Quasi-exact solvability. A new phenomenon in quantum mechanics (algebraic approach)

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A new approach to the problem of quasi-exact solvability in quantum mechanics is proposed. It is shown that quasi-exactly solvable Schrödinger equations (admitting exact solutions only for limited parts of the spectrum) can be obtained from completely integrable Gaudin models based on simple Lie algebras by means of partial separation of the variables.

INTRODUCTION

Despite the rapid changes of taste of modern theoretical physics and of the views of which problems are fundamental, we may with confidence identify a link connecting the majority of "hot spots" of quantum theory. This link is exact methods and their physical applications. Interest in these methods rose sharply at the end of the seventies, when the progress achieved by them in understanding the nature of nonperturbative phenomena and the structure of quantum theories in the strong-coupling region became obvious. A decisive role in the development of exact methods was played by the possibility of using explicit solutions of quantum problems as the zeroth approximations in the construction of perturbation theory. A no less important factor was the observation that some hidden symmetry of the problem could always be found behind the phenomenon of exact solvability. This symmetry did not simply play the part of a necessary attribute of integrability but was, rather, the language in which this phenomenon acquired a clear and transparent mathematical meaning.

In the following years, numerous remarkable methods for constructing exactly solvable models were created. These were the quantum inverse-problem method,[1] the alternative group approach,[2] various modifications of the Bethe method[3] (in quantum field theory) and also the Gel'fand–Levitan–Marchenko equations,[4] Darboux transformations,[5] the supersymmetric approach,[6] the projection method,[7] and numerous analytical methods[8] (in nonrelativistic quantum mechanics). Despite all the advantages of these approaches, one must admit with regret that the number and diversity of exactly solvable models obtained by means of them is relatively small and by no means meets the requirements of modern theoretical physics. There is no doubt about the need to look for new approaches and new ideas, and in this connection it seems to me that there is a very promising direction which arose three years ago[9,10] and is associated with study of models that realize a fundamentally new type of exact solvability in quantum theory. These models, which have been dubbed "quasi-exactly solvable," are characterized by the fact that the spectral problems for them can be solved exactly, but not for the entire spectrum but only for certain limited sections of it. They are important above all because it has been found that they possess all the advantages of ordinary exactly solvable models, i.e., they enable one to model real physical situations and observe the occurrence of numerous nonperturbative phenomena; they can be used as "reference points" in the realization of approximate methods; and they owe their existence to the presence in the spectral equations of deep symmetry properties.[11,12] In addition (and this, surely, is the most important thing), their number is appreciably greater than the number of exactly solvable models, and this makes the problem of studying quasi-exactly solvable problems very promising from the practical point of view.

The last circumstance can be explained in the following transparent manner. If exact solvability is a synonym for explicit diagonalizability of an infinite-dimensional Hamiltonian matrix, then quasi-exact solvability corresponds to block diagonalizability with, moreover, one of the blocks finite. The spectral problem then decomposes into two completely decoupled problems, one of which is finite-dimensional and can be solved purely algebraically (i.e., exactly), while the second is infinite-dimensional and, as before, we know nothing about its solutions. Since it is always much simpler to block-diagonalize any infinite-dimensional matrix than it is to diagonalize the matrix completely, there must be significantly more quasi-exactly solvable models than exactly solvable ones.

Let us consider an example. In the class of even polynomial potentials in one-dimensional quantum mechanics, only one exactly solvable model is known. It is represented by the single-parameter family of harmonic oscillators:

$$V=b^2x^2. \tag{1}$$

As regards quasi-exactly solvable models, there is an infinite set of them in the same class. Their potentials can be realized in the form of polynomials of degree $2+4n$, where $n=1, 2,\ldots$. The simplest of these are given by polynomials of sixth order:[12,13]

$$V=a^2x^6+2abx^4+[b^2-a(4M+3)]x^2. \tag{2}$$

Here, $M=0, 1, 2, \ldots$, and a and b are real numbers, so that for each fixed M we have a two-parameter family of potentials. The non-negative integer M determines the order of quasi-exact solvability, i.e., it tells us how many states in the model can be found exactly. If M is given, then in (2) the first $M+1$ positive-parity levels can be calculated exactly. The solutions lie in the class of functions of the form

$$\psi = P_M(x^2)\exp\left\{-\frac{ax^4}{4}-\frac{bx^2}{2}\right\}, \quad (3)$$

where $P_M(x^2)$ are unknown even polynomials of order $2M$. The linear hull of the functions of the form (3) forms an $(M+1)$-dimensional subspace of the Hilbert space of the model that is invariant with respect to the action of the Hamiltonian. Therefore, the secular equation is an algebraic equation of degree $M+1$. This is why there are $M+1$ exact solutions.

The model (2) is a natural generalization of the model (1)—it can be regarded as a perturbed harmonic oscillator (the parameter of the perturbation is a). This circumstance makes it possible to compare results obtained by perturbation theory with the exact results and to speak of numerous nonperturbative effects, such as the singularities of Bender and Wu, intertwining and repulsion of levels, fortuitous convergence of the perturbation-theory series that results from cancellation of factorially growing contributions and holds only for integer M and only for states that can be exactly calculated, etc.[12]

At the same time, the models (2) can be regarded as approximations to the exactly solvable model of the anharmonic oscillator with potential

$$V = b^2x^2 + gx^4. \quad (4)$$

For this, it is sufficient in (2) to make the substitution $2ab \to g$ and $b^2 - 4aM \to b^2$, letting M go to infinity.[12] This result is a special case of a more general result: Any exactly solvable model of one-dimensional quantum mechanics is the limit of a certain infinite series of quasi-exactly solvable models.[12]

Thus, it can be asserted that quasi-exactly solvable models occupy an intermediate place between models that can and cannot be exactly solved. This circumstance makes it possible to extend any results obtained for the quasi-exactly solvable case to both of these two cases (see, for example, Refs. 11, 12, and 14).

At the present time, there exist two alternative theories which explain the phenomenon of quasi-exact solvability and make it possible to construct large classes of quasi-exactly solvable models. One of these theories (formulated by Turbiner[9] in 1987) is based on the observation that the quasi-exactly solvable models of quantum mechanics possess hidden dynamical symmetry groups whose finite-dimensional representations generate the finite parts of their spectra. For example, for the model (2) such a group is $SL(2)$, or, rather, its finite-dimensional representations with "spin" $j = M/2$. The essence of Turbiner's idea is that since the generators of finite-dimensional representations of Lie algebras can always be realized in the form of first-order differential operators, any operator quadratic in these generators will be a second-order differential operator. On the other hand, because the representation is finite-dimensional, the spectral problem for this operator reduces to a purely algebraic problem and can be exactly solved. By a homogeneity transformation, the operator itself may (although not always) be reduced to Schrödinger form. The only algebra that admits representations in the form of one-dimensional differential operators is the $sl(2)$ algebra. It generates one-dimensional quasi-exactly solvable models. If multidimensional models are to be generated, one needs algebras of higher rank (the multidimensional case was considered in this approach in Ref. 15). To speak in a more "physical" language in Turbiner's method a connection is established between quasi-exactly solvable problems of quantum mechanics and asymmetric quantum tops (rotators) based on finite-dimensional representations of Lie algebras. We do not intend to enter here into the details of this method, since Shifman[11] has written a beautiful review on this theme.

The other theory of quasi-exact solvability (formulated by the present author,[10] also in 1987) is based on the assertion that there exists an intimate connection between quasi-exactly solvable models of quantum mechanics, completely integrable models of magnets based on infinite-dimensional representations of Lie algebras, and the classical many-particle Coulomb problem. The main elements of this theory were presented in my review in Ref. 12. However, soon after this review was written, it became clear to me that the method of exposition chosen in it was far from the best. The fact is that in the review the main attention is devoted to analytic methods for constructing quasi-exactly solvable equations, and unfortunately their algebraic (symmetry) properties, which undoubtedly play a decisive role for the understanding of the nature of quasi-exact solvability, remained in the background. Subsequent attempts[16,17] to formulate this approach differently (taking the symmetry properties as the cornerstone, i.e., taking models of magnets based on Lie algebras as the point of departure) led ultimately to a quite different understanding of the problem. In fact, this was the main reason for writing the present paper, which is an extended exposition of the papers of Refs. 16–18. The main features of the approach presented here are as follows: 1. We consider completely integrable models of magnets based on Lie algebras that are exactly solvable in the framework of the algebraic Bethe ansatz. Since the Hamiltonians of these magnets are quadratic in the generators of the algebras, we arrive, using differential realizations of the generators, at multidimensional spectral differential equations of second order that can be exactly solved in a certain class of functions. 2. The considered models of magnets and, therefore, the equations obtained from them possess global internal symmetry groups, permitting partial separation of the variables in them. As a result of this, the original equation, which has an infinite exactly calculable spectrum, decomposes into an infinite series of equations of lower dimension with a finite number of exactly calculable eigenfunctions and eigenvalues. The obtained series of quasi-exactly solvable equations is parametrized by a set of integers, which in the procedure described above play the part of separation constants. 3. In the final stage, the constructed quasi-exactly solvable equations are reduced to Schrödinger form. It should be mentioned that throughout the paper we restrict ourselves to a detailed consideration of only the first two of the stages listed above: 1) the definition and solution of generalized Gaudin models; 2) the reduction of these models to infinite series of quasi-exactly solvable equations. The third stage, i.e., the procedure of reduction of these last two Schrödinger forms, is not discussed, since it has been well algorithmized in Ref. 12. According to Ref. 12, any D-dimensional quasi-exactly solvable equation of second order can be reduced in infinitely many ways to equations of Schrödinger type on $(D+1)$-dimensional manifolds, which are, in general, curved.

The review is organized as follows. In Sec. 1, we present the idea of the approach for the example of the simplest algebra $sl(2)$. Sections 2 and 3 are devoted to the discussion of some properties of simple Lie algebras that are needed for the formulation of the approach in the general case. In these sections, much attention is devoted to the algorithms for constructing differential realizations of representations of Lie algebras and the choice of a special basis convenient for exact solution of the generalized Gaudin models. In Sec. 4, we give the solution for the models of Gaudin magnets in the framework of the algebraic Bethe ansatz. In Sec. 5, we formulate the method of reduction of the Gaudin models to multidimensional quasi-exactly solvable differential equations. In the final section, Sec. 7, we discuss the possibility of dealgebrization of the proposed approach.

