Quantum effects in the interaction of an atom with radiation

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A review is given of recent results in cavity quantum electrodynamics (cavity QED)—the science of the interaction of single atoms or groups of atoms, localized in traps, with the radiation field. As a prototype of cavity QED, the Jaynes–Cummings model and its generalizations are considered. The main results are presented from the study of photon antibunching, sub-Poissonian radiation statistics, squeezing of quantum fluctuations, coherent trapping of populations, and more. The problem of quantum field fluctuations is investigated.

1. INTRODUCTION

The simplest model of the interaction of an atom or molecule with a field of electromagnetic radiation is associated with the notion of a so-called two-level atom, which was introduced by Einstein in his famous study of radiation kinetics.[1] Subsequently, this notion played an important part in quantum mechanics, in particular, in the problem of the natural line width of radiation, which was solved by Weisskopf and Wigner.[2]

The most obvious realization of the notion of a two-level atom is provided by a spin 1/2 in an external magnetic field **B**. In the case of a field parallel to the z axis, the Hamiltonian of such a system is

$$H=-\hbar\gamma BS^z,$$

where S^z is the corresponding component of the spin operator, and $\gamma=g_s\mu_B$ (g_s is the Lande factor, and μ_B is the Bohr magneton) in the case of electron spin, or γ is the gyromagnetic ratio in the case of nuclear spin. Such a Hamiltonian has just two eigenstates with energies $\pm\hbar\gamma B/2$. Thus, we do indeed have here realization of a "two-level atom" with transition frequency equal to the Zeeman frequency $\omega_z=\gamma B$. Such two-level systems are widely used in the physics of nuclear magnetic resonance,[3] in the theory of masers with a paramagnetic active medium,[4] and in the theory of superradiance.[5–8]

With the development of the laser, the notion of a two-level atom entered firmly into the practice of optics.[9,10] The fact is that, using lasers as sources of electromagnetic radiation, one can act on an atom with a field having frequency close to the transition frequency between any pair of levels (optical resonance). In this case, the influence of the other levels can be ignored, and one can restrict consideration to a two-level atom (in the general case, an atom with a finite number of levels).[11] On the other hand, the use of high-Q cavities has the consequence that an atom placed in such a cavity interacts with only one or a few modes of a field quantized in the volume of the cavity.

A new impulse in the investigation of the model of a two-level atom came from recent progress in quantum optics, which led to the creation of sources of the electromagnetic field in "nonclassical" states with suppression of the quantum fluctuations of the field quadratures (the squeezing phenomenon) or the energy (the phenomenon of sub-Poissonian radiation) and the investigation of other fundamentally new quantum effects (photon antibunching, violation of the classical Cauchy–Schwarz inequality, etc.).[12–18] It is important to emphasize that the study of "nonclassical" photon states is extremely important, since it provides qualitatively new information about nonlinear phenomena in quantum fields and opens up perspectives for different applications, from quantum nondemolition measurements to transmission of information in light tubes with minimum noise level and the creation of optical computers. The interaction of light in nonclassical states with two- or finite-level atoms is of considerable interest in connection with problems of the generation of such states and the enhancement of their characteristic properties, for example, the squeezing of quantum fluctuations.

Another reason explaining the importance of the problem of one atom interacting with a cavity field was the development of radio-frequency traps for ions and optical ("viscous") traps for neutral atoms, and the development of methods of laser cooling.[19,20] These achievements make it possible to investigate experimentally the interaction of cavity fields with an individual atom or with groups of a small (regulated) number of atoms in practically a state of rest (effective temperature $\sim 100\ \mu$K). These new experimental possibilities are of fundamental importance, since they make it possible to verify directly several propositions of quantum theory, including complementarity,[21] quantum jumps,[22] and more. We should also mention here the creation of a single-atom maser[23] using microwave transitions of long-lived highly excited (so-called Rydberg) atoms in high-Q cavities, including cavities with superconducting walls.

The field of quantum optics associated with the investigation of processes of interaction of one or a few atoms with one or a few modes of the quantized electromagnetic field is usually called cavity quantum electrodynamics (cavity QED). The theoretical concepts of cavity QED are associated in the first place with investigation of the Jaynes–Cummings (JC) model[24] and its generalizations.

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The reason for this is that the model describes fairly well the physical processes and at the same time admits an exact solution.

Innumerable studies have been made of the JC model, including the reviews of Refs. 25, 51, and 52. It must be emphasized that in the framework of this model it was possible to predict and describe several new phenomena such as the collapse and revival of Rabi quantum oscillations, coherent trapping of populations,[25] and more. We emphasize that the JC model has various generalizations corresponding to inclusion of multimode fields,[26–30] multiphoton transitions,[29–35] multilevel atoms,[36–39] many-atom interactions,[40–43] finite Q of the resonator,[44–48] a dependence of the coupling constant on the intensity,[49,50] etc.

2. THE JAYNES–CUMMINGS MODEL

The Hamiltonian of the JC model can be obtained from the Hamiltonian of classical electrodynamics in the standard manner.[53] Let us consider a nonrelativistic electron interacting with a potential field $U(\mathbf{r})$ and an electromagnetic field with some vector potential $\mathbf{A}(\mathbf{r})$. The classical Hamiltonian of the system has the form

$$H=H_A+H_F+H_{AF}, \tag{1}$$

where

$$H_A=\frac{p^2}{2m}+U(r), \quad H_F=\frac{E^2+H^2}{8\pi},$$

$$H_{AF}=-\frac{e}{mc}\,\mathbf{pA}+\frac{e^2}{2mc^2}A^2.$$

Here, $\mathbf{p}$ is the momentum of the electron, m is its mass, e is the charge, $\mathbf{E}=-(1/c)\partial\mathbf{A}/\partial t$, $\mathbf{B}=\mathrm{curl}\,\mathbf{A}$, and the electromagnetic field is assumed to be purely transverse.

In the framework of perturbation theory, which is usually employed in quantum electrodynamics,[54] the first two terms in (1) are regarded as the Hamiltonian of the unperturbed system, whereas H_{AF} describes a perturbation. The operator $H_0=H_A+H_F$ is defined on the state space

$$\mathscr{H}=\mathscr{H}_A\otimes\mathscr{H}_F. \tag{2}$$

The space $\mathscr{H}_A$ is constructed from the eigenstates ψ_n of the operator H_A:

$$H_A\psi_n=E_n\psi_n. \tag{3}$$

The space $\mathscr{H}_F$ is formed by the Fock states of the field modes:

$$\mathscr{H}_F=\left\{\prod_k\left|n_k\right\rangle_k\right\}.$$

The notion of a two-level atom is associated with the assumption that in the complete set of discrete states ψ_n in (3) we can make a restriction to just two states $|e\rangle$ and $|g\rangle$ (excited and ground), the transition between which corresponds to the frequency of the resonator mode. In this case, a restriction must also be made to just one mode of the field.

As is well known, the states of an atom are characterized by the values of the angular momentum and the parity, and transitions are determined by the well-known selection rules. In the JC model, one usually considers dipole transitions of electric type,[54] although other transitions may be considered.

The problem of constructing the Hamiltonian of the JC model now reduces to calculating the matrix elements of the operator H_{AF} in the chosen basis of states. The contribution of the last term in H_{AF} is usually ignored. More detailed allowance for the A^2 term leads to a mass renormalization.[55]

In the second-quantization representation, the vector field has the form (for one field mode)

$$\mathbf{A}=\sqrt{\frac{2\pi\hbar}{\omega\vartheta}}\,\mathbf{e}(a_k^\dagger e^{-i\mathbf{kr}}+a_k e^{i\mathbf{kr}}), \tag{4}$$

where $\mathbf{e}$ is the unit polarization vector of the mode, ω is its frequency ($\omega=kc$), and ϑ is the quantization volume. Since the wavelength of optical radiation is much greater than the characteristic dimensions of the atom, and the integration in the calculation of the matrix elements between the atomic states is over a region of space restricted to the size of the atom, the contribution from the exponentials in (4) can be taken into account in the zeroth approximation:

$$e^{\pm i\mathbf{kr}}\cong 1.$$

Therefore, for the matrix elements of H_{AF} in the chosen approximation we have

$$\langle e|H_{AF}|e\rangle=\langle g|H_{AF}|g\rangle=0,$$

$$\langle e|H_{AF}|g\rangle=-\frac{e}{mc}\sqrt{\omega}\,\mathbf{e}\langle e|\mathbf{p}|g\rangle.$$

Further, taking into account the relation

$$[\mathbf{r},H_A]=i\frac{\hbar}{m}\,\mathbf{p},$$

we can replace the matrix elements of the momentum by the corresponding elements of the radius vector. Using the definition $\mathbf{d}=e\mathbf{r}$ of the dipole moment, we obtain, thus, the Hamiltonian in the dipole approximation. It is convenient to introduce the atomic operators

$$R^z=(|e\rangle\langle e|-|g\rangle\langle g|)/2,$$

$$R^+=|e\rangle\langle g|,$$

$$R^-=|g\rangle\langle e|,$$

in terms of which the total Hamiltonian (1) of the system consisting of the atom and the radiation field has the form (without allowance for C-number terms)

$$H=H_A+H_F+H_{AF}, \tag{5}$$

where

$$H_A=\hbar\omega_0R^z, \quad \omega_0=(E_e-E_g)/\hbar, \quad H_F=\hbar\omega a^\dagger a,$$

$$H_{AF}=\hbar g(R^++R^-)(a^\dagger+a), \quad g=\sqrt{\frac{2\pi\hbar}{\omega\vartheta}}\,\omega_0.$$

The final step is associated with the elimination from H_{AF} in (5) of the "antiresonance" terms $R^+A^\dagger$ and R^-a, which describe virtual processes of excitation (deexcitation) of the atom with simultaneous creation (annihilation) of a photon (rotating-wave approximation[3]).

As a result, we arrive at the JC model (the model of the interaction of a two-level atom with a mode of a cavity field in the dipole approximation and in the rotating-wave approximation):

$$H_{\rm JC}=\hbar\omega_0 R^z+\hbar\omega a^\dagger a+\hbar g(R^+a+a^\dagger R^-). \tag{6}$$

We emphasize that a Hamiltonian having a similar operator structure can be obtained by means of the interaction Hamiltonian in the dipole approximation: $H_{AF}=-\mathbf{dE}$. However, a different value of the coupling constant g is then obtained. A detailed discussion of this problem can be found in the review of Ref. 56.

3. EXACT SOLUTION IN THE JAYNES–CUMMINGS MODEL

As already noted in the Introduction, the quantum problem with the Hamiltonian (6) can be solved exactly—both in the Schrödinger representation[24,57] and in the Heisenberg representation.[11,58,59] Here, we shall use the approach associated with the concept of "dressed" states. This approach is convenient for the interpretation of many physical phenomena.

The Hamiltonian (6) contains only combinations of operators that describe the exchange of one excitation quantum between the atom and the field. For example, $R^\dagger a$ decreases the number of photons by unity and simultaneously increases the energy of the atom by carrying it to the excited state. Therefore, the number of excitations in the system is constant, and the operator $R^z+a^\dagger a$ of the number of excitations is an integral of the motion, as is readily verified. It is therefore convenient to express H in the form

$$H=H_{\rm I}+H_{\rm II},\quad H_{\rm I}=\hbar\omega(a^\dagger a+R^z),$$

$$H_{\rm II}=\hbar\Delta R^z+\hbar g(R^+a+R^-a^\dagger), \tag{7}$$

where $\Delta=\omega_0-\omega$. It is also readily seen that $H_{\rm II}$ is also an integral of the motion and

$$[H_{\rm I},H_{\rm II}]=0, \tag{8}$$

which leads to factorization of the time-evolution operator $U(t)$:

$$U(t)=\exp\left(\frac{-iHt}{\hbar}\right)=U_{\rm I}(t)\,U_{\rm II}(t), \tag{9}$$

where

$$U_{\rm I}(t)=\exp\left(\frac{-iH_{\rm I}t}{\hbar}\right),\quad U_{\rm II}(t)=\exp\left(\frac{-iH_{\rm II}t}{\hbar}\right). \tag{10}$$

Since the Hamiltonian (6) couples a state of the type $|e;n\rangle$ only to a state $|g;n+1\rangle$, these states can be used as a basis in a two-dimensional (n)-subspace of the state space, in which the matrix representation for $H_{\rm I}$ and $H_{\rm II}$ has the form

$$H_{\rm I}=\hbar\omega\left(n+\frac{1}{2}\right)I,\quad H_{\rm II}=\begin{pmatrix} H\Delta/2 & \hbar g\sqrt{n+1} \\ \hbar g\sqrt{n+1} & -\hbar\Delta/2\end{pmatrix}. \tag{11}$$

Here I is the unit 2×2 matrix. It is now easy to find the eigenvalues and eigenstates for $H_{\rm II}$:

$$H_{\rm II}|\psi_n^\pm\rangle=\pm\hbar\lambda_n|\psi_n^\pm\rangle,\quad \lambda_n=\sqrt{g^2(n+1)+\Delta^2/4},$$

$$|\psi_n^\dagger\rangle=\left(\frac{\lambda_n-\Delta/2}{2\lambda_n}\right)^{1/2}|n+1;g\rangle+\left(\frac{\lambda_n+\Delta/2}{2\lambda_n}\right)^{1/2}|n;e\rangle, \tag{12}$$

$$|\psi_n^-\rangle=-\left(\frac{\lambda_n+\Delta/2}{2\lambda_n}\right)^{1/2}|n+1;g\rangle+\left(\frac{\lambda_n-\Delta/2}{2\lambda_n}\right)^{1/2}|n;e\rangle,$$

$$n=0,1,2,\ldots,\infty,$$

and also

$$H_{\rm II}|0;g\rangle=-\frac{\hbar\Delta}{2}|0;g\rangle. \tag{13}$$

By means of (12) and (13), we obtain

$$\exp(-iH_{\rm II}t/\hbar)|n;e\rangle=A_{n,e}(t)|n;e\rangle+B_{n,e}(t)|n+1;g\rangle, \tag{14}$$

$$\exp(-iH_{\rm II}t/\hbar)|n+1;g\rangle=A_{n+1,g}(t)|n+1;g\rangle+B_{n+1,g}(t)|n;e\rangle,$$

where

$$A_{n,e}(t)=\cos\lambda_n t-i\frac{\Delta}{2\lambda_n}\sin\lambda_n t,\quad A_{n+1,g}(t)=\cos\lambda_n t+i\frac{\Delta}{2\lambda_n}\sin\lambda_n t,$$

$$A_{0,g}(t)=\exp\left(\frac{i\Delta t}{2}\right),\quad B_{0,g}(t)=0, \tag{15}$$

$$B_{n,e}(t)=B_{n+1,g}(t)=-i\frac{g\sqrt{n+1}}{\lambda_n}\sin\lambda_n t.$$

As was noted in Ref. 25, it is natural to interpret the operator $U_{\rm I}(t)$ as the quantum-field version of the semiclassical unitary operator of transformation from the laboratory system to a rotating frame of reference, and the operator $U_{\rm II}(t)$ describes the time evolution of the system in the new system. It is often convenient to omit $U_{\rm I}(t)$ altogether and work in the intermediate "II representation," which was proposed by Yoo and Eberly.[25] This representation is identical to the ordinary interaction representation in the case of exact resonance, $\Delta=0$.

Using the results (14) and (15), we can readily construct the state of the atom+field system at any time t and investigate the behavior in time and the statistical properties of the observables for different initial conditions.

4. COLLAPSE AND REVIVAL OF QUANTUM OSCILLATIONS OF THE LEVEL POPULATION

Using the exact solution in the JC model, we now investigate the time evolution of the population difference (inversion) of the atom.

