Introduction of Lobachevskiĭ geometry into the theory of gravitation

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Fiz. Elem. Chastits At. Yadra **23**, 1155–1191 (September–October 1992)

A review is given on the occasion of the 200th anniversary of the birth of Lobachevskiĭ. Some brief details about the great scientist and the discovery of Lobachevskiĭ geometry are given. The kind of change that occurs when Lobachevskiĭ geometry is introduced into celestial mechanics and the theory of gravitation is considered.

December 1, 1992 is the 200th anniversary of the birth of the great geometer, astronomer, mechanist, and theoretical physicist N. I. Lobachevskiĭ. He has not only the priority for the discovery of a new geometry but also the priority for introducing the new geometry into astronomy, mechanics, and the theory of gravitation.

1. THE LIFE OF N. I. LOBACHEVSKIĬ

Nikolaĭ Ivanovich Lobachevskiĭ was born in Nizhni Novgorod (now Gorki) on November 20 (new calendar, December 1) 1792. In 1802, he was enrolled in the Kazan gymnasium. In 1805 the Imperial Kazan University was opened on the basis of this gymnasium. Lobachevskiĭ was enrolled there as a student when he completed his gymnasium studies in February 1807.

Lobachevskiĭ's entire activity was associated with the Kazan University. In 1811 he was made a magister, and three years later a junior scientific assistant. In 1816 he became a professor, and in 1827 was chosen as the rector of the Kazan University. He was elected to this post six times and occupied it for 19 years in succession. Lobachevskiĭ's distinguished services in the field of university education and in raising popular enlightenment were recognized by his contemporaries. On January 12, 1855 the Imperial Moscow University celebrated the centenary of its existence, and in this year the council of the university unanimously elected, and the minister of enlightenment confirmed the nomination of, full Councillor of State Nikolaĭ Ivanovich Lobachevskiĭ as an honorary member of the Moscow University (Ref. 1, p. 565).

However, the most important thing in his life, immeasurably more than these services—the discovery of the new geometry—was not understood by his contemporaries. Nevertheless, precisely through this discovery he worthily responded to the patriotic call of Lomonosov:[2]

Be audacious and, encouraged by your
industry, show that the Russian land
can give birth to its own Platos and
Newtons. (Unrhymed translation of
Lomonosov's verse.)

The discovery of the new geometry put Lobachevskiĭ's name alongside those of Archimedes and Newton.

Lobachevskiĭ presented his first paper on the new geometry for publication to the physics and mathematics section of the Kazan University on February 7 (19), 1826. Five days later, he gave a lecture at the university on this paper that was incomprehensible for his colleagues. Lobachevskiĭ published an extract from this lecture in 1828–1830 in the Kazanskiĭ Vestnik, which was published by the Imperial Kazan University under the title "On the foundations of geometry." This extract is now readily found in the complete collection of Lobachevskiĭ's works.[3]

Meeting incomprehension, Lobachevskiĭ agreed to perform the laborious duties of the rector, which gave him the possibility to publish his papers at his own discretion. This was a premeditated step: If official persons of any kind could delay the publication of his works for three or four years, Russia's priority in the discovery of the new geometry would be lost.

Even in the Imperial St. Petersburg Academy of Sciences there was no one who could understand the new geometry. Although not awarded the recognition of this academy, Lobachevskiĭ did become a member of a different academy, a foreign one: On November 11 (23), 1842, at Gauss's initiative, Lobachevskiĭ was elected a corresponding member of the Göttingen Royal Society of the Sciences.

However, this did not yet signify public recognition of the new geometry. The public recognition of Lobachevskiĭ's geometry occurred 12 years after the physical death of its creator.

Lobachevskiĭ died at Kazan on February 12 (24), 1856. He was buried there, in the Arskiĭ cemetery.

This is what was said by Professor F. M. Suvorov about the discovery of the new geometry at the celebration in 1893 by the Imperial Kazan University of the centenary of the birth of Lobachevskiĭ:

"Nikolaĭ Ivanovich reported his new doctrine for the first time in 1826 at a session of the physics and mathematics section of our university and from that time unceasingly attempted to propagandize it. Besides an exposition of his doctrine in lectures, before the auditorium of selected students, he published his works many times.

I risk losing your attention, gentlemen, by listing all the published works of Lobachevskiĭ, but if I am not to accuse him of insufficient effort to publish his new investigations I must at least say that he published his investigations, developed from different points of view, four times in Russian, twice in French, and once in German. But despite all these efforts of Lobachevskiĭ to acquaint his contemporary scientific world with the new investigations, that world did not accept his new doctrine, ana even his studies were subjected to ridicule; it is said that the geometrical investigations of Lobachevskiĭ were compared with the mountain giving birth to a mouse, in other words, Lobachevskiĭ's contemporaries regarded these investigations as negligible compared with the scientific authority that Lobachevskiĭ enjoyed for his works in other branches of mathematics.

The scornful critical responses were obviously extremely painful for the great mathematician, who fully understood the importance of his investigations, as can be seen from his comments on one of the pages of his imaginary geometry. Other papers in which a critic showed understanding for the new doctrine were not published during Lobachevskiĭ's life—he died in the deep knowledge that his doctrine had been unjustifiably ridiculed and without knowing whether a true estimate of his work could be expected soon" (Ref. 4, pp. 81–82).

2. THE OLD GEOMETRY

As a science, geometry developed in deep antiquity. The reasons for its development are to be sought in the many-sided activity of humanity. At all times, work led people to the knowledge of remarkable three-dimensional bodies and two-dimensional figures, to the need to systematize the observations of the motion of the celestial bodies, to perfect geodetic studies and astronomical measurements, and to the representation of figures and bodies. Ultimately, geometry arose through the inquisitiveness of the human mind and through man's inherent tendency to spiritual perfection.

The results of the geometrical studies of the scientists of antiquity were summarized by Euclid in his book, the *Elements*. Euclid (330–275 BC) lived and worked in Alexandria. For many centuries, his *Elements* were the paradigm of mathematical rigor. The best textbooks on geometry were written after the manner of Euclid's book. Therefore, Euclid's geometry is called the old geometry (and it is what we study in high school).

Euclid based geometry on a number of axioms. Among them, the fifth axiom, known as Euclid's postulate on parallel straight lines, was special. Euclid defined two straight lines as parallel if they lay in a common plane but did not intersect each other (Ref. 5, p. 14). The doctrine of parallel lines is the most difficult part of Euclid's geometry.

The fifth axiom is also called Euclid's eleventh axiom, or simply Euclid's postulate. The following three postulates are taught in high school:

1. Only one straight line can be drawn between two points.

2. A straight line is the shortest path between two points.

3. Through a given point on a plane one can draw only one straight line that does not intersect a given line.

The final postulate is equivalent to Euclid's fifth axiom.

Many other postulates equivalent to Euclid's axiom are known. For example: "A line inclined to a straight line and the perpendicular to the line always intersect if extended sufficiently."[6] This formulation is taken from the book of the eminent mathematician Bunyakovskiĭ (1804–1889). From this book, one can deduce the state of Euclid's geometry directly before the discovery of Lobachevskiĭ's geometry (since the book of Ref. 6 is a bibliographic rarity, its opening pages are reproduced in the Appendix).

In Euclid's geometry, one distinguishes the part that consists of propositions whose proofs do not rely on Euclid's axiom. Such propositions were presented by Euclid himself in the first place. They include, for example, the theorem that two lines do not intersect if they are perpendicular to some third line. This part is called absolute geometry. It is taken over in entirety into Lobachevskiĭ's geometry.

The propositions of Euclidean geometry that do not belong to the absolute part rely in one way or another on the fifth axiom. All such propositions were proved conditionally, under the assumption of the truth of the fifth axiom. It only remained to prove the fifth axiom itself. For thousands of years mathematicians attempted to do this, but did not achieve success. As Henri Poincaré (1854–1912) said: "For a long time a proof of the third axiom known as *Euclid's postulate* was sought in vain. It is impossible to imagine the efforts that have been spent in pursuit of this chimera. Finally, at the beginning of the nineteenth century, a Russian and a Bulgarian, Lobatschewsky and Bolyai, showed irrefutably that this proof is impossible. They have nearly rid us of inventors of geometries with a postulate, and ever since the Académie des Sciences receives only about one or two new demonstrations a year" (Ref. 7, p. 46, quoted from the English translation).

The impossibility of proving Euclid's postulate could be established only by discovering a new, non-Euclidean geometry. This was first done by Lobachevskiĭ. The new geometry was discovered independently by the remarkable Hungarian mathematician and military engineer János Bolyai (1802–1860). His original investigation[8] of the problem of parallels was published in 1832 in the form of an appendix to the first volume of a large course of mathematics written and published by his father Farkas Bolyai (1775–1856). Bolyai's investigation therefore entered the history of science with the name Appendix. The first offprint of the Appendix came out in 1831. The Appendix contains an elementary exposition of the foundations of the new geometry. Bolyai wrote this paper in Latin in a style that recalls a modern computer program. For comparison, we note that Lobachevskiĭ wrote his papers on geometry in a style that recalls modern papers on theoretical physics. As can be seen from the complete title,[8] the term "absolute geometry" was proposed by Bolyai.

Bolyai learnt of Lobachevskiĭ's discovery 16 years after the publication of the Appendix. It was never granted to Lobachevskiĭ to learn about Bolyai's discovery.

