Effective chiral Lagrangians and the Nambu–Jona-Lasinio model

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A review is presented of papers devoted to the construction of effective meson Lagrangians on the basis of studies of QCD at low energies. Two different approaches to the solution of this problem are given as an example. In the first case, meson fields are introduced as a chiral phase of quark fields and external sources directly in the QCD Lagrangian. In other papers, mesons are treated as composite quark–antiquark objects. More attention is paid to the Nambu–Jona-Lasinio model, from the second group of models, and to its various physical applications. The main advantage of the last model is its mechanism of spontaneous breaking of chiral symmetry.

1. INTRODUCTION

Quantum chromodynamics is well known to be a successful theory in the description of high-energy interactions of hadrons. On the other hand, the low-energy physics of hadrons is also well described by effective chiral Lagrangians developed much earlier than QCD.[1–3] As at low energies, the direct use of QCD is difficult, owing to a large value of the coupling constant. Attempts have repeatedly been undertaken to obtain such effective chiral Lagrangians directly from QCD rather than from group-theoretical considerations, and to consider them as a low-energy representation of QCD. In recent years, many papers have been devoted to this problem. Here we present a brief review of some of the most characteristic investigations.

All these attempts can be divided into two groups. In the first of them, the authors introduce pseudoscalar meson fields, considering them to be external fields, as a chiral phase of quark fields directly in the QCD Lagrangian.[4–7] In the second group, a consistent bosonization of QCD is carried out, with all types of mesons treated as quark–antiquark bound states.[8–13] We consider here the most typical examples of these approaches.

The main attention in this work is paid to the Nambu–Jona-Lasinio model (NJL model). This model belongs to the second group. The NJL model at the quark level was first studied in 1976 in Ref. 14. In the last decade interest in this model was continuously increasing,[15–19] thus expanding the range of its application in recent years.[20–32] More thorough studies were made of chiral anomalies and Skyrme-like terms with fourth-order derivatives, as well as of intrinsic properties of the model connected with nondiagonal transitions of mesons,[16,18,21,22] mixing of flavors,[13,20–23] dependence of the model parameters on temperature and density,[24–26] gluon condensates,[26,27] phase transitions,[28] relations to QCD,[9,11,13,29] and the description of diquarks,[30,31] baryons, and light nuclei.[11,29,32] This list, being far from complete, shows a sufficiently wide spectrum of possible applications of the NJL model. Here we describe the connection of the NJL model with QCD, the main points of this model, and some of its applications to meson physics.

The paper is organized as follows. In Sec. 2 we describe models of the first type.[4–7] We shall show how the effective chiral Lagrangian can be obtained from QCD in the limit of large N_c and low energies. The next section is devoted to the problem of scalar particles and the conformal anomaly. In Sec. 4 we show how the NJL model is connected with QCD, and in Secs. 5–7 we describe the main points of the NJL model. Finally, in Sec. 8 we present different applications of the NJL model to low-energy meson physics.

2. DERIVATION OF THE EFFECTIVE CHIRAL LAGRANGIAN FROM QCD IN THE LIMIT OF LARGE N_c AND LOW ENERGIES

In 1985–1986 there appeared two papers[4,12] devoted to similar problems of derivation of an effective meson chiral Lagrangian from QCD at large N_c (the number of colors) and low energies. It has been shown that a nonlinear chiral Lagrangian together with the anomalous Wess–Zumino term can be obtained as a low-energy approximation of QCD at large N_c under the assumption that the chiral symmetry in the space of flavors is spontaneously broken. Then QCD is reduced to a pure pseudoscalar theory, provided that heavier scalar, vector, and axial mesons are neglected; in this case pions represent a chiral phase of quark fields or Goldstone modes of the dynamically broken chiral symmetry. These modes are conserved only in the parts of the QCD action without complete local chiral invariance.

Integration over all the color degrees of freedom can be performed under quite general assumptions about the behavior of the resulting potential. As a result, we get a chiral Lagrangian for pion fields and a number of standard formulas for such quantities as F_π (the pion decay constant), m_q (the mass of a constituent quark), and $\langle\bar{q}q\rangle_0$ (the quark condensate).

The model thus obtained leads to the Skyrme model, in which we can study the problem of stability of the soliton.

Now we shall describe the basic features of the model.[4] The effective chiral Lagrangian follows from QCD by using identical transformations within the $1/N_c$ expansion. The change of variables of quark fields is made in the functional integral, which separates the degrees of freedom responsible for the spontaneous breaking of chiral symme-

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try and corresponding to pseudoscalar mesons. Use is also made of the hypothesis of spontaneous breaking of chiral symmetry (SBCS).

The QCD Lagrangian [chiral symmetry $U(n)\times U(n)$ and color symmetry $SU(N_c)$] is of the form

$$\mathscr{L}(\bar{q},q,G)=-\frac{N_c}{4g^2}G^a_{\mu\nu}(x)G^{\mu\nu}_a(x)+i\bar{q}(x)\gamma^\mu \times[\partial_\mu-iG_\mu(x)]q(x). \quad (G_\mu=G^a_\mu T_a). \tag{1}$$

Meson fields are described by operators $\bar{q}\gamma^5 t^a q$, where t^a are the generators of $U(n)$ and $\mathrm{tr}(t^a t^b)=2\delta^{ab}$. Consider the generating functional

$$Z(\eta)=Z_0^{-1}\int d\mu(G)d\bar{q}dq \times\exp\left\{i\int dx[\mathscr{L}(x)+\bar{q}\gamma^5 t^a q\eta_a(x)]\right\}, \tag{2}$$

where $d\mu(G)$ is the measure that includes gauge-fixing terms and ghost fields. To single out the chiral phase of the quark fields, one performs the transformation

$$q^\Omega=(\Omega^+P_R+P_L)q,\quad \bar{q}^\Omega=\bar{q}(\Omega P_L+P_R), \tag{3}$$

where $\Omega(x)=\exp\{i\pi(x)\}$, $\pi=2\pi^a t_a$, $P_{R/L}=\frac{1}{2}(1\pm\gamma^5)$. With the help of the Faddeev–Popov procedure

$$1\equiv\Delta(\bar{q},q)\int d\mu(\Omega)\delta(\bar{q}^\Omega\gamma^5 t^a q^\Omega) \tag{4}$$

for the transformed fields q, Z is written in the form

$$Z(\eta)=Z_0^{-1}\int d\mu(G)d\bar{q}dqd\mu(\Omega)d\varphi J\Delta(\bar{q},q) \times\exp\left\{i\int dx[\mathscr{L}(x)+i\bar{q}(x)\hat{L}P_R q(x) +\bar{q}^{\Omega^{-1}}(x)\gamma^5 t^a q^{\Omega^{-1}}(x)\eta^a(x) +\bar{q}(x)\gamma^5 t^a q(x)\varphi^a(x)]\right\}. \tag{5}$$

Here use was made of the δ function in the exponential form with the Lagrange multiplier φ^a, $\hat{L}=\gamma^\mu L_\mu$, $L_\mu=\Omega^{-1}\partial_\mu\Omega$, and J is the Jacobian of the change of variables, $q\to q^{\Omega^{-1}}$. Owing to the δ function in (4), the pseudoscalar meson fields cannot simultaneously appear as the phase of the quark fields [see Eq. (3)] and as constituent operators $\bar{q}\gamma^5 t^a q$ in the functional $Z(\eta)$. In what follows, they appear only as chiral phases of quark fields. An analogous procedure is employed also in Ref. 12. There the transformation (3) is related to the transition from current quarks to constituent quarks with a large dynamical mass.

The Jacobian J results in the Wess–Zumino terms and Abelian anomaly with a π^0 meson necessary for solving the $U_A(1)$ problems:

$$\ln J=i\mathscr{L}_{\mathrm{WZ}} +i\frac{\sqrt{2\pi}}{8\pi^2}\mathrm{tr}\int dxG_{\mu\nu}(x)G_{\rho\sigma}(x)\varepsilon^{\mu\nu\rho\sigma}\pi^0(x). \tag{6}$$

Now let us consider the remaining part of the functional $Z(\eta)$. We must integrate over all the color variables and obtain the effective action for $\pi^a(x)$. Integration over the gluon fields yields the factor

$$\exp\{iS_0(\bar{q}T^a\gamma^\mu q,\pi^0)\}, \tag{7}$$

and the functional S_0 contains the color-singlet bilocal field combinations $\bar{q}^i_\kappa(x)q^j_\kappa(y)$ (κ is the color index; i and j are flavors).

Now we introduce the color-singlet bilocal collective variable $\xi^{ij}(x,y)$ (Refs. 4 and 10)

$$\exp\{iS_0\}=\int d\xi\delta[N_c\xi^{ij}(x,y)-\bar{q}^i_\kappa(x)q^j_\kappa(y)] \times\exp\{-iV(\xi,\pi^0)\},$$

$$\delta(N_c\xi-\bar{q}q)=\int d\lambda\exp\left\{i\int dxdy\lambda^{ij}(x,y) \times[N_c\xi^{ij}-\bar{q}^i_\kappa(x)q^j_\kappa(y)]\right\} \tag{8}$$

and then integrate over $\bar{q}$ and q:

$$Z(\eta)=Z^{-1}\int d\mu(\Omega)d\xi d\lambda d\varphi\Delta \times\exp\{i[-V(\xi,\pi^0)+\mathscr{L}_{\mathrm{WZ}}(L)+N_c\,\mathrm{tr}\,\lambda\xi -iN_c\,\mathrm{tr}\ln(i\hat{\partial}+i\hat{L}P_R-\lambda+\varphi^a t^a\gamma^5+\dots)]\}. \tag{9}$$

The functional $V(\xi,\pi^0)$ can be split into two parts

$$V(\xi,\pi^0)=V(\xi,0)+V'(\xi,\pi^0), \tag{10}$$

where $V'(\xi,\pi^0)$ includes at least one source, π^0. The fermion lines are associated with the factor $\delta_{\kappa\kappa'}\xi$. At large N_c the behavior of $V'(\xi,\pi^0)$ is determined by purely Yang–Mills diagrams and does not depend on ξ: $V'(\xi,\pi^0) = V'_0(\pi^0)+O(1/N_c)$.

The diagrams corresponding to $V(\xi,0)$ contain at least one quark loop and lead to an expansion of the form

$$V(\xi,0)=N_c\left[V_0(\xi)+\frac{1}{N_c}V_1(\xi)+\dots\right]. \tag{11}$$

To integrate over the remaining variables in the leading order in N_c, one expands the action in the exponential (9) around the stationary point:[1)]

$$\xi^{ij}_{\mathrm{st}}(x-y)=-i\left(\frac{1}{i\hat{\partial}-\lambda_{\mathrm{st}}}\right)^{ij},$$

$$\lambda^{ij}_{\mathrm{st}}(x-y)=\left(\frac{\delta V_0}{\delta\xi_{\mathrm{st}}}\right)^{ij},\quad \varphi^a=\pi^a=0. \tag{12}$$

From the assumption that the $U(n)\times U(n)$ symmetry is broken down to the diagonal $U(n)$ it follows that $\xi^{ij}_{\mathrm{st}} = \delta^{ij}\xi_{\mathrm{st}}$, $\lambda^{ij}_{\mathrm{st}}=\delta^{ij}\lambda_{\mathrm{st}}$. Though explicit solutions ξ and λ are not known, since $V_0(\xi)$ is not known, it is sufficient to assume that $\tilde{\lambda}_{\mathrm{st}}(p)$ at small momenta tends to a nonzero constant, $\tilde{\lambda}_{st}(p)|_{p=0}=c\neq 0$. This implies that quarks acquire a dynamical mass, i.e., chiral symmetry is spontaneously broken [the quark propagator in the leading order in N_c coincides with $\tilde{\xi}_{st}(p)$]. This point is crucial in both Refs. 4 and 12 for constructing effective chiral Lagrangians. Computation of the effective action for the fields π^a is only a technical problem, which is slightly different in the two papers.

Let us return to the paper of Ref. 4. At low energies the leading terms are the terms quadratic in the currents L_μ. In the vicinity of the stationary point the argument of the exponential function in (9) is of the form

$$\mathscr{L}_{WZ}(L)-V_0'(\pi^0)+N_c\,\mathrm{tr}\times\left\{-\frac{1}{2}\hat{L}P_R\phi\hat{L}P_R-\frac{i}{2}\hat{L}P_R\phi\lambda+\frac{1}{2}\lambda\phi\lambda+\tfrac{1}{2}\phi\gamma^5\phi\gamma^5\varphi+\lambda\xi+\tfrac{1}{2}\xi K\xi+\ldots\right\}, \quad (13)$$

where ϕ and K are coefficient functions in the expansion of $\mathrm{tr}\ln(i\hat{D})$ and $V_0(\xi)$.

Then by integrating over ξ, λ, and φ successively we get

$$\mathscr{L}_{\mathrm{eff}}=-\frac{F_\pi^2}{4}\mathrm{tr}\,L_\mu L_\mu-V_0'(\pi^0)-\frac{N_c}{48\pi^2}\int_0^1 d\tau\varepsilon^{\mu\nu\rho\sigma}\times\mathrm{tr}[L_5(\tau)L_\mu(\tau)L_\nu(\tau)L_\rho(\tau)L_\sigma(\tau)], \quad (14)$$

where $F_\pi=93$ MeV is the pion decay constant. Upon expanding the functional $V_0'(\pi^0)$ in powers of π^0 one obtains an extra mass term for the η' meson.

Note that a very important point in all the models considered here is the hypothesis of spontaneous breaking of chiral symmetry. This phenomenon is a natural result of the assumption about the form of the potential $V_0(\xi)$, i.e., of the condition $[\tilde{\lambda}_{st}(p)=\delta V_0/\delta\xi_{st}]_{p=0}=c\neq 0$.

A rather different formulation of that hypothesis is given in Ref. 6, where the authors considered the eigenvalues K of the total Dirac operators $i\hat{D}=i\hat{\partial}+gG+V+\gamma_5 A-S+i\gamma_5 P$ with external vector, axial, scalar, and pseudoscalar fields, V_μ, A_μ, S, and P, and gluon fields G_μ. In the Euclidean space, $i\hat{D}q_{k\alpha}=Kq_{k\alpha}$. These eigenvalues define the determinant of the Dirac operator and thus the generating functional. Then the low-energy region is separated in an asymmetric way:

$$-\Lambda+M\leqslant K\leqslant\Lambda+M(0\leqslant M\leqslant\Lambda), \quad (15)$$

where Λ is a spectral parameter, and M is the mass of a constituent quark. For $M\neq 0$ the quark condensate is nonvanishing:

$$\langle\bar{q}q\rangle_0=-\frac{N_c}{2\pi^2}\left(\Lambda^2 M+\frac{M^3}{3}\right). \quad (16)$$

This expression can be easily derived from the definition $\langle\bar{q}q\rangle_0=iN_c\,\mathrm{tr}\int d^4p G_q(p)$, where $G_q(p)=[1/(2\pi)^4]\times[1/(m-\hat{p})]$ is the momentum representation for the quark propagator, integrated between the limits (15). In this way the spontaneous breaking of chiral symmetry is phenomenologically introduced into the model.

As we see, SBCS in all models cannot be shown to follow directly from the construction of the models, but is rather an extra condition imposed either on an unknown potential[4,12] or on the spectrum of eigenvalues of the Dirac operator.[6] In this regard, the Nambu–Jona-Lasinio (NJL) model to be considered in the next sections is a more self-consistent model, directly reproducing the mechanism of SBCS at low energies.

To conclude this section, we briefly indicate how the parameters Λ and M in the model of Ref. 6 are expressed in terms of the quark and gluon condensates. The quark condensate is described by Eq. (16); to deduce an equation for the gluon condensate we consider "radial" fluctuations of the quark field $q(x)\to\exp(-\sigma(x))q(x)$ with the scalar field $\sigma(x)=\sigma_0(x)+t^a\sigma_a(x)$. Singlet fluctuations change the magnitude of the condensate $\langle\bar{q}q\rangle_0$; therefore they should be suppressed for stability of the region L. In order to investigate the possibility of suppression of these fluctuations these authors construct the effective action $W_{\mathrm{eff}}(\sigma)$ for the field $\sigma(x)$ generated by the conformal triangular anomaly; here gluons play a leading role. Therefore, calculating $W_{\mathrm{eff}}(\sigma)$, one can put $m_q=0$ and $A_\mu=V_\mu=S=P=0$, so that $i\hat{D}=i(\hat{\partial}+g\hat{G})$. Along with $\hat{D}$ there is the conformally transformed operator $\hat{D}_\sigma=e^\sigma\hat{D}e^\sigma$. The corresponding generating functional is of the form

$$Z_q(G,\sigma)=\int d\bar{q}dq\exp\left(-\int d^4x\bar{q}\hat{D}_\sigma q\right).$$

The conformal-invariant part $Z_{\mathrm{inv}}(G)$ follows upon integration over $\sigma(x)$:

$$Z_{\mathrm{inv}}^{-1}=\int d\sigma Z_1^{-1}(G,\sigma).$$

Then the conformal-noninvariant part $Z_{\mathrm{conf}}(G)$ and the effective action $W_{\mathrm{eff}}(\sigma)$ are given by the formula

$$Z_{\mathrm{conf}}(G)=\int d\sigma Z_q(G,1)Z_q^{-1}(G,\sigma)\equiv\int d\sigma e^{-W_{\mathrm{eff}}(\sigma,G)},$$

$$W_{\mathrm{eff}}(\sigma,G)=\int_0^1 ds\int d^4x 2\,\mathrm{tr}[\sigma(x)\langle x|\theta\times(\Lambda^2-(i\hat{D}_{s\sigma}-M)^2|x\rangle]. \quad (17)$$

By using the methods proposed in Ref. 6 for the effective potential one obtains the expression

$$\frac{N_f}{4\pi^2}\left\{\frac{N_f}{8}(e^{-8\sigma_c}-1)(6\Lambda^2M^2-\Lambda^4-M^4)+\frac{\sigma_c}{6}g^2\sum(G_{\mu\nu}^a)^2\right\}. \quad (18)$$

Stability of the low-energy region (15) implies that the effective potential should possess a minimum at $\sigma_c=0$, i.e., at the value of the quark condensate chosen above [see (16)]. As a result, we arrive at the condition

$$6N_c(6\Lambda^2M^2-\Lambda^4-M^4)=\left\langle g^2\sum(G_{\mu\nu}^a)^2\right\rangle_0,$$

$$\left\langle g^2 \sum (G^a_{\mu\nu})^2 \right\rangle_0 > 0. \qquad (19)$$

Thus, the quark and gluon condensates become directly connected with the parameters Λ and M. In Ref. 6 the following estimates were found: for $\langle \bar{q}q \rangle_0 = -[(200\text{–}250)\ \text{MeV}]^3$ and $\langle (g^2/4\pi)\Sigma(G^a_{\mu\nu})^2 \rangle_0 = [(350\text{–}400)\ \text{MeV}]^4$ we have $\Lambda = (475\text{–}610)$ MeV and $M = (250\text{–}300)$ MeV. In the derivation of (19) the authors made use of the low-energy approximation

$$g^2 \sum (G^a_{\mu\nu})^2 \approx \left\langle g^2 \sum (G^a_{\mu\nu})^2 \right\rangle_0. \qquad (20)$$

As a result, the quantum fluctuations of the gluon fields are neglected. However, we note that these quantum gluon fields play an important role in calculating the effective four-quark interaction, as has been shown in Refs. 8–13, 16, and 18. They may produce additional terms in Eq. (19) of the form const M^2/G, where G is the four-quark coupling constant (see, for instance, the gap equation in the NJL model). This, in turn, may somewhat change the values of the parameters M and Λ.[2)]

The nonlinear Lagrangian for the fields V_μ, A_μ, S, and P can be obtained by the method suggested by Andrianov and Bonora.[6]

Here we have for the first time met the notion of conformal symmetry. As it is of much significance for the construction of effective Lagrangians, we will dwell upon its physical consequences in the next section.