Suppose that at $t=0$ the field is in an arbitrary state described by the density matrix $\rho_f=\Sigma_{n,n'}p_{n,n'}|n\rangle\langle n'|$, and the atom is in the upper level. Then the density matrix $\rho(0)$ of the complete atom+field system has the form

$$\rho(0)=\sum_{n,n'}p_{n,n'}|n;e\rangle\langle n';e|. \tag{16}$$

Using (14), we obtain for $\rho(t)$ in the representation II the expression

$$\begin{aligned}\rho(t)&=\exp(-iH_{\rm II}t/\hbar)\rho(0)\exp(iH_{\rm II}t/\hbar)\\&=\sum_{n,n'}p_{n,n'}[A_{n,e}(t)|n;e\rangle+B_{n,e}(t)|n+1;g\rangle]\\&\quad\times[A^*_{n',e}(t)\langle n';e|+B^*_{n',e}(t)\langle n'+1;g|].\end{aligned} \tag{17}$$

The time-dependent expectation value of the atomic inversion can now be readily calculated:

$$\begin{aligned}W(t)&\equiv2\langle R^z(t)\rangle=2\,{\rm Tr}|R^z(0)\rho(t)|\\&=\sum_{n=0}^{\infty}p_{nn}\cos(2gt\sqrt{n+1}).\end{aligned} \tag{18}$$

For simplicity we have here assumed exact resonance: $\Delta=0$. The sum (18) plays a central role in the investigation of the phenomenon of collapse and revival of the Rabi quantum oscillations. For the initial field Fock state $|n\rangle$,

$$W(t)=\cos(2gt\sqrt{n+1}), \tag{19}$$

and this is actually not only qualitatively but also quantitatively identical to the semiclassical result:[11]

$$W(t)=\cos\left(\frac{2d}{\hbar}\mathscr{E}_0t\right), \tag{20}$$

where d is the matrix element of the dipole moment, and $\mathscr{E}_0$ is the amplitude of the applied field. However, if the initial field was the vacuum ($n=0$), then in accordance with the quantum-electrodynamic solution (19) the energy of the atom oscillates with the Rabi frequency $2g$. On the other hand, the semiclassical result (20) corresponding to zero field does not oscillate at all. The quantum-electrodynamic solution differs still more from the semiclassical one if the field at the initial time $t=0$ was in a state with a certain spread over the occupation number. An example of such states is provided by the coherent state with weight factor

$$p_{nn}=\exp(-\bar{n})\frac{\bar{n}^n}{n!}, \tag{21}$$

or the random (thermal) state with weight factor

$$p_{nn}=\frac{\bar{n}^n}{(\bar{n}+1)^{n+1}}, \tag{22}$$

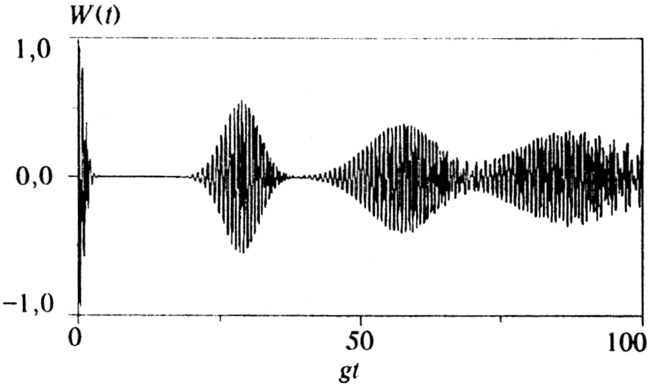

FIG. 1. Time evolution of the inversion of the level populations for the case when the atom is initially in the upper level and the field is in the coherent state. The mean photon number is $\bar{n}=20$; $\Lambda=0$.

where $\bar{n}$ is the mean number of photons for the state. The time behavior of the atomic inversion $W(t)$ is shown in Fig. 1 for an atom initially in the upper level and the field in a coherent state. As can be seen from the figure, the evolution of the inversion is characterized by alternating dampings and revivals of the Rabi oscillations.

Although the collapse of the Rabi quantum oscillations was predicted many years ago,[57,60,61] their revival, or rather, the series of successive collapses and revivals, was discovered and investigated only recently.[62–64] This phenomenon can be explained as follows. The atomic inversion (18) consists of a sum of terms representing the Rabi oscillations and due to a definite number n of photons. At the initial time $t=0$, the system is prepared in a quite definite state, and all these processes are correlated. At $t>0$, because all the terms in (18) oscillate with different frequencies, appreciable phase shifts between them are established, and the averaging of the contribution of these terms on the background of the smooth distribution p_{nn} leads to the collapse of the Rabi quantum oscillations. After a certain time, the phases of the oscillations of neighboring terms in (18) become multiples of 2π for $n\sim\bar{n}$. Then the initial time behavior of the Rabi oscillations of the level population inversion is partly restored, and then they again collapse. This argument enables us to determine the interval T_R between revivals from the simple relation

$$(2g\sqrt{\bar{n}+1}-2g\sqrt{\bar{n}})T_R=2\pi \tag{23}$$

or

$$T_R=\frac{2\pi\sqrt{\bar{n}}}{g}, \tag{24}$$

which agrees with the result of the analytic calculation of Refs. 63 and 64. As was noted above, the revival of the Rabi oscillations is not complete. With increasing interaction time, the peaks of the revivals spread and begin to overlap, and, ultimately, the Rabi oscillations acquire a random nature. The greater the dispersion of the field intensity, the sooner the random regime commences.

An analytic investigation of the phenomenon of collapse and revival of the Rabi oscillations was made in Refs. 63 and 64. The phenomenon has also been investigated for a cavity field initially in random,[65–67] binomial,[68] squeezed,[69] squeezed Fock,[70] displaced Fock,[71] and other states. Although the collapse and revival of the Rabi oscillations have been observed experimentally for an initial random (thermal) state of the cavity field,[72] which does not need a special preparation, the analytic calculation for this case is rather complicated.[67]

In the following section, we consider the influence of the initial state of the atom on the time behavior of the different dynamical variables.

5. INFLUENCE OF ATOMIC COHERENCE ON THE COLLAPSE AND REVIVAL OF RABI OSCILLATIONS

The idea of preparing an atom in a coherent superposition of states before switching on the interaction with the field became popular because of its possible application to the suppression of noise by correlated spontaneous emission,[73] in the investigation of quantum beats,[74] and for other purposes. Knight and his collaborators[75–77] showed that even spontaneous emission from an appropriately phased atom could manifest squeezing. Agarwal and Puri[78,79] and Zaheer and Zubiary[80] found that if the relative phase between the atomic dipole and coherent field was tuned in such a way as to make the initial state of the two-level atom an eigenstate of the semiclassical Hamiltonian,[79] then the level population inversion remains almost unchanged and the spectrum of the resonance fluorescence has an asymmetric two-peak structure instead of a symmetric three-peak structure.

A deeper explanation of the influence of atomic coherence on the Rabi oscillations was given in Ref. 81, in which a study was made of the interaction of an atom with the field in a binomial state:[82]

$$|p,N\rangle=\sum_{n=0}^{N}\left[\frac{N!p^n(1-p)^{N-n}}{(N-n)!n!}\right]^{1/2}|n\rangle,\quad 0<p<1. \tag{25}$$

Such a state is transformed into a coherent state in the limit $p\to 0$, $N\to\infty$, $Np=\text{const}$, and the Fock state $|N\rangle$ in the limit $p\to 1$. It was found that not only the amplitude of the revivals but also their structure are sensitive to the phase of the coherence between the atomic levels. When the atom at the initial time is in a state that is an eigenstate of the semiclassical Hamiltonian, the level inversion [the correlation function $g^{(2)}(t)$] has the form of a doublet (triplet) instead of a singlet (doublet) of revivals. However, this effect was found in Ref. 81 only for a field in a strongly binomial state, i.e., when $p\neq 0$, 1. It was shown in Ref. 83 that such a change in the structure of the revivals also occurs for an initial coherent field.

Suppose that at $t=0$ the atom is in a coherent superposition of the excited and ground states:

$$|\psi_a(0)\rangle=\frac{1}{\sqrt{1+|\mu|^2}}(|e\rangle+\mu|g\rangle), \tag{26}$$

and the cavity field is in the coherent state

$$|\psi_f(0)\rangle=\sum_{n=0}^{\infty}b_n\exp(in\varphi)|n\rangle,\qquad b_n=\exp\left(-\frac{\bar n}{2}\right)\left(\frac{\bar n^n}{n!}\right)^{1/2}. \tag{27}$$

The weight factor p_{nn} is obviously $p_{nn}=b_n^2$. Then for the complete atom + field system

$$|\psi(0)\rangle=\frac{1}{\sqrt{1+|\mu|^2}}\sum_{n=0}^{\infty}b_n\exp(in\varphi)(|n;e\rangle+\mu|n;g\rangle). \tag{28}$$

In the case of exact resonance, the state vector of the system at the time t in the interaction representation can readily be found by means of (14) and (15):

$$\begin{aligned}|\psi(t)\rangle=&\frac{1}{\sqrt{1+|\mu|^2}}\sum_{n=0}^{\infty}b_n\exp(in\varphi)\\&\times\{-i\sin(gt\sqrt{n+1})|n+1;g\rangle\\&+\cos(gt\sqrt{n+1})|n;e\rangle+\mu[\cos(gt\sqrt{n})|n;g\rangle\\&-i\sin(gt\sqrt{n})|n-1;e\rangle]\}.\end{aligned} \tag{29}$$

Then for the population $P_e(t)$ of the upper level we obtain

$$P_e(t)=\tfrac{1}{2}+\langle R^z(t)\rangle, \tag{30}$$

where

$$\begin{aligned}\langle R^z(t)\rangle=&\frac{1}{2(1+|\mu|^2)}\sum_n b_n^2\Big[\cos(2gt\sqrt{n+1})\\&-|\mu|^2\cos(2gt\sqrt{n})+2|\mu|\sin(\varphi+\eta)\\&\times\left(\frac{\bar n}{n+1}\right)^{1/2}\sin(2gt\sqrt{n+1})\Big],\end{aligned} \tag{31}$$

and $\mu=|\mu|\exp(i\eta)$. For $\bar n\gg 1$, replacing the sum in (31) by an integral and integrating the resulting expressions by the method of steepest descent,[64] we obtain the expression

$$\begin{aligned}\langle R^z(t)\rangle\cong&\frac{1}{2(1+|\mu|^2)}\sum_{k=0}\frac{1}{(1+\pi^2k^2)^{1/4}}\Big[I_k+2|\mu|\\&\times\sin(\varphi+\eta)\left(1-\frac{\pi k\varepsilon_k}{1+\pi^2k^2}\right)J_k\Big],\end{aligned} \tag{32}$$

where

$$\begin{aligned}I_k=&\exp(-\bar n\Psi_k\varepsilon_k^2)[\cos(\bar n\Phi_k+\alpha_k+\beta_k\varepsilon_k)\\&-|\mu|^2\cos(\bar n\Phi_k+\alpha_k)],\\J_k=&\exp(-\bar n\Psi_k\varepsilon_k^2)\sin(\bar n\Phi_k+\alpha_k+\beta_k\varepsilon_k),\\\Psi_k=&\frac{2}{1+\pi^2k^2},\quad\Phi_k=2\left(\pi k+2\varepsilon_k+\frac{\pi k\varepsilon_k^2}{1+\pi^2k^2}\right),\\\alpha_k=&-\frac{1}{2}\tan^{-1}(\pi k),\quad\beta_k=\frac{2}{1+\pi^2k^2},\end{aligned} \tag{33}$$

and

$$\varepsilon_k=\tau-k\pi, \quad k=0,1,2,\dots. \tag{34}$$

here, ε_k is a local time near the kth saddle point. The true time t is given by

$$t=\frac{2\tau\sqrt{\bar{n}}}{g}. \tag{35}$$

It can be seen from Eqs. (33) that the behavior of J_k consists of sinusoidal oscillations having as envelope a Gaussian function with a maximum at $\varepsilon_k=0$. To determine the behavior of I_k, we rewrite this function in the form

$$I_k=\exp(-\bar{n}\Psi_k\varepsilon_k^2)\left[(1-|\mu|^2)^2+4|\mu|^2\times\sin^2\left(\frac{\beta_k\varepsilon_k}{2}\right)\right]^{1/2}\sin(\bar{n}\Phi_k+\alpha_k+\theta_k), \tag{36}$$

where

$$\theta_k=\tan^{-1}\left[\frac{|\mu|^2-\cos(\beta_k\varepsilon_k)}{\sin(\beta_k\varepsilon_k)}\right]. \tag{37}$$

Since $\beta_k\ll\bar{n}$, the oscillations of I_k are largely due to the final sinusoidal function in (36). These oscillations have an envelope that is the product of a Gaussian function and a slowly oscillating function. When $|\mu|\neq1$, the envelope for I_k has one maximum at $\varepsilon_k=0$. This is also true of the envelope for J_k. Therefore, $P_e(t)$ exhibits in this case revival singlets.

When $|\mu|=1$, we obtain from (36)

$$I_k=2\exp(-\bar{n}\Psi_k\varepsilon_k^2)\sin\left(\frac{\beta_k\varepsilon_k}{2}\right)\sin(\bar{n}\Phi_k+\alpha_k+\theta_k). \tag{38}$$

The envelope for I_k now possesses two maxima at $\varepsilon_k=\pm1/(2\bar{n}\Psi_k)^{1/2}$ and is equal to zero at $\varepsilon_k=0$. Obviously, if the relative phase between the cavity field and the atomic dipole satisfies the conditions $\varphi+\eta=0$, only I_k makes a contribution, and this leads to revival doublets displaced from the center of the kth revival by intervals $\pm1/(2\bar{n}\Psi_k)^{1/2}$. Note further that, since $\bar{n}\gg1$, the maximum of the envelope for I_k has order $(\bar{n})^{-1/2}$ compared with the maximum at $\varepsilon_k=0$ of the envelope for J_k. Therefore, for $\psi+\eta\neq0$ the contribution from J_k dominates over the contribution of I_k, and this leads to a singlet structure of the revivals.

The revival doublets manifested in the time evolution of $P_e(t)$ for $|\mu|=1$, $\varphi+\eta=0$ are shown in Fig. 2 (curve a) and are compared with the singlet behavior of the revivals in the case when the atom was initially excited (curve b). When the mean number of photons decreases, the neighboring revivals begin to spread in time and ultimately overlap, and this leads to disappearance of the revival structure.

