Nucleus-like states in the $SU(2)$ Skyrme model

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The variational approach to the problem of seeking nucleus-like solitons is reviewed. The solution of this problem is the first step in the construction of a nuclear model of chiral solitons. The structure and some properties of solitons with the quantum numbers of the lightest nuclei in the original Skyrme model are discussed. Theoretical analysis reveals the exclusiveness of each individual state, which is manifested both in the structure of the classical solitons and in the strong dependence of the effective quantum Hamiltonian on the topological sector.

INTRODUCTION

In the infrared region of quantum chromodynamics where the hadron spectrum is formed, effective chiral Lagrangians put at our disposal collective degrees of freedom corresponding to observable fields which can be used to describe phenomena in a limited energy range.

Nonlinear chiral Lagrangians naturally lead to soliton sectors. The classical chiral solitons are already very similar to hadrons. The topological stability makes baryon number conservation experimentally observable. Chiral solitons are essentially extended, strongly interacting objects. The soliton size corresponds to the non-pointlike nature of observable particles characterized by strong and electroweak form factors. They are massive objects: their masses are considerably larger than the masses of the quantum fluctuations of the fields appearing in the Lagrangian, in complete correspondence with the observed large ratio of the baryon mass to the π-meson mass.

It is precisely such objects which can serve as classical prototypes of nucleons and nuclei. All the properties of these solitons are based on the chiral symmetry, the presence of which has been confirmed experimentally in a large number of elementary processes involving strongly interacting particles.

We shall restrict ourselves to discussion of a model which possesses chiral $SU(2)_V \otimes SU(2)_A$ symmetry broken down to the vector symmetry $SU(2)_V$ via the mechanism of chiral symmetry breakdown with the appearance of massless pseudoscalar Goldstone excitations with the pion quantum numbers.

The effective boson Lagrangian describes the dynamics of Goldstone bosons and preserves this symmetry. In order to construct such a Lagrangian, we choose as the fundamental variables the elements U of some $SU(2)$ group, the local coordinates of which are identified with the bosonic fields. Left and right transformations are associated with left and right multiplication by elements of this group. The chiral $SU(2)_L \times SU(2)_R$ group of transformations of the field U will correspond to the direct product of left and right transformations of elements of $SU(2)$: $U \to AUB^+$ with arbitrary constant $SU(2)$ matrices A and B. The chirally invariant action density is usually constructed by means of the left-invariant Cartan form $L_\mu = U^+ \partial_\mu U$.

For example, the $SU(2)$ Skyrme model is defined by the following Lorentz scalar:[1]

$$\mathscr{L} = \frac{F_\pi^2}{16} \mathrm{Tr}(L_\mu L_\mu) + \frac{1}{32e^2} \mathrm{Tr}[L_\mu, L_\nu]^2. \tag{1}$$

Here the currents L_μ are expressed in terms of $SU(2)$ matrices

$$U(x) = \exp\left\{ i \frac{2}{F_\pi} \tau^k \pi^k(x) \right\}, \tag{2}$$

determined by the isospin triplet of pion fields $\pi(x)$ and the Pauli τ matrices. The parameters of the model are the dimensional constant F_π and the dimensionless constant e. The Lagrangian is obviously invariant under the global $SU(2)_L \times SU(2)_R$ chiral group of transformations $U(x) \to AU(x)B^+$ with arbitrary constant unitary unimodular matrices A and B. This symmetry corresponds to conservation of left- and right-handed currents.

The success of the Skyrme model[1] in describing nucleons[2–4] and the deuteron[5–15] as the quantum states of solitons of the chiral field makes it natural to apply the model to heavier objects. Such objects are, for example, the lightest nuclei heavier than the deuteron. The first step in describing nuclei using the Skyrme model was taken using the potential approximation.

In this approach one calculates the Skyrmion interaction potential,[9,10] and nuclei are treated as Skyrmion bound states.[15]

However, the potential (adiabatic) approximation is not the only possible approach to the problem of describing nuclei in the Skyrme model. A more direct method of constructing systems with arbitrary baryon number is to search for solitons of the classical equations for the chiral fields with the corresponding topological charge and to quantize the soliton degrees of freedom. In this approach to the theory of nuclear states it must be assumed that the nucleons themselves can be created only near the nuclear surface and, strictly speaking, do not exist in the interior of the nucleus.

Numerical calculations of toroidal soliton configurations with topological charge $B=2$–4 were undertaken in Ref. 16. Recently, significant success has been achieved in searching for nontoroidal solitons with the minimum mass in sectors with $B\leqslant 6$ (Ref. 17) by direct numerical methods. Here it should be noted that the search for these six configurations required 170 h of CPU time on a Cray-2 supercomputer. Moreover, the accuracy of the calculations fell with increasing topological charge. For example, in the calculation of the configuration with baryon number $B=6$ about 0.22 units of baryon charge were lost.

In connection with this it is necessary to develop a variational approach to solving the problem of seeking nucleus-like configurations. In addition, this approach usually makes it easier to understand the nature of the solutions found and to quantize the classical degrees of freedom.

The variational approach was used in Refs. 16–22. The properties of toroidal configurations with baryon number $B\leqslant 4$ were studied. It was shown that the variational ansatz that was used satisfactorily reproduces the results of direct numerical calculations for toroidal configurations. The physically interesting solutions are described in detail in Ref. 19.

Nontoroidal solutions are sought for the following reasons. First, it has not been possible to find stable toroidal solitons with $B\geqslant 4$. This was rather depressing, since the growth of the soliton mass with increasing baryon number was almost linear. On the other hand, it is difficult to expect that nucleus-like solitons will have an exclusively toroidal form, although comparison of the calculated electromagnetic form factors of α-particle-like objects is consistent with this version.[23]

Second, for the quantum states of toroidal solitons with $k\geqslant 3$ studied in Ref. 24 the order of the picture is spoiled by the presence of a rigorous quantum condition on the 3-projection of the spin and isospin $kT_3^{\text{b.f.}}+S_3^{\text{b.f.}}$ in the body-fixed coordinate system. This, in particular, causes the state with quantum numbers $T=1/2$, $S=3/2$ to be the lowest state, the analog of the tritium nucleus in the model.

It is well known[25,26] that the Skyrme model can be viewed as a "broken" series, which arises in the expansion of an effective nonlocal Lagrangian in powers of the pion-field derivative.[27–30] The effective nonlocal Lagrangian itself originates in the QCD generating function, into which collective pseudoscalar variables are introduced with an integration over the quark and gluon variables.[31–38] At first glance, the quality of the Skyrme-model description could deteriorate with increasing baryon number. If, for example, we dealt only with Skyrme–Witten configurations, we would encounter precisely this situation. The derivative of the profile function would grow with increasing baryon number roughly in proportion to the latter.

However, if the Skyrme model satisfactorily reproduces the saturation of the nuclear forces, the radius of the system will increase with increasing baryon number. This leads us to expect that the derivative of the pion field will not be increased, so that there is no danger of going outside the region of applicability of the Skyrme model. The calculations of nucleus-like states which have been carried out[39] convincingly support this picture.

1. GENERAL FORMULATION OF THE PROBLEM

The trivial (chirally asymmetric) solution $U_0(x)=1$ defines the vacuum state. One of the conditions which any solution must satisfy is that the solution $U(\mathbf{x})$ must tend to the vacuum, $U(\infty)=1$, in the asymptotic region. Therefore, we are dealing with maps $U(\mathbf{x})$ of coordinate space R^3 with identified points at infinity (which is topologically equivalent to the sphere S^3) to the sphere S^3 of parameters of the matrix U.

It is possible to construct a trivally conserved topological (Noether) current[1]

$$I_\mu^B=-\frac{1}{24\pi^2}\,\varepsilon_{\mu\nu\rho\sigma}\,\mathrm{Tr}(L_\nu L_\rho L_\sigma) \tag{3}$$

with charge

$$B=\int I_0^B(x)d\mathbf{x}, \tag{4}$$

classifying such maps. The topological charge, which is a homotopic invariant, is conserved independently of the dynamics of the system. In the Skyrme model the topological charge is identified with the baryon number.

