# Perturbative methods in string theory

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The status of the theory of free massless fields on Riemann surfaces within the general context of perturbative string theory in the first-quantization formalism and the application of this theory to the analysis of various conformal models are described. Some of the results of the free-field theory are presented, including the expressions for the Mumford measure for the lower genera $p\leqslant 4$, the expressions for the determinants of the operators $\bar{\partial}$ on the hyperelliptic surfaces, and the equations for the correlators of the $\beta$ and $\gamma$ fields. The method of transforming to quadratic actions in the corresponding functional integrals is demonstrated for the examples of the Green–Schwartz and Wess–Zumino–Novikov–Wittin models. The GKO reductions and features of models with $W$ algebras are also briefly discussed. In particular, it is shown that there are two sources of breakdown of the Lie-algebra structure for $W_N$ algebras, and that this structure is restored for $N\to\infty$. The topics discussed are mainly those in which the author was directly involved.

## INTRODUCTION

String theory is a special variant of nonlocal quantum field theory with possible future applications to strong-coupling theory, the theory of phase transitions, the theory of spin glasses, the construction of a consistent theory of quantum gravity, and viable models unifying all the fundamental interactions. String theory also has deep and fruitful connections with various areas of mathematics, from number theory and algebraic geometry to the theory of differential equations and catastrophe theory. It is hoped that string theory will lead to a physical construct which unifies these widely differing areas. The conceptual foundation of string theory is the idea of the dynamical selection of certain models from the full manifold of local quantum field theories. It is this which is the fundamental problem from the viewpoint of interaction unification: it is necessary to explain why our world (at least at energies below 100 GeV) is described by the $SU(3)\times SU(2)\times U(1)$ Standard Model, and not some other gauge theory with a different symmetry group and/or different particle content. For this reason, the main content of string theory is the study of dynamics in the space of all possible models of quantum field theory. The main technical method which, at least in principle, allows this problem to be discussed is the first-quantization formalism, which reduces the analysis of a particular class of multidimensional nonlocal field theories to the study of ordinary local field theories in two dimensions. Here the two-dimensional theory is interpreted as the "theory on the world sheet" describing the dynamics of test "strings" (one-dimensional extended objects) in external fields, and the first-quantization formalism itself is a sort of Radon transform in the functional space of field theories. The consideration of one-dimensional test objects instead of zero-dimensional ones ("particles") has proved very fruitful, since it has made it possible to interpret the interactions in the original nonlocal theory in terms of the geometrical properties of the two-dimensional theory. In a certain sense the theory of interacting strings is the direct analog of the theory of free particles, which is what gives it its uniqueness and special mathematical beauty. The fundamental dynamical principle of string theory identifies the classical string equations of motion (specifying the dynamics in "theory space") with the condition of conformal invariance of the theory on the world sheet. A description of the equations of motion in terms of a symmetry principle is thereby obtained. (For comparison, in the case of Yang–Mills theories it is necessary to use the minimum principle in addition to gauge invariance. This principle is good for effective low-energy theories, but does not work at all for the fundamental theory of matter.) So far there is no generally accepted method of describing the quantum equations of motion and the off-shell features of string theory.

The simplest problem of string theory in this context is the study of the classical solutions of the string equations of motion, i.e., two-dimensional conformal models and their neighborhoods. Here, in general, it is necessary to study both "adjacent" points in configuration space—nonconformal models—and the quantum corrections related to fluctuations of the string fields corresponding to the conformal theory itself. In principle, both types of fluctuation are important for describing the string dynamics, but it is advisable to separate them in the first stage of investigations. This review is devoted only to the question of the quantum corrections ("string loops") within the individual conformal models. The simplest model of this type is the theory of a free massless scalar field on a two-dimensional closed orientable surface of arbitrary genus. The study of free fields with nonzero spins on both nonorientable surfaces and surfaces with a boundary is easily reduced to this problem. The direct solution of this problem completely answers the question of perturbative calculations in the 26-dimensional bosonic string model and its simplest compactifications, on tori and orbifolds. The question of perturbative calculations in the models at the next stage of complexity and importance—the Neveu–Schwartz–Ramond (NSR) and Green–Schwartz (GS)

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superstrings—is more complicated. In spite of the importance of these models for using string theory to unify the fundamental interactions, the problem of multiloop calculations here has not yet been completely solved. Finally, the question of universal approaches to the perturbative description of all string models (i.e., string models constructed on the basis of all the conformal theories) is extremely important. In principle, the route to such a description is revealed by the free-field representation for conformal theories treated as various reductions of the Wess–Zumino–Novikov–Witten (WZNW) model, the symmetry of which corresponds to a Kac–Moody (KM) algebra. Such a description is absolutely necessary for future attempts to analyze strings off the mass shell.

A great many scientific groups and individual investigators have taken part in the development of perturbative string theory. An especially important role was played by Knizhnik in the most recent stage of elucidating the role of the complex-analytic aspects of the theory. This review mainly is focused on the results obtained directly by the author in collaboration with A. Belavin, A. Gerasimov, R. Kallosh, V. Knizhnik, D. Lebedev, A. Marshakov, G. Moore, M. Ol'shanetskiĭ, A. Perelomov, A. Roslyĭ, A. Turbiner, S. Shatashvili, and M. Shifman. This development ultimately led to a more or less clear understanding of the corresponding mathematical structures and has recently made it possible to progress to the study of the extremely interesting and rich nonperturbative aspects of the string dynamics. Before turning to the more detailed description of the actual results, let us discuss them within the general context of perturbative string theory.

The first studies on perturbative string theory were carried out in the 1960s and 70s. Among the most important for the present review are the discovery of the world-sheet action;[1,2] the first calculations of the tree,[3] one-loop,[4,5] and multiloop[6] amplitudes; and the formulation of the supersymmetry principles and superstring models.[7,8] The discovery of the low-energy limit and the subsequent proposal that string models be used to unify the interactions (see, for example, Refs. 9 and 10) were of major importance. A systematic approach to the study of strings became possible after the explicit formulation by Polyakov[11] of the idea of replacing the functional integral with the strongly nonlinear Nambu–Goto action[1,2] by an integral with additional averaging over two-dimensional metrics and large gauge invariance. From the viewpoint of physical applications, a fundamental (and *a priori* not obvious) consequence of this formulation, known as the Fradkin–Tseytlin theorem,[12] was the possibility of reformulating the (touched-up) Einstein and Yang–Mills equations (and, in general, many differential equations specifying the space-time dynamics) in terms of a nontrivial symmetry principle: the two-dimensional conformal invariance of the theory on the world sheet. There was a burst of interest in string theory in 1984, when after long preliminary work (see Ref. 13) Green and Schwartz showed[14] that there exists a new, unique string theory without anomalies and with a large $SO(32)$ or $E_8\times E_8$ space-time gauge symmetry. (Actually, the latter version was consistently formulated somewhat later in Ref. 15 as the "heterotic string.") At the present time it is hoped that a unified theory of all the fundamental interactions can be constructed on the basis of this work, and the number of articles on string theory is enormous. The most general results are collected in Ref. 16. A broader view of the position of string theory in physics is given in Ref. 17.

Perturbative calculations within a particular string model amount to summing over all metrics and fields of the model on a two-dimensional surface of any given topology. The first results in this area after Ref. 11 were obtained by Alvarez.[18] In the remarkable studies of Belavin and Knizhnik[19] it was shown for the simplest example of the bosonic string in 26 dimensions that the only important characteristic of the surface (if it is closed and orientable) is its complex structure. (The fact that the critical dimension 26 coincides with the number appearing in the Mumford theorem was noticed somewhat earlier by Manin, who is responsible for one of the first applications of the Belavin–Knizhnik theorem.[20] This was further developed in work with Beilinson.[21] The restriction to closed and orientable surfaces is easily lifted using the doubling technique; see, for example, Ref. 22.) This fact is extremely important for several reasons. First, it establishes the relation of string theory to complex and, in general, algebraic geometry. Second, it indicates the most important generalizable property of the 26-dimensional string model: the holomorphic factorization of correlators. This feature actually proved to be central in the axiomatic definition of conformal models in Ref. 23; this led directly to the currently widely held view of string theory as an interpolation between all possible conformal models. Third, the Belavin–Knizhnik theorem revealed a method of solving the problem of multiloop calculations in various specific string models. From many points of view an exhaustive solution of this problem for the case of the 26-dimensional string has been given in Refs. 24–37. The authors of Refs. 20, 21, and 24–28 used the deep results of Refs. 38–43 on the properties of Riemann surfaces, their moduli spaces, and bundles over them, and a clearer theory of free fields on Riemann surfaces was developed in Refs. 29–37. The main results of this theory (together with the simplest necessary information from the differential geometry of Riemann surfaces) are given in, for example, Sec. 7 of Ref. 44 and repeated in Ref. 45. The results of Refs. 19, 26–28, and 46 on explicit expressions for the 2-, 3-, and 4-loop amplitudes based on more detailed information about the properties of the Mumford measure are given below in Subsec. 1.1. It is difficult to obtain completely explicit expressions for the multiloop amplitudes because the results contain complicated special functions, theta functions on higher-genus surfaces. They are obtained by constraining the theta functions from the Jacobians for maps of the surfaces. The moduli spaces of the Jacobians and the surfaces coincide for $p=1,2,3$ ($p$ is the genus of the surface or the number of loops), so in these cases the formulas are very clear (the Jacobians and surfaces themselves coincide only for $p=1$). For $p>3$ the situation is more complicated. From many points of view it would be most correct to have a description of perturbative string theory in which the Riemann surfaces were specified by algebraic equations. Unfortunately, in general overdetermined systems of equations are needed, and the problem in its general form has not been solved. However, there is a simple and important example: hyperelliptic surfaces. The calculus of free fields on hyperelliptic surfaces was first studied in Refs. 47–55. In Subsec. 1.2 below we follow the exposition of Ref. 48.

In Subsec. 1.3 we give a brief description of the theory of compactification on tori and orbifolds based on Refs. 56 and 57 (the number of articles in which this problem has been independently analyzed is huge; orbifolds are studied in Ref. 57 only in the one-loop approximation; see Refs. 58 and 59 for multiloops). Next, in Subsec. 1.4 we discuss another important area of the theory of free fields on Riemann surfaces: the theory of $\beta$, $\gamma$ systems. They were introduced in a special case in Ref. 60 for describing superghosts in NSR superstrings. The general theory of $\beta$, $\gamma$ fields (bosons with a first-order Lagrangian) was developed in connection with this problem in Refs. 33, 35–37, and 61–65. Recently, it has been shown that the $\beta$, $\gamma$ fields play an important role also in other conformal models, including the WZNW model[66–70] (see Ref. 44 and below). The further development of free-field theory itself is related to the interpretation of results in algebraic terms, primary related to the Virasoro algebra and its various realizations associated with the algebras of vector fields on Riemann surfaces.[71–73] The corresponding results are closely related to the so-called operator formalism for free fields (Refs. 32, 34, 35, and 74–78). In turn, this approach is important for the systematic summation of the perturbation series and the formulation of the results in terms of the universal moduli space.[79–86]

The 10-dimensional supersymmetric and heterotic strings occupy a special place in string theory. In contrast to the 26-dimensional bosonic string, these models describe perturbatively stable solutions of the string equations of motion—they contain no tachyon excitations. For this reason these models must be free of divergences in all orders of perturbation theory (divergences can appear only as nonperturbative instabilities corresponding to tunneling to other string models). Finally, in such models there are additional symmetries both in space-time and on the world sheet. These features make supersymmetric and heterotic string models interesting and important objects of study which possess new features compared with the models discussed in Sec. 1. These models are discussed in Secs. 2 and 3. So far no complete perturbative theory of supersymmetric and heterotic strings has been constructed. The two approaches to such a theory are based on two fundamentally different classical approximations: the Neveu–Schwartz–Ramond[7,8] (NSR) and the Green–Schwartz[87] (GS) approximations. In the NSR approach the starting point is the model of the 10-dimensional fermionic string possessing supersymmetry on the world sheet. The mathematical apparatus is the theory of super-Riemann surfaces and their moduli (supermoduli) spaces. The main difficulties are the extraction of a self-consistent superstring model from the fermionic string (the Hilbert space of one model is embedded in the Hilbert space of another; historically, this is the first example of a situation which turned out to be quite universal in the entire field of conformal and string models). The fundamental idea in this case is the use of the discrete $Z_2$ symmetry—the so-called $G$ parity—present in the fermionic string model. It allows all the excitations to be split into "plus-particles" and "minus-particles," and the elementary three-particle interaction event cannot transform a minus-particle into a pair of plus-particles. It is easy to understand that this property guarantees that any tree diagram for which all the external lines correspond to plus-particles will not contain any minus particles even in the internal lines. This observation allowed the authors of the remarkable study of Ref. 10 to formulate the model of the NSR superstring as a projection of the fermionic string onto plus states. This operation is referred to as the GSO projection, and the argument just given guarantees the tree-level unitarity of the resulting theory. After this it was understood that there also must exist a full-fledged unitary theory, but its relation to the fermionic string at the multiloop level is less obvious: the loop diagrams in the fermionic string model contain intermediate minus particles, even if all the external lines correspond to plus states. It turned out that the GSO projection at the multiloop level presupposes summation over all possible boundary conditions for the fermionic fields (the square root of the tangent bundle contains a finite-fold uncertainty). The main problem in the theory of NSR superstrings is the choice of weights in this sum over theta characteristics: the weights are $\pm 1$ for genus $p = 1$, but much less trivial for higher genera. Important contributions to the study of multiloop calculations for NSR superstrings have been made in Refs. 20, 33, 35–37, 60–65, 76, and 88–100. In Subsecs. 2.1 and 2.2 below we follow the exposition of Refs. 37 and 96–100. See Refs. 101–111 for further developments. We again note that there is still a long way to go before definitively solving the problem to the point reached for 26-dimensional strings.

An alternative formalism for superstring models was proposed by Green and Schwartz.[13,16,87] In contrast to the NSR approach, the GS formalism possesses explicit space-time supersymmetry, and it is not necessary to introduce any Hilbert-space projections. Owing to this, in the GS approach the key features of superstrings are clearly revealed, including the absence of tachyons and nonrenormalization theorems. From the viewpoint of multiloop calculations it is very important that in the GS formalism there are no fields with half-integer spin on the world sheet. However, in return the two-dimensional covariant GS action[87] is not quadratic and possesses a complicated gauge symmetry (the so-called $k$ invariance; see Ref. 112 and references therein regarding the properties of the $k$ symmetry). To develop the GS theory on arbitrary Riemann surfaces it is necessary to change variables in the functional integral, making the action quadratic. Such a local and anomaly-free variable substitution must exist, owing to the conformal invariance of the theory. For the case $p = 0$ it is described in Subsec. 3.1, which is based on Ref. 113. In

Subsec. 3.2, following Ref. 114, we present the generalization to the case of arbitrary $p$ and the proof of the simplest theorems of the nonrenormalization of the 0-, 1-, 2-, and 3-point functions for massless particles, which are a direct consequence of the space-time supersymmetry (the proofs for the 2- and 3-point functions make use of some assumptions about the structure of the vertex operators, and in this sense are not really complete). The further development of these ideas is described in Refs. 120–123. The Hamiltonian foundation of the procedures proposed in Ref. 113 is apparently not yet completely satisfactory (see Ref. 115 for the naive approach and Ref. 116 for an analysis based on the Batalin–Fradkin–Vilkovisky procedure[117–119]). The question of whether or not the NSR and GS formalisms are identical is central to the entire theory of supersymmetric and heterotic strings. So far there is no proof that they are identical which is valid in all orders of perturbation theory (although their parallels can be traced a long way). Up to now there is no reliable expression for the simplest nontrivial quantity: the 2-loop superstring 4-point function; see Refs. 100 and 106–108 for some results in this area).

As already mentioned, the main content of the future string theory must be the unification of various theories, primarily the various two-dimensional conformal models. Here we shall not discuss the ideas about how such a unification might work (see Refs. 81 and 124–130 for some possibilities). In any case, it is clear that the unification must be based on a unified description of all the conformal models which distinguishes the properties common to all conformal theories without emphasizing their differences. It is widely assumed that such a feature of two-dimensional conformal models is their holomorphic factorization:[23] the reduction of any correlation function on a surface of any genus to a bilinear combination of analytic sections of complex bundles over moduli spaces. In simpler language this means that any conformal model can be written in terms of free massless fields on a Riemann surface, or, more precisely, it is a certain local projection (reduction) of a free field theory. Stated differently, the Hilbert space of the theory is embedded in some set of Verma modules of the operator algebra (and differs from this set by possible rejection of 0-vectors). Such representations of the structure of conformal models are reflected in many studies; some of the most important in our opinion are Refs. 23 and 131–141. In Sec. 4 we discuss the representations of free massless fields for various popular classes of conformal models. Special attention is given to the WZNW model, which in some sense is a universal conformal theory: the symmetries of any other conformal model can be interpreted as a reduction of the symmetry of the WZNW model, in which it is embedded. The operator algebra of the WZNW model is a KM algebra, and any other operator algebras can be viewed as fragments of the universal enveloping KM algebra (the best known examples are the Sugawara construction for the Virasoro algebra and similar constructions for the Zamolodchikov $W$ algebras). The WZNW model is discussed in Subsecs. 4.1 and 4.2, in which we follow the discussion of Refs. 44 and 142. (Similar results were obtained in Refs. 143–147 and are developed in numerous studies, in particular, in Refs. 148–154.) The local anomaly-free variable substitution in the functional integral making the nonlinear WZNW action quadratic is a direct generalization of the analogous substitution in the GS model discussed in Sec. 3. In Subsecs. 4.3–4.5 we give a discussion based on Refs. 155–159 of the free-field formalism and its derivation from the expressions for various reductions of the "universal" WZNW model: minimal models (in the Dotsenko–Fateev formalism[132]), theories with $W$ symmetries, and GKO models.[160] Finally, in Subsec. 4.6 we describe the construction proposed in Ref. 161, which is a generalization of the Sugawara embedding of the Virasoro algebra into the universal enveloping KM algebra. See Refs. 162–167 for its further development. The representation of free massless fields is important not only for understanding the general structure of conformal models and for constructing a complete string theory, but also for solving the problem of multiloop calculations in the corresponding string models. In principle, the solution of this problem is given by the free-field representation in conjunction with the theory of such fields on arbitrary Riemann surfaces, which is briefly discussed in Sec. 1. However, to obtain complete clarity and unambiguous explicit expressions for the correlators in rational conformal theories it is necessary to explicitly describe the projection from the Hilbert space of the free fields to the smaller Hilbert space of the particular rational model (see Refs. 139 and 141 regarding such projections). This part of the problem is not yet fully solved.

In arriving at conclusions about the role of the WZNW model, it is also of fundamental importance to take into account the possibility of a completely different interpretation of this model. As shown by Alekseev and Shatashvili,[138] the WZNW Lagrangian arises in the geometrical quantization of the KM algebra as $d^{-1}$ of the Kirillov–Kostant form (see Ref. 168). This result is a generalization of the old suggestion of Faddeev that the Kirillov–Kostant construction be described in terms of functional integration. The results for an arbitrary classical Lie algebra have been obtained by Alekseev, Faddeev, and Shatashvili[169] and extended to the case of Kac–Moody and Virasoro algebras in Ref. 138. Of current importance is the further generalization of the construction to the case of 2-loop algebras, including $\widehat{Sl}(\infty)$, $\widehat{W}_\infty$, and Moyal–Baker–Fairlie algebras.[170–172] The possible relation of this problem to the theory of integrable systems is discussed in Refs. 173 and 174, and to the full string theory in Refs. 129 and 130.

