$

$$\Psi''_m(x_1,x_2,x_3)=[f_s(x_1,x_3)\psi(x_2)-f_s(x_2,x_3)\psi(x_1)]/\sqrt{2}. \quad (133)$$

The probability of observing particles at the points x_1, x_2, and x_3 will be, according to Eqs. (128), (132), and (133),

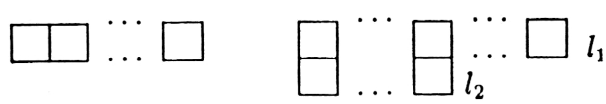
$$W(x_1,x_2,x_3)=[|f_s(x_1,x_2)|^2|\psi(x_3)|^2+|f_s(x_2,x_3)|^2|\psi(x_1)|^2+|f_s(x_3,x_1)|^2|\psi(x_2)|^2]/3. \quad (134)$$

A similar expression can also be obtained for a system of three particles, having a subsystem of two particles in an antisymmetric state.

The example worked out above shows that every state n of identical particles can be characterized with the help of their intrinsic symmetry and the symmetry of the corresponding subsystems of $n-1$, $n-2$, ..., 2 identical particles.[17] The same rule corresponds to the rule for adding the isospins of the subsystems in order to form the total spin.

Obviously, we can now introduce for each particle an *auxiliary* isospin variable ξ, taking on two values: $\xi=\pm 1/2$. The total isospin space for the particles is obtained as the direct product of all single-particle isospin spaces: $\xi^n=\xi\otimes\xi\otimes\ldots\otimes\xi$. The total isospin function, constructed in the form of a product of isospinors for each particle, can be determined in this space.

Transformations of the group $SU(2)$ are defined on spinors, and its irreducible representations are determined by symmetrized combinations of spinor functions in accordance with Young tableaux containing *not more than two rows*:



Each such irreducible representation is characterized by isospin I, determined by Eq. (120), and has $2I+1$ basis vectors, corresponding to different projections of the total isospin:

$$I_3=\sum_{i=1}^{n}\iota_3(\xi_i). \quad (135)$$

It should be noted that the entire irreducible representation of the group $SU(2)$ is associated with each vector of the irreducible representation of S_n, represented by the given Young tableau, and all such representations are equivalent.

It is now clear how to construct a completely antisymmetric (fermion) wave function from the wave functions of parafermions, belonging to a given irreducible representation (l) of the group S_n. The basis vectors of this representation must be multiplied by the basis vectors of the conjugate[18] irreducible representation (l) in the isotopic space ξ^n:

$$\Psi^{(l)}_{I,M}(x_1,\xi_1;\ldots;x_n,\xi_n)=\sum_{i=1}^{N_l}\Psi^{(l,i)}(x_1,\ldots,x_n)\times\tilde{\phi}^i_{\alpha'\ldots\alpha^{(n)}}(\xi_1,\ldots,\xi_n), \quad (136)$$

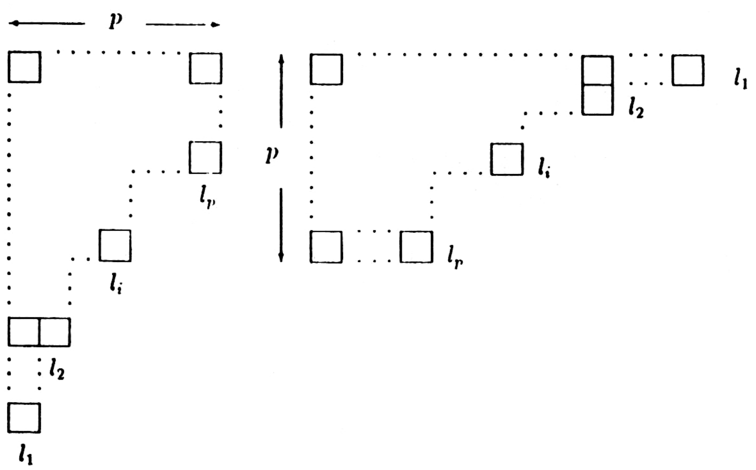
where $M=\alpha'+...+\alpha^{(n)}$. In so doing there arise $2l+1$ functions, differing by the projections of the isospin M and forming the basis of the irreducible representation of the group $SU(2)$ with total isospin $I=(l_1-l_2)/2$. We now fix some (it is not important which one) value of M and average the probability over the auxiliary variables $\xi_1, ..., \xi_n$. Since the basis vectors $\tilde{\phi}$ are orthogonal, we obtain the following expression for the probability:

$$W^{(l)}(x_1,...,x_n)=\sum_{\xi_1,...,\xi_n}|\Psi^{(l)}_{I,M}(x_1,\xi_1;...;x_n,\xi_n)|^2$$
$$=\sum_{i=1}^{N_l}|\Psi^{(l,i)}(x_1,...,x_n)|^2. \quad (137)$$

Thus this expression can be regarded, on the one hand, as the probability of finding *identical parafermions* at the points $x_1,...,x_n$ that is invariant under permutations of these points and, on the other hand, as the probability of finding *fermions* at these points, but having additional isospin over which averaging is performed. Similar expressions can also be written for other observables.

Since the probability (137) does not depend on the projection M of the isospin I, the theory will be invariant under $SU(2)$ transformations in isospin space. The second-order para-Fermi statistics is indeed found to be equivalent to ordinary Fermi statistics in the presence of *exact* $SU(2)$ symmetry in the auxiliary isospin space.

Obviously, our analysis can be easily extended to para-Fermi or para-Bose statistics of arbitrary order p, when the wave functions (100) admit a Young tableau with not more than p columns in the case of para-Fermi statistics



Similarly, for para-Bose statistics diagrams with not more than p rows are allowed.

Every irreducible representation is characterized by p integers

$$(l)=(l_1,l_2,...,l_p), \quad (138)$$

such that

$$l_1\geqslant l_2\geqslant...\geqslant l_p\geqslant 0, \quad l_1+l_2+...+l_p=n. \quad (139)$$

For fixed n we have $p-1$ parameters

$$L_1=l_1-l_2, \quad L_2=l_2-l_3,..., \quad L_{p-1}=l_{p-1}-l_p. \quad (140)$$

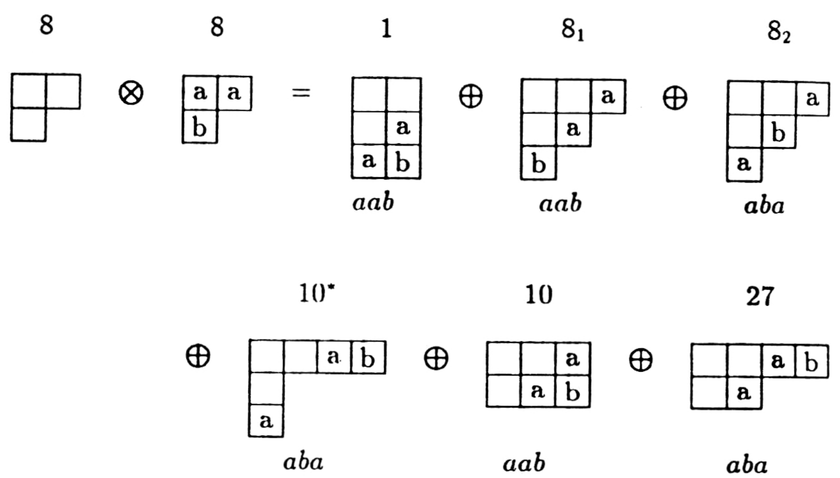
The dimension of the representation is

$$N_l=n!\prod_{j>i=1}^{p}(l_i-l_j-i+j)/[(l_1+p-1)!$$
$$\times(l_2+p-2)!...l_p!].$$

A system of n paraparticles is constructed from two subsystems with n_1 and n_2 paraparticles $(n_1+n_2=n)$ in exactly the same way as for the case $p=2$. Decomposition of the exterior product of the irreducible representations of the groups S_{n_1} and S_{n_2} into irreducible representations of the group S_n is performed according to the same rule. In so doing it is necessary to take into account the fact that only tableaux with not more than p columns in the case of para-Fermi statistics and not more than p rows in the case of para-Bose statistics can appear in such a decomposition.

In the case of para-Fermi statistics we construct the Young tableaux determining for paraparticles the irreducible representations of the group S_n which are the *conjugate* ("flipped") Young tableaux containing not more than p rows, as shown in our tableau (for para-Bose statistics the initial tableaux themselves have this form, and in this case the *same* tableaux must be taken). To these conjugate tableaux we can associate the irreducible representations of the group $SU(p)$ in some auxiliary p-dimensional *unitary space*. Then all *invariants* of the group $SU(p)$ can be expressed in terms of the parameters (142) of the initial tableau.[19] Thus every irreducible representation of paraparticles can be characterized by the values of the invariants—Casimir operators for the conjugate tableau of the group $SU(p)$. We note that the reducible representations can always be represented as a direct product of irreducible representations.

The decomposition of the exterior product of the groups S_{n_1} and S_{n_2} in representations of the group S_n for paraparticles corresponds to decomposition of the *inner* product of representations of the group $SU(p)$ into *its* irreducible representations (this is why such a product is called "inner"). But the latter decomposition is performed *according to the same* rules (see Ref. 58, p. 94) as the initial decomposition of the exterior product of irreducible representations of paraparticles according to representations of S_n, but now the successively (at each step) arising "words" must be read according to the "European" method (from left to right), as shown in the tableau below for the case $SU(3)$ for the product of two octets.



The dimension of the irreducible representation of the group $SU(p)$ is computed with the help of the following formula (see Ref. 59, p. 275):

$$N[l]=\prod_{k,i=1}^{p}(l_i-l_k-i+k)/(k-i).$$

In the case of $SU(3)$ we have

$$N[l]=(L_1+1)(L_2+1)(L_1+L_2+2)/2.$$

These dimensions are indicated above the corresponding Young tableaux.

We conclude that the parastatistics of any order p can be associated to a group $SU(p)$, whose invariants are Casimir operators determined by the numbers (L) and characterize the symmetry of the irreducible representations of the paraparticles, and they are observables. In the general case, there does not exist a simple rule for adding such operators, as happened for the isospin. The values of these operators for a given representation must be determined directly from the type of tableaux obtained when the product of Young tableaux is decomposed into irreducible parts.