3. SCALE SYMMETRY, CONFORMAL ANOMALY, AND EFFECTIVE LAGRANGIAN OF GLUONIUM (DILATON)

Scale symmetry plays an important role in field theory and is employed in constructing effective Lagrangians. Here we shall only review a part of the voluminous literature devoted to this problem.

Let us recall the basic ideas related to scale transformations of fields and Lagrangians.[34] Under a scale transformation of the coordinates

$$x \to \lambda x$$

scalar and spinor fields are transformed as follows:

$$\varphi(\lambda x) \to \lambda^{-1}\varphi(x), \quad \psi(\lambda x) \to \lambda^{-3/2}\psi(x).$$

As a result, the simplest Lagrangian of the form

$$\mathscr{L}(x) = -\tfrac{1}{2}(\partial_\mu \varphi)^2 + i\bar{\psi}\hat{\partial}\psi + g\bar{\psi}\varphi\psi + h\varphi^4$$

is transformed as $\mathscr{L}(\lambda x) \to \lambda^{-4}\mathscr{L}(x)$ and gives rise to a scale-invariant action. It is easy to see that mass terms break scale invariance.

If the masses of the light current quarks are set to zero ($m^0_{u,d,s} = 0$), the QCD Lagrangian should be scale-invariant. However, there is an internal scale with mass dimensionality at the quantum level:[35,36]

$$\mu = M_0 \exp\left\{ -\frac{8\pi^2}{b g_0^2} \right\}, \qquad (21)$$

where M_0 is the mass of an ultraviolet regulator, g_0 is the bare coupling constant $g_0 = g(M_0)$, and $b = (11N_c - 3N_f)/3$. This effect breaks scale invariance. The naive trace of the energy–momentum tensor should equal zero; however, owing to the gluon anomaly,

$$\sigma(x) = \theta_{\mu\mu}(x) = \frac{\beta(\alpha_s)}{4\alpha_s} [G^a_{\mu\nu}(x)]^2, \qquad (22)$$

where $\beta(\alpha_s) = -b\alpha_s^2/2\pi + O(\alpha_s^2)$ is the Gell-Mann–Low function.

In quantum field theory, the classical scale invariance gives rise to the relation (Ref. 37)[3)]

$$\lim_{q\to 0} i \int d^4x e^{iqx} \langle T\{Q(x), \sigma(0)\}\rangle = -d_n \langle Q \rangle, \qquad (23)$$

where $Q(x)$ is an arbitrary local operator constructed from gluons and quarks, and d_n is its normal dimension. In what follows we will utilize the very important particular case of the relation (23) when $Q(x) = \sigma(x) = [\beta(\alpha_s)/4\alpha_s](G^a_{\mu\nu})^2$, i.e., the Ward identity

$$i \int d^4x \langle T\{\sigma(x), \quad \sigma(0)\}\rangle = -4\langle \sigma \rangle. \qquad (24)$$

Now let us determine the low-energy (tree) interaction Lagrangian for $\sigma(x)$ obeying the Ward scale identity (24). Note that the solution is entirely determined by the sign of the vacuum energy; it is stable only when $\varepsilon_{\text{vac}} < 0$.

The normal dimension of $\sigma(x)$ is four. Therefore, if one takes the kinetic term in the form

$$\mathscr{L}_{\text{kin}} = \text{const}[\partial_\mu \sigma(x)]^2 [\sigma(x)]^{-3/2}, \qquad (25)$$

the corresponding part of the action will be scale-invariant, and $\mathscr{L}_{\text{kin}}$ will not contribute to $\theta_{\mu\mu}$.

Let us now construct the potential part $V(\sigma)$ so as to satisfy the condition $\theta_{\mu\mu} = \sigma$. Under an infinitesimal scale transformation, when $\lambda = 1 + \varepsilon$, the field σ and the potential part of the action transform as follows:

$$\sigma \to (1 - 4\varepsilon)\sigma,$$

$$\Delta S_{\text{pot}} = -\Delta \int d^4x V(\sigma) = -\int d^4x \left(4V - 4\sigma \frac{\delta V}{\delta \sigma} \right). \qquad (26)$$

Equating this change of S to the quantity $\int d^4x \theta_{\mu\mu}(x)$, we arrive at an equation that ensures the validity of the identity (24):[38]

$$4V - 4\sigma \frac{\delta V}{\delta \sigma} = \sigma. \qquad (27)$$

It has a simple solution[4)]

$$V = -\frac{\sigma}{4} (\ln \sigma + \text{const}'). \qquad (28)$$

The constants in (25) and (28) can be expressed in terms of the mass of σ, m_σ, and the vacuum energy $\varepsilon_{\text{vac}} \equiv \langle 0|\theta_{00}|0\rangle \equiv \tfrac{1}{4}\langle 0|\theta_{\mu\mu}|0\rangle$. Recall that $\varepsilon_{\text{vac}} < 0$. The field σ can be written in the form $\sigma = 4\varepsilon_{\text{vac}} \exp \chi$; then the gluonium (dilaton) effective Lagrangian acquires the form

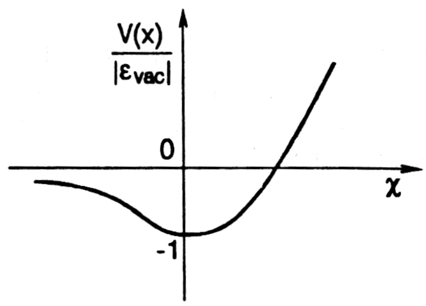

FIG. 1. The potential part of the effective Lagrangian of gluonium (dilaton).

$$\mathscr{L}_{\text{eff}}=-\frac{\varepsilon_{\text{vac}}}{2m_\sigma^2}(\partial_\mu\chi)^2\exp\left(\frac{\chi}{2}\right)+\varepsilon_{\text{vac}}(\chi-1)\exp(\chi). \tag{29}$$

Let us demonstrate how the effective Lagrangian describing both the gluonium and quarkonium fields can be constructed. We will follow Refs. 41 and 42. Besides the dilaton field $d(x)$ (gluonium) we consider the pion fields $\pi_i(x)$ $(i=1, 2, 3)$ and the singlet scalar field $S(x)$. The quarkonium fields π_i and S are described by the chiral $SU(2)\times SU(2)$ sigma model. The gluonium field d is invariant under chiral transformations. The total Lagrangian has the form

$$\mathscr{L}=\tfrac{1}{2}\{(\partial_\mu d)^2+(\partial_\mu S)^2+(\partial_\mu\pi_i)^2\}-V(d,S,\pi_i), \tag{30}$$

where the potential V obeys an equation of the type (27):

$$\sigma=-H_0\left(\frac{d}{f_d}\right)^4=4V-d\frac{\delta V}{\delta d}-S\frac{\delta V}{\delta S}-\pi_i\frac{\delta V}{\delta\pi_i}. \tag{31}$$

Here $H_0=-\langle\sigma\rangle$, and f_d is an analog of the pion constant f_π:

$$-\langle 0|\sigma|d\rangle=m_d^2 f_d,\quad f_d=\langle 0|d|0\rangle. \tag{32}$$

Equation (31) again ensures the validity of the Ward identity (24). The potential V does not contain fields with derivatives and is a function of two $SU(2)\times SU(2)$ invariants, d and $(s^2+\pi_i^2)^{1/2}$.

Then from Eq. (31) one can determine the following form of the potential:

$$V(d,S,\pi_i)=H_0\left(\frac{d}{f_d}\right)^4\ln\frac{d}{C}+d^4 f\left(\frac{\sqrt{S^2+\pi_i^2}}{d}\right), \tag{33}$$

where C is an arbitrary constant of mass dimensionality and f is an arbitrary dimensionless function of the argument $\sqrt{S^2+\pi_i^2}/d$.

Following Ref. 42, the function f can be taken in the form

$$f\left(\frac{\sqrt{S^2+\pi_i^2}}{d}\right)=-\frac{1}{6}\frac{G_0}{f_d^4}\ln\left(\frac{C}{f_\pi}\frac{\sqrt{S^2+\pi_i^2}}{d}\right)+\frac{G_0}{24f_\pi^4}\frac{(S^2+\pi_i^2)^2}{d^4}, \tag{34}$$

where $G_0=(8/b)H_0$. Then the potential V acquires the form

$$V(d,S,\pi_i)=\frac{G_0}{24}\left(\frac{d}{f_d}\right)^4\left[11N_c\ln\frac{d}{C}-4\ln\frac{\sqrt{S^2+\pi_2}}{f_\pi}\right]+\frac{1}{24}\frac{G_0}{f_\pi^4}(S^2+\pi_i^2)^2. \tag{35}$$

If fluctuations of the gluon field are neglected, i.e., if $d=f_d$ in (35), one arrives at the following expression for the potential of the sigma model:

$$V_{\text{SM}}(S,\pi_i)=-\frac{G_0}{6}\ln\frac{\sqrt{S^2+\pi_i^2}}{f_\pi}+\frac{1}{24}\frac{G_0}{f_\pi^4}(S^2+\pi_i^2)^2.$$

An analogous potential was obtained in Ref. 43 directly from QCD with the use of the procedure of bosonization described in the previous section. The resulting Lagrangians allow us to estimate the masses of the gluonium and mesons of the scalar nonet, and the parameters of mixing of the gluonium with the neutral scalar mesons, and to describe the basic decays of these scalar states.[41,42]

4. QCD AND THE NAMBU–JONA-LASINIO MODEL[9]

Recall that the generating functional of QCD (in Euclidean space) takes the form

$$e^{-W}=\int d\bar{q}dqDG\exp\left(-\int dx\left\{\bar{q}(\hat{\partial}+m^0)q+\bar{q}\hat{G}q+\frac{1}{8g^2}\operatorname{tr}(G_{\mu\nu})^2\right\}\right). \tag{36}$$

Here $\hat{G}=\gamma^\mu G_\mu^a T^a$, and m^0 is the current quark mass. Gauge-fixing terms and the Faddeev–Popov factor for the $SU(N_c)$ color gauge symmetry are included in the gluon measure. When $m^0=0$, the Lagrangian has $U(N_f)\times U(N_f)$ global flavor symmetry.

Integrating again over the gluon fields, we have

$$e^{-W}=\int d\bar{q}dq\exp\left(-\left\{\int dx\bar{q}(\hat{\partial}+m^0)q+\sum_{n=2}^{\infty}\frac{1}{n!}\int dx_1\ldots dx_n\Gamma^{a_1\ldots a_n}_{\mu_1\ldots\mu_n}j^{a_1}_{\mu_1}(x_1)\ldots j^{a_n}_{\mu_n}(x_n)\right\}\right), \tag{37}$$

where $j_\mu^a(x)=\bar{q}(x)\gamma_\mu T^a q(x)$ is a chiral singlet local current coupling to G_μ^a; $\Gamma^{a_1\ldots a_n}_{\mu_1\ldots\mu_n}$ are chiral singlets related to one-gluon irreducible Green functions that contain the whole gluon dynamics. Fortunately, the principal features of the low-energy dynamics can be understood without detailed information on the functions Γ.

We introduce a bilocal field $\eta(x,y)$ describing mesons at low energies (see Sec. 2):[4,9,10]

$$1\equiv\int D\zeta D\eta\exp\left(\int dxdy\operatorname{tr}\eta(x,y)[\zeta(y,x)\right.$$

$$-q(y)\bar{q}(x)]\Big).$$

Then

$$e^{-W}=\int d\bar{q}dqD\zeta D\eta\exp\Big(-\Big\{\int dx\bar{q}(\hat{\partial}+m^0)q$$
$$-\int dxdy\eta(\zeta-q\bar{q})\Big\}-\Gamma(\zeta)\Big),$$

where

$$\Gamma(\zeta)=\sum_{n=2}^{\infty}\frac{1}{n!}\int dx_1...dx_n\Gamma^{a_1...a_n}_{\mu_1...\mu_n}(x_1...x_n)$$
$$\times\mathrm{tr}[\gamma^{\mu_1}T_{a_1}\zeta(x_1,x_2)\gamma^{\mu_2}$$
$$\times T_{a_2}\zeta(x_2,x_3)...T_{a_n}\zeta(x_n,x_1)].$$

Integrating over ζ and defining

$$e^{-G(\eta)}\equiv\int D\zeta\exp\{-(\Gamma(\zeta)-\mathrm{tr}\,\eta\zeta)\},$$

we get

$$e^{-W}=\int d\bar{q}dqD\eta\exp\Big\{-\Big(\int dx\bar{q}(\hat{\partial}+m^0)q$$
$$+\int dxdy\bar{q}(x)\eta(x,y)q(y)+G(\eta)\Big)\Big\},\qquad(38)$$

where $G(\eta)$ is a nonlocal functional containing any powers of the meson fields.

Further, use is to be made of a scale parameter Λ when considering low-energy physics.

The following assumptions are then made:

1) Gluon confinement:

$\Gamma^{a_1...a_n}_{\mu_1...\mu_n}(x_1,...x_n)$ should be a color singlet for $|x_n-x_m|>\Lambda^{-1}$, $n\neq m$; glueballs should have masses larger than Λ. Then $\Gamma^{(n)}$ is completely defined by the gluon condensate and has no poles. From Lorentz invariance it follows then that

$$\Gamma^{a_1...a_n}_{\mu_1...\mu_n}(x_1,...,x_n)\approx\delta^{a_1...a_n}\delta_{\mu_1...\mu_n}C_n\Lambda^n,$$

where C_n are constants on the scale $|x_n-x_m|>\Lambda^{-1}$.

2) Quark confinement:

$\eta(x,y)$ and $\zeta(x,y)$ are strongly localized on the scale Λ^{-1}, so that no free quarks would exist at energies lower than Λ. Therefore the following expansions are possible:

$$\Lambda^{-4}\eta(x,y)=\eta(z)f(t)+\eta_\mu(z)t_\mu f'(t)$$
$$+\tfrac{1}{2}\eta_{\mu\nu}(z)t_\mu t_\nu f''(t)+...,$$

$$\zeta(x,y)=\zeta(z)g(t)+...,$$

where $z=\frac{1}{2}(x+y)$ and $t=(\Lambda/2)(x-y)$; f and g vanish rapidly when $t^2\to\infty$; $q(x)$ can also be expanded around z:

$$q(x)=q(z)+\Lambda^{-1}t_\mu\partial^t_\mu q\Big(z+\frac{t}{\Lambda}\Big)\Big|_{t=0}+...\,.$$

Then every term of (38) can be written as follows:

$$\iint dxdy\bar{q}(x)\eta(x,y)q(y)$$
$$=\int dz\bar{q}(z)\eta(z)q(z)\int dtf(t)+...,$$

$$\iint dxdy\zeta(x,y)\eta(x,y)$$
$$=\int dz\zeta(z)\eta(z)\int dtf(t)g(t)+...,$$

$$\Lambda^4\iint dxdy\gamma_\mu\zeta(x,y)\gamma_\mu\zeta(y,x)$$
$$=\int dz(\gamma_\mu\zeta(z))^2\int dtg^2(t)+...,\text{ etc.}$$

Note that only g and the meson wave function f are integrated over momenta larger than Λ.

Apart from the localization of $\eta(x,y)$, one must further require the suppression of disintegration of the meson into a quark–antiquark pair above the threshold (≈0.6 GeV) (quark confinement).

In the spinor space the local fields η are expanded as follows:

$$\eta=\eta^S+i\gamma_5\eta^P+\gamma_\mu\eta^V_\mu+\gamma_5\gamma_\mu\eta^A_\mu$$
$$=\bar{\sigma}+i\gamma^5\bar{\varphi}+\hat{\bar{V}}+\gamma^5\hat{\bar{A}}=M+\gamma_\mu N_\mu,$$

($\bar{a}=a_a\lambda^a$; λ^a are the Gell-Mann matrices).