Therefore, in this review we discuss the main results on perturbative string theory, understood as the theory of conformal models on Riemann surfaces of arbitrary topology. The question of the full-fledged perturbation theory taking into account fluctuations in nonconformal directions and, even more ambitiously, that of nonperturbative phenomena is at present rather less clear and is not dealt with in this review.

## 1. RESULTS FROM THE THEORY OF FREE FIELDS ON RIEMANN SURFACES AND APPLICATIONS TO THE MODEL OF 26-DIMENSIONAL BOSONIC STRINGS

The most reasonable and economical foundation of all perturbative calculations in string theory is the theory of a free scalar field on a Riemann surface of arbitrary genus. This has been constructed independently by many authors, including Knizhnik (Refs. 29, 49, and 175), Alvarez-Gaumé, Bost, Moore, Nelson, and Vafa,[30] and Verlinde and Verlinde.[31,62] The application of this theory to string models is primarily based on the Polyakov formulation of string theory[11] and the Belavin–Knizhnik theorem.[19,175] In this section we give a number of intermediate results related to the formation of this point of view. In Subsec. 1.1 we describe the generalization of the Koba–Nielsen[3] and Shapiro–Virasoro[5] formulas for the tree and one-loop amplitudes in the 26-dimensional string model in terms of the period matrices and holomorphic 1-differentials for the 2-, 3-, and (with certain restrictions) 4-loop cases. Subsection 1.2 is devoted to the calculation of determinants on hyperelliptic surfaces, which up to now has been the clearest application of the general results, usually expressed in too transcendental a form (in terms of theta functions on multidimensional Jacobians). As a very simple example of calculations for models different from the standard 26-dimensional string, in Subsec. 1.3 we give the one-loop expressions for compactification on Abelian orbifolds. Subsection 1.4 contains the preliminary results of Refs. 37 and 176 on the $\beta$- and $\gamma$-field correlators, which are important for analyzing NSR strings and other models. More general expressions have been obtained by Verlinde and Verlinde.[62] A nearly complete solution of the problem of the correlators for the $\beta$, $\gamma$ systems directly in terms of the functional integral has been given recently by Gerasimov.[36]

### 1.1. The string measure for $p=2,3,4$

In the space-time of critical dimension $d=26$ the $p$-loop amplitude for the scattering of $N$ tachyons with momenta $k_\alpha$, the simplest quantity in string theory, is written as a finite-fold integral over the moduli space $\mathfrak{M}_p$ of Riemann surfaces of genus $p$ (Refs. 11 and 19):

$$\int Dg_{ab}Dx^\mu\left\{\exp\left[-\frac{M^2}{2}\int d^2\xi\,\sqrt{g}g^{ab}\partial_a x^\mu\partial_b x^\mu\right]\right\}$$
$$\times\prod_{\alpha=1}^{N}\int d^2\xi_\alpha\sqrt{g(\xi_\alpha)}\exp[ik_\alpha^\mu x^\mu(\xi_\alpha)]$$
$$\sim\int_{\mathfrak{M}_p}dv_p\left[\frac{\det N_0}{\det'\Delta_0}\right]^{13}\det{}'\Delta_{-1}\prod_{\alpha=1}^{N}\int d^2\xi_\alpha$$
$$\times\prod_{\alpha,\beta}^{N}\exp\left[-\frac{1}{2M^2}k_\alpha k_\beta G(\xi_\alpha,\xi_\beta)\right].\tag{1}$$

The fundamental ingredient in finding the Polyakov measure,

$$dv_p\left[\frac{\det N_0}{\det'\Delta_0}\right]^{13}\det{}'\Delta_{-1},\tag{2}$$

is the property of the holomorphic factorization of the determinants of the Laplace operators

$$\mathrm{Det}'\Delta_j=\exp\left[\frac{c_j}{24\pi}S_{\mathscr{L}}\right]\det N_j\det N_{1-j}|\mathrm{Det}\,\bar\partial_j|^2.\tag{3}$$

The chiral determinant $\mathrm{Det}\,\bar\partial$ depends on the complex moduli $y,\bar y$ holomorphically, $\partial/\partial y\mathrm{Det}\,\bar\partial=0$. The Liouville factor in (3) takes into account all anomalies of the regularized determinant, $c_j=2(6j^2-6j+1)$, and these factors cancel in anomaly-free products of determinants, including the Polyakov measure (2). The determinants $\det N_j$ of finite matrices of scalar products of the zero modes of the operators $\bar\partial_j$ arise in (3) for a reason which we shall illustrate by a simple example. Let $\Delta_j=\bar\partial_j^+\bar\partial_j$ not have any zero modes at all (for example, $j<0$ for $p>1$). We also neglect all details related to divergences, the regularization, and anomalies. Then

$$\mathrm{Det}\,\Delta_j=\int Dc D\bar c\exp\int|\bar\partial c|^2$$
$$=\int dbD\bar bDcD\bar c\exp\left(\int b\bar\partial c\right)\exp\left(\int \bar b\bar\partial\bar c\right)$$
$$\times\exp\left(\int\bar bb\right)$$
$$=\int DbD\bar bD\bar cD\bar c\exp\left(\int b\bar\partial c\right)$$
$$\times\exp\left(\int\bar b\bar\partial\bar c\right)\sum_n\frac{1}{n!}\left(\int\bar bb\right)^n$$
$$=\prod_{a=1}^{N_{1-j}}\int d^2\xi_\alpha\left|\int DbDcb(\xi_1)\dots b(\xi_{N_{1-j}})\right.$$
$$\left.\times\exp\left(\int b\bar\partial c\right)\right|^2=\prod_{a=1}^{N_{1-j}}\int d^2\xi_a|\det{}_{(ab)}b_a^{(0)}(\xi_b)|^2$$
$$\times\left|\frac{\int DbDcb(\xi_1)\dots b(\xi_{N_{1-j}})\exp(\int b\bar\partial c)}{\det_{(ab)}b_a^{(0)}(\xi_b)}\right|^2$$
$$=\det N_{1-j}|\mathrm{Det}\,\bar\partial_j|^2.\tag{4}$$

Here $c$ and $b$ are the fields of the $j$ and $(1-j)$ differentials, respectively; $b_a^{(0)}$, $a=1,\dots,N_{1-j}$, denote the linearly independent holomorphic $(1-j)$ differentials, the zero modes of $\bar\partial_{1-j}$. We also note that

$$\mathrm{Det}\,\bar\partial_j\equiv\frac{\int DbDcb(\xi_1)\dots b(\xi_{N_{1-j}})\exp(\int b\bar\partial c)}{\det_{(ab)}b_a^{(0)}(\xi_b)}=\mathrm{Det}\,\bar\partial_{1-j}.\tag{5}$$

This is the expression for the elementary measure $dv_p$ on $\mathfrak{M}_p$ and is needed in the construction of the Polyakov measure. From the viewpoint of the holomorphic properties it is important to use complex coordinates $y,\bar y$ on $\mathfrak{M}_p$. In terms of these coordinates[19]

$$dv_p=\left|\prod_{\alpha=1}^{N_2}dy_\alpha\right|^2/\det N_2.\tag{6}$$

From (3) and (6) we obtain for the Polyakov measure (2)

$$\left|\frac{\text{Det }\bar{\partial}_2}{[\text{Det }\bar{\partial}_0]^{13}}\prod_{\alpha=1}^{N_2} dy_\alpha\right|^2/(\det N_1)^{13}\equiv|d\mu_p|^2/(\det N_1)^{13}. \tag{7}$$

The holomorphic measure $d\mu_p$ is known as the Mumford measure. This measure, which is related to the ratio of the determinants of the Laplace operators on 0- and 1-differentials, the number of zero modes of which is independent of the moduli, cannot have zeros or poles inside $\mathfrak{M}_p$, and, according to Ref. 19, it must have a second-order pole at infinity (i.e., on infinite divisors of $\overline{\mathfrak{M}}_p$).

Explicit expressions for the Green functions $G(\xi,\xi')$ and the determinants entering into (1) and (7) are given in Refs. 29, 43, and 175. However, it turns out that the Mumford measure in the case of the lower genera $p\leqslant 4$ can be written in a much simpler form[26–28] in terms of modular forms on the Siegel upper half-plane. Historically, such expressions for $p=2,3$ were the first examples of explicit multiloop expressions in string theory. The genera $p=1,2,3$ are special because in these cases the moduli spaces can be parametrized by period matrices $T_{ij}=\int_{B_i}\omega_j$ (the canonical 1-differentials are normalized by the conditions $\int_{A_i}\omega_j=\delta_{ij}$, $i,j=1,\dots,p$). The space of such symmetric matrices with positive-definite imaginary part Im $T$, known as the Siegel upper half-plane $\text{Sieg}_p$, has complex dimension $p(p+1)/2$, which for $p=2,3$ coincides with the complex dimension $3p-3$ of the moduli space $\mathfrak{M}_p$, and for $p=4$ exceeds it by unity (they coincide also in the case $p=1$, which we do not consider, since it has been well studied for a long time[4,5]). According to the Torelli theorem,[38] the space $\mathfrak{M}_p$ is analytically embedded in $\text{Sieg}_p$, and the fact that the dimensions coincide allows us to use the $T_{ij}$ as local coordinates on $\mathfrak{M}_p$. But in the case of genus $p=4$, $\mathfrak{M}_4$ is a complex subspace of codimension one in $\text{Sieg}_4$ and can be viewed as a divisor. The association with the period matrix $T_{ij}$ of the Riemann surface is nonunique, owing to arbitrariness in the choice of canonical basis of the cycles $\{A_i,B_j\}$. The arbitrariness is reflected in the modular transformations $T\to(aT+b)/(cT+d)$, where $\left[\begin{smallmatrix}ab\\cd\end{smallmatrix}\right]\in\text{Sp}(p,\mathbb{Z})$, and for $p=2,3$, $\mathfrak{M}_p$ can be identified with Sieg
$(p,\mathbb{Z})$. The Polyakov measure on $\mathfrak{M}_p$ in these cases can be written in terms of $T_{ij}$, but is obliged to be modular-, i.e., $\text{Sp}(p,\mathbb{Z})$-invariant.

In the situation in question, $\Pi dy=\Pi_{i<j}dT_{ij}$ is not modular-invariant; the ratio $|\Pi_{i\leqslant j}^p dT_{ij}|^2/(\det\text{Im }T)^{p+1}$ is. If we choose the canonical 1-differentials as the zero modes of the operator $\bar{\partial}_1$, then $\det N_1=\det_{(ij)}\times\int\omega_i\bar{\omega}_j\sim\det\text{Im}T$, and the requirement that the Polyakov measure (7) be modular-invariant means that the expression $\chi_{12-p}(T)$ $(\partial\chi_{12-p}/\partial\bar{T}=0)$, which is holomorphic in the moduli, in the expression

$$d\mu_p=\prod_{i\leqslant j}^{p} dT_{ij}/\chi_{12-p}(T) \tag{8}$$

for the Mumford measure must be a modular form of weight $12-p$, i.e.,

$$\left|\chi_{12-p}\left(\frac{aT+b}{cT+d}\right)\right|^2=|\det(cT+d)^{12-p}\chi_{12-p}(T)|^2,$$

in order for $|\chi_{12-p}(T)|(\det\text{Im }T)^{12-p}$ to be modular-invariant. The problem of finding the Polyakov measure for the cases $p=2,3$ thereby reduces to the construction of modular forms of the necessary weight. They can be expressed in terms of the corresponding theta constants with half-integer even characteristics:

$$\text{gen }p=2\ \text{(Refs.26–28)}:\ \chi_{10}(T)=\prod_{\text{even }e}^{10}\theta_e^2(0|T); \tag{9}$$

$$\text{gen }p=3\ \text{(Ref.26)}\ :\chi_9^2(T)=\prod_{\text{even }e}^{36}\theta_e(0|T). \tag{10}$$

The number of even half-integer characteristics is $2^{p-1}(2^p+1)$. Under modular transformations the theta constants transform into each other. In addition to the change of the characteristic and the appearance of a phase factor, under a modular transformation each theta constant is multiplied by $[\det(cT+d)]^{1/2}$. It is easy to verify that the products on the right-hand sides of (9) and (10) form modular forms of weight 10 and 18 for $p=2$ and $p=3$, respectively. In the case of genus $p=2$ all the theta constants entering into the product (9) are nonzero for $T$ corresponding to interior points of the moduli space. This can be verified as follows. First we have the relation[19,29]

$$\det{}_e\,\bar{\partial}_{1/2}(\det\bar{\partial}_0)^{1/2}=\theta_e(0|T). \tag{11}$$

According to this expression, the theta constant vanishes only when $\bar{\partial}_{1/2}$ has zero modes with suitable boundary conditions, i.e., when on the Riemann surface there are holomorphic (1/2)-differentials with theta characteristic $e$. According to the Riemann theorem on zeros, the number of such differentials has the same parity as the theta characteristic $e$ [the number $\Sigma_i^p\delta_i\varepsilon_i$(mod2) is called the parity of the half-integer theta characteristic $e=\left[\begin{smallmatrix}\delta_1\dots\delta_p\\\varepsilon_1\dots\varepsilon_p\end{smallmatrix}\right]$]. Therefore, for the theta constant with even characteristic to vanish it is necessary that there be at least two holomorphic (1/2)-differentials on the Riemann surface. Each of these has $2j(p-1)|_{j=1/2}=p-1$ zeros on the surface, and the ratio of two differentials is a meromorphic function with $p-1$ poles. Riemann surfaces on which there are functions with a single simple pole are equivalent to the sphere, i.e., they have genus zero. If there is a function with two poles, it describes the surface as a double covering of the Riemann sphere, i.e., the surface is hyperelliptic, and so on. (A meromorphic function of general configuration on a surface of genus $p\neq 2$ has at least $p+1$ zeros.) This argument proves that the right-hand side of (9) does not have zeros inside the moduli space. We shall return to the case $p=3$ below. The following occurs on the boundary of the moduli space $\mathfrak{M}_2$. If $T_{12}\to 0$, then exactly one even theta constant vanishes, and this is a first-order zero:

$$\theta\begin{bmatrix}1&1\\1&1\end{bmatrix}(0|T)=T_{12}\theta'\begin{bmatrix}1\\1\end{bmatrix}(0|T_{11})\theta'\begin{bmatrix}1\\1\end{bmatrix}(0|T_{22})+O(T_{12}^3),$$

so $\chi_{10}$ has a double zero, as required. For $T_{11}\to i\infty$ the correct variable on the moduli space is $q_{11}=\exp(2\pi i T_{11})$, and it is easily verified that $\chi_{10}$ has a first-order zero for $q_{11}=0$. In addition, $dT_{11}dT_{12}dT_{22}\sim dq_{11}/q_{11}$ and $\det \operatorname{Im} T \sim \ln|q_{11}|$. These asymptotic expressions ensure agreement with the Belavin–Knizhnik theorem.[19] The analysis of the expressions for $p=3$ is similar. The only difference is that $\chi_9$ has zeros not only on the boundaries, but also inside the moduli space $\mathfrak{M}_3$. According to the argument following Eq. (11), these zeros are located on the hyperelliptic divisor in $\mathfrak{M}_3$. The moduli space of hyperelliptic surfaces has complex dimension $2p-1$, i.e., codimension 1 in $\mathfrak{M}_3$. According to (10), $\chi_9^2$ has a double zero on this divisor (owing to the two fermionic zero modes), and $\chi_9$ has a simple zero. This zero, however, is canceled by a zero of the measure $\Pi_{i\leq j}^3 dT_{ij}$ written in the correct coordinates on the space $\mathfrak{M}_3$. The relation between $T_{ij}$ and the coordinates $y_\alpha$ on $\mathfrak{M}_p$, related to the Beltrami $(1,-1)$-differentials $\eta_\alpha$, is given by

$$\partial T_{ij}/\partial y_\alpha = \int \eta_\alpha \omega_i \omega_j. \tag{12}$$

The statement that the Mumford measure does not have zeros inside $\mathfrak{M}_p$ is valid in the coordinates $y_\alpha$ related to the Beltrami differentials $\eta_\alpha = \{N_2^{-1}\cdot \bar{f}\}_\alpha/\rho + \bar{\partial}\varepsilon_\alpha$ ($\rho$ is the metric in the conformal gauge, $f_\alpha$ is the basis of holomorphic quadratic differentials, and $\varepsilon_\alpha$ are vector fields). On the hyperelliptic surface given by the equation

$$y^2 = \prod_{j}^{2p-1} (x-a_j), \tag{13}$$

the holomorphic quadratic differentials have the form (see Subsec. 1.2 for details about hyperelliptic surfaces)

$$f_{1\mu} = x^{\mu-1}(dx)^2/y^2(x), \quad \mu = 1,\dots,2p-1; \tag{14}$$

$$f_{2\mu} = x^{\mu-1}(dx)^2/y(x), \quad \mu = 1,\dots,p-2. \tag{15}$$

At the same time, the canonical linear holomorphic differentials $\omega_i$ are linear combinations of 1-differentials of the form

$$v_i = x^{i-1}dx/y(x), \quad i=1,\dots,p. \tag{16}$$

It follows from these expressions that the $v_i$ are odd under $Z_2$ symmetry transformations of the hyperelliptic surface $y\to -y$; $f_{1\mu}$ are even and $f_{2\mu}$ are odd. For this reason the integrals on the right-hand side of (12) corresponding to $f_{2\mu}$ vanish (the integrands are $Z_2$-odd). Therefore, for $p=3$ the Jacobian of the transformation from $T_{ij}$ to $y_\alpha$ has a first-order zero on the hyperelliptic divisor of the moduli space, which cancels the same zero of $\chi_9$ in the expression for the Mumford measure. The asymptotes on the boundary of the moduli space are studied as in the case $p=2$.

Beginning with $p=4$, the space $\mathrm{Sieg}_p$ of period matrices (with respect to the modulus of $\mathrm{Sp}(p,\mathbb{Z})$ transformations) does not coincide with the moduli space $\mathfrak{M}_p$. For $p=4$ their complex dimensions differ only by unity, and $\mathfrak{M}_4$ can be viewed as a divisor (the Schottky divisor) in the space $\mathrm{Sieg}_4$. This divisor is given by the Schottky equation

$$\chi_8(T)=0, \tag{17}$$

where

$$\chi_8(T) \equiv 2^{-p}\sum_e \theta_e^{16}(0|T) - 2^{-2p}\Big[\sum_e \theta_e^8(0|T)\Big]^2 \tag{18}$$

is a modular form of weight 8. [The form (18) is apparently identically equal to zero on all moduli spaces $\mathfrak{M}_p$ (but not $\mathrm{Sieg}_p$!); this is important, in particular, for the $\Gamma_8\times\Gamma_8$ and $\Gamma_{16}$ string models being identical.] The authors of Refs. 19 and 46 proposed the following expression for the Polyakov measure for $p=4$:

$$\left| \operatorname{res} \frac{\Pi_{i<j}^4 dT_{ij}}{\chi_8} \right|^2 (\det \operatorname{Im} T)^{-13}. \tag{19}$$

So far no complete analysis of this expression has appeared in the literature.