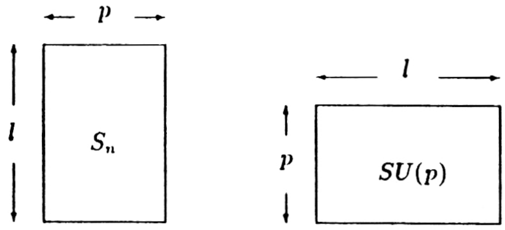
In the general case of arbitrary order, just as in second order, any state of n (identical) paraparticles can be characterized not only with the help of their intrinsic Young symmetry, but also with the help of the symmetry of the corresponding subsystems of $n-1$, $n-2$, ..., 2 (identical) paraparticles. Just as in that case, to such clusterization there corresponds a set of values of the Casimir operators of the group $SU(p)$ which characterize the symmetry of the subsystems of paraparticles.

Moreover, we can once again introduce explicitly additional unitary coordinates for the particles $\xi_1, \xi_2, ..., \xi_n$, each of which can assume p values. Next, introducing the corresponding functions in the space ξ^n, we can construct completely antisymmetric in the case of para-Fermi (or symmetric in the case of para-Bose) statistics wave functions in $x^n \otimes \xi^n$ space. Para-Fermi (para-Bose) statistics arise from the ordinary Fermi (Bose) statistics when the probabilities and average quantities are averaged over the coordinates of the auxiliary space.

We can now compare the results obtained to the results of classification of statistics based on the local algebra of observables.[32–34] We can associate the Casimir operators, which we introduced, of the accompanying parastatistics of the group $SU(p)$ to "generalized charge quantum numbers" and characterize the "superselection sectors" with the help of the values of these operators. Further, the order of the parastatistics must itself be associated to the "parameter of statistics," introduced in these works and simply equal to $-\lambda p^{-1}$ (in our notation $\lambda=-1$ for para-Bose statistics and $\lambda=1$ for para-Fermi statistics).

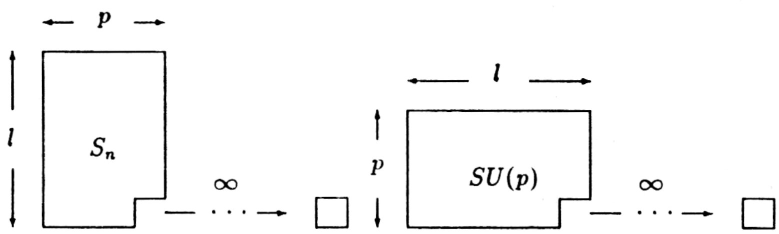
11. NONRELATIVISTIC DEFINITION OF ANTIPARTICLES AND INFINITE STATISTICS

We now consider for parafermions of order p tableaux in the form of rectangles with p columns and an arbitrary number of rows l:



The corresponding conjugate diagrams, shown on the left, will determine the *singlet* representations of $SU(p)$, since in this case all numbers $L_1=L_2=...=L_{p-1}=0$. Since the vacuum state (the state with no particles) is also a singlet state, such tableaux correspond to *vacuum-like* states of parafermions with respect to the "generalized quantum numbers" (Casimir operators). In this case we shall talk about a system of particles consisting of *p shells* completely filled with paraparticles. (We note that with regard to the permutation group we do not have any one-dimensional representations in this case, only if the diagram does not consist of one row or one column.)[20]

Now we remove from some vacuum-like state a single paraparticle and place it in an asymptotically distant (space-like) region, so that its wave function does not overlap with the wave functions of the other particles:



The tableaux obtained correspond to conjugate tableaux, according to which the representations of the group $SU(p)$ transform. The remaining particles correspond to the tableau that is the complement, in this group, to a single-particle tableau. Both are characterized *by the same* "charge quantum numbers" (Casimir operators), since on the whole they form a $SU(p)$-singlet representation. We can say that the removed particle and the remaining particles compensate each other's "charge quantum numbers" and form as a whole a vacuum-like state. With respect to these numbers we can call the system of remaining particles a *hole* in the vacuum-like state or an *antiparticle*.

It should be noted that combining such an *antiparticle* with its independent particle does not necessarily lead to a vacuum-like state: the decomposition of the product of the corresponding Young tableaux into irreducible representations contains both a rectangular tableau and a *hole* with a single-particle tableau attached to it at the bottom. But it is important that such a product always contains a vacuum-like (rectangular) state.

Similarly, several particles, instead of one particle, can be removed to infinity from the vacuum-like state. The remaining tableau can be regarded as a collection of "holes" with respect to the removed particles. If the m removed particles transform according to the representation (l) with respect to the permutation group S_m and according to the representation $[l]$ with respect to the group $SU(p)$, then the remaining particles will transform according to the corresponding complementary[21)] diagrams (l^*) and $[l^*]$. We shall call the paraparticle sector determined by these representations and characterized by the corresponding values of the Casimir operators of the group $SU(p)$ the "complementary," with respect to the sector, particles removed to infinity, and we call the particles remaining in it the "antiparaparticles" with respect to the initial paraparticles. It is obvious that these definitions are interchangeable and also that the antiparaparticles satisfy the same parastatistics as the particles. When the product of tableaux $[l]\times[l^*]$ is decomposed into irreducible parts, the irreducible parts always contain a vacuum-like representation.

Our construction corresponds precisely to the construction of the analogous "conjugate sector" within the local algebra of observables.[32] Now it is also easy to understand the possibility of the existence of such a sector in the case of infinite statistics.[40]

We now consider the limit $p\rightarrow\infty$. In this case no restrictions are imposed on the number of particles in symmetric or antisymmetric states and any Young tableaux are now admissable. The group $SU(p\rightarrow\infty)$, for which the number of "generalized charges"—Casimir operators—becomes infinite, corresponds to the "inner" symmetry. The state of the paraparticles must be characterizing by prescribing an infinite set of values of these operators. In this case it is impossible to supplement any system consisting of a finite number of particles by another system consisting of a finite number of particles in a manner so that for the entire system as a whole the entire infinite sequence of Casimir operators vanishes. In other words, the product of arbitrary finite Young tableaux does not contain a singlet representation. Therefore, in this case the state of the particles which is equivalent to the singlet vacuum state cannot be defined, and correspondingly the particle sectors complementing one another with respect to this state cannot be defined.

Fredenhagen[40] proved that within a local algebra of observables it is always possible to construct on the basis of the postulates of the algebra a "conjugate" (in our terminology "complementary") sector. Since infinite statistics, as we can see, does not permit defining such a sector, such statistics is found to be impossible within a local algebra of observables. This is the content of Fredenhagen's theorem.

In what follows we shall consider the limit of infinite statistics within a local field theory. We shall see that although the field theory corresponding to finite parastatistics is local, as the order of the statistics approaches infinity the theory becomes *nonlocal*, since particles and antiparticles start to behave differently.

We underscore the fact that the nonrelativistic definition, given above, of antiparticles refers only to the properties of the antiparticles with respect to "inner charge quantum numbers," but, of course, it does not mean that the masses of the particles and "holes" are equal. However, by extending this definition to relativistic field theory we can associate to the particles and antiparticles, as usual, positive- and negative-frequency solutions.

As is well known, when Dirac encountered the problem of the appearance of negative-energy states in his equation, he proposed filling the states in advance according to the Pauli principle, with electrons and assuming that the real vacuum is an infinite sea of electrons, filling all negative-energy levels. But, obviously, these levels could have been filled not by one, but rather by 2, 3, and so on, in general, an arbitrary finite number of particles. Then it could be conjectured that the particles satisfy not Fermi statistics but rather para-Fermi statistics of some finite order. The "holes" in such a parafermion Dirac vacuum would play the role of antiparafermions. This scheme is logically no different from the original scheme, and thereby para-Fermi statistics has the same right to exist as Fermi statistics. It is well known that electrons satisfy Fermi statistics, but quarks could be easily regarded, as proposed by Greenberg,[60] not as fermions, degenerate with respect to "color," but as third-order parafermions. A special question here, however, is the possibility of introducing into such a scheme a color gauge group. It was recently found[11] that new paraquantization in this respect could have some advantages over the Green's paraquantization.[43,61]

If it is supposed, however, that the order of the para-Fermi statistics approaches infinity, then, obviously, such filling of negative-energy levels becomes impossible. Correspondingly, antiparticles cannot, in this case, be defined as "holes" in a Dirac vacuum.

Finally, it should be noted that we compared the parastatistics to the inner group $SU(p)$, thereby excluding phase transformations and the associated abelian charges. It is easy to include such transformations by introducing the group $U(p)=U(1)\times SU(p)$ and regarding, as Dirac proposed, the abelian charges of the vacuum particles to be unobservable directly, i.e., by setting the abelian charges of the filled p-shells of the paraparticles equal to zero. Then the absence of particles in such a p-shell will be manifested as a particle with abelian charge of opposite sign together with the presence of additional nonabelian charges in the shell.

We note that everything we have said above is true not only for our choice of class of functions (100), corresponding to the new paraquantization (97), but also for Green's paraquantization (88), for which, however, additional constraints are imposed on the functions, as shown for the case of second-order parastatistics. Such constraints lead to the fact that the generalized charge quantum numbers can be not only the Casimir operators of the group $SU(p)$ but also some additive operators of the group, such as the third projection of the isospin, strangeness, and so on. This opens up the possibility of describing physical symmetries and their breaking on the basis of Green's paraquantization.[9,39]

12. GENERALIZED QUANTIZATION OF FIELDS

We now attempt to answer the question: What type of field theory corresponds to parastatistics?

For definiteness, as an example of a field with integer spin we consider the scalar field

$$\varphi(x)=(2\pi)^{-3/2}\int d^3k(2E_{\mathbf{k}})^{-1/2}(a_{\mathbf{k}}e^{-ikx}+b_{\mathbf{k}}^{+}e^{ikx}), \tag{141}$$

and as a field with half-integer spin we consider the Dirac field

$$\psi(x)=(2\pi)^{-3/2}\int d^3k(m/E_{\mathbf{k}})^{1/2} \times\sum_{\sigma=\pm1/2}[a_{\sigma,\mathbf{k}}u(\sigma,\mathbf{k})e^{-ikx} +b_{\sigma,\mathbf{k}}^{+}v(\sigma,\mathbf{k})e^{ikx}], \tag{142}$$

where k is the momentum, $k_0=E_{\mathbf{k}}=(m^2+\mathbf{k}^2)^{1/2}$, and σ is the spin state; $u(\sigma,\mathbf{k})$ and $v(\sigma,\mathbf{k})$ are known Dirac four-component spinors, corresponding to positive and negative frequency solutions of the Dirac equation, respectively.