Equally, Lobachevskiĭ never learnt of the discovery of the new geometry by Gauss. Gauss (1777–1855) arrived at the idea of the new geometry at the end of the 18th century, but the scientific world learnt about this only after his death.

The titanic labor of many generations on the problem of parallels was not so fruitless as it then appeared. Evidently, it had come to the end by the beginning of the 19th century, but only a genius could understand that.

3. LOBACHEVSKIĬ'S DISCOVERY OF NON-EUCLIDEAN GEOMETRY

Lobachevskiĭ's studies on the systematic exposition of geometry began long before 1826. It is evident that he was concerned with the problem of parallels from an early age. In 1823 he presented to the Kazan University, for publication at public cost, a textbook of geometry for students that he had written. The manuscript was called "Geometry." Lobachevskiĭ's colleagues did not give a positive response, and the manuscript was sent for consideration to the administrator of the Kazan educational district, M. L. Magnitskiĭ. He forwarded the manuscript to Academician Fuss (1755–1826) with a request to give an opinion of it. Fuss gave a decidedly negative response, and Lobachevskiĭ's "Geometry" was not published.

Lobachevskiĭ evidently drew a lesson from this. If I were rector, as is now bruited in the newspapers, then I could publish "Geometry" without regard to people. What is more, they will squander and bury not only the manuscript "Geometry" but also the new science that, without any quotation marks, must be called Lobachevskiĭ's Geometry. Incidentally, this thought can also be read in the following extract from the programmatic "Lecture on the most important problems of education," which the young rector read at the ceremonial gathering of the Kazan University on July 5, 1828: "It is already a year, my dear comrades, since, following your election, I have taken upon myself the duty whose honors, importance, and difficulties are proof of your complimentary confidence in me. I do not dare to complain that you wished to take me away from my beloved occupations, to which I was long disposed. You imposed on me new labors and worries hitherto foreign to me; but I do not dare to grumble because you also gave me new means that were helpful. I answered your call because I respect your opinion; because I did not wish to go against the general desire; because I myself could not justify a person who might be in my position. Finally, your choice was confirmed by the imperial sovereign, and the duties of the new calling were made holy for me" (Ref. 9, p. 16).

For many years, it was believed that the manuscript of "Geometry" had been lost, but in 1898 Professor N. P. Zagoskin discovered it among old things in the archive of the administrative chancellery. As a result, the manuscript was published for the first time in 1909,[10] then in 1911,[11] once more in 1949,[12] and then in 1956.[13]

Already in this work, Lobachevskiĭ strongly doubted the possibility of proving Euclid's postulate, which, in fact, he called neither a postulate nor an axiom: "The measurement of planes[1)] is based on the fact that two lines converge when they stand on a third on one side and when one is perpendicular and the other inclined at an acute angle toward the perpendicular.... A rigorous proof of its truth could not yet be found. What has been found can only be called explanations, but they do not deserve to be called in the full sense Mathematical Proofs." Moreover, he noted: "Some Mathematicians wanted to take the impossibility of determining lines by means of angles as the foundation of geometry, but such a foundation is inadequate, because quantities[2)] may depend on each other." Thus, already then Lobachevskiĭ allowed that such different things as angles and lines could depend on each other, but this directly contradicts Euclid's postulate (see the Appendix, Propositions 3c and 3d, which are equivalent to Euclid's postulate).

The recollections of Lobachevskiĭ with which he begins the paper of Ref. 14 belong to the time of writing of the composition we are discussing: "It is known to all that in Geometry the theory of parallel lines has hitherto remained incomplete. The fruitless striving since the time of Euclid, continuing for 2000 years, made me suspect that the very concepts did not yet contain the truth that it was hoped to prove and which can be proved, like other physical laws, only by experiments such as, for example, Astronomical observations. Being finally convinced of the correctness of my guess, and regarding the difficult question as completely resolved, I wrote a discourse on this in 1826.[3)] An application of the new theory to analytical geometry can also be found in my papers under the title 'On the foundations of Geometry' published in the Kazanskiĭ Vestnik for 1829 and 1830. The main conclusion at which I arrived with the assumption of a dependence of lines on angles admits the existence of a Geometry in a more extended sense than it was first presented to us by Euclid. In this extensive form I named the science *Imaginary Geometry*, which contains as a special case *Common Geometry...*".

By the beginning of 1826, the decisive step had been taken: Lobachevskiĭ established that in the new geometry there could pass through a given point on a plane not only one but infinitely many lines that do not intersect a given line. In the family of such lines, there is a pair of limiting lines, which are called lines parallel to the given line. If from the beginning of a straight ray one drops the perpendicular onto the line parallel to it, then the angle Π formed by the ray and the perpendicular in Euclid's planimetry is, as is well known, $\pi/2$. But in Lobachevskiĭ's planimetry it is less than $\pi/2$ and depends on the height p from which the perpendicular is dropped. This dependence contains a characteristic length k. It is called Lobachevskiĭ's constant. Lobachevskiĭ called the dependence $\Pi = \Pi(p/k)$ the angle of parallelism and showed that

$$\operatorname{tg} \tfrac{1}{2}\Pi(x) = e^{-x}. \qquad (1)$$

The key to the proof of this formula was the intrinsic geometry of the orisphere. This was the name that Lobachevskiĭ gave to the surface orthogonal to the sheaf of parallel lines, and he proved that the intrinsic geometry of the orisphere was identical to Euclid's planimetry. It is here worth placing an exclamation mark: Having rejected Euclid's fifth axiom for the plane, Lobachevskiĭ proved it for the orisphere!

The constant k occurs in all formulas of Lobachevskiĭ's geometry in the part in which it differs from Euclid's geometry. Besides the formula given above for the dependence of the parallelism angle on the altitude of the perpendicular, let us consider the formula for the area of a triangle. As is well known, in Euclid's planimetry, the sum of the angles A, B, C of every triangle is π. In Lobachevskiĭ's geometry, the sum is less than π, and the area of the triangle is

$$F=k^2(\pi-A-B-C). \tag{2}$$

To verify this formula, Lobachevskiĭ took the largest of the triangles accessible to him, the base of which was the diameter of the earth's orbit, the opposite vertex of which was a "fixed" star.

Lobachevskiĭ created a new, hyperbolic trigonometry. He derived the following four formulas for an arbitrary triangle (in which a, b, c are the sides of the triangle opposite to its angles A, B, C):

$$\operatorname{ch}\frac{c}{k}=\operatorname{ch}\frac{a}{k}\operatorname{ch}\frac{b}{k}-\operatorname{sh}\frac{a}{k}\operatorname{sh}\frac{b}{k}\cos C,$$

$$\operatorname{sh}\frac{b}{k}\sin A=\operatorname{sh}\frac{a}{k}\sin B,$$

$$\operatorname{ctg} A\sin C+\operatorname{ch}\frac{b}{k}\cos C=\operatorname{cth}\frac{a}{k}\operatorname{sh}\frac{b}{k},$$

$$\cos C+\cos A\cos B=\sin A\sin B\operatorname{ch}\frac{c}{k},$$

and the following six formulas for a right-angled triangle (in which $C=\pi/2$):

$$\operatorname{ch}\frac{c}{k}=\operatorname{ch}\frac{a}{k}\operatorname{ch}\frac{b}{k},\quad \operatorname{th}\frac{b}{k}=\operatorname{th}\frac{c}{k}\cos A,$$

$$\operatorname{sh}\frac{a}{k}=\operatorname{sh}\frac{c}{k}\sin A,\quad \cos A=\operatorname{ch}\frac{a}{k}\sin B,$$

$$\operatorname{th}\frac{a}{k}=\operatorname{sh}\frac{b}{k}\operatorname{tg} A,\quad \operatorname{ch}\frac{c}{k}\operatorname{tg} A\operatorname{tg} B=1.$$

On the basis of these formulas, one can solve any trigonometric problem in Lobachevskiĭ's geometry.

On the basis of the hyperbolic geometry, Lobachevskiĭ developed an analytic and differential geometry of a space that was previously unknown.

It is expedient to begin the consideration of the difference between the geometrical theories of Lobachevskiĭ and Euclid for the example of a sphere of some radius ρ. According to Euclid's third axiom: "From every center and with every distance a circle can be drawn" (Ref. 5, p. 14). This means that not only in Euclid's geometry but also in Lobachevskiĭ's geometry the radius ρ can take all values from zero to infinity. Lobachevskiĭ proved that the intrinsic geometry of the sphere does not depend on Euclid's fifth axiom. Therefore, both in Euclid's geometry and in Lobachevskiĭ's geometry we can describe on a sphere "parallels" and "meridians" and introduce polar coordinates θ and φ. For this, we denote by $2\pi r$ the length of the "equator." We set the distance traversed along the meridian from the "north pole" to a point $(0,\varphi)$ lying on the sphere equal to θr, and the distance traversed along the parallel $\theta=\text{const}$ from the zero meridian to the same point is set equal to $\varphi r\sin\theta$. In the usual manner, we obtain the metric form

$$r^2(d\theta^2+\sin^2\theta\, d\varphi^2) \tag{3}$$

of the sphere and the element of area

$$r^2\sin\theta\, d\theta\, d\varphi \tag{4}$$

on the sphere. The area of the complete sphere is $4\pi r^2$.