### 1.2. Calculations in the hyperelliptic case

The derivation of explicit expressions for the Mumford measure in the case $p>4$ is made difficult by the absence in all the relations of a suitable parametrization of the moduli space $\mathfrak{M}_p$. The problem is clearly seen in the example of the Knizhnik determinant formulas.[29] The problem with them is that they involve both theta functions and Riemann classes $\Delta = \Sigma \mathbf{R}_i$. These two classes of objects are not independent—they are related by the Riemann zero theorem—and in this sense the formulas obtained are not completely explicit; instead, they are formulas with ratios. In Subsec. 1.1 we showed how this difficulty can be overcome in the simple cases $p\leqslant 4$ by parametrizing the moduli space not by the points $R_i$, but by period matrices and not directly using the determinant formulas from Refs. 29, 45, and 175 at all. In the case of arbitrary $p$ this possibility is not available. If we use a parametrization of $\mathfrak{M}_p$ in terms of $R_i$, it is necessary to get rid of the explicit theta functions in the formulas for the Mumford measure. The points $R_i$ naturally arise in the description of a Riemann surface as a branched covering of the Riemann sphere. An independent derivation of the determinant formulas on the basis of such a description is possible.[49,175] It is also possible to directly trace the "disappearance" of the theta functions from the expressions.[29] Unfortunately, the problem has not yet been solved in its full generality. Complete answers have been obtained in the case of the so-called Abelian $N$-sheeted coverings of the Riemann sphere possessing $Z_N$ symmetry. The simplest examples of this type are hyperelliptic surfaces, $N=2$. Here we follow Refs. 37 and 48. The technique of hyperelliptic calculations was also developed in Refs. 49 and 50 and in numerous subsequent publications.

The determinant formulas for $\Lambda_j \equiv (\det'\partial_0)^{1/2}\det'\delta_j$ following from the expressions[54] for the correlators of the $b$ and $c$ fields have the form[29]

$$\Lambda_0 = \theta_{,i}^{*}\hat{\omega}_i''(R_1)/\det[\hat{\omega}_i''(R_1)\hat{\omega}_i(R_1)\dots\hat{\omega}_i(R_{p-1})]; \tag{20a}$$

$$\Lambda_2 = \theta_{,i}^{*}\hat{\omega}_i''(R_1)/\det[\hat{f}_\alpha(R_1)\hat{f}_\alpha''(R_1)\hat{f}_\alpha'''(R_1)\hat{f}_\alpha(R_2)$$

$$\times\hat{f}'_\alpha(R_2)\hat{f}''_\alpha(R_2)\dots\hat{f}_\alpha(R_{p-1})$$
$$\times\hat{f}'_\alpha(R_{p-1})\hat{f}''_\alpha(R_{p-1})]; \qquad (20b)$$
$$\Lambda_{1/2}[e]=\theta[e]; \qquad (20c)$$
$$\Lambda_{3/2}[e]=\theta[e]/\det_{(\mu j)}[\hat{\zeta}_\mu(R_j)\hat{\zeta}'_\mu(R_j)]. \qquad (20d)$$

Owing to anomalies, the expressions for the individual determinants necessarily contain an additional dependence on the metric (and not only on the complex structure) and the choice of coordinates on the surface. For a suitable choice of metric this information can be reduced to a minimum: if the metric is the squared modulus of a holomorphic 1-differential, the answer involves only its zero (the metric singularities, the number of which is $2p-2$). In Eq. (20) the metric $\rho=|\theta^*_{,i}\omega_i|^2$ is constructed from the holomorphic 1-differential $v_*^2\equiv\Sigma_{i=1}^p\theta^*_{,i}\omega_i$ with double zeros at the points $R_1,\dots,R_{p-1}$; $\theta^*_{,i}\omega'_i(R_k)=\theta^*_{,i}\omega'_i(R_k)=0$. The quantity $\theta^*$ is the theta function on the surface with nonsingular odd characteristic $e_*$ and $\theta^*_{,i}$ is its derivative with respect to the $i$th argument at zero:

$$\theta^*_{,i}\equiv\frac{\partial}{\partial z_i}\theta[e_*](z_1\dots z_p|T)|_{z_1=\dots=z_p=0}.$$

The dependence on $e_*$ (and on the metric $\rho$ in general) must cancel in anomaly-free combinations like $\Lambda_2\Lambda_0^{-9}$ or $\Lambda_2\Lambda_0^{-5}\Lambda_{-1/2}^5\Lambda_{3/2}^{-1}$ entering into the expressions for the Mumford measure and its superanalog. Finally, in accordance with the general meaning of the determinant of a non-self-adjoint operator,[42] it depends on the choice of basis in the space of zero modes of the operator itself and its adjoint. In the case of the operators $\bar{\partial}_j$ these zero modes are holomorphic $j$ and $(1-j)$ differentials. Therefore, Eqs. (20) involve the holomorphic 1-differentials $\omega_1\dots\omega_p$, the holomorphic 2-differentials $f_1\dots f_{3p-3}$, and the holomorphic (3/2)-differentials $\zeta_1\dots\zeta_{2p-2}$. For simplicity we assume that the boundary conditions on half-integer differentials are such that holomorphic (1/2)-differentials are absent (the theta characteristic $e$ is nonsingular). We note that only the number of holomorphic (1/2)-differentials is not determined uniquely by the Riemann–Roch index theorem in conjunction with the theorem on the absence of holomorphic $j$ differentials with $j<0$ on Riemann surfaces of genus $p>1$. The residues of the differentials are marked with hats, and their derivatives with primes. If $\xi$ is a local coordinate in the vicinity of the point $\xi_0$, then $\omega(\xi)=[\hat{\omega}(\xi_0)+(\xi-\xi_0)\hat{\omega}(\xi_0)+\dots]d\xi$ and so on. A final comment about the notation: the letter $\omega$ will always be used to denote canonical 1-differentials normalized by the conditions $\oint_{A_i}\omega_j=\delta_{ij}$.

As already noted, the use of Eqs. (20) is complicated by the fact that these equations explicitly involve numerous holomorphic differentials and points $R_k$, about which there is not sufficient information in the case of Riemann surfaces of arbitrary genus. However, all of these objects are easily constructed if the Riemann surface is specified explicitly. An important class of such surfaces is formed by the hyperelliptic surfaces, defined by the equation

$$y^2(x)=\prod_{n=1}^{2p+2}(x-a_n). \qquad (21)$$

The aim of this section is to analyze Eqs. (20) in the case of hyperelliptic surfaces. We are not interested in determining the numerical factor in front of the correlators, which depends only on the genus (on the topology), so for the rest of this subsection the equalities hold only up to such factors. Hyperelliptic surfaces form a $(2p-1)_{\mathbb{C}}$-dimensional subspace $\mathscr{D}_p^H$ in $\mathfrak{M}_p$, which coincides with the entire moduli space only for $p=1,2$ (i.e., all Riemann surfaces are hyperelliptic only for the genera $p=1,2$). However, the constraints of the string measures on $\mathscr{D}_p^H$ can also be of interest for $p\geqslant3$. See Refs. 49 and 50 regarding generalizations to Abelian $Z_N$ coverings, and Ref. 175 on attempts to study surfaces of general form in analogous terms.

### *Holomorphic 1-differentials*

We shall assume that the surface is given by Eq. (21). The Riemann surface of this equation is the set of two Riemann spheres glued together along the $(p+1)$ cuts running between the branch points, for example, from $a_1$ to $a_2$, from $a_3$ to $a_4$,..., and from $a_{2p+1}$ to $a_{2p+2}$. The hyperelliptic surface obtained as a result of this gluing obviously has $p$ handles, i.e., it is a surface of genus $p$. The parameter $x$ is a coordinate in the vicinity of any finite point not coinciding with any of the branch points $a_n$. Near the $a_n$ the local parameter is $\xi=\sqrt{x-a_n}$. Finally, near infinitely separated points the local coordinate is $\xi=1/x$ if all the $a_n\neq\infty$ and $\xi=1/\sqrt{x}$ if some $a_n=\infty$. Below we shall assume that all the $a_n\neq\infty$. Then the 1-differential $dx$ has simple zeros at all the points $a_n$ and double poles at the two points with infinite separation (on two sheets of the Riemann surface). The function $y(x)$ also has simple zeros at all the $a_n$ and poles of order $p+1$ at infinity. Therefore, $v_j(x)\equiv x^{j-1}dx/y(x)$ for $1<j<p$ are holomorphic differentials—they have no poles on the entire hyperelliptic surface. However, the $p$ holomorphic 1-differentials $v_1,\dots,v_p$ do not coincide with the canonical differentials $\omega_1,\dots,\omega_p$. Instead, we have a linear relation: $v_i=\sigma_{ij}\omega_j$ and $\sigma_{ij}=\oint_{A_j}v_i$. The integral over the Riemann surface is

$$\int\omega_i\bar{\omega}_j=\sum_{k=1}^p\left(\oint_{A_k}\omega_i\oint_{B_k}\bar{\omega}_j-\oint_{B_k}\omega_i\oint_{A_k}\bar{\omega}_j\right)$$
$$=-2i\,\mathrm{Im}T_{ij},$$

and so $\det\mathrm{Im}T\sim\det_{(ij)}\int\omega_i\bar{\omega}_j$. In exactly the same way,

$$G\equiv|\det\sigma|^2\det\mathrm{Im}T\sim\det_{(ij)}\int v_i\bar{v}_j$$
$$=\frac{1}{p!}\int\dots\int\left|\prod_{\substack{i<j\\i,j=1}}^p(x_i-x_j)\prod_{i=1}^p\frac{dx_i}{y(x_i)}\right|^2. \qquad (22)$$

### *θ characteristics and spinor structures (Refs. 39 and 40)*

Each Riemann surface can be associated with $2^{2p}$ half-integer $\theta$ characteristics. Each $\theta$ characteristic $e$ is a pair of $p$ vectors $\delta=(\delta_1...\delta_p)$ and $\varepsilon=(\varepsilon_1...\varepsilon_p)$ composed of the numbers 0 and 1. The (not necessarily half-integer) $\theta$ characteristic corresponds to the $\theta$ function

$$\omega[e](z|T)=\sum_{n\in\mathbb{Z}^p}\exp\Big\{i\pi\sum_{i,j=1}^{p}(n+\delta/2)_iT_{ij}\times(n+\delta/2)_j+2\pi i\sum_{i=1}^{p}(n+\delta/2)_i\times(z+\varepsilon/2)_i\Big\}. \quad (23)$$

The half-integer $\theta$ characteristics split into even and odd ones, depending on the parity of the function $\theta[e](z)$ or, equivalently, on the parity of the sum $\Sigma_{i=1}^{p}\delta_i\varepsilon_i$. There are $2^{p-1}(2^p+1)$ even and $2^{p-1}(2^p-1)$ odd $\theta$ characteristics. Under modular transformations $T\to(aT+b)/(cT+d)$, $\theta$ functions with characteristics of the same parity transform into each other. The $\theta$ characteristics are interesting for string theory primarily because they are in one-to-one correspondence with the boundary conditions on the fermionic fields—half-integer differentials. In the case of hyperelliptic surfaces there is an additional relation between the half-integer $\theta$ characteristics and the branch points (for $N$-sheeted coverings the analogous relation arises for the $1/N$ characteristics). Each splitting of the $2p+2$ branch points into two sets of $p+1$ points is associated with one of the $\frac{1}{2}C_{2p+2}^{p+1}=(2p+2)!/2[(p+1)!]^2$ even $\theta$ characteristics, which are termed nonsingular (index 0). The splitting into two sets of $p-1$ and $p+3$ points determines one of the $C_{2p+2}^{p-1}$ "nonsingular" odd $\theta$ characteristics (index 1). All the other $\theta$ characteristics are termed singular and are associated with splittings of the branch points into sets of $p+1-2m$ and $p+1+2m$ elements, where for even "indices" $m$ the $\theta$ characteristics are even, and for odd $m$ they are odd. The total number of odd characteristics is

$$\sum_{\substack{\text{odd } m\\ 1\leqslant m\leqslant(p+1)/2}}C_{2p+2}^{p+1-2m}=2^{p-1}(2^p-1), \quad (24)$$

and the number of even ones is

$$\tfrac{1}{2}C_{p+2}^{p+1}+\sum_{\substack{\text{even } m\\ 1\leqslant m\leqslant(p+1)/2}}C_{2p+2}^{p+1-2m}=2^{p-1}(2^p-1). \quad (25)$$

The relation of the $\theta$ characteristics to the splitting of the branch points is determined in terms of the $\theta$ constants: the values of the $\theta$ functions and their derivatives with respect to $\mathbf{z}$ at zero, $\mathbf{z}=0$. (We shall denote these simply as $\theta[e]$, $\theta_{,i}[e]$, $\theta_{,ij}[e]$...). In the case of even characteristics all the odd derivatives at zero vanish for any period matrices [since $\theta(\mathbf{z})$ is an even function], and in the case of odd characteristics all the odd derivatives at zero vanish. On the subspace $\mathscr{D}_p^H$ of hyperelliptic surfaces in the moduli space $\mathfrak{M}_p$ not all the even $\theta$ constants $\theta[e]$ are nonzero, but only those which correspond to nonsingular characteristics. In general, for a $\theta$ characteristic of index $m$ only the $m$th derivative $|\theta_{,i_1,...,i_m}[e]|_{\mathscr{D}_p^H}$ is the first nonzero one. Singular $\theta$ characteristics appear for $p\geqslant 3$, when $\mathscr{D}_p^H$ ceases to coincide with the entire moduli space $\mathfrak{M}_p$.

Let the nonsingular even $\theta$ characteristic $e$ correspond to the splitting of the branch points into two sets of $p+1$ elements $\{a_q\}$ and $\{a_{\tilde q}\}$. Then the Thomae formula is valid:

$$|\theta[e]^4|_{\mathscr{D}_p^H}=\det\sigma^2\prod_{q<r}(a_q-a_r)\prod_{\tilde q<\tilde r}(a_{\tilde q}-a_{\tilde r}). \quad (26)$$

Analogous representations exist in all the other cases: for all $|\theta_{,i_1,...,i_m}[e]|_{\mathscr{D}_p^H}$ of index $m$. We shall also need the expression in the case of a nonsingular odd characteristic $e_*$. It corresponds to the set of $p-1$ points of the $2p+2$ branch points. These points are exactly the double zeros $R_1^*,...,R_{p-1}^*$ of the 1-differential:

$$\left.\begin{aligned}&\theta^*_{,i}\omega_i(z)=c\prod_{k=1}^{p-1}(z-R_k^*)\frac{dz}{y(z)};\\&c^4=\det\sigma^4\prod_{m<n}^{2p+2}(a_m-a_n)\prod_{k<l}^{p-1}(R_k^*-R_l^*)\prod_{k}^{p-1}\hat y(R_k^*)^{-2};\\&\hat y(R_k^*)^2=\prod_{n\neq k}^{2p+2}(R_k^*-a_n).\end{aligned}\right\} \quad (27)$$

By integrating along the cycle $A_i$ we easily find[177] that

$$\theta^*_{,i}=c\sum_{k=0}^{p-1}(-)^kS_k\{R\}\sigma_{p-k,i}. \quad (28)$$

Here $\sigma_{p-k,i}=\int_{A_i}v_{p-k}$, and $S_k\{R\}$ is a symmetric polynomial of degree $k$ composed from the points $R$: $S_0=1$, $S_1=\Sigma_kR_k$, $S_2=\Sigma_{k<l}R_kR_l$,... .

Let us now turn to the calculation of Eq. (20a). The numerator in this expression is already known from Eq. (27). The determinant in the denominator is

$$(\det\sigma)^{-1}\det[v_i(z)\hat v_i(R_1)...\hat v_i(R_{p-1})]=(\det\sigma)^{-1}\frac{dz}{y(z)}\times\Big[\det[z^{i-1},R_1^{i-1},...,R_{p-1}^{i-1}]/\prod_k^{p-1}\hat y(R_k)\Big]. \quad (29)$$

The Vandermondès determinant remaining in the numerator is equal to $\Pi_{k<l}(R_k-R_l)\Pi_k(z-R_k)$. Therefore,

$$\Lambda_0=c(\det\sigma)\Big[\prod_k\hat y(R_k)/\prod_{k<l}(R_k-R_l)\Big]=(\det\sigma)^{3/2}\prod(a)^{1/4}\prod(R)^{-1/2}Y(R)^{1/2}. \quad (30)$$

Here we have introduced convenient abbreviated notation:

$$\prod(a)=\prod_{m<n}^{2p+2}(a_m-a_n);$$

$$\prod(R)=\prod_{k<l}^{p-1}(R_k-R_l);$$

$$Y(R)=\prod_{k}^{p-1}\hat{y}(R_k);$$

$$c=(\det\sigma)^{1/2}\prod(a)^{1/4}\prod(R)^{1/2}Y(R)^{-1/2}.$$

To calculate $\Lambda_2$ it is necessary to also know the basis of holomorphic quadratic differentials associated with the branch points as coordinates on the moduli space. More precisely, as the coordinates $y_1,...,y_{2p-1}$ we take $a_1,...,a_{2p-1}$ (variations of the other moduli $y_{2p},...,y_{3p-3}$ lead out from the subspace of hyperelliptic curves to the moduli space), and we take the three points $a_{2p}=a'$, $a_{2p+1}=a''$, $a_{2p+2}=a'''$ to be fixed (changes of them correspond to linear-fractional coordinate transformations). The correct linear combinations of the quadratic differentials $\tilde{f}_\alpha=z^{\alpha-1}dz^2/y^2(z)$, $\alpha=1,...,2p-1$, corresponding to changes of the moduli preserving the hyperellipticity [shifts related to $F_\gamma=z^{\gamma-1}dz^2/y(z)$, $\gamma=1,...,p-2$ violate it] have the form

$$f_\alpha=\frac{dz^2(a_\alpha-a')(a_\alpha-a'')(a_\alpha-a''')}{(z-a_\alpha)(z-a')(z-a'')(z-a''')}. \tag{31}$$

One of the ways of verifying the validity of this expression is based on the general relation[19,48]

$$\partial T_{ij}/\partial y_\alpha=\int\eta_\alpha\omega_i\omega_j \tag{32}$$

and the elementary formula

$$\partial T_{ij}/\partial a_\alpha=\hat{\omega}_i(a_\alpha)\hat{\omega}_j(a_\alpha) \tag{33}$$

As a result, for $\Lambda_2$ defined by Eq. (20b) we have

$$\Lambda_2=(\det\sigma)^{1/2}\prod(a)^{-3/4}\prod(R)^{-1/2}Y(R)^{1/2}$$

$$\times(a'-a'')(a''-a''')(a'''-a'), \tag{34}$$

and for the Mumford measure

$$d\mu_{\rm bos}^{(p)}|_{\mathscr{D}_p^H}=\Lambda_0^{-9}\Lambda_2\prod_{\alpha}^{3p-3}dy_\alpha$$

$$=(\det\sigma)^{-13}\prod(a)^{-3}\left(\prod_{n=1}^{2p+2}da_n/d\Omega\right)dV_\perp. \tag{35}$$

Here

$$d\Omega=\frac{da'da''da'''}{(a'-a'')(a''-a''')(a'''-a')},$$

and $dV_\perp=\Pi_{\alpha=2p}^{3p-3}dy_\alpha$ is the measure on the subspace orthogonal to $\mathscr{D}_p^H$ in the moduli space. See Ref. 48 for the detailed derivation of Eqs. (32)–(35) and also the expressions for $\Lambda_{1/2}$ and $\Lambda_{3/2}$ on $\mathscr{D}_p^H$. The formulation of the Riemann identities in the hyperelliptic case and useful generalizations of them as polynomial identities for the branch points $a_\alpha$ are given in Refs. 37 and 178.