It is known that the stationary solution in the sector with unit topological charge is the spherically symmetric solution or "hedgehog" configuration[40]

$$U(\mathbf{x})=\exp\{i\boldsymbol{\tau}\cdot\mathbf{N}F(r)\}, \tag{5}$$

where $r=|\mathbf{r}|$ is the modulus of the radius vector $\mathbf{r}$ and $\mathbf{N}=\boldsymbol{\pi}/|\boldsymbol{\pi}|=\mathbf{r}/|\mathbf{r}|$ is the unit isotopic vector correlated with the space vector $\mathbf{n}=\mathbf{r}/|\mathbf{r}|$. This configuration is referred to as the Skyrme–Witten ansatz. We choose the field of the isotopic vector $\mathbf{N}$ in a form sufficiently general for our purposes:

$$\mathbf{N}=\begin{cases}\sin T(\theta)\cos\Phi(\theta,\varphi),\\ \sin T(\theta)\sin\Phi(\theta,\varphi),\\ \cos T(\theta),\end{cases} \tag{6}$$

where the functions Φ and T are arbitrary functions of the angles θ and φ in spherical coordinates determining the local orientation of the isotopic vector $\mathbf{N}$.

In Refs. 14 and 20 this ansatz was used to search for toroidal solitons. It was shown that this ansatz leads to a series of physically interesting solutions not only in the baryon sector, but also in the topologically trivial sector (Refs. 19, 20, 41, and 42).

The computation of the Lagrangian

$$\mathscr{L}=-\frac{F_\pi^2}{8}[(\partial_\mu\mathbf{N}\cdot\partial_\mu\mathbf{N})\sin^2F(x)+\partial_\mu F(x)\partial_\mu F(x)]$$
$$-\frac{1}{4e^2}[(\partial_\mu\mathbf{N}\cdot\partial_\mu\mathbf{N})^2-(\partial_\mu\mathbf{N}\cdot\partial_\nu\mathbf{N})(\partial_\mu\mathbf{N}$$
$$\cdot\partial_\nu\mathbf{N})]\sin^4F(x)-\frac{1}{4e^2}[(\partial_\mu\mathbf{N}\cdot\partial_\mu\mathbf{N})\partial_\nu F(x)$$
$$\cdot\partial_\nu F(x)-(\partial_\mu\mathbf{N})\cdot(\partial_\nu\mathbf{N})\partial_\mu F\partial_\nu F]\sin^2F \qquad (7)$$

does not involve any serious difficulties. Therefore, here we restrict ourselves to several formal remarks. This representation is valid for any fields of the unit vector **N** and arbitrary shape function $F(\mathbf{x})$. The constants F_π and e are treated as arbitrary parameters, although F_π in the boson sector has the meaning of the pion decay constant, and e determines the amplitude of the $\rho\to2\pi$ transition.

The expression $(\partial_k\mathbf{N}\cdot\partial_k\mathbf{N})$ written in terms of the functions F, T, and Φ, reduces to the following:

$$(\partial_k\mathbf{N}\cdot\partial_k\mathbf{N})=\frac{1}{r^2}\left\{\left(\frac{\partial T}{\partial\theta}\right)^2+\left(\frac{\partial\Phi}{\partial\theta}\right)^2\sin^2T+\frac{\sin^2T}{\sin^2\theta}\left(\frac{\partial\Phi}{\partial\varphi}\right)^2\right\}, \qquad (8)$$

whereas for the square of the vector product $[\partial_k\mathbf{N}\times\partial_i\mathbf{N}]^2$ we have

$$[\partial_i\mathbf{N}\times\partial_k\mathbf{N}]^2=\frac{2}{r^4}\frac{\sin^2T}{\sin^2\theta}(\Phi')^2(T')^2, \qquad (9)$$

and, since the vector **N** is independent of the variable r,

$$n_i(\partial_i\mathbf{N}\cdot\partial_k\mathbf{N})n_k=0. \qquad (10)$$

Therefore, for the Lagrangian of the static field configurations we obtain

$$\mathscr{L}=\int\mathscr{L}_2d^3x+\int\mathscr{L}_4d^3x, \qquad (11)$$

where

$$\mathscr{L}_2=-\frac{F_\pi^2}{8}\left\{\left(\frac{\partial F}{\partial r}\right)^2+\left[\frac{\sin^2T}{\sin^2\theta}\left(\frac{\partial\Phi}{\partial\varphi}\right)^2+\left(\frac{\partial T}{\partial\theta}\right)^2+\left(\frac{\partial\Phi}{\partial\varphi}\right)^2\sin^2T\right]\frac{\sin^2F}{r^2}\right\} \qquad (12)$$

and

$$\mathscr{L}_4=-\frac{1}{2e^2}\frac{\sin^2F}{r^2}\left\{\frac{\sin^2T}{\sin^2\theta}\left(\frac{\partial T}{\partial\theta}\right)^2\left(\frac{\partial\Phi}{\partial\varphi}\right)^2\frac{\sin^2F}{r^2}+\left[\frac{\sin^2T}{\sin^2\theta}\left(\frac{\partial\Phi}{\partial\varphi}\right)^2+\left(\frac{\partial T}{\partial\theta}\right)^2+\sin^2T\left(\frac{\partial\Phi}{\partial\theta}\right)^2\right]\times\left(\frac{\partial F}{\partial r}\right)^2\right\}. \qquad (13)$$

Let us minimize approximately the functional that we have obtained. From this point of view as the functions $\Phi(\theta,\varphi)$ we shall consider only functions of the form

$$\Phi(\theta,\varphi)=k(\theta)\varphi+\text{const}, \qquad (14)$$

where

$$k(\theta)=\sum_i k_i\{\Theta(\theta_{i+1}-\theta)-\Theta(\theta_i-\theta)\}. \qquad (15)$$

Here Θ is the Heaviside step function. These functions obviously satisfy the equation

$$\partial^2\Phi/\partial\varphi^2=0 \qquad (16)$$

in the regions $\theta_i<\theta<\theta_{i+1}$. We immediately find the following condition for the boundary values:

$$T(\theta_i)=\pi l_i. \qquad (17)$$

This is obtained from the condition that the coefficients of the singular functions vanish. In general, the function $k(\theta)$ must satisfy the condition that the vector **N** be single-valued throughout space. In order to ensure that the solution $\mathbf{N}(\theta,\varphi)$ is single-valued, we must require that the function $k(\theta)$ be an integer except, perhaps, at points θ_i, where the function $T(\theta)$ is a multiple of π. The points $T=\pi i$ correspond to the south or north pole of the sphere in the isotopic coordinate system. At these points the vector **N** has components $\{0,0+1\}$ or $\{0,0,-1\}$, and the function $k(\theta)$ effects a rotation of the vector **N** only about the 3-axis. Therefore, $k(\theta)$ is arbitrary at the points θ_i. The range $[0,\pi]$ in which the functions $T(\theta)$ and $k(\theta)$ are defined is therefore divided into segments $[\theta_0,\theta_1),(\theta_1,\theta_2),\dots,(\theta_l-1,\theta_l]$, where the boundary points are determined from the conditions

$$T(\theta_i)=\pi i,\quad 0\leqslant i\leqslant l,\quad \theta_0=0,\quad \theta_k=\pi. \qquad (18)$$

The piecewise-constant function

$$k(\theta)=\begin{cases}k_1, & 0\leqslant\theta<\theta_1,\\ k_2, & \theta_1<\theta<\theta_2,\\ \dots\dots\dots & \\ k_l, & \theta_{l-1}<\theta\leqslant\pi\end{cases} \qquad (19)$$

is determined by the set of integers k_i.

Let us consider the terms in the functional $\mathscr{L}$ containing $(\partial\Phi/\partial\theta)^2$:

$$\sim\int_0^\pi\sin^2T(\theta)\left(\frac{\partial\Phi}{\partial\theta}\right)^2\sin\theta d\theta. \qquad (20)$$

This integral vanishes if

$$T|_{\theta\to\theta_i}\sim\pi i+\alpha(\theta_i-\theta)^m,\quad m\geqslant1. \qquad (21)$$

It is not difficult to find the equations for the functions $T_i(\theta)$ on each interval (θ_{i-1},θ_i): dropping the terms in $\mathscr{L}$ proportional to the integral (20), we obtain

$$\mathscr{L}_2=-\frac{F_\pi^2}{8}\left\{\left(\frac{\partial F}{\partial r}\right)^2+\left[k^2(\theta)\frac{\sin^2 T}{\sin^2\theta}+\left(\frac{\partial T}{\partial\theta}\right)^2\right]\frac{\sin^2 F}{r^2}\right\}, \tag{22}$$

$$\mathscr{L}_4=-\frac{1}{2e^2}\frac{\sin^2 F}{r^2}\left\{k^2(\theta)\frac{\sin^2 T}{\sin^2\theta}\left(\frac{\partial T}{\partial\theta}\right)^2\frac{\sin^2 F}{r^2}+\left[k^2(\theta)\frac{\sin^2 T}{\sin^2\theta}+\left(\frac{\partial T}{\partial\theta}\right)^2\right]\left(\frac{\partial F}{\partial r}\right)^2\right\} \tag{23}$$

for the function $k(\theta)$ of the form (19). Integrating (22) and (23) over the volume and changing to the dimensionless variable $x=F_\pi er$, we obtain the mass of the stationary configuration for a particular value of the number l:

$$M=\sum_{i=1}^{l}M_i=\pi\frac{F_\pi}{e}\sum_{i=1}^{l}(a_1A_i+b_iB_i+C_i). \tag{24}$$