Here the different terms correspond to the composite operators $(\bar{q}\lambda^a q)$, $(\bar{q}\gamma_5\lambda^a q)$, $(\bar{q}\gamma_\mu\lambda^a q)$, $(\bar{q}\gamma_5\gamma_\mu\lambda^a q)$ with quantum numbers 0^{++}, 0^{-+}, 1^{--}, 1^{++}, respectively. The gluon potential $G(\eta)$ then takes the form

$$G(\eta)=\int dx\Big(-E_0+\Big\{\frac{\mu^2}{4}\mathrm{tr}\Big(\frac{1}{2}\hat{M}M-N_\mu N_\mu\Big)+...\Big\}$$
$$+\Big\{\frac{\varepsilon}{4}\mathrm{tr}\Big[(\hat{M}M)^2-N_\mu N_\mu\hat{M}M+\frac{1}{2}\hat{M}N_\mu MN_\mu$$
$$+8(N_\mu N_\mu)^2\Big]+...\Big\}+O\Big(\frac{\eta^6}{\Lambda^2}\Big)\Big),\qquad(39)$$

where E_0 is the vacuum energy density. The function $G(\eta)$ is locally chiral-invariant, does not contain derivatives, and is even in η. Terms in $G(\eta)$ are of the order of N_c^2, 1, $1/N_c$,..., respectively.

In the first approximation, where the ε terms are omitted, $G(\eta)$ contains only one parameter μ^2, and this construction may generate four meson multiplets. As will be seen, we can achieve a quantitative description of meson physics without a detailed analysis of the gluon dynamics. Integrating over η, from (38) and (39) we get

$$e^{-W}=\int[d\bar{q}dq]_\Lambda\exp\Big(-\int dx\Big\{\bar{q}(\hat{\partial}+m^0)q$$
$$-\frac{G_1}{2}[(\bar{q}\lambda^a q)^2+(\bar{q}i\gamma_5\lambda^5 q)^2]+\frac{G_2}{2}[(\bar{q}\gamma_\mu\lambda^a q)^2$$

$$+(\bar{q}\gamma_5\gamma_\mu\lambda^a q)^2]\Big\}\Big), \tag{40}$$

where $G_1=4/\mu^2$, $G_2=2/\mu^2$. This is just the NJL model. The resulting four-quark interaction is the Fierz transformation of the interaction $(\bar{q}\gamma_\mu\lambda^a q)^2$, i.e., a consequence of the vector-like structure of QCD. However, the relation $G_1=2G_2$ for the coupling constants of the scalar (pseudoscalar) and vector (axial-vector) channels may be changed if either the functions f or ε terms are taken into consideration (different for π and ρ) [see (39)].

For a special treatment of the $U(1)$ problem it is necessary to include the $U(1)$ anomaly in $G(\eta)$, for instance, as a term with π_s^2 (see Sec. 2).[5)]

5. NJL MODEL AND LINEAR σ MODEL

The basic independent quantities of the NJL model are the masses of the constituent quarks connected with the quark condensate, the cutoff parameter Λ determining the boundary of the region of spontaneous breaking of chiral symmetry (SBCS), and the four-quark interaction constants G_1 and G_2. The purpose of the present section is to show how the standard linear σ model describing the scalar and pseudoscalar mesons can be obtained from the effective four-quark interaction of the NJL type.

Let us consider the following effective quark Lagrangian of the strong interactions, which is invariant under $SU(3)_{\text{color}}\otimes U(3)_L\otimes U(3)_R$ symmetry (for the case $m^0=0$):

$$\mathscr{L}(\bar{q}q)=\bar{q}(\mathrm{i}\hat{\partial}-m^0)q+2G_1 \times\sum_{a=0}^{n^2-1}\left\{\left(\bar{q}\frac{\lambda_a}{2}q\right)^2+\left(\bar{q}\mathrm{i}\gamma^5\frac{\lambda_a}{2}q\right)^2\right\}-2G_1 \times\sum_{a=0}^{n^2-1}\left\{\left(\bar{q}\gamma^\mu\frac{\lambda_a}{2}q\right)^2+\left(\bar{q}\gamma^\mu\gamma^5\frac{\lambda_a}{2}q\right)^2\right\}. \tag{41}$$

Here summation over the color indices is implicit; λ_a are generators of the flavor $U(n)$ group; $m^0=\mathrm{diag}(m_1^0,m_2^0,\ldots,m_n^0)$ is the bare quark mass matrix, which explicitly breaks chiral and diagonal $U(n)$ flavor symmetry; G_1 and G_2 are universal quark coupling constants with the dimension of length squared. The meson fields can be introduced and a phenomenological meson Lagrangian can be derived by a standard procedure on the basis of generating functionals $Z(\eta,\bar{\eta})$:[14–18]

$$Z(\eta,\bar{\eta})=\frac{1}{N}\int d\bar{q}dq\exp\left\{\mathrm{i}\int d^4x[\mathscr{L}(\bar{q},q)+\eta\bar{q}+\bar{\eta}q]\right\}$$
$$=\frac{1}{N'}\int d\bar{q}dq\prod_{a=0}^{8}d\tilde{S}_a dP_a dV_a dA_a \times\exp\left\{\mathrm{i}\int d^4x[\mathscr{L}'(\bar{q},q,\tilde{S},P,V,A)+\eta\bar{q}+\bar{\eta}q]\right\}$$
$$=\frac{1}{N''}\int\prod_{a=0}^{8}dS_a dP_a dV_a dA_a \times\exp\left\{\mathrm{i}\int d^4x\left[\mathscr{L}''(S,P,V,A)+\mathrm{i}\int d^4y\bar{\eta}(x)D^{-1}(x,y)\eta(y)\right]\right\}, \tag{42}$$

where N, N', N'' are normalization constants, η, $\bar{\eta}$ are external sources, and

$$\mathrm{i}D(x,y)=[\mathrm{i}\hat{\partial}-m+S+\mathrm{i}\gamma_5P+\hat{V}+\hat{A}\gamma_5]\delta^{(4)}(x-y)$$
$$=[\mathrm{i}\hat{\partial}+A_R+M]P_R+[\mathrm{i}\hat{\partial}+\hat{A}_L+M^+]P_L-m. \tag{43}$$

Here $m=\mathrm{diag}(m_1,m_2,\ldots,m_n)$ are the masses of the constituent quarks, $P_{R/L}=\frac{1}{2}(1\pm\gamma_5)$ are projection operators, V and A are vector and axial-vector fields, respectively,

$$\hat{V}=\sum_{a=0}^{n^2-1}V_\mu^a\lambda_a\gamma^\mu,\quad \hat{A}=\sum_{a=0}^{n^2-1}A_\mu^a\lambda_a\gamma^\mu, \tag{44}$$

$(A_{R/L})_\mu=V_\mu\pm A_\mu$, and S and P are scalar and pseudoscalar fields, with

$$S=\sum_{a=0}^{n^2-1}S^a\lambda_a,\quad P=\sum_{a=0}^{n^2-1}P^a\lambda_a,\quad M=S+\mathrm{i}P. \tag{45}$$

The Lagrangians $\mathscr{L}'$ and $\mathscr{L}''$ are given by

$$\mathscr{L}'(\bar{q},q,\tilde{S},P,V,A)=\bar{q}[\mathrm{i}\hat{\partial}-m^0+\tilde{S}+\mathrm{i}\gamma_5P+\hat{V}+\hat{A}\gamma_5]q -\frac{\mathrm{tr}(\tilde{S}^2+P^2)}{4G_1}+\frac{\mathrm{tr}(V_\mu^2+A_\mu^2)}{4G_2}, \tag{46}$$

$$\mathscr{L}''(S,P,V,A)=-\frac{\mathrm{tr}(\tilde{S}^2+P^2)}{4G_1}+\frac{\mathrm{tr}(V_\mu^2+A_\mu^2)}{4G_2} -\mathrm{i}\,\mathrm{tr}\ln[\mathrm{i}\hat{D}(x-y)]|_{x=y}. \tag{47}$$

In what follows we consider only the case of three flavors $(n=3)$, and in this section our considerations will be restricted to scalar and pseudoscalar fields, S and P.

Only identical transformations were employed for introducing boson fields into the functional $Z(\eta,\bar{\eta})$. However, when one passes from the Lagrangian $\mathscr{L}'$ to $\mathscr{L}''$, there occurs an important phenomenon caused by rearrangement of the vacuum due to spontaneous breaking of chiral symmetry. As a result, the light masses of the current quarks, m^0, change to heavier masses of the constituent quarks, m. This is a result of the nonzero vacuum expectation values of the originally introduced fields $\tilde{S}$ $(\tilde{S}_0,\tilde{S}_3,\tilde{S}_8)$ in (46). In terms of diagrams, these expecta-

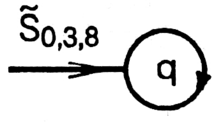

FIG. 2. A loop, or tadpole.

tion values are described by loop diagrams, or tadpoles (see Fig. 2). Redefining the quark mass matrix m, one can go over to the physical scalar fields S with zero vacuum expectation values,

$$-m^0+\tilde{S}=-m+S\to\langle S_0,S_3,S_8\rangle_0=0. \quad (48)$$

In this way there appear masses of the constituent quarks.

Equation (48) represents an analog of the gap equation in a superconductor. Upon calculating the vacuum expectation value of both sides of this equation, the latter acquires the form[16]

$$m_i^0=m_i[1-8G_1I_1(m_i)]=m_i+2G_1\langle\bar{q}_iq_i\rangle_0. \quad (49)$$

Here $\langle\bar{q}_iq_i\rangle_0$ is the vacuum condensate of the quarks and is a quadratically divergent quark loop (Fig. 2). We will use an invariant cutoff Λ in the Euclidean region of momenta determining the boundary of the region of SBCS:

$$I_1(m_i)=\frac{N_c}{(2\pi)^4}\int\frac{d_e^4k\theta(\Lambda^2-k^2)}{m_i^2+k^2}=\frac{3}{(4\pi)^2}\left[\Lambda^2-m_i^2\ln\left(\frac{\Lambda^2}{m_i^2}+1\right)\right],$$

$$I_2(m_i,m_j)=\frac{N_c}{(2\pi)^4}\int\frac{d_e^4k\theta(\Lambda^2-k^2)}{(m_i^2+k^2)(m_j^2+k^2)}=\frac{3}{(4\pi)^2}\frac{1}{(m_i^2-m_j^2)}\left[m_i^2\ln\left(\frac{\Lambda^2}{m_i^2}+1\right)-m_j^2\ln\left(\frac{\Lambda^2}{m_j^2}+1\right)\right]. \quad (50)$$

Let us now show how one can derive a standard sigma model that describes masses and interactions of scalar and pseudoscalar mesons from the Lagrangian (47). The functional (42) can be written (without external sources)

$$Z=(\det \mathrm{i}\,D)^N\int\prod_0^8 dS_adP_a \times\exp\left\{-\mathrm{i}\int d^4x\frac{\mathrm{tr}[(S-m+m^0)^2+P^2]}{4G_1}\right\}, \quad (51)$$

where $\det \mathrm{i}\,D=\det(\mathrm{i}D_0+\tilde{M})=\exp\,\mathrm{tr}\ln(\mathrm{i}D_0+\tilde{M})$ is the quark determinant arising upon integration over the quark fields, $\mathrm{i}D_0=\mathrm{i}\hat{\partial}-m$, and $\tilde{M}=MP_R+M^+P_L=S+\mathrm{i}\gamma_5P$. There exist various methods for computing this fermion determinant (see, for instance, Ref. 18). In this section we will use the simplest expansion in powers of the external meson lines corresponding to the one-loop quark approximation,

$$-\mathrm{i}\,\mathrm{tr}\ln[\mathrm{i}D_0+\tilde{M}]=\mathrm{i}\,\mathrm{tr}\sum_{n=1}^{\infty}\frac{1}{n}[\mathrm{i}D_0^{-1}\tilde{M}]^n. \quad (52)$$

This method was employed in Refs. 14–17. To obtain the standard sigma model, it suffices to consider divergent quark loops of the four types drawn in Fig. 3. The complete set of these diagrams together with the quadratic meson terms entering into (47) is described by the Lagrangian

$$\mathscr{L}^{(4)}=-\frac{\mathrm{tr}[(S-m+m^0)^2+P^2]}{4G_1}+\mathrm{tr}\{[p^2I_2+2(I_1+m^2I_2)][(S-m)^2+P^2]-I_2([(S-m)^2+P^2]^2-[(S-m),P]_-^2\}. \quad (53)$$

The first term is the kinetic term (p is the meson momentum). Let us renormalize the fields so as to give the kinetic term the correct coefficient:

$$S_a=g_aS_a^R,\quad P_a=g_aP_a^R,$$
$$g_a=[4I_2(m_i,m_j]^{-1/2}\ [a\equiv(i,j)]. \quad (54)$$

Then the Lagrangian (53) acquires the form corresponding to the standard sigma model:

$$\mathscr{L}(S,P)=\frac{1}{4}\mathrm{tr}\left\{(\partial_\mu S)^2+(\partial_\mu P)^2+g_a^2\left[\frac{2}{g_a}\left(\frac{m-m^0}{G_1}-8mI_1(m)\right)S-\left(\frac{1}{G_1}-8I_1(m)\right)(S^2+P^2)-\left(S^2-\frac{1}{g_a}\{m,S\}_++P^2\right)^2+\left[\left(S-\frac{m}{g_a}\right),P\right]_-^2\right]\right\}-\mathrm{i}\,\mathrm{tr}\ln\left\{1+\frac{g_a}{\mathrm{i}\hat{\partial}-m}(S+\mathrm{i}\gamma_5P)\right\}'=\frac{1}{4}\mathrm{tr}\left\{(\partial_\mu S)^2+(\partial_\mu P)^2-(M_{S_a}S_a\lambda^a)^2-(M_{P_a}P_a\lambda^a)^2+4mgS(S^2+P^2)-g^2\left[(S^2+P^2)^2-\left[\left(S-\frac{m}{g}\right),P\right]_-^2\right]\right\}-\mathrm{i}\,\mathrm{tr}\ln\left\{1+\frac{g}{\mathrm{i}\hat{\partial}-m}[S+\mathrm{i}\gamma_5P]\right\}'. \quad (55)$$

The index R is omitted here and in what follows; M_{S_a} and M_{P_a} are the masses of the scalar and pseudoscalar mesons. Using the condition of a minimum of $\mathscr{L}$ with respect to the variable S,

$$\left.\frac{\delta\mathscr{L}}{\delta S}\right|_{S,P=0}=0,$$

or the absence of terms linear in S in the Lagrangian $\mathscr{L}(S,P)$, we again arrive at the gap equation (49). We shall discuss the masses of the scalar and pseudoscalar mesons somewhat later, upon consideration of nondiagonal $P\to A$ transitions. The prime on the last term in (55) means that this term contains the convergent parts of the quark loop diagrams (including loop diagrams with an arbitrary number of external meson lines).

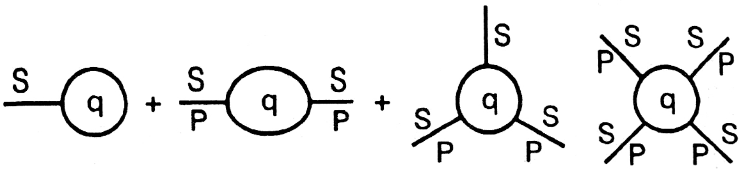

FIG. 3. Expansion of the fermion determinant. The mesons interact through quark loops.

Now let us fix the parameters m and Λ. To this end we construct the axial current on the basis of the Lagrangian (55):

$$J_{5\mu}^{(\pi^-)} = S_0 \partial_\mu \pi^- - \pi^- \partial_\mu S_0 + \frac{m_u}{g_u} \partial_\mu \pi^- + \dots . \qquad (56)$$

If we then apply it to the decay $\pi^- \to \mu\bar{\nu}$, we arrive at the Goldberger–Treiman identity

$$\frac{m_u}{g_u} = F_\pi = 93 \text{ MeV (pion decay constant)}. \qquad (57)$$

In the next section, we will show that after renormalization of the vector fields we have the following relation between the constants g_u and g_ρ (g_ρ is the decay constant for the ρ meson, $g_\rho^2/4\pi \approx 3$):

$$g_\rho = \sqrt{6} g_u. \qquad (58)$$

Using this relation, the experimental value of g_ρ taken from the decay $\rho \to 2\pi$, and the Goldberger–Treiman identity (57), we estimate the parameters of our model as

$$m_u = 234 \text{ MeV}, \quad \Lambda = 1.05 \text{ GeV}, \qquad (59)$$

$$\langle \bar{q}_u q_u \rangle_0 = -4 m_u I_1(m_u) = -(255 \text{ MeV})^3,$$

and for the masses of the pion and the σ particle we obtain

$$m_\pi^2 = \frac{g_u^2}{G_1} [1 - 8 G_1 I_1(m_u)] = \frac{1}{G_1} \frac{m_u^0}{m_u} \frac{m_u^2}{F_\pi^2} \approx -2 \frac{m_u^0}{F_\pi^2} \langle \bar{q}_u q_u \rangle_0,$$

$$m_\sigma^2 = m_\pi^2 + 4 m_u^2. \qquad (60)$$

These formulas represent the known Gell-Mann–Oakes–Renner formula for the pion mass[44] and the standard model mass of the σ particle, $m_\sigma \approx 500$ MeV. The constant of the four-quark interaction G_1 influences only the masses of the mesons but not their interaction constants. From (60) we get $G_1 \approx 7 \text{ GeV}^{-2}$, and from (49) we derive an estimate for the current mass, $m_u^0 = 5$ MeV. The masses of the remaining members of the pseudoscalar nonet are also well described in the model when the mass of the strange quark is introduced and the gluon anomaly and $P \to A$ transitions are taken into account.[16,18]

6. VECTOR AND AXIAL-VECTOR MESONS AS COMPOSITE $q\bar{q}$ STATES

In the previous section, it was shown how the NJL model leads to the linear sigma model describing interactions of scalar and pseudoscalar mesons. To this end, vector and axial-vector fields were neglected in the fermion determinant, and the one-loop approximation with divergent quark loops was considered.

For a general description of scalar, pseudoscalar, vector, and axial-vector fields it is convenient to employ the so-called "heat-kernel" technique used and expounded in detail in Ref. 18. The merits of this technique consist in the fact that, on the one hand, it allows one to preserve the explicit dependence of the theory on the important physical parameter Λ related to the scale of SBCS,[6)] and, on the other hand, it ensures gauge invariance of the theory. The latter plays an important role for ensuring vector dominance of our theory upon introducing electroweak interactions and external electromagnetic fields, W and Z bosons, and also the inclusion of gluon background fields producing gluon condensates.