We assume that the coefficients a and b^+ are particle *annihilation* and antiparticle *creation* operators, respectively. We define for the fields commutation relations which correspond to the commutation relations (29) and (30) for particle creation and annihilation operators. For the scalar field we postulate

$$[[\varphi(x),\varphi^+(y)]_{-q},\varphi(z)]_-=i\rho\Delta(z-y)\varphi(x), \tag{143}$$

$$[[\varphi(x),\varphi^+(y)]_{-q},\varphi^+(z)]_-=-i\rho\Delta(x-z)\varphi^+(y), \tag{144}$$

where, as previously,

$$[\varphi(x),\varphi^+(y)]_{-q}=\varphi(x)\varphi^+(y)-q\varphi^+(y)\varphi(x), \tag{145}$$

the field $\varphi^+(x)$ is the hermitian conjugate of the field $\varphi(x)$, and $\Delta(x)$ is the well-known (odd) function that is singular on the light cone and vanishes outside the light cone (see, for example, Ref. 62):

$$\Delta(x)=\frac{-i}{2(2\pi)^3}\int\frac{d^3k}{E_{\mathbf{k}}}(e^{-ikx}-e^{ikx}). \tag{146}$$

The commutator of the two forms (145) is defined by the expression[22)]

$$[[\varphi(x),\varphi^+(y)]_{-q},\ \ [\varphi(z),\varphi^+(u)]_{-q}]_-$$
$$=i\rho\Delta(z-y)[\varphi(x),\varphi^+(u)]_{-q}-i\rho\Delta(x-u)$$
$$\times[\varphi(z),\varphi^+(y)]_{-q}. \tag{147}$$

It vanishes if the points z and y and also the points x and u are separated by space-like intervals. Thus if the currents and other observables have the form (145), then the theory will be local in the sense of commutation of two observables, belonging to space-like separated regions of space-time.

Similarly, for the spinor field we postulate the relations

$$[[\psi(x),\tilde{\psi}(y)]_{-q},\psi(z)]_-=-i\rho S(z-y)\psi(x), \tag{148}$$

$$[[\psi(x),\tilde{\psi}(y)]_{-q},\tilde{\psi}(z)]_-=i\rho S(x-z)\tilde{\psi}(y), \tag{149}$$

where $\tilde{\psi}=\psi^+\gamma_0$ and $S(x)$ is the well-known singular function for the Dirac field:

$$S(x)=-(i\gamma^\mu\partial_\mu+m)\Delta(x). \tag{150}$$

In the general case the generalized relations (143) and (148) are noninvariant under charge conjugation:

for the scalar field

$$\varphi(x)\to\varphi_c(x)=\varphi^+(x),\quad \varphi^+(x)\to\varphi_c^+(x)=\varphi(x), \tag{151}$$

for the spinor field

$$\psi(x)\to\psi_c(x)=C\tilde{\psi}^T(x),$$
$$\tilde{\psi}(x)\to\tilde{\psi}_c(x)=[C^{-1}\psi(x)]^T, \tag{152}$$

where ψ^T is the transposed (bi)spinor and C is the charge-conjugation matrix. Indeed, under such a transformation the order of the products in the inner $(-q)$-mutator is reversed, and we obtain *different* (compared to the initial) relations, if only $q^2\neq 1$. The theory remains C-invariant only in the particular case of Green's quantization, when $q^2=1$ $(q=\pm 1)$.

Thus it turns out that the $q^2\neq 1$ theory under consideration is unsuitable for describing neutral fields, such as the neutral scalar field or the Majorana spinor field. For such fields the only suitable quantization turns out to be Green's method.

On the other hand, the theory remains invariant under spatial reflections:

$$\varphi(x)\to\varphi(i_sx),\quad \psi\to\eta_P\gamma^0\psi(i_sx),$$
$$\eta_P^2=\pm 1,\quad i_sx=(x_0,-\mathbf{x}).$$

But the invariance under *antiunitary* time reversals

$$T(i_t)\varphi(x)T^{-1}(i_t)=\varphi(i_tx),$$
$$T(i_t)\psi(x)T_{-1}(i_t)=\eta_tC^{-1}\gamma_5\psi(i_tx),\quad i_t=(-x_0,\mathbf{x})$$

also does not hold. As any local theory, this theory exhibits CPT invariance, but, in the general case it turns out to be invariant only with respect to combined CT parity: Under time reversal particles and antiparticles must be interchanged. The only exception is Green's quantization, invariant with respect to all three operations C, P, and T, separately.

Finally, we note that the commutation relations (143) and (148), which we postulated above, ensure that the field theory is canonically self-consistent. We demonstrate this for a Dirac field, writing its Hamiltonian in the form

$$\mathscr{H}_{\text{Dir.}}=-\rho^{-1}\int d^3x[(-i\gamma\cdot\nabla+m)_{\mu\nu}$$
$$\times\psi_\nu(x),\tilde{\psi}_\mu(x)]_{-q}+\text{const.} \tag{153}$$

If this Hamiltonian is now substituted as the generator of time translation into Heisenberg's equation

$$-i\partial\psi(x)/\partial t=[\mathscr{H}_{\text{Dir.}},\psi(x)]_-,$$

then, by virtue of the relations (148) and the projection property of the singular function

$$\psi_\kappa(x)=i\int d^3yS_{\kappa\mu}(x-y)\gamma^0_{\mu\nu}\psi_\nu(y),$$

we obtain the free Dirac equation

$$(-i\gamma^\mu\partial_\mu+m)\psi(x)=0.$$

A similar conclusion is also valid for the scalar field, if the Hamiltonian of the field likewise has the form of a $(-q)$-mutator, just as all other observables.

13. RELATIONS FOR PARTICLE AND ANTIPARTICLE OPERATORS

Substituting the expansions of the free fields (141) into Eqs. (143) and (144) and Eq. (142) into Eqs. (148) and (149) gives, using the representation (146) for the scalar Δ function, the following commutation relations for the particle and antiparticle creation and annihilation operators:

$$[[a_k,a_{k'}^+]_{-q},a_{k''}]_-=\rho\delta_{k'k''}a_k, \tag{154a}$$

$$[[a_k,a_{k'}^+]_{-q},a_{k''}^+]_-=-\rho\delta_{k'k''}a_{k'}^+, \tag{154b}$$

$$[[a_k,b_s]_{-q},a_{k'}]_-=0, \tag{154c}$$

$$[[b_s^+,a_k^+]_{-q},a_{k'}^+]_-=0, \tag{154d}$$

$$[[b_s^+,a_k^+]_{-q},a_{k'}]_-=\rho\delta_{kk'}b_s^+, \tag{154e}$$

$$[[a_k,b_s]_{-q},a_{k'}^+]_-=-\rho\delta_{kk'}b_s, \tag{154f}$$

$$[[b_s^+,b_{s'}]_{-q},a_k]_-=0, \tag{154g}$$

$$[[b_s^+,b_{s'}]_{-q},a_k^+]_-=0, \tag{154h}$$

$$[[a_k,a_{k'}^+]_{-q},b_s^+]_-=0, \tag{154i}$$

$$[[a_k,a_{k'}^+]_{-q},b_s]_-=0, \tag{154j}$$

$$[[a_k,b_s]_{-q},b_{s'}^+]_-=\mp\rho\delta_{ss'}a_k, \tag{154k}$$

$$[[b_s^+,a_k^+]_{-q},b_{s'}]_-=\pm\rho\delta_{ss'}a_k^+, \tag{154l}$$

$$[[b_s^+,a_k^+]_{-q},b_{s'}^+]_-=0, \tag{154m}$$

$$[[a_k,b_s]_{-q},b_{s'}]_-=0, \tag{154n}$$

$$[[b_s^+,b_{s'}]_{-q},b_{s''}^+]_-=\mp\rho\delta_{s's''}b_s^+, \tag{154o}$$

$$[[b_s^+,b_{s'}]_{-q},b_{s''}]_-=\pm\rho\delta_{ss''}b_{s'}. \tag{154p}$$

The upper sign in Eqs. (154k, l, o, p) refers to the scalar field and the lower sign refers to the spinor field. These signs will play an important role in establishing the connection of spin to parastatistics. We assume that the system is located in a bounded three-dimensional volume and the momenta assume discrete values. In what follows we designate states with the indices r', r'', r''', and so on for particles and indices s', s'', s''', and so on for antiparticles, and we include the spin states in these indices.

Comparing Eq. (154a) to Eq. (29) shows that for the particles we have once again obtained our initial commutation relations. As one can see by comparing Eqs. (154a) and (154p), however, the relations for the particles and antiparticles are *different*, if only $q^2\neq 1$.

14. FOCK REPRESENTATION FOR ANTIPARTICLES AND CONNECTION OF SPIN TO PARASTATISTICS

We now postulate the existence of a *unique* vacuum vector for both particles and antiparticles, requiring that this vector satisfy not only the condition (35) but also the condition that it not contain antiparticles:

$$b_s|0\rangle=\langle 0|b_s^+=0 \text{ for all stages of } s. \tag{155}$$

We can now prove that, similarly to Eq. (36),

$$b_s b_{s'}^+|0\rangle=p_c\delta_{ss'}|0\rangle, \tag{156}$$

where p_c is a numerical parameter, which we designated with a suffix c in order to indicate that it belongs to antiparticles. However, the proof of the relation (156) with $q=0$ is different from the proof for $q\neq 0$, and we shall examine these cases separately.

The case $q\neq 0$. The relations (154o, p) for antiparticles can be rewritten in a form similar to Eqs. (29) and (30):

$$[[b_s,b_{s'}^+]_{-q_c},b_{s''}]_-=\rho_c\delta_{s's''}b_s, \tag{157}$$

$$[[b_s,b_{s'}^+]_{-q_c},b_{s''}^+]_-=-\rho_c\delta_{ss''}b_{s'}^+, \tag{158}$$

where we have introduced the notation

$$q_c=1/q, \quad \rho_c=\mp\rho/q. \tag{159}$$

As previously, the upper sign in Eq. (159) refers to the scalar case and the lower sign refers to the spinor case.