### 1.3. Compactification on a torus and an orbifold (the case $p=1$)

The Lagrangian $\partial X\bar{\partial}X$ for the scalar field $X$ admits the imposition of various auxiliary conditions on the global behavior of this field. Up to now we have considered the case where such conditions are absent and the field $X$ takes values in the noncompact line $\mathbb{R}$. However, other cases when instead of a straight line we have a circle, a ray, or a segment are also interesting. In the general case of a multicomponent scalar field $X^\mu$ this can be interpreted as a model of a string compactified on a torus or an orbifold.

The analysis of compactification on a torus carried out in Ref. 30 in studying the "bosonization" of $b$, $c$ systems is very well known. This trivial calculation has been carried out independently by many different authors. Here we shall follow Ref. 56. The functional integral for the case of toroidal compactification contains an additional $2p$-fold sum over the lattice $\Gamma^{(D)}$ specifying the toroidal structure in dimension $D\leqslant d=26$. Accordingly, it is necessary to take into account the contributions of the various sectors differing by the results of going around the $2p$ noncontractible cycles on the world sheet. The field $X^\alpha$, $\alpha=1,...,D$, in a given sector has the form

$$X^\alpha(z,\bar{z})=\tilde{X}^\alpha(z,\bar{z})+(2i)^{-1}\Big[(\Lambda_{1j}^\alpha-\Lambda_{2i}^\alpha\bar{T}_{ij})$$

$$\times(1/\mathrm{Im}\,T)_{jk}\int^z\omega_k-(\Lambda_{1j}^\alpha-\Lambda_{2i}^\alpha T_{ij})$$

$$\times(1/\mathrm{Im}\,T)_{jk}\int^{\bar{z}}\bar{\omega}_k\Big]. \tag{36}$$

Here $\tilde{X}$ is a periodic (quantum) field, and the classical ("instanton") solution is defined by specifying the $2p$ vectors $\Lambda_{1i}$, $\Lambda_{2i}$, $i=1,...,p$, taking values in $\Gamma^{(D)}$. As a result, the two-dimensional action takes the form

$$\int\partial X^\alpha\bar{\partial}X^\alpha=\int\partial\tilde{X}^\alpha\bar{\partial}\tilde{X}^\alpha+\tfrac{1}{4}(\Lambda_{1j}^\alpha-\Lambda_{2i}^\alpha\bar{T}_{ij})$$

$$\times(1/\mathrm{Im}\,T)_{ik}(\Lambda_{1k}^\alpha-\Lambda_{2l}^\alpha T_{lk})$$

(owing to the periodicity of $\tilde{X}$ there are no cross terms), and the measure on the moduli space corresponding to toroidal compactification contains the additional ("instanton") factor

$$\mathscr{F}_\Gamma^{(p)}=\sum_{\Lambda_{1i},\Lambda_{2i}\in\Gamma}\exp\{-\pi(\Lambda_1-\Lambda_2\bar{T})(1/\mathrm{Im}\,T)$$

$$\times(\Lambda_1-\Lambda_2T)\}. \tag{37}$$

After Fourier-transforming in the variable $\Lambda_1\to M$ (known as the Poisson transform), from Eq. (37) we obtain

$$\mathscr{F}_\Gamma^{(p)}=(\det\mathrm{Im}\,T)^{D/2}\sum_{\substack{\Lambda_i\in\Gamma\\ M_i\in\Gamma^*}}\exp\Big\{\frac{i\pi}{2}(M+\Lambda)$$

$$\times T(M+\Lambda)\Big\}\exp\Big\{-\frac{i\pi}{2}(M-\Lambda)\bar{T}(M-\Lambda)\Big\}. \tag{38}$$

The dependence on the moduli in the instanton factor reduces to a dependence on the period matrices, so (38) can be rewritten as a bilinear combination of lattice $\theta_\Gamma$ functions. In the case of even self-dual lattices this combination contains a finite number of terms. The factor in front of the summation sign in (38) partially cancels $(\det \mathrm{Im} T)^{-d/2}$ in the Polyakov measure, in agreement with the factorization requirement—the degree of the logarithmic (dilaton) singularity on the boundary of the moduli space is determined by the number $d-D$ of noncompactified dimensions.[19] (See Ref. 56 for more details.)

Following Ref. 57, let us briefly describe the new elements arising in orbifold compactification (restricting ourselves to the case of an Abelian orbifold and genus $p=1$). Now the space of values of the field $X$ is obtained by factorizing $\mathbb{R}^D$ with respect to the action of the affine group of transformations $S$:

$$S \circ X = (g,\lambda) \circ X = gX + \lambda, \quad \lambda \in \Gamma^D, \quad g \in G \subset O(D). \tag{39}$$

Multiplication in the group $S$ is defined in the obvious manner:

$$s_1 \circ s_2 = (g_1,\lambda_1) \circ (g_2,\lambda_2) = (g_1 g_2, g_1\lambda_1 + \lambda_2),$$

i.e., $S$ is the semi-direct product of the stationary subgroup $G$ and the translation subgroup. If the group $G$ is finite, its elements have finite order. We assume that the original space of the fields $X$ is even-dimensional and equivalent to $\mathbb{C}^{D/2}$, and that $G \subset SU(D/2)$. Let us consider transformations generated by a fixed element $g$ of order $h$: $g, g^2, \dots, g^h = 1$. In some basis in $\mathbb{C}^{D/2}$ they are simultaneously diagonalizable:

$$g^k = \varepsilon^k = \mathrm{diag}(\varepsilon_1^k, \dots, \varepsilon_{D/2}^k), \quad k = 1, \dots, h. \tag{40}$$

Here

$$\varepsilon_\mu = \exp\left(\frac{2\pi i}{h} m_\mu\right), \quad \sum_{\mu=1}^{D/2} m_\mu = 0 (\mathrm{mod}\, h). \tag{41}$$

The last equation follows from the fact that $g$ belongs to the group $SU(D/2)$.

Let us consider one-loop configurations of a closed bosonic string lying on the orbifold. Then

$$X(z+1) = g_1 X(z) + \lambda_1, \quad X(z+\tau) = g_2 X(z) + \lambda_2. \tag{42}$$

For these boundary conditions to be compatible we must require that

$$[g_1, g_2] = 0, \quad g_2\lambda_2 - \lambda_1 = g_1\lambda_2 - \lambda_2. \tag{43}$$

The first of the conditions (43) was first discussed in Ref. 179. It determines the general form of the transformations $g_1 g_2$: $g_1 = \varepsilon^k$, $g_2 = \varepsilon^s$, $k,s \in \mathbb{Z} (\mathrm{mod}\, h)$, where $\varepsilon$ has the form (40). Configurations with $s=k=0$ form the untwisted sector, which was actually dealt with in the first half of this subsection. The other configurations form the twisted sector, which requires separate analysis. The compatibility conditions (43) already make the one-loop case special: these conditions have a simple form, and their solution is easily classified only for $p=1$. It is important that in the twisted sector the determinants of the Laplace operators differ from their periodic analogs and, in particular, depend on $s$ and $k$. In the case $p=1$ there is no difficulty in explicitly writing out all the eigenfunctions and calculating the determinants by multiplying together the eigenvalues.[57] The contribution of the sector with fixed $k$ and $s$ not simultaneously equal to zero to the measure on the moduli space differs from the case of the noncompactified string by the factor

$$(\mathrm{Im}\, \tau)^{D/2} \prod_{\mu=1}^{D/2} \frac{\eta^3(q) \exp[-i\pi(k m_\mu/h)^2 \tau]}{\theta[^1_1][u_\mu(k,s;\tau)|\tau]}, \tag{44}$$

where $u_\mu(k,s;\tau) \equiv (k\tau - s) m_\mu/h$. In the multiloop case for analyzing the simplest $Z_2$ orbifold it is necessary to use the Prym-manifold technique (see Ref. 58) and its further generalization for $Z_n$ orbifolds.[59] Up to now no technique has been developed for multiloop calculations for non-Abelian orbifolds. This problem is closely related to the development of the theory of free fields on Riemann surfaces specified as branched coverings on $CP^1$ (the hyperelliptic case is described in Subsec. 1.2, and the case of other Abelian $Z_n$ coverings is discussed in Refs. 49 and 50). For a discussion of the importance of this representation for the development of string theory, see, for example, Sec. 12 of the review by Knizhnik.[175]

### 1.4. The $\beta,\gamma$ fields on Riemann surfaces

Three fundamentally different theories of massless free fields can be defined on Riemann surfaces. Two of them are obvious: a free scalar field with Lagrangian $\partial\phi\bar\partial\phi$, either without any additional constraints, or taking values on a circle of radius $r$, i.e., related by the equivalence condition $\phi \sim \phi + 2\pi r$. The second of these theories, in particular, is sufficient for describing $b,c$ systems—Grassmann fields of $j$- and $(1-j)$-differentials with the first-order Lagrangian $b\bar\partial c$. (It is also possible to distinguish the case where $\phi$ takes values on a ray or a segment. This is in fact the orbifold compactification discussed in Subsec. 1.3.) The third theory of free fields is more complicated. It describes bosonic fields of $j$- and $(1-j)$-differentials with the first-order Lagrangian $\beta\bar\partial\gamma$. These fields were first introduced by Friedan, Martinec, and Shenker[60] for describing superghosts in the NSR superstring model. Later in Ref. 44 it was shown that they are also important for describing the WZNW model and other conformal models. The theory of the $\beta,\gamma$ fields was developed in Refs. 33, 37, and 62; see also Refs. 35, 63, and 110. A complete review of the entire theory of free fields is given in Ref. 45. Recently, Gerasimov[36] proposed a direct derivation of the correlators of the $\beta,\gamma$ fields from the functional integral (i.e., all the necessary reductions of the integration space were determined). Here in this subsection we discuss the early stage of development of the theory of $\beta,\gamma$ fields: following Ref. 37, we describe the assumption for the correlator which is most important in the NSR model; by now it has been proven rigorously. We also discuss the direct derivation, using the methods of Refs. 62 and 63, of the important Lechtenfeld formula[110] published in Ref. 111.

According to Ref. 60, the $\beta,\gamma$ fields can be viewed as a subsector in the theory of a free scalar field $\phi$ and the $b,c$ system 0 (traditionally denoted as $\xi,\eta$). Here

$$\beta = v_*^{2j-1} e^{-\phi}\partial\xi, \quad \gamma = v_*^{1-2j} e^{\phi}\eta \tag{45}$$

[such a bosonization assumes that the singular metric $|v_*(z)|^4$, or at least $|\Omega(z)|^2$ in the case of half-integer $j$, is fixed on the surface]. For superghosts in the NSR model, $j=3/2$. (The current understanding of $\beta,\gamma$ systems[36] is based on the accurate separation of this subsector.) The most important correlator in the NSR theory (see Subsec. 2.1 below) has the form

$$G_e(x|z) = \langle \xi(x_0)\xi(x_1)\ldots\xi(x_{n_j})\gamma(x_1)\ldots\gamma(x_{n_j})\rangle_e \tag{46}$$

(for half-integer $j$ it depends on the spinor structure $e$), and $n_j = (2j-1)(p-1)$. As usual in the two-dimensional theory of free fields, the zero modes play a fundamental role in the analysis of this correlator. The Grassmann scalar field $\xi$ has exactly one—constant—zero mode, and it must be absorbed by one of the $\xi$ fields in the correlator (46). In the case of the bosonic $(1-2j)$-differential (for $j>1/2$, $p\geqslant 2$), its zero modes would lead to divergences of the functional integral over the $\beta,\gamma$ fields, Since $1-2j<0$, $\gamma$ can have zero modes only owing to simple poles at the points $x_0,\ldots,x_{n_j}$ dictated by the operator expansion $\xi(z)\gamma(z')\sim(z-z')^{-1}$ following from (45). In fact, one of these points must be excluded, since one of the fields $\xi$ is required for absorption of the zero mode $\xi=\text{const}$. As a result, the correlator (46) breaks up into a sum:

$$G_e(x|z) = \sum_{\alpha=0}^{n_j} g_e(x_0,\ldots,x_\alpha,\ldots,x_{n_j}), \tag{47}$$

and in the $\alpha$th term the field $\gamma$ has simple poles at all points $x$ except $x_\alpha$. The zero mode $\gamma_0(z)$ must be the corresponding meromorphic differential. If the point $x_\alpha$ were not excluded from the set of poles, then $\gamma_0(z) = \langle c(z)\Pi_{\beta=0}^{n_j} b(x_\beta)\rangle$, where $b$ and $c$ are components of the Grassmann $b,c$ system with the same spin $j$. In the case (47) the pole at $z=x_\alpha$ must be absent, i.e., the residue $\langle \Pi_{\beta\neq\alpha}^{n_j} b(x_\beta)\rangle = 0$. In other words, one of the zeros of $\gamma_0(z)$ [the locations of which are given by the equation $\theta_e(\Sigma_{\beta=0}^{n_j}\mathbf{x}_\beta - z - (2j-1)\Delta_* = 0)$] must coincide with $x_\alpha$ [i.e., $\theta_e(\Sigma_{\beta\neq\alpha}^{n_j}\mathbf{x}_\beta - (2j-1)\Delta_*) = 0$]. Therefore, the field $\gamma$ has a zero mode, and the correlator (46) is singular only for a very special choice of points $x$. The result for $G_e(x|z)$ suggested in Ref. 37 on the basis of these arguments has the form

$$G_e(x|z) = \frac{\Pi_{i=1}^{n_j}\langle c(z_i)\Pi_{\beta=0}^{n_j} b(x_\beta)\rangle_e}{\Pi_{\alpha=0}^{n_j}\langle \Pi_{\beta\neq\alpha}^{n_j} b(x_\beta)\rangle_e}. \tag{48}$$

The correlators of the $b,c$ fields are known from Ref. 29 and are expressed in terms of $\theta$ functions.

The main drawback of this expression, which is central to multiloop calculations in the NSR model, is that it does not give compact expressions for the components $g_e$ in (47). This deficiency is partially corrected by the more general expression for the correlators discovered by Lechtenfeld:[110]

$$\Gamma_e(x;y;z) = \left\langle \prod_{i=1}^{n}\beta(x_i)\prod_{j=1}^{n}\gamma(y_i)\prod_{k=1}^{n_j}\delta[\beta(z_k)]\right\rangle_e. \tag{49}$$

Lechtenfeld's derivation was based on analysis of the $\theta$ functional formulas for correlators of the type (48) and is not completely convincing. In Ref. 111 a direct derivation is given on the basis of the technique of Ref. 62; it was developed independently and in greater detail by Losev.[63] The idea of Ref. 63 is that the Green functions $G_e(\ldots)$ for the $\beta,\gamma$ and $b,c$ fields are completely identical. Therefore, the correlators of the operators $e^{p\beta}$ and $e^{q\gamma}$ reducing to $\exp(pq\cdot G)$ are easily expressed in terms of the characteristics of the $b,c$ system. The original correlator (49) is obtained after this by integration or differentiation with respect to $p$ and $q$. Therefore, we easily obtain

$$\Gamma_e(x;y;z) = [\det G_e(z,Q)]^{-n-1}\prod_{i,j=1}^{n}\det \times \begin{bmatrix} G_e(x_i,y_j) & G_e(z,y_j) \\ G_e(x_i,Q) & G_e(z,Q)\end{bmatrix}. \tag{50}$$

Here $Q = \{Q_1,\ldots,Q_{n_j}\}$ are the zeros of the holomorphic $(2j-1)$-differential $\Omega(z)$ specifying the bosonization of the fields according to the rule $\beta^{(j)} = \Omega_{2j-1}^{1/2}\beta^{(1/2)}$, $\gamma^{(j)} = \Omega_{2j-1}^{-1/2}\gamma^{(1/2)}$. The second determinant in (50) is the determinant of an $(n_j+1)\times(n_j+1)$ matrix, $G_e(z,z') \equiv \theta_e(z-z')/[\theta_e(0)E(z,z')]$, and $E(z,z')$ is the Prime form.[40] The merit of Eq. (50) compared to (48) is that it does not contain a special transcendental dependence on $x$ and $y$ in the denominator [more precisely, in the denominator $x$ and $y$ enter only into the Prime form, but not into the more complicated $\theta$ functions as in (48)]. However, the advantages of Eq. (50) have not yet been fully used in analyzing the NSR model.

## 2. TWO-LOOP AND MULTILOOP CALCULATIONS FOR NSR SUPERSTRINGS

The model of 10-dimensional superstrings actually proposed in Refs. 7 and 8 and definitely formulated as a very promising object of study by Gliozzi, Olive, and Scherk[10] occupies a special place in the history of string theory. Being essentially a nonlocal field theory, the perturbative string theory is more or less naturally free of the usual ultraviolet (UV) divergences. However, instead, many string models have severe infrared problems related to the presence of tachyons and massless particles in the string excitation spectrum. In particular, the simplest model, that of 26-dimensional bosonic strings, contains a tachyon, which makes the final expressions for the scattering amplitudes devoid of physical meaning, since they describe perturbation theory about a classically unstable vacuum. From the formal point of view this corresponds to divergence of the integral of the Mumford measure over moduli space. (More precisely, owing to the modular invariance such divergences can also be interpreted as "ultraviolet" ones related to the fast growth of the number of virtual states—intermediate particles. However, the "infrared" interpretation is clearer and more useful from the viewpoint of seeking string models free from this type of divergence arising from instability of the vacuum states.) Therefore, the model of the 26-dimensional string, which has played an important role in understanding the importance of the first-quantization method in going from particles to strings and in understanding the basic mathematical constructions (complex geometry) determining the structure of perturbation theory in string models, turns out to be completely uninteresting from the viewpoint of physical applications to problems like interaction unification. The superstring model was historically the first example of a string model free from tachyons and of immediate physical interest. Here we shall not dwell on the mathematical formalism related to multiloop calculations in this model. See Ref. 180 and references therein for applications of the superstring model to the construction of a grand unification theory. At present, for physical applications the superstring model is being displaced by the richer set of so-called 4-dimensional string models; see Ref. 181. There is apparently much promise in the idea of constructing a unification theory based on a "complete" string theory which dynamically unites all the string models, including the 26-dimensional bosonic string, the 10-dimensional superstring, 4-dimensional strings, and many others (for possible ways of doing this, see, for example, Refs. 81, 129, and 130).

As already noted in the Introduction, the idea of eliminating the tachyon is based on the possibility of introducing into the fermionic string model (the simplest variant of which possesses two-dimensional $N=1$ supersymmetry on the world sheet and has critical dimension 10) a new conserved quantum number, the so-called $G$ parity, and excluding $G$-odd states, among which is the tachyon. After this the theory must acquire a 10-dimensional $N=1$ supersymmetry in space-time. The technical realization of this idea is known as the GSO projection. It involves summation (with certain weights) over all the boundary conditions imposed on the fermionic fields on a non-simply connected Riemann surface (over the $\theta$ characteristics). It is *a priori* not obvious that this procedure (originally formulated at the 0- and 1-loop level) admits a generalization to the multiloop case which, as for fewer loops, excludes (a) tachyons and (b) divergences. (In general, these two conditions are not the same, owing to the presence of massless excitations, which can also generate infrared divergences.) In this section we give some results on the NSR superstring theory obtained by the author in collaboration with G. Moore and A. Perelomov. Among these results are:

The idea of divergence cancellation as a consequence of the Riemann identities for $\theta$ functions (first stated in Ref. 95).