The difference between the two geometrical theories is manifested in the actual form of the dependence of r on ρ: in Euclid's theory $r=\rho$, but in Lobachevskiĭ's theory

$$r=k\operatorname{sh}\frac{\rho}{k}. \tag{5}$$

In the limit $k\to\infty$, Lobachevskiĭ's geometry goes over into Euclid's geometry. But if $k<\infty$, in a sphere of radius ρ one can use Euclid's geometry only if the ratio ρ/k is small.

In both geometrical theories, the radius is perpendicular to the sphere. Therefore, the metric form of space in both theories has the form

$$ds^2=d\rho^2+r^2(d\theta^2+\sin^2\theta\, d\varphi^2). \tag{6}$$

The volume element is

$$dV=r^2\sin\theta\, d\rho\, d\theta\, d\varphi. \tag{7}$$

In the equatorial plane $\theta=\pi/2$, the metric form is

$$ds^2=d\rho^2+r^2\, d\varphi^2, \tag{8}$$

and the element of area is

$$d\Sigma=r\, d\rho\, d\varphi. \tag{9}$$

4. THE ATTITUDE OF THE ST. PETERSBURG ACADEMY

On August 19, 1832, at the initiative of Lobachevskiĭ, the council of the Kazan University sent a copy of his work "On the foundations of geometry" to the Imperial St. Petersburg Academy of Sciences. In its turn, on September 5 of the same year, the academy forwarded this paper for consideration to Academician M. V. Ostrogradskiĭ (1801–1861). On November 7, 1832, he responded that the paper did not warrant the attention of the academy.[15]

Ostrogradskiĭ's answer stimulated an enthusiastic outcry of the "Boeotians" (Gauss's expression: "Boeotians are ignorant, malign, and aggressive people). The Boeotians went into battle: In 1834 there was published in issue No. 41 of the journal Syn Otechestva a very insulting and entirely unjustified criticism[16] of the paper. Thus it happened that these "busybodies" drew the attention of their readers to something that, in their own opinion, did not warrant the attention of the academy. Lobachevskiĭ sent the editor of the journal an answer, but the Boeotians love to shout and do not like to listen. Lobachevskiĭ's answer was not published in the Syn Otechestva.

In 1835, Lobachevskiĭ published his answer to this criticism in the Uchenye Zapiski Kazanskogo Universiteta in a paper "Imaginary Geometry." In the same paper, Lobachevskiĭ gave a worthy answer to Ostrogradskiĭ. This paper is now published in Ref. 17.

However, Ostrogradskiĭ kept his opinion, as can be seen from his report to the First Section of the Imperial Academy of Sciences made on June 10, 1842,[18] not long before Lobachevskiĭ was elected to the Göttingen Royal Society of the Sciences: "The academy instructed me to examine the memoir on the convergence of series and give a report on it. The author of this memoir, Mr. Lobachevskiĭ, the rector of the Kazan University, is already well known, to speak the truth, from a rather unfavorable side, the new geometry, which he calls imaginary, a rather voluminous treatise on algebra, and some dissertations on various questions of mathematical analysis. The memoir presented for my consideration does not help to change the reputation of the author...".

It must be supposed that Ostrogradskiĭ's response was one reason why Lobachevskiĭ's geometry was not understood at that time by the St. Petersburg Academy. As was noted in 1893 by Professor A. B. Vasil'ev at the celebration by the Kazan University of the centenary of the birth of Lobachevskiĭ, even Academician V. Ya. Bunyakovskiĭ in his paper "Parallel lines" published in 1853 did not mention Lobachevskiĭ's investigations (Ref. 4, p. 133). After this, one must not be surprised how the well-known revolutionary democrat D. I. Pisarev (1840–1868) admonished teachers of mathematics (Ref. 19): "Mathematicians may be excellent teachers, bothering not at all about the shallow mathematical memoirs that are presented every year by industrious scientists to different European academies. One can truly say that the time of great discoveries and radical revolutions has finally passed for mathematicians, that the present generation of this science in its essential features remains unshakably firm on the ancient times, that the shallow memoirs of modern geometers do not destroy in this science anything old and do not construct in it anything new, and that after all these circumstances the conscientious teacher of mathematics in no case risks being backward in the matter of his speciality."

In the matter of the lack of understanding, Lobachevskiĭ could seek comfort only in the recognition of his services in the field of teaching.

If Academician Fuss had been favorably inclined to the manuscript sent to him for comment by the administrator Magnitskiĭ, then the work would, of course, have been published. Then Academician Ostrogradskiĭ, having read the "Geometry," might have looked into the "On the foundations of Geometry" and written a positive report. In its turn, this would have attracted the careful attention of Academician Bunyakovskiĭ to Lobachevskiĭ's geometry. The recognition of these two eminent academicians (one of them a future vice president of the Imperial St. Petersburg Academy of Sciences, which is what Bunyakovskiĭ was from 1864 to 1889) would, undoubtedly, have ensured the election of Lobachevskiĭ to the Academy. However, such a chain of events did not occur, and it turned out that the most worthy scientist of Russia did not become a Russian academician. At the same time, in order to read Lobachevskiĭ's papers in the original, Gauss attempted to overcome the language barrier. In this case, neither Ostrogradskiĭ nor Bunyakovskiĭ had to overcome a barrier.

Without question, among Lobachevskiĭ's contemporaries Gauss was the most authoritative mathematician. Reckoning on his understanding and support, Lobachevskiĭ in 1840 published his paper "Geometrische Untersuchungen zur Theorie der Parallellinien," in German.[20] Gauss rated these investigations so highly that he started to study Russian in order to read Lobachevskiĭ's writings published in Russian. This explains why Gauss proposed the candidature of Lobachevskiĭ to the Göttingen Society.

The paper of Ref. 20 can now be found in a translation into Russian in the Complete Collected Works.[21] It was translated into Russian for the first time by Professor A. V. Letnikov and published by him in 1868.[22]

5. EXTRACTS FROM GAUSS'S LETTERS

The extracts from Gauss's letters relating to the discovery of non-Euclidean geometry have exceptional interest. They were published after Gauss's death. Letnikov began publication of them in a translation into Russian[23] in 1868, together with Lobachevskiĭ's paper of Ref. 22. [*Translation Editor's note.* The translations given here have been checked by the translator against the German original.]

In a letter[24] to Gerling, Gauss reported: "It is easy to show that if Euclidean geometry is not the true geometry, then similar figures do not exist at all; in an equilateral triangle, the angles vary with the magnitude of the side, in which I find nothing absurd. In this case, the angle is a function of the side, and the side is a function of the angle, however, naturally, a function that also contains the constant line. It appears somewhat paradoxical that there can also be a straight line specified *a priori*; but in this I find nothing contradictory. It would even be desirable for Euclid's geometry not to be true, because we would then possess a general measure *a priori*" (April 11, 1816).

At the beginning of 1819, Gerling sent Gauss a letter,[25] in which he wrote: "In the last year, I learnt that my colleague Schweikart (Professor of Jurisprudence, now prorector) has long been occupied with mathematics and has in fact written on parallel lines ... he has not ceased to be concerned with this subject and is now almost convinced that Euclid's postulate cannot be proved without some additional data and thinks it is not improbable that our geometry is only the head of a different, more general..."

In the same letter, Gerling presented a request of Schweikart to Gauss to let him know his opinion on the attached note. In it, Ferdinand Karl Schweikart (1780–1859) asserts: "There exists a twofold geometry: geometry in the narrow sense of the word—Euclidean—and an astral doctrine of magnitudes.

Triangles of this last geometry have the property that the sum of three angles is not equal to two right angles.

If this is assumed, one can prove with complete rigor the following: a) the sum of the three angles in a triangle is less than two right angles; b) this sum is the smaller, the larger the area of the triangle; c) the height of a right-angled isosceles triangle does indeed always increase with increasing lateral sides but cannot exceed a certain line, which I call the constant."

To this, in a letter[26] to Gerling, Gauss answered: "...The note of Professor Schweikart gave me a great deal of pleasure, and I wish to express to him in this connection as much as I can that is good. Almost all of this is written from my own heart... . Be so good as to pass this on to Schweikart" (March 16, 1819).

Gauss is now convinced of the need to continue the development of non-Euclidean geometry and begins to write down results on this subject. The very expression "non-Euclidean geometry" was proposed by Gauss. The term is encountered for the first time in his letter[27] to Taurinus: "...The assumption that the sum of the three angles of a triangle is less than 180° leads to a geometry quite different from our (Euclidean) geometry; this geometry is completely consistent, and I developed it for myself to complete satisfaction; in this geometry, I have the possibility of solving any problem, except for the determination of a certain constant, the value of which cannot be established *a priori*. The larger the value that we give to this constant, the closer we approach Euclidean geometry, and an infinitely large value of the constant makes the two systems identical. The assumptions of this geometry appear in part paradoxical and incongruous to the uninitiated, even absurd; but a rigorous and quiet meditation shows that the assumptions contain nothing that is impossible. For example, all three angles of a triangle can be made arbitrarily small as long as the sides are taken sufficiently large; the area of the triangle cannot exceed, indeed it cannot reach a certain limit, however large its sides... If the non-Euclidean geometry were true and the constant mentioned above bore a finite relation to the quantities that are accessible to our measurements in the sky or on the earth, then it could be determined *a posteriori*. I have therefore sometimes jokingly expressed the wish that Euclidean geometry should not be true, because we would then possess an *a priori* absolute measure of length..." (November 8, 1824).