1. THE IDEA OF THE APPROACH. THE CASE OF THE ALGEBRA *sl*(2)

The Gaudin algebra and its representations

Let $I^{\pm}(\lambda)$, $I^0(\lambda)$ be the generators of the infinite-dimensional Gaudin algebra; they depend on the complex parameter λ and satisfy the commutation relations[3]

$$\left.\begin{aligned} &[I^-(\lambda),I^+(\mu)]=\frac{2}{\mu-\lambda}\{I^0(\lambda)-I^0(\mu)\};\\ &[I^0(\lambda),I^{\pm}(\mu)]=\pm\frac{1}{\mu-\lambda}\{I^{\pm}(\lambda)-I^{\pm}(\mu)\}.\end{aligned}\right\} \tag{5}$$

We define representations of this algebra in accordance with the formulas

$$\left.\begin{aligned} &I^-(\lambda)|0\rangle=0;\\ &I^0(\lambda)|0\rangle=F(\lambda)|0\rangle.\end{aligned}\right\} \tag{6}$$

Here $|0\rangle$ is the lowest (vacuum) vector, and $F(\lambda)$ is a function that plays the part of the lowest weight. The space of the representation $W\{F(\lambda)\}$, infinite-dimensional in the general case, is defined as the linear hull of all possible vectors of the form

$$I^+(\lambda_1)...I^+(\lambda_M)|0\rangle, \tag{7}$$

where $M=0, 1, 2, \ldots$, and $\lambda_1, \ldots, \lambda_M$ are arbitrary complex numbers.[18]

Gaudin spectral problem

We consider the operators

$$K(\lambda)=I^0(\lambda)I^0(\lambda)-\tfrac{1}{2}I^+(\lambda)I^-(\lambda) -\tfrac{1}{2}I^-(\lambda)I^+(\lambda). \tag{8}$$

Using the commutation relations (5), we can readily verify that these operators form a commutative family,

$$[K(\lambda),K(\mu)]=0, \tag{9}$$

for all λ and μ.[3] We shall call the spectral problem

$$K(\lambda)\phi=E(\lambda)\phi, \quad \phi\in W\{F(\lambda)\} \tag{10}$$

the generalized Gaudin problem. We use the adjective "generalized" here because Gaudin himself and other authors[3,19,20] considered only the case corresponding to a nondegenerate rational function $F(\lambda)$. A special form of the operators $I^a(\lambda)$, $a=\pm, 0$, was also considered. Here, no restrictions are imposed on the form of the function $F(\lambda)$ and the operators $I^a(\lambda)$. By virtue of the commutation relations (9), the eigenfunctions of the operators $K(\lambda)$ do not depend on λ. The problem (10) is completely integrable and, as in the special, Gaudin case, can be solved exactly in the framework of the algebraic Bethe ansatz.[3,21]

Bethe solution of the Gaudin problem

Using the commutation relations (5) of the Gaudin algebra, we can readily verify (see, for example, Refs. 21 and 27) that the solution of the Gaudin problem has the form

$$\left.\begin{aligned} &\phi=\phi_M(\xi)=I^+(\xi_1)...I^+(\xi_M)|0\rangle;\\ &E(\lambda)=E_M(\lambda,\xi)=F'(\lambda)+F^2(\lambda)+2\\ &\qquad\times\sum_{m=1}^{M}\frac{F(\lambda)-F(\xi_m)}{\lambda-\xi_m},\end{aligned}\right\} \tag{11}$$

where $M=0, 1, 2,\ldots$, and the numbers ξ_m, $m=1,\ldots, M$, satisfy the system of numerical equations

$$\sum_{m=1}^{M}{}'\frac{1}{\xi_n-\xi_m}+F(\xi_n)=0, \quad n=1,\ldots,M. \tag{12}$$

We shall show below that this is merely a special solution.

Symmetry of the Gaudin model and general solution

For definiteness, we assume that the limit

$$F=\lim_{\lambda\to\infty}\lambda f(\lambda) \tag{13}$$

exists. Then by virtue of (5) and (6) there must also exist the limits

$$I^a=\lim_{\lambda\to\infty}\lambda I^a(\lambda), \quad a=\pm,0. \tag{14}$$

The operators (14) satisfy the commutation relations of the algebra $sl(2)$:

$$[I^-,I^+]=2I^0, \quad [I^0,I^+]=\pm I^{\pm}, \tag{15}$$

and commute with all the operators $K(\lambda)$:

$$[K(\lambda),I^a]=0, \quad a=\pm,0. \tag{16}$$

This means that the Gaudin model possesses global $sl(2)$ symmetry. Therefore, if $\phi_M(\xi)$ is a solution of Eq. (10), then any vector of the form $f(I^+,I^0,I^-)\phi_M(\xi)$, where f is an arbitrary function, will also be a solution with the same eigenvalue $E_M(\lambda,\xi)$. The degeneracy of each eigenvalue is obviously infinite. Thus, proceeding from the special solution (11),(12) of the Gaudin problem, and using its global symmetry, we can construct the general solution.[21]

Properties of Bethe solutions

The Bethe solutions (11),(12) possess the remarkable property

$$\left.\begin{aligned} &I^-\phi_M(\xi)=0,\\ &I^0\phi_M(\xi)=(F+M)\phi_M(\xi),\end{aligned}\right\} \tag{17}$$

which can be verified by means of (5), (13), and (14). This enables us to regard the Bethe solutions as the lowest vectors of representations of the symmetry algebra with lowest weights $F+M$, $M=0, 1, 2, \ldots$. By virtue of what was said above, these representations are infinite-dimensional.[21]

From generalized Gaudin models to models of magnets

An important special case is realized when $F(\lambda)$ is a rational function. In this case, the Gaudin models reduce to models of magnets based on finite-dimensional Lie algebras. To see this, it is sufficient to consider only nondegenerate functions $F(\lambda)$ of the form

$$F(\lambda)=\sum_{A=1}^{N}\frac{f_A}{\lambda-\sigma_A}. \tag{18}$$

All other (degenerate) rational functions $F(\lambda)$ can be obtained from (18) by merging all or some of the simple poles σ_A. There is no point in taking a pole to infinity, since by hypothesis $F(\lambda)$ must be regular at infinity. If (18) is to be compatible with the expressions (5) and (6), the operators $I^a(\lambda)$ must be sought in a form analogous to (18):

$$I^a(\lambda)=\sum_{A=1}^{N}\frac{I_A^a}{\lambda-\sigma_A}. \tag{19}$$

Substitution of (19) in (5) leads to a commutation relations directly for the operators I_A^a:

$$\left.\begin{aligned} &[I_A^-,I_A^+]=2I_A^0,\quad [I_A^0,I_A^\pm]=\pm I_A^\pm,\\ &[I_A^a,I_B^b]=0;\quad a,b=\pm,0;\quad A\neq B.\end{aligned}\right\} \tag{20}$$

We see that they form the algebra $sl(2)\oplus\ldots\oplus sl(2)$ (N times). If $F(\lambda)$ is a degenerate rational function, then Eq. (19) and the relations (20) become more complicated. However, they still describe a certain finite-dimensional Lie algebra that arises as a result of contraction of the algebra (20).[21,22]

Substituting the expansion (19) in Eq. (8), we obtain

$$K(\lambda)=\sum_{A,B=1}^{N}\frac{I_A^0I_B^0-\frac{1}{2}I_A^+I_B^- -\frac{1}{2}I_A^-I_B^+}{(\lambda-\sigma_A)(\lambda-\sigma_B)}. \tag{21}$$

Thus, we arrive at a commutative family of integrals of the motion of the magnet based on the finite-dimensional algebra (20). This (nondegenerate) case was considered in detail in Refs. 3, 19, and 20. Magnets based on the contracted algebras (20) were considered in Refs. 21 and 22.

Differential realizations

It is well known[9,24,25] that the generators of the algebra $sl(2)$ can be realized in the form of first-order differential operators. For the case (20),

$$I_A^-=\frac{\partial}{\partial t_A},\quad I_A^0=t_A\frac{\partial}{\partial t_A}+f_A,\quad I_A^+=t_A^2\frac{\partial}{\partial t_A}+2t_Af_A. \tag{22}$$

These operators realize representations with the lowest weights f_A. The unit function plays the part of the lowest vector: $|0\rangle\equiv 1$.

Substituting (22) in (21), we see that $K(\lambda)$ take the form of N-dimensional differential operators of second order:

$$K(\lambda)=\sum_{A,B=1}^{N}\times\frac{(t_A-t_B)^2\frac{\partial^2}{\partial t_A\partial t_B}+2(t_A-t_B)\left(f_A\frac{\partial}{\partial t_B}-f_B\frac{\partial}{\partial t_A}\right)}{(\lambda-\sigma_A)(\lambda-\sigma_B)}. \tag{23}$$

Accordingly, the exactly solvable Gaudin equation (10) becomes a differential equation. In the considered nondegenerate case, its Bethe solutions take the form

$$\phi=\phi_M(\xi)=\prod_{m=1}^{N}\left(\sum_{A=1}^{N}\frac{t_A^2(\partial/\partial t_A)+2f_At_A}{\xi_m-\sigma_A}\right)1, \tag{24}$$

i.e., they are homogeneous polynomials in $t_1,\ldots,t_N$ of degree M.[16]

Separation of invariant subspaces

We denote by $\Phi_M\{F\}$ the spaces that consist of the vectors $\phi\in W\{F(\lambda)\}$ that satisfy the conditions

$$I^-\phi=0, \tag{25a}$$

$$I^0\phi=(F+M)\phi. \tag{25b}$$

It follows from the commutativity of the operators $K(\lambda)$ and I^a that if $\phi\in\Phi_M\{F\}$, then also $K(\lambda)\phi\in\Phi_M\{F\}$. Therefore, the spaces $\Phi_M\{F\}$ are invariant with respect to the action of the operators $K(\lambda)$. Accordingly, the spectral problems

$$K(\lambda)\phi=E(\lambda)\phi,\quad \phi\in\Phi_M\{F\}, \tag{26}$$

are defined for all $M=0,1,2,\ldots$.

Finite dimensionality of invariant subspaces

It is readily seen that the spaces $\Phi_M\{F\}$ are finite-dimensional. To prove this, we consider the auxiliary spaces $\tilde{\Phi}_M\{F\}$, which are formed from vectors that satisfy only Eq. (25b). It is obvious that a basis in $\Phi_M\{F\}$ is supplied by vectors of the form

$$I_{A_1}^{+}\cdot\ldots\cdot I_{A_M}^{+}|0\rangle,\quad A_1,\ldots,A_M=1,\ldots,N. \tag{27}$$

The number of such vectors determines the dimension of $\tilde{\Phi}_M\{F\}$, which is

$$\dim\,\tilde{\Phi}_M\{F\}=\frac{(M+N-1)!}{(N-1)!M!}. \tag{28}$$

Since

$$\Phi_M\{F\}\subset\tilde{\Phi}_M\{F\}, \tag{29}$$

it follows that

$$\dim\,\Phi_M\{F\}<\dim\,\tilde{\Phi}_M\{F\}. \tag{30}$$

Using the second condition (25a), we find that

$$\dim\,\Phi_M\{F\}=\dim\,\tilde{\Phi}_M\{F\}-\dim\,\tilde{\Phi}_{M-1}\{F\}=\frac{(M+N-2)!}{(N-2)!M!}. \tag{31}$$

It follows from this formula that for each M the spectral equations (26) have precisely $(M+N-2)!\times[(N-2)!M!]^{-1}$ solutions.

