The geometry of a rotating disk

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A scientific approach of the great Russian theoretical physicist Academician N. N. Bogolubov to the solution of theoretical problems is given in the form of an imaginary dialogue.

In memory of N. N. Bogolubov

I met N. N. Bogolubov for the first time in 1955 at the first great conference where after the Stalin era Soviet physicists could invite people from the West. It was a remarkable experience for me to see the two great masters of Russian theoretical physics, N. N. Bogolubov and L. D. Landau, in action. Their ways of thinking complemented each other, one having all the mathematical tools at his fingertips, the other having the physical situation before his mind. I shall try to illustrate how the two approaches can fertilize each other by an imaginary dialogue (B. and L. should only be taken as representatives of the two schools of thought).

L. One of the great ideas in physics was Einstein's recognition that gravity determines the geometry of space-time. In his popular writings[1] he motivates that by his famous argument of the rotating disk. There, radial measuring rods move perpendicular to their direction and show no Lorentz contraction but the ones in the azimuthal direction do. Thus, when one measures the ratio circumference/radius, one finds a number $>2\pi$ and concludes that the geometry is non-Euclidean. The equivalence principle then tells us that also gravity must change the geometry. On second thought the argument becomes less clear for the following reason, already recognized by Einstein.[2] Einstein has taught us that to measure the length of a moving object simultaneity has first to be defined. But how does one define simultaneity on the rotating disk, where the velocity depends on the radius, and thus also the rate of clocks? As geometry is something intrinsic, the rotating observer will not accept any outside suggestion as to what is simultaneous.

B. Like all problems in relativity one should be able to settle it by rational means. But since the analysis promises to become delicate, I propose that we first get the uniform motion in one space dimension straight. The general linear transformation of (x,t) involves four parameters, but in physics one usually considers only the one-parameter group of Lorentz transformations. What happened to the other three parameters?

L. This one understands very well. Two parameters fix the length scale of the new coordinates $(\bar{x},\bar{t})$, and they have to be set equal to one if one wants $\bar{x}$ and $\bar{t}$ to correspond to distances and time intervals as measured by real measuring rods and clocks. Formally, this means that when the metric $g=dx^2-dt^2$ is expressed by $(\bar{x},\bar{t})$ the coefficients of $d\bar{x}^2$ and $d\bar{t}^2$ must be $(1,-1)$. The other two parameters determine the angles between the $(\bar{x},\bar{t})$ axis and the (x,t) axis. The $\bar{t}$ axis $(\bar{x}=0)$ gives the velocity of the motion, and the $\bar{x}$ axis $(\bar{t}=0)$ gives the definition of simultaneity in the new system. The Lorentz transformations are the ones where the two angles are equal, which implies that in g the mixed term $d\bar{x}d\bar{t}$ vanishes. Physically, this means synchronization of clocks by the Einstein convention that light emitted by the two events arrives at the same time in the middle. In this case $d\bar{x}\pm d\bar{t}$ are light-like directions, and $g=d\bar{x}^2-d\bar{t}^2$.

B. We had better accept this convention of simultaneity, otherwise the length of a moving object and thus geometry is not defined. When turning to nonuniform motion and thus to nonlinear coordinate transformation, I propose to proceed slowly and first to consider uniform acceleration in one dimension.

L. I agree, since this can physically be realized by a constant electric field, which is a beautiful example in relativistic dynamics.[3] Thus, let us imagine our measuring rods and our clocks to be electrically charged, and by a constant electric field E we set the system into motion. Then the dynamics tells us exactly how our equipment will behave. For $\bar{x}$, I take the initial position, and for $\bar{t}$ the proper time, which gives me the time reading on the clocks. The well-known solution[4] tells me

$$x=\bar{x}+\nu(\cosh \bar{t}/\nu-1),\quad t=\nu \sinh \bar{t}/\nu,\quad \nu=eE/m.$$

B. I am sorry, but you did not respect the rules of our game. I calculate

$$dx^2-dt^2=d\bar{x}^2+2 \sin \bar{t}/\nu d\bar{x}d\bar{t}-d\bar{t}^2.$$

Thus, the basis $(d\bar{x},d\bar{t})$ is not orthogonal, and I cannot accept the events with $\bar{t}=$constant as simultaneous.