For simplicity, we will consider here a more trivial Pauli–Villars regularization with two subtractions. The parameter Λ will be the mass of the subtracted field.[7)] The main purposes of this section are as follows: derivation of the Yang–Mills structure of the Lagrangian for the composite quark–antiquark vector and axial-vector fields from the NJL model, renormalization of these fields leading to the universal physical coupling constant g_ρ that, in particular, describes the decay $\rho \to 2\pi$, and obtaining the important relation

$$g_\rho = \sqrt{6} g \quad (g \equiv g_u) \qquad (61)$$

used in the previous section for determining the parameters m_u and Λ. We will follow the papers by Kikkawa[14] and Volkov.[16] The corresponding part of the fermion determinant $\det(\mathrm{i}\hat{D})$ will be determined within the one-loop approximation with the above regularization of the divergent integrals.

Summing the divergent quark loops with two, three, and four external vector mesons (as was done in the previous section), we arrive at the expression

$$-\tfrac{1}{3} \mathrm{tr}\{ I_2(m) (V_{\mu\nu} - \mathrm{i}[V_\mu, V_\nu]_-)^2 \}, \qquad (62)$$

where $V_{\mu\nu} = \lambda_a [\partial_\mu V_\nu^a - \partial_\nu V_\mu^a]$ and $[V_\mu, V_\nu]_-$ is the commutator of the operators V_μ. Upon renormalization

$$V_\mu^{ij} = \sqrt{\frac{3}{8 I_2(m_i, m_j)}} V_\mu^{Rij} = \frac{g_{V_{ij}}}{2} V_\mu^{Rij} \qquad (63)$$

for obtaining the right coefficient of the kinetic term, the vector part of the Lagrangian $\mathscr{L}''$ is reduced to the form

$$\mathscr{L}(V) = \frac{1}{4} \mathrm{tr} \left\{ M_{V_{ij}}^2 V_\mu^2 - \frac{1}{2} \left(V_{\mu\nu} - \mathrm{i} \frac{g_{V_{ij}}}{2} [V_\mu, V_\nu]_- \right)^2 \right\}$$

$$- \mathrm{i} \, \mathrm{tr} \ln \left\{ 1 + \frac{1}{\mathrm{i}\hat{\partial} - m} \frac{g_{V_{ij}}}{2} \hat{V} \right\}', \qquad (64)$$

where $g_{V_{ij}} = [\frac{2}{3}I_2(m_i,m_j)]^{-1/2}$ and $M^2_{V_{ij}} = (g_{V_{ij}})^2/4G_2$.[8)] Hence it is seen that if for the scalar (pseudoscalar) and vector mesons one uses regularization with the same cutoff parameter Λ (SBCS scale), the constants g_ρ and g are related by (61) [see Eqs. (63) and (54) derived in the previous section].

From (64) and (47), for the masses of the vector mesons we get

$$M^2_\rho = M^2_\omega = \frac{3}{8G_2 I_2(m_u)}, \quad M^2_\varphi = M^2_\rho \frac{I_2(m_u)}{I_2(m_s)},$$

$$M^2_{K*} = M^2_\rho \frac{I_2(m_u)}{I_2(m_u,m_s)} + \frac{3}{2}(m_s - m_u)^2. \tag{65}$$

Numerical estimates for the meson masses will be given in the next section, upon consideration of nondiagonal transitions $P \to A$ and calculation of the final values for the parameters $m_u = m_d$, m_s, G_2, and Λ.

For the axial-vector mesons we will in the same way arrive at the Lagrangian

$$\mathscr{L}(A) = \frac{1}{4}\mathrm{tr}\left\{(M^2_{V_{ij}} + 6m^2)A^2_\mu - \frac{1}{2}A^2_{\mu\nu} + \frac{g^2_{V_{ij}}}{8}[A_\nu, A_\mu]^2_-\right\}$$
$$-\mathrm{i}\,\mathrm{tr}\ln\left\{1 + \frac{1}{\mathrm{i}\hat\partial - m}\frac{g_{V_{ij}}}{2}\hat{A}\gamma_5\right\}, \tag{66}$$

from which for their masses we obtain

$$M^2_{a_1} = M^2_{A_u} = M^2_\rho + 6m^2_\rho,$$

$$M^2_{A_{1/2}} = M^2_{K*} + 6m_u m_s, \quad M^2_{A_s} = M^2_\varphi + 6m^2_s. \tag{67}$$

Selecting all the divergent quark loops with scalar, pseudoscalar, vector, and axial-vector meson lines in the fermion determinant $\det(\mathrm{i}\hat{D})$, we get the Lagrangian for the interaction of these fields,

$$\mathscr{L}(S',P,V,A)$$
$$= -\frac{1}{2G_1}\mathrm{tr}(gm^0S') - \frac{1}{4}\mathrm{tr}\{\mu^2(S'^2 + P^2)$$
$$- M^2_V(V^2_\mu + A^2_\mu) + g^2[(S'^2 + P^2)^2 - [S',P]^2_-]$$
$$- (D_\mu S')^2 - (D_\mu P)^2 + \tfrac{1}{2}G^{\mu\nu}_V G_{V\mu\nu} + \tfrac{1}{2}G^{\mu\nu}_A G_{A\mu\nu}\}$$
$$- \mathrm{i}\,\mathrm{tr}\ln\left\{1 + \frac{1}{\mathrm{i}\hat\partial - m}\left[g(S + \mathrm{i}\gamma_5 P)\right.\right.$$
$$\left.\left. + \frac{g_V}{2}(\hat{V} + \hat{A}\gamma_5)\right]\right\}', \tag{68}$$

where

$$S' = S - \frac{m}{g}, \quad \mu^2 = g^2\left[\frac{1}{G_1} - 8(I_1(m) + m^2 I_2(m))\right],$$

$$D_\mu S' = \partial_\mu S' - \mathrm{i}\frac{g_V}{2}[V,S']_- - \frac{g_V}{2}\{A_\mu, P\}_+,$$

$$D_\mu P = \partial_\mu P - \mathrm{i}\frac{g_V}{2}[V,P]_- + \frac{g_V}{2}\{A_\mu, P\}_+, \tag{69}$$

$$G^{\mu\nu}_V = \partial^\mu V^\nu - \partial^\nu V^\mu - \mathrm{i}\frac{g_V}{2}([V^\mu, V^\nu]_- + [A^\mu, A^\nu]_-),$$

$$G^{\mu\nu}_A = \partial^\mu A^\nu - \partial^\nu A^\mu + \mathrm{i}\frac{g_V}{2}([A^\mu, V^\nu]_- + [V^\mu, A^\nu]_-).$$

The last term of the Lagrangian (68), written in the form $-\mathrm{i}\,\mathrm{tr}\ln\{\text{-"-}\}'$, describes a finite part of the determinant $\det(\mathrm{i}\hat{D})$ connected with anomalous quark loops resulting in Wess–Zumino terms, with loops having five or more external meson lines, with finite parts of divergent diagrams, etc.

The Lagrangian (68) contains nondiagonal terms describing $S \to V$ and $P \to A$ transitions:

$$\Delta\mathscr{L}_{SV} = \mathrm{i}\sqrt{\tfrac{3}{8}}\,\mathrm{tr}\{\partial_\mu S[V_\mu, m]_-\},$$

$$\Delta\mathscr{L}_{PA} = -\sqrt{\tfrac{3}{8}}\,\mathrm{tr}\{\partial_\mu P[A_\mu, m]_+\}. \tag{70}$$

The terms of the first type are proportional to the mass difference of the constituent quarks $(m_i - m_j)$ and arise for $n_f = 3$ only in the description of strange mesons.[18] A much more important role in chiral meson Lagrangians is attributed to the second-type transitions in (70), to be thoroughly analyzed in the next section.

7. NONDIAGONAL TRANSITIONS IN THE NJL MODEL

The nondiagonal terms that describe $S \to V$ and $P \to A$ mixing appear in the NJL model because of the SBCS and dynamical masses acquired by the quarks, m_i. As is well known, these transitions play an important role in chiral phenomenological Lagrangians (see the review by Gasiorowicz and Geffen[45]). Here we will show how these terms affect the values of the basic parameters of our model (m_i, Λ, G_1, G_2).

The first-type transitions $(S \to V)$ when $n_f = 3$ appear only for strange particles and result in an insignificant additional renormalization of the scalar fields of the form (Ref. 18)[9)]

$$S'_{ij} = Y^{-1/2}_{ij} S_{ij}, \quad Y_{ij} = \left[1 - \frac{3}{2}\frac{(m_i - m_j)^2}{M^2_{V_{ij}}}\right]^{-1}. \tag{71}$$

These renormalizations amount to about 10%, which does not go beyond the accuracy of the model, and they do not lead to essential physical consequences. Therefore we shall neglect them in what follows.

The transitions of the second type $(P \to A)$ are more important and deserve more accurate consideration.

Let us present one more diagonalization of the Lagrangian through the introduction of physical fields A'_μ (Refs. 16, 18, and 46):

$$A^{ij}_\mu = A'^{ij}_\mu + \sqrt{\frac{3}{2}}\frac{m_i + m_j}{M^2_{A_{ij}}}\partial_\mu P^{ij}. \tag{72}$$

Then the pseudoscalar fields acquire an extra renormalization,

$$P'_{ij}=Z_{ij}^{-1/2}P_{ij},\ Z_{ij}=\left[1-\frac{3}{2}\frac{(m_i+m_j)^2}{M^2_{A_{ij}}}\right]^{-1}, \tag{73}$$

and the constant g_P will now differ from the constant g_S by the factor $Z^{1/2}$:

$$g_P^{ij}=Z_{ij}^{1/2}g_S^{ij}\,(g_S^{ij}=g^{ij}). \tag{74}$$

Using the Goldberger–Treiman identity, $g_P^{ij}=(m_i+m_j)/2F_{ij}$, and the relation $g_V=\sqrt{6}\,g_S$, we arrive at an equation for the masses of the constituent quarks,

$$\left(\frac{m_i+m_j}{2F_{ij}}\right)^2\left(1-\frac{3}{2}\frac{(m_i+m_j)^2}{M^2_{A_{ij}}}\right)=\frac{g^2_{V_{ij}}}{6}, \tag{75}$$

from which we get, for the mass of the u quark,

$$m_u^2=\frac{M^2_{a_1}}{12}\left[1-\sqrt{1-\left(\frac{2g_\rho F_\pi}{M_{a_1}}\right)^2}\right],$$

$$Z^{-1}=\frac{1}{2}\left[1+\sqrt{1-\left(\frac{2g_\rho F_\pi}{M_{a_1}}\right)^2}\right]. \tag{76}$$

Here the parameters m_u and Z are expressed in terms of the physical observables, g_ρ, F_π, and M_{a_1} (M_{a_1} is the mass of the axial-vector meson a_1).

Note that of all the above observables, the mass M_{a_1} has now been measured with the least accuracy. In fact, there are large discrepancies between experiments to measure the mass and width of the a_1 meson made in 1981 in hadron reactions $\pi N\to 3\pi N$ and experiments performed in 1986 on the study of τ-lepton decays. In the first experiments it was established that the mass and width of the a_1 meson have the values[47]

$$M_{a_1}=1275\pm 28\ \text{MeV},\quad \Gamma_{a_1}=316\pm 45\ \text{MeV}. \tag{77}$$

At the same time, the analysis of the decay $\tau\to\nu_\tau 3\pi$ gave values in wider and completely different limits:[48]

$$1046\ \text{MeV}\leqslant M_{a_1}\leqslant 1194\ \text{MeV},$$

$$400\ \text{MeV}\leqslant \Gamma_{a_1}\leqslant 520\ \text{MeV}. \tag{78}$$

From (76) it follows that the masses m_u and M_{a_1} are strongly correlated with each other and are subject to the constraints

$$1-\left(\frac{2g_\rho F_\pi}{M_{a_1}}\right)^2\geqslant 0,\quad M_{a_1}\geqslant 2g_\rho F_\pi=1140\ \text{MeV},$$

$$m_u\leqslant 330\ \text{MeV}. \tag{79}$$

The constraints on the mass of the constituent u quark are consistent with the standard ideas. The constraints on the mass of the a_1 meson indicate that in the experiments of Ruckstuhl *et al.*[48] and Albrecht *et al.*[48] in which the values $M_{a_1}=1056$ MeV and $M_{a_1}=1046$ MeV were obtained, the data were probably not analyzed quite accurately. Indeed, a subsequent analysis[49] in which new ideas on the form of

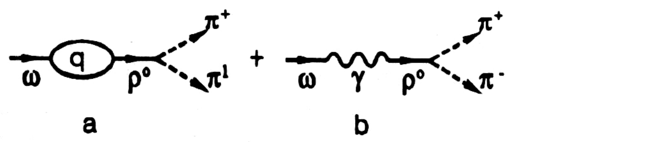

FIG. 4. Diagrams describing the decay $\omega\to 2\pi$.

the $a_1\rho\pi$ vertex were used demonstrated the validity of this assertion and gave results for the experimental data of 1986 that were closer to the original values (77):

$$M_{a_1}\approx 1250\ \text{MeV}. \tag{80}$$

The inequality (79) resulting from (76) allows us to understand under what conditions the Weinberg sum rule, $M^2_{a_1}=2M^2_\rho$ (Ref. 50), and the KSFR relation, $M^2_\rho=2g^2_\rho F^2_\pi$ (Ref. 51), are valid. They can be derived with the minimal mass of the a_1 meson, $M_{a_1}=2g_\rho F_\pi$ [see (79)] and by using the formula $M^2_{a_1}=M^2_\rho+6m^2_u$ for the a_1-meson mass obtained in the previous section. Then, inserting the value $m^2_u=M^2_{a_1}/12$ obtained from (76) into the formula for the a_1-meson mass, we immediately arrive at the above relations.[10)]

Now we have all the formulas necessary for fixing the model parameters, $m_u\approx m_d$, m_s, Λ, G_1, and G_2, and we can proceed to calculate the meson masses and other physical quantities (like F_K, F_s, m_u^0, m_s^0, etc.).

From (76) we get $m_u=280$ MeV, which corresponds to $M_{a_1}=1260$ MeV.[52] Using the formula obtained in the previous section, $g^2_\rho I_2(m_u)=3/2$, we estimate the cutoff parameter Λ as

$$\Lambda=1280\ \text{MeV}. \tag{81}$$

From the expression (65) of the previous section for the ρ-meson mass we can evaluate the constant G_2:

$$G_2=\left(\frac{g_\rho}{2M_\rho}\right)^2\approx 16\ \text{GeV}^{-2}. \tag{82}$$

If the strange-quark mass is taken to be 460 MeV, for the masses of the vector mesons φ and K^* we get from the same formulas

$$M_\varphi=1025\ \text{MeV}\quad\text{and}\quad M_{K^*}=916\ \text{MeV},$$

in satisfactory agreement with experiment,

$$M_\varphi=1020\ \text{MeV}\quad\text{and}\quad M_{K^*}=892\ \text{MeV}.$$

Now let us determine the mass difference for the u and d quarks by using, for instance, the decay $\omega\to 2\pi$.[53]

The amplitude of the decay $\omega\to 2\pi$ is described by the two diagrams shown in Fig. 4 and has the form

$$T_{\omega\to 2\pi}=C(p^+-p^-)^\mu\omega_\mu\pi^+\pi^-. \tag{83}$$

Here, p^+ and p^- are the π^+ and π^- momenta, and the constant $C=C_1+C_2$ consists of two parts; C_1 describes the process of the strong transition $\omega\to\rho^0$ (Fig. 4a), which takes place on account of the mass difference of the u and d quarks,

$$C_1=\frac{8(\pi\alpha_\rho)^{3/2}M_\omega^2}{3(M_\rho^2-M_\omega^2+iM_\rho\Gamma_\rho)}\left[I_2(m_u)-I_2(m_d)\right.$$
$$\left.\approx\frac{6}{(4\pi)^2}\ln\frac{m_d}{m_u}\right],$$

and C_2 describes the process of the electromagnetic transition $\omega\to\rho^0$ (Fig. 4b). It has the opposite sign to C_1:

$$C_2=-\sqrt{\frac{\pi}{\alpha_\rho}}\frac{2\alpha M_\rho^2}{3(M_\rho^2-M_\omega^2+iM_\rho\Gamma_\rho)}\left(\alpha=\frac{1}{137}\right).$$

Using the experimental $\omega\to 2\pi$ decay width, which is 286 keV, we obtain for the mass difference of the u and d quarks the value

$$\Delta=m_d-m_u=4.5\ \text{MeV}. \qquad (84)$$

Now we shall proceed to describe the masses of the pseudoscalar mesons. From the formula (55) for the Lagrangian $\mathscr{L}(S,P)$ we arrive at the following expressions for the masses of the pseudoscalar mesons:[16]

$$M_{\pi^0}^2=\frac{1}{2}(C_{uu}+C_{dd}),\quad M_{\pi^\pm}^2=C_{ud}+(m_d-m_u)^2,$$
$$M_{K^\pm}^2=C_{us}+(m_s-m_u)^2,\quad M_{K^0}^2=C_{ds}+(m_s-m_d)^2, \qquad (85)$$
$$M_{\eta \atop \eta'}^2=\frac{1}{2}\left[C_{uu}+C_{dd}+d\mp\right.$$
$$\left.\times\sqrt{\left(d-\frac{C_{ss}-C_{uu}}{3}\right)^2+\frac{8}{9}(C_{ss}-C_{uu})^2}\right],$$

where

$$C_{ij}=\frac{Z_{ij}}{4I_2(m_i,m_j)}\left[\frac{1}{G_1}-4(I_1(m_i)+I_1(m_j))\right].$$

The term $d=0.8\ \text{GeV}^2$ is due to the gluon anomalies that are taken into account.[16] It causes mixing of the singlet–octet components of the pseudoscalar mesons η and η'. With the value $d=0.8\ \text{GeV}^2$ we get the mixing angle $\theta=-18°$.[11)]

With the use of the π^0-meson mass we can fix the last free parameter, $G_1=4.7\ \text{GeV}^{-2}$. Then for the constituent-quark masses $m_u=280$ MeV, $m_d=284.5$ MeV, $m_s=460$ MeV we get

$$M_{K^\pm}=493\ \text{MeV},\quad M_{K^0}=497\ \text{MeV},$$
$$M_\eta=520\ \text{MeV},\quad M_{\eta'}=1027\ \text{MeV}. \qquad (86)$$

The corresponding experimental values are as follows:[52]

$$M_{K^\pm}=493.6\ \text{MeV},\quad M_{K^0}=497.7\ \text{MeV},$$
$$M_\eta=549\ \text{MeV},\quad M_{\eta'}=958\ \text{MeV}. \qquad (87)$$

The agreement is quite satisfactory.