Here M_i are the contributions to the total classical mass from the energy concentrated in the spatial region

$$\{0\leqslant r<\infty,\ 0\leqslant\varphi<2\pi,\ \theta_{i-1}<\theta<\theta_i;\ i=1,\dots,l\}.$$

The quantities a_i, b_i and A_i, B_i, C_i are given by the integrals

$$a_i=\int_{\theta_{i-1}}^{\theta_i}\left[k_i^2\frac{\sin^2 T_i}{\sin^2\theta}+(T'_i)^2\right]\sin\theta d\theta,$$

$$b_i=k_i^2\int_{\theta_{i-1}}^{\theta_i}\frac{\sin^2 T_i}{\sin^2\theta}(T'_i)^2\sin\theta d\theta, \tag{25}$$

$$A_i=\int_0^\infty\sin^2 F_i\left[\frac{1}{4}+(F'_i)^2\right]dx,$$

$$B_i=\int_0^\infty\frac{\sin^4 F_i}{x^2}dx, \tag{26}$$

$$C_i=\int_0^\infty(F'_ix)^2dx,\quad i=1,\dots,l,\quad \theta_0=0,\quad \theta_l=\pi.$$

The prime is used to denote the following derivatives:

$$\Phi'_i=\frac{\partial\Phi_i}{\partial\varphi},\quad T'_i=\frac{\partial T_i}{\partial\theta},\quad F'_i=\frac{\partial F_i}{\partial x}. \tag{27}$$

From the condition that the classical mass be finite we can conclude that the functions $F_i(x)$ must satisfy the conditions

$$F_i(0)=\pi n_i,\quad F_i(\infty)=0. \tag{28}$$

We therefore obtain the equations for the functions F_i and T_i following from the minimization of this functional:

$$[x^2+2a_i\sin^2 F_i]F''_i+2xF'_i+\left[a_i(F'_i)^2-\frac{a_i}{4}-2b_i\frac{\sin^2 F_i}{x^2}\right]\sin(2F_i)=0,\quad 0\leqslant x\leqslant\infty \tag{29}$$

$$2\left[A_i+k_i^2B_i\frac{\sin^2 T_i}{\sin^2\theta}\right]T''_i-k_i^2[A_i-B_i(T'_i)^2]\frac{\sin(2T_i)}{\sin^2\theta}+2T'_i\cot\theta\left[A_i-k_i^2B_i\frac{\sin^2 T_i}{\sin^2\theta}\right]=0,$$

$$\theta_{i-1}<\theta<\theta_i,\quad i=1,\dots,l. \tag{30}$$

It is not difficult to determine the behavior of the function $F_i(x)$ for $x\to 0$ and $x\to\infty$, of $T_1(\theta)$ for $\theta\to 0$, and of $T_e(\theta)$ for $\theta\to\pi$. For example, near the origin

$$F(x)|_{x\to 0}\sim\pi n-\alpha x^p, \tag{31}$$

and $F(x)$ behaves asymptotically as a polynomial (for $m_\pi=0$) (Ref. 18): $F(x)|_{x\to\infty}\sim\beta/x^{p+1}$ with the same value

$$p=\frac{\sqrt{1+2a}-1}{2}. \tag{32}$$

For the function $T(\theta)$ near the boundaries of its domain of definition we have

$$T(\theta)|_{\theta\to 0}\sim\theta^k, T(\theta)|_{\theta\to\pi}\sim\pi l-(\pi-\theta)^k. \tag{33}$$

Thus, all solutions $U_{\{n,k\},l}(\mathbf{x})$ will be characterized by a set of integers $n_1,\dots,n_l$. Let us now calculate the baryon-number density distribution:

$$J_0^B(r,\theta,\varphi)=-\frac{1}{2\pi^2}\frac{\sin^2 F(r)}{r^2}\frac{\partial F(r)}{\partial r}\frac{\sin T}{\sin\theta}\frac{\partial T}{\partial\theta}\frac{\partial\Phi}{\partial\varphi}(\theta,\varphi). \tag{34}$$

For the solutions discussed above we immediately obtain

$$J_0^B(r,\theta,\varphi)=-\frac{1}{2\pi^2}\frac{\sin^2 F(r)}{r^2}\frac{\partial F}{\partial r}k(\theta)\frac{\sin T(\theta)}{\sin\theta}\frac{\partial T(\theta)}{\partial\theta}. \tag{35}$$

Integrating Eq. (35) over the coordinates r, θ, and φ, we obtain the baryon number for solutions characterized by the number l:

$$B=-\sum_{m=1}^{l}(-1)^m n_m k_m. \tag{36}$$

In deriving the last expression we have used the boundary conditions on the function $F(r)$ and the form of the function $k(\theta)$ (19).

Let us discuss Eq. (36) in more detail. To simplify matters we take $n_m=n$ $(m=1,\dots,l)$:

$$B=-n\sum_{m=1}^{l}(-1)^m k_m. \tag{37}$$

It is obvious that for $k_m=k$ $(m=1,\dots,l)$ the baryon number can be written as

$$B=\begin{cases}nk & \text{for odd } l,\\ 0 & \text{for even } l\end{cases} \tag{38}$$

in agreement with Ref. 18. Therefore, setting $k_m<0$ for even m and $k_m>0$ for odd m, we obtain configurations with positive baryon charge. Obviously, in the general case (36) we must require that

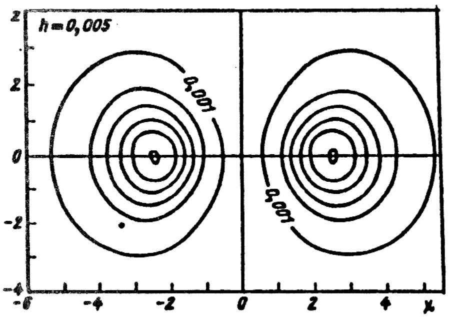

FIG. 1. Contour lines of the baryon-number density of the $3S$ configuration.

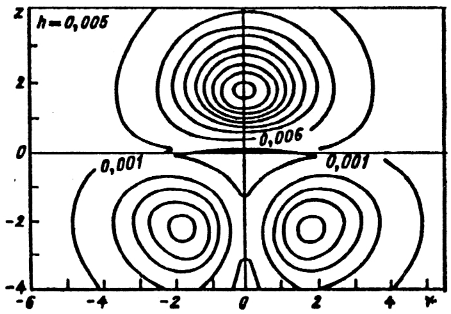

FIG. 2. Contour lines of the baryon-number density of the $S2S$ configuration.

$$n_m k_m > 0 \quad \text{for} \quad \text{odd } l,$$

$$n_m k_m < 0 \quad \text{for} \quad \text{even } l. \tag{39}$$

Below, however, we shall consider only the $n_m = 1$ solutions ($m = 1, \ldots, l$), since we are interested in solutions with a linear (or nearly linear) dependence of the classical mass on the baryon number.

2. MASSES OF CLASSICAL NUCLEUS-LIKE CONFIGURATIONS

We see from the system of equations (29) and (30) that the sign of k does not affect the solutions $F_i(x)$ and $T_i(x)$. Therefore, the classical masses of the resulting soliton configurations will correspond to the classical masses of soliton–antisoliton configurations from Refs. 20 and 24, in which all (or some) of the antisolitons are replaced by solitons. In addition, the boundary points θ_i for the classical analogs of nuclei in this case will be determined by the expression

$$\theta_i = \pi i / l, \quad i = 1, \ldots, l-1. \tag{40}$$

In general, when $\neq |k_m|$, the points θ_i cannot be found from Eq. (40). They can be determined by using the condition for a minimum of the mass functional, i.e., they must take values for which the mass of the classical configuration is a minimum for solutions of the system of equations (29) and (30). However, below we will not find the exact values of the boundary points, but will restrict ourselves to only qualitative arguments, taking Eq. (40) as our first approximation. In this case the geometric shapes of the solitons are changed only quantitatively but not qualitatively. The masses of the classical configurations which will be obtained below should be viewed as estimates obtained by variational analysis. If for these solutions $|k_m| = \text{const}$, for all $i = 1, \ldots, l$, Eq. (40) is exact, so that the accuracy of the masses will be determined solely by the accuracy of the numerical integration.