L. Well, they are simultaneous in the old (x,t) frame, since the clocks with world-lines $\bar{x}=$constant all experience the same acceleration. So after a time t the same proper time $\bar{t}$ has elapsed for all of them. But you are right that they are no longer Einstein-synchronized and we have to look for other variables. Do you have a $\bar{t}$ variable which is Lorentz-orthogonal to hyperbolas $\bar{x}=$constant?

B. Well, we could take

$$x=\bar{x}\cosh \bar{t},\quad t=\bar{x}\sinh \bar{t},$$

and then

$$dx^2-dt^2=d\bar{x}^2-\bar{x}^2d\bar{t}^2.$$

L. I do not like your lapse function $\bar{x}^2$, which tells you that $\bar{t}$ does not give directly the time elapsed on the hyperbola $\bar{x}=$constant, and although you have an orthogonal basis $(d\bar{x},\bar{x}d\bar{t})$, the clocks at $\bar{t}=0$ are not really Einstein-

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synchronized, since light cones emerging from $\bar{x}=1/2$ and $\bar{x}=3/2$ do not intersect at the hyperbola going through $\bar{x}=1$.

B. In differential geometry we always have to distinguish three levels where a property can hold. The infinitesimal level (on the tangent space), the local level (in a suitable neighborhood), and the global level (on all of space-time). The Einstein synchronization belongs to the infinitesimal level, and that is the best we can hope for. As to the lapse function $\bar{x}^2$, I cannot get rid of it altogether. You know very well that $dx^2-dt^2=d\bar{x}^2-d\bar{t}^2$ implies that $(\bar{x},\bar{t})$ are related to (x,t) by a Lorentz transformation which is linear. So by going to accelerated frames we can never have $dx^2-dt^2=d\bar{x}^2-d\bar{t}^2$.

L. This I knew, but I just wanted the time coordinate clean. Can't you get

$$dx^2-dt^2=f^2(\bar{x},\bar{t})d\bar{x}^2-d\bar{t}^2?$$

B. This is also impossible by the following general theorem:[3,4] "If one member of an orthogonal basis is exact, then the associated vector field is geodesic." An exact covector field is of the form $d\bar{t}$, where $\bar{t}$ is some locally defined function. Thus, the theorem applies to your case, and therefore the integral curves of $d\bar{t}$ which are the curves $\bar{x}=$const have to be geodesic and cannot correspond to an accelerated observer. In our previous metric $d\bar{x}$ is exact and orthogonal to the rest, and thus its integral curves $L\bar{t}=$const must be geodesic. Indeed, they are the straight lines $t/x=\tanh\bar{t}$.

L. So what we can do is to turn things around and consider in the region $|x|<t$ the coordinates.

$$\begin{matrix} x=\bar{t}\sinh\bar{x} \\ t=\bar{t}\cosh\bar{x} \end{matrix} \Rightarrow dx^2-dt^2=\bar{t}^2d\bar{x}^2-d\bar{t}^2.$$

Now $\bar{x}=$const are the straight lines $x=t\tanh\bar{x}$. This means that in the new coordinate systems the observers emerge from the origin with different velocities $\tanh\bar{x}$. The curves of equal time are the hyperbolas $t^2-x^2=\bar{t}^2$. One easily sees that there points even a finite distance $\bar{x}$ apart are Einstein-synchronized. Though the velocities of the new system do not change with $\bar{t}$, they change with $\bar{x}$; this situation promises to illustrate the point Einstein wanted to make.

B. That is true, except that we will have to add another space dimension. So far the pure space part $t^2-x^2=\bar{t}^2$ is a one-dimensional submanifold, and in one dimension there is never an intrinsic curvature

$$x=\bar{t}\cos\bar{\varphi}\sinh\bar{x},\quad y=\bar{t}\sin\bar{\varphi}\sinh\bar{x},\quad t=\bar{t}\cosh\bar{x},$$

$$dx^2+dy^2-dt^2=d\bar{x}^2\bar{t}^2+d\varphi^2\bar{t}^2\sinh\bar{x}^2-d\bar{t}^2.$$