We can prove once again Eq. (156) by operating on the vacuum vector by both sides of Eq. (157) and using the fact that the vacuum vector is unique. Next, requiring that the norm of the state vector of a single antiparticle be positive-definite we obtain the condition

$$p_c\geqslant 0. \tag{160}$$

Moreover, making the assumption that the parameter p_c is bounded, we can prove, exactly as for particles, the theorem I for antiparticles: *for finite p_c the number of particles in a λ_c-symmetric (symmetric with $\lambda_c=1$ and antisymmetric with $\lambda_c=-1$) state must be finite and cannot exceed some integer M_c. In addition, since the norm of the antiparticle state vectors in Fock space is positive-definite, the only admissible values of the parameter $x_c=\lambda_c q_c$ are 0 and 1.*

We note immediately that the case $x_c=q_c=0$ reduces to the case $x=q=0$, and for this reason we shall not consider it.[23)]

Thus for finite p_c the only possibility is

$$x_c=\lambda_c q_c=\lambda_c/q=1, \tag{161}$$

where we made the substitution (159). Hence follows immediately

$$q=\lambda_c=q_c, \tag{162}$$

since $q_c^2=1$. Next, according to theorem I for particles with $q\neq 0$ we have the single value

$$x=\lambda q=1 \text{ and } q=\lambda. \tag{163}$$

We conclude from Eqs. (162) and (163) that

$$q=q_c=\lambda=\lambda_c, \tag{164}$$

i.e., the type of *finite* statistics determined by the value of the parameter λ (parafermions with $\lambda=1$ and parabosons with $\lambda=-1$) is the same for particles and antiparticles.

Moreover, we can now establish the connection of the type of parastatistics to the spin of the particles. Indeed, according to the formula for the parameter ω_c, similar to the formula (48), and using also Eqs. (159) and (164), we obtain

$$\omega_c\equiv\lambda_c\rho_c=\mp\rho\geqslant 0. \tag{165}$$

Therefore, since the upper sign corresponds to the scalar case and the lower sign corresponds to the spinor case, we obtain

$$\begin{aligned}&\rho\leqslant 0 \quad \text{for scalar field,}\\&\rho\geqslant 0 \quad \text{for spinor field.}\end{aligned} \tag{166}$$

But for particles the condition (49) means that

$$\omega=\lambda\rho\geqslant 0. \tag{167}$$

According to the conditions (166) and (167), we obtain

$$\begin{aligned}&\lambda=-1 \quad \text{for scalar field,}\\&\lambda=+1 \quad \text{for spinor field,}\end{aligned} \tag{168}$$

which means that scalar particles satisfy para-Bose statistics and spinor particles satisfy para-Fermi statistics.

It remains for us to prove, however, that the orders of the paraparticles are the same for particles and antiparticles:

$$M=M_c. \tag{169}$$

On the basis of Eqs. (165) and (167), and using Eq. (164) and the result (168) obtained above, we can write

$$\omega_c=\omega. \tag{170}$$

In our case $x_c=x=1$ and, according to Eq. (84),

$$\omega=2p/M=2p_c/M_c. \tag{171}$$

It is convenient for us to set, as in Eq. (85), the parameter ω, being arbitrary, to 2. Then

$$p=M \text{ and } p_c=M_c. \tag{172}$$

In order to prove Eq. (169) it remains for us to prove that the values of the parameters p and p_c appearing in Eqs. (36) and (156) are the same. This can be done by making use of the mutual commutation relations (154c–n) for particle and antiparticle operators. The proof of this relation is given in Appendix 2.

The spin-parastatistics which we have obtained can be extended to fields with arbitrary spin: *particles with integer spin correspond to para-Bose statistics while fields with half-integer spin correspond to para-Fermi statistics.* Thus we have extended to finite-order parastatistics Pauli's famous spin-statistics theorem.

For Green's quantization which we are now considering this theorem was proved in Ref. 63, where, however, a special representation was employed for the parafields—so-called Green's ansatz, in which the parafield is constructed as a sum of ordinary Fermi or Bose fields, obeying anomalous (inverse) mutual commutation relations. In our proof we did not employ any special representation of the parafields, though for Green-quantized free fields such a representation can always be employed.[7] We now turn to the case $q=0$, when such a representation is inapplicable, but the spin-parastatistics theorem remains valid.[10]

The case $q=0$ was investigated in detail in Ref. 10. Here we only present the results briefly.

In contrast to Green quantization, the scheme based on the relations (143) and (148) for scalar and spinor fields, respectively, with $q=0$ is not C-invariant from the outset. For this reason, in it the Fock representation for the antiparticle operators differs from the representation for particle operators.

The condition (155) remains in force, but the result (156) now differs somewhat from the previous result for the case $q\neq 0$.

We can employ Eq. (154k) and, applying it to vacuum, obtain

$$a_k b_s b_{s'}^+|0\rangle=0. \tag{173}$$

On the basis of our assumption about the uniqueness of the vacuum vector *for both particles and antiparticles*, we must set in this case also

$$b_s b_{s'}^+|0\rangle=\chi_{ss'}|0\rangle, \tag{174}$$

where $\chi_{ss'}$ is a number. Operating on the vacuum vector by the relation (154o) we obtain

$$\chi_{s's''}b_s^+|0\rangle=\mp\rho\delta_{s's''}b_s^+|0\rangle \tag{175}$$

and

$$\chi_{s's''}=\mp\rho\delta_{s's''}. \tag{176}$$

According to Eqs. (174) and (175) the expression for the norm of the antiparticle vector assumes the form

$$\Big\|\sum_s\Psi_s b_s^+|0\rangle\Big\|^2=\sum_{s,s'}\Psi_s^*\Psi_{s'}\langle b_s b_{s'}^+|0\rangle=\mp\rho\sum_s|\Psi_s|^2. \tag{177}$$

The requirement that this norm always be positive means that

$$\mp\rho\geqslant 0, \tag{178}$$

where, as previously, the upper sign refers to the scalar case and the lower sign refers to the spinor case. Hence follow once again the conditions (166) and, as previously, we obtain, on the basis of the condition (167) for particles, the correct connection of spin to parastatistics (168).

Taking this connection into account and using Eq. (94) (with $p=M$), we obtain for Eq. (174) simply

$$b_s b_{s'}^+|0\rangle=\delta_{ss'}|0\rangle. \tag{179}$$

A characteristic feature of this scheme is that if the state vector contains only antiparticles, then we cannot calculate its norm, with the exception of the case, which we have just considered, of a single antiparticle. The point is that in the case $q=0$ the antiparticle annihilation operator cannot be advanced to the right toward the vacuum with the help of the relation (154o). However, this can be done for a particle—antiparticle *pair* with the help of the relation (154k). In this theory *the admissible states* (i.e., states for which the norm and, generally, any matrix elements can be calculated) *are states with an arbitrary number of particles, but the number of antiparticles cannot exceed by more than one the number of particles.*[24)] The corresponding admissible state vectors can be of two types:

$$\begin{aligned}&b_{s_1}^+a_{r_1}^+b_{s_2}^+a_{r_2}^+\dots b_{s_m}^+a_{r_m}^+a_{r_{m+1}}^+\dots a_{r_{m+n}}^+|0\rangle,\\&b_{s_1}^+a_{r_1}^+b_{s_2}^+a_{r_2}^+\dots b_{s_m}^+a_{r_m}^+a_{r_{m+1}}^+\dots a_{r_{m+n}}^+b_{s_{m+1}}^+|0\rangle.\end{aligned} \tag{180}$$

With the help of Eqs. (154c, d, m, o) it can be shown that such vectors will be *symmetric* with respect to permutations of the pairs $b_{s_k}^+a_{r_k}^+(k=1,\dots,m)$. By calculating the norm of the vectors (180) we can show that for given $p(=M)$ under the condition (36) for particles the number of antiparticles in the symmetric with $\rho=\lambda=1$ or in the antisymmetric with $\rho=\lambda=-1$ state cannot exceed M, but it can equal M, $M-1$, $M-2$, and so on. Thus in this case also *particles and antiparticles satisfy the same finite parastatistics.*

15. INFINITE STATISTICS FOR PARTICLES AND ANTIPARTICLES

In order to establish the complete classification of possible statistics of identical particles, described parafield commutation relations of the type (143) or (148), it remains for us to examine the limiting case of infinite statistics ($M\to\infty$), when our theorem I does not work. In order to establish this limit, we must once again start with the indicated relations and, correspondingly, the relations (154) for particle and antiparticle creation and annihilation operators with *arbitrary* parameters q and ρ as well as the relations for the vacuum vector (35) and (36) for particles and (155) and (156) for antiparticles.

According to our previous analysis, we have for the particle creation and annihilation operators in this limit the bilinear relations (103) or, after renormalization (104), the relations (105) with the restriction (114) on the possible values of the parameter q.

Once again we consider first the case $q\neq 0$. In this case we can employ for the antiparticles the relations (157) and (158) with the parametrization (159). Now we can examine the limit

$$p,p_c\to\infty. \tag{181}$$

We can repeat for particles and their operators, satisfying Eqs. (157) and (158) with $q_c\neq 0(q\neq\infty)$, all arguments that we made for particles, and we arrive for them at the bilinear relations similar to the relations (103):

$$b_sb^+_{s'}-q_cb^+_{s'}b_s=p_c\delta_{ss'}. \tag{182}$$

By virtue of the limits (181) we can always perform renormalization of the type (104) for both particle operators and antiparticle operators. Then we can write for the renormalized operators

$$b_sb^+_{s'}-q_cb^+_{s'}b_s=\delta_{ss'}. \tag{183}$$

Next, from the condition that the norm of λ_c-symmetric vectors for antiparticles is positive-definite, we obtain, similarly to the condition (114) for particles, a restriction on the possible values of q_c for antiparticles

$$-1\leqslant q_c\leqslant 1. \tag{184}$$

But, according to Eq. (159), $q_c=1/q$, and now we obtain from Eq. (184) new restrictions on the possible values of the parameter q

$$q\leqslant -1 \text{ and } 1\leqslant q. \tag{185}$$

Comparing Eqs. (114) and (185) we conclude that if $q\neq 0$, then the only possible values of the parameter q in the limit $p=p_c\to\infty$ are ± 1, and $q=q_c$.