We now turn to the calculation of the second-order correlation function

$$g^{(2)}(t)=\frac{\langle a^{\dagger2}a^2\rangle}{\langle a^\dagger a\rangle^2}-1. \tag{39}$$

Using the state vector (29), we obtain

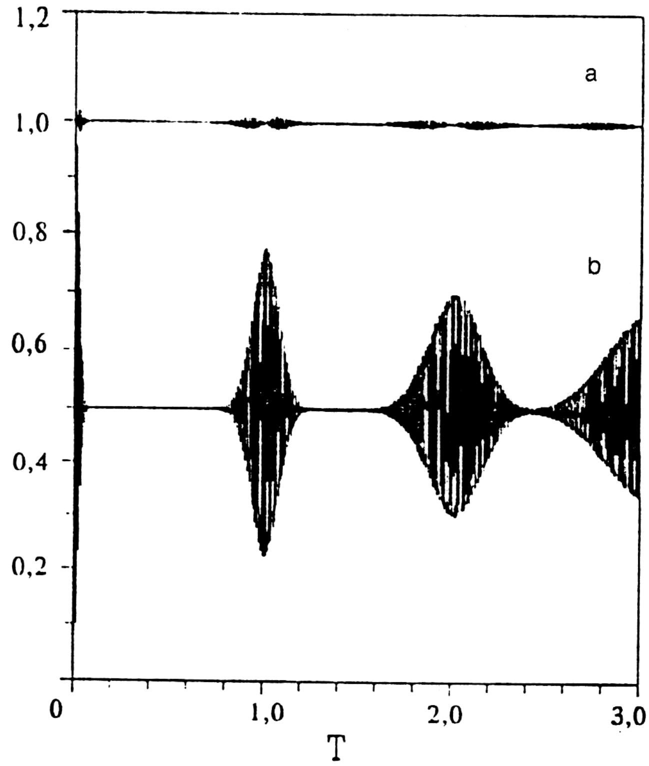

FIG. 2. Evolution of the probability for finding the atom in the upper state $P_e(t)$ for a) $|\mu|=1$, $\eta+\varphi=0$ [$P_e(t)+0.5$] and b) $|\mu|=0$ [$P_e(t)$].

$$\langle a^\dagger a\rangle=\bar{n}+\frac{1}{1+|\mu|^2}-P_e(t), \tag{40}$$

$$\langle a^{\dagger2}a^2\rangle=\frac{1}{1+|\mu|^2}\Big\{\bar{n}^2(1+|\mu|^2)+\bar{n}(1-|\mu|^2)+|\mu|^2-\sum_n b_n^2\Big[n\cos(2gt\sqrt{n+1})-|\mu|^2(n-1)\times\cos(2gt\sqrt{n})+2|\mu|\sin(\varphi+\eta)\times n\left(\frac{\bar{n}}{N+1}\right)^{1/2}\sin(2gt\sqrt{n+1})\Big]\Big\}. \tag{41}$$

Equation (40) can be obtained directly from the fact that the operator $\hat{N}=a^\dagger a+|e\rangle\langle e|$ is an integral of the motion. For large $\bar{n}$, we can write

$$\frac{\langle a^{\dagger2}a^2\rangle}{\langle a^\dagger a\rangle^2}\simeq\frac{\langle a^{\dagger2}a^2\rangle}{\bar{n}^2}\left(1-\frac{1-|\mu|^2}{\bar{n}(1+|\mu|^2)}+2\frac{\langle R^z\rangle}{\bar{n}}\right). \tag{42}$$

Substituting (31) and (41) in (42) and using the method of steepest descent, we obtain

$$\frac{\langle a^{\dagger2}a^2\rangle}{\langle a^\dagger a\rangle^2}\simeq\frac{1}{\bar{n}^2(1+|\mu|^2)}\Big\{\bar{n}^2(1+|\mu|^2)+\frac{3|\mu|^2-1}{1+|\mu|^2}-\sum_k\frac{1}{(1+\pi^2k^2)^{1/4}}[F_k+2|\mu|\times\sin(\varphi+\eta)G_k]$$

$$+\text{other terms of order } \bar{n}^0$$

$$\left.+\text{terms of order } (\bar{n})^{-1} \text{ and lower}\right\}, \tag{43}$$

where

$$F_k=\exp(-\bar{n}\Psi_k\varepsilon_k^2)\times\left\{\frac{2\bar{n}\pi k\varepsilon_k}{1+\pi^2k^2}\sqrt{(1-|\mu|^2)^2+4|\mu|^2\sin^2\left(\frac{\beta_k\varepsilon_k}{2}\right)}\times\sin(\bar{n}\Phi_k+\alpha_k+\theta_k)+\frac{1}{1+|\mu|^2}\left[(1+|\mu|^2)^2-4|\mu|^2(1-2|\mu|^2)+8|\mu|^2(1-|\mu|^2)\times\sin^2\left(\frac{\beta_k\varepsilon_k}{2}\right)\right]^{1/2}\sin(\bar{n}\Phi_k+\alpha_k+\varphi_k)\right\}, \tag{44}$$

$$G_k=\exp(-\bar{n}\Psi_k\varepsilon_k^2)\,\frac{2\bar{n}\pi k\varepsilon_k}{1+\pi^2k^2}\sin(\bar{n}\Phi_k+\alpha_k+\beta_k\varepsilon_k), \tag{45}$$

$$\varphi_k=\tan^{-1}\left[\frac{(1-|\mu|^2)\cos(\beta_k\varepsilon_k)-2|\mu|^2}{(1-|\mu|^2)\sin(\beta_k\varepsilon_k)}\right]. \tag{46}$$

It is easy to see that G_k takes the form of sinusoidal oscillations with an envelope having two maxima at the points $\varepsilon_k=\pm 1/(2\bar{n}\Psi_k)^{1/2}$. The magnitude of these maxima is proportional to $(\bar{n})^{1/2}$. This is also true of the first term in F_k, when $|\mu|\neq 1$, whereas the envelope for the second term in F_k in the given situation has only one maximum of order $\bar{n}^0$ at $\varepsilon_k=0$. Thus, for $|\mu|\neq 1$ the contributions to $g^{(2)}(t)$ from G_k and from the first term in F_k are dominant, and this leads to a doublet structure of the revivals in the time behavior $g^{(2)}(t)$. This is shown in Fig. 3 (curve b), which gives $g^{(2)}(t)$ as a function of the effective time

$$T\equiv\frac{t}{T_R}=\frac{gt}{2\pi\sqrt{\bar{n}}} \tag{47}$$

for an atom that at the initial time is in the excited state ($|\mu|=0$).

For $|\mu|=1$, the expression (44) for F_k takes the form

$$F_k=\exp(-\bar{n}\Psi_k\varepsilon_k^2)\left\{\frac{2\bar{n}\pi k\varepsilon_k}{1+\pi^2k^2}\sin\left(\frac{\beta_k\varepsilon_k}{2}\right)\times\sin(\bar{n}\Phi_k+\alpha_k+\theta_k)+\sin(\bar{n}\Phi_k+\alpha_k+\varphi_k)\right\}. \tag{48}$$

For $\varepsilon_k=\pm 1/(2\bar{n}\Psi_k)^{1/2}$, since the envelope for the first term F_k has a maximum of order $\bar{n}^0$, it is necessary to take into account other terms of the same order in (43). However, it can be shown that the terms already retained in (43) make a nonvanishing contribution to F_k, whereas the remainder make a contribution of order $(\bar{n})^{-1}$. The envelope for F_k in this case has three peaks at the points $\varepsilon_k=0$, $\varepsilon_k=\pm 1/(2\bar{n}\Psi_k)^{1/2}$. For $\varphi+\eta=0$, only F_k contributes, and a triplet structure of revivals is manifested in the time evolution $g^{(2)}(t)$, as can be seen from Fig. 3 (curve a) and Fig. 4. In Fig. 4, the revival of the oscillations $g^{(2)}(t)$ corresponding to $k=1$, $|\mu|=1$, $\varphi+\eta=0$ is shown on an increased scale to exhibit the triplet structure of the revivals more clearly. In the situation in which $|\mu|=1$ and $\varphi+\eta\neq 0$ both terms—G_k and F_k—contribute. However, as was said above, the contribution from G_k has the order $(\bar{n})^{1/2}$, and the contribution from F_k has the order $\bar{n}^0$. Since the envelope for G_k has two maxima at the points $\varepsilon_k=\pm 1/(2\bar{n}\Psi_k)^{1/2}$ and the contribution from G_k is dom-

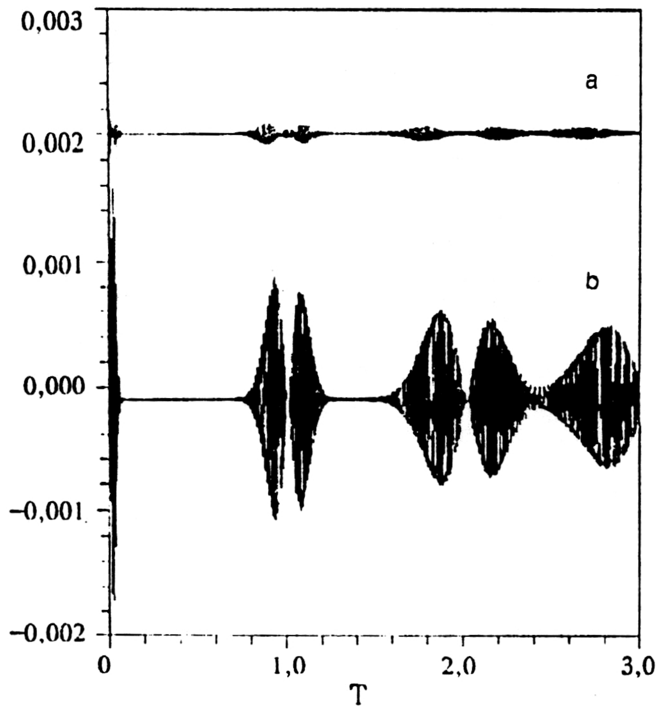

FIG. 3. Dependence of the correlation function $g^{(2)}(t)$ on the dimensionless time T for a) $|\mu|=1$, $\eta+\varphi=0$ [$g^{(2)}(t)+0.0018$] and b) $|\mu|=0$ [$g^{(2)}(t)$]. The mean photon number is $\bar{n}=50$.

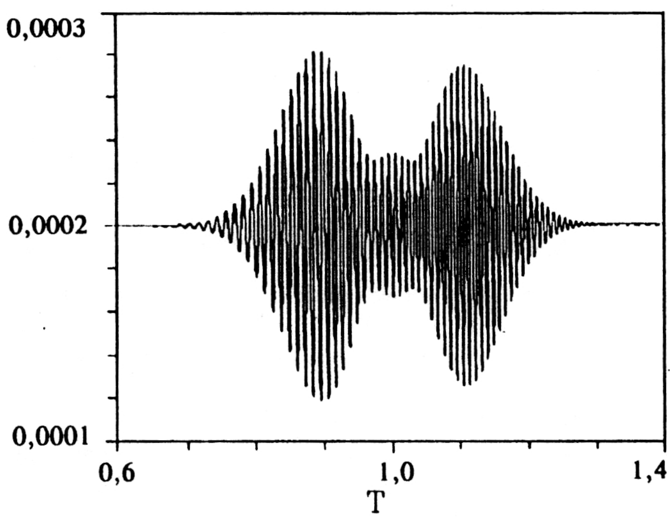

FIG. 4. Triplet of revivals of $g^{(2)}(t)$ on an enlarged scale for $|\mu|=1$, $\eta+\varphi=0$; $\bar{n}=50$.

inant, the time behavior of $g^{(2)}(t)$ exhibits revival doublets with centers at the points $\varepsilon_k=0$ (see Fig. 3, curve b).

6. SQUEEZED LIGHT IN THE JAYNES–CUMMINGS MODEL

Recently, there has been considerable study of the phenomenon of squeezing of the quantum fluctuations of the quadratures of the electromagnetic field. The essence of the phenomenon is that in some correlated states which arise in nonlinear processes of photon generation the quantum fluctuations of one of the field quadratures have a value less than in the vacuum state, which determines the shot-noise limit.[12–18]

A squeezed coherent state of a single-mode radiation field can be obtained from the vacuum $|0\rangle$ as[84]

$$|\alpha,z\rangle \equiv D(\alpha)S(z)|0\rangle, \tag{49}$$

where $S(z)$ is the squeezing operator,

$$S(z)=\exp\left[\frac{1}{2}(z^*a^2-za^{\dagger 2}\right], \tag{50}$$

and $D(\alpha)$ is the displacement operator:

$$D(\alpha)=\exp(\alpha a^\dagger-\alpha^* a). \tag{51}$$

For $z=0$ we have a coherent state; for $\alpha=0$, a squeezed vacuum. The distribution over the Fock states of the squeezed coherent state is given by the expression[85,86]

$$P_{nn}=|q_n|^2,\quad q_n\equiv\langle n|\alpha,z\rangle$$
$$=\frac{1}{\sqrt{n!\mu}}\left(\frac{\nu}{2\mu}\right)^{n/2}H_n\left[\frac{\beta}{\sqrt{2\nu\mu}}\right]\exp\left(-\frac{1}{2}|\beta|^2+\frac{\nu^*}{2\mu}\beta^2\right), \tag{52}$$

where $\mu=\cosh|z|$, $\nu=(z/|z|)\cosh|z|$, $\beta=\mu\alpha+\nu\alpha^*$, and H_n is a Hermite polynomial of degree n. Note that the state (49) is an eigenstate of the operator of annihilation of the "new" Bose field $b=\mu a+\nu a^\dagger$ (Refs. 87 and 88) obtained from a and $a^\dagger$ by means of a Bogolyubov canonical transformation:[89]

$$b|\alpha,z\rangle=\beta|\alpha,z\rangle. \tag{53}$$

In what follows, we shall assume that α and z are real and positive.

We consider the appearance of squeezing in the Jaynes–Cummings model with many-photon transitions.[91] The effective Hamiltonian for such a system has the form

$$H=\hbar\omega a^\dagger a+\hbar\omega_0 R^z+\hbar g(R^+a^m+R^-a^{\dagger m}), \tag{54}$$

where ω and $\omega_0=m\omega$ (we consider the case of resonance) are the frequencies of the field and the atom, g is the coupling constant, m is the photon multiplicativity of the transition, R^z and $R^\pm$ are atomic operators, and $a^\dagger$ and a are field creation and annihilation operators.

We assume that the atom is initially in the ground state $|g\rangle$ and the initial field is in the coherent state $|\alpha\rangle$:

$$|\alpha\rangle=\exp(-|\alpha|^2/2)\sum_{n=0}^{\infty}\frac{\alpha^n}{\sqrt{n!}}|n\rangle. \tag{55}$$

Then the wave function of the system in the interaction representation has the form

$$|\psi(t)\rangle=\sum_{n=0}^{\infty}(|g;n\rangle A_-^{(n)}(t)-i|e;n-m\rangle$$
$$\times A_+^{(n)}(t))\frac{\alpha^n}{\sqrt{n!}}e^{-|\alpha|^2/2}, \tag{56}$$

where

$$A_-^{(n)}(t)=\cos\left(gt\sqrt{\frac{n!}{(n-m)!}}\right),$$
$$A_+^{(n)}(t)=\sin\left(gt\sqrt{\frac{n!}{(n-m)!}}\right). \tag{57}$$

Therefore, the values of the photon number $\langle a^\dagger a\rangle$, the photon amplitudes $\langle a\rangle$, and the square of the photon amplitudes $\langle a^2\rangle$ can be readily obtained in the form

$$\langle a^\dagger a\rangle\equiv\sigma_0=\bar{n}-m\sum_{n=0}^{\infty}p_nA_+^{(n)2},$$
$$e^{i\omega t}\langle a\rangle\equiv\alpha\sigma_1$$
$$=\alpha\sum_{n=0}^{\infty}p_n\left(A_-^{(n)}A_-^{(n+1)}\right.$$
$$\left.+A_+^{(n)}A_+^{(n+1)}\sqrt{1-\frac{m}{(n+1)}}\right), \tag{58}$$
$$e^{2i\omega t}\langle a\rangle\equiv\alpha^2\sigma_2$$
$$=\alpha^2\sum_{n=0}^{\infty}p_n\left(A_-^{(n)}A_-^{(n+2)}\right.$$
$$\left.+A_+^{(n)}A_+^{(n+2)}\sqrt{\left(1-\frac{m}{(n+2)}\right)\left(1-\frac{m}{(n+1)}\right)}\right).$$

Here, p_n is the Poisson distribution (21) corresponding to the initial coherent field state (55).

We introduce two slowly varying Hermitian quadrupole components of the field:

$$a_1=\frac{1}{2}(ae^{i(\omega t-\theta)}+a^\dagger e^{-i(\omega t-\theta)}),$$
$$a_2=\frac{1}{2i}(ae^{i(\omega t-\theta)}-a^\dagger e^{-i(\omega t-\theta)}), \tag{59}$$

where θ is a certain phase angle. The condition of squeezing in the quadrupole components can be written as[12]

$$S_{1,2}<0, \tag{60}$$

where the so-called squeezing factor has the form

$$S_{1,2}=\frac{(\Delta a_{1,2})^2-(\Delta a_{1,2})^2_{\text{vac}}}{(\Delta a_{1,2})^2_{\text{vac}}}=4\langle(a_{1,2}-\langle a_{1,2}\rangle)^2\rangle-1. \tag{61}$$

Note that states for which the condition (60) is satisfied need not be states with minimum uncertainty or, in other words, are not squeezed coherent states.