In accordance with Eqs. (17), the Bethe solutions (11)–(12) of the Gaudin equation (10) belong to the spaces $\tilde{\Phi}_M\{F\}$ and are therefore simultaneously solutions of the finite-dimensional spectral problems (26). To resolve the question of the completeness of these solutions, it is sufficient to count the number of admissible sets of numbers ξ_n that satisfy the equations of the Bethe ansatz:

$$\sum_{m=1}^{M}{}' \frac{1}{\xi_n-\xi_m}+\sum_{A=1}^{N}\frac{f_A}{\xi_n-\sigma_A}=0,\quad n=1,\ldots,M. \tag{32}$$

It follows from analysis of (32) by the Coulomb-analogy method (see, for example, Ref. 12) that for given M the required number is also equal to $(M+N-2)!\times[(N-2)!M!]^{-1}$. From this we conclude that for all M all solutions of Eqs. (26) are completely described by the Bethe formula (11)–(12).[16]

Partial separation of the variables

It follows from the invariance of the spaces $\Phi_M\{F\}$ with respect to the action of the operators $K(\lambda)$ that it is meaningful to speak of the projection of $K(\lambda)$ onto $\Phi_M\{F\}$. To construct this projection, we rewrite Eqs. (25) in the differential form

$$\left(\sum_{A=1}^{N}\frac{\partial}{\partial t_A}\right)\phi=0, \tag{33a}$$

$$\left(\sum_{A=1}^{N}t_A\frac{\partial}{\partial t_A}\right)\phi=M\phi. \tag{33b}$$

We have here used the fact that $F=\Sigma_{A=1}^{N}f_A$. The general solution of the system can be represented in the factorized form

$$\phi=\phi_M\psi, \tag{34}$$

where

$$\phi_M=(t_{N-1}-t_N)^M \tag{35}$$

is a particular solution of the system (33), and

$$\psi=\psi\left(\frac{t_1-t_N}{t_{N-1}-t_N},\ldots,\frac{t_{N-2}-t_N}{t_{N-1}-t_N}\right) \tag{36}$$

is the general solution of the homogeneous system. As we see, this solution depends effectively on only $N-2$ variables:

$$x_A=\frac{t_A-t_N}{t_{N-1}-t_N},\quad A=1,\ldots,N-2. \tag{37}$$

The fact that (34) belongs to $\Phi_M\{F\}$ restricts the arbitrariness in the choice of the functions ψ. These functions must be polynomials of degree M in the variables x_A. We denote the space of such polynomials by Ψ_M. Since $\Phi_M\{F\}$ is invariant with respect to the action of the operators $K(\lambda)$, the result of applying $K(\lambda)$ to $\phi_M\psi$ (where $\psi\in\Psi_M$) must again have the form $\phi_M\tilde{\psi}$ (where $\tilde{\psi}\in\Psi_M$). Therefore

$$K(\lambda)(\phi_M\psi)=\phi_M K_M(\lambda)\psi, \tag{38}$$

where $K_M(\lambda)$ is some $(N-2)$-dimensional differential operator of second order that acts only on the variables x_A and is the required projection of $K(\lambda)$ onto $\Phi_M\{F\}$. Irrespective of the case—degenerate or nondegenerate—that is considered, this operator can be represented in the form

$$K_M(\lambda)=\sum_{A,B=1}^{N-2}P_{AB}(\lambda,M,x)\frac{\partial^2}{\partial x_A\partial x_B}+\sum_{A=1}^{N-2}Q_A(\lambda,M,x)\frac{\partial}{\partial x_A}+R(\lambda,M,x), \tag{39}$$

where P, Q, and R are polynomials in $x_1,\ldots,x_{N-2}$ of degrees 3, 2, and 1, respectively. Substituting (38) and (34) in (26), we find that the spectral problems (26) are equivalent to differential spectral equations:[16]

$$K_M(\lambda)\psi=E_M(\lambda)\psi,\quad \psi\in\Psi_M. \tag{40}$$

Transition to quasi-exactly solvable equations

We denote by Ψ the space of all analytic functions of the variables $x_1,\ldots,x_{N-2}$. Then it is obvious that the equations

$$\left\{\sum_{A,B=1}^{N-2}P_{AB}(\lambda,M,x)\frac{\partial^2}{\partial x_A\partial x_B}+\sum_{A=1}^{N-2}Q_A(\lambda,M,x)\frac{\partial}{\partial X_A}+R(\lambda,M,x)\right\}\psi(x)=E_M(\lambda)\psi(x),\quad \psi(x)\in\Psi, \tag{41}$$

can be regarded as quasi-exactly solvable equations. For each $M=0, 1, 2,\ldots$, they have $(M+N-2)!\times[(N-2)!M!]^{-1}$ exact solutions, which lie in the class of polynomials of degree M in $x_1,\ldots,x_{N-2}$ and are described by the Bethe formulas (11)–(12).

Connection with the Turbiner–Shifman approach

The transition from the original variables t_A ($A=1,\ldots,N$) to the variables $x_A=(t_A-t_N)/(t_{N-1}-t_N)$ ($A=1,\ldots,N-2$) can be made in two stages. First, we introduce the variables ξ_A in accordance with the formula

$$\xi_A=t_A-t_N,\quad A=1,\ldots,N-1;\quad \xi_N=t_N. \tag{42}$$

Then we have

$$\frac{\partial}{\partial t_A}=\frac{\partial}{\partial \xi_A},\quad A=1,\ldots,N-1;\quad \frac{\partial}{\partial t_N}=\frac{\partial}{\partial \xi_N}-\sum_{A=1}^{N-1}\frac{\partial}{\partial \xi_A}. \tag{43}$$

Since the functions (34), on which the operators $K(\lambda)$ act, are translationally invariant (in the space of the variables t_A), the derivatives with respect to ξ_N in the expression for the operator $K(\lambda)$ can be omitted by writing

$$K(\lambda)=\sum_{A,B=1}^{N}\frac{H_{AB}}{(\lambda-\sigma_A)(\lambda-\sigma_B)}, \tag{44}$$

where

$$H_{AB}=(\xi_A-\xi_B)^2\frac{\partial^2}{\partial\xi_A\partial\xi_B}+2(\xi_A-\xi_B)\times\left(f_A\frac{\partial}{\partial\xi_B}-f_B\frac{\partial}{\partial\xi_A}\right),\quad A,B=1,\ldots,N-1; \tag{45a}$$

$$H_{AN}=H_{NA}=-\xi_A^2\frac{\partial}{\partial\xi_A}\left(\sum_{B=1}^{N-1}\frac{\partial}{\partial\xi_B}\right)-2\xi_A\times\left(f_A\sum_{B=1}^{N-1}\frac{\partial}{\partial\xi_B}+f_N\frac{\partial}{\partial\xi_A}\right),\quad A=1,\ldots,N-1. \tag{45b}$$

This actually means that the operators (44) can be represented in the form

$$K(\lambda)=\sum_{A,B,C,D=1}^{N-1}P_{ABCD}(\lambda)L_{AB}L_{CD}+\sum_{A,B=1}^{N-1}Q_{AB}(\lambda)L_{AB}, \tag{46}$$

where

$$L_{AB}=\xi_A\frac{\partial}{\partial\xi_B},\quad A,B=1,\ldots,N-1, \tag{47}$$

are the generators of the algebra $gl(N-1)$. Remembering that

$$gl(N-1)=gl(1)\oplus sl(N-1), \tag{48}$$

we can divide the set of generators L_{AB} into two groups, namely, the unique generator of the algebra $gl(1)$,

$$J=\sum_{A=1}^{N-1}\xi_A\frac{\partial}{\partial\xi_A}, \tag{49a}$$

and the $(N-1)^2-1$ generators of the algebra $sl(N-1)$]:

$$\{J_a\}=\left\{\xi_A\frac{\partial}{\partial\xi_B},\ A\neq B;\ \xi_A\frac{\partial}{\partial\xi_A}-\xi_B\frac{\partial}{\partial\xi_B},\ B-A=1\right\}. \tag{49b}$$

Then in place of (46) we have

$$K(\lambda)=\sum_{a,b=1}^{N(N-2)}P_{ab}(\lambda)J_aJ_b+\sum_{a=1}^{N(N-2)}[Q_a(\lambda)+R_a(\lambda)J]J_a+S(\lambda)J+T(\lambda)J^2. \tag{50}$$

These operators (50) act on the spaces of homogeneous polynomials in $\xi_1,\ldots,\xi_{N-1}$ of degree M. None of the generators J_a takes us out of these spaces, and therefore they realize representations of the algebra $sl(N-1)$ of dimensions $(M+N-2)![(N-2)!M!]^{-1}$. These are representations with signatures $(M,0,\ldots,0)$.

We now make the final transition from the variables $\xi_A(A=1,\ldots,N-1)$ to the variables $x_A=\xi_a/\xi_{N-1}$ ($A=1,\ldots,N-2$). This transition is essentially a projection of the operators (50) onto the class of homogeneous polynomials in ξ_A of degree M, i.e., onto the class of functions that are eigenfunctions with respect to the operator J (with eigenvalue M). Because J commutes with all J_a, the projection operation is defined for each of the J_a separately. As a result of this operation, the generator J is replaced by its eigenvalue M, and the $(N-1)$-dimensional operators J_a are transformed into $(N-2)$-dimensional operators $J_a(M)$, which depend explicitly on M. They now act on the space of polynomials in $x_1,\ldots,x_{N-2}$ of degree M, realizing, as before, a finite-dimensional representation of the algebra $sl(N-1)$. With regard to the operators $K(\lambda)$, they are replaced by

$$K_M(\lambda)=\sum_{a,b=1}^{N(N-2)}P_{ab}(\lambda)J_a(M)J_b(M)+\sum_{a=1}^{N(N-2)}[Q_a(\lambda)+R_a(\lambda)M]J_a(M)+S(\lambda)M+T(\lambda)M^2. \tag{51}$$

If we ignore the unimportant c-number correction, then we have Hamiltonians of quantum tops based on the algebra $sl(N-1)$ or, more precisely, its finite-dimensional representations with dimensions $[(M+N-1)!]\times[(N-2)!M]^{-1}$ and signatures $(M,0,\ldots,0)$.[22]

Note that, in contrast to Turbiner (see, for example, Ref. 11), we have obtained at once an entire family of mutually commuting Hamiltonians of quantum tops. This circumstance reflects the complete integrability of the original Gaudin problem.

Coulomb analogy

The spectra of all the quantum problems considered above are associated with solutions of the system of numerical equations

$$\sum_{m=1}^{M}{}'\frac{1}{\xi_n-\xi_m}+F(\xi_n)=0,\quad n=1,\ldots,M. \tag{52}$$

Since the function $F(\lambda)$ is, in general, complex, the numbers ξ_m must also be regarded as complex. Introducing the vector quantities

$$\xi_m=(\mathrm{Re}\,\xi_m,\mathrm{Im}\,\xi_m),\quad m=1,\ldots,M, \tag{53}$$

and the function

$$U(\xi)\equiv\mathrm{Re}\int F(\xi)d\xi, \tag{54}$$

we can readily see that Eqs. (52) can be obtained from the condition of an extremum of the function

$$U(\xi_1,\ldots,\xi_M)=-\sum_{n,m=1}^{M}{}'\ln|\xi_n-\xi_m|-\sum_{n-1}^{M}U(\xi_n), \tag{55}$$

which is the potential of a system of M two-dimensional Coulomb particles with unit charges and with coordinates $\xi_n(n=1,\ldots,M)$ placed in the external electrostatic field with potential $U(\xi)$.[12]

Summary

Thus, we have presented the main elements of an approach that makes it possible to associate each rational function $F(\lambda)$ with some infinite class of quasi-exactly solvable models parametrized by the continuous parameter λ and the discrete parameter $M=0, 1, 2, \ldots$. If the total number of poles of the function $F(\lambda)$, which is regular at infinity, is N, then the model can be formulated in an $(N-2)$-dimensional space and for each fixed M has $(M+N-2)![(N-2)!M!]^{-1}$ exact solutions. These models are equivalent to completely integrable models of magnets based on the algebras $sl(2)\oplus\ldots\oplus sl(2)$ (N times) (in the nondegenerate case) or on their contractions (in the degenerate case). In this language, the reason for the quasi-exact solvability is the presence in the Gaudin model of the global symmetry group $sl(2)$ and the fact that the lowest subspaces of its representations with weights $M+\lim_{\lambda\to\infty}\lambda F(\lambda)$ are finite-dimensional. On the other hand, the same quasi-exactly solvable models are related to the Hamiltonians of quantum tops based on representations of dimension $(M+N-2)![(N-2)!M!]^{-1}$ of the algebras $sl(N-1)$ with signatures $(M, 0, \ldots, 0)$. Finally, there exists an equivalence between the considered quantum quasi-exactly solvable models and the two-dimensional classical system of M Coulomb particles in an external field determined by the function $F(\lambda)$.