In this case the bilinear relations for both particles (105) and antiparticles (183) become the ordinary commutators or anticommutators. Thus the *limiting statistics for $p(=p_c)\to\infty$ (if $q\neq 0$) are ordinary Bose and Fermi statistics.* The existence of antiparticles in our theory forbids different, so-called quon statistics, corresponding to arbitrary bilinear relations for q taking on values in the interval (114) and proposed by Greenberg[16] for formulation of small violation of ordinary statistics.

In the case of Green quantization ($q^2=1$) in the limit $p\to\infty$, as shown in Ref. 54, para-Fermi statistics becomes Bose statistics and para-Bose statistics becomes Fermi statistics. We find that the scalar fields, satisfying the relations (143) with $q=-1$ (and ρ negative) in the limit $p\to\infty$ satisfy anticommutator relations and, conversely, spinor fields, satisfying the relations (148) with $q=1$ (and ρ positive), satisfy in this limit commutator relations. Thus in this limit the correct connection of spin to statistics breaks down. But the theory itself becomes nonlocal, and this is the reason for the breakdown. In Sec. 9 it was shown that anomalous statistics of this kind can arise from normal statistics when the normal statistics contain degeneracy with respect to some hidden degree of freedom, taking on an infinite number of values, but having for the system as a whole a single definite value.

It remains for us to examine the behavior of antiparticles in the limit $p\to\infty$, with $q=0$.[10] As follows from Ref. 179, the vacuum condition for antiparticles does not contain the parameter p. For this reason, for antiparticles, in contrast to particles, renormalization of the type (104) should not be performed in this limiting case. The relations (154e, f) become commutators (since $\rho/p\to 0$)

$$[b^+_sa^+_r,a_{r'}]_-=0, \tag{186}$$

$$[a_rb_s,a^+_{r'}]_-=0, \tag{187}$$

while the relations (154k, l) remain unchanged. We can see that in this limiting case, eventhough the product a_rb_s contains a_r it nonetheless commutes with a^+_r. Moreover, introducing the pair operators

$$A_{rs}=a_rb_s,\quad A^+_{sr}=b^+_sa^+_r, \tag{188}$$

we obtain, by virtue of Eqs. (115), (154c, d, k, l) (with $q=0$), (186), and (187),

$$[A_{rs},A^+_{s'r'}]_-=|\rho|\delta_{rr'}\delta_{ss'}, \tag{189}$$

$$[A_{rs},A_{s'r'}]_-=[A_{sr},A^+_{s'r'}]_-=0, \tag{190}$$

i.e., *Bose-like* relations. As a consequence of Eqs. (186) and (187) these operators commute with the particle operators a and a^+. The latter operators, however, as before, satisfy Eqs. (115). Thus in this case we obtain in the limit $p\to\infty$ a collection of unpaired particles, satisfying infinite statistics, and a collection of particle—antiparticle pairs, and these two collections, appearing in the admissible vectors (180), behave as two *independent subsystems which commute with one another!*

Our initial theory, based on trilinear relations (with $q=0$), was local. It is interesting to trace how such a theory, which is local for any arbitrarily large but finite value of the parameter p, changes into a nonlocal theory in the limit $p\rightarrow\infty$. In contrast to cases with finite values of the parameter p, in this limiting theory particles and antiparticles separate, the particles satisfied the relations (115) (these conditions are not satisfied for arbitrary finite p), but the antiparticles satisfy, as before, the trilinear relations (154o, p). Taking into consideration also the new relations (186) and (187), we obtain, instead of the initial relations, for example, the relation (143), for a scalar field (setting $|\rho|=1$)

$$[\varphi(x)\varphi^{+}(y),\varphi(z)]_{-}=-i\Delta^{(-)}(z-y)\varphi(x). \qquad (191)$$

The singular function $\Delta^{(-)}$ on the right-hand side does not vanish outside the light cone, and the theory becomes explicitly nonlocal. It can be shown, however, that the theory is still *CPT* invariant.[10]

Our analysis of the field theory corresponding to parastatistics is summarized by the following theorem:

Theorem III. *A local theory of a free field satisfying q-deformed trilinear paracommutation relations (of the type (143) for a scalar field or (148) for a spinor field), under the conditions that*

a) *there exists a unique vacuum for particles and antiparticles and*

b) *the norm of the state vectors of particles and antiparticles is positive-definite admits only three values of the parameter* q*:*$q=\pm 1$ *and 0.*

Both admissible theories—C-invariant Green's paraquantization ($q^2=1$*) and the new charge-asymmetric paraquantization (*$q=0$*)—describe finite parastatistics of particles and antiparticles with the correct connection of spin to parastatistics: particles with integer spin correspond to para-Bose statistics and particles with half-integer spin correspond to para-Fermi statistics.*

In the limit of infinite-order statistics the trilinear relations pass into q-deformed bilinear relations, but the very existence of antiparticles limits them to the values $q=\pm 1$ *and 0. The theory then becomes nonlocal.*

In such a theory small violation of ordinary Fermi and Bose statistics is impossible either on the basis of local theory of finite parastatistics or on the basis of infinite statistics.

The commutation relations admissable within such a theory can be written in the final form

$$[[\varphi(x),\varphi^{+}(y)]_{q},\varphi(z)]_{-}=-i|\rho|\Delta(z-y)\varphi(x), \qquad (192)$$

$$[[\varphi(x),\varphi^{+}(y)]_{q},\varphi(z)^{+}]_{-}=i|\rho|\Delta(x-z)\varphi^{+}(x) \qquad (193)$$

for a scalar field and

$$[[\psi(x),\tilde{\psi}(y)]_{-q},\psi(z)]_{-}=-i|\rho|S(z-y)\psi(x), \qquad (194)$$

$$[[\psi(x),\tilde{\psi}(y)]_{-q},\tilde{\psi}(z)]_{-}=i|\rho|S(x-z)\tilde{\psi}(y) \qquad (195)$$

for a Dirac field. In these relations the parameter q assumes only two values: $q=1$ and $q=0$. In these cases it is convenient to set the arbitrary parameter $|\rho|$ equal to 2 and 1, respectively.

We note that in the case $q=0$ the form of the commutation relations is itself identical for scalar and Dirac fields and the difference lies only in the spin structure of these fields.

The vacuum conditions have the form

$$a_r|0\rangle=b_s|0\rangle=0. \qquad (196)$$

For particles

$$a_r a_{r'}^{+}|0\rangle=p\delta_{rr'}|0\rangle, \qquad (197)$$

where p is an integer equal to 1,2,..., is always satisfied. But for antiparticles the analogous condition is satisfied only in the case of Green's paraquantization ($q^2=1$):

$$b_s b_{s'}^{+}|0\rangle=p\delta_{ss'}|0\rangle, \qquad (198)$$

while in the case $q=0$ we have

$$b_s b_{s'}^{+}|0\rangle=\delta_{ss'}|0\rangle. \qquad (199)$$

In the limit of infinite order ($p\rightarrow\infty$) under Green's paraquantization para-Bose statistics pass into Fermi statistics and para-Fermi statistics pass into Bose statistics, and at the same time the correct connection of spin to statistics breaks down and the local theory becomes nonlocal.

In the case $q=0$ the analogous limit leads to the relation for particles

$$a_r a_{r'}^{+}=\delta_{rr'}, \qquad (200)$$

and parastatistics passes into classical Maxwell–Boltzman statistics. Antiparticles can enter only in particle–antiparticle pairs, satisfying Bose statistics and not depending on existing particles.

We wish to make a remark about bilinear q-mutator relations. The same relations that led us to forbid all trilinear relations other than relations with $q^2=1$ and 0 can be applied to the case when one starts from *bilinear* relations for the fields.[17,18] Adopting first the general form of the q-deformed bilinear relations for the fields, we found, on the basis of only the requirement that the norm of the state vectors of particles and *antiparticles* in Fock space be positive-definite, that these relations are restricted to only commutators and anticommutators, though the theory was initially not assumed to be local. We note that here the C-invariance of the commutation relations, as proposed previously in Ref. 64 (p. 196), was not employed.

16. CONCLUSIONS

Thus, on the basis of our definitions of "identity" (symmetry of the density matrix) and "elementarity" (transition operators which are bilinear in the particle creation and annihilation operators), we have arrived at possible generalizations of ordinary statistics in the form of para-Fermi and para-Bose statistics of finite orders and infinite statistics.

The conditions that the vacuum state be unique and the norm of the state vectors in Fock space be positive-definite led to a strong restriction on possible quantization schemes. There are only two possible schemes: Green's charge-symmetric and the new charge-asymmetric trilinear relations. Both admissable schemes describe finite-order para-Fermi and para-Bose statistics.

Since in such schemes it is impossible to pass continuously from statistics of one order to statistics of a different order, small violation of ordinary Fermi and Bose statistics, in particular, the Pauli principle, is impossible within such a theory. We note that in order to obtain this result on the basis of *finite-order* statistics we had to invoke locality of the corresponding fields and the existence of antiparticles.

The possibility of small violation of ordinary statistics within infinite statistics is not precluded, since infinite statistics is described by q-deformed *bilinear* commutation relations and admits a continuous change of statistics between Fermi statistics, corresponding to $q=-1$ and Maxwell–Boltzman statistics, corresponding to $q=0$, and between the latter and Bose statistics, corresponding to $q=1$, as recently proposed by Greenberg.[16] The norms of the state vectors remain positive-definite.

We also obtained bilinear relations as limiting relations for parastatistics as their order approaches infinity. However, on transferring to the field theory and incorporating in the analysis antiparticles described by the same field, due to the existence of the antiparticles the deformation parameter q, once again, can no longer vary continuously; it can take on only the three indicated values, corresponding to Fermi, Bose, and Maxwell–Boltzman statistics. Thus, even within the field theory corresponding to infinite statistics (as a limiting field theory for finite statistics), small violation of ordinary statistics once again becomes impossible. We also note that on passing from finite to infinite order statistics the corresponding field theory passes from local to nonlocal in complete agreement with Fredenhagen's general theorem[40] on the impossibility of formulating infinite statistics within a local algebra of observables.