In terms of the photon operators, we can readily find that

$$S_1=2\langle a^\dagger a\rangle+2\mathrm{Re}\langle a^2 e^{2i(\omega t-\theta)}\rangle-4(\mathrm{Re}\langle a e^{i(\omega t-\theta)}\rangle)^2,$$
$$S_2=2\langle a^\dagger a\rangle-2\mathrm{Re}\langle a^2 e^{2i(\omega t-\theta)}\rangle-4(\mathrm{Im}\langle a e^{i(\omega t-\theta)}\rangle)^2. \quad (62)$$

It can be seen from (58) and (62) that the values of σ_0, σ_1, and σ_2 are real numbers, and therefore the optimum choice of θ to achieve maximum squeezing is $\theta=\varphi$ or $\theta=\varphi+\pi/2$, where φ is the phase of α, i.e., $\alpha=\bar{n}^{1/2}\exp(i\varphi)$. By virtue of the periodicity $S_1(\theta+\pi/2)=S_2(\theta)$, we shall consider below only the case $\theta=\varphi$. Then Eqs. (62) are transformed to

$$S_1=2\sigma_0+2\bar{n}\sigma_2-4\bar{n}\sigma_1^2, \quad (63)$$

$$S_2=2\sigma_0-2\bar{n}\sigma_2. \quad (64)$$

For short times, $gt\ll 1$, using series expansions of the sine and cosine, we find from (63), (57), and (58) the asymptotic expressions

$$S_1\cong\begin{cases}-\frac{1}{3}\bar{n}(gt)^4 & \text{in the case } m=1;\\ -m(m-1)\bar{n}^{m-1}(gt)^2 & \text{in the case } m\geqslant 2.\end{cases} \quad (65)$$

These negative expressions indicate the appearance of squeezing in a_1 for any photon multiplicity n of the transition and any initial nonzero intensity $\bar{n}$ directly after the atom + field interaction has been switched on. If the atom is in the excited level,[90] then such behavior is absent. Similarly, we find asymptotic expressions S_2 at small times $gt\ll 1$ from (64), (57), and (58) in the form

$$S_2\cong\begin{cases}\frac{1}{3}\bar{n}(gt)^4 & \text{in the case } m=1;\\ m(m-1)\bar{n}^{m-1}(gt)^2 & \text{in the case } m\geqslant 2.\end{cases} \quad (66)$$

The positive expressions indicate the absence of squeezing in a_2 at the beginning of the interaction for any photon multiplicity m of the transition and any initial field intensity $\bar{n}$. Note that for the special case $m=1$ the first expression in (65) and the appearance of squeezing in the component a_1 immediately after the interaction has been switched on agree with the results of Ref. 92 for cooperative systems of Dickie type.

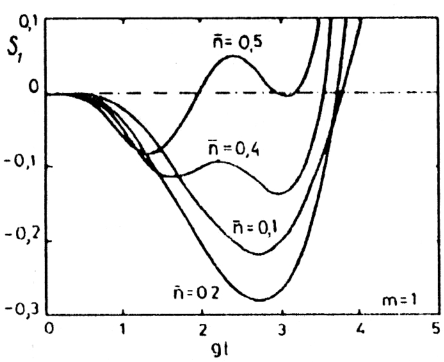

FIG. 5. Evolution of S_1 for $m=1$ when $gt<5$.

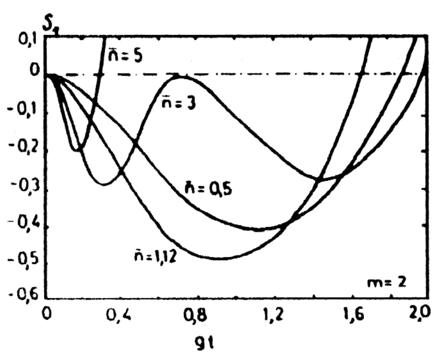

FIG. 6. Evolution of S_1 for $m=2$ when $gt<2$.

Figures 5–8 show the evolution in time of the squeezing factor S_1 found numerically from (57), (58) and (63), (64) for different intensities $\bar{n}$ of the initial coherent field and different photon multiplicity m. As can be seen from the graphs, for $t>0$ we observe a nonclassical negative value of S_1. Table I gives the numerically found maximum values of the squeezing at the beginning of the interaction for values of the photon multiplicity from $m=1$ to $m=9$ with corresponding intensities of the initial field. It can be seen from Table I that with increasing m the intensity $\bar{n}$ of the initial field needed to obtain maximum squeezing increases. It may also be noted that with increasing multiplicity of the transition up to $m=3$ the squeezing factor increases up to 57%, but with further increase in the multiplicity the squeezing factor steadily decreases. As time passes, S_1 begins to oscillate and reaches positive values. The behavior of S_1 at large times is characterized by revival of the squeezing. Squeezing in a_1 appears, disappears, and later reappears (Fig. 8). The maximum value of the restored squeezing ($\cong$52% for $gt=17.28$, $\bar{n}=1.12$, $m=2$; see Fig. 8) is greater than the maximum squeezing in the short time interval ($\cong$49% for $gt=0.92$). As was shown in Ref. 93, in the successive appearances of the squeezing it is possible to achieve the limiting squeezing up to 100% with increasing initial intensity. It can be seen from the figures

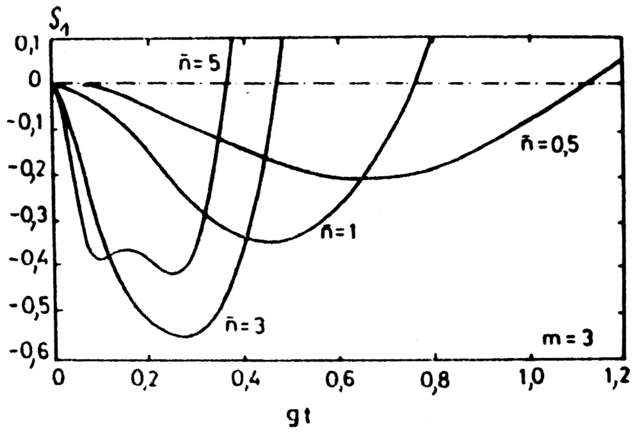

FIG. 7. Evolution of S_1 for $m=3$ when $gt<1.2$.

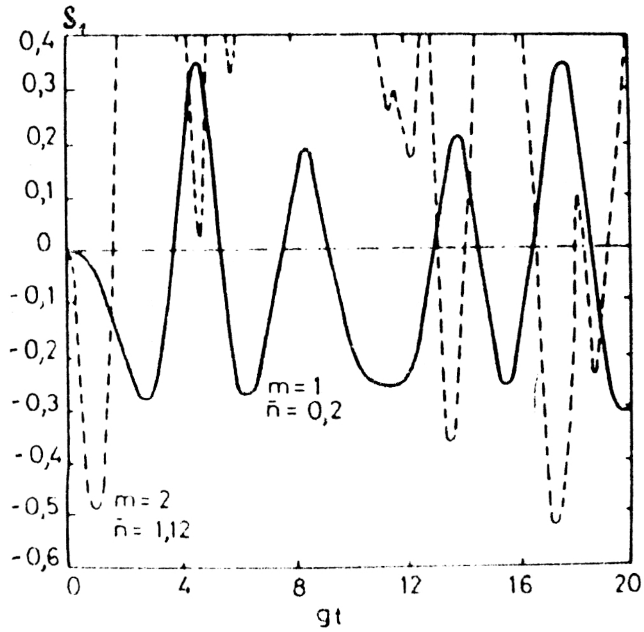

FIG. 8. Evolution of S_1 at large times for $m=1$, $\bar{n}=0.2$ (continuous curve) and $m=2$, $\bar{n}=1.12$ (broken curve).

that for high intensities $\bar{n}$ the duration of the first squeezing is shorter (except for the case of overlapping of the regions of the first and second squeezing).

Squeezing is also found in the quadrature component a_2. The squeezing appears with a certain time delay.

The possibility of squeezing of light in the Jaynes–Cummings model with an initial coherent field was demonstrated in Ref. 90. When the atom is in the upper level, squeezed light is generated in the system a certain time after the interaction has been switched on, and the squeezing factor may reach 20%. The presence of the time delay and low degree of squeezing is responsible for the experimental difficulties in detecting the squeezing effect in such systems. In the case of interaction of a two-level atom in a coherent superposition of the lower and upper levels with a vacuum field[77] light manifesting periodic squeezing up to 25% is generated. It was shown that squeezing is absent in the case of an initial vacuum field for an atom completely in the upper or lower state. The influence of the superposition state on the generation of squeezed light was studied in Ref. 75 for two- and three-level atoms with an initial squeezed field. In the case of the three-level atom, the interaction with one or two cavity modes was investigated; in the latter case, the squeezing in the superposition of two modes may reach 40%. The maximum values of the squeezing in the various generalizations of the Jaynes–Cummings model with an initial coherent field reach 42% for the two-level two-photon case,[94] 31% for the three-level single-mode case,[95] 36% for the four-level single-mode case,[96] and 52% for the ten-level single-mode case[97] (in the last study, squeezing for an atomic dipole was considered). The study of Ref. 93 demonstrated the possibility of obtaining large degrees of squeezing (up to 100%) in the single-photon Jaynes–Cummings model with an initial coherent field of high intensity, and an asymptotic expression was found for the squeezing factor in the limit $\bar{n}_0 \to \infty$. For nonresonance cases, the values of the squeezing are greater than in the resonance case. Allowance for damping in the cavity shows that the revivals of the quantum oscillations of the population disappear earlier than the squeezing. The behavior of the squeezing in the three-level single-mode JC model with an initial coherent field was investigated in Refs. 95 and 98 with allowance for detuning from resonance.

In Ref. 92, a study was made of the generation of squeezed states when a system of two-level atoms of Dickie type interacts with a single-mode coherent field. The cases in which the atoms are originally in excited or ground states were considered as well. For short interaction times, analytic expressions were obtained for the squeezing, and for large times numerical calculations were made. These show that when the atoms are in the lower states squeezing appears immediately after the interaction is switched on, but in the other case it appears a certain time after the interaction has been switched on.

For complete representation of the behavior of the squeezing in the JC models we give below the results of Ref. 98, which analyzed in more detail the evolution of the squeezing in a three-level cascade model with one or two cavity modes. Figure 9 shows the $\bar{n}_2$ dependence of the squeezing factor $S_1^{(1)}$ for mode 1 and $S_1^{(2)}$ for mode 2, respectively, for $\bar{n}_1=6$ and single-photon resonance, $\Delta_1=\Delta_2=0$. After the interaction, $S_1^{(1)}$ and $S_1^{(2)}$ oscillate and at certain periods become less than zero, thus indicating the appearance of squeezing in the corresponding mode. The maximum squeezing in both modes is different for different values of $\bar{n}_2$. The change of mode 2 acts on mode 1 in two ways; first, the two-photon transitions are changed; second, the detuning of mode 1 is changed because of the splitting of the central level by the Stark effect. When $\bar{n}_2$ increases, this leads to an increased role of the two-photon transitions, and the squeezing of mode 1 increases somewhat as long as the contribution of the two-photon transitions is greater than the contribution associated with the change in the detuning of mode 1. After $\bar{n}_2=3$, the squeezing in mode 1 decreases because the contribution from the change in the detuning of mode 1 be-

TABLE I. Maximum degrees of squeezing $S_1(m,\bar{n},gt)$ in the region of the first squeezing.

m	1	2	3	4	5	6	7	8	9
$\bar{n}$	0,2	1,12	3,0	5,4	8,2	11,75	16,1	21,1	23,3
$-S^{(1)}_{\max}$	0,28	0,49	0,5659	0,5498	0,5177	0,4873	0,4614	0,4393	0,4034
gt	2,75	0,92	0,27	0,048	$6{,}9\cdot 10^{-3}$	$8\cdot 10^{-4}$	$7{,}5\cdot 10^{-5}$	$6{,}2\cdot 10^{-6}$	$7{,}4\cdot 10^{-7}$

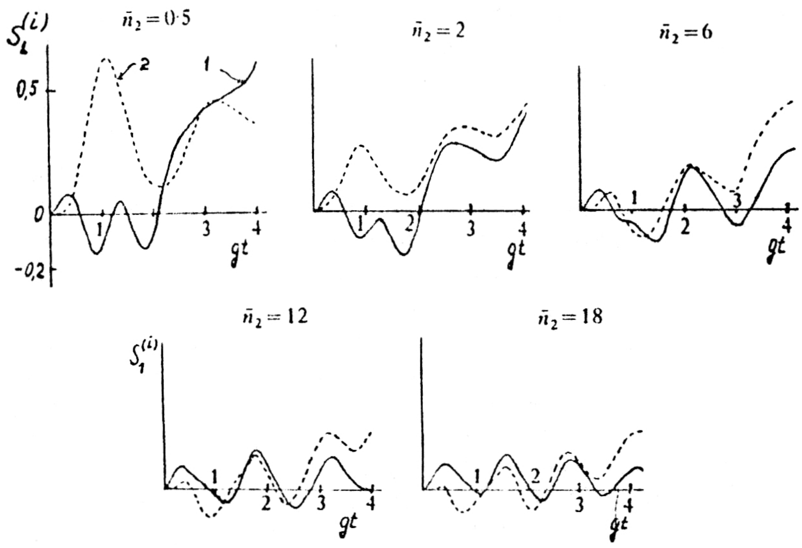

FIG. 9. Evolution of S_1 for different values of $\bar{n}_2$, where $\bar{n}_1=6$, $\Delta_1=\Delta_2=0$ for mode 1 (continuous curve) and mode 2 (broken curve).

comes dominant. When $\bar{n}_2$ becomes sufficiently large, the squeezing disappears altogether, and the oscillations of the amplitude $S_1^{(1)}$ become very small. With increasing $\bar{n}_2$ the squeezing in mode 2 appears at $\bar{n}_2=4$, increases up to the value $\bar{n}_2=8$, and then decreases.

The influence of the detuning Δ on the evolution of $S_1^{(1)}$ for two-photon resonance, $\Delta_1=\Delta_2=\Delta$, was investigated. With increasing detuning, the squeezing decreases. For $\Delta=0$, the squeezing has the largest value. In this case, both single-photon and two-photon transitions contribute to the squeezing. For $\Delta>0$, the contribution of the single-photon transitions to the squeezing decreases. Even at very large detunings of the single-photon resonance the squeezing does not disappear, since the two-photon transitions are in resonance, and this makes a corresponding contribution to the squeezing.

Investigation of the influence of the single-photon detuning Δ_2 on $S_1^{(1)}$ for $\Delta_1=0$ showed that with increasing Δ_2 the squeezing in mode 1 initially decreases, but after $|\Delta_2|=2$ it increases. Increase of $|\Delta_2|$ reduces the two-photon transitions and leads to an additional detuning of mode 1. For small values of $|\Delta_2|$, the first effect is greater than the second, and increase of $|\Delta_2|$ reduces the squeezing until the second effect begins to become more important than the first. In the limit $\Delta_2\to\infty$, single- and two-photon transitions to the upper level are absent and there are only single-photon transitions between the lower and central level. The system becomes equivalent to the two-level Jaynes–Cummings model.

When $\Delta_2=0$, the squeezing in mode 1 decreases monotonically with increasing Δ_1, and after $|\Delta_2|\geqslant 4$ the squeezing disappears. This is due to the decrease of the single- and two-photon transitions.