Example

In our exposition, we have paid almost no attention to the degenerate case, restricting ourselves to general comments. It is therefore worth considering as an example the degenerate case and giving all the calculations to the end.

The simplest degenerate function $F(\lambda)$ has the form

$$F(\lambda)=\frac{\tilde f_1}{\lambda}+\frac{\tilde f_2}{\lambda^2}+\frac{\tilde f_3}{\lambda^3}. \tag{56}$$

The generators of the Gaudin algebra should be sought in an analogous form:

$$I^a(\lambda)=\frac{\tilde I_1^a}{\lambda}+\frac{\tilde I_2^a}{\lambda^2}+\frac{\tilde I_3^a}{\lambda^3},\quad a=\pm,0, \tag{57}$$

where $\tilde I_1^a$, $\tilde I_2^a$, $\tilde I_3^a$ are certain operators for which the commutation relations can be established by substituting (57) in (5). They have the form

$$\begin{aligned}
&[\tilde I_1^a,\tilde I_1^b]=\Gamma_c^{ab}\tilde I_1^c,\quad [\tilde I_1^a,\tilde I_2^b]=\Gamma_c^{ab}\tilde I_2^c,\\
&[\tilde I_1^a,\tilde I_3^b]=\Gamma_c^{ab}\tilde I_3^c,\quad [\tilde I_2^a,\tilde I_2^b]=\Gamma_c^{ab}\tilde I_3^c,\\
&[\tilde I_2^a,\tilde I_3^b]=0,\qquad [\tilde I_3^a,\tilde I_3^b]=0,
\end{aligned} \tag{58}$$

where Γ_c^{ab} are the structure constants of the algebra $sl(2)$. The expressions (58) describe one of the contractions of the algebra $sl(2)\oplus sl(2)\oplus sl(2)$. To obtain differential realizations of the operators I_A^a, we shall proceed from the nondegenerate case. Note that the function (56) can be obtained from the nondegenerate function

$$F(\lambda)=\frac{f_1}{\lambda-\sigma_1}+\frac{f_2}{\lambda-\sigma_2}+\frac{f_3}{\lambda-\sigma_3} \tag{59}$$

by the substitution

$$f_A=\frac{\tilde f_1\sigma_{A+1}\sigma_{A+2}-\tilde f_2(\sigma_{A+1}+\sigma_{A+2})+\tilde f_3}{(\sigma_{A+1}-\sigma_A)(\sigma_{A+2}-\sigma_A)},$$
$$A+3\equiv A,\quad A=1,2,3, \tag{60}$$

and subsequent passage to the limit $\sigma_1=\sigma_2=\sigma_3=0$. Similarly, the degenerate operator functions (57) must be obtained from the nondegenerate functions:

$$I^-(\lambda)=\frac{\frac{\partial}{\partial t_1}}{\lambda-\sigma_1}+\frac{\frac{\partial}{\partial t_2}}{\lambda-\sigma_2}+\frac{\frac{\partial}{\partial t_3}}{\lambda-\sigma_3}; \tag{61a}$$

$$I^0(\lambda)=\frac{t_1\frac{\partial}{\partial t_1}+f_1}{\lambda-\sigma_1}+\frac{t_2\frac{\partial}{\partial t_2}+f_2}{\lambda-\sigma_2}+\frac{t_3\frac{\partial}{\partial t_3}+f_3}{\lambda-\sigma_3}; \tag{61b}$$

$$I^+(\lambda)=\frac{t_1^2\frac{\partial}{\partial t_1}+2f_1t_1}{\lambda-\sigma_1}+\frac{t_2^2\frac{\partial}{\partial t_2}+2f_2t_2}{\lambda-\sigma_2}+\frac{t_3^2\frac{\partial}{\partial t_3}+2f_3t_3}{\lambda-\sigma_3}. \tag{61c}$$

For this, it is sufficient to postulate the form of $I^-(\lambda)$ (in the degenerate case),

$$I^-(\lambda)=\frac{\frac{\partial}{\partial\tilde t_1}}{\lambda}+\frac{\frac{\partial}{\partial\tilde t_2}}{\lambda^2}+\frac{\frac{\partial}{\partial\tilde t_3}}{\lambda^3}, \tag{62}$$

and determine the connection between the old and new variables by means of formulas that are exactly analogous to (60):

$$\frac{\partial}{\partial t_A}=\frac{\frac{\partial}{\partial\tilde t_1}\sigma_{A+1}\sigma_{A+2}-\frac{\partial}{\partial\tilde t_2}(\sigma_{A+1}+\sigma_{A+2})+\frac{\partial}{\partial\tilde t_3}}{(\sigma_{A+1}-\sigma_A)(\sigma_{A+2}-\sigma_A)},$$

$$A+3\equiv A,\quad A=1,2,3. \tag{63}$$

After this, making the substitution (60) and (63) in (61b) and (61c), and going to the limit $\sigma_1=\sigma_2=\sigma_3=0$, we find the final differential representations for the degenerate operators $\tilde{I}^a_A$:

$$\left.\begin{aligned}
&\tilde{I}_1^-=\frac{\partial}{\partial\tilde{t}_1};\quad \tilde{I}_2^-=\frac{\partial}{\partial\tilde{t}_2};\quad \tilde{I}_3=\frac{\partial}{\partial\tilde{t}_3};\\
&\tilde{I}_1^0=\tilde{t}_1\frac{\partial}{\partial\tilde{t}_1}+\tilde{t}_2\frac{\partial}{\partial\tilde{t}_2}+\tilde{t}_3\frac{\partial}{\partial\tilde{t}_3}+\tilde{f}_1;\\
&\tilde{I}_2^0=\tilde{t}_1\frac{\partial}{\partial\tilde{t}_2}+\tilde{t}_2\frac{\partial}{\partial\tilde{t}_3}+\tilde{f}_2;\quad \tilde{I}_3^0=\tilde{t}_1\frac{\partial}{\partial\tilde{t}_3}+\tilde{f}_3;\\
&\tilde{I}_1^+=\tilde{t}_1^2\frac{\partial}{\partial\tilde{t}_1}+2\tilde{t}_1\tilde{t}_2\frac{\partial}{\partial\tilde{t}_2}+2\tilde{t}_1\tilde{t}_3\frac{\partial}{\partial\tilde{t}_3}+\tilde{t}_2^2\frac{\partial}{\partial\tilde{t}_3}\\
&\quad+2\tilde{f}_1\tilde{t}_1+2\tilde{f}_2\tilde{t}_2+2\tilde{f}_3\tilde{t}_3;\\
&\tilde{I}_2^+=\tilde{t}_1^2\frac{\partial}{\partial\tilde{t}_2}+2\tilde{t}_1\tilde{t}_2\frac{\partial}{\partial\tilde{t}_3}+2\tilde{f}_2\tilde{t}_1+2\tilde{f}_3\tilde{t}_2;\\
&\tilde{I}_3^+=\tilde{t}^2\frac{\partial}{\partial\tilde{t}_3}+2\tilde{f}_3\tilde{t}_1.
\end{aligned}\right\} \tag{64}$$

We are now ready to write down the explicit form of the operator $K(\lambda)$:

$$K(\lambda)=\frac{K_2}{\lambda^2}+\frac{K_3}{\lambda^3}+\frac{K_4}{\lambda^4}+\frac{K_5}{\lambda^5}+\frac{K_6}{\lambda^6}. \tag{65}$$

Here

$$\left.\begin{aligned}
K_2=&\tilde{t}_2^2\frac{\partial^2}{\partial\tilde{t}_2^2}+\tilde{t}_3^2\frac{\partial}{\partial\tilde{t}_3^2}+2\tilde{t}_2\tilde{t}_3\frac{\partial^2}{\partial\tilde{t}_2\partial\tilde{t}_3}-\tilde{t}_2^2\frac{\partial^2}{\partial\tilde{t}_1\partial\tilde{t}_3}+2\tilde{f}_1\tilde{t}_2\frac{\partial}{\partial\tilde{t}_2}\\
&+2\tilde{f}_1\tilde{t}_3\frac{\partial}{\partial\tilde{t}_3}-2\tilde{f}_2\tilde{t}_2\frac{\partial}{\partial\tilde{t}_1}-2\tilde{f}_3\tilde{t}_3\frac{\partial}{\partial\tilde{t}_1}+\tilde{f}_1(\tilde{f}_1-1);\\
K_3=&\tilde{t}_2^2\frac{\partial^2}{\partial\tilde{t}_2\partial\tilde{t}_3}+2\tilde{t}_2\tilde{t}_3\frac{\partial^2}{\partial\tilde{t}_3^2}+2\tilde{f}_1\tilde{t}_2\frac{\partial}{\partial\tilde{t}_3}+2\tilde{f}_2\tilde{t}_3\frac{\partial}{\partial\tilde{t}_3}\\
&-2\tilde{f}_3\tilde{t}_2\frac{\partial}{\partial\tilde{t}_1}-2\tilde{f}_3\tilde{t}_3\frac{\partial}{\partial\tilde{t}_2}+2\tilde{f}_2(\tilde{f}_1-1);\\
K_4=&2\tilde{f}_1\tilde{f}_3+\tilde{f}_2^2-3\tilde{f}_3;\quad K_5=2\tilde{f}_2\tilde{f}_3;\quad K_6=\tilde{f}_3^2.
\end{aligned}\right\} \tag{66}$$

Using (56) and (57), we find

$$F\equiv\tilde{f}_1,\quad I^a=\tilde{I}_1^a,\quad a=\pm,0. \tag{67}$$

By virtue of (67), the equations that determine the finite-dimensional invariant subspaces $\Phi_M\{F\}$ take the form

$$\begin{aligned}
&\frac{\partial}{\partial\tilde{t}_1}\phi(\tilde{t}_1,\tilde{t}_2,\tilde{t}_3)=0,\\
&\left(\tilde{t}_1\frac{\partial}{\partial\tilde{t}_1}+\tilde{t}_2\frac{\partial}{\partial\tilde{t}_2}+\tilde{t}_3\frac{\partial}{\partial\tilde{t}_3}\right)\phi(\tilde{t}_1,\tilde{t}_2,\tilde{t}_3)=M\phi(\tilde{t}_1,\tilde{t}_2,\tilde{t}_3).
\end{aligned} \tag{68}$$