Thus small violation of ordinary statistics is impossible, both within a local field theory, corresponding to finite-order statistics, and a nonlocal field theory, corresponding to the limiting infinite-order statistics. In the second case the positive-definiteness of the norms of state vectors not only for particles but also for *antiparticles* plays a significant role. As is well known, the existence of antiparticles played a decisive role in Pauli's establishing the connection of spin to statistics within the ordinary field theory. Now, their existence is found to be necessary in order to forbid small violation of statistics within the generalized field theory. On the other hand, the positive-definiteness of the norm of state vectors of particles and antiparticles is directly connected to the possibility of interpreting wave functions as probability amplitudes of the corresponding states. In this sense the arrangement of the universe in accordance with Pauli's principle is found to be connected to the possible observability of the universe according to quantum-mechanical laws!

It is the indicated prohibitions that form the content of the "theorem on the statistics of identical particles."

For admissible generalizations of the quantization scheme we determined the classes of wave functions corresponding to parastatistics of given order. It turned out that within the new paraquantization ($q=0$) an internal $SU(p)$ symmetry can be associated to each such finite-order statistics and the parastatistics can be regarded as ordinary statistics with the particles being degenerate with respect to this symmetry. The Casimir operators of this group play the role of generalized observables of nonabelian charges, whose values characterize completely the symmetry of any given state of the paraparticles. On this basis a nonrelativistic definition of "antiparticles" as "holes" in a filled p-shell of paraparticles and transforming according to the complementary representations of $SU(p)$ was derived. This approach elucidates the meaning of Fredenhagen's theorem: in the limit $p\to\infty$ it is impossible to construct from systems with a finite number of paraparticles a singlet representation of $SU(p)$, since in this case it is impossible to fill the p-shells completely. For this reason it is impossible to introduce "particles" and "antiparticles" as objects which transform according to the complementary representations of $SU(p)$. The reason why Maxwell–Boltzman statistics arises in this limit also becomes clear: degeneracy of ordinary fermions or bosons with respect to an internal coordinate, assuming an infinite number of values, which, in principle, can be used to distinguish the particles.

In the case of Green quantization the observables are not only Casimir operators of the group $SU(p)$, but they are also some generators of the group of the type third projection of isospin, strangeness, and so on. Due to the fact that such states can be separated, in the limit $p\to\infty$ para-Fermi statistics passes into Bose statistics and, analogously, para-Bose statistics passes into Fermi statistics.[54]

Paracommutation relations could thus provide the intriguing possibility of describing naturally internal symmetries with their help. This approach to internal symmetries, however, leaves many other questions unresolved: the possibility of formulation of gauge symmetries on the basis of parafields;[43,61] associating groups of internal symmetries, arising in this approach, to physical symmetries;[9,39] and so on. In this respect the restrictions on such a choice of symmetries, arising in this approach on the basis of the locality requirement,[61] are interesting. Further generalization of the paraquantization scheme by constructing nonassociative parafields on the basis of the algebra of octonions is also possible.[66,67] Such a specific realization of parafields, even though it contains features which are characteristic only for it, can serve at the same time as a foundation for incorporating adequately into the mathematical construction the physical generations of leptons and quarks together with an explanation of the integer valued charges of the leptons and fractional value charges of the quarks.[67] However, consideration of this possibility falls outside the scope of the present review.

It is a great honor for me to participate in the publication of a journal dedicated to Nikolai Nikolaevich Bogolyubov. N. N. Bogolyubov's fundamental works on statistical methods of describing complex systems[2] and the group-theoretic approach to describing physical symme-

tries of elementary particles[68] have always been an inspiring example for me.

APPENDIX 1. DESCRIPTION OF ORDINARY STATISTICS BY TRILINEAR RELATIONS

The standard Fermi and Bose statistics can be studied on the basis of the general relations (29) and (30) and the conditions (35) and (36), imposing a restriction on the number of particles in the λ-symmetric (symmetric with $\lambda=1$ and antisymmetric with $\lambda=-1$) state $M=1$. According to this condition we have the relation (71), which, recalling the notations (53) and (58), we can rewrite in the form

$$p(\lambda+q)=\rho. \tag{A1.1}$$

Hence $qp-\rho=-\lambda\rho$ and, as one can see from Eqs. (39c, d), the two- and three-particle projections become $(-\lambda)$-symmetric, i.e., they indeed correspond to Fermi and Bose statistics. Moreover, using Eq. (A1.1) we can rewrite the relation (37) as follows:

$$\begin{aligned} a_r a^+_{r_1} a^+_{r_2} \dots a^+_{r_n}|0\rangle = p\{\delta_{rr_1} a^+_{r_2}\dots a^+_{r_n}|0\rangle \\ -\sum_{k=2}^{n}\delta_{sr_k}\Big[\lambda q^{k-2} a^+_{r_1}a^+_{r_2}\dots a^+_{r_{k-1}} \\ +\sum_{l=1}^{k-2}(q+\lambda)q^{k-l-2}a^+_{r_1}a^+_{r_2}\dots a^+_{r_{k-l-2}} \\ \times a^+_{r_{k-l}}a^+_{r_{k-l+1}}\dots a^+_{r_{k-1}}a^+_{r_{k-l-1}}\Big] \\ \times a^+_{r_{k+1}}\dots a^+_{r_n}|0\rangle\}. \end{aligned} \tag{A1.2}$$

Now we can check that in this representation the bilinear relation holds identically:

$$a_r a_{r'} + \lambda a_{r'} a_r = 0. \tag{A1.3}$$

Correspondingly, the hermitian-conjugate relation must also hold:

$$a^+_r a^+_{r'} + \lambda a^+_{r'} a^+_r = 0. \tag{A1.4}$$

Now, using Eqs. (A1.2) and (A1.4) it can be shown that the relation

$$a_r a^+_{r'} + \lambda a^+_{r'} a_r = p\delta_{rr'} \tag{A1.5}$$

also holds. Setting the arbitrary parameter $p=1$, we obtain the standard commutation relations for fermions ($\lambda=1$) and bosons ($\lambda=-1$).

It is also easy to verify that when the relations (A1.3)–(A1.5) hold the initial relations (29) and (30) are satisfied identically. Thus under the indicated restriction (A1.1) the initial trilinear relations are necessary and sufficient conditions for describing ordinary statistics.

Setting $p=1$ the relation (A1.1) assumes the form

$$\rho=\lambda+q. \tag{A1.6}$$

The parameter q remains arbitrary: q can assume any real value and the general scheme will nonetheless be identical to the scheme described on the basis of quantization with the help of commutators or anticommutators. The meaning of this arbitrariness in the choice of the parameter q is as follows. For fermions and bosons observables can be written as an arbitrary bilinear combination of the form (26). All such combinations will be equivalent to within a constant, the constant depending on q. This is a well-known arbitrariness in the definition of the product of operators for fermions and bosons.

APPENDIX 2. PROOF THAT THE ORDERS OF PARASTATISTICS ARE THE SAME FOR PARTICLES AND ANTIPARTICLES ($q^2=1$)

First we prove a property of the vacuum vector:

$$a_k b^+_s|0\rangle = b_s a^+_k|0\rangle = 0. \tag{A2.1}$$

Operating on the vacuum vector by both sides of Eq. (154f) and using Eqs. (36) and (155) we obtain

$$a_k b_s a^+_{k'}|0\rangle = 0, \tag{A2.2}$$

whence, by virtue of the uniqueness of the vacuum vector, we conclude

$$b_s a^+_k|0\rangle = \phi_{sk}|0\rangle, \tag{A2.3}$$

where ϕ_{sk} is some number.

Operating on the vacuum vector by both sides of Eq. (154h) and using Eqs. (155), (156) and (A2.3) we obtain

$$\begin{aligned} qb_{s'}b^+_s a^+_k|0\rangle = \phi_{s'k}b^+_s|0\rangle \\ +\delta_{ss'}p_c q a^+_k|0\rangle. \end{aligned} \tag{A2.4}$$

We now operate on this equation with the operator $b_{s''}$. Using Eq. (154p) in the form

$$qb_{s''}b_{s'}b^+_s = -b^+_s b_{s'}b_{s''} + qb_{s'}b^+_s b_{s''} + b_{s''}b^+_s b_{s'} \pm \rho\delta_{ss''}b_{s'} \tag{A2.5}$$

and applying Eqs. (A2.3), (155), and (156), we rewrite Eq. (A2.4) in the form

$$\begin{aligned} \delta_{ss'}p_c q\phi_{s''k}|0\rangle + \delta_{ss''}\phi_{s'k}(p_c\pm\rho)|0\rangle \\ =\delta_{ss''}p_c\phi_{s'k}|0\rangle + \delta_{ss'}p_c q\phi_{s''k}|0\rangle. \end{aligned} \tag{A2.6}$$

Canceling identical terms on the left- and right-hand sides of this equation we are left with the single term

$$\pm\rho\delta_{ss'}\phi_{s'k}|0\rangle = 0. \tag{A2.7}$$

Setting $s=s''$ and $\rho\neq 0$, we obtain

$$\phi_{s'k}=0, \tag{A2.8}$$

which proves the second equality (A2.1). We note that this relation is also valid for $q=0$; then it follows directly from (A2.4).

The right-hand equality in Eq. (A2.1) is also proved similarly. Starting from Eq. (154k) we obtain

$$qb_s a_k b^+_{s'}|0\rangle = 0. \tag{A2.9}$$

If $q\neq 0$, then, by virtue of the uniqueness of the vacuum vector,

$$a_k b^+_s|0\rangle = \chi_{ks}|0\rangle, \tag{A2.10}$$

where χ_{ks} is a number. Next, on the basis of Eq. (154i) we obtain

$$a_k a_{k'}^+ b_s^+ |0\rangle = q\chi_{ks} a_{k'}^+ |0\rangle + \delta_{kk'} p b_s^+ |0\rangle. \quad (A2.11)$$

Operating on this equality with the operator $a_{k''}$ and using Eq. (154a) in the form

$$a_{k''} a_k a_{k'}^+ = a_k a_{k'}^+ a_{k''} - q a_{k'}^+ a_k a_{k''} + q a_{k''} a_{k'}^+ a_k - \rho \delta_{k'k''} a_k, \quad (A2.12)$$

and also (A2.10), (35), and (36) we obtain

$$\delta_{kk'} p \chi_{k''s} |0\rangle + \delta_{k'k''}(pq-\rho)\chi_{ks}|0\rangle = \delta_{k'k''} pq\chi_{ks}|0\rangle + \delta_{kk'} p\chi_{k''s}|0\rangle. \quad (A2.13)$$

Canceling identical terms on the left- and right-hand sides we obtain

$$\delta_{k'k''}\rho\chi_{ks}|0\rangle = 0, \quad (A2.14)$$

whence follows

$$\chi_{ks} = 0$$

and the first equality in Eq. (A2.1). We note that in deriving this equality we assumed that $q \neq 0$, so that the results obtained below are valid only in this case.