It can be seen that the term "non-Euclidean geometry" proposed by Gauss means "absolute geometry + denial of Euclid's fifth axiom," and not simply a geometry that is not Euclidean. As has been noted acutely, there are many geometries which are not Euclidean (spherical, affine, projective, Riemannian, pseudo-Euclidean, semi-Euclidean, and so forth), but non-Euclidean geometry (like Euclidean geometry itself) is unique. The expression "non-Euclidean geometries" encountered in books (see, for example, Ref. 28) is not due to Gauss, but to the authors of these books.

In a letter[29] to Schumacher, Gauss reports: "In non-Euclidean geometry, there is never similarity of figures without equality. For example, the angles of an equilateral triangle are not merely different from 2/3 of a right angle, but they can also change with the length of the sides, and if the sides increase without limit, then they can be made arbitrarily small. Therefore, there will be here a contradiction with the very desire to represent a triangle by means of a smaller triangle... .

In non-Euclidean geometry, the semicircumference of a circle of radius r is

$$-\tfrac{1}{2}\pi k(e^{r/k}-e^{-r/k}),$$

where k is a constant that experience tells us is extremely large compared to anything that we can measure. In Euclidean geometry, this constant becomes infinite" (July 12, 1831).

Soon after this Gauss learnt about the Appendix of Janos Bolyai. In their youth, Gauss and Farkes Bolyai (the father of Janos) were friends. They were both interested in the foundations of geometry and developed ideas at that time concerning Euclid's axiom. Naturally, the father and son Bolyai sent Gauss one of the first copies of the Appendix. Here is a response of Gauss to the Appendix and its author in a letter[30] to Gerling: "...In the last days I received from Hungary a small brochure on non-Euclidean geometry in which I find all my own ideas and results presented with great elegance, though admittedly in a form that, on account of its concentration, will not be understood without difficulty by anyone foreign to the subject. The author is a very young Austrian officer, the son of a friend of my youth, with whom I often spoke of this subject in 1798, although at that time my ideas were much further from the development and maturity that they have obtained as a result of the meditations of this young man. I regard this young geometer von Bolyai a genius of the first order..." . (February 14, 1832).

It was not so enthusiastically that Gauss responded[31] to his old friend: "...now something about the work of your son.

You will probably be shocked for a moment when I begin by saying *that I cannot praise it*, but I cannot do anything else, since to praise it would be to praise myself. The whole content of the paper, the path that your son has taken, and the results to which he has been led, agree almost everywhere with my own meditations, which have occupied me in part already for 30–35 years. Indeed, I am extremely astonished. My intention was during my life to publish nothing of my own work, of which, in fact, little has been put down on paper up to the present time. The majority of people do not at all have the correct understanding for the questions that are here under discussion: I have found only a few people who have responded with particular interest to what I have said to them on this subject. To be in a position to master this, it is necessary, above all, to have felt strongly that which is actually missing here; and this is quite unclear to the majority of people. However, I had the intention with time to put all this down on paper in a form such that these ideas at least would not perish with me.

Hence I am very surprised that now I have been saved the trouble, and I am very glad that it is exactly the son of my old friend who has preceded me in such a remarkable way" (March 6, 1832).

Eight years later, Gauss learnt about Lobachevskiĭ. In a letter[32] to Encke, he wrote: "...I begin to read Russian quite successfully and find great pleasure in this. Herr Knorre sent me a small memoir of Lobachevskiĭ (in Kazan) written in Russian, and both this memoir and a small book on parallel lines in German (it received a very unsatisfactory review in the Gersdorfs Repertorium) stimulated in me a strong desire to know more about this sharp-witted mathematician. As Knorre told me, many of his papers are published in Russian in the Zapiski Kazanskogo Universiteta" (February 1, 1841).

Gauss reread this paper of Lobachevskiĭ many times. Five year later, in a letter[33] to Schumacher, he reported "...Recently I had occasion to reread the small essay of Lobachevskiĭ ("Geometrische Untersuchungen zur Theorie der Parallellinien," Berlin, 1840, G. Fincke, four printed pages). This essay contains the foundations of the geometry that should exist, and could exist as a rigorously consistent whole, if Euclidean geometry is not true. A certain Schweikart[5)] gave the name "astral geometry" to this geometry. Lobachevskiĭ calls it "imaginary geometry"; you know that already for 54 years (since 1792) I have held the same views with a certain development of them, which I do not wish to mention here; thus, in Lobachevskiĭ's essay I did not find for myself anything actually new. But in the development of the subject, the author did not follow the path that I had taken; it is done magisterially by Lobachevskiĭ in a truly geometrical spirit. I feel it a duty to draw your attention to this essay, which, certainly, will give you most exquisite pleasure..." (November 28, 1846).

In a letter[34] Gauss recommends this paper of Lobachevskiĭ to Farkes Bolyai too: "The works of the Russian geometer are largely published in the Russian Zapiski Kazanskogo Universiteta. However, I believe that you will find it easier to obtain the small excellent essay "Geometrische Untersuchungen zur Theorie der Parallellinien" von Nicolaus Lobatschewsky, Berlin, 1840, published by Fincke..." (April 20, 1848).

Janos Bolyai learnt of Lobachevskiiĭ's discovery from this letter too. He made comments on the paper cited in the letter, and these were published, but only in 1902. Had he sent his comments directly to Lobachevskiĭ, one must suppose that there would have been a most interesting correspondence.

As for Gauss, to Lobachevskiĭ he reported neither the work of Bolyai nor his own ideas. Altogether, Gauss assumed that public opinion was not ready to receive such new ideas. Therefore, he did not publish his own investigations on non-Euclidean geometry, did not recommend publication of Schweikart's note, and did not respond publically to the paper of Bolyai or the works of Lobachevskiĭ. Moreover, he not only regarded public opinion as unprepared but also foresaw a stormy reaction to such unexpected ideas.

Thus, in a letter[35] to Gerling, he warned: "...I am glad that you have the courage to express yourself as if you recognized the possible falsity of our theory of parallels, and with it all our geometry. But the wasps whose nest you disturb will fly to your head..." (August 25, 1818).

In a letter[36] to Bessel he admitted: "...One further subject, which in my case already has a 40-year history, by which I mean the foundations of geometry... . However, it will still be a long time before I come to rework for publication my very extensive investigations on this question and, perhaps, this will never happen in my life, since I fear the cry of the Boeotians if I should express my views entirely" (January 29, 1829).

Moreover, as we know, the cry of the Boeotians did go up, so that Gauss could be persuaded of his perspicacity. Admittedly, he did not foresee that this would happen in St. Petersburg. In a letter[37] to Gerling, Gauss reported: "Lobachevskiĭ's paper appears in Crelles Journal in Vol. 17, p. 295ff. I find that it represents only a free translation of the Russian paper "Voobrazhaemaya Geometriya," which appeared in the "Uchenye Zapiski Kazanskogo Universiteta" for 1835... The preceding paper ... will be the same paper that is mentioned in the Russian text under the heading "On the foundations of geometry"; it appeared in the Kazanskiĭ Vestnik for 1829–1830. On this there is a note that a very sharp criticism of this paper was published in issue No. 41 of another Russian journal, Syn Otechestva, in 1843, which is apparently published in St. Petersburg; Lobachevskiĭ sent a refutation of this criticism, which, however, had not been published by the beginning of 1835..." (February 8, 1844).

6. THE PUBLIC RECOGNITION OF LOBACHEVSKIĬ'S GEOMETRY

The posthumous publication of Gauss's correspondence with his colleagues and scientists attracted the serious attention of the scientific world to non-Euclidean geometry. In Russia, the first public recognition of Lobachevskiĭ's geometry was a publication[38] of A. V. Letnikov.

The exceptional impression made then by Lobachevskiĭ's discovery is revealed by the following extract from the novel The Brothers Karamazov. This is what Ivan Karamazov said to his younger brother: "If God exists and if He really did create the world, then, as we well know, He created it in accordance with Euclidean geometry, and the human mind with an understanding of only three dimensions of space. However, there have been and there are even now geometers and philosophers, even the most outstanding, that doubt whether the entire universe or, more extensively, all being was created only in accordance with Euclidean geometry; they even dare to dream that two parallel lines, which according to Euclid can never meet on the earth, could meet somewhere at infinity. My dear chap, I decided that if I could not even understand this, then how could I understand anything about God. I freely recognize that I have no ability to resolve such questions, my mind is Euclidean, terrestrial, and therefore how can I settle what does not belong to this world... . Suppose even parallel lines do meet and I see this: I see it and I say that they meet, but I still do not accept it. That's the heart of my matter, Alesha, that's my thesis" (Ref. 39, pp. 294–296).

Thus, it was Fedor Mikhaĭlovich Dostoevskiĭ (1821–1881) who first responded in literature (not only Russian but in the whole world) to the discovery of Nikolaĭ Ivanovich Lobachevskiĭ. Undoubtedly, this drew the attention of many readers of the novel to the new geometry and its creator.

The discovery of the new geometry had a fruitful influence on the development of the physical and mathematical sciences. Naturally, in the first place this applied to the mutual applications of Geometry and Analytical Geometry.