The explicit realization of this idea in the proof that the 2-loop correction to the partition function vanishes,[37,97] and the development for this of a technique for doing calculations on hyperelliptic surfaces (similar calculations have been carried out by Knizhnik[61]).

Noting the problem of the choice of the odd moduli as the main source of ambiguities in analyzing multiloop calculations for the NSR string[96] (this problem was also noted by Verlinde and Verlinde[62] and by Atick, Rabin, and Sen;[64] later articles on this topic are those of Refs. 65 and 94, but these do not give a complete solution of the problem).

The construction of a prescription for multiloop calculations (the successful (?) fixing of all arbitrary quantities[96]) and examples of its application to proving the theorem that renormalizations are absent at the 2-loop level.[98,99]

Recently these ideas (sometimes with different emphasis) have been used to analyze the 3- and multiloop amplitudes[101–110] and to obtain explicit results for the 4-point functions in the 2-loop approximation.[106–108] However, there is a long way to go to the final solution of the problem of the perturbative theory of the NSR string.

### 2.1. An example of a 2-loop calculation: The contribution to the cosmological constant

One of the main problems in the perturbative theory of the NSR superstring and the heterotic string is understanding what the GSO projection—the rule for summing over spinor structures or $\theta$ characteristics—means. As already noted, this operation ensures the elimination of the tachyon and dilaton singularities occurring in the fermionic string, and in the end must lead to the construction of a finite theory. One of the necessary conditions is that the partition function (vacuum diagrams) vanish, in addition to diagrams with no more than three external massless excitations.[60,182] Of all the manifestations of the space-time supersymmetry it is these properties which are simplest to formulate and study by first-quantization methods. At the one-loop level the question is quite clear: the problem of the partition functions has already been solved in Ref. 13, and bosonic and fermionic amplitudes with $N\leqslant 3$ external massless particles have been studied in Refs. 33 and 183. The vanishing of these one-loop expressions is ensured by the Riemann identities for the $\theta$ functions.

The situation regarding multiloop expressions is more complicated. Let us begin with a discussion of the partition functions. The complexity in the multiloop case is related to the fact that the answer is expressed not only in terms of the obvious combination of determinants

$$d\mu_e^{ss}=(\det\bar{\partial}_0)^{-5}(\det{}_e\bar{\partial}_{1/2})^{+5}\det\bar{\partial}_2(\det{}_e\bar{\partial}_{3/2})^{-1} \tag{51}$$

(the index $e$ denotes the $\theta$ characteristic or the spinor structure). To obtain the $p$-loop partition function, the product $d\mu_e^{ss}$ and the correlator of the $2p-2$ supercurrents is integrated over the corresponding moduli space. The origin of this correlator is most simply understood starting from the superfield approach to the construction of fermionic strings (see Refs. 93 and 176). The path integral $\int\mathcal{D}X\mathcal{D}g\,e^{-\mathcal{A}(X,g)}$ defining the bosonic string model is in this approach replaced by an integral over superfields $\hat{X}=X+\vartheta\psi$, $\hat{g}$. Whereas in the critical dimension only the

$(3p-3)$-fold (for $p \geqslant 2$) integral over the moduli of the Riemann surfaces $\mathscr{D}g \to \Pi_{v=1}^{3p-3} dy_v$ remains nontrivial in the integral over metrics in the bosonic case, after supersymmetrization there also remains an integral over odd moduli, the $2p-2$ Grassmann variables $\alpha_i$, $i=1,...,2p-2$: $\mathscr{D}\hat{g} \to \Pi_{v=1}^{3p-3} dy_v \Pi_{i=1}^{2p-2} d\alpha_i$. The integral over $\alpha_i$ is nonzero, owing to the dependence of the action $\hat{\mathscr{A}}(\hat{X},\hat{g})$ on the odd moduli. Since in a sense the $\alpha_i$ are the components of the metric superpartner (the gravitino), the derivatives of the action $\partial\hat{\mathscr{A}}/\partial\alpha_i$ are related to the superpartner of the energy–momentum tensor $\partial\hat{\mathscr{A}}/\partial g$—the supercurrent. More precisely, $\partial\hat{\mathscr{A}}/\partial\alpha_i = \int \chi_i S$, where $S$ is the supercurrent and the $\chi_i$ are Beltrami $(-1/2,+1)$-differentials. As a result, after integrating over the odd moduli $\alpha_i$, we obtain the expectation value of a product of the type $\langle \Pi_i \int \chi_i S \rangle_e \equiv K_e$ (it depends on the spinor structure $e$). Therefore, in the supersymmetric case we are interested in an expression of the form

$$\sum_e c_e d\mu_e^{ss} \cdot K_e, \tag{52}$$

where the coefficients $c_e$ in the sum over spinor structures determine the exact meaning of the GSO projection. Naively, as in the 1-loop case, they should be taken to be $\pm 1$. The expression (52) is the measure on moduli space. For the partition function to vanish it must be the divergence of some local expression on the moduli space.

In fact, $d\mu_e^{ss} \cdot K_e$ strongly depends on the method of introducing the odd moduli, and for some (natural?) choice of term the sum over $e$ with coefficients $\pm 1$ can turn out to be not simply a divergence, but identically (as a function of the ordinary moduli) zero. The existence of such a choice must be closely related to the existence of an "explicitly unitary" first-quantized formulation of the NSR superstring (for which reason the corresponding choice is more and more frequently termed "unitary" in the current literature). In this formulation the entire dependence of $d\mu_e^{ss} \cdot K_e$ on the spinor structure must reduce to an $e$ dependence of the determinants for the transverse ("physical") degrees of freedom, i.e., $d\mu_e^{ss} \cdot K_e \sim (\det_e \bar{\partial}_{1/2})^4 \sim \theta_e^4$, and the vanishing of the partition function in this case follows from the Riemann identity $\Sigma\, e(\pm) e\theta_e^4 = 0$ (this scenario for the vanishing of the partition function was proposed in Ref. 95). At the present time the question of the choice of coordinates on the ordinary moduli space leading to the explicitly unitary formulation of the bosonic string model is more or less clear.[184] The situation in the case of NSR superstrings has so far been more complicated. (One of the obvious problems here is the absence of modular-invariant Riemann identities in the case $p \geqslant 2$. However, it is clear that the modular invariance is partially violated by the choice of odd moduli, and the statement that the unitary choice violates modular invariance may prove more or less acceptable.)

Below we briefly describe one of the first results in this area: the explicit and detailed calculation of the 2-loop partition function carried out in Ref. 37 (it is close to the analysis of Knizhnik in Ref. 61). In this case it is possible to carry out a completely explicit calculation in hyperelliptic coordinates.

It is most convenient to start from the expression for $\Phi_e \equiv d\mu_e^{ss} \cdot K_e$ in the form proposed by Martinec in Ref. 88:

$$\Phi_e = \langle\langle \xi(x_0) Q\xi(x_1) ... Q\xi(x_{2p-2}) \rangle\rangle. \tag{53}$$

The double angle brackets mean that the path integral is to be computed over all fields [except for the Liouville field, since we are dealing with strings in the critical dimension and this field is already included in the fields $X^\mu$ (Ref. 185)]. The averaging over bosonic fields $X$, fermionic fields $\psi$, ghost fermions $b,c$, and bosonic superghosts $\beta,\gamma$ enters here. The correlators of the $X$, $\psi$, and $c$ fields are known from the free field theory (see the review of Ref. 45). The least trivial correlators in the $\beta,\gamma$ system [including the definition and correlators of the fields $\xi = \mathscr{H}(\beta)$, where $\mathscr{H}$ is the Heaviside function] are briefly discussed in Subsec. 1.4 above. The BRST operator $Q$ acts as follows:

$$Q\xi(x) = c\partial\xi(x) + \left[ \oint_{z \text{ around } x} (\gamma\psi\partial X + \tfrac{1}{4}\gamma^2 b)(z) \right] \xi(x). \tag{54}$$

The points $x_i$ in (53) determine the special choice of odd moduli associated with the Beltrami differentials $\chi_i(z) \sim \delta(z-x_i)$. In accordance with the selection rule for correlators of the fields $b$ and $c$, only correlators with the same number of operators $c\partial\xi$ and $\xi\oint\gamma^2 b$ contribute to (53). It is easily verified that, as a result, Eq. (53) becomes a sum of terms, each containing the same number, $2p-2$, of fields and a single correlator of the $\beta,\gamma$ fields, which must be known for calculating the partition function; it is

$$G_e(x|z) = \langle \xi(x_0) ... \xi(x_{2p-2}) \gamma(z_1) ... \gamma(z_{2p-2}) \rangle, \tag{55}$$

and was found in Sec. 1.4. Expanding the double angle brackets in (53), we obtain

$$\begin{aligned}\Phi_e = \prod_{v=1}^{3p-3} dy_v \int \eta_v \Bigg\{ \prod_{i=1}^{2p-2} \oint_{z_1 \text{ around } x_i} &\times \langle b(w_1) ... b(w_{3p-3}) \rangle \\ &\times G_e(x|z) \langle \psi(z_1) ... \psi(z_{2p-2}) \rangle_e \langle \partial X(z_1) ... \partial X(z_{2p-2}) \rangle \\ &+ \oint_{(z_1=z_2) \text{around } x_2} \prod_{i=3}^{2p-2} \oint_{z_1 \text{ around } x_1} \\ &\times \langle b(w_1) ... b(w_{3p-3}) b(z_2) c(x_1) \rangle \frac{\partial}{\partial x_1} G_e(x|z) \\ &\times \langle \psi(z_3) ... \psi(z_{2p-2}) \rangle_e \langle \partial X(z_3) ... \partial X(z_{2p-2}) \rangle + ... \Bigg\}.\end{aligned} \tag{56}$$

(Altogether there are $p$ terms in this sum.) All the correlators in this expression are known from the free field theory.

In order to explain the source of the difficulties which follow, let us briefly analyze the first term in (56). Using Wick's theorem for the fields $\psi$ (also known in the mathematical literature as the Fay identity[40]), the dependence of this term on $e$ can be reduced to

$$\theta_e^4(0)\,\frac{\theta_e(\Sigma_{i=1}^{2p-2}u_i-\Sigma_{i=1}^{2p-2}v_i)}{\theta_e(\Sigma_{i=1}^{2p-2}u_i-\Sigma_{i=1}^{2p-2}v_i-2\Delta_*)}, \tag{57}$$

where the points $u_i$ and $v_i$ coincide with points of the set $\{x_i...x_{2p-2}\}$. In the case of correlators with no more than three external massless lines, the fourth power of the theta constant here is replaced by $\theta_e(0)\theta_e(\zeta_1-\zeta_2)\times\theta_e(\zeta_2-\zeta_3)\theta_e(\zeta_3-\zeta_1)$. If it were possible to choose the points $u_i$, $v_i$ such that $\Sigma_i\mathbf{u}_i=\Sigma_i\mathbf{v}_i=\Delta_*$, then the vanishing of the partition functions would be ensured by the Riemann identities

$$\sum_e \langle e_*,e\rangle\theta_e^4(0)=0. \tag{58}$$

(Here $\langle e_1,e_2\rangle$ is the modular-invariant product of $\theta$ characteristics.) Unfortunately, the situation is not this simple, since for $\Sigma_i\mathbf{u}_i=\Sigma_i\mathbf{v}_i=\Delta_*$ not depending on $e$ the coefficients of (57) have poles, so that identities stronger than (58) are needed to make the coefficients of all the singularities vanish. In the case $p=2$ it was shown in Ref. 97 that it is possible to go to the limit $u=v=R_*$ (in this case $\Delta_*=\mathbf{R}_*$), and an adequate allowance is ensured by the Riemann identity

$$\sum_e \langle e_*,e\rangle\theta_e^4(\mathbf{z})=2^{1-p}\theta_*^4(z/2).$$

In the hyperelliptic case (and always for $p=2$) there is, however, yet another possibility: $u_i$ and $v_i$ can tend to different points of the $2p-2$ branch points. Here the product (57) again becomes $\theta_e^4(0)$, but perhaps with different sign. Nevertheless, the new signs are such that the sum over $e$ again vanishes. This limit is considerably less singular than the previous one, and it was the one originally analyzed in Refs. 37 and 61. The greatest computational difficulties are associated with the determination of the highest-order terms in (56), containing ghost contributions to the supercurrents. The details for the $p=2$ case, where all the calculations have been completed, can be found in Ref. 37. An important feature of the hyperelliptic case is the fact that many contributions to (56) separately cancel.

### 2.2. Lessons of the 2-loop calculation

The example we have given of estimating the 2-loop contribution to the cosmological constant shows that the choice of odd moduli, associated with

$$\chi_i(z)\sim\delta(z-x_i), \tag{59}$$

and of the coefficients $c_e=\pm 1$ in the sum over spinor structures is consistent with the nonrenormalization theorems, but it requires that the choice of $c_e$ be explicitly modular-noninvariant:

$$c_e=\langle e_*,e\rangle, \tag{60}$$

and depend on the choice of the odd $\theta$ characteristic $e_*$. This leads to the real danger that in cases where the final result is nonzero (for example, in the calculation of 4-point functions), the use of such a recipe will give a result which is not modular-invariant and therefore incorrect. This problem was stated in Ref. 96 (see also Refs. 62 and 63). An attractive modification of the above recipe was also presented in that study, and later developed further in Ref. 98. The actual cause of the difficulties can be described as follows.

In principle, it is necessary to calculate the integral over the supermoduli space of a well defined quantity like the Mumford supermeasure. This integration splits up into two stages: first we integrate over odd moduli, then over the usual even ones. It is understood that the result of the intermediate integration depends on the choice of splitting into even and odd moduli. However, this dependence reduces to total derivatives with respect to the even moduli and must vanish after the second integration. Unfortunately, there is a serious flaw in this argument. It would be faultless if the original integral over the entire supermoduli space were defined. Unfortunately, owing to the tachyon and dilaton singularities in the fermionic string model, the integral of the Mumford supermeasure diverges near the boundaries of the moduli space. An NSR superstring model free from these (at least, the tachyon) divergences has not yet been formulated directly in terms of the supermoduli space: it differs from the fermionic string, owing to the GSO projection, which can be formulated as a sum over spinor structures. The supermoduli spaces with different spinor structures are simply different spaces, and it does not appear to be possible to sum measures from different spaces. It is possible to sum only measures on a single space—the space of even moduli, i.e., the quantities obtained by integration over the odd moduli. However, after this it is not clear whether or not the result is independent of how the intermediate integration was done. The argument that the ambiguity reduces to the total derivatives remains true (and is explicitly verified in, for example, Ref. 62), but there is no reason to think that the integrals of these derivatives always vanish. Therefore, the definition of the NSR superstring model at the multiloop level is a quite special and nontrivial problem. This definition (which so far does not exist) must satisfy at least several criteria:

It must give unique (in particular, modular-invariant) results for the amplitudes.

It must not violate the nonrenormalization "theorem" (Refs. 60 and 182) for the 0-, 1-, 2-, and 3-point correlators of massless excitations.

It must give results which are consistent with space-time supersymmetry.

It must obey the factorization property (equivalent to space-time unitarity of the theory).

Strictly speaking, we must require that the theory satisfy only the first and last of these criteria; after this the other two are theorems. Unfortunately, as mentioned above, the NSR superstring model is not yet fully constructed, and the main obstacle is the proof of the compatibility of the factorization property and any proposed recipe for multiloop calculations. Up to now the main approach to the problem has been to seek a recipe satisfying the first two requirements, and then to try to prove the last two. Some progress has been made in this way, but we are far from complete success (see Refs. 37, 61, 62, 64, 65, and 92–111). Below we give a recipe for constructing multiloop amplitudes proposed some time ago in Refs. 96 and 97. It associates the vanishing of the corrections to the cosmological constant with the requirement of modular invariance.

On a surface of genus $p$ it is necessary to specify an odd spinor structure (the $\theta$ characteristic) $e_*$ and to carry out all the calculations using the metric $|v_*|^4$ with double zeros at the points $R_1^*,...,R_{p-1}^*$, the Beltrami superdifferentials $\chi_\alpha \propto \delta(z-R_\alpha^*)$, $\chi_\alpha \propto \delta'(z-R_\alpha^*)$, $\alpha = 1,...,p=1$. The weights in the sum over spinor structures $e$ should be taken to be $\langle e,e_*\rangle$. In fact, for the regularization of the possible divergences it is suitable to begin with the 1-differential $\Omega$ with pairwise differing zeros, the metric $|\Omega|^2$, and the Beltrami differentials localized at the zeros of $\Omega$, and only then, right before summing over the characteristics, take the limit $\Omega = v_*^2$. In general, this procedure leads to a result $\Phi_*$ which depends on $e_*$. The final result is taken to be $\Phi = \Sigma_* \Phi_*$.

As already mentioned, this recipe was formulated on the basis of the analysis of the 2-loop calculation of the cosmological constant (the 0-point function). This procedure obviously leads to a modular-invariant result. However, it is not known whether or not this recipe is consistent with the factorization requirement in the general case. In Ref. 99 it was shown that it ensures the absence of 2-loop corrections also to the 1-, 2-, and 3-point functions (a comparison with more naive recipes is also given in that study). The technique developed in Ref. 98 was later used in Refs. 104 and 110 and other articles (with varying success) to analyze the 3- and multiloop cases. The ideas of Ref. 96 (and of the independent study of Ref. 64) were developed slightly differently in the two long papers of Ref. 65. Other points of view are reflected in Refs. 89 and 94. In Ref. 100 the first attempt was made to guess the answer for the simplest nontrivial quantity in the NSR model: the 2-loop contribution to the 4-point function, and several problems were pointed out. In particular, it was noted (in parallel with the arguments of Ref. 186 based on completely different considerations) that the suggestion that the multiloop amplitudes are finite must be taken not completely literally, but rather in the sense of analytic continuation. More precisely, the naive integrals over moduli space are finite only in a certain range of the momenta, and are obtained in other ranges by analytic continuation. Here the situation is fundamentally different from the 1-loop case. The potential divergences in question are related to dilaton singularities, and from the technical point of view they are absent at the one-loop level only because $\mathrm{Im}\tau$ enters into the integrands with the power $-(d/2+1) = -6$ instead of $-d/2 = -5$ for $p \geqslant 2$. The direct calculation of the 2-loop 4-point function (using a particular recipe based on the hyperelliptic technique of Refs. 49 and 48) has so far been carried out by only a single group.[106–108] The analysis of the complicated result obtained in these articles (needed to check its modular-invariance, finiteness, and so on) is very difficult and has not yet been done.

Therefore, the perturbative theory of the NSR string is still far from completion, although many of its important features and regularities have been revealed. In connection with this, alternative (of course, not at all simpler) approaches to the superstring model are of interest. The most interesting of them is the Green–Schwartz superstring model.