The general solution of this system is the function

$$\phi(\tilde{t}_1,\tilde{t}_2,\tilde{t}_3)=\tilde{t}_2^M\psi\left(\frac{\tilde{t}_3}{\tilde{t}_2}\right). \tag{69}$$

Introducing the new variable

$$x=\tilde{t}_3/\tilde{t}_2 \tag{70}$$

and projecting $K(\lambda)$ onto $\Phi_M\{F\}$, we find

$$K_M(\lambda)=\frac{K_{M2}}{\lambda^2}+\frac{K_{M3}}{\lambda^3}+\frac{K_{M4}}{\lambda^4}+\frac{K_{M5}}{\lambda^5}+\frac{K_{M6}}{\lambda^6}, \tag{71}$$

where

$$\begin{aligned}
K_{M2}=&(M+\tilde{f}_1)(M+\tilde{f}_1-1);\\
K_{M3}=&x\frac{\partial^2}{\partial x^2}+(M-1+2\tilde{f}_1+2\tilde{f}_2x+2\tilde{f}_3x^2)\frac{\partial}{\partial x}\\
&-2\tilde{f}_3Mx+2\tilde{f}_2(\tilde{f}_1-1);\\
K_{M4}=&2\tilde{f}_1\tilde{f}_3+\tilde{f}_2^2-3\tilde{f}_3;\quad K_{M5}=2\tilde{f}_2\tilde{f}_3;\quad K_{M6}=\tilde{f}_3^2.
\end{aligned} \tag{72}$$

From the expressions for $E_M(\lambda)$, we obtain

$$E_M(\lambda)=\frac{E_{M2}}{\lambda^2}+\frac{E_{M3}}{\lambda^3}+\frac{E_{M4}}{\lambda^4}+\frac{E_{M5}}{\lambda^5}+\frac{E_{M6}}{\lambda^6}, \tag{73}$$

where

$$\begin{aligned}
&E_{M2}=(M+\tilde{f}_1)(M+\tilde{t}_1-1);\\
&E_{M3}=2\tilde{f}_2(\tilde{f}_1-1)-2\tilde{f}_3\sum_{m=1}^{M}\frac{1}{\xi_m};\\
&E_{M4}=2\tilde{f}_1\tilde{f}_3+\tilde{f}_2^2-3\tilde{f}_3;\quad E_{M5}=2\tilde{f}_2\tilde{f}_3;\quad E_{M6}=\tilde{f}_3^2.
\end{aligned} \tag{74}$$

The only nontrivial part of the spectral equation for $K_M(\lambda)$ has the form

$$\left\{x\frac{\partial^2}{\partial x^2}+(M-1+2\tilde{f}_1+2\tilde{f}_2x+2\tilde{f}_3x^2)\frac{\partial}{\partial x}-2\tilde{f}_3Mx\right\}\psi_M(x,\xi)=E_M(\xi)\psi_M(x,\xi). \tag{75}$$

This is the required quasi-exactly solvable equation. Its exact solutions belong to the class of polynomials of degree M in x, and the eigenvalues are determined in accordance with (74) by the formulas

$$E_M(\xi)=-2\tilde{f}_3\sum_{m=1}^{M}\frac{1}{\xi_m}, \tag{76}$$

where ξ_m $(m=1,\dots,M)$ are the solutions of the system of algebraic equations

$$\sum_{m=1}^{M}{}'\frac{1}{\xi_n-\xi_m}+\frac{\tilde{f}_1}{\xi_n}+\frac{\tilde{f}_2}{\xi_n^2}+\frac{\tilde{f}_3}{\xi_n^3}=0,\quad n=1,\dots,M. \tag{77}$$

For given M, Eqs. (77), and, therefore, (75) have $M+1$ different solutions. Thus, we are dealing with a quasi-exactly solvable model of $(M+1)$th order.

Equations (75) can also be rewritten in the form

$$\left\{J^0(M)J^-(M)+2\tilde{f}_3J^+(M)+2\tilde{f}_2J^0(M)+\left(\frac{3M}{2}-1+2\tilde{f}_1\right)J^-(M)+2\tilde{f}_2M\right\}\psi_M=E_M\psi_M, \tag{78}$$

where

$$J^-(M)=\frac{\partial}{\partial x},\quad J^0(M)=x\frac{\partial}{\partial x}-\frac{M}{2},$$

$$J^+(M)=x^2\frac{\partial}{\partial x}-Mx \qquad (79)$$

are the generators of the $(M+1)$-dimensional representation of the algebra $sl(2)$ with "spin" $j=M/2$. (In our treatment of this example, we have followed Ref. 16.)

2. SOME PROPERTIES OF SIMPLE LIE ALGEBRAS

Cartan–Weyl basis

Let $\mathscr{L}_r$ be a simple Lie algebra of rank r and dimension d_r, and $I_a, a\in\Omega_r$ be basis elements of it satisfying the commutation relations

$$[I_a,I_b]=\sum_{c\in\Omega_r}\Gamma^c_{ab}I_c;\quad a,b\in\Omega_r. \qquad (80)$$

From the practical point of view, it is most convenient to take the Cartan–Weyl basis, which is based on the decomposition

$$\mathscr{L}_r=\mathscr{L}_r^-\oplus\mathscr{L}_r^0\oplus\mathscr{L}_r^+. \qquad (81)$$

We denote the elements of the $(d_r-r)/2$-dimensional subalgebras $\mathscr{L}_r^\pm$ by I_α, $\alpha\in\Delta_r^\pm$, where $\Delta_r^\pm$ are the sets of positive and negative roots α of the algebra $\mathscr{L}_r$. We denote the elements of the r-dimensional Cartan subalgebra $\mathscr{L}_r^0$ by I_i, $i\in N_r$, where N_r is the set of numbers 1, ... , r. Thus, the d_r-element set of indices that label the basis elements of the algebra $\mathscr{L}_r$ can be represented in the form $\Omega_r=\Delta_r^-\cup N_r\cup\Delta_r^+$.

The most important commutation relations in the chosen basis are

$$[I_i,I_\alpha]=(\alpha,\pi_i)I_\alpha,\quad i\in N_r,\quad \alpha\in\Delta_r^\pm. \qquad (82)$$

Here, π_i, $i\in N_r$ are simple roots.

On the algebra $\mathscr{L}_r$, we introduce the bilinear form $\langle I_a, I_b\rangle$, which satisfies the subsidiary condition

$$\langle[I_a,I_b],I_c\rangle=\langle I_a,[I_b,I_c]\rangle. \qquad (83)$$

Such a definition fixes the form only up to a factor. We choose this factor in such a way as to satisfy the requirements

$$\langle I_i,I_k\rangle=\gamma_{ik},\quad \langle I_\alpha,I_\beta\rangle=\varepsilon_{\alpha\beta}, \qquad (84)$$

where $\gamma_{ik}\equiv(\pi_i,\pi_k)$ is the matrix of scalar products of the simple roots (it is nondegenerate by virtue of the linear independence of the set π_i), and $\varepsilon_{\alpha\beta}$ is the matrix whose elements are 1 for $\alpha+\beta=0$ and 0 in all the remaining cases. In what follows, the matrix

$$g_{ab}\equiv\langle I_a,I_b\rangle \qquad (85)$$

will play the part of a metric tensor (obviously, nondegenerate), which will be used to raise and lower the indices that label the elements of the algebra $\mathscr{L}_r$. For example,

$$I_a=\sum_{b\in\Omega_r}g_{ab}I^b. \qquad (86)$$

If Eq. (86) is written out in terms of components, it takes the form

$$I_i=\sum_{k\in N_r}\gamma_{ik}I^k,\quad I_{\pm\alpha}=I^{\mp\alpha}. \qquad (87)$$

The metric tensor (85) (which differs from the Killing–Cartan tensor only by a factor) is convenient in that its components g_{ab} do not depend on whether the generators I_a and I_b are regarded as elements of the algebra $\mathscr{L}_r$ or some subalgebra of it. It is obvious that this does not apply to the inverse tensor and to entities with superscripts. Bearing this in mind, but not wishing to burden the text with redundant notation, we shall retain for the elements conjugate to I_a the same notation I^a irrespective of the method of conjugation that is used—with respect to the algebra or a subalgebra of it. Of course, we shall attempt to make it clear from the context which particular case we have.

Equations (82)–(84) make it possible to recover uniquely all the commutation relations in the algebra $\mathscr{L}_r$ not yet given. They have the form

$$[I_i,I_k]=0,\quad i,k\in N_r; \qquad (88a)$$

$$[I_\alpha,I_{-\alpha}]=\sum_{i\in N_r}(\alpha,\pi_i)x^i,\quad \alpha\in\Delta_r^\pm; \qquad (88b)$$

$$[I_\alpha,I_\beta]=\Gamma_{\alpha\beta}I_{\alpha+\beta},\quad \alpha,\beta\in\Delta_r^\pm,\quad \alpha+\beta\in\Delta_r^\pm, \qquad (88c)$$

where $\Gamma_{\alpha\beta}$ are calculable structure constants.[24]

We define the quadratic Casimir operator[1)] as follows:

$$K_r=\sum_{a\in\Omega_r}I^aI_a, \qquad (89)$$

or, with allowance for the commutation relations (88b),

$$K_r=\sum_{i\in N_r}I^i\left[I_i+\sum_{\alpha\in\Delta_r^+}(\alpha,\pi_i)\right]+2\sum_{\alpha\in\Delta_r^+}I^\alpha I_\alpha. \qquad (90)$$

Since

$$\sum_{\alpha\in\Delta_r^+}\alpha=\sum_{i\in N_r}\nu^i\pi_i \qquad (91)$$

(where ν^i is a certain set of non-negative integers that characterize the algebra $\mathscr{L}_r$), we have

$$K_r=\sum_{i\in N_r}(I^i+\nu^i)I_i+2\sum_{\alpha\in\Delta_r^+}I^\alpha I_\alpha. \qquad (92)$$

The representations of the algebras $\mathscr{L}_r$ with highest weight are determined as follows:

$$I_i|0\rangle=\Lambda_{0i}|0\rangle,\quad i\in N_r; \qquad (93a)$$

$$I_\alpha|0\rangle=0,\quad \alpha\in\Delta_r^+. \qquad (93b)$$

The set Λ_{0i} is called the highest weight, and $|0\rangle$ is called the highest vector. Let M^i, $i\in N_r$, be a set of non-negative integers. We denote by $|M\rangle$ the linear hull of vectors of the form

$$I_{\alpha_1}\cdot\ldots\cdot I_{\alpha_K}|0\rangle,\quad \alpha_1,\ldots,\alpha_K\in\Delta_r^-, \qquad (94)$$

in which $\alpha_1 + ... + \alpha_K = -\Sigma_{i\in N_r} M^i \pi_i$. Obviously, the spaces $|M\rangle$ are eigenspaces with respect to the elements of the Cartan subalgebra:

$$I_i|M\rangle = (\Lambda_{0i} - M_i)|M\rangle. \quad (95)$$

The linear hull of all such spaces,

$$W\{\Lambda_0\} = \oplus_{M\geqslant 0} |M\rangle, \quad (96)$$

forms the representation space of the algebra $\mathscr{L}_r$ with highest weight Λ_0.