The gap equation (49) of Sec. 5 gives the following values for the masses of the current quarks:

$$m_u^0=3\ \text{MeV},\quad m_d^0=4\ \text{MeV},\quad m_s^0=90\ \text{MeV}. \qquad (88)$$

Diagonalization of the P–A terms leads not only to renormalization of a number of fields and constants and to a change of the basic model parameters, but also to the appearance of extra diagrams that are important for obtaining self-consistent results. Recall that under the redefinition of the field A_μ in the form

$$A_\mu^{ij}=A_\mu'^{ij}+\frac{g_\rho F_{ij}Z_{ij}}{M_{A_{ij}}^2}\partial_\mu P'^{ij} \qquad (72')$$

diagrams with external lines A_μ^{ij} can be transformed into diagrams with gradients of pseudoscalar mesons, $\partial_\mu P'^{ij}$.

As an illustration, we present the calculation of two typical strong vertices with corrections due to P–A transitions.[55] These are the $\varepsilon\pi\pi$ and π^4 vertices that play an important role in the description of $\pi\pi$ scattering in the sigma model (ε is a scalar isoscalar meson consisting of u and d quarks). If $\pi\to a_1$ transitions are taken into account, the above vertices acquire the form

$$\mathscr{L}_{\varepsilon\pi\pi}=2m_u g_\pi Z^{1/2}\varepsilon\left\{\pi^2+\frac{1}{2m_u^2}\left(\frac{Z-1}{Z}\right)\right.$$
$$\left.\times\left[\pi\partial_\pi^2\pi+\left(\frac{Z+1}{Z}\right)(\partial_\mu\pi)^2\right]\right\},$$
$$\mathscr{L}_{\pi^4}=-\frac{g_\pi^2}{2}Z\left\{\pi^4-\left(\frac{Z-1}{Z}\right)^2\frac{(\pi\partial_\mu\pi)^2}{m_u^2}+\left(\frac{Z-1}{Z}\right)^4\right.$$
$$\left.\times\frac{(\partial_\mu\pi\partial_\mu\pi)^2-(\partial_\mu\pi\partial_\nu\pi)^2}{12m_u^4}\right\}. \qquad (89)$$

Let us show that the formula obtained here describes $\pi\pi$ scattering in complete agreement with the low-energy theorems.

The diagrams describing $\pi\pi$ scattering in the tree approximation are shown in Fig. 5.

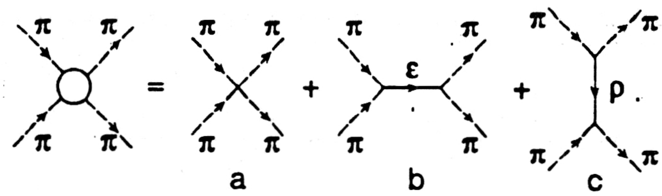

FIG. 5. Diagrams of $\pi\pi$ scattering in the tree approximation.

Using the formula $M_\varepsilon^2=M_\pi^2+4m_u^2$ and retaining only terms with the lowest derivatives, we get, for diagrams 5a and 5b from (89),

$$\mathscr{L}'_{\pi^4}=\frac{g_\pi^2}{2m_u^2 Z}[(\pi\partial_\mu\pi)^2+(Z-1)\pi^2(\partial_\mu\pi)^2].$$

Using the Lagrangian $\mathscr{L}_{\rho\pi\pi}=g_\rho(\pi\times\pi)\rho$, we get, for diagram 5c,

$$\mathscr{L}''_{\pi^4}=\frac{g_\rho^2}{2M_\rho^2}[(\pi\partial_\mu\pi)^2-\pi^2(\partial_\mu\pi)^2]$$
$$=-\frac{g_\rho^2}{2M_\rho^2}(\pi\times\partial_\mu\pi)^2.$$

TABLE I. Partial widths of radiative decays $P\to\gamma\gamma$, $P\to V\gamma$, and $V\to P\gamma$ of low-lying mesons.

Decay	Numerical values of partial widths (keV)	
	Theory ($\theta=-18°$)	Experiment (Ref. 52)
$\pi^0\to\gamma\gamma$	$7.6\cdot10^{-3}$	$(7.5\pm0.3)\cdot10^{-3}$
$\eta\to\gamma\gamma$	0.63	0.46 ± 0.01
$\eta'\to\gamma\gamma$	4.5	4.5 ± 0.35
$\eta'\to\rho^0\gamma$	67	62 ± 3
$\eta'\to\omega\gamma$	7.5	6.2 ± 0.6
$\omega\to\pi^0\gamma$	830	720 ± 40
$\rho\to\pi\gamma$	87	118 ± 30
$\rho\to\eta\gamma$	65	57 ± 10
$\omega\to\eta\gamma$	8.5	$3.9^{+1.9}_{-1.5}$
$\varphi\to\pi^0\gamma$	5.3	5.8 ± 0.6
$\varphi\to\eta\gamma$	69	56 ± 3
$\varphi\to\eta'\gamma$	0.56	<1.8
$K^{*+}\to K^+\gamma$	52	50 ± 4
$K^{*0}\to K^0\gamma$	130	120 ± 10

As we saw earlier, in the region where the low-energy sum rules hold, it must be assumed that $M_{a_1}=2g_\rho F_\pi$, $M_\rho^2=6m_u^2$, $Z=2$. Then, using these relations and Eq. (75), we obtain for the total scattering amplitude the expression

$$\mathscr{L}_{\pi A}=\mathscr{L}'_{\pi A}+\mathscr{L}''_{\pi A}=\frac{g_\pi^2}{2m_u^2Z}[2(\pi\partial_\mu\pi)^2+(Z-2)\pi^2(\partial_\mu\pi)^2]$$
$$=\frac{g_\pi^2}{2m_u^2}(\pi\partial_\mu\pi)^2, \qquad (90)$$

satisfying all the requirements of the low-energy theorems for pion–pion scattering.

A thorough analysis of the role of $P\to A$ transitions in the NJL model was performed by Wakamatsu.[56]

8. APPLICATIONS

In the previous sections it has been shown that all known phenomenological chiral Lagrangians describing the low-energy physics of scalar, pseudoscalar, vector, and axial-vector mesons can easily be constructed within the NJL model. Besides, this model allows one to describe the deviations from the exact chiral group $U(n)\times U(n)$, which can be seen in calculations of the mass spectrum of the mesons and the constants F_π, F_K, and F_s. This is possible because the NJL model describes meson vertices in the one-loop quark approximation, and the difference of masses of the constituent quarks (u,d) and s, which breaks the chiral group, can be taken into account. In what follows, we will show a number of cases where the difference of the masses m_u and m_s plays an important role in the description of meson interactions. The NJL model allows us to describe all the basic decays of pseudoscalar, vector, and axial-vector mesons (for $n_f=3$) and their intrinsic properties, in particular, electromagnetic and weak radii, polarizabilities, and scattering lengths. Since a detailed account of applications of the model is given in the review papers of Refs. 16 and 57, here we shall only demonstrate the most typical and interesting examples.

1) Strong decays of vector mesons. The basic decays of vector mesons occur in a strong channel of the *VPP* type and are specified by constants g_ρ, g_{K^*}, and g_φ defined by Eqs. (63) and (65) of Sec. 6. The expressions (65) can be written in the form

$$\frac{g_\varphi^2}{g_\rho^2}=\frac{\alpha_\varphi}{\alpha_\rho}=\frac{M_\varphi^2}{M_\rho^2},\quad \frac{g_{K^*}^2}{g_\rho^2}=\frac{\alpha_{K^*}}{\alpha_\rho^2}=\frac{M_{K^*}^2}{M_\rho^2}-\frac{3}{2}\frac{(m_s-m_u)^2}{M_\rho^2},$$

from which we obtain, for the above constants, the following ratios:

$$g_{K^*}^2=1.26g_\rho^2,\quad g_\varphi^2=1.75g_\rho^2. \qquad (91)$$

Then one can calculate the partial widths of the decays $V\to P_1+P_2$:

$$\Gamma(\rho\to\pi\pi)=\frac{\alpha_\rho M_\rho}{12}\left[1-\left(\frac{2M_\pi}{M_\rho}\right)^2\right]^{3/2}=156\text{ MeV},$$
$$\Gamma(K^*\to K\pi)=\frac{\alpha_{K^*}M_{K^*}}{16}\left\{\left[1-\left(\frac{M_K-M_\pi}{M_{K^*}}\right)^2\right]\times\left[1-\left(\frac{M_K+M_\pi}{M_{K^*}}\right)^2\right]\right\}^{3/2}=54\text{ MeV},$$
$$\Gamma(\varphi\to\bar KK)=\frac{\alpha_\varphi M_\varphi}{12}\left[1-\left(\frac{2M_K}{M_\varphi}\right)^2\right]^{3/2}=3.4\text{ MeV}. \qquad (92)$$

The theoretical values of these partial widths are in good agreement with experiment (Ref. 52):[12)]

$$\Gamma_{\rho\to\pi\pi}=(149\pm3)\text{ MeV},\quad \Gamma_{K^*\to K\pi}=(50\pm1)\text{ MeV},$$
$$\Gamma_{\varphi\to\bar KK}=(3.7\pm0.2)\text{ MeV}.$$

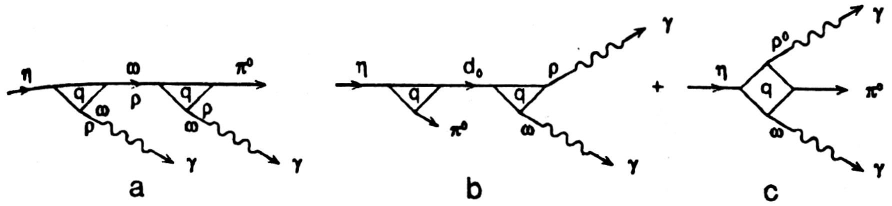

FIG. 6. Diagrams describing the decay $\eta\to\pi^0\gamma\gamma$.

2) Radiative decays of pseudoscalar and vector mesons. The most typical radiative decays of pseudoscalar and vector mesons are two-particle decays of the type $P\to\gamma\gamma$, $P\to V\gamma$, and $V\to P\gamma$. All these processes are well described by anomalous triangular quark diagrams.[58] These diagrams give the Wess–Zumino terms connected with the imaginary part of the determinant $\det(i\hat{D})$ or with the last term of the Lagrangian (68) from Sec. 6. These processes are thoroughly described in the review papers of Refs. 16 and 57; here we only cite Table I giving the theoretical and experimental values for those decays. Note in particular the decays $K^*\to K\gamma$, for which our model gives better agreement with experiment, due to the difference m_s-m_u [the deviation from the group $U(3)\times U(3)$].

The partial width of the decay $V\to P\gamma$ is of the form

$$\Gamma(V\to P\gamma)=\frac{\alpha\alpha_V}{6}\frac{C_{VP}^2}{F_P^2}\left(\frac{M_V^2-M_P^2}{4\pi M_V}\right)^3. \qquad (93)$$

The coefficients C_{VP} are, respectively,

$$C_{\rho\pi}=1,\quad C_{\omega\pi}=3,\quad C_{\omega\eta}=\sin\bar{\theta},\quad C_{\rho\eta}=3\sin\bar{\theta},$$

$$C_{\varphi\eta}=2\cos\bar{\theta},\quad C_{\varphi\eta'}=2\cos\bar{\theta},\quad C_{\varphi\pi^0}=3\sin\beta.$$

Here $\bar{\theta}=\theta_0-\theta$, where θ is the singlet–octet mixing angle ($\theta=-18°$); θ_0 is the ideal mixing angle ($\sin\theta_0=1/\sqrt{3}$); $\beta=3°$ is the ω–φ mixing angle. For the decays $K^*\to K\gamma$ the coefficients C_{K^*K} depend on the quark mass difference:

$$C_{K^{*+}K^+}=\frac{\lambda^2-6\lambda-1}{2(\lambda^2-1)}+\frac{\lambda(2\lambda^2+1)}{(\lambda^2-1)}\ln\lambda^2=1.22,$$

$$C_{K^{*0}K^0}=1+\frac{\lambda\ln\lambda^2}{\lambda^2-1}=1.97,$$

where $\lambda=m_s/m_u=1.64$ ($m_u=280$ MeV, $m_s=460$ MeV). As can be seen from Table I, quite good agreement is achieved with experiment. In particular, for the ratio of these decays we get

$$\frac{\Gamma(K^{*0}\to K^0\gamma)}{\Gamma(K^{*+}\to K^+\gamma)}=2.6,$$

whereas the experimental value is 2.4. If one neglects the quark mass difference, i.e., if one puts $\lambda=1$, then $C_{K^{*+}K^+}=1$, $C_{K^{*0}K^0}=2$, and this ratio will differ significantly from the experimental value,

$$\frac{\Gamma(K^{*0}\to K^0\gamma)}{\Gamma(K^{*+}\to K^+\gamma)}=4\quad(\lambda=1).$$

3) The decay $\eta\to\pi^0\gamma\gamma$. The decay $\eta\to\pi^0\gamma\gamma$ is of interest, since it clearly demonstrates the merits of the linear sigma model as compared with nonlinear chiral Lagrangians in describing some processes. Actually, in the nonlinear model it is difficult to get a correct result for the width of this decay.[59] Let us see what is the reason for this.

The decay $\eta\to\pi^0\gamma\gamma$ is described both by anomalous vertices like Wess–Zumino terms (vertices $\eta\omega\omega$, $\eta\rho^0\rho^0$, and $\pi^0\rho^0\omega$; Fig. 6a) linked by intermediate vector mesons and by strong vertices corresponding to divergent quark diagrams and entering into the main Lagrangian of the sigma model [see the Lagrangian (68) of Sec. 6] [vertices $\eta\pi^0\rho^0\omega$, $a_0(980)\eta\pi^0$, and $a_0(980)\omega\rho^0$; Figs. 6b and 6c]. The last two vertices are connected by intermediate scalar mesons $a_0(980)$.

If one passes to the nonlinear chiral model, the contribution of the diagrams of Figs. 6b and 6c disappears, and only the diagrams with intermediate vector mesons survive.[13)] In terms of current algebra this means that the commutator of the neutral currents vanishes. However, the contribution from diagram 6a amounts to only 30% of the experimental value. If one takes for the $a_0(980)$ mass not a theoretical but the experimental value, diagrams 6b and 6c do not cancel each other, which leads to the violation of chiral symmetry that occurs in reality. Then the contributions of these diagrams are comparable with those of diagram 6a, and together with the interference terms they will produce a quite satisfactory result for the width of the decay $\eta\to\pi^0\gamma\gamma$,[60]

$$\Gamma^{(a)}_{\eta\to\pi^0\gamma\gamma}=0.4\ \text{eV},\quad \Gamma^{(b+c)}_{\eta\to\pi^0\gamma\gamma}=0.3\ \text{eV},$$

$$\Gamma^{(\text{int})}_{\eta\to\pi^0\gamma\gamma}=0.4\ \text{eV},\quad \Gamma_{\eta\to\pi^0\gamma\gamma}=1.1\ \text{eV},$$

whereas the experimental value is

$$\Gamma^{(\text{exp})}_{\eta\to\pi^0\gamma\gamma}=(0.85\pm0.26)\ \text{eV}.$$

4) Axial-vector mesons. The $a_1\rho\pi$ vertex and the decays $a_1\to\rho\pi$, $a_1\to\pi\gamma$, $\pi^-\to e\bar{\nu}\gamma$, and $\tau\to\nu 3\pi$. The most typical decay of axial mesons is a decay of the type $A\to VP$. The appropriate vertex is contained in the Lagrangian (68) of Sec. 6 and is shown in Fig. 7a. Upon P–A diagonaliza-

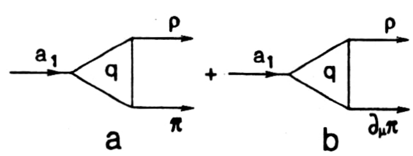

FIG. 7. Diagrams describing the $a_1\to\rho\pi$ vertex. Diagram (b) results from $\pi\to a_1$ diagonalization.

tion, the divergent part of diagram 7a cancels the divergent part of diagram 7b [the vertex $PA(A\to\partial_\mu\pi)$], and we are left with the remaining convergent part that enters into the last term of the Lagrangian (68). Therefore we should make use of the momentum expansion of quark loops and consider at least the q^2 terms of those vertices. Strictly speaking, this step is beyond the scope of the approximation employed earlier and requires special substantiation. Nevertheless, we will try to consider some physical consequences of this approximation, which, as will be seen below, gives reasonable results.[14)]

Diagram 7a along with the divergent part and the q^2 terms result in the amplitude[64]

$$T^{(a)}_{a_1\to\rho\pi}=ig_\rho^2F_\pi\{Zg^{\mu\nu}+\kappa[p^\mu q^\nu-g^{\mu\nu}pq+g^{\mu\nu}(q^2+p^2)]\}\times\varepsilon_\mu(Q)\varepsilon_\nu(q). \quad (94)$$

Here $\kappa=(8\pi^2F_\pi^2)^{-1}$, $Q=p+q$, where p and q are the momenta of the pion and the ρ meson, and $\varepsilon_\mu(Q)$ and $\varepsilon_\nu(q)$ are the polarization vectors of the a_1 and ρ mesons. Diagram 7b gives the additional contribution[15)]

$$T^{(b)}_{a_1\to\rho\pi}=ig^2F_\pi Z\frac{Q^2-q^2}{M^2_{a_1}}g^{\mu\nu}\varepsilon_\mu(Q)\varepsilon_\nu(q). \quad (95)$$

Summing (94) and (95), one can easily see that on the mass shell the contribution from the divergent part of diagram 7a [the first term of (94)] cancels the corresponding contribution from diagram 7b, giving the result [we put $p^2=M_\pi^2$ and neglect the last small term in (94)]

$$T_{a_1\to\rho\pi}=-ig_\rho^2F_\pi\left\{Z\frac{M_\rho^2}{M^2_{a_1}}g^{\mu\nu}+\kappa[M_\rho^2g^{\mu\nu}+(q^\mu p^\nu-g^{\mu\nu}qp)]\right\}\varepsilon_\mu(Q)\varepsilon_\nu(q)$$
$$=-ig_\rho^2F_\pi\kappa\{q^\mu p^\nu-g^{\mu\nu}qp+cg^{\mu\nu}q^2\}\varepsilon_\mu(Q)\varepsilon_\nu(q)\times\left(C=1+\frac{Z}{\kappa M^2_{a_1}},\quad q^2=M_\rho^2\right). \quad (96)$$

In the low-energy limit ($Z=2$, $M^2_{a_1}=2M_\rho^2$) the factor of the first term is $ZM_\rho^2/M^2_{a_1}=1$; thus, we obtain the known chiral-symmetric result for the local part of the given meson vertex. The expression in parentheses coincides with the κ term proposed in Ref. 61 on the basis of the chiral symmetry as well. All this indicates a close relation between our approach and the phenomenology based on the chiral Lagrangian, which has shown itself to be good.