Now we consider the identity

$$[[b_s^+, a_k]_{-q}, [b_{s'}, a_{k'}^+]_{-q}]_- \equiv [b_s^+, [a_k, [b_{s'}, a_{k'}^+]_{-q}]_-]_{-q} + [[b_s^+, [b_{s'}, a_{k'}^+]_{-q}, a_k]_{-q}. \quad (A2.15)$$

In order to calculate the right-hand side we employ a Jacobi-type identity:[25)]

$$[[a_k, [b_s, a_{k'}^+]_{-q}]_- \equiv q[[b_s, a_k]_{-1/q}, a_{k'}^+]_- + [[a_{k'}^+, a_k]_{-q}, b_s]_-. \quad (A2.16)$$

In our case $q^2 = 1$, and we can make the substitution $q = 1/q$. Then

$$[b_s, a_k]_{-1/q} = -(1/q)[a_k, b_s]_{-q} = -q[a_k, b_s]_{-q}, \quad (A2.17)$$

$$[a_{k'}^+, a_k]_{-q} = -q[a_k, a_{k'}^+]_{-1/q} = -q[a_k, a_{k'}^+]_{-q}. \quad (A2.18)$$

Substituting these expressions into Eq. (A2.16) and using next Eq. (154f) and (154j) we obtain the relation

$$[a_k, [b_s, a_{k'}^+]_{-q}]_- = \rho\delta_{kk'} b_s. \quad (A2.19)$$

The second term on the right-hand side of Eq. (A2.15) can be calculated with the help of the identity

$$[b_s^+, [b_{s'}, a_{k'}^+]_{-q}]_- \equiv q[[b_{s'}, b_s^+]_{-1/q}, a_{k'}^+]_- + [[a_{k'}^+, b_s^+]_{-q}, b_{s'}]_- \quad (A2.20)$$

and the relations

$$[b_{s'}, b_s^+]_{-1/q} = -(1/q)[b_s^+, b_{s'}]_{-q} = -q[b_s^+, b_{s'}]_{-q}, \quad (A2.21)$$

$$[a_{k'}^+, b_s^+]_{-q} = -q[b_s^+, a_{k'}^+]_{-1/q} = -q[b_s^+, a_{k'}^+]_{-q}. \quad (A2.22)$$

Substituting the latter two relations into the right-hand side of Eq. (A2.20) and using Eqs. (154h) and (154l) we obtain the relation

$$[b_s^+, [b_{s'}, a_{k'}^+]_{-q}]_- = \mp \rho q \delta_{ss'} a_{k'}^+. \quad (A2.23)$$

Substituting Eqs. (A2.19) and (A2.23) into the initial identity (A2.15) gives the relation

$$[[b_s^+, a_k]_{-q}, [b_{s'}, a_{k'}^+]_{-q}]_- = \rho\delta_{kk'}[b_s^+, b_{s'}]_{-q} \mp \rho q \delta_{ss'}[a_{k'}^+, a_k]_{-q}. \quad (A2.24)$$

We now operate on the vacuum by both sides of this equation. By virtue of the relations (A2.1) proved above we obtain zero for the left-hand side. For the right-hand side, according to Eqs. (35), (36), (155), and (156), we obtain

$$-\rho q(p_c \pm qp)\delta_{ss'}\delta_{kk'}|0\rangle = 0. \quad (A2.25)$$

Setting $s = s'$ and $k = k'$ and using the fact that $\rho \neq 0$ and $q \neq 0$ we obtain

$$p_c = \mp qp. \quad (A2.26)$$

But, according to the previously proved connection of spin to statistics in the scalar case (upper sign) $q = -1$ and in the spinor case (lower sign) $q = 1$. Therefore we always have

$$p = p_c \quad (A2.27)$$

and, according to Eq. (172), $M = M_c$ for the orders of the particles and antiparticles.

[1)] Lango[35] established the connection between the "statistics parameter" (the inverse of the order of parastatistics) and the square root of the Jones index for subfactors of the algebra of observables.

[2)] The operator $\hat{Q}^+$, the hermitian conjugate of $\hat{Q}$, is defined by the relation

$$\int dx dx' [\check{\mathcal{Q}}^+(x')\rho^*(x',x)]^*\delta(x-x') = \int dx \int dx' \check{\mathcal{Q}}(x)\rho(x,x')\delta(x,x').$$

If $\hat{Q}^+ = \hat{Q}$, then this operator is hermitian.

[3)] This condition can be associated to the condition that the product $\rho_1^0\rho_2^0 \ldots {}^0\rho_N$ of the morphisms $\rho_1, \rho_2, \ldots, \rho_N$ of local algebras of observables, localized in mutually space-like regions be order-independent.[32] Morphisms are transformations on operators which do not change the algebraic operations on the images of these operators.

[4)] It is also obvious that the property (4) will hold for the previously mentioned anyons, for which the wave function acquires under permutation of the arguments an arbitrary phase factor. It is well known, however, that with an appropriate choice of the gauge the particles satisfying such "anyon statistics" can be described as particles with ordinary statistics but interacting with a Chern—Simons gauge field (see, for example, Ref. 44).

[5)] They could also assume infinite values with a finite value of the ratio ρ/q. This case, however, reduces to the case $q = 0$ by the simple transformation $a^+ \Leftrightarrow a$.

[6)]The parameters q and ρ introduced here correspond to the parameters α and ε of Ref. 5 as follows: $q=-1/\varepsilon$, $\rho=-\alpha/\varepsilon$.

[7)]The following identity must be used to derive them:[5,9]
$[[A,B]_\varepsilon,[C,D]_\eta]_-\equiv[A,[B,[C,D]_\eta]_-]_\varepsilon+[[A,[C,D]_\eta]_-,B]_\varepsilon$,
setting $\varepsilon=\eta=-q$.

[8)]In Okun's scheme[49] of small violation of Pauli's principle, being an extension of the Ignat'ev–Kuz'min scheme[12] to a multilevel (but not field!) system, this property is not satisfied: For operators referring to the same level the Ignat'ev–Kuz'min relations are assumed, while for arbitrarily close but not coincident states the ordinary anticommutation relations are assumed.

[9)]We shall often employ below the term "λ-symmetric vector," meaning a vector that is symmetric with $\lambda=1$ or antisymmetric with $\lambda=-1$.

[10)]The fact that these combinations are orthogonal, i.e.,

$$\sum_{r_1,r_2,r_3}\Psi'_m(r_1,r_2,r_3)\Psi''_m(r_1,r_2,r_3)=0.$$

is easily proved, rewriting in the sum $r_1\Leftrightarrow r_3$ and using the fact that with the reverse rearrangement to the initial order the function (50) acquires the factor $-\lambda$ and the function (51) acquires the factor λ, where $\lambda^2=1$.

[11)]For even M we have the following lower limits: $M=2$, $x=-0.5$; $M=4$, $x=-0.722...$; $M=6$, $x=-0.806$, and so on. The sequence of these values approaches -1 as $M\to\infty$.

[12)]We note that this formulation differs from the formulation of the analogous theorem proved in Ref. 5. In the previous formulation vanishing of the norms of all vectors containing more than $M+1$ particles, among which there are $M+1$ particles in a λ-symmetric state, was posited as a *requirement*. In the present formulation, however, this condition arises as a *consequence* of the more general requirement that the norm of the state vectors be positive-definite. This was previously proved in the case $M=2$.[14]

[13)]In the case of Green quantization, in contrast to ordinary quantization, there exists, besides the Fock representation, an infinite set of other separable representations, based on vacuum-like vectors, which, however, now contain some initial number of particles.[39,51] It is in these representations that all other states of internal degree of freedom of the particles are realized. In the presence of only one type of paraparticle, however, transitions between different irreducible representations of the Green relations are impossible. In order for them to mix several types of paraparticles of the same order must be present.[9]

[14)]Kuryshkin[52] proposed such relations for a single-level system, and they were later studied in Ref. 53. Now, however, we are interested in the *statistics* of particles and for this reason we examine from the outset an infinite-level system.

[15)]This question was recently investigated in Ref. 65.

[16)]Due to such sorting of all particles such a direct product of irreducible representations of S_{n_1} and S_{n_2} is said to be "exterior," since it now gives the representation of the complete group S_n.

[17)]We underscore the fact that when separate particles are systematically removed the remaining particles are still identical and can in no way differ from one another. This property is called the "cluster law." In Refs. 31 it was used as the basis for classification of the statistics of identical particles.

[18)]Conjugate schemes are schemes for which the columns are replaced by rows and vice versa.

[19)]In the case of $SU(3)$ these invariants have the form:[57]

$$F^2=(L_1^2+L_1L_2+L_2^2)/3+L_1+L_2,G_3+[2(L_1+L_2)^2+L_1L_2]$$
$$\times(L_1-L_2)/9+(2L_1+L_2)(L_1+2).$$

[20)]The arguments for parabosons are completely analogous, only tableaux with p-rows must be considered and they must be associated not to complementary groups but rather the groups $SU(p)$ themselves.

[21)]We cannot call these tableaux "conjugate" since this term has already been employed for tableaux in which the rows and columns have been interchanged.

[22)]In order to perform this calculation the identity presented in footnote 7 must be used.