Now $x^2+y^2-t^2=-\bar{t}^2$, and thus $\bar{t}=$const is a hyperboloid, and one knows that the Minkowski metric when reduced to this submanifold makes it a space of constant curvature.[3,4]

L. Since in this case the Einstein synchronization works even locally, we have exemplified Einstein's claim that for moving coordinates the geometry of space will be non-Euclidean. But let us now turn to the rotating disk and see what happens in reality when the disk is set in rotation. We can model this by a betatron mechanism where an increasing magnetic field generates an azimuthal electric field. If our equipment is again charged, it will be set into circular motion, and the equations of motion will tell us how it behaves. If in cylindrical coordinates (ρ,φ,z,t) I have a vector potential

$$A=(0,\rho\sqrt{\mu^2(\rho-a)^2+\nu^2t^2},0,0)\ \text{with}\ a,\mu,\nu,\in\mathbf{R}^+,$$

it generates a B_z and E_φ such that the equations of motion are solved by[4]

$$(\rho,\varphi,z,t)=\left(a,\varphi_0+\frac{m}{a\nu}\cosh\frac{\nu s}{m},0,\frac{m}{\nu}\sinh\frac{\nu s}{m}\right),$$

where s is the proper time along the orbit. If I use the new time coordinate and the initial angle σ_0 as the other coordinate, I find for fixed $\rho=a$ and $z=0$

$$ds^2=\rho^2d\varphi^2-dt^2=a^2d\bar{\varphi}^2+2a\sinh\frac{\nu\bar{t}}{m}d\bar{\varphi}d\bar{t}-d\bar{t}^2.$$

Thus, in the same way as for the linear acceleration the Einstein synchronization gets lost in the acceleration process.

B. Because of that you also lost the Lorentz contraction of the measuring rods in the φ direction: For the circumference of the disk I calculate, for $\bar{t}=$const, $\int_{\rho=a,0<\bar{\varphi}<2\pi}ds=2\pi$, contrary to Einstein's argument.

L. Let us forget the acceleration process and let B_z become constant after an angular velocity $\omega=\nu/m$ has been reached. Then we synchronize the clocks (for $\rho=a,z=0$) with Einstein's prescription, which must mean that we make a Lorentz transformation

$$\bar{\varphi}=\frac{\varphi-\omega t}{\sqrt{1-\omega^2a^2}},\quad \bar{t}=\frac{t-\omega a^2\varphi}{\sqrt{1-\omega^2a^2}}$$

and $a^2d\varphi^2-dt^2=a^2d\bar{\varphi}^2-d\bar{t}^2$.

B. Now you got Lorentz contraction in the φ direction but at the expense of $\bar{t}$ not being globally defined. If I change φ by 2π, I do not come back to the same value of $\bar{t}$. Since Einstein's argument presupposes that you can go smoothly around the disk, I have to insist on a global definition for $\bar{t}$.

L. So the situation is worse than I thought originally. I anticipated trouble in synchronizing clocks with different ρ and thus different velocities. Now I find that if I Einstein-synchronize clocks with $\rho=a$ and I go around the disk, I do not come back to the same $\bar{t}$.

B. Though φ is not smoothly defined all over the disk, $d\varphi$ is. Therefore I suggest that we do not worry about coordinate systems but only about the basis in the cotangent space. Since they define directions, they also contain the information about how the system is moving. If we use

$$e_1=d\rho,\quad e_2=\frac{\rho d\varphi-\rho\omega dt}{\sqrt{1-\omega^2\rho^2}},$$

$$e_3=dz,\quad e_0=\frac{dt-\omega\rho^2d\varphi}{\sqrt{1-\omega^2\rho^2}},$$

then

$$d\rho^2+\rho^2 d\varphi^2+dz^2-dt^2=e_1^2+e_2^2+e_3^2-e_0^2.$$

Thus, we bypassed the difficulty of global synchronization but kept the Lorentz contraction in the φ direction.

L. But can't we now look for hypersurfaces Σ_t for which e_0 vanishes, $e_0|_{\Sigma_{\bar{t}}}=0$? They would represent pure space for the rotating observer and generalize our surfaces $\bar{t}=0$.