It may be noted that in the two-mode case the largest squeezings are less than in the two-level single-mode JC model. This indicates that coupling between the two modes has an unfavorable effect on the squeezing.

For the single-mode case, study of the behavior of $S_1^{(1)}$ shows that in this case the squeezing is deeper than in the two-mode case. For two-photon resonance, $\Delta_1=\Delta_2=\Delta$, and far from single-photon resonance, $|\Delta|\gg 1$, only two-photon transitions have an effect, and squeezing is obtained as a consequence of these transitions. For large Δ, the maximum squeezing is almost independent of Δ, since the contributions of the single-photon transitions are negligibly small. For single-photon resonance ($\Delta_1=0$) and far from two-photon resonance ($\Delta_2=10$), the single-photon transitions predominate over the two-photon transitions, and the latter can be ignored. For large Δ_2, the squeezing is due to single-photon transitions. For $\Delta_1=\Delta_2=0$, both single- and two-photon transitions contribute to the squeezing. This leads to the largest squeezing 32% for $\bar{n}=6.5$, this being greater than the 19% for the two-level JC model.[90]

Figure 10 shows the minimum values of S_1 for different $\bar{n}$. Squeezing is absent when $\bar{n}<3.5$. When $\bar{n}$ increases, the squeezing initially increases strongly, and then slowly decreases. The maximum squeezing occurs when $\bar{n}=6.5$.

Thus, we have shown the possibility of generating squeezed light in the multiphoton two-level Jaynes–Cummings model, and we have considered the behavior of the squeezing in the three-level JC model with detuning from resonance when the initial field is coherent. The behavior of the squeezing factor when the initial field is in squeezed states with minimum uncertainty is also of great theoretical and practical interest. The behavior of the atomic populations for the single-photon JC model interacting with light in vacuum squeezed states was discussed in Ref. 69. In Ref. 37 a study was made of the damping and

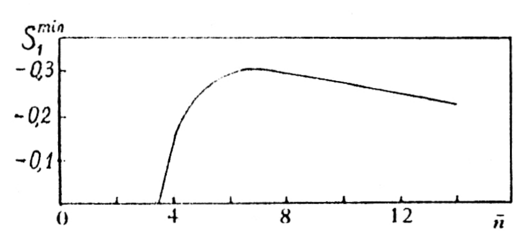

FIG. 10. Variation of the minimum value of S_1 as a function of $\bar{n}$.

revival of the quantum oscillations of the populations in a three-level atom with an initial squeezed field. The interaction of a squeezed vacuum with the two-photon generalized JC model was studied in Ref. 99, in which it was shown that the initial squeezing disappears with the passage of time and may appear at later interaction times. The greater the initial squeezing, the more regular the oscillations of the squeezing. The interaction of a squeezed vacuum with the single-photon JC model was also discussed in Ref. 100. The results show that, first, there is no general dependence on the phase difference between the atomic dipole and the squeezed field and, second, the squeezing disappears rapidly after the interaction is switched on.

We now consider the JC model with an initial squeezed coherent field $|\alpha,z\rangle$ and show that the squeezing can be enhanced if the atom is initially in a coherent superposition of the upper and lower states.[101] However, the greater the initial degree of squeezing, the less it can be enhanced. In the rotating-wave approximation the corresponding Hamiltonian has the form (6).

The initial state of the atom in a coherent superposition of the upper and lower states is[77]

$$|\psi_a(0)\rangle=\cos\left(\frac{\xi}{2}\right)|e\rangle+e^{i\eta}\sin\left(\frac{\xi}{2}\right)|g\rangle \tag{67}$$

[(67) is another form of expression of (26) but is more convenient for this problem]. We assume that the initial field is in the squeezed coherent state

$$|\psi_f(0)\rangle=|\alpha,z\rangle=\sum_{n=0}^{\infty} q_n|n\rangle, \tag{68}$$

where q_n are determined in (52).

Using (6) and (68), we can readily find the state vector of the complete system at the time t in the interaction representation:

$$|\psi(t)\rangle=\sum_{n=0}^{\infty} q_n\left\{\cos\left(\frac{\xi}{2}\right)\left[\cos(\sqrt{n+1}gt)|n;e\rangle-i\times\sin(\sqrt{n+1}gt)|n+1;g\rangle\right]+e^{i\eta}\sin\left(\frac{\xi}{2}\right)[-i\times\sin(\sqrt{n}gt)|n-1;e\rangle+\cos(\sqrt{n}gt)|n;g\rangle]\right\}. \tag{69}$$

For simplicity, we consider here the resonance case $\Delta=0$.

Using (61) and (69), we find approximately for short times $gt\ll 1$

$$S_1=[\exp(-2z)-1]+g^2t^2[\exp(-2z)\cos\xi+1-\sin^2\xi\sin^2\eta], \tag{70}$$

$$S_2=[\exp(2z)-1]+g^2t^2[\exp(2z)\times\cos\xi+1-\sin^2\xi\cos^2\eta]. \tag{71}$$

Suppose $z>0$; then $S_1(t=0)<0$, i.e., squeezing is initially present in the first quadrature component of the field. To enhance it, the coefficient of g^2t^2 must be less than zero. It can be verified that for $\eta=\pi/2$ and $\cos\xi=-1/2\exp(-2z)$ this coefficient takes its minimum value. This is the condition for optimum squeezing enhancement. Then from (70) we obtain

$$S_1=[\exp(-2z)-1]-\tfrac{1}{2}g^2t^2\exp(-4z). \tag{72}$$

After the interaction has been switched on, the squeezing factor grows quadratically in time (g^2t^2), and the greater the initial squeezing, the less the relative increase of the squeezing. These properties are also true for the case of an initial vacuum squeezed state, since in Eqs. (70)–(72) there is no dependence on α. Numerical calculations showed that the largest relative increase in the squeezing after the switching on of the interaction is achieved in fact in the case $\alpha=0$. We now consider two limiting cases of the initial state of the atom—when it is either completely in the excited state or completely in the ground state. For the first case, we have $\xi/2=0$, and, therefore, from (70) and (71) we obtain

$$\begin{aligned}S_1&=[\exp(-2z)-1]+g^2t^2[\exp(-2z)+1],\\ S_2&=[\exp(2z)-1]+g^2t^2[\exp(2z)+1].\end{aligned} \tag{73}$$

For the second case $\xi/2=\pi/2$, and Eqs. (70) and (71) are transformed to

$$\begin{aligned}S_1&=[\exp(-2z)-1]+g^2t^2[1-\exp(-2z)],\\ S_2&=[\exp(2z)-1]+g^2t^2[1-\exp(2z)].\end{aligned} \tag{74}$$

We see from (73) and (74), first, that in both cases there is no dependence on the initial phase between the levels of the atom; second, the squeezing factor does not increase after the interaction has been switched on. These results agree well with the results of Ref. 98.

The expansions of the squeezing factor S_i to fourth order in $gt\ll 1$ have a rather complicated form. In particular, in the case of an initially unexcited atom ($\xi=\pi$), choosing the phase $\eta=\pi/2$, we find, as before,

$$\begin{aligned}S_1&=[\exp(-2z)-1](1-g^2t^2)+\tfrac{1}{3}g^4t^4\{[|\beta|^2|3\\&\quad\times\exp(-4z)-4\exp(-2z)]+\tfrac{1}{4}[3\exp(-4z)-4\\&\quad\times\exp(-2z)+1]\},\\ S_2&=[\exp(2z)-1](1-g^2t^2)+\tfrac{1}{3}g^4t^4\{|\beta|^2+\tfrac{1}{4}\\&\quad\times[3\exp(4z)-4\exp(2z)+1]\}.\end{aligned} \tag{75}$$

On the right-hand side of (75) there now appears a dependence of the squeezing factors S_i on α through the parameter β. For an initial coherent field (when $z=0$), Eqs. (75) simplify strongly:

$$S_1=-\tfrac{1}{3}\bar{n}g^4t^4,\quad S_2=\tfrac{1}{3}\bar{n}g^4t^4, \tag{76}$$

where $\bar{n}$ is the mean number of photons. The relations (76) agree with the result of Ref. 94 for the case of an isolated atom in a cavity and with the expressions (65) and (66) for a single-photon transition,[91] which show the appearance of squeezing in the initial stage of the time evolution of the first quadrature for an initial coherent field.

Figure 11 shows the time evolution of the squeezing factor S_1 determined by means of numerical calculations

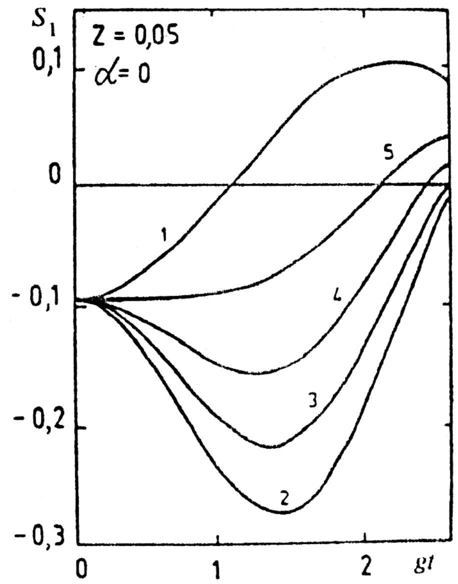

FIG. 11. Evolution of S_1 for $\eta=\pi/2$, $z=0.05$, $\alpha=0$: *1)* $\cos\xi=-1$; *2)* $\cos\xi=-e^{-2z/2}$; *3)* $\cos\xi=-0.2$; *4)* $\cos\xi=-0.1$; *5)* $\cos\xi=-0$.

for the values $\cos\xi=-1$, $-1/2e^{-2z}$ ($\cong -0.45$ for $z=0.05$), -0.2, -0.1, 0. Here and in what follows, we choose $\eta=\pi/2$. Obviously, when $\cos\xi$ satisfies the condition $-\exp(-2z)<\cos\xi<0$ [then the coefficient of g^2t^2 in Eq. (70) is negative] when the interaction is switched on, the degree of squeezing begins to increase up to some maximum value. The optimum enhancement is attained when $\cos\xi=-1/2\exp(-2z)$.

Figure 12 shows the time behavior of the squeezing factor S_1 for fixed initial degree of squeezing and different field intensities. As was noted above, the first minimum of the field fluctuations in the first quadrature attains the lowest value for $\alpha=0$, corresponding to a squeezed vacuum. The greater the initial squeezing, the more difficult it is to enhance the squeezing in the initial stage of the interaction. With the passage of time, a strong dependence on α appears in the behavior of S_1.

Investigations for large values of the time reveal a manifestly random dependence of S_1 on the time due to the random statistical properties of the squeezed vacuum. When α is increased, the behavior of S_1 becomes similar to the behavior of the squeezing in Refs. 92, 91, and 37 for an initial coherent field. Large values of α lead to strong squeezing (up to 100%) at large interaction times.

7. COHERENT TRAPPING OF POPULATIONS

In recent years, there has been much study of the phenomenon of coherent trapping of populations,[25,102–107] which is actively used in high-resolution spectroscopy. When there is coherent trapping, the population of some level remains constant despite the existence of a radiation field and transitions to other levels. This is explained by the interference between different transition channels. Until recently, it was asserted that coherent trapping of populations could occur only in systems with three or more levels and was completely absent in two-level systems.[25] However, in the recently published studies of Refs. 80, 108, and 109 cases of coherent trapping of populations have been found in two-level systems. As was shown in Refs. 80, 108, and 109, coherent trapping of populations can also arise for two-level atoms if the atom is in a coherent superposition of the lower and upper levels at the initial time. In this case there exists thermodynamic equilibrium and interference between the atom and the field. For a three-level atom, coherent trapping occurs because of interference between two transitions, and this interference can be described in the semiclassical approach; in the case of a two-level atom, the interference occurs between the atomic dipole and the cavity eigenmodes, i.e., it has a quantum origin. It was shown in Ref. 80 that if the atom is in a coherent superposition of the lower and upper states and the initial field is coherent, then the dynamics of the atom and the spectrum of the field depend on the relative phase between the

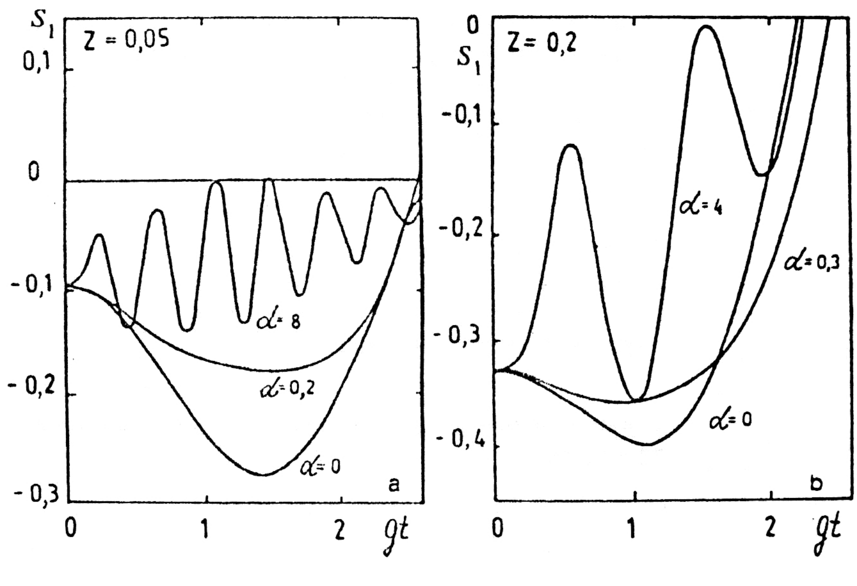

FIG. 12. Evolution of S_1 for different values of α. The initial squeezing and the maximum value achieved after the interaction has been switched on are approximately: a) 9.5% and 27% for $z=0.05$; b) 33% and 40% for $z=0.2$, respectively.

atomic dipole and the cavity field. For a definite choice of this phase there is coherent trapping of the two-level atom, and the spectrum goes over from a three-peak picture to an asymmetric two-peak structure. The dynamics of the level populations in the two-level Jaynes–Cummings model when the atom is initially in a coherent superposition of the lower and upper levels was investigated in Ref. 108. For certain values of the relative phase between the atomic dipole and the field, the atomic populations remain almost constant, and, therefore, there is partial coherent trapping of populations. If the initial state of the field is an eigenstate of the Susskind–Glogower phase operator,[110] then there is exact coherent trapping of populations. The degree of squeezing depends strongly on the initial phase of the atomic dipole at short times, but at large times this is not observed. At low field intensities it is found that the squeezing appears periodically in time. At high field intensities, the oscillations of S_1 after the disappearance of the initial squeezing occur in the region of positive values of the squeezing factor irrespective of the initial phase of the field, disagreeing with the results of Ref. 100. The reason for this is that at low field intensities the atom and field are almost decoupled; the atomic dipoles and the field parameters oscillate with the Rabi frequency. The amplitudes of these oscillations are not very large, and therefore the initial situation is repeated after each oscillation. However, at high intensities of the initial field the atomic dipole and the field are strongly coupled through their phases, and the presence of such coupling leads to destruction of the field squeezing.