## 3. MULTILOOP CALCULATIONS FOR THE GREEN–SCHWARTZ STRING

The approach to superstring theory developed by Green and Schwartz (see Ref. 13) is fundamentally different from the NSR approach, since attention is focused on the space-time supersymmetry. The main difficulty in this approach is the fact that it is not at all obvious how to find the two-dimensional covariant formulation of the theory, which is needed for the systematic application of the first-quantization method. This formulation—the Green–Schwartz action—was found only in 1984, in Ref. 87. Here the action, in contrast to the usual situation in string models, turned out to be nonquadratic in the matter fields. In addition, the GS model does not contain explicit traces of the two-dimensional supersymmetry; the fermionic fields in it are two-dimensional scalars (this situation is rather reminiscent of the recently introduced concept of "twisted supersymmetry"). For these reasons the question of whether or not the GS and NSR superstring models coincide (i.e., coincide at tree level, so that the unitarized theories can also be made identical) is completely nontrivial, and completely different properties of the superstrings are manifested in the different formulations. Naturally, the GS formulation is more convenient for seeing the properties related to space-time supersymmetry (and in Subsec. 3.2 below we shall see that, for example, it is quite easy to prove the nonrenormalization theorems in this formalism). However, the NSR formulation is considerably simpler and more comprehensible from the two-dimensional viewpoint (in the case of the GS model, the principal two-dimensional symmetry, the so-called $k$ invariance, turns out to be rather difficult to study and interpret). Obviously, at first glance the advantage of the NSR model is the fact that the two-dimensional action is quadratic. However in Refs. 115 and 113 (see Subsec. 3.1) it was shown that in fact the GS action is dynamically equivalent to the quadratic action. The method used in Ref. 113 was subsequently applied in Ref. 44 for the analogous operation of transforming to free fields (Darboux variables) in the Wess–Zumino–Novikov–Witten model, and then also in other interesting string models. In a certain sense any conformal theory admits being written in terms of free massless fields (the Feigin–Fuchs[131] or Dotsenko–Fateev[132] representation), and the GS model is not an exception.

In Subsec. 3.1 we give the basic principles of multiloop calculations in the GS model. The central feature is the choice of Lorentz gauge $\theta^+ = 0$ and the nonlinear variable substitution transforming the action to quadratic form,

$$\int d^2z \left[ \partial x^\mu \bar{\partial} x^\mu \eta \frac{k}{z} \partial \theta_k + \text{ghosts} + \text{right-handed fields} \right].$$

We show how to include the Jacobian arising from the variable substitution.

In Subsec. 3.2 we discuss the factors which can hinder the choice of the Lorentz gauge in the GS theory and their effect on the structure of multiloop corrections. It is shown that here an ambiguity arises which is very similar to that in the NSR case, and in the final expressions structures appear which are similar in form and interpretation to the insertions containing supercurrents ("image-changing" operators). It is shown that owing to the space-time symmetry of the original action, the appearance of such insertions cannot get rid of at least four of the eight zero modes of the field $\theta$, which is sufficient to prove the vanishing of the corrections to the 0-, 1-, and, most importantly, 2-, and 3-point correlators of massless particles (in the case of the 2- and 3-point functions the structure of the vertex operators is still important).

### 3.1. The Green–Schwartz action

To understand the role of the GS formalism and the limitations of the NSR approach, it is useful to go to the formula for the 1-loop contribution to the 4-point function

$$\int \frac{d^2\tau}{(\mathrm{Im}\,\tau)^6} \int d^2z_1 ... \int d^2z_4 \langle e^{ip_1X(z_1)} ... e^{ip_4X(z_4)} \rangle \quad (61)$$

(the kinematical factor has been dropped). The standard derivation of this result in the NSR formalism is quite lengthy, since it starts from the expressions for the spinor correlators and includes the sum over spinor structures. Only after using the Riemann identities does the awkward original expression acquire the simple form (61), which does not contain any traces of the spinors on the world sheet. In this sense the existence of the GS formalism, which does not involve two-dimensional spinors at all, appears very natural.

The Green–Schwartz action[87] (in the simplest, the heterotic, case) has the form

$$S_{\mathrm{GS}} = \int d^2z\{ -\tfrac{1}{2}\sqrt{g} g^{\alpha\beta}\Pi^\mu_\alpha\Pi^\mu_\beta + \varepsilon^{\alpha\beta}\partial_\alpha X^\mu \bar{\theta}\gamma^\mu\partial_\beta\theta + \mathscr{L}\}. \quad (62)$$

Here $\mathscr{L}$ describes the gauge degrees of freedom of the heterotic string,

$$\Pi^\mu_\alpha \equiv \partial_\alpha X^\mu - i\bar{\theta}\gamma^\mu\partial_\alpha\theta, \quad (63)$$

and $\theta$ is the anticommuting 10-dimensional 16-component Majorana–Weyl spinor, which is a 2-dimensional scalar. The Lagrangian quantization of this theory in the general case encounters problems of the infinite sequence of "ghosts for ghosts" and nonclosure of the symmetry algebra off the mass shell. However, the analysis carried out in Ref. 113 suggests a possible way out (reducing essentially to supplementing the transformation rules for an infinite ghost chain). Specifically, we fix the gauge, for example,

$$\gamma^+\theta = 0, \quad g_{\alpha\beta} = \rho g^{(y)}_{\alpha\beta}, \quad (64)$$

where $g^{(y)}_{\alpha\beta}$ is some fixed metric depending on the moduli $y_1 ...$ . After fixing the gauge the functional integral takes the form

$$\int Dy \int \mathscr{D}X^\mu \mathscr{D}\theta \mathscr{D}b \mathscr{D}c (\mathrm{Det}\, u_{\bar{z}})^{-4} \times \exp\Big\{ -\int d^2z(\partial X^\mu \bar{\partial} X^\mu + \bar{\theta}\bar{\gamma}\bar{\partial}X^+\partial\theta + \mathscr{L} + b\bar{\partial}c + \bar{b}\partial\bar{c})\Big\}. \quad (65)$$

Here $b$, $c$, $\bar{b}$, and $\bar{c}$ are ordinary reparametrization ghosts related to fixing the conformal gauge for the metric. We have also introduced the notation $u_{\bar{z}} = \bar{\partial}X^+$. The appearance of the factor $(\mathrm{Det}\, u_{\bar{z}})^{-4}$ in the local metric is related to the choice of the Lorentz gauge for $\theta$. The choice of the degree of the determinant is mainly justified by the invariance of the answer of how the symmetry is fixed, but can also be related to the naive definition of the product of an infinite number of ghost determinants. We note that, in contrast to the light-cone gauge, in (65) $X^+$ is not identified with the coordinate on the world sheet such that $u_{\bar{z}} = 1$ (this is possible only on a cylindrical surface). Instead of this (impossible) fixing of the two-dimensional reparametrization invariance, we can attain practically the same result in a different manner: by variable substitution.

We note that the $SO(8)$ spinor $\theta$, $\gamma^+\theta = 0$, can be split into two $SU(4)$ spinors, $\eta^k \equiv \theta^k$ and $\theta_k$, $k = 1,...,4$. Then the cubic term in (65) can be rewritten as $\bar{\theta}\bar{\gamma}u_{\bar{z}}\partial\theta \Rightarrow 2\theta^k u_{\bar{z}}\partial\theta_k - \theta^k\theta_k\partial u_{\bar{z}}$, and the second of the resulting terms $\theta^k\theta_k\partial\bar{\partial}X^+$ is easily eliminated by shifting $X^-$ by $\theta^k\theta_k$ (i.e., by going to the "chiral" field $X^-$ in the standard supersymmetric formulation). Now we still need to make the variable substitution $\eta^k_{\bar{z}} = \eta^k u_{\bar{z}}$ to transform (65) into

$$\int Dy \int \mathscr{D}X^\mu \mathscr{D}\eta^k_{\bar{z}} \mathscr{D}\theta_k \mathscr{D}b \mathscr{D}c (\mathrm{Det}\, u_{\bar{z}})^{-4} \times \exp\Big\{ -\int d^2z(\partial X^\mu \bar{\partial} X^\mu + \eta^k_{\bar{z}}\partial\theta_k + \mathscr{L} + b\bar{\partial}c + \bar{b}\partial\bar{c})\Big\}. \quad (66)$$

In studying the anomalies in the resulting theory it is necessary to take into account the fact that the 1-differential $\eta_{\bar{z}}$ has nonstandard norm

$$\|\eta_{\bar{z}}\|^2 = \|\theta\|^2 = \int d^2z\sqrt{g}|\theta|^2 = \int d^2z \frac{\sqrt{g}}{|u_{\bar{z}}|^2}\eta_{\bar{z}}\eta_z$$

(instead of the naive one $\int d^2z\eta_{\bar{z}}\eta_z$). The Liouville action is slightly changed by the presence of this nontrivial norm. In particular, the anomaly in the determinant of the Laplace operator $\Delta_{f,h} = f(z,\bar{z})\partial h(z,\bar{z})\bar{\partial}$ [i.e., in the functional integral over the field $\phi$ with action $\int d^2zh(z,\bar{z})|\bar{\partial}\phi|^2$ and norm $\|\phi\|^2 = \int d^2zf(z,\bar{z})|\phi|^2$] is given by the expression[113]

$$\exp\left\{ -\frac{1}{48\pi}\int\left[\frac{|\partial f|^2}{f^2} - \frac{4\partial f\bar{\partial}h}{fh} + \frac{|\partial h|^2}{h^2}\right]\right\}. \quad (67)$$

In particular, for the ordinary operator $\bar{\partial}_j$ we have $f = \rho^{j-1}$, $h = \rho^{-j}$, and from (67) we obtain the standard result[11] $\exp - \{(c_j/48\pi)\int|\partial(\log\rho)|^2\}$ with $c_j = (-j)^2$

$-4(-j)(1-j)+(1-j)^2=6j^2-6j+1$. In the field case we have instead of this

$$\exp\left\{-\frac{1}{48\pi}\int[-2|\partial(\log\rho)|^2-(2\log\rho+\log|u_{\bar z}|^2)\partial\bar\partial(\log|u_{\bar z}|^2)]\right\}. \tag{68}$$

The coefficient of the first term is equal to $-2$ instead of $c_0=c_1=+1$ for ordinary 1-differentials. This is exactly what is required to cancel the anomalies between the contributions of the $X$, $\theta$, $b$, and $c$ fields: $-(d/2)(+1)+4(-2)+13=0$ ($d=10$). This anomaly cancellation rule was first formulated by Carlip[115] on the basis of less clear arguments. The other terms in (68) do not have analogs in the simplest known string models. Since the contributions are proportional to $\partial(\ln|u_{\bar z}|^2)\sim\partial\bar\partial X^+$, they can be eliminated by a shift of the field $X^-$ (at least, in those cases where correlators not containing $X^-$ fields are calculated).

This method of bringing the GS action to quadratic form reveals a way of investigating the GS model by standard methods. In Subsec. 3.2 we give the simplest application to proving the theorems on the absence of renormalization proposed in Ref. 112. The literature devoted to further study of the GS model is quite extensive (see, for example, Refs. 112 and 120–123). A significant fraction of the studies is devoted to developing the work of Kallosh and Rahmanov[112] on analyzing the GS model in a gauge which does not violate the 10-dimensional supersymmetry. As assumed in such cases,[187,188] here the model contains an infinite number of unphysical fields, and its analysis is quite complicated, particularly from the viewpoint of multiloop calculations.

### 3.2. Proof of the theorems that renormalizations are absent

In Sec. 2 we already mentioned that the 0-, 1-, 2-, and 3-point functions for massless excitations in the superstring model do not need to be renormalized, at least in perturbation theory.[60,182] In the NSR formalism studied in Sec. 2 this is a complicated theorem which is difficult to prove even in the simplest nontrivial case $p=2$. However, in the GS formalism it is almost obvious. Here the nonrenormalization theorems are related to the fact[13] that the fields $\theta_k$, $k=1,...,4$ (but not $\eta^k_{\bar z}$), are two-dimensional scalars, and so in the model with the action (66) there are always at least 4 zero modes. In order to "absorb" these zero modes at least four vertex operators, each containing one field $\theta$, are needed. In this subsection, following Ref. 114, we give more precise arguments. They also imply a close analogy to the NSR model, which, however, so far has not been shown in detail.

A critical complication which in principle can ruin the validity of the above argument is the obstacle to choosing the gauge $\theta^+\equiv\gamma^+\theta=0$ on surfaces of higher genus. The point is that gauge transformations act on $\theta^+$ according to the rule $\delta\theta^+=\Pi^+\gamma^-k$ ($\gamma^+k=0$), and the 1-differential $\Pi^+$ must have at least $2p-2$ zeros (in fact, even more). As a result, instead of $\theta^+=0$ we can impose the weaker condition

$$\theta^+(z,\bar z)=\sum_{m=1}^{M}\alpha_m\delta^{(2)}(z-Q_m), \tag{69}$$

where $Q_m$ are the zeros of $\Pi^+$. As a result, the complete action differs from (66) by the addition of extra terms of the form

$$\sum_{m=1}^{M}\alpha_m\int\delta^{(2)}(z-Q_m)\mathcal{O}_{(1)}+\sum_{m,n=1}^{M}\alpha_m\alpha_n\int\int\delta^{(2)}\times(z-Q_m)\delta^{(2)}(z'-Q_n)\mathcal{O}_{(2)}, \tag{70}$$

where $\mathcal{O}_{(1,2)}$ are easily determined combinations of the $\theta$, $\eta$, and $X$ fields. After integrating over the Grassmann variables $\alpha_m$ (the direct analogs of the supermoduli in the NSR approach) we obtain a certain insertion of the operators $\mathcal{O}_{(1,2)}$ (analogous to the supercurrent insertion). The danger of this for the nonrenormalization theorems is that the insertion (which is present even in the calculation of the partition function) could absorb the zero modes of the scalar fields $\theta_k$. However, this does not occur. The zero modes turn out to be directly related to the space-time supersymmetry, and the theorems about the absence of renormalization must be valid as long as perturbation theory does not spoil this important symmetry of the GS model (we again note that such a perturbation theory has not yet been constructed in the complete volume).

In order to determine the relation between the zero modes and the supersymmetry, we note that before the gauge fixing there was invariance under substitution of the fields

$$\delta\theta^-=\varepsilon^-=\text{const},\quad \delta X^\mu=i\bar\varepsilon^-\gamma^\mu\theta,$$

$$\text{i.e., } \delta X^+=0,\quad \delta X^-=i\bar\varepsilon^-\gamma^-\eta^-,\quad \delta X^i=i\,\bar\varepsilon^-\gamma^i\eta^+$$

(and $\delta\theta^+=\delta\eta^+=\delta\eta^-=0$). We can always make a shift inside the path integral:

$$X^\mu\to\tilde X^\mu{:}\tilde X^+\equiv X^+,$$

$$\tilde X^-\equiv X^--i\theta^-\gamma^-\eta^-,$$

$$\tilde X^\mu\equiv-i\theta^-\gamma^1\eta^+.$$

The Jacobian of this transformation is equal to unity. Now the action is invariant under the shift $\delta\theta^-=\varepsilon^-=\text{const}$, where $\delta\tilde X^\mu=\delta\theta^+=\delta\eta^+=\delta\eta^-=0$. Therefore, in the field space there are well defined constant harmonics

$$\theta_0^-=\text{const},\quad \tilde X_0^\mu=0,\quad \theta_0^+=0,\quad \eta_0^+=0,\quad \eta_0^-=0$$

(the ghost fields are equal to zero), which are obviously orthogonal to all the other harmonics and here do not contribute to the action. Therefore, they are true zero modes, and there are exactly four of them, since $\theta_k^-$ has 4 components: $k=1,...,4$. (We note that the action is also invariant under the transformations

$$\delta\eta^-=\zeta^-=\text{const},\quad \delta X^\mu=i\zeta^-\gamma^\mu\theta,$$

$$\delta\theta^+=\delta\eta^+=\delta\theta^-=0,$$

but they act nontrivially on $X$ and therefore do not necessarily lead to the existence of zero modes: the mode $\eta^{-}=\text{const}$, $\tilde{X}^{\mu}=\text{const}\ \gamma^{\mu}\theta$ is not necessarily orthogonal to the other harmonics of the Laplace operator.)

The results of this section show that available methods can shed light on the GS model. It seems that further real progress in understand the superstring must involve a synthesis of the results obtained in the NSR and GS formulations, and a proof that these formulations are equivalent (finding and studying the explicit variable substitution).

## 4. THE FREE-FIELD REPRESENTATION FOR GENERAL CONFORMAL THEORIES

The analysis of the GS superstring model given in Sec. 3 contains an idea important for the entire perturbative theory of strings. As mentioned in the Introduction, an object of study in this area is two-dimensional conformal models on Riemann surfaces. Usually such models, if they are specified in the Lagrangian form, have nonquadratic actions (or, almost equivalently, they come with constraints). However, the free-field formalism—the only technique presently available—is applicable only to theories with quadratic actions. The example of the GS model illustrates the most important general rule of all conformal models: they can all be written in terms of free massless fields; there is always a corresponding change of variables which makes the action quadratic. This statement remains to be proved in its full generality (the main problem is that the allowed degree of nonlocality in the variable substitution is not clear; the question of what should be the starting point of such a proof is also open, and it is not impossible that the free-field representation itself can serve as a suitable device for defining conformal models, an alternative to the definition given in Ref. 23). In this section we discuss the transformation to free fields in one of the fundamental conformal models: the WZNW model. This theory is important because it is directly related to the KM algebras associated with the fundamental symmetries of two-dimensional conformal theories. In evaluating the results we obtain we should bear in mind the following. Up to now the most popular examples of conformal models are representatives of the so-called rational theories. These models are special because they contain a finite number of conformal blocks and the approach of Ref. 23 works particularly well for them. (See Ref. 137 on the general theory of rational conformal models.) In contrast, the free-field representation is particularly simple to construct for nonrational theories (the theory of a single free field, like the conformal models arising in the analysis of 26-dimensional bosonic strings or 10-dimensional superstrings, is not rational). The description of rational models in the free-field formalism requires an additional reduction: the imposition of additional conditions on the free fields compatible with the equations of motion and the holomorphic structure of the correlation functions. The simplest reductions transform nonrational theories into other nonrational ones (for example, in the construction of the so-called coset models from WZNW models in Subsec. 4.3). For reductions leading to rational models a central role is played by the screening operators actually introduced by Feigin and Fuchs[131] in analyzing the representations of the Virasoro algebra and by Fateev and Dotsenko[132] in studying the free-field representation for minimal models. At the present time these reductions are well understood, owing to the BRST cohomology of Feigin, Fuchs, and Felder.[131,139] The free-field representation, augmented by these new results, can now be used to construct explicit expressions for the multiloop amplitudes in rational conformal models.[140,141]

Our discussion in this section reflects the previous stage of development: the establishment of the generality and value of the free-field representation for a broad class of conformal models. In addition to the example of the GS model discussed in Sec. 3, in Subsecs. 4.1 and 4.2 we analyze the WZNW model (without the Felder reduction existing for integer, and, most likely, rational values of the central charge $k$). In particular, we obtain the free-field representation for any KM algebra with any (not necessarily integer) central charge. In Subsecs. 4.3 and 4.4 we consider the simplest (non-Felder) reductions leading to nonrational coset models, in particular, to nonrational parafermions. Subsection 4.5 is devoted to studying $W$ algebras from the viewpoint of the free-field representation. This viewpoint is convenient, in particular, for understanding the structure of the $W_\infty$ algebra, which may still play an important role in the further development of string theory. Finally, Subsec. 4.6 contains a brief discussion of the generalized Sugawara construction.