[23)]Indeed, according to Eq. (159), the case $q_c=0$ corresponds to the limit $q\to\infty$ with finite $\rho_c=\mp\rho/q$, i.e., $\rho\to\infty$ (in other words $\rho_c\to 0$). But in this limit the relation (154a) for particles (after dividing by q and passing to the limit $q\to\infty$) acquires the form

$$[a^+_{r'}a_r,a_{r''}]_-=\pm\rho_c\delta_{r'r''}a_r,$$

while for antiparticles, according to Eq. (157), we have the relation

$$[b_sb^+_{s'},b_{s''}]_-=\rho_c\delta_{s's''}b_s.$$

Comparing the first of these relations to Eq. (154p) and the second to (154a) we can see that they are identical (with $q=0$), if we simply interchange a and b.

[24)]As shown in Ref. 11, the vacuum vector itself is not unique in this theory. The ground state can also be a state in which a finite number of particles have been placed in advance. However, such "vacuum" antiparticles will behave as ordinary fermions or bosons.

[25)]This identity can be written in a general form as

$$\alpha[[A,B]_\varepsilon,C]_\eta\equiv-\varepsilon\eta[[A,C]_{-\alpha/\varepsilon},B]_{-\alpha/\eta}+\alpha^2[[B,C]_{\varepsilon\eta/\alpha},A]_{1/\alpha}.$$

[1] L. B. Okun', Usp. Fiz. Nauk **158**, 293 (1989) [Sov. Phys. Usp. **32**, 543 (1989)]; Comments Nucl. Part. Phys. **19**, 99 (1989). E. Ramberg and G. A. Snow, Phys. Lett. B **238**, 483 (1990). V. A. Novikov *et al.*, Phys. Lett. B **240**, 227 (1990).

[2] N. N. Bogolyubov, Selected Works [in Russian], Naukova dumka, Kiev, 1970, Vol. II.

[3] A. B. Govorkov, J. Phys. A **13**, 1673 (1980).

[4] A. Bach, Phys. Lett. A **151**, 1 (1990).

[5] A. B. Govorkov, Teor. Matem. Fiz. **54**, 361 (1983).

[6] H. S. Green, Phys. Rev. **90**, 170 (1953).

[7] O. W. Greenberg and A. M. L. Messiah, Phys. Rev. B **138**, 1155 (1965).

[8] Y. Ohnuki and S. Kamefuchi, *Quantum Field Theory and Parastatistics*, Springer-Verlag, New York, 1982.

[9] A. B. Govorkov, Fiz. Elem. Chastits At. Yadra **14**, 1229 (1983) [Sov. J. Part. Nucl. **14**, 520 (1983)].

[10] A. B. Govorkov, Teor. Matem. Fiz. **85**, 222 (1990).

[11] A. B. Govorkov, Nucl. Phys. B. **365**, 381 (1991).

[12] A. Yu. Ignat'ev and V. A. Kuz'min, Yad. Fiz. **46**, 786 (1987) [Sov. J. Nucl. Fiz. **46**, 444 (1987)].

[13] O. W. Greenberg and R. N. Mohapatra, Phys. Rev. Lett. **59**, 2507 (1987).

[14] A. B. Govorkov, Phys. Lett. A **137**, 7 (1989).

[15] O. W. Greenberg and R. N. Mohapatra, Phys. Rev. Lett. **62**, 712 (1989).

[16] O. W. Greenberg, Phys. Rev. D **43**, 4111 (1991); Physica A **180**, 419 (1992).

[17] M. V. Cougo-Pinto, Phys. Rev. D **46**, 858 (1992).

[18] A. B. Govorkov, Mod. Phys. Lett. A **7**, 2383 (1992).

[19] O. W. Greenberg, Phys. Rev. Lett. **64**, 705 (1990).

[20] W. Pauli, Phys. Rev. **58**, 116 (1940).

[21] N. Burgoyne, Nuovo Cimento **8**, 604 (1958).

[22] G. Luders and B. Zumino, Phys. Rev. **110**, 1450 (1958).

[23] R. F. Streater and A. S. Wightman, *PCT, Spin and Statistics, and All That*, Benjamin, N.Y., 1964.

[24] D. V. Volkov, Zh. Eksp. Teor. Fiz. **36**, 1560 (1959); **38**, 518 (1960) [Sov. Phys. JETP **9**, 1107 (1959); **11**, 375 (1960)].

[25] W. Pauli, *Wave Mechanics*, MIT Press, Cambridge, MA, 1973.

[26] P. A. M. Dirac, *The Principles of Quantum Mechanics*, Clarendon Press, N.Y., Oxford, 1987.

[27] G. Gentile, Nuovo Cimento **17**, 493 (1940); **19**, 106 (1942).

[28] I. E. McCarthy, Proc. Camb. Phil. Soc. **51**, 131 (1955). I. E. Guenault and D. K. C. McDonald, Mol. Phys. **5**, 525 (1962). S. I. Ben-Abraham, Am. J. Phys. **38**, 1335 (1970).

[29] T. Okayama, Prog. Theor. Phys. **7**, 517 (1952).

[30] S. Kamefuchi and Y. Takahashi, Nucl. Phys. **36**, 177 (1962).

[31] J. B. Hartle and J. R. Taylor, Phys. Rev. **178**, 2043 (1969). J. B. Hartle, R. H. Stolt, and J. B. Taylor, Phys. Rev. D **2**, 1759 (1970). R. H. Stolt and J. R. Taylor, Nucl. Phys. B. **19**, 1 (1970); Nuovo Cimento A **5**, 185 (1971).

[32] S. Doplicher, R. Haag, and J. E. Roberts, Commun. Math. Phys. **23**, 199 (1971); **35**, 49 (1974).

[33] R. Haag in *Progress in Quantum Field Theory*, edited by H. Ezawa and S. Kamefuchi, North-Holland, New York, 1986.

[34] D. Buchholz and K. Fredenhagen, Commun. Math. Phys. **84**, 1 (1982).

[35] R. Longo, Commun. Math. Phys. **126**, 217 (1989).

[36] A. Galindo and F. Indurain, Nuovo Cimento **30**, 1040 (1963).

[37] M. Yamada, Nucl. Phys. B **6**, 596 (1968).

[38] N. A. Chernikov, Acta Phys. Polonica **21**, 51 (1962).

[39] A. B. Govorkov, Zh. Eksp. Teor. Fiz. **54**, 1785 (1968) [Sov. Phys.

JETP 27, 960 (1968)]; Ann. Phys. (N.Y.) **53**, 349 (1969); Int. J. Theor. Phys. **7**, 49 (1973).
[40] K. Fredenhagen, Commun. Math. Phys. **79**, 141 (1981).
[41] D. Buchholz, M. Porrmann, and U. Stein, Phys. Lett. B **267**, 377 (1991).
[42] S. Doplicher and J. E. Roberts, Commun. Math. Phys. **131**, 51 (1990).
[43] O. W. Greenberg and K. I. Macrae, Nucl. Phys. B **219**, 358 (1983).
[44] R. Iengo and K. Lechner, Phys. Rep. **213**, 179 (1992).
[45] L. D. Landau and E. M. Lifshitz, *Quantum Mechanics*, Pergamon Press, N.Y.
[46] N. N. Bogolyubov, A. A. Logunov, A. I. Oksak, and I. I. Todorov, *General Principles of the Quantum Field Theory* [in Russian], Nauka, Moscow, 1987.
[47] P. Erenfest and J. R. Oppenheimer, Phys. Rev. **37**, 333 (1931).
[48] K. Akama *et al.*, Phys. Rev. Lett. **68**, 1826 (1992).
[49] L. B. Okun', Pis'ma Zh. Eksp. Teor. Fiz. **46**, 420 (1987) [JETP Lett. **46**, 529 (1987)]; Yad. Fiz. **47**, 1182 (1988) [Sov. J. Nucl. Phys. **47**, 752 (1988)].
[50] I. Bialynicki-Birula, Nucl. Phys. **49**, 605 (1963).
[51] A. J. Bracken and H. S. J. Green, Math. Phys. **14**, 1784 (1973).
[52] V. Kuryshkin, Annales de la Fondation Louis de Broglie **5**, 111 (1980).
[53] A. Jannussis, G. Brodimas, and D. Soulas, Lett. Nuovo Cimento **30**, 123 (1981); **31**, 177 (1981); **34**, 375 (1982). A. Jannussis, Haadronic J. **14**, 257 (1991).
[54] O. W. Greenberg and A. M. L. Messiah, J. Math. Phys. **6**, 500 (1965).
[55] S. Stanciu, Preprint, BONN-HE-92-04, University of Bonn (1992).
[56] M. Hamermesh, *Group Theory and Its Applications to Physical Problems*, Addison-Wesley, N.Y., 1962.
[57] J. De Swart, Usp. Fiz. Nauk **84**, 651 (1964) 2, [*sic*].
[58] D. E. Littlewood, *The Theory of Group Characters*, Oxford, 1950.
[59] H. Weyl, *The Classical Groups*, Princeton University Press, Princeton, N.J., 1946.
[60] O. W. Greenberg, Phys. Rev. Lett. **13**, 598 (1964).
[61] A. B. Govorkov, Teor. Matem. Fiz. **53**, 283 (1982); **55**, 3 (1983).
[62] N. N. Bogolyubov and D. V. Shirkov, *Introduction to Quantum Field Theory* [in Russian], Nauka, Moscow, 1984.
[63] G. F. Dell'Antonio, O. W. Greenberg, and E. C. G. Sudarshan in: *Group Theoretical Concepts and Methods in Elementary Particle* Physics, Gordon and Breach, New York, 1964.
[64] Yu. B. Rumer and A. I. Fet, *Group Theory and Quantum Fields* [in Russian], Nauka, Moscow, 1977.
[65] R. Ramanathan, Phys. Rev. **45**, 4706 (1992).
[66] M. Gunaydin and F. Gursey, Phys. Rev. D **9**, 3387 (1974).
[67] A. B. Govorkov, Kratkie soobshcheniya (Brief Communications), Joint Institute for Nuclear Research, Dubna, No. 7-85, 1985; Proceedings of the 8th International Conference on Problems in Quantum Field Theory, Alushta, 1987. Joint Institute for Nuclear Research, D2-87-798, Dubna, 1987, p. 262. JINR, E2-90-379, Dubna, 1990.
[68] N. N. Bogolyubov in: *Proceedings of the International School on the High Energy Physics and the Theory of Elementary Particles* [in Russian], Naukova dumka, Kiev, 1987, p. 5.

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