We now consider coherent trapping of populations in a three-level atom of Λ type with multiphoton transitions. The Hamiltonian of the system has the form

$$H=H_A+H_F+H_{AF}, \tag{77}$$

where

$$H_A=\sum_{j=1}^{3}\hbar\Omega_j R_{jj};$$

$$H_F=\sum_{k=1}^{2}\hbar\omega_k a_k^\dagger a_k;$$

$$H_{AF}=\hbar\sum_{k=1}^{2}g_k(R_{3k}a_k^{m_k}+R_{k3}a_k^{\dagger m_k}).$$

We shall say that the state $|\phi\rangle$ of the system is a state of coherent trapping if $|\phi\rangle$ satisfies the conditions

$$|\phi\rangle=|\phi_A\rangle\otimes|\phi_F\rangle \tag{78}$$

and

$$|\phi(t)\rangle=\exp(-iHt/\hbar)|\phi\rangle$$
$$=\exp(-iH_At/\hbar)|\phi_A\rangle\otimes\exp(-iH_Ft/\hbar)|\phi_F\rangle, \tag{79}$$

where $|\phi_A\rangle$ is an atomic state consisting of a linear superposition of only the lowest levels 1 and 2, and $|\phi_F\rangle=|\phi_{F_1},\phi_{F_2}\rangle$ is a state of the field that does not contain the Fock states $|n_1,n_2\rangle$ with $n_1<m_1$ or $n_2<m_2$.

In the case of multiphoton resonance, when

$$\Omega_3-\Omega_k=m_k\omega_k,\quad k=1,2, \tag{80}$$

it is easy to show that the condition (79) is equivalent to the equation

$$H_{AF}|\phi\rangle=0. \tag{81}$$

Since level 3 does not occur in $|\phi_A\rangle$, we can write

$$|\phi_a\rangle=u_1|1\rangle_A-u_2|2\rangle_A,\quad |u_1|^2+|u_2|^2=1. \tag{82}$$

Using Eqs. (77), (78), and (82), from (81) we obtain

$$g_1u_1a_1^{m_1}|\phi_F\rangle=g_2u_2a_2^{m_2}|\phi_F\rangle. \tag{83}$$

Equation (83) determines the states of the field $|\phi_F\rangle$ and the values of u_1 and u_2, and, thus, determines the atomic states $|\phi_A\rangle$. Solving this equation, we find for the system of Λ type states of coherent trapping of the field:

$$|\phi_F\rangle=|Z_1\rangle_{\text{coh}}\otimes|Z_2\rangle_{\text{coh}}. \tag{84}$$

Here, $|Z_1\rangle_{\text{coh}}$ and $|Z_2\rangle_{\text{coh}}$ are Glauber coherent states of the modes 1 and 2, respectively. The amplitudes u_1 and u_2 of the corresponding atomic states of the coherent trapping have the form

$$u_1=\frac{g_2Z_2^{m_2}}{\sqrt{|g_1Z_1^{m_1}|^2+|g_2Z_2^{m_2}|^2}},$$
$$u_2=\frac{g_1Z_1^{m_1}}{\sqrt{|g_1Z_1^{m_1}|^2+|g_2Z_2^{m_2}|^2}}. \tag{85}$$

We introduce

$$Z_1=\sqrt{\bar{n}_1}e^{i\theta_1},\ Z_2=\sqrt{\bar{n}_2}e^{i\theta_2},\quad u_1/u_2=\sqrt{r_{12}}e^{i\varphi_{12}}. \tag{86}$$

Here, the angles θ_1 and θ_2 are the phases of the field in the modes, $\bar{n}_1=|Z_1|^2$ and $\bar{n}_2=|Z_2|^2$ are the mean numbers of photons in the modes, r_{12} reflects the ratio of the populations of the trapped levels, and the angle φ_{12} is the phase difference of the atomic states $|1\rangle$ and $|2\rangle$ in the coherent superposition (82). Applying (85) and (86), we obtain

$$r_{12}=g_2^2\bar{n}_2^{m_2}/g_1^2\bar{n}_1^{m_1};\quad \varphi_{12}=m_2\theta_2-m_1\theta_1. \tag{87}$$

As can be seen from (87), the presence of multiplicativity of the transitions, $m>1$, leads to a very strong dependence of the level-trapping parameter r_{12} on the mean photon numbers $\bar{n}_1$ and $\bar{n}_2$ and the differences of the atomic phases from the phases of the field modes θ_1 and θ_2. In the special case $m_1=m_2=m$, the relations (87) become

$$r_{12}=(g_2^2/g_1^2)(\bar{n}_2/\bar{n}_1)^m;\quad \varphi_{12}=m(\theta_2-\theta_1). \tag{88}$$

This means that r_{12} and φ_{12} are determined by the photon-number ratio $\bar{n}_2/\bar{n}_1$ and the phase difference $\theta_2-\theta_1$ of the field modes. For systems of cascade and V type, coherent-trapping states are not found.

8. ANTIBUNCHING AND SUB-POISSONIAN PHOTON STATISTICS

As is well known, the statistical properties of an electromagnetic field, for example, coherent or thermal light, can be described by means of a technique similar to classical probability theory by expanding the density operator with respect to projection operators $|\alpha\rangle\langle\alpha|$, where $|\alpha\rangle$ is a coherent state of the field (Glauber—Sudarshan P representation[111,112]):

$$\rho = \int P(\alpha)|\alpha\rangle\langle\alpha|d^2\alpha. \tag{89}$$

In the P representation, the statistical mean value of any normally ordered product of photon creation and annihilation operators of the type $(a^\dagger)^n a^m$ reduces to the simple mean of $(\alpha^*)^n \alpha^m$ taken with respect to the weight function $P(\alpha)$:

$$\mathrm{Tr}[\rho(a^\dagger)^n a^m] = \int P(\alpha)\langle\alpha|(a^\dagger)^n a^m|\alpha\rangle d^2\alpha. \tag{90}$$

However, the weight function $P(\alpha)$ may have a strong singularity, and, generally speaking, it cannot be regarded rigorously as a probability density distribution. If $P(\alpha)$ has singularities stronger than the Dirac δ function or is not positive definite, the corresponding state of the light is called nonclassical.

As an example, we take a state of light with sub-Poissonian photon statistics. By definition, this is a state with intensity fluctuations less than in a coherent state,

$$\langle(\Delta n)^2\rangle < \langle n\rangle, \tag{91}$$

which has Poissonian photon statistics with variance $\langle(\Delta n)^2\rangle = \bar{n}$. We rewrite (91) in the form

$$\langle a^\dagger a^\dagger a a\rangle - \langle a^\dagger a\rangle^2 < 0, \tag{92}$$

which means in the P representation,

$$\int P(\alpha)|\alpha|^4 d^2\alpha - \left[\int P(\alpha)|\alpha|^2 d^2\alpha\right]^2 = \frac{1}{2}\int\int d^2\alpha d^2\beta P(\alpha,\beta)(|\alpha|^4 + |\beta|^4 - 2|\alpha|^2|\beta|^2) < 0, \tag{93}$$

where $P(\alpha,\beta) \equiv P(\alpha)P(\beta)$. Since

$$|\alpha|^4 + |\beta|^4 - 2|\alpha|^2|\beta|^2 \geqslant 0, \tag{94}$$

the function $P(\alpha)$ must also have negative values if the condition (93) is to be satisfied.

Nonclassical field states can be defined by other criteria. As is well known, the normalized second-order correlation function between two space–time points of a light ray with intensity $I(x)$,

$$g^{(2)}(x_1,x_2) = \frac{\langle I(x_1)I(x_2)\rangle}{\langle I(x_1)\rangle\langle I(x_2)\rangle}, \tag{95}$$

satisfies in the semiclassical theory the inequalities[113–115]

$$g^{(2)}(x,x) \geqslant 1, \tag{96}$$

$$g^{(2)}(x_1,x_2) \leqslant [g^{(2)}(x_1,x_1)g^{(2)}(x_2,x_2)]^{1/2}. \tag{97}$$

In quantum theory $\langle:I(x_1)I(x_2):\rangle$ is the mean rate of counting of double coincidences for ideal photodetectors situated at the points x_1 and x_2. In the case of a single-mode field and purely time correlations, the quantum analog of the normalized second-order correlation function has the form

$$g^{(2)}(t,t+\tau) = \frac{\langle a^\dagger(t)a^\dagger(t+\tau)a(t+\tau)a(t)\rangle}{\langle a^\dagger(t)a(t)\rangle\langle a^\dagger(t+\tau)a(t+\tau)\rangle}. \tag{98}$$

The coincidence counting rate $g^{(2)}(t,t+\tau)$ plays a central role in the definition of the phenomenon of photon antibunching. In the literature, there exist two definitions (Refs. 115 and 116): 1) $g^{(2)}(t,t+\tau) < 1$ when $\tau=0$; 2) the derivative with respect to τ of $g^{(2)}(t,t+\tau)$ at the point $\tau=0$ is positive. In the first case, the inequality (96) is violated; in the second, (97) is violated. If by antibunching one understands the effect consisting of the fact that when a light beam is incident on a photodetector photons are more often detected separately than in a pair, then we must use the second definition.[117]

In this section, we shall study sub-Poissonian statistics and the phenomenon of photon antibunching in the Jaynes–Cummings model.[118] To investigate the antibunching, we use the definition based on positivity of the derivative with respect to τ of $g^{(2)}(t,t+\tau)$ at $\tau=0$. We show that sub-Poissonian statistics is not related to photon antibunching: When the cavity field has sub-Poissonian statistics, the photons may exhibit either bunching or antibunching and, conversely, photon antibunching may be accompanied by super- or sub-Poissonian statistics. Considering the influence of the initial atomic conditions on the time evolution of Mandel's Q factor[119] and $(\partial/\partial\tau)g^{(2)}(t,t+\tau)|_{\tau=0}$, we show that for exact resonance when the atom is placed in the cavity in the ground state an initial coherent field becomes sub-Poissonian in the initial stage of evolution. Later, however, super-Poissonian behavior is predominant. Conversely, when the atom is placed in the cavity in the excited state, the field in the early stage of evolution manifests super-Poissonian behavior, but later sub-Poissonian behavior is predominant. Thus, the atom in the ground state is more effective for the generation of sub-Poissonian light at small values of the time but is inferior to the atom in the excited state at large times. We also show that immediately after the interaction has been switched on there is photon antibunching when the atom is initially excited but no antibunching for an initially unexcited atom. Further, we compare the time behavior of Mandel's Q factor and $(\partial/\partial\tau)g^{(2)}(t,t+\tau)|_{\tau=0}$ for a field initially in coherent and random states. We find that for both initial field states photon antibunching occurs and that, in contrast to squeezing and sub-Poissonian photon statistics, the antibunching effect does not disappear when the field intensity in the random state is increased.

Suppose that at the initial time the atom and field are not coupled and the field is in an arbitrary state $\rho_j = \sum_{n,n'}^{\infty} p_{n,n'}|n\rangle\langle n'|$; then the density matrix of the complete system at $t=0$ is

$$\rho(0) = \sum_{n,n'}^{\infty} p_{n,n'}|n;e\rangle\langle n';e| \tag{99}$$

for an initially excited atom and

$$\rho(0)=\sum_{n,n'}^{\infty} p_{n,n'}|n;g\rangle\langle n';g| \qquad (100)$$

for an initially unexcited atom.

A good measure of the field's being in states with sub-Poissonian statistics is the Q factor

$$Q\equiv\frac{\langle(\Delta n)^2\rangle}{\langle n\rangle}-1, \qquad (101)$$

which was introduced by Mandel.[119] The smaller Q in the region of negative values, the more strongly the field statistics is sub-Poissonian. Mandel's Q factor (101) can be rewritten in the form

$$Q=\frac{\langle a^\dagger(t)a^\dagger(t)a(t)a(t)\rangle-\langle a^\dagger(t)a(t)\rangle^2}{\langle a^\dagger(t)a(t)\rangle}. \qquad (102)$$

We also need to know the derivative with respect to τ of $g^{(2)}(t,t+\tau)$ at $\tau=0$:

$$[g^{(2)}(t,t+\tau)]'_\tau|_{\tau=0}$$
$$=\{[\langle a^\dagger(t)a^\dagger(t+\tau)a(t+\tau)a(t)\rangle]'_\tau|_{\tau=0}\langle a^\dagger(t)a(t)\rangle$$
$$-\langle a^\dagger(t)a^\dagger(t)a(t)a(t)\rangle[\langle a^\dagger(t+\tau)$$
$$\times a(t+\tau)\rangle]_\tau'|_{\tau=0}\}\langle a^\dagger(t)a(t)\rangle^{-3}. \qquad (103)$$

Using the solutions (12)–(15), we readily obtain

$$\langle a^\dagger(t)a(t)\rangle=\bar{n}+\sum_{n=0}^{\infty} p_{nn}\frac{g^2(n+1)}{\lambda_n^2}\sin^2\lambda_n t,$$
$$\langle a^\dagger(t)a^\dagger(t)a(t)a(t)\rangle$$
$$=\overline{n^2}-\bar{n}+\sum_{n=0}^{\infty} p_{nn}\frac{2g^2(n+1)n}{\lambda_n^2}\sin^2\lambda_n t, \qquad (104)$$
$$[\langle a^\dagger(t+\tau)a(t+\tau)\rangle]'_\tau|_{\tau=0}=\sum_{n=0}^{\infty} p_{nn}\frac{g^2(n+1)}{\lambda_n}\sin 2\lambda_n t,$$
$$[\langle a^\dagger(t)a^\dagger(t+\tau)a(t+\tau)a(t)\rangle]'_\tau|_{\tau=0}$$
$$=\sum_{n=0}^{\infty} p_{nn}\frac{g^2(n+1)n}{\lambda_n}\sin 2\lambda_n t$$

for the initial condition (99) and

$$\langle a^\dagger(t)a(t)\rangle=\bar{n}-\sum_{n=0}^{\infty} p_{nn}\frac{g^2 n}{\lambda_{n-1}^2}\sin^2\lambda_{n-1}t,$$
$$\langle a^\dagger(t)a^\dagger(t)a(t)a(t)\rangle=\overline{n^2}-\bar{n}$$
$$-\sum_{n=0}^{\infty} p_{nn}\frac{2g^2 n(n-1)}{\lambda_{n-1}^2}\sin^2\lambda_{n-1}t, \qquad (105)$$
$$[\langle a^\dagger(t+\tau)a(t+\tau)\rangle]'_\tau|_{\tau=0}=-\sum_{n=0}^{\infty} p_{nn}\frac{g^2 n}{\lambda_{n-1}}\sin 2\lambda_{n-1}t,$$
$$[\langle a^\dagger(t)a^\dagger(t+\tau)a(t+\tau)a(t)\rangle]'_\tau|_{\tau=0}$$
$$=-\sum_{n=0}^{\infty} p_{nn}\frac{g^2 n(n-1)}{\lambda_{n-1}}\sin 2\lambda_{n-1}t$$

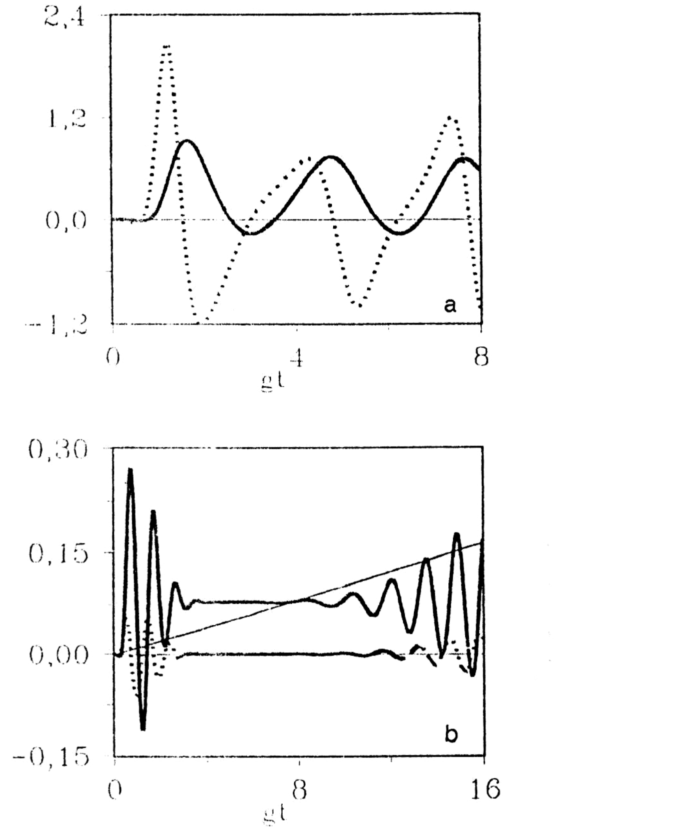

FIG. 13. Evolution of Q (continuous curves) and $(\partial/\partial\tau)g^{(2)}(t,t+\tau)|_{\tau=0}$ (broken curves) when the atom at the initial time is in the lower level and the field is in a coherent state in which: a) $\bar{n}=1$; b) $\bar{n}=10$.

for the initial condition (100). Substitution of (104) and (105) in (102) and (103) immediately gives explicit expressions for Mandel's Q factor and $(\partial/\partial\tau)g^{(2)}(t,t+\tau)|_{\tau=0}$. Light for which $Q>0$ (respectively, <0) has greater (lesser) dispersion than for a Poisson distribution and is said to be super (sub)-Poissonian; the lower limit of Q is -1 and corresponds to a Fock state, i.e., a state with exactly defined photon number. Photon bunching or antibunching is revealed by the behavior of the normalized two-time correlation function $g^{(2)}(t,t+\tau)$ in the neighborhood of $\tau=0$. If $g^{(2)}(t,t+\tau)$ decreases with increasing τ (negative derivative), then there is bunching. If $g^{(2)}(t,t+\tau)$ increases with increasing τ (positive derivative), there is antibunching.