Let us consider one of the possible solutions with baryon charge $B = 3$. This solution is characterized by the numbers

$$n = 1, \quad l = 2, \quad k_1 = 1, \quad k_2 = -2 \quad (\text{or } k_1 = 2, \quad k_2 = -1).$$

In this case the function $k(\theta)$ has the form

$$k(\theta) = \begin{cases} 1, & 0 \leqslant \theta < \theta_1, \\ -2, & \theta_1 < \theta \leqslant \pi, \end{cases} \tag{41}$$

where θ_1 is the boundary point, which we will determine from the condition (40): $\theta_1 = \pi/2$. In the terminology of Ref. 24, this configuration is referred to as the $S2S$ configuration. The baryon-number density distributions in the x,y plane are shown in Figs. 1–3 for three configurations $3S$, $S2S$, and SSS with $B = 3$. The masses of these solitons calculated from Eqs. (24)–(26) are $M_{\text{cl}} = 34.585$ for $3S$, $M_{\text{cl}} = 35.947$ for $S2S$, and $M_{\text{cl}} = 46.332$ for SSS. In the $B = 4$ sector there are six different possible soliton shapes. Their classification and the corresponding masses are given in Table I. In Figs. 4–9 we show the contour lines of the baryon-number density in the (x,z) plane for all these configurations.

The kS configuration shown in Fig. 4 for $k = 4$ is a spread-out torus. The more complicated SkS configuration consists of a single deformed soliton with baryon number $B = 1$ and a soliton with toroidal form of the baryon-number distribution with $B = k$. In Fig. 5 we show such a configuration for $k = 3$. As seen from Eq. (35), all solitons are assumed to be axially symmetric. Here all of space is split into regions bounded by cones embedded in each other. The distributions of baryon charges concentrated in interior regions of space always have the shape of a (deformed) spread-out torus. In the inner regions of the soliton they have the shape of a torus only for $k \neq 1$. For example, the $SS2S$ configuration (see Fig. 7) in the upper

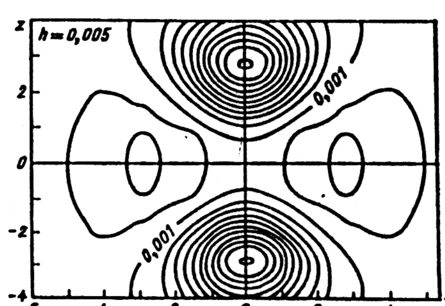

FIG. 3. Contour lines of the baryon-number density of the SSS configuration.

TABLE I. Classical masses of solitons with topological charge $B=4$.

Structure	Classification in the numbers	Classical mass in units of $\pi E_{\pi}/e$
$4S$	$n=1, l=1, k=4$	47,675
$S3S$	$n=1, l=2, k=\{1, -3\}$	46,529
$2S2S$	$n=1, l=2, k=\{2, -2\}$	45,536
$SS2S$	$n=1, l=3, k=\{1, -1, 2\}$	54,521
$S2SS$	$n=1, l=3, k=\{1, -2, 1\}$	57,175
$SSSS$	$n=1, l=4, k=\{1, -1, 1-1\}$	71,169

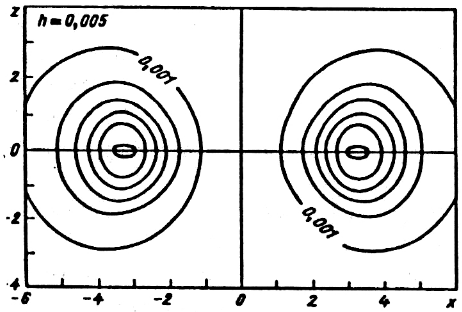

FIG. 4. Contour lines of the baryon-number density of the $4S$ configuration.

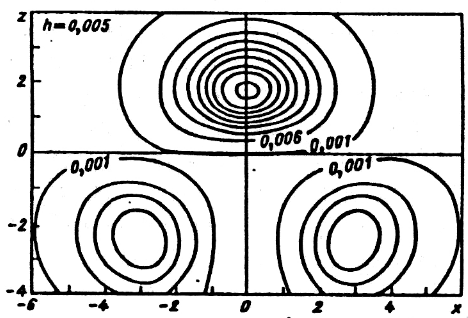

FIG. 5. Contour lines of the baryon-number density of the $S3S$ configuration.

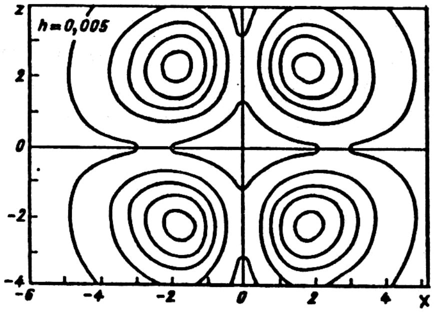

FIG. 6. Contour lines of the baryon-number density of the $2S2S$ configuration.

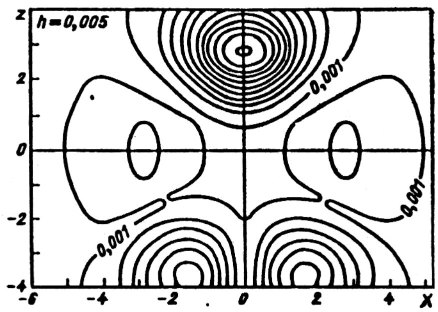

FIG. 7. Contour lines of the baryon-number density of the $SS2S$ configuration.

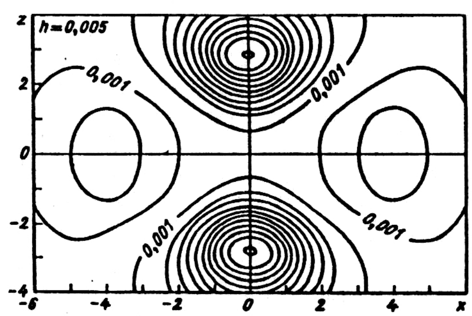

FIG. 8. Contour lines of the baryon-number density of the $S2SS$ configuration.

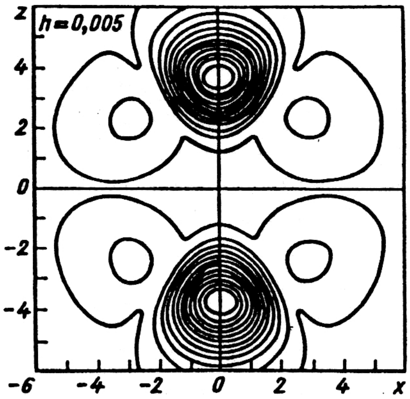

FIG. 9. Contour lines of the baryon-number density of the $SSSS$ configuration.

TABLE II. Classical masses of toroidal Skyrmions (in units of $\pi F_\pi/e$).

B	1	2	3	4	5
M	11,605	22,458	34,585	47,675	61,569

region of space has the shape of a deformed sphere (S), and in the lower region it has the shape of a deformed torus ($2S$).

According to our calculations, the masses of solitons with identical topological charge differ strongly. We see from Table I that for $B=4$ the object of lowest mass is that consisting of two spread-out tori $2S2S$. This configuration is stable with respect to breakup into components with a baryon-number $B\leqslant 3$. The quantum state of the $2S2S$ soliton is a possible candidate for an α particle.

The masses of stationary configurations with a baryon-number distribution of toroidal shape are given in Table II. In Table III we give the lowest masses of Skyrmions with baryon-number up to $B=12$. It is easy to see that they correspond to an almost linear dependence of the classical mass on the baryon number. It might be thought that, upon being quantized, the solitons whose masses are given in Table II can be interpreted as physical nuclear states. Some of the states can have isomer states differing in the shape of the baryon-number distribution (and also in the shape of the energy distribution). The rms radii given in Table III are seen to grow with increasing baryon-number roughly as $B^{1/2}$.

3. CLUSTERS IN THE NUCLEAR MODEL OF CHIRAL SOLITONS

With the development of accelerators and experimental techniques for studying nuclei far from the β-stability line, greater possibilities have been opened up for studying nuclei with an unusual ratio of the number of protons to the number of neutrons. It has become possible to study highly excited collective nuclear states "buried" in the continuum. The ability to obtain such unstable nuclei and to separate, identify, and scatter them on stable nuclei makes it possible to study nuclei of anomalously large dimensions together with weakly coupled structures on their surface. Experimental data on the fragmentation of incident nuclei will probably lead to conclusions about the general structure of nuclei.