As in the method of chiral Lagrangians, we have not yet obtained rigorous theoretical proofs for the q^2 approximation used here. Therefore we shall dwell on indirect arguments. The $a_1\rho\pi$ vertex has some properties similar to those of the anomalous $\omega\rho\pi$ vertex (see Sec. 8.2). Calculated in terms of the quark loop, this vertex has no divergent part and begins directly with the q^2 terms. In this case the q^2 approximation describes a large class of decays in the energy interval from 100 MeV to 1 GeV [$\pi^0\to\gamma\gamma$, $\rho\to\pi\gamma$, $\omega\to\pi\gamma$, $(\eta,\eta')\to\gamma\gamma$, etc.; see Table I]. When the $a_1\rho\pi$ vertex is calculated after taking into account the two diagrams (see Fig. 7), the expansion also begins with the q^2 terms. The amplitudes of the radiative decays $\rho\to\pi\gamma$ and $a_1\to\pi\gamma$ associated with these meson vertices look similar, even superficially:

$$L=\frac{e}{4}g_\rho\kappa F_\pi(\varepsilon^{\mu\nu\alpha\beta}\rho^-_{\alpha\beta}-2ia^{-\mu\nu})F_{\mu\nu}\pi^++\text{H.c.}$$
$$=\frac{e}{4}g_\rho\kappa F_\pi\varepsilon^{\mu\nu\alpha\beta}(\rho^-_{\alpha\beta}+ia^{*-}_{\alpha\beta})F_{\mu\nu}\pi^++\text{H.c.}, \quad (97)$$

where $a^*_{\alpha\beta}=\frac{1}{2}\varepsilon_{\alpha\beta\mu\nu}a^{\mu\nu}$ is the tensor dual to $a^{\mu\nu}=\partial^\mu a^\nu-\partial^\nu a^\mu$ ($a^-\equiv a_1^-$).

In the region of energies near the a_1 and ρ masses they give a good description of the decay $a_1\to\pi\gamma$,

$$\Gamma_{a_1\to\pi\gamma}=400\ \text{keV},$$

$$\Gamma^{\text{exp}}_{a_1\to\pi\gamma}=640\pm240\ \text{keV (Ref. 65)},$$

and the decay $\rho\to\pi\gamma$ (see Table I).

In the low-energy region the Lagrangian (97) allows a good description of the axial and vector form factors of the radiative decay $\pi^-\to e\bar\nu\gamma$.[64] The structure part of the $\pi^-\to e\bar\nu\gamma$ amplitude has the form

$$T_{\pi^-\to e\bar\nu\gamma}=eG_F\cos\theta_c[h_V\varepsilon^{\mu\nu\alpha\beta}p_\alpha q_\beta+ih_A(g^{\mu\nu}pq-p^\mu q^\nu)]l_\nu^{(+)}\varepsilon_\mu(q). \quad (98)$$

Here G_F is the Fermi constant, θ_c is the Cabibbo angle, p and q are the momenta of the pion and the photon, $l_\nu^{(+)}$ is the lepton current, and h_V and h_A are the vector and axial-vector form factors. Then for h_V we have the standard value

$$h_V=\frac{1}{8\pi^2F_\pi}. \quad (99)$$

For h_A, using (97) and the axial-vector dominance of the weak interaction,[66]

$$L_W=\frac{G_F}{g_\rho}\cos\theta_c[M_\rho^2\rho_\mu^++Z^{-1}M^2_{a_1}a^+_{1\mu}-g_\rho F_\pi\partial_\mu\pi^+]l^\mu+\text{H.c.}, \quad (100)$$

we get

$$h_A=\frac{1}{8\pi^2F_\pi Z}.$$

Then the ratio of these form factors is

$$\gamma=\frac{h_A}{h_V}=Z^{-1}=\frac{1}{2}\left[1+\sqrt{1-\left(\frac{2g_\rho F_\pi}{M_{a_1}}\right)^2}\right]. \quad (101)$$

For $M_{a_1}=1.2$ GeV we get $\gamma=0.65$. The experimental values are $\gamma=0.52\pm0.06$ (Ref. 67), $\gamma=0.67\pm0.04\pm0.16$ (Ref. 68).

Similarly, we can describe the form factors of the decays $K^-\to e\bar\nu\gamma(\mu\bar\nu\gamma)$ and $\pi^-\to e\bar\nu e^+e^-$.[64,69,70]

Now let us calculate the $a_1\to\rho\pi$ decay width, using the amplitude (96):

TABLE II. The parameters a_1 obtained by different collaborations with the $a_1\rho\pi$ amplitude taken as a constant, and the results obtained with our representation of the amplitude.*

Source (Refs. 47 and 48)	$T_{(a_1\to\pi\rho)}$ = const		$T_{(a_1\to\pi_\rho)}$ given by (96) (Ref. 49)	
	a_1 mass (MeV)	a_1 width (MeV)	a_1 mass (MeV)	a_1 width (MeV)
DELCO (Ruckstuhl 1986)	1056±20±15	476^{+132}_{-120}±54	1242±37	465^{+228}_{-143}
MARK II (Schmidke 1986)	1194±14±10	462±56±30	1260±14	298^{+40}_{-34}
ARGUS (Albrecht 1986)	1046±11	521±27	1250±9	488±32
$\pi^-p\to p\pi^+\pi^-\pi^-$ (Daum 1981)	1280±30	300±50		
$\pi^-p\to p\pi^+\pi^-\pi^0$ (Dankowych 1981)	1240±80	380±100		

*The results of Ref. 49 have been included in the tables of the Particle Data Group (1992).

$$\Gamma_{a_1\to\pi\rho}=\frac{2}{3\pi M_{a_1}}\left(\frac{M_\rho^2 g_\rho^2}{32\pi^2 F_\pi}\right)^2\left(1-\frac{M_\rho^2}{M_{a_1}^2}\right)$$
$$\times\left\{\frac{M_{a_1}^2}{M_\rho^2}\left(1+\frac{M^4}{2M_\rho^2M_{a_1}^2}\right)+(1+C)^2\right.$$
$$\left.\times\left(2+\frac{M^4}{4M_\rho^2M_{a_1}^2}\right)-3(1+C)\frac{M^2}{M_\rho^2}\right\}$$
$$\times(M^2=M_\rho^2+M_{a_1}^2). \qquad (102)$$

Then for M_{a_1} = 1.2 GeV we get

$$\Gamma_{a_1\to\pi\rho}=300 \text{ MeV}.$$

An interesting property of the amplitude (96) is that in the calculation of the $a_1\to\rho\pi$ decay width there appears a large negative interference term, which reduces the sum of the independent contributions of the gradient $(p^\mu q^\nu-g^{\mu\nu}pq)$ and the q^2 (with $g^{\mu\nu}$) parts of the amplitude by almost an order of magnitude. It also turns out that in the interval 1.1 GeV $\leqslant M_{a_1}\leqslant$ 1.4 GeV the $a_1\to\rho\pi$ decay width decreases with increasing a_1-meson mass.

If these properties of the $a_1\to\rho\pi$ amplitude are included in the analysis of the experimental data on the decay $\tau\to\nu 3\pi$, the results of different groups[48] giving very different masses and widths of the a_1 meson become comparable with each other and also with data obtained earlier in analyzing the processes $\pi N\to 3\pi N$ (Ref. 47) (see Ref. 49 and Table II).

5) Strong, weak, and electromagnetic kaon decays and the $\Delta I=1/2$ rule. Here we shall consider kaon decays of the types $K\to\gamma\gamma$, $K_L\to\pi^0\gamma\gamma$, $K\to 2\pi$, $K\to 3\pi$ and present a theoretical explanation of the phenomenological $\Delta I=1/2$ rule.

The above-mentioned decays are described by weak interactions with W-boson exchange, strong high-energy corrections with gluon exchange, and strong low-energy interactions. The result of weak interactions with W-boson exchange and strong high-energy corrections with gluon exchange can be described by a local low-energy Lagrangian $\mathscr{L}_{\rm eff}^{|\Delta S|=1}$ (Refs. 71–73):

$$\mathscr{L}_{\rm eff}^{|\Delta S|=1}=\frac{G_D}{\sqrt{2}}s_1c_1c_3\sum_{i=1,2,3,5,6}C_i(\mu)Q_i(\mu)$$
$$=\frac{G_F}{\sqrt{2}}s_1c_1c_3Q. \qquad (103)$$

Here $G_Fs_1c_1c_3=2.5$ GeV^{-2}; s_1, c_1, c_3 are elements of the Kobayashi–Maskawa matrix;[74] $C_i(\mu)$ are numerical coefficients; $Q_i(\mu)$ are the four-quark operators of Gilman and Wise,[72] while $Q_6(\mu)$ is the "penguin" operator;[71] μ is a normalization point. The appearance of the normalization point μ is due to the inclusion of the high-energy QCD interaction. The dependence on μ is contained in terms of the current quark–gluon coupling constant $\alpha_3(\mu)=(2\pi/9)\ln^{-1}(\mu/\Lambda_3)$, calculated for three quark flavors, where Λ_3 is the QCD parameter ($\Lambda_3\approx0.12$ GeV). The numerical values of the coefficients C_i/μ are functions of μ and the masses of the heavy quarks (m_c, m_b, and m_t). For the standard normalization $\alpha_3(\mu)=1$ (μ=0.24 GeV) we have[73]

$$C_1=1,\quad C_2=-1.6,\quad C_3=0.033,$$
$$C_5=-0.02,\quad C_6=0.1.$$

The effective Lagrangian (103) satisfies the selection rules $\Delta S=1$, $\Delta I=1/2$ and 3/2.

The amplitudes of the considered kaon decays contain matrix elements of the $Q_i(\mu)$ operators: $\langle x|Q_i(\mu)|K\rangle$, where $|x\rangle$ is any low-energy state. The values of the matrix elements $\langle x|Q_i(\mu)|K\rangle$ are determined by the strong low-energy interactions. Thus, the calculation of $\langle x|Q_i(\mu)|K\rangle$ is an extremely involved problem, which admits at present only model solutions.

Using the results of Ref. 75, we can represent the matrix elements $\langle x|Q_i(\mu)|K\rangle$ as the sum of two terms

$$\langle x|Q_i(\mu)|K\rangle=\langle x|Q_i(\mu)|K\rangle_{\rm NPC}+\langle x|Q_i(\mu)|K\rangle_{\rm PC},$$

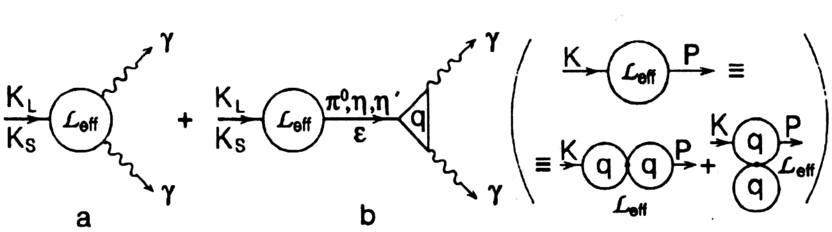

FIG. 8. Contact and pole diagrams of the decays $K_{L,S}\to\gamma\gamma$ [$\varepsilon \equiv f_0$ (700)].

where $\langle x|Q_i(\mu)|K\rangle_{\rm NPC}$ is the nonperturbative contribution (NPC), which is independent of μ and is defined by the SBCS.

The term $\langle x|Q_i(\mu)|K\rangle_{\rm PC}$ corresponds to the perturbative contribution (PC); the quantity $\langle x|Q_i(\mu)|K\rangle_{\rm PC}$ is defined by quantum fluctuations with energy less than μ. For sufficiently small μ ($\mu \sim 0.24$ GeV) the nonperturbative contributions become essential. Thus, all perturbative contributions for $\mu \sim 0.24$ GeV can be neglected.

In our model $\langle x|Q_i(\mu)|K\rangle$ is naturally approximated by quark loops with virtual constituent quarks and the cutoff parameter $\Lambda = 1.28$ GeV, which characterizes the scale of the chiral symmetry breaking.

Decays $K\to\gamma\gamma$

The $K\to\gamma\gamma$ amplitudes are determined by the contact and pole diagrams in Fig. 8. The pole diagrams are due to exchange of the pseudoscalar mesons π^0, η, η' and the scalar isoscalar meson $\varepsilon(f_0)$. Within our model accuracy (20–25%) the contribution of the contact diagrams can be neglected.[76]

In the pole approximation the $K\to\gamma\gamma$ decay amplitudes contain matrix elements of the operators $Q_i(\mu)$ calculated between $|K^0\rangle$ and $|x\rangle$ states, where $X=\pi^0$, η, η', or $\varepsilon(f_0)$. Since the values of the matrix elements of the abovementioned transitions depend mainly on the operator Q_6, we write down these elements in explicit form only for the "penguin" operator Q_6:

$$Q_6=\left[\bar{s}_a\gamma^\nu(1-\gamma^5)d_b \sum_{q=u,d,s} [\bar{q}_b\gamma_\nu(1-\gamma^5)q_a]\right].$$

Here a, $b=1$, 2, 3 are color indices. Then

$$\langle\pi^0|Q_6|K^0\rangle=\rho X,$$

$$\langle\eta|Q_6|K^0\rangle=\left[-\left(\frac{2}{3}+\rho\right)\sin\bar\theta + \sqrt{2}\frac{F_s}{F_\pi}\left(\frac{1}{3}+\rho'\right)\cos\bar\theta\right]X,$$

$$\langle\eta'|Q_6|K^0\rangle=\left[-\left(\frac{2}{3}+\rho\right)\cos\bar\theta - \sqrt{2}\frac{F_s}{F_\pi}\left(\frac{1}{3}+\rho'\right)\sin\bar\theta\right]X. \qquad (104)$$

Here $X=\langle\pi^0|Q_1|K^0\rangle=3.5\cdot10^{-3}$ GeV^4. The parameters ρ and ρ' have the values

$$\rho=64(1+\lambda)\left(\frac{Zm_uF_\pi}{M_KF_K}\right)^2\left[1-\frac{\lambda F_K^2}{2(1+\lambda)F_\pi^2}\right]\approx 50,$$

$$\rho'=64\lambda(1+\lambda)\left(\frac{Zm_uF_\pi^2}{M_KF_KF_s}\right)^2\left[1-\frac{F_K^2}{2(1+\lambda)F_\pi^2}\right]\approx 60. \qquad (105)$$

Here $\lambda=m_s/m_u$ and M_K is the K-meson mass.

It can be seen from the above formulas that in the case of exact $SU(3)$ symmetry ($m_u=m_s$, $F_\pi=F_K=F_s$) Eqs. (104) and (105) result in the usual $SU(3)$-symmetric relation between the matrix elements $\langle K^0|Q_6|\pi^0,\eta,\eta'\rangle$ used in some papers.[77,78]

Using the formulas given in Refs. 76 and 79, one can obtain the following values for the matrix elements of the transitions $K^0\to\pi^0$, η, η' for two different values of the angle θ:

$$\theta=-18^\circ:\quad \langle\pi^0|Q|K^0\rangle=4.9X;$$

$$\langle\eta|Q|K^0\rangle=3X;\quad \langle\eta'|Q|K^0\rangle=-10.6X.$$

$$\theta=-20^\circ:\quad \langle\pi^0|Q|K^0\rangle=4.9X;$$

$$\langle\eta|Q|K^0\rangle=2.6X;\quad \langle\eta'|Q|K^0\rangle=-10.7X.$$

For the $K_L\to\gamma\gamma$ decay amplitude we have

$$T_{K_L\to\gamma\gamma}=\frac{\alpha G_F s_1c_1c_3}{3\pi F_\pi}\left\{\frac{3\langle\pi^0|Q|K^0\rangle}{M_K^2-M_\pi^2} + \left(5\sin\bar\theta-\sqrt{2}\frac{F_\pi}{F_s}\cos\bar\theta\right)\frac{\langle\eta|Q|K^0\rangle}{M_K^2-M_\eta^2} + \left(5\cos\bar\theta+\sqrt{2}\frac{F_\pi}{F_s}\sin\bar\theta\right)\frac{\langle\eta'|Q|K\rangle}{M_K^2-M_{\eta'}^2}\right\}$$

$$=\begin{cases}4.5\cdot10^{-9}\ \text{GeV}^{-1} & (\theta=-18^\circ)\\ 3.4\cdot10^{-9}\ \text{GeV}^{-1} & (\theta=-20^\circ)\end{cases}. \qquad (106)$$

The experimental values are

$$\Gamma_{K_L\to\gamma\gamma}=(7.24\pm0.35)\cdot10^{-12}\ \text{eV},$$

$$T_{K_L\to\gamma\gamma}=3.4\cdot10^{-9}\ \text{GeV}^{-1}.$$

We see that at $\theta=-20^\circ$ one can obtain good agreement between the theoretical and experimental data.