For simplicity, we assume in what follows exact resonance: $\Delta=0$.

We consider a coherent state of the field. Even after substitution of the weight factor P_{nn} (21) in (102)–(105) it is not possible to sum (102) and (103) explicitly. Therefore, we use numerical calculations. The results are given in Figs. 13 and 14.

Figure 13 shows the dependence of Mandel's Q factor (continuous curves) and the derivative $(\partial/\partial\tau)g^{(2)}(t,t+\tau)$ at $\tau=0$ (broken curves) on the dimensionless time gt for an initially unexcited atom and for two values of the mean photon number: $\bar{n}=1$ (Fig. 13a) and $\bar{n}=10$ (Fig. 13b). When the interaction is switched on, the curves represent-

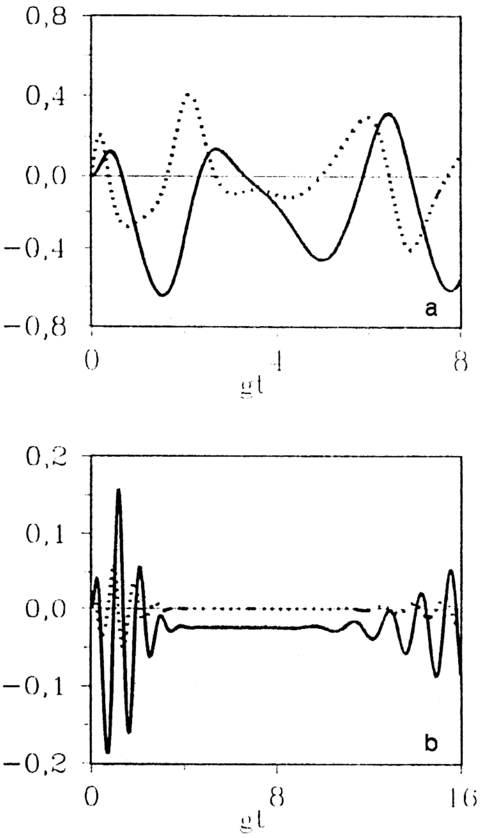

FIG. 14. Evolution of Q (continuous curves) and $(\partial/\partial\tau)g^{(2)}(t,t+\tau)|_{\tau=0}$ (broken curves) when the atom at the initial time is in the upper level and the field is in a coherent state in which: a) $\bar{n}=1$; b) $\bar{n}=10$.

ing the Q factor turn downward, indicating sub-Poissonian field statistics. After a certain time, these curves turn upward, and Q attains positive values. The light now has super-Poissonian statistics. In the process of the time evolution, Q oscillates near the initial zero value, and the field statistics changes from sub-Poissonian to super-Poissonian and vice versa. However, as can be seen from the figures, the field during the interaction process possesses super-Poissonian statistics for most of the time.

When the field intensity is increased, the amplitude of the Q oscillations decreases. This agrees with the conclusions drawn in Ref. 120—that the higher the intensity of the initial coherent field, the "weaker" the sub-Poissonian (or super-Poissonian) statistics.

When the interaction commences, the value of $(\partial/\partial\tau)g^{(2)}(t,t+\tau)|_{\tau=0}$ (which is zero at the initial time) decreases, i.e., there is photon bunching. With increasing interaction time, $(\partial/\partial\tau)g^{(2)}(t,t+\tau)|_{\tau=0}$ begins to oscillate around zero. This means that bunching and antibunching alternate in time. When the field statistics is sub-Poissonian, the photons may exhibit bunching or antibunching, and, conversely, photon antibunching may be accompanied by super- or sub-Poissonian statistics. It can also be seen from the figures that when the field has super-Poissonian statistics for most of the time there is no predominance of photon bunching or antibunching with respect to each other.

The results for an initially excited atom are shown in Fig. 14. When the interaction between the atom and field is switched on, Q increases from an initial value of zero. This tells us that the field becomes super-Poissonian. Thus, at short times an unexcited atom is more effective for the generation of a field with sub-Poissonian statistics than an excited atom. It can be seen that the field has sub-Poissonian statistics for most of the time, in contrast to the case of an initially unexcited atom.

As for an initially unexcited atom, $(\partial/\partial\tau)g^{(2)}(t,t+\tau)|_{\tau=0}$ often changes sign during the evolution. An important difference occurs near $t=0$: In the case of an excited atom, there is antibunching, whereas in the case of an unexcited atom there is bunching.

When the field intensity is enhanced, collapses and revivals of the Rabi oscillations[62] are manifested in the time behavior of Q and $(\partial/\partial\tau)g^{(2)}(t,t+\tau)|_{\tau=0}$. Taking into account the distribution of (21) around its peak at $\bar{n}$ in the case of strong pumping, $\bar{n}\gg 1$, we find for Q the quasisteady value

$$Q_{\text{quasi-steady}}\sim\frac{3}{4\bar{n}}, \tag{106}$$

which is attained in the collapse region for the initial state $|\psi_a(0)\rangle=|g\rangle$ of the atom and

$$Q_{\text{quasi-steady}}\sim-\frac{1}{4\bar{n}} \tag{107}$$

for the initial state $|\psi_a(0)\rangle=|e\rangle$. At these values of the time, $(\partial/\partial\tau)g^{(2)}(t,t+\tau)|_{\tau=0}$ oscillates almost with zero amplitude, which indicates that the bunching and antibunching effects are less clearly expressed. This is entirely understandable, since in a quasisteady regime the mean photon number remains almost unchanged.

We now consider a chaotic state of the field, which has diagonal density matrix with weight factor (22), variance $\langle(\Delta n)^2\rangle=\bar{n}(\bar{n}+1)$ and Mandel Q factor $Q=\bar{n}$. Since $Q>0$, the chaotic state is super-Poissonian, and this property is enhanced when the mean photon number is increased. At the same time, it is natural to expect that only a chaotic field with very small photon number can become sub-Poissonian as a result of interaction with a two-level atom.[121,120]

Figure 15 shows the time evolution of Q and $(\partial/\partial\tau)g^{(2)}(t,t+\tau)|_{\tau=0}$ for a cavity field initially in a chaotic state and the atom in the excited state. It can be seen that for $\bar{n}=1$ (Fig. 15a) sub-Poissonian statistics still occurs, but for $\bar{n}=3$ (Fig. 15b) it does not. We also note that when the sub-Poissonian behavior disappears with increasing mean photon number the sign of $(\partial/\partial\tau)g^{(2)}(t,t+\tau)|_{\tau=0}$ changes from positive to negative, indicating that photon antibunching is replaced by bunching and vice versa. The fact that antibunching still occurs despite the broadening of the distribution with respect to the Fock states is largely due to the definition employed. Indeed, defining photon antibunching in accordance with positivity of the derivative $(\partial/\partial\tau)g^{(2)}(t,t+\tau)$ at $\tau=0$, it is important for us to know how the value of the two-dimensional correlation function $g^{(2)}(t,t+\tau)$ changes in the neighborhood of $\tau=0$;

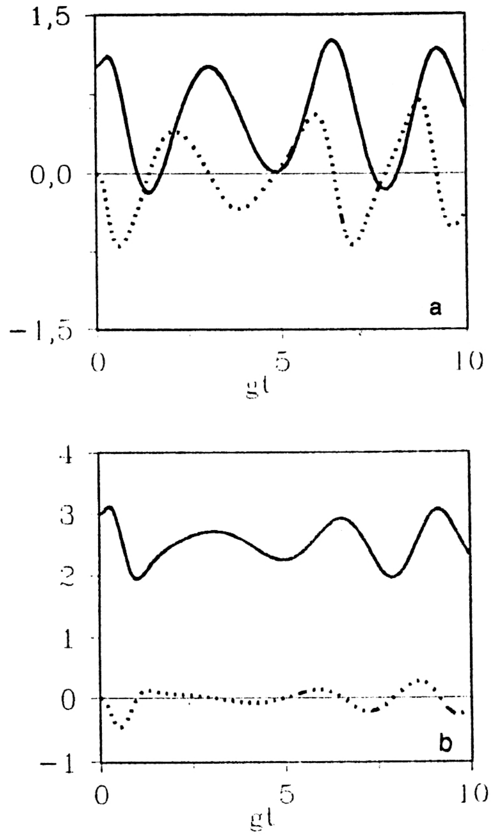

FIG. 15. Evolution of Q (continuous curves) and $(\partial/\partial\tau)g^{(2)}(t,t+\tau)|_{\tau=0}$ (broken curves) when the atom at the initial time is in the upper level and the field is in a coherent state in which: a) $\bar{n}=1$; b) $\bar{n}=3$.

we do not need to know the value of $g^{(2)}(t,t+\tau)$ itself at the point $\tau=0$. It is obvious that the variation of $g^{(2)}(t,t+\tau)$ as a function of τ is determined to a greater degree by the atom, which absorbs and emits photons, and not by the properties of the field statistics. On the other hand, the field statistics also influences the time behavior of $g^{(2)}(t,t+\tau)$. For example, comparison of Figs. 14 and 15 (for the case of an initially excited atom) shows that after the interaction has been switched on there is photon bunching for a field in a chaotic state but antibunching for a coherent field.

9. COLLAPSE AND REVIVAL OF RABI OSCILLATIONS AND EVOLUTION OF THE FIELD PHASE

Although quantum electrodynamics was created forty years ago, there exist fundamental problems that have not yet been fully resolved. One of them is the question of the existence of a Hermitian operator for the phase of a harmonic oscillator (or a single-mode electromagnetic field). The classical electromagnetic field is described by an amplitude, i.e., the square root of the field intensity, and a phase. In the quantum theory, the field amplitude is proportional to the square root of the photon-number operator, but it remains an open question how the operator of the phase is to be defined. In his pioneering paper[124] on the quantization of the electromagnetic field, Dirac first postulated the existence of a Hermitian phase operator $\hat{\phi}$ conjugate to the photon-number operator. He suggested that the photon-number operator and the phase operator must satisfy the canonical commutation relation

$$[\hat{N},\hat{\phi}]=i, \tag{108}$$

and that the operators of annihilation a and creation $a^\dagger$ of the single-mode electromagnetic field can be represented in the polar form

$$a=\exp(i\hat{\phi})\sqrt{\hat{N}},\quad a^\dagger=\sqrt{\hat{N}}\exp(-i\hat{\phi}). \tag{109}$$

The difficulties of this approach, which were subsequently recognized by Dirac himself,[125] are the following. First, applying the Heisenberg uncertainty relation to the commutator (108), we obtain

$$\Delta N\Delta\phi\geqslant\tfrac{1}{2}. \tag{110}$$

It can be seen from (110) that a state with well-defined photon number will have a phase uncertainty greater than 2π. Second, it follows from the commutator (108) that the matrix elements of the operator $\hat{\phi}$ in the basis of states with definite photon number are not defined:[126]

$$(n'-n)\langle n'|\hat{\phi}|n\rangle=i\delta_{nn''}. \tag{111}$$

Finally, the exponential operator $\exp(i\hat{\phi})$ obtained by means of this approach is not unitary.[127]

Susskind and Glogower[110] emphasized that the main difficulty in the correct definition of the phase operator arises because the eigenvalue spectrum of the photon-number operator is bounded below. There are two possible ways of overcoming this difficulty. The first is to extend the Hilbert space of the normal harmonic oscillator by adjoining to it states with negative photon number and to assume that the states with negative energy are separated from the positive-energy ground state.[128] However, it was shown that this approach is not free of certain contradictions, which arise because the state space is unbounded.[128] Another way of resolving the problem of the semibounded spectrum of the harmonic oscillator is to assume that the spectrum of the harmonic oscillator is bounded, i.e., to consider a Hilbert space of the harmonic oscillator of finite dimension. This was done by Garrison and Wong in 1970,[129] and also by Popov and Yarunin in 1973[130] (see also Ref. 131). A Hermitian phase operator $\hat{\phi}_r$ and a corresponding unitary exponential operator $\hat{U}_r=\exp(i\phi_r)$ were constructed in the finite-dimensional subspace $(l_2)_r$ generated by the r first vectors of the basis $|n\rangle$. The operator $\hat{\phi}$ obtained after passage to the limit $r\to\infty$ possesses some of the properties needed for a phase operator. For example the commutator $[\hat{N},\hat{\phi}]$ is equal to i on a dense set of functions in the space with basis $|n\rangle$. Eigenvalues and eigenvectors were found for $\hat{\phi}$. However, they have a very complicated mathematical structure, making it difficult to use them to study the phase properties of the electromagnetic field.

We shall use the formalism of the Pegg–Barnett Hermitian phase operator[122,123] to investigate the dynamical properties of the phase of a coherent field interacting with a two-level atom in a cavity. We find an interesting connection between the time behavior of the field phase in the cavity and the collapses and revivals of the Rabi oscillations. We also calculate the distribution function and the phase fluctuations in the JC model with a constant interaction that depends on the field intensity.[49]

We begin by giving some of the basic details of the Pegg–Barnett formalism needed for the subsequent investigation of the phase properties of a cavity field.

Proceeding from the existence of a state with an exactly defined phase,

$$|\theta\rangle = \lim_{s\to\infty}(s+1)^{-1/2}\sum_{n=0}^{s}\exp(in\theta)|n\rangle, \tag{112}$$

Pegg and Barnett continued to work initially in an $(s+1)$-dimensional space Ψ (where s can be arbitrarily large) and only went to the limit $s\to\infty$ when all the physical quantities had been calculated. Although the parameter θ in the phase state (112) can take any real value, distinguishable phase states $|\theta\rangle$ exist for all values of θ only in the given interval $\theta_0-(\theta_0+2\pi)$, where θ_0 is the relative phase. It is easy to show that states with values of θ differing by $2\pi/(s+1)$ multiplied by an integer are orthogonal. Therefore, specifying the relative state

$$|\theta_0\rangle = \frac{1}{\sqrt{s+1}}\sum_{n=0}^{s}\exp(in\theta_0)|n\rangle, \tag{113}$$

one can find a complete set of $s+1$ orthonormal phase states:[132–134]

$$|\theta_m\rangle = \frac{1}{\sqrt{s+1}}\sum_{n=0}^{s}\exp(in\theta_m)|n\rangle,\quad m=0,1,\dots,s, \tag{114}$$

where the $s+1$ values θ_m are given by

$$\theta_m \equiv \theta_0 + \frac{2\pi m}{s+1},\quad m=0,1,\dots,s. \tag{115}$$

The set of phase states $|\theta_m\rangle$ can be used as an orthonormal basis in the space Ψ.