The theoretical approaches to the interpretation of these data fall into two categories. In one, already existing approximations are used to describe the data. In the other, new models are proposed for them. For example, there are some indications that traditional approaches like the shell model cannot reproduce these data. These models underestimate the nuclear cross sections and radii obtained from measurements on light targets for neutron-enriched nuclei.[43–45]

In Refs. 46 and 47 microscopic calculations were carried out for the ^{6}He, ^{6}Li, and ^{6}Be nuclei using the $\alpha+2N$ model with pair potentials fitted to low-energy two-particle scattering.

Two possible configurations of the nuclear-density spatial distributions have been obtained. The two-nucleon type of distribution corresponds to a compact pair of neutrons located on one side of the α particle. The other cigar-shaped distribution corresponds to neutrons localized on opposite sides of the α particle.

The rms radii of the nucleon distributions are approximately 2.6 F, i.e., they reproduce the data of Ref. 48.

The variational approach formulated in the preceding section can be used to study possible structures for Skyrmions with $B=6$.

Analysis shows that the toroidal Skyrmions with $B\leqslant 4$ studied earlier can be represented in the structure of heavier Skyrmions. However, the structure of even light objects with $B=4$ and $B=6$ is fairly complicated. An object with $B=4$ similar to the α particle can consist of, for example, one toroidal soliton with $B=2$ and two solitons with $B=1$ (see Fig. 7). We denote this configuration by $U_{13\{1,1,2\}}$. Another possible configuration is $U_{12\{1,3\}}$, which consists of one toroid with $B=3$ and a soliton with $B=1$ (see Fig. 5). The other three possible configurations are $U_{11\{4\}}$ (see Fig. 4), consisting of one soliton with $B=4$, $U_{14\{1,1,1,1\}}$, consisting of two apple-shaped solitons and two toroidal solitons, each with $B=1$ (see Fig. 9), and $U_{12\{2,2\}}$ with the lowest mass, consisting of two toroidal solitons with $B=2$ (see Fig. 6).

The rms radii of these configurations are given in Table IV.

About 15 different configurations can be manifested in the structure of the six-baryon system. We shall consider only the most interesting of them, which are given in Table V.

Just as for $B=4$, the configuration represented by two toroids $U_{12\{3,3\}}$ is the most compact. Its rms radius differs greatly from those of other configurations containing non-toroidal solitons. It should be noted that toroidal solitons

TABLE III. Smallest masses in units of $\pi F_\pi/e$ and rms radii of solitons for topological sectors with $B\leqslant 12$ [in units of $(F_\pi e)^{-1}$].

B	1	2	3	4	6	8	9	12
M	11,605	22,458	34,585	45,536	66,701	89,310	103,08	134,45
$\langle r^2\rangle^{1/2}$	2,12	2,829	3,497	3,841	4,613	5,353	5,610	6,336

TABLE IV. Rms radii of the baryon-number density distributions in units of $(F_\pi e)^{-1}$) for configurations with $B=4$.

$U_{nl\{k(d)\}}$	11{4}	12{1,3}	1,2{2,2}	13{1,1,2}	14{1,1,1,1}
$<r^2>^{1/2}\cdot B^{1/2}$	8,26	8,54	7,68	8,50	8,19

with $B=1$ appear in the interior of multi-Skyrmions. It is especially important to stress the fact that two forms of α particle are manifested in the interior of objects with $B=6$. One is the most compact of the two toroids, and the other is the "broad" toroid with $B=4$.

The dimensions of the configurations $U_{13\{1,1,4\}}$ (of the dinucleon type) and $U_{13\{1,4,1\}}$ (of the cigar-shaped type) are practically the same. Their rms radii calculated with the constants $F_\pi=109.45$ MeV and $e=4.138$ are considerably closer to those measured experimentally in Ref. 48 than is the radius of the configuration $U_{14\{1,1,2,2\}}$. This result can be interpreted as evidence that the α particle in the Li structure can be in an excited state. The values of the constants F_π and e given above correspond to the values at which the smallest masses of the classical solitons with $B=4$ and $B=12$ correspond to the masses of the ^{4}He and ^{12}C nuclei.

4. QUANTUM PROPERTIES OF NUCLEUS-LIKE STATES

Let us obtain the effective quantum-mechanical Hamiltonian by the method of introducing Bogolyubov collective variables.[49] We restrict ourselves to configurations with a symmetric distribution of the energy (mass) density in the (x,y) plane. This means that from the class of all solutions considered, characterized by the numbers $n,l,\{k\}_1^l$, we choose only the solutions satisfying the condition

$$k_i=k_{l+1-i},\quad i=1,...,(l-1)/2 \quad \text{for odd } l,$$
$$k_i=-k_{l+1-i},\quad i=1,...,l/2 \quad \text{for even } l. \tag{42}$$

For such configurations no additional difficulties arise with the separation of the center-of-mass motion. These configurations exhaust almost the entire class of nucleus-like states.

It should be noted[50] that the energy of the system is not changed if the static solution $U_0(\mathbf{x})=\sigma(\mathbf{x})+\mathrm{i}[\boldsymbol{\tau}\cdot\boldsymbol{\phi}(\mathbf{x})]$ is transformed as

$$U_0(\mathbf{x})\rightarrow\sigma(\mathbf{x}')+\mathrm{i}\tau^i I^{ij}\phi^j(\mathbf{x}'),\quad x_n'=(R^{-1})_{nm}x_m, \tag{43}$$

where I and R are constant orthogonal 3×3 matrices. The matrix I specifies the rotation in isotropic space, and R specifies that in coordinate space. The parameters of the rotation matrices I and R can serve as collective variables describing the rotational degrees of freedom. Therefore, the time-dependent field configurations including these motions are[51,52]

$$U(\mathbf{x},t)=\sigma(\mathbf{x}')+\mathrm{i}\tau^i I^{ij}(t)\phi^j(\mathbf{x}'),$$
$$x_n'(t)=(R^{-1})_{nm}(t)x_m. \tag{44}$$

The introduction of the collective variable corresponding to monopole vibrations is somewhat less well justified. However, the first calculations including monopole vibrations already revealed that the latter have a very important influence on the baryon properties.[18,24]

The collective variable corresponding to monopole vibrations is introduced by the scale transformation $\mathbf{x}\rightarrow\mathbf{x}\exp\{\lambda(t)\}$ in the stationary field configuration $U(\mathbf{x})$. Therefore, it is more general to assume that the form of the time-dependent solution is that in (44), where

$$x_n'(t)=(R^{-1})_{nm}(t)x_m\mathrm{e}^{\lambda(t)}. \tag{45}$$

The time-dependent scalar parameter $\lambda(t)$ corresponds to uniform density fluctuations of the observed quantities.

Let us calculate the Lagrangian in which the time components of the currents L_0 now play an important role:

$$L=-M(\lambda)+\frac{F_\pi^2}{16}\int\mathrm{Tr}(L_0L_0)d\mathbf{r}$$
$$+\frac{1}{16e^2}\int\mathrm{Tr}[L_0,L_k]^2d\mathbf{r}. \tag{46}$$

The part of the Lagrangian independent of the time derivatives has the form

$$-M(\lambda)=-\mathrm{e}^{-\lambda}M_2-\mathrm{e}^{\lambda}M_4, \tag{47}$$

where M_2 and M_4 are defined by (12) and (13).

For the time-dependent components it is easy to obtain the expressions for $(\mathbf{L}_0\cdot\mathbf{L}_0)$ and $[\mathbf{L}_0\times\mathbf{L}_k]^2$ in terms of the fields σ and $\boldsymbol{\phi}$:

$$(\mathbf{L}_0\cdot\mathbf{L}_0)=\partial_0\sigma\partial_0\sigma+(\partial_0\boldsymbol{\phi}\cdot\partial_0\boldsymbol{\phi}), \tag{48}$$

TABLE V. Calculated soliton masses in units of $\pi F_\pi/e$ and rms radii of the baryon-number distribution in units of $(F_\pi e)^{-1}$ for solitons with $B=4$.