"The new Geometry, the foundation of which is presented here," explained Lobachevskiĭ, "...opens up a new extensive field for mutual applications of Geometry and Analytical Geometry" (Ref. 3, p. 209). In this field, Lobachevskiĭ himself achieved outstanding results: "For a wide circle of mathematicians, physicists, and engineers it is usually unknown that in many of the handbooks and tables of definite integrals that they use there are formulas obtained by Lobachevskiĭ using the methods of his "imaginary geometry." The fact is that almost all reference books use to a considerable extent material from the Tables of Definite Integrals of Bierens de Haan, which were the first extensive collection of definite integrals and in which many of Lobachevskiĭ's results were included.

The publication of these tables began in 1853, i.e., while Lobachevskiĭ was still alive, but ended only in 1858—two years after his death—so that Lobachevskiĭ was not able to see his integrals included in these tables (Ref. 40, p. 413).

Applying Lobachevskiĭ's geometry, Poincaré (1854–1912) created in 1882 the theory of automorphic functions. In this, the idea of a conformal mapping of the Euclidean half-plane onto the Lobachevskiĭ plane was a great service. In this connection, he wrote: "I cannot stay silent on the connection between the foregoing concepts and Lobachevskiĭ's non-Euclidean geometry... .

If these renamings are accepted, then the theorems of Lobachevskiĭ are true, i.e., to these new quantities all the theorems of ordinary geometry apply except those that are a consequence of Euclid's postulate.

This terminology was a great assistance to me in my investigations, but, to avoid every obscurity, I shall not use it here" (Ref. 41, p. 306).

Through the felicitous use of Lobachevskiĭ's planimetry, Poincaré, in creating the theory of automorphic functions, overtook his strongest rival F. Kline (1849–1925).

Reading the book of Ref. 7, one can understand how intensively and deeply Poincaré studied Lobachevskiĭ's geometry. Because of this, he not only achieved success in creating the theory of automorphic functions but also constructed in 1905 a rigorous theory of the Lorentz group. However, we have seen above how carefully Poincaré spoke of his sympathies toward Lobachevskiĭ's geometry. Evidently it was not "every obscurity" that he avoided but, like Gauss, he feared the "cry of the Boeotians." His excessive caution, as in the case of Gauss, did not serve him at all well. In 1910, Klein took his revenge, proving that the Lorentz group is isomorphic to the group of isometries of Lobachevskiĭ's space.[42] It is probable that this result did not appear new to Poincaré. Introducing the concept of quadratic geometry, he had proven in 1887: "If the fundamental surface is a two-sheeted hyperboloid, then the quadratic geometry does not differ from Lobachevskiĭ's geometry" (Ref. 43, p. 390).

All the results follow from Eqs. (5) and (6). Indeed, the four functions

$$x=r\sin\theta\cos\varphi,\quad y=r\sin\theta\sin\varphi,$$

$$z=r\cos\theta,\quad u=\mathrm{ch}\,\frac{\rho}{k} \tag{10}$$

of the spherical coordinates ρ, θ, φ define, in the four-dimensional centroaffine space (Ref. 44, p. 46) with Cartesian coordinates x, y, z, u, a three-dimensional surface. This surface is the sheet of the hyperboloid

$$k^2u^2-x^2-y^2-z^2=k^2 \tag{11}$$

on which $u>0$. In Cartesian coordinates, it is given by the equation

$$u=\sqrt{1+(x^2+y^2+z^2)/k^2}. \tag{12}$$

We have in fact just obtained a one-to-one (or, as one now says, bijective) mapping of Lobachevskiĭ's space onto this surface. Differentiating the functions (5) and (10), we obtain

$$dx^2+dy^2+dz^2=dr^2+r^2(d\theta^2+\sin^2\theta\ d\varphi^2),$$

$$k^2du=rd\rho,\quad dr=ud\rho. \tag{13}$$

From this we find that in the coordinates x, y, z the Lobachevskiĭ metric (6) is

$$ds^2=dx^2+dy^2+dz^2-k^2du^2, \tag{14}$$

where du is the differential of the function (12), so that

$$(k^2+x^2+y^2+z^2)k^2du^2=(xdx+ydy+zdz)^2. \tag{15}$$

Therefore, the intrinsic geometry of the surface (12) in the pseudo-Euclidean space with metric (14) is identical to Lobachevskiĭ's geometry. Isometric transformations of Lobachevskiĭ's space are represented by linear transformations of the coordinates x, y, z, u that preserve the quadratic form on the left-hand side of Eq. (11) and do not change the sign of the coordinate u. In accordance with Poincaré's definition (Ref. 45, p. 54), such transformations constitute the Lorentz group.

We arrive at the concept of the velocity space of a particle by assuming that Lobachevskiĭ's constant k is equal to the speed of light c, and the distance ρ is the particle's rapidity s. In such a case, the functions (10) are equal to the components of the particle's 4-velocity. The components of the ordinary velocity of the particle are

$$v_1=c\,\mathrm{th}\,\frac{s}{c}\sin\theta\cos\varphi,\quad v_2=c\,\mathrm{th}\,\frac{s}{c}\sin\theta\sin\varphi,$$

$$v_3=c\,\mathrm{th}\,\frac{s}{c}\cos\theta. \tag{16}$$

In cosmic rays, one observes, and in modern accelerators reaches (for example, at the JINR, Dubna and even greater at the Institute of High Energy Physics at Protvino) rapidities that greatly exceed the speed of light. Therefore, it is impossible to avoid using Lobachevskiĭ's geometry in high-energy physics. The application of Lobachevskiĭ's geometry to the mechanics of the collisions of elementary particles is considered in the review of Ref. 46.

However, we have to hear: "Why do we need to study Lobachevskiĭ if we can calculate everything using the well-known formulas of relativity theory?" To this I answer, following F. Klein (Ref. 42, p. 144): "The modern principle of relativity of the physicists is the same thing as the theory of the Lorentz group. In its turn, the Lorentz group is isomorphic to the group of isometries of Lobachevskiĭ's space." To this, I usually add that the study of Lobachevskiĭ's geometry is repaid a hundredfold, since it leads to a deep understanding of relativity theory, endowing it with methods that are strong and unknown even to the theory and establishes an intimate connection between the theory of relativity and the millennial history of the problem of parallels.[47] Indeed, physicists, often without knowing it themselves, still use Lobachevskiĭ's geometry, but in a poor exposition, i.e., a surrogate. But no advantage is gained—it is better to learn the subject itself rather than its surrogate.

7. INTRODUCTION OF LOBACHEVSKIĬ'S GEOMETRY INTO MECHANICS AND INTO THE LAW OF UNIVERSAL GRAVITATION

Lobachevskiĭ considered the new geometry not only in its logical aspect. He also thought much about its applications in physics, astronomy, and mechanics, and as a result he posed two entirely new problems—the astronomical testing of the geometry of the world seen by us and investigation of the changes that result from the introduction of the new geometry into mechanics. On the basis of data on the parallaxes of stars, he established that in a sphere whose radius is equal to the distance from the earth to these stars the foundations of Euclid's geometry can "... be regarded as rigorously proved. At the same time, one cannot but be attracted to the opinion of Laplace that the stars and Milky Way that we see belong to just one collection of heavenly bodies like those that can be discerned as faintly flickering spots in the constellations of Orion, Andromeda, Capricornus, and others. Thus, quite apart from the fact that in our imagination space can be extended without limit, Nature herself opens up before our eyes distances in comparison with which even the distances from our earth to the fixed stars are as nothing" (Ref. 3, p. 209).

Having allowed the possibility that the new geometry could be realized in the contingent world, Lobachevskiĭ posed the problem of the mechanics of bodies moving in a space hitherto unknown: "It would remain to investigate what kind of changes result from the introduction of the imaginary Geometry into Mechanics and whether we do not find here concepts about the nature of things that are accepted already as indubitable but force us to limit or even not allow at all a dependence between lines and angles. In fact, one can already anticipate that the changes in Mechanics resulting from the new foundations of Geometry will be of the same kind as Laplace demonstrated (Mećanique Céleste. Vol. I, Book I, Chap. II) in assuming that there can be any dependence of the velocity on the force or, if we express ourselves more correctly, in assuming forces that can be measured always by a velocity and satisfy a different law in combination than the adopted composition of them" (Ref. 3, p. 261).

In a following paper, Lobachevskiĭ develops this thought: "... but if there is one thing that one cannot doubt, it is that forces all produce the same thing: motion, velocity, time, mass, even distances and angles... when it is sure that forces depend on the distance, lines can also be in a dependence with angles. At the least, the difference is the same in the two cases, the difference between which resides not so much in the concepts themselves as in the fact that we recognize one dependence from experiments, while the other, observations being inadequate, must be assumed intellectually, either beyond the limits of the visible world or in the intimate sphere of molecular attractions" (Ref. 14, pp. 159–160).

At this point Lobachevskiĭ poses the problem of the *introduction of the new geometry into the theory of gravity*. He assumed that the forces of attraction become weaker by the dispersal of their effect over a sphere, as a consequence of which they must decrease as the area of the sphere, and he pointed out how this area depends on the radius of the sphere in the new geometry. In this way, he introduced a new expression into Newton's law of universal gravitation.