Realization of representations of Lie algebras in the form of differential operators

We have already mentioned that in both our scheme and the scheme of Turbiner and Shifman a decisive role is played by the possibility of realizing representations of Lie algebras in the form of first-order differential operators. There exists an opinion (which is even stated in Refs. 11 and 15) that the construction of such realizations is a rather complicated matter. This is not at all so. The general principles of the construction are rather simple, although the algorithms described in the literature[24,25] are not always carried through to explicit formulas, especially for the higher algebras. We shall attempt to fill this gap by constructing explicit expressions for differential operators which realize arbitrary representations of arbitrary simple Lie algebras.

Let G_r be the Lie group of the Lie algebra $\mathscr{L}_r$ and let

$$g(x)\in G_r, \quad x\in R_{d_r} \quad (97)$$

be elements of the group, parametrized by the vectors $x\equiv\{x^a\}$, $a\in\Omega_r$, of the d_r-dimensional space R_{d_r}. We choose the parametrization in such a way that the following equations hold:

$$g(0)=1, \quad (98a)$$

$$\left.\frac{\partial g(x)}{\partial x}\right|_{x=0} = I, \quad (98b)$$

where $I\equiv\{I_a\}$, $a\in\Omega_r$, are the generators of the algebra $\mathscr{L}_r$ that we introduced earlier. One of the possible methods of parametrization consists of the choice

$$g(x) = \exp(x\cdot I), \quad (99)$$

but this method is not unique and, as we shall see later, not the most convenient. Actually, the general scheme that will be presented in this section does not depend on the particular method of parametrization.

For the elements x, y of the space R_{d_r} we define the binary operation $x \dot{+} y$ in accordance with the formula

$$g(x)g(y) = g(x\dot{+}y). \quad (100)$$

This operation, for which we have chosen the symbol $\dot{+}$ (not to be confused with the direct sum, for which the symbol $\oplus$ is reserved), possesses the following properties:

a) for all $x, y, z\in R_{d_r}$

$$x\dot{+}(y\dot{+}z) = (x\dot{+}y)\dot{+}z; \quad (101a)$$

b) for all $x\in R_{d_r}$ there exists a zero element $0\in R_{d_r}$ such that

$$x\dot{+}0 = 0\dot{+}x = x; \quad (101b)$$

c) for all $x\in R_{d_r}$ there exists a unique inverse element $(\dot{-}x)\in R_{d_r}$ such that

$$x\dot{+}(\dot{-}x) = (\dot{-}x)\dot{+}x = 0. \quad (101c)$$

Remarks

In what follows, in place of $x\dot{+}(\dot{-}y)$ we shall simply write $x\dot{-}y$. It is readily verified that

$$\dot{-}(x\dot{+}y) = \dot{-}y\dot{-}x. \quad (102)$$

If the conditions (98) are satisfied, the point 0 corresponds to the usual zero of the space R_{d_r}. In the general case, $-x\neq\dot{-}x$, although for some parametrizations [see, for example, (99)] the equation $-x=\dot{-}x$ can hold. For commutative groups, we have $x\dot{+}y=x+y$, i.e., in this case the binary operation that we have introduced can be identified with ordinary addition of vectors in R_{d_r}.

Let Φ_r be the space of functions on the group G_r and let

$$\phi(x)\in\Phi_r, \quad x\in R_{d_r} \quad (103)$$

be the elements of this space. We define the operators $\hat{g}(\varepsilon)$, $\varepsilon\in R_{d_r}$, which are linear on Φ_r, by means of the formulas

$$\hat{g}(\varepsilon)\phi(x) = \phi(x\dot{+}\varepsilon). \quad (104)$$

It follows from this definition that

$$\tilde{g}(\varepsilon_2)\tilde{g}(\varepsilon_1) = \hat{g}(\varepsilon_1\dot{+}\varepsilon_2), \quad (105)$$

i.e., the operators $\hat{g}(\varepsilon)$, $\varepsilon\in R_{d_r}$, form a representation of the group G_r. In accordance with the formulas (98), we have

$$\hat{g}(0) = \hat{1}; \quad (106a)$$

$$\left.\frac{\partial\hat{g}(x)}{\partial x}\right|_{x=0} = \hat{I}, \quad (106b)$$

where $\hat{I}=\{\hat{I}_a\}$, $a\in\Omega_r$, are the generators of the corresponding representation of the algebra $\mathscr{L}_r$. Differentiating (104) with respect to ε and setting $\varepsilon=0$, we find

$$\hat{I} = \hat{T}(x)\frac{\partial}{\partial x}, \quad (107)$$

where

$$\hat{T}(x) = \frac{\partial}{\partial\varepsilon}\otimes(x\dot{+}\varepsilon)|_{\varepsilon=0}. \quad (108)$$

For further simplification of Eqs. (107) and (108), we note that any vector of the space R_{d_r} can be represented in the form of the expansion

$$x = x^-\dot{+}x^0\dot{+}x^+, \quad (109)$$

where $x^\pm\equiv\{x^\alpha\}$, $\alpha\in\Delta_r^\pm$, and $x^0\equiv\{x^i\}$, $i\in N_r$, are the vectors of dimensions $(d_r-r)/2$ and r associated with the generators $I_\pm\equiv\{I_\alpha\}$, $\alpha\in\Delta_r^\pm$, and $I_0\equiv\{I_i\}$, $i\in N_r$, of the algebra $\mathscr{L}_r$ in the Cartan–Weyl basis. Eqs. (107) and (108) can now be rewritten in the form

$$\hat{I}=\hat{T}^-(x)\frac{\partial}{\partial x^-}+\hat{T}^0(x)\frac{\partial}{\partial x^0}+\hat{T}^+(x)\frac{\partial}{\partial x^+}, \quad (110)$$

where

$$\hat{T}^{\pm}(x)=\frac{\partial}{\partial\varepsilon}\otimes(x+\varepsilon)^{\pm}|_{\varepsilon=0};\quad \hat{T}^0(x)=\frac{\partial}{\partial\varepsilon}\otimes(x+\varepsilon)^0|_{\varepsilon=0}. \quad (111)$$

It follows from the obvious equation

$$x+\varepsilon=\{x^-+[x^0+(x^++\varepsilon)^-]^-\}+\{x^0+(x^++\varepsilon)^0\}+\{(x^++\varepsilon)^+\} \quad (112)$$

that

$$\left.\begin{aligned}&(x+\varepsilon)^+=(x^++\varepsilon)^+;\\&(x+\varepsilon)^0=x^0+(x^++\varepsilon)^0;\\&(x+\varepsilon)^-=x^-+[x^0+(x^++\varepsilon)^-]^-.\end{aligned}\right\} \quad (113)$$

Therefore

$$\left.\begin{aligned}&\hat{T}^+(x)=\frac{\partial}{\partial\varepsilon}\otimes(x^++\varepsilon)^+|_{\varepsilon=0};\\&\hat{T}^0(x)=\frac{\partial}{\partial\varepsilon}\otimes(x^++\varepsilon)^0|_{\varepsilon=0};\\&\hat{T}^-(x)=\frac{\partial}{\partial\varepsilon}\otimes\{x^-+[x^0+(x^++\varepsilon)^-]^-\}|_{\varepsilon=0}.\end{aligned}\right\} \quad (114)$$

We see that the matrices $\hat{T}^+(x)$ and $\hat{T}^0(x)$ depend only on the variables x^+. Therefore, considering the action of the operators (110) on the class of functions of the form

$$\phi(x)=\exp(x^0\Lambda_0)\psi(x^+), \quad (115)$$

we obtain

$$\hat{I}\exp(x^0\Lambda_0)\psi(x^+)=\exp(x^0\Lambda_0)\hat{I}(\Lambda_0)\psi(x^+). \quad (116)$$

The operators

$$\hat{I}(\Lambda_0)=\hat{t}^+(x^+)\frac{\partial}{\partial x^+}+\hat{t}^0(x^+)\Lambda_0, \quad (117)$$

in which

$$\hat{t}^+(x^+)=\frac{\partial}{\partial\varepsilon}\otimes(x^++\varepsilon)^+|_{\varepsilon=0},$$

$$\hat{t}^0(x^+)=\frac{\partial}{\partial\varepsilon}\otimes(x^++\varepsilon)^0|_{\varepsilon=0} \quad (118)$$

act on the space of functions of the $(d_r-r)/2$ variables x^+, are inhomogeneous first-order differential operators, and realize a certain representation of the algebra $\mathscr{L}_r$. To identify this representation, we note that by virtue of the obvious formulas

$$\frac{\partial}{\partial\varepsilon^+}\otimes(x^++\varepsilon^+)^0\bigg|_{\varepsilon^+=0}=0,$$

$$\frac{\partial}{\partial\varepsilon^0}\otimes(x^++\varepsilon^0)^0\bigg|_{\varepsilon^0=0}=1, \quad (119)$$

we have

$$\hat{I}_+(\Lambda_0)=\hat{t}^+_+(x^+)\frac{\partial}{\partial x^+}; \quad (120a)$$

$$\hat{I}_0(\Lambda_0)=\hat{t}^+_0(x^+)\frac{\partial}{\partial x^+}+\Lambda_0; \quad (120b)$$

$$\hat{I}_-(\Lambda_0)=\hat{t}^+_-(x^+)\frac{\partial}{\partial x^+}+\hat{t}^0_-(x^+)\Lambda_0. \quad (120c)$$

Therefore, the operators (120) describe a representation of the algebra $\mathscr{L}_r$ with highest weight Λ_0. The unit function $|0\rangle\equiv 1$ here plays the part of the highest vector. It is readily seen that the operators (120) can also be obtained as infinitesimal operators of the representation $G_r(\Lambda_0)$ of the group G_r determined by the formula

$$\hat{g}(\varepsilon,\Lambda_0)\psi(x^+)=\exp[(x+\varepsilon)^0\Lambda_0]\psi[(x^++\varepsilon)^+]. \quad (121)$$

In what follows, we shall frequently need to work with the operators

$$\hat{I}(\Lambda_0^1,\ldots,\Lambda_0^N)\equiv\sum_{A=1}^N\hat{I}(\Lambda_0^A)=\sum_{A=1}^N\left\{\hat{t}^+(x_A^+)\frac{\partial}{\partial x_A^+}+\hat{t}^0(x_A^+)\Lambda_0^A\right\}, \quad (122)$$

which realize a representation of the algebra $\mathscr{L}_r$ on the class of functions that depend on N vector variables. By analogy with (121), these operators can be interpreted as the infinitesimal operators of the following group transformations:

$$\hat{g}(\varepsilon,\Lambda_0^1,\ldots,\Lambda_0^N)\psi(x_1^+,\ldots,x_N^+)=\exp\left\{\sum_{A=1}^N(x_A^++\varepsilon)^0\Lambda_0^A\right\}\times\psi[(x_1^++\varepsilon)^+,\ldots,(x_N^++\varepsilon)^+]. \quad (123)$$