$U_{nl\{k(d)\}}$	12{3,3}	12{2,4}	13{1,1,4}	13{1,4,1}	14{1,1,2,2}
M_U	66,69	67,41	72,41	82,30	88,88
$<r^2>^{1/2}\cdot B^{1/2}$	11,3	12,00	13,7	13,67	12,45

$$[\mathbf{L}_0\times\mathbf{L}_k]^2=\partial_0\sigma\partial_0\sigma(\partial_k\boldsymbol{\phi}\cdot\partial_k\boldsymbol{\phi})+\partial_k\sigma\partial_k\sigma(\partial_0\boldsymbol{\phi}\cdot\partial_0\boldsymbol{\phi})$$
$$-2\partial_0\sigma\partial_k\sigma(\partial_0\boldsymbol{\phi}\times\partial_k\boldsymbol{\phi})+[\partial_0\boldsymbol{\phi}\times\partial_k\boldsymbol{\phi}]^2. \quad (49)$$

Differentiation immediately gives

$$\partial_0\sigma=\mathbf{i}F'\sin Fs\dot{\lambda}, \quad (50)$$

$$\partial_0\boldsymbol{\phi}=-\mathbf{i}\{[\mathbf{N}\times\boldsymbol{\omega}]+\bar{\partial}_i\mathbf{N}(-\varepsilon_{ijk}\Omega_k+\delta_{ij}\dot{\lambda})S_j\}\sin F$$
$$-\mathbf{i}F'\cos F\mathbf{N}S\dot{\lambda}. \quad (51)$$

Here $\bar{\partial}_i=\partial/\partial s_i$, $s_i=(R^{-1})_{ik}x_k e^{\lambda}$, $s=\sqrt{s_is_i}$, $F'=\partial F/\partial s$, the dot denotes differentiation with respect to the time, and the frequencies of isotopic rotations ω^k and spatial rotations Ω_k are given by the equations

$$\dot{I}^{ik}(I^{-1})^{kj}=\varepsilon^{ijk}\omega^k, \quad (\dot{R}^{-1})_{ik}R_{kj}=-\varepsilon_{ijk}\Omega_k. \quad (52)$$

Calculating the scalar product $(\partial_0\boldsymbol{\phi}\cdot\partial_0\boldsymbol{\phi})$ and substituting it into Eq. (48), we obtain

$$(\mathbf{L}_0\cdot\mathbf{L}_0)=(F')^2s^2\dot{\lambda}^2+\{[\mathbf{N}\times\boldsymbol{\omega}]^2+([\mathbf{N}\times\boldsymbol{\omega}]\cdot\bar{\partial}_i\mathbf{n})\dot{S}_i$$
$$+\dot{S}_i(\bar{\partial}_i\mathbf{N}\cdot\bar{\partial}_j\mathbf{N})S_j\}\sin^2 F, \quad (53)$$

where

$$\dot{S}_i=(-\varepsilon_{ijk}\Omega_k+\delta_{ij}\dot{\lambda})S_j. \quad (54)$$

The expression for the vector product is somwhat more complicated:

$$[\mathbf{L}_0\times\mathbf{L}_k]^2=-(F')^2\sin^4F\{(\bar{\partial}_i\mathbf{N}\cdot\bar{\partial}_i\mathbf{N})s^2\dot{\lambda}^2$$
$$+2([\mathbf{N}\times\boldsymbol{\omega}]\cdot\partial_i\mathbf{N})\dot{S}_i+\dot{S}_i(\partial_i\mathbf{N}\cdot\partial_j\mathbf{N})S_j\}$$
$$-\{(F')^2s^2\dot{\lambda}^2\cos^2F+([\mathbf{N}\times\boldsymbol{\omega}]^2$$
$$+2([\mathbf{N}\times\boldsymbol{\omega}]\cdot\bar{\partial}_i\mathbf{N})\dot{S}_i+\dot{S}_i(\bar{\partial}_i\mathbf{N}$$
$$\cdot\bar{\partial}_j\mathbf{N})\dot{S}_j)\sin^2F\}\{(F')^2+(\bar{\partial}_i\mathbf{N}\cdot\bar{\partial}_i\mathbf{N})\}$$
$$+\{([\bar{\partial}_j\mathbf{N}\times\mathbf{N}]\boldsymbol{\omega})^2\sin^2F+\dot{S}_i(\bar{\partial}_i\mathbf{N}$$
$$\cdot\partial_j\mathbf{N})\sin^2F+(F')^2\cos^2F\dot{\lambda}s_j\}^2. \quad (55)$$

Substituting the explicit form of the isotopic vector $\mathbf{N}$ into the last expression and integrating over all space, after simple but tedious manipulations we can write the time-dependent part of the Lagrangian in the form

$$L_t=\frac{\pi}{F_\pi e^3}\dot{\lambda}^2\int_0^\infty x^4dx(F')^2\left\{\frac{e^{-3\lambda}}{2}\right.$$
$$\left.+e^{-\lambda}\frac{\sin^2F}{x^2}\int_0^\pi\sin\theta d\theta\left(k^2\frac{\sin^2T}{\sin^2\theta}+(T')^2\right)\right\}$$
$$+\frac{\pi}{2F_\pi e^3}\int_0^\infty x^2dx\int_0^\pi\sin\theta d\theta\left\{\sin^2F\left[\frac{e^{-3\lambda}}{4}\right.\right.$$
$$\left.+e^{-\lambda}\left((F')^2+\left[k^2\frac{\sin^2T}{\sin^2\theta}+(T')^2\right]\frac{\sin^2F}{x^2}\right)\right]$$
$$\otimes\left[(\bar{\omega}_1^2+\bar{\omega}_2^2)(1+\cos^2T)+2\bar{\omega}_3^2\sin^2T\right.$$
$$-4k\bar{\omega}_3\Omega_3\sin^2T+(T')^2(\Omega_1^2+\Omega_2^2)$$
$$\left.+k^2\frac{\sin^2T}{\sin^2\theta}[(\Omega_1^2+\Omega_2^2)\cos^2\theta+2\Omega_3^2\sin^2\theta]\right\}$$
$$-e^{-\lambda}\frac{\sin^4F}{x^2}\left[k^2\frac{\sin^2T}{\sin^2\theta}\{(\bar{\omega}_1^2+\bar{\omega}_2^2)\cos^2T\right.$$
$$+2\bar{\omega}_3^2\sin^2T\}+(T')^2(\bar{\omega}_2^2+\bar{\omega}_2^2)$$
$$-4k^3\bar{\omega}_3\Omega_3\frac{\sin^4T}{\sin^2\theta}+(T')^4(\Omega_1^2+\Omega_2^2)$$
$$\left.\left.+k^4\frac{\sin^4T}{\sin^4\theta}[(\Omega_1^2+\Omega_2^2)\cos^2\theta+2\Omega_3^2\sin^2\theta]\right]\right\}. \quad (56)$$

Here $\bar{\omega}_i=(I^{-1})_{ij}\omega_j$ and $x=sF_\pi e$ is a dimensionless coordinate. In the calculation of the integrals over the angle φ in spherical coordinates we need to calculate integrals of the type

$$\int_0^{2\pi}\sin(\varphi)\sin(k\varphi)d\varphi=\int_0^{2\pi}\cos(\varphi)\cos(k\varphi)d\varphi$$
$$=\begin{cases}0, & k\neq1,\\ \pi, & k=1.\end{cases} \quad (57)$$

Equation (56) is valid for configurations with $k\neq1$. Configurations with $k\neq1$ will be studied in what follows.

Dividing the range of integration over θ into intervals limited by the boundary points θ_i, we write the part of the Lagrangian describing the rotational degrees of freedom as

$$L^{\rm rot}=\sum_{m=1}^{l}L_m^{\rm rot}$$
$$=\frac{1}{2}\sum_{m=1}^{l}[Q_s^m(\Omega_1^2+\Omega_2^2)+Q_T^m(\bar{\omega}_1^2+\bar{\omega}_2^2)$$
$$+Q^m(\bar{\omega}_3^2-2k_m\bar{\omega}_3\Omega_3+k_m^2\Omega_3^2)]. \quad (58)$$