The problem of the motion of bodies moving in Lobachevskiĭ's space was considered in the review of Ref. 48. It was shown there that on the introduction of Lobachevskiĭ's geometry into the visible world the principle of kinematic relativity remains valid, but the special relativity principle ceases to hold. This means that the introduction, after the manner of Newton, of a space absolutely at rest under the Newtonian condition of absoluteness of the time is equivalent to denying Euclid's axiom in the visible world.

8. SOLUTION OF POISSON'S EQUATION IN LOBACHEVSKIĬ'S SPACE

As was shown in Ref. 49, Lobachevskiĭ actually gave the fundamental solution of Poisson's equation in the space with the metric (6) under the condition (5). (In this connection, see also Refs. 50 and 51.).

We shall assume that the attractive center in Lobachevskiĭ's space is absolutely at rest. Let us call it the sun, and the attracted body a planet or comet. We denote the mass of the sun by m. The mass of the attracted body is assumed to be small compared with the mass of the sun and does not occur in Poisson's equation. This equation contains a single parameter, $\alpha=\gamma m$, where γ is Newton's gravitational constant.

In spherical coordinates, Poisson's equation can be written as follows:

$$\Delta U=4\pi\alpha\delta(x)\delta(y)\delta(z), \tag{17}$$

where U is the Newtonian potential (considered in the given case in Lobachevskiĭ's space), Δ is the Laplacian,

$$\Delta=\frac{1}{r^2}\frac{\partial}{\partial\rho}r^2\frac{\partial}{\partial\rho}+\frac{1}{r^2}\left(\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\sin\theta\frac{\partial}{\partial\theta}+\frac{1}{\sin^2\theta}\frac{\partial^2}{\partial\varphi^2}\right),$$

and δ is a generalized Dirac function of the arguments x, y, z equal to (10). The Newtonian potential satisfies the condition

$$\lim_{\rho\to\infty} U=0. \tag{18}$$

In accordance with Lobachevskiĭ's remark, we find that the covector of the force of attraction has the components

$$F_1=-\frac{\partial U}{\partial \rho}=-\alpha/r^2,\quad F_2=-\frac{\partial U}{\partial \theta}=0,\quad F_3=-\frac{\partial U}{\partial \varphi}=0.$$

Therefore

$$U=\frac{\alpha}{k}\left(1-\operatorname{cth}\frac{\rho}{k}\right). \tag{19}$$

The Lagrangian of the attracted body is

$$L=\tfrac{1}{2}v^2-U, \tag{20}$$

when, in accordance with (6),

$$v^2=\dot\rho^2+(\dot\theta^2+\sin^2\theta\dot\varphi^2), \tag{21}$$

and the dot denotes the derivatives with respect to the absolute time t.

9. SOLUTION OF KEPLER'S PROBLEM IN LOBACHEVSKIĬ'S SPACE

In accordance with (19)–(21), we form the Lagrangian equations of motion of the attracted body:

$$\frac{d}{dt}\dot\rho-rr'(\dot\theta^2+\sin^2\theta\dot\varphi^2)+U'=0,$$

$$\frac{d}{dt}(r^2\dot\theta)-r^2\sin\theta\cos\theta\dot\varphi^2=0,$$

$$\frac{d}{dt}(r^2\sin^2\theta\dot\varphi)=0, \tag{22}$$

where

$$r'=\operatorname{ch}\frac{\rho}{k},\quad U'=\alpha/r^2. \tag{23}$$

Since the Lagrangian (20) does not depend explicitly on the time, we have conservation of the energy

$$E=\tfrac{1}{2}(\dot\rho^2+r^2\dot\theta^2+r^2\sin^2\theta\dot\varphi^2)+U, \tag{24}$$

and by virtue of the spherical symmetry we also have conservation of the angular momentum of the attracted body, with components

$$M_1=y\dot z-z\dot y=-r^2(\sin\theta\cos\theta\cos\varphi\dot\varphi+\sin\varphi\dot\theta),$$

$$M_2=z\dot x-x\dot z=-r^2(\sin\theta\cos\theta\sin\varphi\dot\varphi+\cos\varphi\dot\theta),$$

$$M_3=x\dot y-y\dot x=r^2\sin^2\theta\dot\varphi. \tag{25}$$

In fact, this conservation can be readily established directly by differentiating the functions (24) and (25) with respect to the time.

In the given case, the conservation of the angular momentum means that the attracted body moves in a Lobachevskiĭ plane passing through the center of attraction. Without loss of generality, we can assume that this is the equatorial plane $\theta=\pi/2$. For such a choice $M_1=0$, $M_2=0$, $M_3=M$, where

$$M=r^2\dot\varphi, \tag{26}$$

and the energy integral takes the form

$$E=\tfrac{1}{2}(\dot\rho^2+r^2\dot\varphi^2)+U. \tag{27}$$

Substituting (26) in (27), we obtain the following differential equation for the radius ρ as a function of the angle φ:

$$E=\frac{M^2}{2r^2}\left[\left(\frac{d\rho}{d\varphi}\right)^2+r^2\right]+U. \tag{28}$$

To integrate this equation, we write

$$k\operatorname{th}\frac{\rho}{k}=\frac{1}{w}. \tag{29}$$

Since

$$dw=-\frac{1}{r^2}d\rho,\quad w^2=\frac{1}{r^2}+\frac{1}{k^2},$$

Eq. (28) is transformed to

$$E=\frac{M^2}{2}\left[\left(\frac{dw}{d\varphi}\right)^2+w^2-\frac{1}{k^2}\right]+\frac{\alpha}{k}-\alpha w. \tag{30}$$

The solution of this equation is

$$w=\frac{\alpha}{M^2}+\sqrt{\left(\frac{\alpha}{M^2}-\frac{1}{k}\right)^2+\frac{2E}{M^2}}\cos\varphi. \tag{31}$$

The constant of integration is chosen so that the angle $\varphi=0$ corresponds to the largest value of w.

Introducing the notation

$$p=\frac{M^2}{\alpha},\quad \varepsilon=\sqrt{\left(1-\frac{p}{k}\right)^2+\frac{2pE}{\alpha}}, \tag{32}$$

we write the equation of the trajectory of the attracted body in the form

$$k\operatorname{th}\frac{\rho}{k}=\frac{p}{1+\varepsilon\cos\varphi}. \tag{33}$$

It follows from (30) that the radicand in Eq. (32) for ε cannot be negative. Indeed

$$E\geqslant\frac{M^2}{2}(w^2-k^{-2})+\alpha(k^{-1}-w)\geqslant-\frac{\alpha}{2p}\left(1-\frac{p}{k}\right)^2.$$

Therefore, ε is a real number.

We consider in detail finite motion. We shall call the attracted body a planet, and the trajectory of the planet the orbit. It is to finite motion, of course, that the classical laws of Kepler correspond.

In accordance with the definition, for finite motion the radius ρ is bounded for every value of the angle φ. In accordance with (33), this means that for every value of the angle φ the following inequality must hold for finite motion:

$$k>\frac{p}{1+\varepsilon\cos\varphi}. \quad (34)$$

In particular, setting here $\varphi=\pi/2$, we obtain

$$p<k. \quad (35)$$

However, if we set $\varphi=\pi$, we obtain

$$p<k(1-\varepsilon). \quad (36)$$

Thus, finite motion is possible either in the case

$$\varepsilon=0, \quad 0<p<k, \quad (37)$$

or in the case

$$\varepsilon>0, \quad p>0, \quad p+k\varepsilon<k. \quad (38)$$

In the first case, the orbit of the planet is a circle, and in the second it is an ellipse.

10. CIRCULAR MOTION OF A PLANET IN LOBACHEVSKIĬ'S SPACE

The conditions (37) for circular motion are equivalent to the conditions

$$2E=-\alpha^2M^{-2}\left(1-\frac{M^2}{\alpha k}\right)^2, \quad 0<M^2<\alpha k. \quad (39)$$

They can be readily obtained directly from the equations of motion (22). In the equatorial plane, these reduce to the two equations

$$\frac{d}{dt}\dot{\rho}-ru\dot{\varphi}^2+\frac{\alpha}{r^2}=0,$$

$$\frac{d}{dt}(r^2\dot{\varphi})=0. \quad (40)$$

It follows directly from the second equation that the angular momentum (26) is conserved, so that

$$\dot{\varphi}=Mr^{-2}. \quad (41)$$

Substituting this expression in the first of the equations of motion (40) and multiplying the result by $\dot{\rho}$, we obtain the conservation law for the energy

$$E=\tfrac{1}{2}(\dot{\rho}^2+M^2r^{-2})+U. \quad (42)$$

If ρ does not depend on the time (circular motion), then everything simplifies greatly. In accordance with the first of Eqs. (40), in this case

$$\dot{\varphi}^2=\alpha/(ur^3). \quad (43)$$

It then follows from (42) that

$$E=\tfrac{1}{2}M^2r^{-2}+U. \quad (44)$$

From (41) and (43) we find

$$M^2=\frac{\alpha r}{u}=\alpha k\,\mathrm{th}\,\frac{\rho}{k}. \quad (45)$$

For variation of the argument x in the range $0<x<\infty$, the hyperbolic tangent varies in the range $0<\tanh x<1$, so that in accordance with (45) the second of the conditions (39) is satisfied. The first of the conditions (39) follows from Eqs. (5), (19), (44), and (45). It means that in the case of circular motion

$$E=\frac{\alpha}{k}\left(1-\mathrm{cth}\,\frac{2\rho}{k}\right). \quad (46)$$

It is interesting to compare the last formula with Eq. (19).