### 4.1. The WZNW model: The $SL(2)_k$ case

The Lagrangian of the WZNW model is a nonlocal nonquadratic functional and has the form

$$4\pi\mathscr{L}=-k\,\mathrm{tr}\left[|g^{-1}\partial g|^{2}+\frac{i}{3}d^{-1}(g^{-1}dg)^{3}\right]. \qquad (71)$$

Let us first consider the simplest non-Abelian WZNW theory corresponding to the current algebra $sl(2)_k$. The free-field representation for this algebra was first constructed by Wakimoto,[143] Kolokolov,[189] and Zamolodchikov.[144] (Even earlier, Witten[68] studied the special case $k=1$, when a representation in terms of a single free field is possible. Usually three are needed. Another special case, $k=4$, when two fields are sufficient, is studied in Ref. 163.) Below we give the systematic Lagrangian derivation proposed in Ref. 44. To diagonalize the action (71) we parametrize an element of the $SL(2)$ algebra using the Gauss decomposition

$$\begin{aligned}g&=g_U(\psi)g_D(\varphi)g_L(\chi)\\&=\begin{bmatrix}1&\psi\\0&0\end{bmatrix}\begin{bmatrix}e^{\varphi}&0\\0&e^{-\varphi}\end{bmatrix}\begin{bmatrix}1&0\\\chi&1\end{bmatrix}\\&=\begin{bmatrix}\psi\eta e^{-\varphi}+e^{\varphi}&\psi e^{-\varphi}\\\chi e^{-\varphi}&e^{-\varphi}\end{bmatrix}.\end{aligned} \qquad (72)$$

The idea of the role of the Gauss representation in the study of the WZNW model is apparently due to Polyakov. Shortly before Ref. 44, this representation was used in a very similar context in Ref. 138. The invariant norm of the field $g$ in this parametrization has the form

$$\|\delta g\|^2=\tfrac{1}{2}\int G\,\mathrm{tr}(g^{-1}\partial g)^2$$
$$=\int G[(\delta\varphi)^2+e^{-2\varphi}\delta\psi\delta\chi], \quad (73)$$

where $G$ is the two-dimensional metric in the conformal gauge. The current matrix can be defined as

$$kJ=kg^{-1}dg$$
$$=\begin{bmatrix} \tilde{w}\chi+d\varphi & \tilde{w} \\ -\tilde{w}\chi^2-2\chi d\varphi+d\chi & -\tilde{w}\chi-d\varphi \end{bmatrix}$$
$$=\tilde{J}^+\sigma_+ +\tilde{J}^-\sigma_- +\tilde{H}\sigma_3 \quad (74)$$

for

$$w=k\tilde{w}=ke^{-2\varphi}d\psi. \quad (75)$$

It is easily shown that

$$\frac{k}{2}\mathrm{tr}(g^{-1}\partial_\mu g)^2=k(\partial_\mu\varphi)^2+w_\mu\partial_\mu\chi, \quad (76)$$

$$k\,\mathrm{tr}(g^{-1}dg)^3=kd(e^{-2\varphi})d\chi d\psi=d(wd\chi), \quad (77)$$

and the Wess–Zumino term has the form

$$\frac{k}{3}d^{-1}(g^{-1}dg)^3=\varepsilon_{\mu\nu}w_\mu\partial\chi_\nu. \quad (78)$$

Now from (71) we obtain

$$\mathscr{L}_{\mathrm{cl}}=-\frac{1}{4\pi}[k|\partial\varphi|^2+w\bar{\partial}\chi], \quad (79)$$

where the 1-differential $w$ is defined by the equation $w=ke^{-2\varphi}\partial\psi$.

It follows from Eqs. (73) and (75) that the naive measure in the path integral coincides with the free measure for the fields $w$, $\chi$, and $\varphi$. The nontrivial factor $e^{-2\varphi}$ in the measure defined by (73) cancels after substitution of the variables (75). However, in order to make the change of variables (75) correctly it is necessary to take into account the functional determinant $\mathrm{Det}(e^{-2\varphi}\partial)$, which, of course, is anomalous. The corresponding Laplace operator has the form

$$G^{-1}e^{2\varphi}\bar{\partial}e^{-2\varphi}\partial \quad (80)$$

(this time $G$ is the two-dimensional metric on the Riemann surface). The anomaly of the determinant is now calculated according to the general rule of Ref. 113, given in Subsec. 3.1. It "shifts" the Lagrangian (79) by $\frac{1}{48\pi}[24|\partial\varphi|^2+12\varphi\partial\bar{\partial}\log G]$, and, accordingly, the quantum Lagrangian has the form

$$4\pi\mathscr{L}_q=-[w\bar{\partial}\chi+(k+2)|\partial\varphi|^2+R\varphi] \quad (81)$$

($R=\bar{\partial}\partial\ln G$ is the curvature 2-form in two dimensions). In terms of the free fields $w$, $\chi$, and $\varphi$ the presence of the Kac–Moody symmetry is not obvious. The simplest way of revealing this symmetry is to explicitly construct the current operators forming the KM algebra. In principle, the needed operators are given by Eq. (74). However, in these expressions it is still necessary to do the normal ordering accurately. It is useful to change from the field $\varphi$ in (74) to the "renormalized" scalar field $\phi=i\sqrt{2}q\varphi$ with the operator expansion $\phi(z)\phi(0)=-\ln z+\ldots$ to obtain the correctly normalized kinetic term in the Lagrangian. In terms of $\phi$ the Lagrangian (81) is rewritten as

$$4\pi\mathscr{L}_q=-w\bar{\partial}\chi+\frac{1}{2}|\partial\phi|^2+\frac{i}{\sqrt{2}q}R\phi \quad (82)$$

($q^2=k+2$). The Noether energy–momentum tensor is quadratic in the fields:

$$T=w\partial\chi-\frac{1}{2}(\partial\phi)^2-\frac{i}{\sqrt{2}q}\partial^2\phi. \quad (83)$$

The current algebra $sl(2)_k$ written in terms of the correctly normalized quantum fields has the form[143,189]

$$J^+=w;\quad H=w\chi-\frac{i}{\sqrt{2}}q\partial\phi;$$
$$J^-=w\chi^2-i\sqrt{2}\chi\partial\phi-k\partial\chi, \quad (84)$$

where the operator expansion for the fields $\phi$ is given above, the fields $w$ and $\chi$ are the bosonic $\beta,\gamma$ system with unit spin (see Subsec. 1.4), and

$$w(z)\chi(0)=\frac{1}{z}+\ldots. \quad (85)$$

Here the energy–momentum tensor (83) coincides with the Sugawara tensor ($T\sim$: tr $J^2$:) for the algebra (84).

However, this is still not a complete description of the theory. The variable substitution (75)

$$d\psi=\tilde{w}e^{2\varphi}=\frac{1}{k}we^{-i\sqrt{2}\phi/q} \quad (86)$$

also implies additional constraints on the functional integration. These constraints are a reflection of the fact that the original variables $\psi$, $\chi$, and $e^{\varphi}$ must be viewed as single-valued functions on the Riemann surface. Therefore, for any cycle on the surface

$$\oint we^{2\varphi}=\oint we^{-i\sqrt{2}\phi/q}=k\oint d\psi=0. \quad (87)$$

(Owing to the holomorphic properties of the correlators, (87) yields only a finite set of constraints, the number of them equal to the number of nonhomologous noncontractible contours on the surface.) This means that the $sl(2)_k$ WZNW model defined as a functional integral over the free fields $w$, $\chi$, and $\phi$ (the latter taking values on a circle of radius determined by the single-valuedness of the exponential $e^{\varphi}$, i.e., $\varphi\sim\varphi+2\pi i$, $\phi\sim\phi-2\pi\sqrt{2}q$) with $\delta$-functional insertions $\Pi_C\delta[\oint_C we^{-i\sqrt{2}\phi/q}]$ for the set of nonhomologous cycles $C$. Each $\delta$ function can be represented as $\int d\lambda\exp[i\lambda\oint we^{-i\sqrt{2}\phi/q}]$. Expanding the exponential in a series, we see that the conformal blocks in the WZNW model are linear combinations of the free-field correlators with additional insertions of the operators $Q = \oint w e^{-i\sqrt{2}\phi/q}$, the so-called screening operators.[132]

### 4.2. The WZNW model: general case

The general algorithm for constructing the free-field representation for an arbitrary KM algebra $\hat{G}$ is formulated in Ref. 44 as follows:

1. We fix the system of positive roots $\Delta_+$ and introduce two fields $\psi_\alpha$, $\chi_\alpha$ for each $\alpha \in \Delta_+$.

2. We use the Gauss decomposition, $g = g_U(\psi) g_D(\varphi) g_L(\chi)$ to write the current matrix $g^{-1}\partial g$ in the form

$$\tilde{J} = g^{-1}\partial g = g_L^{-1}(\chi)\tilde{J}_{(0)}(\psi;\varphi) g_L(\chi) + g_L^{-1}(\chi)\partial g_L(\chi). \tag{88}$$

We automatically get $g_D^{-1}(\varphi)\partial g_D(\varphi)$ on the diagonal of the upper triangular matrix $J_{(0)}$, and the other elements determined by the functions $\psi$ and $\varphi$ are denoted as $\tilde{w}_\alpha$.

3. We introduce new fields $w_\alpha = w_\alpha(\psi,\chi;\varphi)$ linearly depending on $\{\tilde{w}_\alpha\}$ according to the rule

$$k \operatorname{tr} g_L^{-1}\bar{\partial} g_L \tilde{J}_{(0)} = \sum_{\alpha\in\Delta_+} w_\alpha \bar{\partial}\chi_\alpha. \tag{89}$$

4. The 1-form $d^{-1}\Omega = k \operatorname{tr} g_L^{-1}\bar{\partial} g_L \tilde{J}_{(0)}$ should be viewed as a symplectic structure ($\Omega \sim dp dq$, $d^{-1}\Omega \sim p dq$) which determines the operator decompositions for the fields $w$ and $\chi$. In particular, in (89) it is understood that $w$ and $\chi$ are Darboux variables, more precisely, a $\beta,\gamma$ system with spin 1, and $w_\alpha(z)\chi_\beta(0) = \delta_{\alpha\beta}/z + \text{reg}$.

5. Now it is necessary to express $J$ (originally defined in terms of $\psi$, $\chi$, and $\varphi$) in terms of the variables $w$ accurately taking into account the normal-ordering rules. Then $k\tilde{J}(\tilde{w},\chi,\varphi) = J(w,\chi,\varphi) + F(\chi)$, where $F(\chi)$ is some matrix function of $\chi$.

The fields $\varphi_i$ are also free (the index $i$ actually numbers the linearly independent weights of the fundamental representation $\mu_i$ lying in the Cartan subalgebra). The operator expansions of $\varphi_i$ have the form $\varphi_i(z)\varphi_j(0) = \mu_i\mu_j(q^2/k^2)\log + \text{reg}$, where $q^2 = k + g_V$ and $g_V$ is the dual Coxeter number (the squared Casimir operator in the adjoint representation). The free scalar fields $\vec{\phi}$ with the usual operator decomposition, $\vec{\alpha}\vec{\phi}(z)\vec{\beta}\vec{\phi}(0) = -(\vec{\alpha}\vec{\beta})\log z + \text{reg}$, are related to $\varphi$ by the transformation $\vec{\mu}_i\vec{\phi} = i\frac{k}{q}\varphi_i$.

Then the currents $J(w,\chi,\phi)$ form a KM algebra with central charge $k$, and the Sugawara energy–momentum tensor is quadratic in the fields:

$$\begin{aligned} T &= \frac{1}{2q^2} :\operatorname{tr} J^2: \\ &= \frac{1}{2q^2} \sum_{\alpha\in\Delta_+} :\left(J_{-\alpha}J_\alpha + J_\alpha J_{-\alpha} - \frac{1}{C_V} H_\alpha H_\alpha\right): \\ &= \sum_{\alpha\in\Delta_+} w_\alpha \partial\chi_\alpha - (\partial\vec{\phi})^2 - \frac{i}{q}\vec{\rho}\partial^2\vec{\phi}. \end{aligned} \tag{90}$$

Here $\vec{\rho} = \frac{1}{2}\Sigma_{\alpha\in\Delta_+}\vec{\alpha}$. This operation corresponds to the diagonalization of the Novikov–Witten Lagrangian (71) in terms of free fields. In the conformal gauge taking into account the anomaly the Lagrangian has the form

$$4\pi\mathscr{L}_q = -\sum_{\alpha\in\Delta_+} w_\alpha\bar{\partial}\chi_\alpha + \frac{1}{2}|\partial\vec{\phi}|^2 + \frac{i}{q}R\vec{\rho}\vec{\phi}. \tag{91}$$

Unfortunately, the derivation of the quantum measure (the anomaly) in the general case is less elegant than for $Sl(2)$ (see Ref. 44 for details). Apparently, in the general case the measure from the very beginning is different from the Haar measure. Explicit examples of the use of the algorithm for the $A_2$, $A_3$, and $B_2$ algebras are given in Ref. 44. The free-field representation for the $Sl(3)$ current algebra was first proposed in Ref. 142, and later in Ref. 148. A fragment of the answer in the general case (obtained by developing the general ideas of Feigin and Fuchs) was published by Feigin and Frenkel in 1988 (Ref. 145), and a complete analysis was given recently in Ref. 146. Similar results are also contained in Ref. 147. An important role in the analysis of the corresponding representation theory is played by non-Abelian analogs of the Feigin–Fuchs operator, also studied in Ref. 146. In the language of conformal models they are known as screening operators and are discussed in Ref. 44. There is an enormous number of later studies on this topic (see, for example, Refs. 148–150 and 152–154).

### 4.3. Parafermions and the coset constructions related to them

Having available the description of WZNW models in terms of fields with quadratic action, we can hope to formulate an arbitrary two-dimensional conformal theory in analogous terms. This is because theories possessing symmetry related to the current algebra, as do all two-dimensional conformal theories, are special, and because it is widely believed that practically any conformal field theory can be obtained by a reduction of the WZNW model. In this subsection we demonstrate the possibility of performing this reduction in such a way that at each stage we are dealing with a theory of free fields.

We begin by describing the so-called coset construction,[160] which allows the construction of the so-called coset model $G/H$ with central charge $c_G - c_H$ from WZNW models with algebra $G$ and its subalgebra $H$. In order to obtain such a coset model in the language of free fields, it is necessary to "bosonize" the original $G$ theory using the fields $\phi_i$ ($i = 1,\ldots,\text{rank}\,G$), $u_\alpha$, and $v_\alpha$ [$\alpha = 1,\ldots,|\Delta_+| = \frac{1}{2}(\dim G - \text{rank}\,H)$]; these fields are related to the $w_\alpha$, $\chi_\alpha$ pair for each positive root $\alpha\in\Delta_+$ by Eqs. (89) and the additional bosonization of the $\xi$, $\eta$ system in terms of the scalar field $v$. It is then necessary to separate the free variables describing the group $G$ by linear transformations into two mutually orthogonal (in the sense of operator decomposition) subsystems, one of which describes the WZNW model constructed for the subalgebra $H$, while the other describes the coset theory $G/H$.

Owing to the linear nature of the transformations and the orthogonality of these two subsystems, the energy–momentum tensor splits into two parts:

$$T_G = T_H + T_{G/H}, \tag{92}$$

which will be quadratic in the fields. This procedure is essentially the same for theories constructed in terms of arbitrary semisimple algebras $G$ and subalgebras $H$.

Following Ref. 155, let us discuss the example of the theory of $\mathbb{Z}_k$ parafermions,[134,190] which can be viewed as the coset $Sl(2)_k/U(1)_k$. We rewrite the bosonization formulas (84) in terms of the fields $\phi$, $u$, and $v$:

$$J^+ = \partial v e^{-u+iv}; \quad J^0 = \frac{i}{\sqrt{2}} q\partial\phi + \partial u;$$

$$J^- = [\sqrt{2}q\partial\phi - iq^2\partial u + (1-q^2)\partial v]e^{u-iv}. \tag{93}$$

We factorize this algebra with respect to the Cartan current $J^0$. Here it should be noted that the current $J^0$ can be written in the form which is standard for the $U(1)_k$ bosonization by introducing a new linear combination of the original fields: $h = (1/\sqrt{k})(q\phi - i\sqrt{2}u)$ and $J^0 = i(\sqrt{k/2})\partial h$. Now, following the usual ideology, we must transform to new, mutually orthogonal variables, one of which is the field $h$. These variables will be $h$, $v$, and a new field $\mathcal{U}$ defined as the other linear combination of $u$ and $\phi$: $(\mathcal{U} = 1/\sqrt{k})(qu + i\sqrt{2}\phi)$. In the new variables the energy–momentum tensor will have the form

$$\begin{aligned} T &= T_u + T_v + T_\phi \\ &= T_{u(1)}(h) + T_{\mathrm{pf}}(\mathcal{U}, v) \\ &= -\frac{1}{2}(\partial h)^2 + \left[ -\frac{1}{2}(\partial\mathcal{U})^2 - \frac{1}{2}\frac{\sqrt{k}}{q}\partial^2\mathcal{U} \right. \\ &\quad \left. -\frac{1}{2}(\partial v)^2 + \frac{i}{2}\partial^2 v \right]. \end{aligned} \tag{94}$$

In Eq. (94) it is easy to separate the part corresponding to the $U(1)$ subgroup, and then the two fields $\mathcal{U}$ and $v$ describing parafermions remain. The central charge for the parafermion system is $c_{\mathrm{pf}} = c_{G/H} = 3k/(k+2) - 1 = 2(k-1)/(k+2)$. The two other $SL(2)$ currents can be rewritten in terms of the new variables as

$$J^+ = \psi_1(\mathcal{U}, v)\exp\left[i\sqrt{\frac{2}{k}}h\right],$$

$$J^- = \psi_1^+(\mathcal{U}, v)\exp\left[-i\sqrt{\frac{2}{k}}h\right], \tag{95}$$

where $\psi_1$ and $\psi_1^+$ are the parafermionic currents.[134] The screening operator is

$$Q = \oint \partial v e^{-u+iv-i\sqrt{2}\phi/q} = \oint \partial v e^{-\sqrt{k}\mathcal{U}/q+iv}. \tag{96}$$

A similar analysis of more general parafermionic and coset models can be found in Refs. 44 and 151.