If we choose as the highest vector the unit function $|0\rangle\equiv 1$, then the highest weight will be $\Lambda_0^1+\ldots+\Lambda_0^N$. However, in what follows we shall need to consider representations of the algebra of operators (122) with arbitrary higher weights, which can be written in the form $\Lambda_0^1+\ldots+\Lambda_0^N-M_0$. In this case, which form can the highest vector $|0\rangle$ have? To answer this question, we must solve the system of equations

$$\left.\begin{aligned}&\hat{I}_+(\Lambda_0^1,\ldots,\Lambda_0^N)\psi=0;\\&\hat{I}_0(\Lambda_0^1,\ldots,\Lambda_0^N)\psi=\left(\sum_{A=1}^N\Lambda_0^A-M_0\right)\psi\end{aligned}\right\} \quad (124)$$

for the functions ψ. Essentially, we must find functions ψ invariant with respect to transformations of the subgroup $\hat{g}(\varepsilon^+,0,\ldots,0)$ and transforming homogeneously [with addition of the factor $\exp(-\varepsilon^0 M_0)$] under transformations of the subgroup $\hat{g}(\varepsilon^0, 0, \dots, 0)$:

$$\psi[(x_1^+ \dotplus \varepsilon^+)^+, \dots, (x_N^+ \dotplus \varepsilon^+)^+] = \psi(x_1^+, \dots, x_N^+); \quad (125a)$$

$$\psi[(x_1^+ + \varepsilon^0)^+, \dots, (x_N^+ + \varepsilon^0)^+] = e^{-\varepsilon^0 M_0}\psi(x_1^+, \dots, x_N^+). \quad (125b)$$

To solve (125a), we use the equation

$$(x_A^+ \dotplus \varepsilon^+)^+ = x_1^+ \dotplus \varepsilon^+, \quad (126)$$

from which it follows that

$$(x_A^+ \dotplus \varepsilon^+) \dot{-} (x_B^+ \dotplus \varepsilon^+) = x_A^+ \dotplus \varepsilon^+ \dot{-} \varepsilon^+ \dot{-} x_B^+ = x_A^+ \dot{-} x_B^+. \quad (127)$$

Thus, the functions $x_A^+ \dot{-} x_B^+$ are invariant with respect to the transformations $g(\varepsilon^+, 0, \dots, 0)$. The following combinations of them are functionally independent:

$$\xi_A = x_A^+ \dot{-} x_N^+, \quad A = 1, \dots, N-1. \quad (128)$$

Therefore, the most general solution of Eq. (125a) has the form

$$\psi = \psi(\xi_1, \dots, \xi_{N-1}). \quad (129)$$

We now consider how the components of the vectors $\xi_A = \{\xi_A^\alpha\}$, $\alpha \in \Delta_r^+$, change under a transformation of the subgroup $g(\varepsilon^0, 0, \dots, 0)$. For the components $x_A = \{x_A^\alpha\}$, we have

$$(x_A^+ \dotplus \varepsilon^0)^\alpha = \dot{-}\varepsilon^0 \dotplus x_A^\alpha \dotplus \varepsilon^0 = \exp\{-\varepsilon^0(\pi_0,\alpha)\}x_A^\alpha, \quad (130)$$

where $\pi_0 \equiv \{\pi_i\}$, $i \in N_r$. The components of the vectors $\xi_A = x_A \dot{-} x_N$ also transform homogeneously:

$$\xi_A^\alpha \to \exp\{-\varepsilon^0(\pi_0,\alpha)\}\xi_A^\alpha. \quad (131)$$

This means that all quantities of the form $[\xi_{A_1}^{\alpha_1} \cdot \dots \cdot \xi_{A_K}^{\alpha_K}] \times [\xi_{B_1}^{\beta_1} \cdot \dots \cdot \xi_{B_L}^{\beta_L}]^{-1}$, where $\alpha_1 + \dots + \alpha_K = \beta_1 + \dots + \beta_L$, $\alpha_K, \beta \in \Delta_r^+$, are invariant with respect to the transformations $g(\varepsilon^0, 0, \dots, 0)$. As functionally independent variables, we can choose the following $(N-2)(d_r - r)/2$ variables:

$$\eta = \left\{ \frac{\xi_A^\alpha}{\prod_{i \in N_r} (\xi_{N-1}^{\pi_i})^{(\alpha,\pi^i)}} \right\}, \quad \alpha \in \Delta_r^+, \quad A = 1, \dots, N-2, \quad (132)$$

and also $(d_r - r)/2$ variables of the form

$$\nu = \left\{ \frac{\xi_{N-1}^\alpha}{\prod_{i \in N_r} (\xi_{N-1}^{\pi_i})^{(\alpha,\pi^i)}} \right\}, \quad \alpha \in \Delta_r^+, \quad \alpha \neq \pi_i, \quad i \in N_r \quad (133)$$

Introducing the notation

$$\xi^i \equiv \xi_{N-1}^{\pi_i}, \quad i \in N_r, \quad (134)$$

we find that the functions

$$\psi = \prod_{i \in N_r} (\xi^i)^{M^i} \psi(\eta, \nu) \quad (135)$$

realize the most general solution of the system (125). Any of these functions can play the part of the highest vector for representation of the algebra of operators (122) with highest weight $\Lambda_0^1 + \dots + \Lambda_0^N - M_0$.

We now obtain explicit expressions for the matrices $\hat{t}^+(x^+)$ and $\hat{t}^0(x^+)$ under the assumption that the parametrization is Gaussian:

$$g(x) = \exp\left\{ \sum_{\alpha \in \Delta_r^-} x^\alpha I_\alpha \right\} \exp\left\{ \sum_{i \in N_r} x^i I_i \right\} \exp\left\{ \sum_{\alpha \in \Delta_r^+} x^\alpha I_\alpha \right\}. \quad (136)$$

In this case, the equation for the matrices $\hat{t}^+(x^+)$ and $\hat{t}^0(x^+)$ takes the form

$$\exp\left\{ \sum_{i \in N_r} x^i I_i \right\} \exp\left\{ \sum_{\alpha \in \Delta_r^+} x^\alpha I_\alpha \right\} \exp\{\varepsilon^a I_a\} = \exp\left\{ \sum_{i \in N_r} (x^i I_i + \varepsilon^a t_a^i(x^+) I_i) \right\} \exp\left\{ \sum_{\alpha \in \Delta_r^+} (x^\alpha I_\alpha + \varepsilon^a t_a^\alpha(x^+) I_\alpha) \right\}. \quad (137)$$

Applying the well-known formula

$$\exp(A+B) = \exp A\left[T\exp\int_0^1 d\tau \times \exp(-\tau A) B \exp(\tau A) \right] \quad (138)$$

to the right-hand side of (37), expanding both sides in series in powers of ε^a, and retaining only the first powers in ε^a, we obtain after trivial manipulations

$$t_a^i(x^+) = \left\langle \exp\left(\mathrm{ad} \sum_{\alpha \in \Delta_r^+} x^\alpha I_\alpha \right) I_a, I^i \right\rangle, \quad i \in N_r. \quad (139)$$

With regard to the matrix $t_a^\alpha(x^+)$, it can be found from the system of algebraic equations

$$t_a^\alpha(x^+) \int_0^1 d\tau \left\langle \exp\left(\mathrm{ad}\, \tau \sum_{\beta \in \Delta_r^+} x^\beta I_\beta \right) I_\alpha, I^\gamma \right\rangle = \left\langle \exp\left(\mathrm{ad} \sum_{\beta \in \Delta_r^+} x^\beta I_\beta \right) I_a, I^\gamma \right\rangle, \quad (140)$$

$$a \in \Omega_r, \quad \gamma \in \Delta_r^+.$$

We now note that the matrix

$$\int_0^1 d\tau \left\langle \exp\left(\mathrm{ad}\, \tau \sum_{\beta \in \Delta_r^+} x^\beta I_\beta \right) I_\alpha, I^\gamma \right\rangle, \quad (141)$$

which occurs in Eq. (140), will have triangular form if the roots α and γ are arranged in nondecreasing order of their heights. On the principal diagonal of (141) there will be units. The determinant of such a matrix is equal to unity, and therefore its inverse matrix consists of the minors.

Since each minor is a finite polynomial in x^{α} [this also applies to the right-hand side of (140)], the matrices $\hat{t}^{+}(x^{+})$ and $\hat{t}^{0}(x^{+})$ will also be polynomial in x^{α}.

The differential realizations of the representations of the algebra $\mathscr{L}_r$ that we have obtained are not particularly convenient from the practical point of view, since some work is needed to reduce them to explicit form. This is because the Gaussian decomposition that we have used (and is also mainly discussed in the literature[25,26]) is not completely suitable for this purpose. In the following sections, we shall consider more convenient decompositions, and on their basis we shall derive explicit expressions suitable for any simple Lie algebra.

To conclude this section, we consider how the matrices $\hat{t}^{+}(x^{+})$ and $\hat{t}^{0}(x^{+})$ transform under homogeneous transformations of the components of the parameter x^{+}:

$$x^{\alpha}\to\exp\{-\varepsilon^{0}(\pi_0,\alpha)\}x^{\alpha}. \tag{142}$$

Since each component x^{α} acquires a factor $\exp[-\varepsilon^{0}(\pi_0,\alpha)]$, the only nonvanishing terms in the decomposition (141) will be the terms that acquire the factor $\exp\{-\varepsilon^{0}[\pi_0,(\gamma-\alpha)]\}$. This means that the components of the considered matrices will transform as

$$\left.\begin{aligned}&t^{\alpha}_{\beta}(x^{+})\to\exp\{-\varepsilon^{0}[\pi_0,(\alpha-\beta)]\}t^{\alpha}_{\beta}(x^{+});\\&t^{\alpha}_{i}(x^{+})\to\exp\{-\varepsilon^{0}(\pi_0,\alpha)\}t^{\alpha}_{i}(x^{i}).\end{aligned}\right\} \tag{143}$$

Similarly, we can show that

$$t^{i}_{\alpha}(x^{+})\to\exp\{\varepsilon^{0}(\pi_0,\alpha)\}t^{i}_{\alpha}(x^{+}). \tag{144}$$

It follows from this that the differential operators $\hat{I}_{\alpha}(\Lambda_0)$, $\alpha\in\Delta_r^{\pm}$, and $\hat{I}_i(\Lambda_0)$, $i\in N_r$, which realize representations of the algebra $\mathscr{L}_r$ with the highest weight Λ_0, transform in accordance with the rules

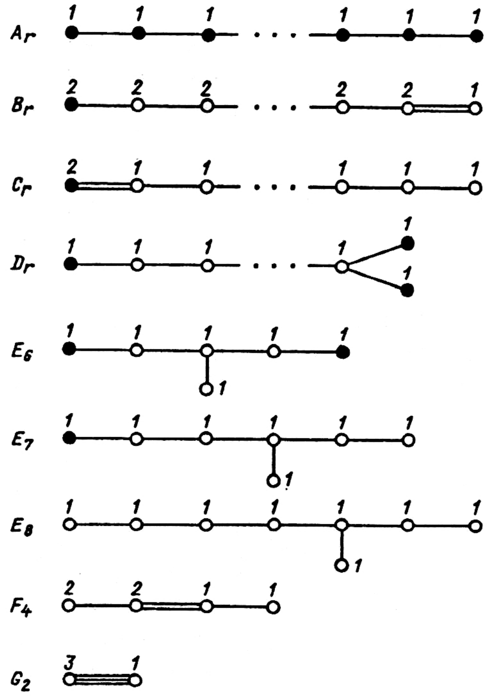
$$\left.\begin{aligned}&\hat{t}_{\alpha}(\Lambda_0)\to\exp\{\varepsilon^{0}(\pi_0,\alpha)\}\hat{I}_{\alpha}(\Lambda_0),\quad \alpha\in\Delta_r^{\pm},\\&\hat{I}_i(\Lambda_0)\to\hat{I}_i(\Lambda_0),\quad i\in N_r.\end{aligned}\right\} \tag{145}$$

3. SPECIAL DECOMPOSITION IN SIMPLE LIE ALGEBRAS

Construction of basis

We shall say that a simple root of the algebra $\mathscr{L}_r$ is singular if it occurs in the decomposition of any root with coefficient whose modulus does not exceed unity.