Note that in accordance with (41) the period of revolution of the planet around the circle is

$$T=\frac{2\pi}{M}r^2. \quad (47)$$

In accordance with (45), we obtain the following expression for the square of the period:

$$T^2=\frac{4\pi^2}{\alpha}r^3u. \quad (48)$$

11. ELLIPTIC MOTION OF A PLANET IN LOBACHEVSKIĬ'S SPACE

The conditions (38) for elliptic motion are equivalent to the conditions

$$-\alpha^2M^{-2}\left(1-\frac{M^2}{\alpha k}\right)^2<2E<0, \quad 0<M^2<\alpha k, \quad (49)$$

as can be readily verified.

It is interesting that the orbit of the planet can be determined in absolute planimetry, i.e., on the Lobachevskiĭ plane in the same way as on the Euclidean plane. In the case of circular motion this is obvious. In the case of elliptic motion, we define the planet's orbit as the set of points from which the sum of the distances to two specified points is given. In absolute geometry, we call such a set an ellipse. We call the specified points foci and denote them by F and $\tilde{F}$. We denote the given sum of distances by $2a$ and call it the length of the major axis of the ellipse. We denote the distances from a point M on the ellipse to the foci by ρ and $\tilde{\rho}$, so that $\rho+\tilde{\rho}=2a$. We denote the distance between the foci by $2c$. We denote the central point between the foci by O and call it the center of the ellipse. The major axis of the ellipse passes through its foci. The minor axis of the ellipse passes through its center at right angles to the major axis. On the minor axis, $\rho=\tilde{\rho}=a$. We denote the length of the minor axis of the ellipse by $2b$. We emphasize once more that with such a definition we have completely left out of consideration the question of the parallelism of straight lines.

We now consider the following question.

From the triangle $MF\tilde{F}$ we find in the case of Euclidean geometry

$$(2a-\rho)^2=(2c)^2+\rho^2-2(2c)\rho\cos(\pi-\varphi), \quad (50)$$

and in the case of Lobachevskiĭ's geometry

$$\mathrm{ch}\,\frac{2a-\rho}{k}=\mathrm{ch}\,\frac{2c}{k}\,\mathrm{ch}\,\frac{\rho}{k}-\mathrm{sh}\,\frac{2c}{k}\,\mathrm{sh}\,\frac{\rho}{k}\cos(\pi-\varphi). \quad (51)$$

After simple manipulations, we obtain in the first case the well-known equation

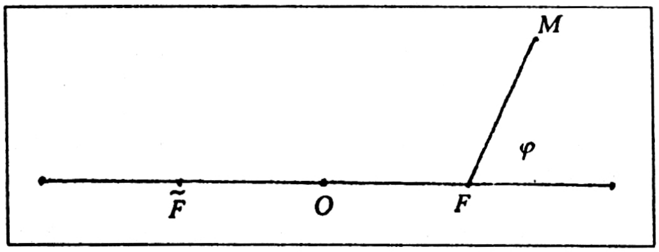

FIG. 1. Major axis of the ellipse with two foci and the center on the axis. The point M lies on the ellipse.

$$\rho=\frac{a^2-c^2}{a+c\cos\varphi}, \tag{52}$$

and in the second the equation (33) considered above, in which

$$p\,\mathrm{sh}\,\frac{2a}{k}=k\left(\mathrm{ch}\,\frac{2a}{k}-\mathrm{ch}\,\frac{2c}{k}\right),\quad \varepsilon\,\mathrm{sh}\,\frac{2a}{k}=\mathrm{sh}\,\frac{2c}{k}. \tag{53}$$

It is easy to show that the p and ε defined in this manner satisfy the conditions (38). However, it is obvious that $p>0$ and that $\varepsilon>0$. Moreover, it is obvious that $\varepsilon<1$. It remains to verify that $p/k=\varepsilon-1<0$. This last inequality follows from the fact that $\cosh x-\sinh x=\mathrm{e}^{-x}$.

It is also interesting to note the following. When the point M is on the minor axis, $MF\tilde{F}$ becomes an isosceles triangle, and MOF a right-angled triangle. From this we find

$$\mathrm{ch}\,\frac{a}{k}=\mathrm{ch}\,\frac{b}{k}\,\mathrm{ch}\,\frac{c}{k}. \tag{54}$$

Substituting this equation in Eq. (53) for p, we obtain

$$p\,\mathrm{th}\,\frac{a}{k}=k\,\mathrm{th}^2\,\frac{b}{k}. \tag{55}$$

It is interesting that the major axis of the ellipse depends only on the energy of the planet. We denote by ρ_1 and ρ_2 the smallest and largest distance from the sun to the orbit of the planet. Obviously, they correspond to the angles $\varphi=0$ and $\varphi=\pi$. In accordance with (33),

$$k\,\mathrm{th}\,\frac{\rho_1}{k}=\frac{p}{1+\varepsilon},\quad k\,\mathrm{th}\,\frac{\rho_2}{k}=\frac{p}{1-\varepsilon}. \tag{56}$$

Since $\rho_1+\rho_2=2a$, it follows that

$$k\,\mathrm{th}\,\frac{2a}{k}=\frac{2p}{1-\varepsilon^2+p^2/k^2}=\frac{\alpha}{\alpha/k-E}, \tag{57}$$

i.e.,

$$E=\frac{\alpha}{k}\left(1-\mathrm{cth}\,\frac{2a}{k}\right). \tag{58}$$

A special case of the formula just obtained is (46).

We now find the period of revolution of the planet around the ellipse. In accordance with (26), it is

$$T=M^{-1}\int_0^{2\pi}r^2(\varphi)\,d\varphi. \tag{59}$$

We find the integrand from (5) and (33):

$$r^2(\varphi)=\frac{p^2}{(1+\varepsilon\cos\varphi)^2-p^2/k^2}=\frac{pk}{2}\left(\frac{1}{1+\varepsilon\cos\varphi-p/k}-\frac{1}{1+\varepsilon\cos\varphi+p/k}\right). \tag{60}$$

The integral can be calculated by the substitution $\xi=\tan(\varphi/2)$. We have

$$\frac{d\varphi}{m+n\cos\varphi}=\frac{2}{\sqrt{m^2-n^2}}\,d\,\mathrm{arctg}\left(\frac{\sqrt{m-n}}{m+n}\,\mathrm{tg}\,\frac{\varphi}{2}\right),$$

$$\int_0^{2\pi}\frac{d\varphi}{m+n\cos\varphi}=\frac{2\pi}{\sqrt{m^2-n^2}}\quad\text{for}\quad m>|n|.$$

Therefore,

$$MT=\pi pk\left(\frac{1}{\sqrt{(1-p/k)^2-\varepsilon^2}}-\frac{1}{\sqrt{(1+p/k)^2-\varepsilon^2}}\right). \tag{61}$$

Substituting (32) in (61), we find

$$T=\frac{\pi k}{\sqrt{2}}\left(\frac{1}{\sqrt{-E}}-\frac{1}{\sqrt{-E+2\alpha/k}}\right), \tag{62}$$

so that the period depends on the energy but does not depend on the angular momentum M. Substituting (57) in (62), we obtain for the square of the period

$$T^2=\frac{4\pi^2}{\alpha}\left(k\,\mathrm{sh}\,\frac{a}{k}\right)^3\mathrm{ch}\,\frac{a}{k}. \tag{63}$$

Let us summarize. The introduction of Lobachevskiĭ's geometry into celestial mechanics does not alter Kepler's first law and introduces readily understood corrections in accordance with Eq. (26) into his second law and in accordance with Eq. (63) into his third law.

Turning now to the final part of the review, we mention that A. A. Fridman (Friedman, Friedmann, 1888–1925), having proposed the model of a nonstationary universe, applied Lobachevskiĭ's idea of introducing non-Euclidean geometry into the theory of gravitation.[52] It is true that he did this in a different scheme, not the one that was conceived by Lobachevskiĭ himself and has been realized in this review.

12. CELEBRATION OF LOBACHEVSKIĬ'S JUBILEE

In 1893, the centenary of the birth of Lobachevskiĭ, the Imperial Kazan University honored this event with a three-day celebration. The program of the festivities was printed in Russian and in French and sent, with an invitation to participate in the festivities, to Russian and foreign universities and other higher establishments of learning, scientific societies, many scientists in both Russia and abroad, and also to distinguished persons in Kazan.

"The Imperial Academy of Sciences, the Academies of Science in Berlin and Vienna, all Russian universities and other higher establishments of learning, all Russian scientists, many foreign universities and societies, gymnasia, and modern schools, and private persons, including some students of Lobachevskiĭ, honoring with reverence his memory, have responded to the invitation of the Kazan University, which has thus had the joy of seeing the universal sincere sympathy with which its intention to honor the memory of the great Russian Geometer has been met" (Ref. 4, p. 11).

To mark the occasion, Henri Poincaré and Felix Klein sent greetings to the Kazan University.

We give the complete text of the following four telegrams and one letter that were received then at the Kazan University.

Telegram (Ref. 4, p. 69)

"*From the Honorary Member of the University, Professor D. I. Mendeleev.*

Geometrical understanding formed the basis of all exact science, and the originality of Lobachevskiĭ's geometry was the dawn of the independent development of the sciences in Russia. The seeds sown by science will be a harvest for the people! Mendeleev."