Now let us consider the decay $K_S\to\gamma\gamma$. Here the main diagram is the pole diagram 8b with the ε meson [f_0(700–800)]. The expressions for the corresponding matrix elements are[79]

$$\langle\varepsilon|Q_1|K^0\rangle=\langle\varepsilon|Q_2|K^0\rangle=\langle\varepsilon|Q_3|K^0\rangle=0,$$

$$\langle\varepsilon|Q_5|K^0\rangle=\frac{1}{3}\langle\varepsilon|Q_6|K^0\rangle=\frac{i}{3}\rho''X.$$

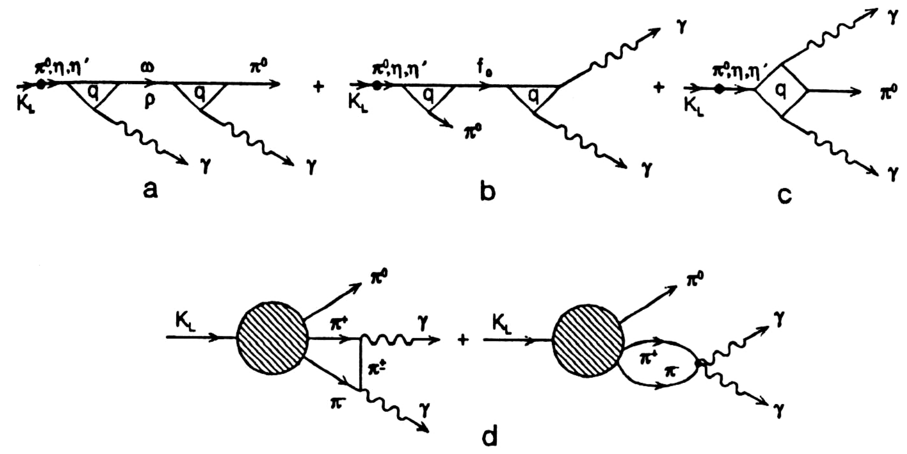

FIG. 9. Pole, contact, and loop diagrams that describe the decay $K_L \to \pi^0\gamma\gamma$.

The parameter ρ'' is defined by the expression

$$\rho'' = 64(1+\lambda)Z^{3/2}\left(\frac{m_u F_\pi}{F_K M_K}\right) \approx 70.$$

The $K_S \to \gamma\gamma$ decay amplitude is

$$T_{K_s\to\gamma\gamma} = -\frac{10}{9}\frac{\alpha}{\pi F_\pi Z^{1/2}}(G_F s_1 c_1 c_3) \times \frac{i\langle\varepsilon|Q|K^0\rangle}{M_K^2 - M_\varepsilon^2}\cos\delta_\varepsilon(M_K)\exp i\delta_\varepsilon(M_K) = \begin{cases} 2.7\cdot 10^{-9}\exp(i60^\circ)\ \text{GeV}^{-1} & (m_\varepsilon = 0.7\ \text{GeV}) \\ 2.3\cdot 10^{-9}\exp(i45^\circ)\ \text{GeV}^{-1} & (m_\varepsilon = 0.8\ \text{GeV}), \end{cases} \tag{107}$$

where

$$\delta_\varepsilon(M_K) = \arctan\left(\frac{M_K\Gamma_\varepsilon(M_K)}{M_\varepsilon^2 - M_K^2}\right),$$

$$\Gamma_\varepsilon(M_K) = \frac{3g^2_{\varepsilon\pi\pi}}{32\pi M_K}\left[1 - \frac{4M_\pi^2}{M_K^2}\right]^{1/2},$$

$$g_{\varepsilon\pi\pi}(M_K) = \frac{4m_u^2 Z^{1/2}}{F_\pi}.$$

Here $\Gamma_\varepsilon(M_K)$ and $g_{\varepsilon\pi\pi}(M_K)$ are the partial width and effective coupling constant of the decay $\varepsilon \to \pi\pi$ of the virtual meson $\varepsilon(700\text{–}800)$ with energy M_K.

The experimental value is[52]

$$T_{K_s\to\gamma\gamma} = (5.0\pm 1.3)\cdot 10^{-9}\ \text{GeV}^{-1}.$$

Decay $K_L \to \pi^0\gamma\gamma$

If the matrix elements of the transitions $K_L \to \pi^0$, η, η' are known, one can describe the decay $K_L \to \pi^0\gamma\gamma$ by analogy with the decay $\eta \to \pi^0\gamma\gamma$ (see Sec. 8.3). This decay is described by diagrams with intermediate vector ω and ρ mesons, scalar $f_0(700\text{–}800)$, $f_0(975)$, and $f_0(1400)$ mesons, and contact diagrams (see Fig. 9). Besides, as was shown in Refs. 77 and 78, a noticeable contribution to this process comes from diagrams with pion loops (see Fig. 9d).

The decay $K_L \to \pi^0\gamma\gamma$ is also of interest because it is essential to take into account the mass difference of the s and (u,d) quarks when the transitions $K_L \to \pi^0$, η, η' are calculated. If this difference is neglected and use is made of the relation for the matrix elements $\langle K_L|\pi^0,\eta,\eta'\rangle$ following from the group $SU(3)$, the relative contributions of the diagrams with an intermediate η meson are considerably reduced for physical values of the (η_0,η_8) mixing angle $(-20^\circ \leqslant \theta \leqslant -18^\circ)$. Therefore, we are left with the diagrams with intermediate π^0 and η' mesons, which have the same sign and reinforce each other.[77,78] Normalization to the matrix element with an intermediate π^0 meson gives the following value for the amplitude with vector mesons (Fig. 9a):

$$T_{K_L\to\pi^0\gamma\gamma} = T^{(\pi^0)}_{K_L\to\pi^0\gamma\gamma}[\overset{(\pi^0)}{1} \overset{(\eta)}{-0.09} + \overset{(\eta')}{0.21} = 1.1] \times (\theta = -18^\circ,\ m_u = m_s).$$

However, when the mass difference of the constituent s and (u,d) quarks is taken into account [see Eqs. (104) and (105)], the contribution of the diagram with an intermediate η meson becomes much larger; since it is opposite in sign to the contributions from π^0 and η' mesons, the total value of the amplitude decreases drastically. Therefore, when the width of the decay $K_L \to \pi^0\gamma\gamma$ is computed, one can neglect the influence of the diagrams with intermediate vector mesons:[80]

$$T_{K_L\to\pi^0\gamma\gamma} = T^{(\pi^0)}_{K_L\to\pi^0\gamma\gamma}[\overset{(\pi^0)}{1} \overset{(\eta)}{-1.27} + \overset{(\eta')}{0.266} = -0.04] \times (\theta = -18^\circ,\ m_u \neq m_s).$$

A similar situation holds for the contact diagrams (Fig. 9c).

As a result, appreciable contributions to the amplitude of the decay $K_L \to \pi^0\gamma\gamma$ come only from the diagrams with intermediate scalar mesons and pion loops[16)] (Figs. 9b and 9d). Theoretical estimates are as follows:[80]

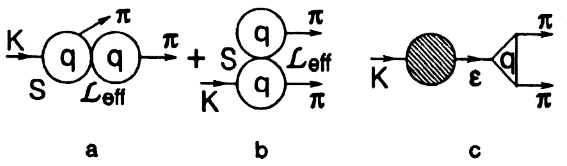

FIG. 10. Contact and pole diagrams of the decay $K\to 2\pi$ [$\varepsilon\equiv f_0$ (700)].

$$\Gamma^{(M_\varepsilon=0.7\text{ GeV})}_{K_L\to\pi^0\gamma\gamma}=\begin{cases}3.9\cdot 10^{-14}\text{ eV} & (\theta=-18°),\\ 3.3\cdot 10^{-14}\text{ eV} & (\theta=-20°);\end{cases}$$

$$\Gamma^{(M_\varepsilon=0.8\text{ GeV})}_{K_L\to\pi^0\gamma\gamma}=\begin{cases}3.0\cdot 10^{-14}\text{ eV} & (\theta=-18°),\\ 2.5\cdot 10^{-14}\text{ eV} & (\theta=-20°).\end{cases}$$

The experimental value is

$$\Gamma_{K_L\to\pi^0\gamma\gamma}=(2.7\pm0.8)\cdot 10^{-14}\text{ eV}.$$

Decay $K\to 2\pi$

a) $\Delta I=3/2$ *transitions.* The effective Lagrangian, obeying the selection rules $\Delta S=1$ and $\Delta I=3/2$, takes the form[79]

$$\mathcal{L}^{|\Delta I|=3/2}_{\text{eff}}=\frac{G_F}{\sqrt{2}}s_1c_1c_1\cdot 0.2\cdot Q_{|\Delta I|=3/2}, \tag{108}$$

where

$$Q_{|\Delta I|=3/2}=(\bar{s}_aL^\mu d_a)(\bar{u}_bL_\mu u_b)-(\bar{s}_aL^\mu d_a)(\bar{d}_bL_\mu d_b)+(\bar{s}_aL^\mu u_a)(\bar{u}_bL_\mu d_b),$$

$L^\mu=\gamma^\mu(1-\gamma^5)$, and a and b are color indices. The decay amplitude $T^{3/2}(K\to 2\pi)$ is

$$T^{3/2}(K\to 2\pi)=3.5\cdot 10^{-7}\mathrm{i}\langle 2\pi|Q_{|\Delta I|=3/2}|K\rangle\text{ GeV}^{-2}.$$

In our model the matrix elements $\langle 2\pi|Q_{|\Delta I|=3/2}|K\rangle$ are determined by the contact quark diagrams shown in Figs. 10a and 10b.

Let us write down the results of the calculation:

$$\langle\pi^+\pi^-|Q_{|\Delta I|=3/2}|K^0\rangle=-\frac{1}{2}\langle\pi^0\pi^0|Q_{3/2}|K^0\rangle=\frac{\sqrt{2}}{3}\langle\pi^+\pi^0|Q_{3/2}|K^+\rangle,$$

$$\mathrm{i}\langle\pi^+\pi^0|Q_{3/2}|K^+\rangle=\frac{8M_K^2F_K}{1+\lambda}\left[1-\left(\frac{1+\lambda}{2}\right)^2\left(\frac{M_\pi F_\pi}{M_KF_K}\right)^2-(Z-1)\left(\frac{1+\lambda}{2}\right)\times\left(1+\frac{F_\pi^2}{F_K^2}\right)\right]=6.6\cdot 10^{-2}\text{ GeV}^3.$$

The numerical values of the $K\to 2\pi$ decay amplitudes and partial widths due to the $\Delta I=3/2$ transitions are presented in Table III.

b) $\Delta I=1/2$ transitions. The $\Delta I=1/2$ transitions take place in the decays $K^0\to\pi^+\pi^-$ and $K^0\to\pi^0\pi^0$. The effective Lagrangian satisfying the selection rules $\Delta I=1/2$ and $\Delta S=1$ can be obtained by subtracting (108) from (103):

$$\mathcal{L}^{|\Delta I|=1/2}_{\text{eff}}=\mathcal{L}^{|\Delta S|=1}_{\text{eff}}-\mathcal{L}^{|\Delta I|=3/2}_{\text{eff}}=\frac{G_F}{\sqrt{2}}s_1c_1c_3Q_{|\Delta I|=1/2}. \tag{109}$$

The $K^0\to 2\pi$ decay amplitudes are determined defined by both the contact and pole diagrams. The main contribution comes from the pole diagram with ε-meson exchange [f_0(700–800)]. Within the model accuracy, the contribution of the contact diagrams and pole diagrams with exchange of other resonances can be neglected in comparison with the ε-meson contribution.

Then the $K^0\to 2\pi$ decay amplitudes due to the $\Delta I=1/2$ transitions are

$$A^{1/2}(K^0\to 2\pi)=\frac{G_F}{\sqrt{2}}s_1c_1c_3\mathrm{i}\langle\varepsilon|Q_{1/2}|K^0\rangle\frac{g_{\varepsilon\pi\pi}(M_K)}{M_\varepsilon^2-M_K^2}\cos\delta_\varepsilon(M_K)\times\exp[\mathrm{i}\delta_\varepsilon(M_K)]=3\cdot 10^{-7}\exp(\mathrm{i}\delta_\varepsilon)\text{ GeV}. \tag{110}$$

Here $\delta_\varepsilon=60°$ is the phase of the amplitude $A^{1/2}(K^0\to 2\pi)$ [see Eq. (107)]. The numerical values of the amplitudes and partial widths of the decays $K^0\to 2\pi$ due to $\Delta I=1/2$ transitions are presented in Table III.

The theoretical values of the $K\to 2\pi$ decay amplitudes are in satisfactory agreement with the experimental data and confirm the phenomenological $\Delta I=1/2$ rule. Enhancement of $\Delta I=1/2$ transitions in the linear σ model

TABLE III. Numerical values of the $K\to 2\pi$ amplitudes and partial widths.

Decay	Theory				Experiment	
	$\Delta I=3/2$		$\Delta I=1/2$			
	T	Γ	T	Γ	T	Γ
$K^+\to\pi^+\pi^0$	2.3	1.8	0	0	1.84	1.13
$K^0\to\pi^+\pi^-$	1.1	0.4	30	300	27.7	253
$K^0\to\pi^0\pi^0$	2.2	0.8	30	150	26.3	116

Here T is the absolute value of the $K\to 2\pi$ amplitude units 10^{-8} GeV, and Γ is the partial width of the decay in units 10^{-17} GeV.

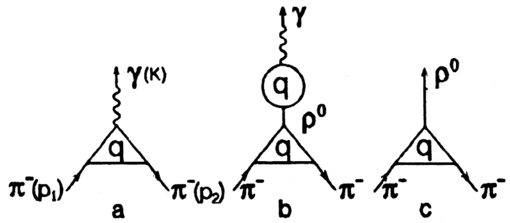

FIG. 11. Diagrams describing the pion electromagnetic radius.

takes place by exchange of the scalar meson $f_0(700\text{–}800)$. This scalar meson is a broad resonance which can be identified with the $S_1(910)$ state of Au, Morgan, and Penington[81] (see also the new data in Ref. 81).

Decay $K\to 3\pi$

Nonleptonic decays $K\to 3\pi$ proceed with a small energy release (~ 25 MeV per particle of the decay); thus, a soft-meson technique (a low-energy limit) is a good approach for their description. The $K\to 3\pi$ amplitudes can be related to the $K^0\to 2\pi$ amplitudes.[82] For instance, in the case of the decay $K^+\to\pi^+\pi^+\pi^-$ one obtains

$$T[K^+\to\pi^+\pi^-\pi^+(q_+)]|_{q_+=0}$$

$$=\frac{i}{\sqrt{2}F_\pi}T^{1/2}(K^0\to\pi^+\pi^-).$$

The variant of the NJL model proposed here gives a good description of the decays $K_L\to\pi^+\pi^-\gamma$, $\eta\to\pi^+\pi^-\gamma$, $\omega\to 3\pi$, $\varphi\to 3\pi$,[83] the form factors of the decay K_{l4},[84] and so on.

6) Internal structure of mesons. In addition to various decays of mesons, the NJL model can describe their intrinsic properties, i.e., electromagnetic radii, polarizabilities, $\pi\pi$ and πK scattering lengths, etc.[16] As an illustration, here we will calculate the pion electromagnetic radius.

We demonstrate two versions of this calculation, one with the use of vector dominance and the second as a direct calculation. We start with the latter. The pion electromagnetic radius is described by the two diagrams shown in Figs. 11a and 11b. The $\rho^0\gamma$ transition through a quark loop leads to the expression[16]

$$\frac{1}{2}\frac{e}{g_\rho}F_{\mu\nu}\rho^0_{\mu\nu}.$$

Using then the local approximation for all divergent quark loops, we obtain the formula

$$F_{\pi^-}(k)=-ek^\nu\left(1+\frac{k^2}{M_\rho^2-k^2}\right)A_\nu\approx -ek^\nu\left(1+\frac{k^2}{M_\rho^2}\right)$$

$$\times A_\nu(k=p_1-p_2),$$

which gives the well known result

$$\langle r^2\rangle_{\pi^-}=\frac{6}{M_\rho^2},\quad \sqrt{\langle r^2\rangle_{\pi^-}}=0.63\ \text{fm},$$

in good agreement with experiment,

$$\sqrt{\langle r^2\rangle_{\pi^-}}=(0.663\pm 0.023)\ \text{fm}.$$

An analogous expression for the pion radius follows from our model when use is made of vector dominance. In this case, apart from diagrams 11a and 11b, one should consider diagram 11c which leads to the expression $(g_\rho/2)k^\nu\rho^0_\nu$. Upon making a change of the vector fields, $\rho_\nu=\tilde{\rho}_\nu+(e/g_\rho)A_\nu$, the contributions of diagrams 11a and 11c cancel, and the remaining diagram 11b again gives the standard result corresponding to the vector-dominance model:

$$F_{\pi^-}(k)=-ek^\nu\left(1-1+\frac{M_\rho^2}{M_\rho^2-k^2}\right)A_\nu$$

$$=-ek^\nu\left(1+\frac{k^2}{M_\rho^2}\right)A_\nu.$$

The radii of the K^0 and $K^\pm$ mesons are described in the same manner.

The model can also satisfactorily describe the polarizabilities of pions and kaons[85,86] and the $\pi\pi$ and πK scattering lengths.[87]

The validity of the q^2 expansions used in the model should be proved more rigorously, which, in particular, requires the solution of the very complicated problem of quark confinement. Successful attempts along this line were undertaken in Ref. 11.