The moments of inertia Q_s^m, Q_T^m, and Q^m are given by

$$Q_T^m(\lambda)=\frac{\pi}{F_\pi e^3}\int_0^\infty x^2dx\int_{\theta_{m-1}}^{\theta_m}\sin\theta d\theta$$
$$\times\left\{-e^{-\lambda}\frac{\sin^4F}{x^2}\left(k^2\frac{\sin^2T}{\sin^2\theta}\cos^2T\right.\right.$$
$$\left.+(T')^2\right)+\sin^2F\left[\frac{e^{-3\lambda}}{4}+e^{-\lambda}\left((F')^2\right.\right.$$
$$\left.\left.+\left[k^2\frac{\sin^2T}{\sin^2\theta}+(T')^2\right]\frac{\sin^2F}{x^2}\right)\right](1$$
$$\left.+\cos^2T)\right\}, \quad (59)$$

$$Q_s^m(\lambda)=\frac{\pi}{F_\pi e^3}\int_0^\infty x^2dx\int_{\theta_{m-1}}^{\theta_m}\sin\theta d\theta$$
$$\times\Bigg\{-e^{-\lambda}\frac{\sin^4 F}{x^2}\left[k^4\frac{\sin^4 T}{\sin^4\theta}\cos^2\theta+(T')^4\right]$$
$$+\sin^2 F\Bigg[\frac{e^{-3\lambda}}{4}+e^{-\lambda}\Bigg((F')^2+\Bigg[k^2\frac{\sin^2 T}{\sin^2\theta}$$
$$+(T')^2\Bigg]\frac{\sin^2 F}{x^2}\Bigg)\Bigg]$$
$$\times\left(k^2\frac{\sin^2 T}{\sin^2\theta}\cos^2\theta+(T')^2\right)\Bigg\}, \qquad (60)$$

$$Q^m(\lambda)=\frac{2\pi}{F_\pi e^3}\int_0^\infty x^2dx\int_{\theta_{m-1}}^{\theta_m}\sin\theta d\theta$$
$$\times\Bigg\{-e^{-\lambda}\frac{\sin^4 F}{x^2}k^2\frac{\sin^4 T}{\sin^2\theta}+\sin^2 F\Bigg[\frac{e^{-3\lambda}}{4}$$
$$+e^{-\lambda}\Bigg((F')^2+\Bigg[k^2\frac{\sin^2 T}{\sin^2\theta}$$
$$+(T')^2\Bigg]\frac{\sin^2 F}{x^2}\Bigg)\Bigg]\sin^2 T\Bigg\}. \qquad (61)$$

Equation (58), like Eqs. (59)–(61), is valid only for solutions with $k\neq 1$.

Using the condition (42), we obtain

$$L^{\rm rot}=\frac{1}{2}\sum_{m=1}^{l}[Q_S^m(\Omega_1^2+\Omega_2^2)+Q_T^m(\bar\omega_1^2+\bar\omega_2^2)$$
$$+Q^m(\bar\omega_3^2-2k_m\bar\omega_3\Omega_3+k_m^2\Omega_3^2)]. \qquad (62)$$

The momenta conjugate to the collective variables Ω_i and ω_i are

$$S_i^{\rm b.f.}=\sum_{m=1}^{l}Q_S^m(\lambda)\Omega_i \quad \text{for} \quad i=1.2,$$
$$S_3^{\rm b.f.}=\sum_{m=1}^{l}k_mQ^m(\lambda)\{k_m\Omega_3-\omega^3\}, \qquad (63)$$

$$T_i^{\rm b.f.}=\sum_{m=1}^{l}Q_T^m(\lambda)\bar\omega^i \quad \text{for} \quad i=1.2,$$
$$T_3^{\rm b.f.}=\sum_{m+1}^{l}Q^m(\lambda)\{\bar\omega_3-k_m\Omega_3\}. \qquad (64)$$

Calculating the Hamiltonian using the expressions

$$H=\omega^k\frac{\partial L}{\partial\omega^k}+\dot\lambda\frac{\partial L}{\partial\lambda}-L, \qquad (65)$$

and requiring that the conjugate momenta satisfy the canonical commutation relations, we arrive at the expression

$$\hat H=M(\lambda)+\frac{P^2}{2m(\lambda)}+\frac{T^2}{2Q_T(\lambda)}+\frac{S^2}{2Q_S(\lambda)}+H_1, \qquad (66)$$

where the momentum P conjugate to λ is defined canonically as

$$P=\frac{\partial L}{\partial\dot\lambda}=m(\lambda)\dot\lambda,$$

and for the mass $m(\lambda)$ we have

$$m(\lambda)=\frac{2\pi}{F_\pi e^3}\int_0^\infty (F')^2\left\{\frac{e^{-3\lambda}}{2}+e^{-\lambda}\frac{\sin^2 F}{x^2}\right.$$
$$\left.\times\int_0^\pi\left[k^2\frac{\sin^2 T}{\sin^2\theta}+(T')^2\right]\sin\theta d\theta\right\}x^4dx. \qquad (67)$$

The rotational moments of inertia Q_T and Q_S are given by

$$Q_T(\lambda)=\sum_{m=1}^{l}Q_T^m(\lambda), \quad Q_S(\lambda)=\sum_{m=1}^{l}k_m^2Q_S^m(\lambda). \qquad (68)$$

The part H_1 of the Lagrangian depending on the internal quantum variables is

$$H_1=\frac{1}{2}\left[\frac{Q_1}{Q_1Q_2-Q_0^2}-\frac{1}{Q_T}\right](\hat T_3^{\rm b.f.})+\frac{1}{2}\left[\frac{Q_1}{Q_1Q_2-Q_0^2}-\frac{1}{Q_S}\right]$$
$$\times(\hat S_3^{\rm b.f.})+\frac{Q_0}{Q_1Q_2-Q_0^2}T_3^{\rm b.f.}S_3^{\rm b.f.} \quad \text{for odd } l \qquad (69)$$

and

$$H_1=\frac{1}{2}\left[\frac{1}{Q_2}-\frac{1}{Q_T}\right](\hat T_3^{\rm b.f})+\frac{1}{2}\left[\frac{1}{Q_1}-\frac{1}{Q_S}\right](\hat S_3^{\rm b.f})^2 \quad \text{for even } l. \qquad (70)$$

Here

$$Q_1(\lambda)=\sum_{m=1}^{l}k_m^2Q^m(\lambda), \quad Q_2(\lambda)=\sum_{m=1}^{l}Q^m(\lambda),$$
$$Q_3(\lambda)=\sum_{m=1}^{l}k_mQ^m(\lambda). \qquad (71)$$

We should point out that now the 3-components of the spin and isospin are in general not coupled together in the body-fixed coordinate system.

The result of calculating the spectrum of quantum states for the nucleus with $B=12$(^{12}C) with the constants $F_\pi=142$ MeV and $e=5.5$ is shown in Fig. 10.

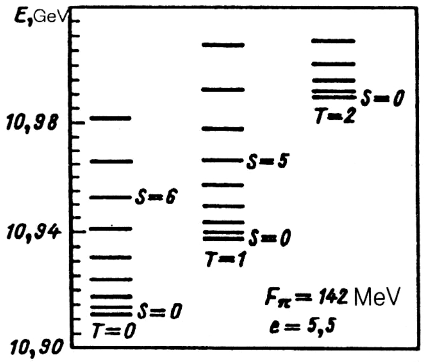

FIG. 10. Spectrum of quantum states of Skyrmions with $B=12$ (^{12}C).

The rotational band with isospin one unit larger than that of the ground state is embedded in the spectrum at a height of about 40 MeV.

5. COMPRESSIBILITY OF SKYRMION MATTER

Among the various collective nuclear excitations, those of the monopole or "breather" mode are of particular interest. They are interesting in connection with the discovery of giant resonances of the collective type in the inelastic scattering of electrons and hadrons on nuclei. It should also be noted that such excitations probably significantly affect the one-nucleon structure function of nuclei, which determines the EMC effect.[53]

It is commonly assumed that the breather mode corresponds to nuclear-matter density oscillations and characterizes the compressibility of nuclear matter. In Ref. 54 this interpretation was proposed also for the Roper resonance. In this version the nucleon bag radius is assumed to be a dynamical variable, and the energy of the Roper resonance corresponds to the first excited state of the quantum radial motion of the surface. The bag energy as a function of the bag radius R plays the role of the potential energy of the motion "along" the collective variable R. There is some difficulty in defining the kinetic-energy operator in the corresponding Schrödinger equation for this model. It is natural to assume that this operator is proportional to the second derivative of the wave function with respect to the radius R. The wave function determines the amplitude of fluctuations of the system size. A similar equation can be obtained in the Hill–Wheeler–Griffin approximation,[55] and was also introduced by Dirac[56] in the theory of the electron. The common difficulty of these approaches is the calculation of the effective mass entering into the kinetic-energy operator. The effective mass is usually a phenomenological parameter of the model which is estimated from heuristic considerations.

An interpretation of the Roper resonance as an excitation of the breather mode of solitons of the chiral field in the Skyrme model has recently been proposed.[51] The remarkable feature of this model is the unity of the description of nucleons and nuclei as topologically nontrivial solitons. The dynamical variables in terms of which nucleons and nuclei are described are the boson fields satisfying the Euler–Lagrange equations corresponding to the chirally invariant Lagrangian. Topologically nontrivial configurations of these fields are interpreted as baryons. Configurations which are topologically equivalent to the vacuum correspond to mesons.

These arguments suggest that the nucleon breather mode becomes the nucleus breather mode.