Letter (Ref. 4, p. 61)

"*From the Russian Turgenev Library in Paris.*

The Russian Turgenev Library in Paris regards itself as fortunate, if only in letter, to participate in the universal celebration of science and express its wonder before the genius of N. I. Lobachevskiĭ, whose memory is honored today by the Kazan University and with it not only the reading, teaching, and learning in Russia but also the entire scientific community of the old and new world.

On behalf of its members, the Library sends from afar warm greetings to all those gathered in the walls of the University to honor the memory of the great compatriot and sincerely wishes greater and greater enlightenment to the Great School, which does not cease to bring forth from among its alumni great men of science.

Director of the Reading Room at the Turgenev Library, Medical Student of the University of Paris, M. Kazanskiĭ."

Telegram (Ref. 4, p. 36)

"*From the Professors of the University of Paris.*

Lobatchefsky a laissé en géométrie une trace glorieuse, imperissable. Tous nous nous asscions a l'honneur rendu à sa mémoire. Nous offrons à cette occasion nos plus cordiales sympathies à l'Université de Kasan et à la science russe.

Hermite, Darboux, Tisserand, Boussinesq, Picard, Poincaré, Appell, Wolf Professeurs à la Sorbonne, Paris."

Telegram (Ref. 4, p. 62)

"*From Honorary Member of the University, Professor of the Göttingen University, F. Klein.*

Zur Centennärfeier des grossen Geometers Lobatschewsky sendet der Universität Kasan Gruss und Dank. Professor Klein."

Telegram (Ref. 4, p. 47)

"*From the Göttingen Royal Society of Sciences.*

Zur Centennärfeier Lobatschewsky's, des meisterhaften Reformators der Euklidischen Geometrie, sendet achtungsvollen Gruss die Königliche Gesellschaft der Wissenschaften zu Göttingen."

It can be seen that the book (frequently cited here) about the celebration of the centenary of the birth of Lobachevskiĭ[4] is of great interest. In view of the bicentenary that is now occurring, this book should be reprinted.

We also mention that Lobachevskiĭ dreamed of our time, and in his "speech on the most important subjects in education" said the following words: "We all live a third or a quarter less than nature stipulated. This is proved by examples—a certain Eccleston lived for 143 years, and a Henry Jenkins for 169 years. Naturalists who compare the time of growing of humans and animals come to the same conclusion—we should, they say, live for about 200 years. But alas, the vital force collects sustenance in vain; it is consumed by the fire of passion and concerns and destroyed by ignorance" (Ref. 9, p. 20).

Lobachevskiĭ will soon be 200 years old. The entire scientific world is preparing to meet this event worthily. From 18 through 22 August, 1992 the International Geometrical Conference dedicated to the 200th anniversary of the birth of Lobachevskiĭ took place at Kazan. The organizing committee included scientists of our country and some foreign countries. The same date was honored by an event that took place at the Joint Institute for Nuclear Research at Dubna from 16 through 18 May, 1992: The fifth seminar "Gravitational Energy and Gravitational Waves," at which new results on the application of Lobachevskiĭ geometry to the modern theory of gravitation were to be presented.[53,54]

APPENDIX: PARALLEL LINES

(Essay of Academician V. Bunyakovskiĭ, St. Petersburg, 1853)

"In the work presented here, I have had the intention of acquainting lovers of Geometry with the steady development and present status of the fundamental question of the theory of parallel lines, which is so important for science. Having followed critically the more or less unsatisfactory attempts of earlier geometers on this subject, I have dwelt on the most recent investigations, which correspond more closely to the aim, and I have subjected them to careful analysis. For completeness of the exposition, I have also reported my own investigations with considerations on this theory, which in part are already known from three individual memoirs that I have written and in part have appeared here for the first time.

1. As the touchstone of Geometry, the theory of parallel lines has continually attracted the attention of geometers. However, despite all efforts to establish it on a completely firm basis, the proofs that have been devised, from Euclid to our times, give rise to objections, which it does not appear to be easy to eliminate entirely. In his Geome-

try, Euclid adopted as an axiom (axiom 11) that when two lines intersect a third and the sum of the internal angles, on one side of the secant, is not equal to two right angles, then the two lines intersect. From this assumption, which is not as obvious as one would require of an axiom, the entire theory of parallel lines can be deduced.

To give the further exposition the greatest possible clarity, we propose here to list some assumptions from each of which one can, with greater or lesser simplicity, arrive at a complete proof of the truths that constitute the doctrine of parallel lines. First, we must agree on their very definition. By *parallel lines* we shall understand *straight lines perpendicular to a given straight line and forming one plane with it.* For brevity, we shall call the part of this given line bounded by the two parallel lines the base of the parallel lines. By virtue of such a definition, we directly conclude that *parallel lines, no matter how far extended, never intersect each other.*

This definition can also be presented in the following form: *two straight lines that lie in one plane and form with a third internal angles whose sum is equal to two right angles are said to be parallel*, and we can then conclude that these lines never meet. It is obvious that the two definitions are equivalent.

We now turn to the propositions that lead to the rigorous proof of all properties of parallel lines; the number of these truths is very appreciable; we restrict ourselves to listing the main ones. The characteristic propositions that we shall discuss can be divided, essentially, into three kinds, namely, propositions relating to: 1) the mutual intersection of straight lines; 2) the properties of angles; 3) the properties of lines in the discussion of their definite length.

Propositions of the first kind.

a) When two lines are intersected by a third, and the sum of the internal angles, on one side of the secant, is not equal to two right angles, then these two lines meet. Euclid's 11th axiom, or Euclid's postulate, consists in this property.

b) Two lines, one inclined to and one perpendicular to a given straight line, always meet if extended sufficiently far.

c) Through a given point one can describe only one line parallel to a given straight line.

d) When a straight line intersects one of two mutually parallel lines, it necessarily intersects the other.

e) The perpendicular erected from any point of one of two mutually parallel straight lines necessarily intersects the other.

f) Through every point taken inside a definite angle one can describe a straight line that intersects both sides of the angle itself.

Propositions of the second kind.

a) A line that cuts one of two mutually parallel lines in a right angle will also be perpendicular to the other.

b) When two parallel lines are intersected obliquely by a third straight line, out of the eight angles formed by the intersection: 1) the four acute angles will be equal to each other; 2) the four obtuse angles will also be equal; 3) each acute angle will be the supplement of two right angles.

c) The sum of the three angles of any right-angled triangle is equal to two right angles.

d) One can construct a triangle such that the sum of its angles will be equal to two right angles.

e) The sum of the three angles of a triangle is constant.

f) There exists a four-sided rectilinear figure in which either all four angles are right angles or their sum is equal to four right angles.

g) Two angles of a triangle determine the third.

Propositions of the third kind.

a) The distances between parallel lines are always equal to each other, or, equivalently, if from all points of a straight line equal perpendiculars are erected, then their ends will also be on one straight line.

b) The distances between two straight lines cannot first increase and then decrease or vice versa.

c) The intersection of two straight lines, or a given plane angle, cannot lead to a line of definite length.

d) The length of a straight line cannot determine an angle.

e) For a given acute angle, one can construct a right-angled triangle such that its side belonging to this angle and to the right angle will be arbitrarily large.

f) The side of a regular hexagon inscribed in a circle is equal to its radius.

We also draw the attention of our readers to the circumstance that in the proofs of the theory of parallel lines it is usual to present two cases, of which one is solved in all rigor, while the other, in whatever form it is represented, always gives rise to difficulties. In a careful examination of the different methods presented below, we recognize that the case whose proof does not give rise to objections invariably depends on the following property: *a straight line A joining the ends of two equal perpendiculars erected to a given line cannot be less than the line B joining their bases*; this truth can be proved in all rigor. In contrast, the proof that *the straight line A cannot be greater than the line B always presented especial difficulties.* These two truths can be replaced by others equivalent to them. For example, one can prove with geometrical accuracy that *the sum of the angles of any right-angled triangle cannot be greater than two right angles*; in contrast, the proof that *this sum cannot be less than two right angles presents the same difficulties as the second of the propositions relating to lines A and B.*

After these preliminary explanations, we turn to examination of the different proofs of the theory of parallel lines. We shall indicate first, in brief words, the essence of the most important methods relating to the time more or less distant from us. The devices of which we shall speak, judging from the responses that have survived to them, were regarded in their time as generally satisfactory from the point of view of rigor. We shall see that none of them can withstand thorough analysis."

[1] I.e., the areas of two-dimensional figures. (Ed.) (Ref. 12, p. 70).

[2] At this point, Lobachevskiĭ uses an obsolete Russian word (*kolikoe*) which the Russian editor explains has a meaning usually rendered by the Russian word for a quantity (*velichina*) (Ref. 12, p. 69).

[3)]Exposition succinte des principes de la Géometrie, avec une démonstration rigoureuse du théorème des parallèles,[4] read at the session of the Physics and Mathematics Section of the Kazan University on February 12, 1826 but never published (note of Lobachevskiĭ) (Ref. 14, p. 147).

[4)]I.e. "Brief exposition of the principles of geometry with a rigorous proof of the theorem of parallel lines" (Ed.) (Ref. 14, p. 455).

[5)]Previously at Marburg, now Professor of Jurisprudence at Königsberg (note by Gauss) (November 28, 1846).

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Translated by Julian B. Barbour