B. Unfortunately, this does not work. Mr. Frobenius has told us that surfaces $\Sigma_{\bar{t}}$ with $e_0|_{\Sigma_{\bar{t}}}=0$ exist if and only if $e_0\wedge de_0=0$ (d=exterior derivative, $\wedge$=exterior product). Here we have

$$e_0\wedge de_0=d\varphi\wedge dp\wedge dt\frac{d}{d\rho}\frac{\omega\rho^2}{\sqrt{1-\omega^2\rho^2}}\neq 0.$$

L. I can see the "only if" part because a surface $\Sigma_{\bar{t}}$ must be defined by some function $\bar{t}=0$, and an e_0 with $e_0|_{\Sigma_{\bar{t}}}=0$ must be of the form $f d\bar{t}$, where f is some function. Thus, $e_0\wedge de_0=f d\bar{t}\wedge df\wedge d\bar{t}=0$. I take your word for the "if" part, since we don't use it. In any case I have to admit defeat and don't see how to define the space-like hypersurface Σ_t, on which supposedly the Minkowski metric induces a non-Euclidean geometry.

B. Perhaps we should not despair but just retreat from the local to the infinitesimal level. After all, curvature is defined by a limiting procedure and belongs to this level. So we have to see whether we can project out the purely spatial part of the curvature even if e_0 is not surface-forming.

L. It certainly cannot be simple projection. After all, we are working in Minkowski space where the curvature is altogether zero, and no matter how you project zero it remains zero.

B. This is certainly correct, and so we have to go through the procedure of how curvature is defined. For simplicity, let us set $z=0$, so that space is spanned by e_1,e_2. Generally, for an orthogonal basis e_k the connection 1-forms are given by

$$\omega_{kj}=i_k de_j-i_j de_k+\frac{1}{2}i_k i_j(de^m\wedge e_m),$$

$$i_k=\text{inner product with } e_k$$

and the curvature 2-forms are

$$R_{kj}=d\omega_{kj}+\omega_k^m\wedge\omega_{mj}.$$

Since our space is 2-dimensional, only $\omega_{12}=-\omega_{21}=i_1 de_2-i_2 de_1$ and $R_{12}=-R_{21}=d\omega_{12}$ remain. Now in this calculation all terms involving e_0 should be deleted if we consider pure space. However, it turns out that this case does not arise, as we have

$$de_1=0, i_1 de_2$$
$$=(d\varphi-\omega dt)\frac{d}{d\rho}\frac{\rho}{\sqrt{1-\omega^2\rho^2}}=e_2\frac{d}{d\rho}\ln\frac{\rho}{\sqrt{1-\omega^2\rho^2}}$$

and

$$R_{12}=d(i_1 de_2)$$
$$=d\rho_1(d\varphi-\omega dt)\frac{d^2}{d\rho^2}\frac{\rho}{\sqrt{1-\omega^2\rho^2}}=e_1 e_2\cdot R,$$

where the curvature scalar is

$$R=\frac{3\omega^2}{(1-\omega^2\rho^2)^2}.$$

L. This result seems reasonable. In particular, one anticipates from Einstein's argument that things become singular for $\omega\rho=1$, where the disk reaches the velocity of light. But note that if one simply says that Einstein's argument suggests that the metric on the disk is $d\rho^2+\rho^2 d\varphi^2/(1-\omega^2\rho^2)$, one gets exactly this curvature and for the circle $\rho=a$ a ratio circumference/radius$=2\pi/\sqrt{1-\omega^2 a^2}$. Thus, naive as the argument may be, it was good enough to show Einstein the right way to general relativity. This proves once again that for a great discovery in physics a good intuition is more important than all the mathematical sophistication.

B. Though I agree with that, I think that our discussion has shown that what you call mathematical sophistication brings to the fore the subtle features of the underlying physics.

[1]A. Einstein, *Über die spezielle und die allgemeine Relativitätstheorie* (Vieweg, 1919).

[2]A. Einstein, Ann. Phys. (Leipzig) **49**, 769 (1916).

[3]L. D. Landau and E. M. Lifshitz, *The Classical Theory of Fields* (Addison-Wesley, Reading, Mass., 1962).

[4]W. Thirring, *Classical Dynamical Systems and Classical Field Theory*, 2nd ed. (Springer-Verlag, New York, 1992).

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