On the basis of the orthonormal phase states (114), one can construct the phase operator as follows:

$$\hat{\phi}_\theta \equiv \sum_{m=0}^{s}\theta_m|\theta_m\rangle\langle\theta_m|. \tag{116}$$

Obviously, $\hat{\Phi}_\theta$ is Hermitian and satisfies the equation

$$\hat{\phi}_\theta|\theta_m\rangle = \theta_m|\theta_m\rangle. \tag{117}$$

Suppose that at the initial time the atom is in the lower level and the field is in the coherent state (27), whose phase properties are well known.[134,135] Then the initial state of the complete atom + field system has the form

$$|\psi(0)\rangle = \sum_{n=0}^{\infty} b_n e^{in\varphi}|n;g\rangle. \tag{118}$$

It is obvious that φ is the expectation value of the field phase.

Using the solution (14), we can readily find the state vector of the system at time t in the representation II:

$$\begin{aligned}|\psi(t)\rangle &= U_{\rm II}(t)\psi(0)\rangle\\ &= \sum_{n=0}^{\infty} b_n e^{in\varphi}[A_{n,g}(t)|n;g\rangle\\ &\quad + B_{n,g}(t)|n-1;e\rangle],\end{aligned} \tag{119}$$

where $A_{n,g}(t)$ and $B_{n,g}(t)$ are defined in (15). Using the expansion (114) of the phase state in the basis $|n\rangle$, we can arrive at the following expression for the distribution of the phase probability:

$$\begin{aligned}|\langle\theta_m|\psi\rangle|^2 = \frac{1}{s+1}\Big(1+2\sum_{n>k} b_n b_k\{\cos[(n-k)(\theta_m-\varphi)]\\ \times \mathrm{Re}\,U \sin[(n-k)(\theta_m-\varphi)]\mathrm{Im}U\}\Big),\end{aligned} \tag{120}$$

where

$$\begin{aligned}U &\equiv U_{n,k}(t)\\ &= A_{n,g}(t)A^*_{k,g}(t) + B_{n,g}(t)B^*_{k,g}(t)\\ &= \cos(\lambda_{n-1}t)\cos(\lambda_{k-1}t)\\ &\quad + \frac{g^2\sqrt{nk}+\Delta^2/4}{\lambda_{n-1}\lambda_{k-1}}\sin(\lambda_{n-1}t)\sin(\lambda_{k-1}t) + \frac{i\Delta}{2}\\ &\quad \times\Big[\frac{1}{\lambda_{n-1}}\sin(\lambda_{n-1}t)\cos(\lambda_{k-1}t)\\ &\quad - \frac{1}{\lambda_{k-1}}\sin(\lambda_{k-1}t)\cos(\lambda_{n-1}t)\Big].\end{aligned} \tag{121}$$

Choosing the initial point of the phase window

$$\theta_0 = \varphi - \frac{\pi s}{s+1}, \tag{122}$$

we obtain from Eq. (115) for θ_m the result

$$\theta_m \equiv \theta_0 + \frac{2\pi m}{s+1} = \varphi + \frac{\pi\mu}{s+1},\quad m=0,1,\dots,s, \tag{123}$$

where $\mu\equiv m-s/2$ takes values from $-s/2$ to $s/2$ with integer step. The phase distribution (120) then becomes symmetric with respect to μ. In the limit in which s tends to infinity, one can introduce a continuous phase variable by replacing $\mu 2\pi/(s+1)$ by θ and $2\pi/(s+1)$ by $d\theta$ (Ref. 133), and this leads to the continuous phase probability distribution

$$\begin{aligned}P(\theta,t) = \frac{1}{2\pi}\Big(1+2\sum_{n>k} b_n b_k\{\cos[(n-k)\theta)]\mathrm{Re}\,U\\ + \sin[(n-k)\theta)]\mathrm{Im}\,U\}\Big)\end{aligned} \tag{124}$$

with normalization

$$\int_{-\pi}^{\pi} P(\theta,t)d\theta = 1. \tag{125}$$

We consider the case of exact resonance: $\Delta=0$. Then $\mathrm{Im}\,U=0$ and $P(\theta,t)$ takes the form

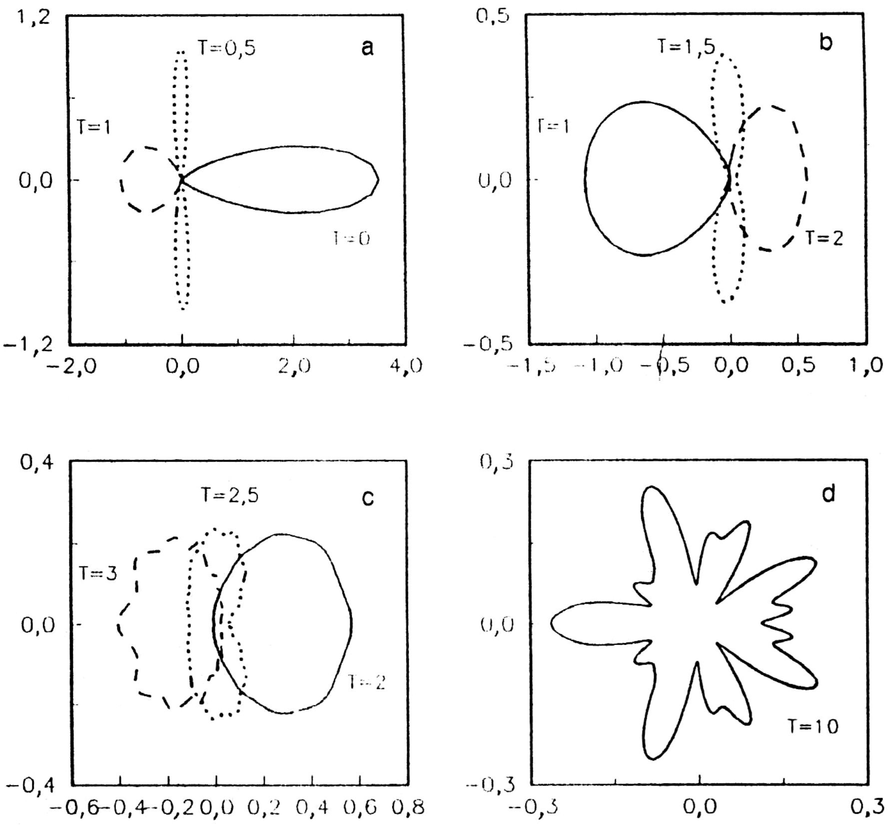

FIG. 16. Distribution of phase probability density $P(\theta,t)$ in polar form at different instants of the dimensionless time $T = gt/(2\pi\sqrt{\bar{n}})$, $\bar{n}=20$.

$$P(\theta,t)=\frac{1}{2\pi}\left\{1+2\sum_{n>k}^{\infty} b_n b_k \times\cos[(n-k)\theta]\cos[(\sqrt{n}-\sqrt{k})gt]\right\}. \quad (126)$$

Replacing the sum in

$$\langle\psi(t)|\hat{\phi}_\theta|\psi(t)\rangle=\sum_m \theta_m|\langle\theta_m|\psi(t)\rangle|^2 \quad (127)$$

by a suitable integral and noting that $\theta_m=\theta+\varphi$, we find the expectation value of the phase:

$$\langle\psi(t)|\hat{\phi}_\theta|\psi(t)\rangle=\varphi, \quad (128)$$

i.e., the expectation value of the phase does not change in time despite the atom–field interaction. This can be immediately seen from Eq. (126); for at any time t the phase probability distribution is an even function with respect to θ, and therefore integration from $-\pi$ to π leads to a zero mean value of θ, so that the expectation value of the phase is always equal to the value φ at the initial time. Note that in a more general situation, for example, when $\Delta\neq 0$ or when the atom is placed in a cavity in a coherent superposition of the ground and excited states, $\langle\hat{\phi}_\theta\rangle$ varies in time.

We now consider the properties of the phase probability distribution. Despite the obvious simplicity of (126), it is difficult to predict the form of $P(\theta,t)$, but if we rewrite P in the form

$$P(\theta,t)=\tfrac{1}{2}[P_+(\theta,t)+P_-(\theta,t)],$$
$$P_\pm(\theta,t)=\frac{1}{2\pi}\left\{1+2\sum_{n>k}^{\infty} b_n b_k \cos\left[(n-k)\left(\theta\mp\frac{gt}{\sqrt{n}+\sqrt{k}}\right)\right]\right\}, \quad (129)$$

then it can be seen that when the time increases the phase probability distribution is split into two equal parts. If they are drawn in a polar coordinate system (θ as polar angle and P as radius), these two parts rotate in opposite directions with equal velocity and at all times are situated symmetrically on the two sides of the line $\theta=0$ (see Fig. 16). After a certain interval of time, the two oppositely rotating distributions "collide." They completely overlap when the expectation value $\langle\hat{\phi}_\theta\rangle_+$ (respectively, $\langle\hat{\phi}_\theta\rangle_-$), taken in accordance with the probability distribution $P_+(\theta,t)[P_-(\theta,t)]$, increases by π ($-\pi$). For strong coherent fields, for which the distribution function with respect to the Fock states has a single sharp peak at $\bar{n}$, this time interval is approximately

$$t=\frac{2\pi\sqrt{\bar{n}}}{g}, \quad (130)$$

in exact agreement with the revival period of the Rabi oscillations (24). The time behavior of $P(\theta,t)$ is illustrated in Fig. 16 in a polar coordinate system. The time scale is changed in accordance with $T=gt/(2\pi\sqrt{\bar{n}})$ to make the revival of the Rabi oscillations occur at $T=1, 2, 3\dots$. At the initial time $T=0$, the phase probability distribution has

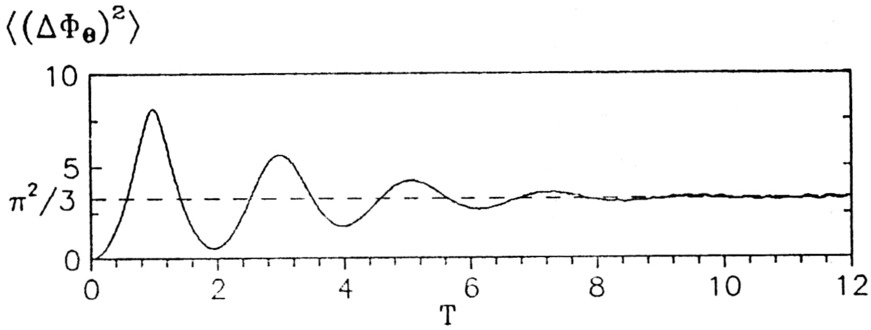

FIG. 17. Phase variance as a function of T; $\bar{n}=20$.

the shape of an elongated leaf corresponding to the coherent field state.[134] When the interaction commences, the phase probability distribution gradually splits into two leaves. At this time, the Rabi oscillations collapse. When the two components of the phase distribution are clearly separated from each other, there is a quasisteady regime, in which the atomic inversion hardly oscillates. At $T=1$, the two distribution curves are completely mixed and the Rabi oscillations are renewed in full force. After this, the phase probability distribution again splits into two peaks moving to the right-hand side of the figure, where they again collide. In the process of evolution, these peaks expand and the splitting of the phase probability into individual parts becomes harder and harder to recognize; this corresponds to a smearing of the signals of the revivals of the Rabi oscillations in time.

The phase variance

$$\langle(\Delta\hat{\phi}_\theta)^2\rangle = \sum_m (\theta_m - \langle\hat{\phi}_\theta\rangle)^2 |\langle\theta_m|\psi(t)\rangle|^2, \qquad (131)$$

like the phase probability distribution, carries information about the collapses and revivals of the Rabi oscillations. Replacing the sum in (131) by an integral, and using (124), we obtain

$$\langle(\Delta\hat{\phi}_\theta)^2\rangle = \frac{\pi^2}{3} + 4\sum_{n>k} b_n b_k \frac{(-1)^{n-k}}{(n-k)^2} \times\cos[(\sqrt{n}-\sqrt{k})gt]. \qquad (132)$$

The evolution of $\langle(\Delta\hat{\phi}_\theta)^2\rangle$ is illustrated in Fig. 17 for $\bar{n}=20$. Initially, the phase variance increases and reaches a maximum at $T=1$. At the first glance, it might appear that this contradicts the single-peak structure of the phase distribution density that exists at this time (Fig. 16a). However, as Pegg and Barnett showed,[134] one must take care in interpreting the results obtained for a definite value of the relative phase θ_0. Here, θ_0 is chosen to minimize the phase fluctuations in the initial coherent field state. Such a choice of θ_0 no longer minimizes the phase fluctuations at $T=1$. At this time, the distribution density $P(\theta,t)$ splits into two symmetric peaks separated by π from the coordinate origin. If the phase window is placed at π, the phase variance will again be minimized. Thus, for the choice (122) of the relative phase one can draw the conclusion that the maxima and minima of the phase variance correspond to revivals of the Rabi oscillations, and the deeper the extremal points, the more clearly the revivals are seen. At large times, $|\langle(\Delta\hat{\phi}_\theta)^2\rangle$ oscillates chaotically around $\pi^2/3$, the phase variance of the field state with randomly distributed phase. This reflects the existence of quasi-irreversibility inherent in the JC model.[62–64]

In the JC model, because of the square root in the expressions for the eigenvalues of the Hamiltonian, the collapse and revival of the Rabi oscillations, and also the splitting of the phase probability distribution do not occur completely. However, there exist various modifications of the JC model whose dynamical behavior is strictly periodic. One such modification is the model with coupling constant that depends on the field intensity proposed by Buck and Sukumar.[49] The Hamiltonian of this model is

$$H^{\mathrm{BS}} = \hbar\omega(a^\dagger a + R^z) + \hbar g(R^\dagger s + R^- s^\dagger), \qquad (133)$$

where $s = a\sqrt{a^\dagger a}$, $s^\dagger = \sqrt{a^\dagger a}\,a^\dagger$. It is interesting to compare the two models from the point of view of the phase properties. Repeating the procedure used above, we obtain for the phase distribution density the expression

$$P^{\mathrm{BS}}(\theta,t) = \tfrac{1}{2}[P_+^{\mathrm{BS}}(\theta,t) + P_-^{\mathrm{BS}}(\theta,t)],$$
$$P_\pm^{\mathrm{BS}}(\theta,t) = \frac{1}{2\pi}\left\{1 + 2\sum_{n>k}^{\infty} b_n b_k \cos[(n-k)(\theta\mp gt)]\right\}. \qquad (134)$$

It can be seen from this that, in contrast to the standard JC model, the phase distribution density is a periodic function of the time with period $2\pi/g$. The two oppositely rotating distributions $P_\pm^{\mathrm{BS}}(\theta,t)$ are none other than the phase distribution densities for a coherent field state with the phase θ replaced by $(\theta\mp gt)$, respectively. They overlap completely after each time interval π/g, which matches exactly the revival period of the Rabi oscillations.[49] The phase variance has the form

$$\langle(\Delta\hat{\phi}_\theta)^2\rangle^{\mathrm{BS}} = \frac{\pi^2}{3} + 4\sum_{n>k} b_n b_k \frac{(-1)^{n-k}}{(n-k)^2} \times\cos[(n-k)gt], \qquad (135)$$

and oscillates with the same period as the phase distribution density. The strict periodicity of the phase characteristics in the JC model with coupling constant that depends on the field intensity is not surprising, since in this model the revivals of the Rabi oscillations of the atomic inversion exactly restore its initial value.[49]

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