### 4.4. The Hamiltonian reduction in the $SL(2)$ case (minimal models)

Let us now describe the realization, proposed in Refs. 156 and 159, of the Drinfeld–Sokolov Hamiltonian reduction,[191] which in the $SL(2)$ case is accomplished by imposing the constraints

$$J^+ = 1, \quad J^0 = 0. \tag{97}$$

Here the third current $J^-$ becomes a generator of the $W$ algebra (Refs. 134–136 and 192), $W_2 = -J^-/q^2$, which in this case coincides with the Virasoro algebra generated by the energy–momentum tensor of the reduced theory. This statement is easily understood directly once it is noted that $T_{sl(2)} \sim :\mathrm{tr}\, J^2:$ is transformed into $J^-$ if $J^+ = 1$ and $J^0 = 0$. The constraints $J^+ = 1$ and $J^0 = 0$ can be solved explicitly for the fields $w$ and $\chi$:

$$w = 1, \quad \chi = \frac{i}{\sqrt{2}} q\partial\phi, \tag{98}$$

and the reduced theory is therefore described by a single scalar field $\phi$. Substituting the expression in (98) into the equation for $J^-$, we obtain

$$W_{2,\mathrm{cl}(\phi)} = -\frac{1}{q^2}J^-_{\mathrm{cl}} = -\frac{1}{2}(\partial\phi)^2 + \frac{ik}{\sqrt{2}q}\partial^2\phi. \tag{99}$$

Naturally, these expressions describe only the classical approximation to the exact quantum answer; in solving nonlinear operator equations of the type $\chi w = i\partial\phi/\sqrt{2}$ it is necessary to take into account normal-ordering effects. Therefore, Eq. (99) differs from the exact quantum expression by corrections of order $1/k$; in fact, the correct quantum expression contains $(k+1)/q$ instead of $k/q$ in the second term on the right-hand side of (99). However, this calculation is the simplest way of reproducing the classical expressions for the $W$ generators $W_2, \ldots, W_{r+1}$ in the case of $ADE$ groups of higher rank. In order to find the exact quantum expressions for $W$, it is necessary to work in the $u$, $v$, $\phi$ representation, in which the constraints (97) reduce to linear conditions on the fields $u$, $v$, and $\phi$, so that no problems with normal ordering arise.

We shall use a modification of the representation for $w$ and $\chi$ used above in Subsec. 4.3: $w = \exp(-u - iv)$, $\chi = i\partial v \exp(u + iv)$. Now the $Sl(2)$ currents expressed in terms of the fields $u$, $v$, and $\phi$ have the form

$$J^+ = e^{-u-iv};$$

$$J^0 = -\partial\left(u + \frac{iq}{\sqrt{2}}\phi\right);$$

$$\begin{aligned} J^- &= [(q^2-1)(\partial v)^2 - iq^2\partial u\partial v \\ &\quad + \sqrt{2}q\partial v\partial\phi - i(q^2-1)\partial^2 v]e^{u+iv}, \end{aligned} \tag{100}$$

and the energy–momentum tensor is given by the expression

$$T = T_{u,v} - \frac{1}{2}(\partial\phi)^2 - \frac{i}{q\sqrt{2}}\partial^2\phi. \tag{101}$$

The constraints $J^+ = 1$, $J^0 = 0$ are transformed into the linear equations

$$u + \mathrm{i}v = 0, \quad u + \mathrm{i}q\phi/\sqrt{2} = 0, \tag{102}$$

which make it easy to express $u$ and $v$ in terms of $\phi$: $u = -\mathrm{i}v = -\mathrm{i}q\phi/\sqrt{2}$. Substituting these $u$ and $v$ into the expression for $J^-(u,v,\phi)$, we obtain the quantum energy–momentum tensor in the reduced theory:

$$W_2 = -\frac{1}{q^2}J^- = -\frac{1}{2}(\partial\phi)^2 + \frac{\mathrm{i}}{\sqrt{2}}\left(q - \frac{1}{q}\right)\partial^2\phi. \tag{103}$$

From the viewpoint of the usual ideology described at the beginning of this subsection, these calculations imply that it is necessary to make an orthogonal transformation in the space of fields $u$, $v$, and $\phi$ and introduce new mutually commuting variables

$$U = \frac{1}{\sqrt{2k}}(q^2u + \mathrm{i}kv + \mathrm{i}q\sqrt{2}\phi),$$

$$H = -\mathrm{i}\sqrt{\frac{2}{k}}\left(u + \frac{iq}{2}\phi\right), \quad \Phi = \phi - \frac{\mathrm{i}q}{\sqrt{2}}(u + \mathrm{i}v).$$

The last field $\Phi$ was chosen to be orthogonal to the two constraints, while $U$ and $H$ are linear combinations of them. In these terms the Hamiltonian reduction implies that it is necessary to simply restrict ourselves to vertex operators depending only on the field $\Phi$, i.e., to set

$$U = 0, \quad H = 0. \tag{104}$$

Naturally, the linear transformation is the same in both the quantum and the classical cases. The expression for the energy–momentum tensor in terms of the fields $U$, $H$, and $\Phi$ has the form

$$\begin{aligned} T_{sl(2)} &= -\frac{1}{2}[(\partial U)^2 + (\partial H)^2 + (\partial\Phi)^2] - \frac{\sqrt{k}}{2}\partial^2 U \\ &+ \frac{\mathrm{i}}{\sqrt{2}}\left(q - \frac{1}{q}\right)\partial^2\Phi \\ &= T_{\mathrm{red}}(\Phi) + T(U,H), \end{aligned} \tag{105}$$

and the first term on the right-hand side is the quantum energy–momentum tensor in the reduced theory:

$$T_{\mathrm{red}}(\Phi) = -\frac{1}{2}(\partial\Phi)^2 + \frac{\mathrm{i}}{\sqrt{2}}\frac{k+1}{q}\partial^2\Phi. \tag{106}$$

Accordingly, the central charge is

$$c_{\mathrm{red}} = 1 - 12\left[\frac{k+1}{\sqrt{2}q}\right]^2 = \frac{3k}{k+2} - 6k - 2.$$

Equation (106) can also be rewritten as

$$\begin{aligned} &T_{\mathrm{red}}(\Phi) - \frac{1}{2}(\partial U)^2 - \frac{1}{2}(\partial H)^2 \\ &= T_{sl(2)} - \partial J^0 - \frac{k}{2}\partial^2(u + \mathrm{i}v) \end{aligned} \tag{107}$$

[we note that $\partial J^0 = -\mathrm{i}\sqrt{(k/2)}\partial^2 H$]. This explains the appearance of the "deformation" of the Sugawara tensor introduced in Ref. 135 and discussed in Refs. 148, 193, and 194. We note that the term $\partial^2(u + \mathrm{i}v)$ was omitted in, for example, Ref. 148 for the simple reason that the condition $(u + \mathrm{i}v) = 0$ (or $J^+ = 1$) is a first-class constraint, in contrast to the second-class constraint $J^0 = 0$ leading to the equation $H = 0$. We can therefore set $(u + \mathrm{i}v) = 0$ from the very beginning, whereas the term $\partial J^0$ requires more careful treatment.

The two screening operators of the $SL(2)_k$ WZNW model have the form

$$\begin{aligned} &Q_\pm = \int \tilde{J}_\pm, \quad \tilde{J}_+ = w \exp\left(-\frac{\mathrm{i}\sqrt{2}}{q}\phi\right), \\ &\tilde{J}_- = w^{-(k+2)}\exp(\mathrm{i}\sqrt{2}q\phi). \end{aligned} \tag{108}$$

(Only the first admits interpretation as the result of variable substitution; see Subsec. 4.1. The interpretation of the second operator, which actually arises as a nonsingular operator for $k \geqslant -2$ only after reduction, is not yet completely clear.) If the operators $Q_\pm$ are rewritten in terms of the fields $U$, $H$, and $\Phi$, they will depend only on the field $\Phi$:

$$Q_+ = \oint \exp\left(-\frac{\mathrm{i}\sqrt{2}}{q}\Phi\right),$$

$$Q_- = \oint \exp(\mathrm{i}\sqrt{2}q\Phi).$$

The substitution $q^2 = m/n$ transforms the reduced $sl(2)_k$ theory into an ordinary minimal model $M_{m,n}$ (Ref. 23) with central charge $c = 1 - 6(q - 1/q)^2 = 1 - 6((m-n)^2/mn)$. The field $\Phi$ is the scalar Dotsenko–Fateev field,[132] and the operators $Q_\pm$ are the screening operators in this representation.

Further detail about the Drinfeld–Sokolov reduction and the Lagrangian interpretation of classical $W$ algebras can be found in Refs. 157–159, 195, and 196. In particular, a conformal model in which the $W$ generators are Noether currents is discussed in Ref. 157. The $Sl(2)$ version of this model arises in the geometrical quantization of the Virasoro algebra, and after the variable substitution it is transformed into the Liouville theory.[138] These results show that also nonlinear symmetries in a two-dimensional quantum conformal field theory can be viewed as the symmetries of some action and described in terms of free fields. This opens up the possibility of a Lagrangian formulation of string models with extended nonlinear conformal symmetry, which have recently become known as $W$ strings and $W$ gravity.

### 4.5. On the concept of the $W_\infty$ algebra

Here let us briefly discuss the structure of $W_N$ algebras from the viewpoint of the free-field representation (without considering the additional global constraints on these fields implied by the more fundamental geometrical description in Refs. 138 and 157). A similar description of $W$ algebras in terms of free fields was essentially formulated in Refs.

134, 135, and 192, and all the details are worked out in Ref. 136. According to Ref. 156, this description is particularly convenient for understanding the reasons behind the breakdown of the algebraic structure for $W$ algebras. It also led in Ref. 156 to one of the first indications that this symmetry is restored in the limit $N=\infty$. This indication was based on the revelation of two sources of nonlinearity of classical $W_N$ algebras.

The (classical) $W$ algebra[192] associated with the Kac–Moody algebra $\widehat{G}$ can be viewed as a fragment of its universal enveloping algebra. To be specific, we restrict ourselves to the algebra of Hermitian matrices $Sl(N)$. In Ref. 151 it was noted that the operators $W_n$ are directly related to the operators $\mathrm{Tr}J^n$, but only after a certain reduction. It is simplest to *ab initio* restrict ourselves to the Cartan subalgebra in $G$ and free fields $\phi=\{\phi_i\}$, $i=1,...,N$, with operator decomposition $\phi_i(z)\phi_j(0)=z^{-2}\delta_{ij}+\text{reg}$. Then the current matrix $J$ is diagonal and its components are $i\vec{\mu}_i\partial\vec{\phi}$ where $\vec{\mu}_i$ are the fundamental weights of $Sl(N)$, which possess two properties important for what follows:

$$\vec{\mu}_a\vec{\mu}_b=\delta_{ab}-1/N,\quad \sum_{a=1}^{N}\vec{\mu}_a=0. \tag{109}$$

We consider the operators

$$W_n\equiv\mathrm{Tr}\,J^n|_{\mathrm{Cart}}=\sum_{a=1}^{N}(i\vec{\mu}_a\partial\vec{\phi})^n. \tag{110}$$

Together with

$$W_{(n_1...n_k)}\equiv\sum_{a=1}^{N}(i\vec{\mu}_a\partial\vec{\phi})^{n_1}...(i\vec{\mu}_a\partial^k\vec{\phi})^{n_k} \tag{111}$$

they form a closed operator algebra, for example:

$$W_n(z)W_m(0)=\sum_{a,b=1}^{N}(i\vec{\mu}_a\partial\vec{\phi})^n(z)(i\vec{\mu}_a\partial\vec{\phi})^m(0)$$
$$=\sum_{a,b=1}^{N}\sum_{k=1}^{\min(n,m)}k!G_k^nG_k^m\frac{(\vec{\mu}_a\vec{\mu}_b)^k}{z^{2k}}:$$
$$(i\vec{\mu}_a\partial\vec{\phi})^{n-k}(z)(i\vec{\mu}_b\partial\vec{\phi})^{m-k}(0):. \tag{112}$$

Equation (112) involves the binomial coefficients $G_k^n=n!/k!(n-k)!$. Furthermore,

$$(\mu_a\mu_b)^k=(A\delta_{ab}+B)^k$$
$$=(\delta_{ab}-1/N)^k=a_k^{(N)}\delta_{ab}+b_k^{(N)}, \tag{113}$$

where $a_k^{(N)}=(A+B)^k-B^k=(1-1/N)^k-(-1/N)^k$ and $b_k^{(N)}=B^k=(-1/n)^k$. Therefore,

$$W_n(z)W_m(0)=\sum_{k=0}^{\min(n,m)}\frac{k!C_k^n\cdot C_k^m}{z^{2k}}[a_k^{(N)}W_{n-k,m-k}(z,0)$$
$$+b_k^{(N)}:W_{n-k}(z)W_{m-k}(0):]. \tag{114}$$

Here we have introduced new notation for the bilocal $W$ operator:

$$W_{n,m}(z,0)\Pi\sum_{a,b=1}^{N}:(i\vec{\mu}_a\partial\vec{\phi})^n(z)(i\vec{\mu}_a\partial\vec{\phi})^m(0):$$
$$=W_{n+m}(0)+z\cdot n\cdot W_{(n+m-1,1)}(0)+z^2$$
$$\times\left[n(n-1)\cdot W_{(n+m-2,2)}(0)\right.$$
$$\left.+\frac{n}{2}W_{(n+m-1,0,1)}(0)\right]+\dots.$$

It is important that $W_{n,m}(z,0)$ is a linear combination of the original local $W$ operators (111). In the classical limit the operator algebra (110) closes on itself. In a similar manner we can also find the operator decompositions of any other operators (111). Here Eq. (113) always guarantees that two structures arise in this operator decomposition: one linear in $W$ with coefficient proportional to $a_k^{(N)}$, and one bilinear in $W$ with coefficient $b_k^{(N)}$. In the limit $N\to\infty$, $b_k^{(N)}\to 0$, and the quadratic terms disappear. This essentially explains the restoration of the Lie algebra structure for $W_\infty$. In fact, in addition to this source of nonlinearity [related to considering the algebra $G=Sl(N)$ instead of $Gl(N)$], there is another source, even simpler, at least for the discussion at the classical level.[158]

The point is that, having at our disposal only $N$ variables $F_a=i\vec{\mu}_a\partial\vec{\phi}$ (actually, we also have $\Sigma_aF_a=0$), viewed as $c$-numbers, we can express any $W_n=\Sigma_{a=1}^{N}F_a$ with $n>N$ in terms of $W_1,...,W_N$:

$$W_{n>N}=\mathscr{F}_n^{(N)}[W_1,...,W_N]. \tag{115}$$

The functions $\mathscr{F}_n^{(N)}$ can be determined recursively:

$$\mathscr{F}_n^{(0)}\equiv 0;$$

$$\mathscr{F}_{n>N}^{(N)}=\mathscr{F}_n^{(N-1)}+\frac{1}{N}(W_N-\mathscr{F}_N^{(N-1)})W_{n-N} \tag{116}$$

and are strongly nonlinear. As a result, the commutator of the operators $W_n$ and $W_m$ containing $W_l$ with $l>N$ can be rewritten as a nonlinear combination of $W_1,...,W_N$. Clearly, this source of nonlinearity also vanishes in the limit $N\to\infty$. We can transform to a new basis $\{\widetilde{W}_n\}$ nonlinearly related to $\{W_n\}$ such that the conditions (115) acquire the form

$$\widetilde{W}_{N+1},\widetilde{W}_{N+2}...=0. \tag{117}$$

The variables $\widetilde{W}$ are the standard Zamolodchikov operators. In Refs. 136 and 192 it was shown by direct calculation that the algebra of the operators $\widetilde{W}_1,...,\widetilde{W}_N$ remains closed also at the quantum level [which, as we have seen, does not occur in the case of $W_1,...,W_N$; other operators (111) also arise in the operator decompositions, and the total number of such operators is infinite]. So far there is no clear understanding of the reasons for which the quantum $\widetilde{W}_N$ algebra closes.

It is simplest to write the explicit relation between the sets of variables $\{W_n\}$ and $\{\widetilde{W}_n\}$ in terms of the generating functions $\widetilde{W}(t)\equiv\Sigma_{n=1}^{\infty}\widetilde{W}_nt^n$ and $W(t)\equiv\Sigma_{n=1}^{\infty}W_nt^n$:

$$W(t)=\frac{td\widetilde{W}(t)/dt}{1-W(t)},$$

$$\widetilde{W}(t)=1-\exp\left[-\int^t \frac{W(t)dt}{t}\right]. \qquad (118)$$

In these terms the identities (115) and (117) imply that $\widetilde{W}(t)$ is a polynomial of degree $N$ in $t$: $(d/dt)^{N+1}\widetilde{W}(t)=0$.

### 4.6. The generalized Sugawara construction

The standard Sugawara construction[197,198] allows the generators of the Virasoro algebra (the energy–momentum tensor) to be constructed from the generators of the KM algebra $\hat{G}$ (the currents): $T(z)=[2(k+g_G)]^{-1}:\mathrm{Tr}\,J^2(z):$, where $k$ is the central charge of the algebra $\hat{G}$ and $g_G$ is the dual Coxeter number. It is natural to ask the question of what is the most general relation of this type. Since (up to anomalies) $J(z)$ are 1-differentials and $T(z)$ is a quadratic differential, such a relation must naturally be quadratic (simple corrections of the type $\partial J$ are also possible): $T(z)=G_{ab}J^aJ^b(z)$. The conditions on the coefficients $C_{ab}$, $a,b=1,...,\dim G$, were found independently in Refs. 161 and 162:

$$C^{ab}C^{cd}[f^{ace}f^{bme}\delta^{dn}-f^{ace}f^{dme}\delta^{bn}-f^{acm}f^{bdn} + k\delta^{ac}\delta^{bm}\delta^{dn}+(m\leftrightarrow n)]=C^{mn}. \qquad (119)$$

The central charge of the Virasoro algebra here is

$$c=2kC^{aa}. \qquad (120)$$

The simplest solutions of the system of equations (119) are

for the Sugawara solution, $C^{ab}=\delta^{ab}/2(k+g_G)$,

and for the GKO solution,

$$C^{ab}=\delta_\perp^{ab}/2(k+g_G) + \delta_\parallel^{ab}[1/2(k+g_G)\text{-}1/2(k+g_H)]$$

(here $\perp$ and $\parallel$, respectively, denote the subspaces orthogonal and parallel to the subalgebra $H\subset G$ with respect to the Killing form). A nontrivial one-parameter family of solutions of (119) was found in Ref. 116 for the KM algebra $Sl(2)_{k=4}$. It corresponds to the value $c=1$, and in Ref. 163 it was demonstrated that it is related to the standard family of conformal models with central charge 1 [usually described in terms of a free bosonic field with values in a circle or a segment (a $Z_2$ orbifold) of arbitrary length]. The explicit form of the solution is given in Ref. 161. Many other solutions of Eqs. (119) were later found in Refs. 164–167 and other studies. Their complete classification is not yet known. One of the merits of this approach to the construction of conformal models is the possibility of obtaining theories known to be unitary starting from unitary representations of the KM algebra. The relation of this "generalized Sugawara construction" to "quasi-exactly solvable" quantum-mechanical models is discussed in Refs. 191–201, and the relation to rational conformal theories is dealt with in Ref. 161.

## CONCLUSION

In this review we have described the results on perturbative calculations in string theory. As mentioned in the Introduction, this is a very limited area of string theory (but at present it is the only area which is well developed). The actual meaning of taking into account the field fluctuations of only a single specific string model (i.e., multiloop calculations for a particular conformal model) within the framework of the complete string theory is not completely clear: such an approach would be reasonable only if there were a small parameter suppressing quantum transitions between different string models. Therefore, the primary use of this formalism from the viewpoint of string theory is to find a suitable language which makes further generalizations more or less natural. At present there are three fruitful directions of development of the first-quantization formalism which may lead to the creation of a complete string theory. The first is the study of the infinite-dimensional universal moduli space,[81–86] the second is the study of integrable models[173,174] and two-loop algebras,[129,130] and the third, which is closely related to the two preceding ones, is based on the so-called random-triangulation formalism or matrix models[125–128] (see also Ref. 202). All these approaches have much in common,[203,204] which leads us to hope that they will prove complementary and powerful. However, a discussion of these possibilities lies outside the range of this review.


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