Let us formulate our goal more explicitly by considering the simpler case of the breather mode of a piece of nuclear matter. In general there is no single definition of the incompressibility K of a finite system with mass number A. The incompressibility can be defined, for example, as the second derivative of the energy per particle, with respect to the radius R at the equilibrium point R_0:

$$K_A=R_0^2\frac{\partial^2(E/A)}{\partial R^2}\bigg|_{R_0}. \tag{72}$$

However, in order to estimate this expression we need to know the dependence of the energy E on the radius R. On the other hand, the scale transformation $r\to\lambda r$ applied to the single-particle ground-state wave function leads to the so-called "scaling incompressibility"

$$K_A=\frac{\partial^2(E/A)}{\partial\lambda^2}\bigg|_{\lambda=1}. \tag{73}$$

The kinetic-energy density in the neighborhood of the point r is given by

$$\frac{mv^2}{2}=\frac{m}{2}\left(\frac{r}{R}\right)^2\dot{R}^2, \tag{74}$$

if we assume that the breather mode preserves the uniformity of the system. Then the total kinetic energy of the system containing A particles with particle distribution density

$$\rho(r)=\frac{3A}{4\pi R^3}\,\theta(R-r), \tag{75}$$

is

$$T=\int\rho(r)T(r)d^3r=\frac{1}{2}\dot{R}^2\frac{3}{5}m_NA. \tag{76}$$

The last equation corresponds to the effective mass

$$m_{\rm eff}=\frac{3}{5}m_NA. \tag{77}$$

Here m_N is the nucleon mass. It is Eq. (77) for $A=1$ that was taken as the approximation to the effective mass of the nucleon breather mode in Ref. 54.

For the energy of monopole vibrations we have

$$\hbar\omega=\sqrt{\frac{K}{m_{\rm eff}}}=\frac{1}{R_0}\sqrt{\frac{K_\infty}{(3/5)m_N}}=\frac{1}{r_0A^{1/3}}\sqrt{\frac{K_\infty}{(3/5)m_N}}. \tag{78}$$

To simplify the discussion we can drop the contribution of rotations, since it is of order $1/N_c$ compared with the terms of order N_c for the classical mass and N_c^0 for vibrations. We restrict ourselves to the scaling transformation $U(\mathbf{r})\to U(\mathbf{r}e^{\lambda(t)})$ for the field U in the Skyrme Lagrangian. This transformation does not change the baryon-number, but corresponds to a change of the system size. Now the simplified Lagrangian takes the form[51,52]

$$L=\frac{1}{2}\dot{\lambda}^2e^{-3\lambda}Q_2+\frac{1}{2}\dot{\lambda}^2e^{-\lambda}Q_4-M_2e^{-\lambda}-M_4e^{\lambda}. \tag{79}$$

After the canonical transformation and the quantization procedure we obtain the quantum Hamiltonian

$$\hat{H}=\frac{\hat{P}^2}{2(Q_2+Q_4)}+\frac{1}{2}\lambda^2(M_2+M_4)+M_2+M_4. \tag{80}$$

For the incompressibility K of the soliton we find

$$K=M_2+M_4=M, \tag{81}$$

which is equal to the soliton mass.

The frequencies $\hbar\omega$ of the breather mode in light systems with baryon-number B are given in Table VI (Ref. 57) in units of eF_π.

TABLE VI. Frequencies $\hbar\omega$ of the breather mode in light systems.

B	1	2	3	4	6	8	9	12
$\hbar\omega$	0,31	0,27	0,24	0,23	0,20	0,18	0,17	0,15

Owing to the linear dependence of the classical soliton mass on the baryon-number B, we can easily determine the soliton incompressibility:

$$K=M\cong M'B. \quad (82)$$

Our calculations give $M'\cong M|_{B=1}$. For the nuclear incompressibility, following the definition (82), we find

$$K_A=K/B=M'\cong M|_{B=1}=K_{B=1}. \quad (83)$$

The calculation gives the value 800 MeV for K_A. This is consistent with the idea that the nuclear incompressibility is of the order of the nucleon incompressibility, which, in turn, is of the order of the nucleon mass.

It follows from our discussion that the nucleon breather mode is transformed into the nucleus breather mode. In other words, these two phenomena have the same origin. It seems that so far the Skyrme model is the only model in which this idea is realized. Although the calculated frequencies obey the dependence $\hbar\omega\sim B^{-1/3}$, they lie considerably higher than the energy of the giant monopole resonances calculated in traditional approaches. For example, in the hyperspherical-functions method[58] the energies of the giant monopole resonances lie in the range from 20 to 35 MeV for mass numbers $4\leqslant A\leqslant 16$.

In principle, in heavy nuclei there must be two 0^+ vibrational modes, since there are at least two dimensional parameters in the problem. These are the nuclear radius R and the diffusion parameter b. The compressibilities and, accordingly, the frequencies corresponding to time variations of R and b are different. Of course, these two modes are not mutually orthogonal, but this is not important for us.

In light nuclei one might also expect the existence of a hard and a soft breather mode. However, at first glance, in light nuclei one of the dimensional parameters becomes meaningless. But it is precisely in such nuclei that the nonpointlike nature of the nucleon becomes important. We again have two dimensional parameters, the nuclear radius and the nucleon radius, which become comparable in this range of mass numbers.

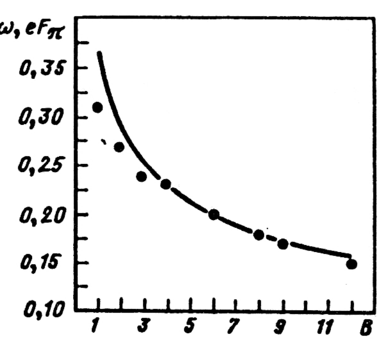

FIG. 11. Dependence of the frequency $\hbar\omega$ of the soliton breather mode on the soliton baryon-number. The points are from our calculation. The solid line is the approximation $B^{-1/3}$.

It seems to us that the hyperspherical-functions method describes vibrations of the locations of pointlike nucleons and therefore encompasses only the soft mode. The scaling transformation in the Skyrme model changes not only the nucleon position, but also the nucleon size. Therefore, the Skyrme model describes a mode lying above the harder mode, which may be a manifestation of the nucleon size in light nuclei.

Some indications of the existence of narrow 0^+ resonances in light nuclei at energies of about 45 MeV have been obtained in experiments on neutron and deuteron scattering on helium nuclei.[59,60]

IN PLACE OF A CONCLUSION

Let us try to summarize the results covered in this review. In the low-energy region of quantum chromodynamics, effective chiral Lagrangians put at the disposal of theoreticians "convenient" colorless degrees of freedom—bosonic fields for describing events in strong-interaction physics. Although the use of chiral Lagrangians is usually restricted to the semiclassical approximation, the success attained by the Skyrme model in the last few years in describing nucleons as quantum states of chiral solitons provides justification for using it or generalizations of it in nuclear physics.

It is important to develop a nuclear model of chiral solitons using a formalism and, even more importantly, variables which are the same as those used in the current nonlinear field theory of nucleons.

The construction of a nuclear model of chiral solitons is also important in connection with existing and projected experimental programs to study the structure of light nuclei far from the β-stability line. The Skyrme model stresses the uniqueness of the structure of each individual nucleus, in contrast to potential models. The Skyrme model gives a new view of such problems as the existence of compound nuclear states involving antinucleons in their structure, shape isomers, and high-lying 0^+ vibrations in light nuclei.

It is also important to develop variational approaches to solving the nonlinear problems which arise. The variational approach to understanding the physics of various phenomena requires considerably less time for numerical calculations than the direct search for solutions, even when supercomputers are used. The exclusivity of each individual state in the chiral soliton model is also emphasized by the strong dependence of the effective quantum Hamiltonian on the topological sector. Not only the inertial parameters of the effective Hamiltonian, which are functionals of the solutions in a given sector, but also the actual form of the effective Hamiltonian depend on the baryon-number. Here we should point out the difficulty with the overbinding of states with the quantum numbers of the lightest nuclei. It should also be noted that reproduction of the details of the electromagnetic form factors obviously requires studies with generalized Skyrme models including the additional scalar (dilaton) and gauge vector fields of a hidden symmetry, although the Skyrme model already qualitatively reproduces the existing experimental data on such details as the deuteron polarization tensor in electron–deuteron scattering.

It seems to us that the model of chiral-field solitons will be a useful tool in the future theory of light nuclei which will allow the study of the familiar problems of the theory of few-particle systems from a completely new point of view. And a new view of a problem implies new solutions.

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