9. THE QUARK AND GLUON CONDENSATES IN THE NJL MODEL

The purpose of this section is to investigate a QCD-motivated NJL model containing a nonperturbative gluon condensate. We will then show how the basic parameters and model equations of the resulting chiral σ model change when this quantity is taken into account.[27]

Let us start with QCD and decompose the gluon field G^a_μ into a condensate field G^a_μ and the quantum fluctuations g^a_μ around it:

$$G^a_\mu(x)=G^a_\mu(x)+g^a_\mu(x). \tag{111}$$

By assumption, the first part of the field yields a nonvanishing gluon condensate

$$\left\langle \text{vac}\left|\frac{g^2}{4\pi}:G^a_{\mu\nu}(0)G^{\mu\nu}_a(0):\right|\text{vac}\right\rangle$$

$$=\left\langle \text{vac}\left|\frac{g^2}{4\pi}G^a_{\mu\nu}(0)G^{\mu\nu}_a(0)\right|\text{vac}\right\rangle, \tag{112}$$

where $G^a_{\mu\nu}$ is the field-strength tensor. Integrating in the generating functional of QCD over the quantum field $g^a_\mu(x)$ and approximating the (unknown) nonperturbative gluon propagator by a δ function, we get an effective chiral four-quark interaction of the NJL type. In this case, the condensate field $G^a_\mu(x)$ enters into the standard Lagrangian of the NJL model through the covariant derivative of the quark field,

$$D_\mu q=\left(\partial_\mu+\mathrm{i}g\frac{\lambda_a}{2}G^a_\mu\right)q, \tag{113}$$

where g is the QCD coupling constant and $\lambda_a/2$ are the generators of the color group $SU(N_c)$.

The effective chiral quark Lagrangian describing interactions of composite scalar and pseudoscalar mesons in the presence of condensate gluons can be written as[16,18]

$$\mathscr{L}(q,G)=\bar{q}(\mathrm{i}\hat{D}-m^0)q+\frac{\kappa}{2}[(\bar{q}\tau^\alpha q)^2+(\bar{q}\mathrm{i}\gamma_5\tau^\alpha q)^2], \tag{114}$$

where $\hat{D}=\gamma^\mu D_\mu$, D_μ is the covariant derivative (113), τ^α are the Pauli matrices of $SU(2)_F$ ($\tau^0\equiv\mathbb{1}$; summation over α is understood), and q are the fields of the current quarks with mass m^0.[17)] Upon introducing meson fields, the Lagrangian (114) takes the equivalent form

$$\mathscr{L}'(q,G,\tilde{\sigma},\varphi)=-\frac{(\tilde{\sigma}^2_\alpha+\varphi^2_\alpha)}{2\kappa}+\bar{q}(\mathrm{i}\hat{D}-m^0+\tilde{\sigma}+\mathrm{i}\gamma_5\varphi)q \tag{115}$$

with $\tilde{\sigma}=\sigma_\alpha\tau^\alpha$, $\varphi=\varphi_\alpha\tau^\alpha$. The vacuum expectation value of the isoscalar scalar field $\tilde{\sigma}_0$ turns out to be nonzero ($\langle\tilde{\sigma}_0\rangle\neq 0$). To pass to a physical field σ_0 with $\langle\sigma_0\rangle=0$ one usually performs a field shift leading to a new quark mass m to be identified with the constituent-quark mass [see (48)]:

$$-m^0+\tilde{\sigma}_0=-m+\sigma_0;\quad \tilde{\sigma}_\alpha=\sigma_\alpha\ (\alpha=1,2,3), \tag{116}$$

where m is determined from the gap equation (see below).

Let us for a moment neglect the gluon condensate in (115) ($\hat{D}\to\hat{\partial}$). Integrating in the generating functional associated with the Lagrangian (115) over the quark fields, evaluating the resulting quark determinant by a loop expansion, and including thereby only second-order field derivatives, we obtain an expression corresponding to the linear σ model [see (53)]:

$$\mathscr{L}''=-\frac{\tilde{\sigma}^2_\alpha+\varphi^2_\alpha}{2\kappa}+\mathrm{Tr}\{[p^2I_2+2(I_1+m^2I_2)]\times[(\sigma-m)^2+\varphi^2]-I_2[(\sigma-m)^2+\varphi^2]^2\}. \tag{53'}$$

Here I_1 and I_2 are divergent integrals, regularized with a cutoff parameter Λ [see (50)].

Now let us see how the Lagrangian (53′) changes when condensate gluon fields are taken into account. Here the effect of gluon condensate corrections will be calculated with an accuracy up to squared terms in the field strength $G^a_{\mu\nu}(x)$. For evaluating the quark determinant with external fields we shall use the "heat-kernel" technique proposed in Refs. 18 and 88. Then, instead of (53′), we get[18)]

$$\mathscr{L}^G=\mathrm{Tr}\left\{-\frac{\varphi^2+(\sigma-m+m^0)^2}{4\kappa}+\left[p^2\left(I_2+\frac{1}{96}\frac{G^2}{m^4}\right)+2\left[I_1+m^2I_2+\frac{1}{48}\frac{G^2}{m^2}\left(1+\frac{1}{2}\right)\right]\right][(\sigma-m)^2+\varphi^2]-\left(I_2+\frac{1}{96}\frac{G^2}{m^4}\right)[(\sigma-m)^2+\varphi^2]^2\right\}, \tag{117}$$

where

$$G^2=\frac{\alpha}{\pi}\langle(G^a_{\mu\nu})^2\rangle,\quad \alpha=\frac{g^2}{4\pi}.$$

Now let us determine the physical quantities and model parameters. To ensure that the Lagrangian (117) takes its usual form (with the standard coefficient of the kinetic term) one should perform the following field renormalizations:

$$\sigma_\alpha=g_\sigma\sigma^R_\alpha,\quad \varphi_\alpha=g_\varphi\varphi^R_\alpha,\quad g_\sigma=g_\varphi=\frac{1}{2}\left(I_2+\frac{G^2}{96m^4}\right)^{-1/2}. \tag{118}$$

Then from the requirement that the terms linear in σ vanish we obtain a modified "gap" equation

$$m=m^0+8\kappa mI_1+\kappa\frac{G^2}{6m}. \tag{119}$$

After renormalization of the meson fields we obtain, for the quadratic part of the Lagrangian (117) determining the mass terms (omitting the field index R),

$$\mathscr{L}^G_m=-\frac{g^2_\sigma}{4}\mathrm{Tr}\left\{\left(\frac{1}{\kappa}-8I_1-\frac{G^2}{6m^2}\right)(\sigma^2+\varphi^2)+(4m)^2\left(I_2+\frac{G^2}{96m^4}\right)\sigma^2\right\}=-\frac{g^2_\sigma m^0}{4\kappa m}\mathrm{Tr}(\sigma^2+\varphi^2)-m^2\,\mathrm{Tr}\,\sigma^2, \tag{120}$$

where in the last part of Eq. (120) we have used the gap equation (119). We then get for the meson masses the well known equations

$$m^2_\pi=\frac{g^2_\varphi m^0}{\kappa m}=\frac{m^0m}{\kappa F^2_\pi}\approx-\frac{2m^0\langle\bar{q}q\rangle}{F^2_\pi}$$

× (Gell-Mann–Oakes–Renner formula),

$$m^2_\sigma=m^2_\pi+4m^2. \tag{121}$$

Here we have used the Goldberger–Treiman identity $g_\varphi=m/F_\pi$ and the expression for the total quark condensate $\langle\bar{q}q\rangle$ which will be derived below [see Eq. (127)]. The Goldberger–Treiman identity leads to the following expression for the pion decay constant F_π:

$$F^2_\pi=\frac{m^2}{g^2_\varphi}=m^2\left(4I_2+\frac{G^2}{24m^2}\right). \tag{122}$$

Finally, the effective coupling constants of the meson interactions can be read off from the interaction terms

$$\mathscr{L}_{\sigma\varphi^2}=g_\sigma m\,\mathrm{Tr}(\sigma\varphi^2),$$

$$\mathscr{L}_{(\sigma^2+\varphi^2)^2}=-\frac{g^2_\sigma}{4}\mathrm{Tr}(\sigma^2+\varphi^2)^2. \tag{123}$$

Thus, the form of the meson interactions remains unchanged after inclusion of the gluon condensate fields [see (55)]. Only the expression for the coupling constant $g_\sigma(g_\varphi)$ changes.

Until now we have not considered the vector and axial-vector mesons. However, one should bear in mind that since axial-vector mesons do exist, nondiagonal transitions of the type $\pi \to a_1$ play an essential role in the NJL model (see Sec. 7).

Then [see (76)] the constituent-quark mass is expressed as

$$m_u^2 = \frac{M_{a_1}^2}{12}\left[1 - \sqrt{1 - \left(\frac{2g_\rho F_\pi}{M_{a_1}}\right)^2}\right].$$

It can be seen from (76) that $1 - (2g_\rho F_\pi / M_{a_1})^2 \geqslant 0$, and hence the minimal mass of the a_1 meson is $M_{a_1} = 2g_\rho F_\pi = \sqrt{2}M_\rho = 1.1$ GeV, where in the last step the KSFR relation has been used. This is just Weinberg's relation. From this the estimates $m_u = 330$ MeV and $Z = 2$ follow.

The value of the gluon condensate is taken from the data on the hadronic process $e^+e^- \to$ hadrons (see Ref. 90):

$$G^2 = \frac{\alpha}{\pi}\langle G^a_{\mu\nu} G^{\mu\nu}_a \rangle = [(410 \pm 80)\ \text{MeV}]^4. \qquad (124)$$

Taking into account the additional renormalization constant Z due to π–a_1 mixing, Eq. (122) together with Eq. (50) gives for F_π the value

$$F_\pi^2 = \frac{N_c m^2}{(2\pi)^2 Z}\left[\ln\left(\frac{\Lambda^2}{m^2} + 1\right) - \left(1 + \frac{m^2}{\Lambda^2}\right)^{-1} + \frac{\pi^2}{6N_c m^4} G^2\right]. \qquad (125)$$

Hence, using the values $F_\pi = 93$ MeV, $N_c = 3$ and Eq. (124), we find for the parameter Λ the estimate

$$\Lambda = 700\ \text{MeV}.$$

Let us first calculate the part of the quark condensate $\langle \bar{q}q \rangle'$ which does not explicitly contain the gluon condensate G^2:

$$\langle \bar{q}q \rangle' = \text{Tr}\left(\frac{1}{i\hat{\partial} - m}\right) = -4mI_1 = -(200\ \text{MeV})^3.$$

The gap equation (119) can be rewritten in terms of the quark and gluon condensates as

$$m = m^0 - 2\kappa\langle \bar{q}q \rangle' + \frac{\kappa}{6m}\frac{\alpha}{\pi}\langle G^a_{\mu\nu} G^{\mu\nu}_a \rangle \equiv m^0 - 2\kappa \langle \bar{q}q \rangle, \qquad (126)$$

where we have introduced the notion of the total quark condensate $\langle \bar{q}q \rangle$, which includes also gluon-condensate corrections,

$$\langle \bar{q}q \rangle = \langle \bar{q}q \rangle' - \frac{1}{12}\frac{\alpha}{\pi}\frac{\langle G^a_{\mu\nu} G^{\mu\nu}_a \rangle}{m} = -(245\ \text{MeV})^3. \qquad (127)$$

We see that this number is close to the standard value of the quark condensate.

With Eqs. (121) and (126) we find the constant κ:

$$\kappa^{-1} = \left(\frac{m_\pi F_\pi}{m}\right)^2 - \frac{2\langle \bar{q}q \rangle}{m} = 0.091\ \text{GeV}^2, \quad \kappa = 11\ \text{GeV}^{-2}.$$

Finally, we can determine the current quark mass m^0:

$$m^0 = \frac{m_\pi^2 F_\pi^2 \kappa}{m} = m + 2\kappa\langle \bar{q}q \rangle \approx 5\ \text{MeV}.$$

This is also a standard value. Thus, our model gives a reasonable self-consistent description of the most important model parameters and physical quantities.

The above investigation shows that corrections due to the gluon condensate, which arises naturally in our QCD-motivated NJL model, give quite reasonable results. In particular, the gluon condensate $(\alpha/\pi)\langle G^a_{\mu\nu} G^{\mu\nu}_a \rangle$ turns out to contribute to various quantities, like g_φ, F_π, m, and $\langle \bar{q}q \rangle$, without changing thereby the form of the meson mass formulas and the interaction terms in the effective meson Lagrangian. Thus, in the considered approximation the main effects of the gluon condensate are the decrease in the value of the low-energy cutoff scale Λ from 1.25 GeV to 700 MeV and the increase in the coupling constant of the effective four-quark interaction, κ, from 5 to 11 GeV^{-2}.

It has been argued in the literature that in models with a gluon condensate a dynamical gluon mass could appear. For example, the authors of Ref. 91 presented a description of the gluon condensate based on an analogy with the Landau–Ginzburg theory of superconductivity. As a result of that analysis they predicted a gluon mass of the order of 600 MeV. In our approach, in the case of a massive gluon, we would have the relation $\kappa = (g^2/4\pi)(4\pi/2N_c M_G^2)$. At low energies we have $g^2/4\pi \simeq 1$, which leads to an estimate $M_G \approx \sqrt{2\pi/N_c\kappa} \approx 440$ MeV, which is not too different from the above value.

The effect of the gluon condensate in nonlinear chiral Lagrangians was studied in Ref. 92 as well, particularly for estimating its influence on the low-energy coefficients. The coefficients of the G^2 terms obtained in their expressions for F_π and in the quark condensate coincide with the coefficients of the present paper. However, a definite advantage of the present approach is the existence of an inherent mechanism for spontaneous breaking of chiral symmetry and the appearance of constituent quarks and meson masses on the basis of a simple effective four-quark interaction arising from gluon exchange. In the above-mentioned paper the mechanism of spontaneous breaking of chiral symmetry is introduced from outside by using additional assumptions. Another essential difference between the two approaches is that we cannot neglect the quark condensate $\langle \bar{q}q \rangle'$, since it is an important contribution of the four-quark interaction caused by quantum fluctuations of the gluon field. Finally, let us mention that our bosonization approach is in some sense complementary to the more phenomenological approach of QCD sum rules.[89] Indeed, in our case composite hadrons arise as a result of two combined nonperturbative effects: *i*) by the ladder summation of (soft) gluon-mediated four-quark interactions; *ii*) the nonperturbative contributions of the quark and gluon condensates.

Here we have shown how it is possible to take into account the G^2 gluon corrections in the NJL model. Let us recall that by using scale symmetry (see Sec. 3) we can introduce a gluon condensate in our model by a more general method.

In conclusion I would like to express my gratitude to all my coauthors, especially to Prof. D. Ebert, for the intensive and fruitful collaboration.

[1)] An analogous procedure is also employed in Ref. 12.

[2)] We have just become acquainted with a recent paper[33] in which four-quark interactions were taken into account, which led to a larger value $\Lambda=750$ MeV. This value is close to the value used in the NJL model.

[3)] A simple derivation of the relation (23) can be found in the paper by Shifman.[36]

[4)] Note that an expression like (28) was obtained by Leutwyler[39] and by Voloshin and Ter-Martirosyan[40] on the basis of the one-loop (rather than tree) approximation of gluon fluctuations in a condensate (background) field.

[5)] Recently, the NJL model has also been derived from a relativistic version of the potential model based on QCD.[13]

[6)] If one uses regularizations without explicit dependence on the parameter Λ (for instance, dimensional regularization), then an important equation of the model, like the gap equation, becomes distorted, and incorrect relations are established between the masses of current and constituent quarks.

[7)] Two fields with equal masses, $M_1=M_2=\Lambda$, are subtracted so as to annihilate the quadratic divergence in the loop diagram with two-meson lines; in this case a constant gauge-non-invariant term that corresponds to the quadratic divergence drops out. Note that all three regularizations mentioned here (cutoff at the upper limit Λ, Pauli–Villars, and heat-kernel technique) give similar expressions for the divergent parts of loop diagrams up to terms of the order $O(m_i^2/\Lambda^2)$ (see Refs. 16 and 18).

[8)] Recall once more that as a result of the employed Pauli–Villars regularization the expressions for I_1 and I_2 up to terms of the order of $O(m_i^2/\Lambda^2)$ coincide with I_1 and I_2 derived in the previous section.[16]

[9)] It is assumed that $m_u \approx m_d \neq m_s$. The renormalization (71) results from diagonalization of the Lagrangian via introduction of the physical fields V'_μ, S'

$$V_\mu^{ij}=V_\mu'^{ij}+\sqrt{\frac{3}{2}}\frac{(m_i-m_j)}{M_{V_{ij}}^2}Y_{ij}^{1/2}\partial_\mu S'_{ij}.$$

[10)] It is interesting to notice that the formula $M_{a_1}^2 = M_\rho^2 + 6m_u^2$ (76) is consistent with the formula only when $M_{a_1} = 2g_\rho F_\pi$.

[11)] The $U_A(1)$ anomaly could be taken into account more accurately by introducing the 't Hooft determinant that breaks chiral symmetry and leads to mixing of flavors.[20,54]

[12)] The value $\alpha_\rho \approx 3$ is taken according to the old experimental data $\Gamma_{\rho\to\pi\pi}=153$ MeV (Particle Data Group, 1988).

[13)] This can easily be verified by using the formula $M_{a_0}^2 = M_\pi^2 + 4m_u^2$, expressing the $a_0^0\eta\pi^0$ vertex in terms of the mass m_u and F_π, and taking the limit $M_{a_0} = \infty$, taking the mass m_u to infinity.

[14)] Note that the heat-kernel technique described in Ref. 18 is also related to the momentum expansion of quark loops. The q^2 approximation has been considered by many authors (Refs. 3, 10, 59, and 62–64).

[15)] One must take two q^2 expansion steps in diagram 7a and only one step in diagram 7b in order to stay within the approximation used here.

[16)] For the decay $\eta\to\pi^0\gamma\gamma$, the contribution of the diagrams with pion loops to the decay amplitude is very small and can thus be neglected.

[17)] This type of interaction results from the (current×current) interaction of quarks due to gluon exchange after applying a Fierz transformation to the color and Dirac indices. For simplicity, we shall omit here the vector and axial-vector channels and consider an unbroken flavor group $SU(2)_F$ with equal quark masses $m_u^0=m_d^0$.

[18)] Higher-order terms of the form $g^3 f_{abc}\langle G_{\mu\nu}^a G_{\nu\sigma}^b G_{\sigma\mu}^c\rangle/m^6+\ldots$ give less important contributions of about several percent[89] and will therefore, be neglected. Note that proper-time regularized integrals are replaced here by momentum cut-off regularized integrals I_1, I_2.

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