We give the Dynkin diagrams of the simple Lie algebras and identify with black circles the vertices associated with singular simple roots:



(146)

We see that for the algebras A_r all the simple roots are singular, while for the algebras E_8, F_4, and G_2 there are no such roots. In what follows, we shall consider only algebras that have at least one singular root. These are the algebras A_r $(r\geqslant1)$, B_r $(r\geqslant2)$, C_r $(r\geqslant3)$, D_r $(r\geqslant4)$, E_6, and E_7. We shall call them singular algebras.

The removal from the algebra $\mathscr{L}_r$ of the extreme singular root (together with all nonsimple roots containing it) transforms the algebra into the simple subalgebra $\mathscr{L}_{r-1}$. For example,

$$\left.\begin{aligned}&A_r\to A_{r-1},\quad B_r\to B_{r-1},\quad C_r\to A_{r-1},\\&D_r\quad \begin{matrix}D_{r-1}\\A_{r-1}\end{matrix},\quad E_7\to E_6,\quad E_6\to D_5.\end{aligned}\right\} \tag{147}$$

In what follows, we shall ascribe the number r to the eliminated singular root. We denote the sets of roots containing in the decomposition a root with coefficients ±1 by $\Sigma_r^{\pm}$.

Singular roots are remarkable in that the elements I_{α} associated with the roots $\alpha\in\Sigma_r^{\pm}$ form commutative subalgebras, which we shall denote by $\mathscr{E}_{\pm r}$. The commutativity of $\mathscr{E}_{\pm r}$ follows directly from the definition of a simple root. It is obvious that

$$\dim\,\mathscr{E}_{\pm r}=\tfrac{1}{2}(d_r-d_{r-1}-1). \tag{148}$$

The algebras $\mathscr{E}_{\pm r}$ are conjugate with respect to the bilinear form (85). For the elements $E_{\pm r}+\{E_{\pm r,\alpha}\}$, $\alpha\in\Sigma_r^{\pm}$, of these subalgebras the following normalization conditions are satisfied:

$$\langle E_{\pm r}\otimes E_{\mp r}\rangle=\hat{I}_r, \tag{149}$$

where $\hat{I}_r$ is the unit matrix of dimension $\dim\,\mathscr{E}_r$

Since the dimensions of the Cartan subalgebras $\mathscr{L}_r^0$ and $\mathscr{L}_{r-1}^0$ differ by unity,

$$\mathscr{L}_r^0=\mathscr{L}_{r-1}^0\otimes H_r, \tag{150}$$

where H_r is an element of $\mathscr{L}_r^0$ that does not belong to $\mathscr{L}_{r-1}^0$. For unique determination of this element, we require it to be self-adjoint with respect to the form (85), and this is equivalent to the conditions

$$\langle H_r I_i\rangle=0,\quad i\in N_{r-1}; \tag{151a}$$

$$\langle H_r H_r\rangle=1. \tag{151b}$$

It is obvious that the element H_r must commute with all elements of the subalgebra $\mathscr{L}_{r-1}$.

Summarizing what was said above, we can conclude that for all singular Lie algebras the following decomposi tion holds:

$$\mathscr{L}_r=(\mathscr{E}_{-r}\oplus H_r\oplus\mathscr{E}_{+r})\oplus\mathscr{L}_{r-1}, \tag{152}$$

where $\mathscr{E}_{\pm r}$ are commutative conjugate subalgebras of dimensions $(d_r-d_{r-1}-1)/2$, H_r is a self-adjoint element of the Cartan subalgebra, and $\mathscr{L}_{r-1}$ is a simple Lie algebra.

The nonvanishing commutation relations in (152) have the form

$$[\mathscr{L}_{r-1},\mathscr{L}_{r-1}]=\mathscr{L}_{r-1}; \tag{153}$$

$$[\mathscr{L}_{r-1},\mathscr{E}_{\pm r}]=\mathscr{E}_{\pm r}; \tag{154}$$

$$[H_r,\mathscr{E}_{\pm r}]=\mathscr{E}_{\pm r}; \tag{155}$$

$$[\mathscr{E}_{\pm r},\mathscr{E}_{\mp r}]=H_r\oplus\mathscr{L}_{r-1}. \tag{156}$$

We now write out these relations in more detail. For (153), we have

$$[I_a,I_b]=\sum_{c\in\Omega_{r-1}}\Gamma_{ab}^c I_c,\quad a,b\in\Omega_{r-1}. \tag{157}$$

The relations (154) can be rewritten in the form

$$[I_a,E_{\pm r}]=-\hat{I}_a(\mathscr{E}_{\pm r})E_{\pm r},\quad a\in\Omega_{r-1}, \tag{158}$$

where $\hat{I}_a(\mathscr{E}_{\pm r})$ are matrices that play the part of structure constants. Using the Jacobi identity

$$[[I_a,E_{\pm r}],I_b]+[[E_{\pm r},I_b],I_a]+[[I_b,I_a],E_{\pm r}]=0 \tag{159}$$

and Eqs. (157) and (159), we find

$$[\hat{I}_a(\mathscr{E}_{\pm r}),\hat{I}_b(\mathscr{E}_{\pm r})]=\sum_{c\in\Omega_{r-1}}\Gamma_{ab}^c\hat{I}_c(\mathscr{E}_{\pm r}),\quad a,b\in\Omega_{r-1}. \tag{160}$$

Thus, the matrices $\hat{I}_a(\mathscr{E}_{\pm r})$, $a\in\Omega_{r-1}$, form representations of the algebra $\mathscr{L}_{r-1}$ of dimension dim $\mathscr{E}_{\pm r}$ and realized on the spaces of the commutative subalgebras $\mathscr{E}_{\pm r}$. It follows from the simplicity of the algebra $\mathscr{L}_{r-1}$ that

$$\mathrm{Sp}\ \hat{I}_a(\mathscr{E}_{\pm r})=0. \tag{161}$$

In addition, it is readily seen that

$$\hat{I}_a(\mathscr{E}_{+r})+\tilde{\hat{I}}_a(\mathscr{E}_{-r})=0, \tag{162}$$

where the tilde denotes the transpose.

In what follows, we shall need only representations realized by the matrices $\hat{I}_a(\mathscr{E}_{+r})$ on the subalgebras $\mathscr{E}_{+r}$. For their identification, we consider Eq. (158) with the plus sign. Labeling the roots $\alpha\in\Delta_{r-1}^+$ in order of nondecrease of their heights, we see that in this case the matrices $\hat{I}_\alpha(\mathscr{E}_{+r})$, $\alpha\in\Delta_{r-1}^+$, have an upper triangular form, while the matrices $\hat{I}_i(\mathscr{E}_{+r})$, $i\in N_{r-1}$, are diagonal. Therefore, the role of highest vector will be played by a vector of dimension dim $\mathscr{E}_r$ for which only the first component is nonzero. This means that the corresponding highest weight must be determined by the result of commutation of the elements I_i, $i\in N_{r-1}$, with the element $\mathscr{E}_r$ corresponding to the positive root of the lowest possible height. This is the root π_r. From this we conclude that the highest vector of the representation (160) will be an $(r-1)$-dimensional vector γ_{ir}, $i\in N_{r-1}$.

Further, using Eqs. (151), we find for the commutation relations (155)

$$[H_r,E_{\pm r}]=\pm Q_r E_{\pm r}, \tag{163}$$

where

$$Q_r=\langle H_r I_r\rangle. \tag{164}$$

With regard to the relations (156), they can be written in the form

$$[E_{\pm r}\otimes E_{\mp r}]=\pm Q_r H_r\hat{1}_r-\sum_{a\in\Omega_{r-1}}I^a\hat{I}_a(\mathscr{E}_{\pm r}). \tag{165}$$

Here, I^a is understood as the element that is the adjoint of the element I_a with respect to the bilinear form of the subalgebra $\mathscr{L}_{r-1}$. Applying to (165) the relation (161), we obtain

$$[E_{\pm r}\cdot E_{\mp r}]=\pm R_r H_r, \tag{166}$$

where

$$R_r=Q_r\ \dim\ \mathscr{E}_r. \tag{167}$$

We also introduce the important formula

$$Q_r^2\hat{1}_r^{(1)}\otimes\hat{1}_r^{(2)}+\sum_{a\in\Omega_{r-1}}\hat{I}_a(\mathscr{E}_{\pm r}^{(1)})\otimes\hat{I}^a(\mathscr{E}_{\pm r}^{(2)})=Q_r^2\hat{1}_r^{(2)}\otimes\hat{1}_r^{(1)}+\sum_{q\in\Omega_{r-1}}\hat{I}_a(\mathscr{E}_{\pm r}^{(2)})\otimes\hat{I}^a(\mathscr{E}_{\pm r}^{(1)}). \tag{168}$$

Here, the indices (1) and (2) identify the numbers of the spaces on which the operators $\hat{I}_r$ and $\hat{I}_a(\mathscr{E}_{\pm r})$ act. Finally, we write out the form of the Casimir operator in the decomposition (152):

$$K_r=H_r^2+R_rH_r+E_{-r}\cdot E_{+r}+K_{r-1}. \tag{169}$$

We now turn to the procedure for eliminating the singular simple roots that separates from the algebra $\mathscr{L}_r$ the simple subalgebra $\mathscr{L}_{r-1}$ in accordance with the scheme

(147). Each resulting subalgebra contains its own singular root, and therefore the elimination procedure can be continued. As a result, we arrive at the following possible change:

$$\left.\begin{array}{l}
A_r \to A_{r-1} \to A_{r-2} \to \dots \to A_3 \to A_2 \to A_1;\\
B_r \to B_{r-1} \to B_{r-2} \to \dots \to B_3 \to B_2 \to A_1;\\
C_r \to A_{r-1} \to A_{r-2} \to \dots \to A_3 \to A_2 \to A_1;\\
D_r \to A_{r-1} \to A_{r-2} \to \dots \to A_3 \to A_2 \to A_1;\\
\quad \lfloor\!\!\longrightarrow D_{r-1} \longrightarrow \dots \longrightarrow D_4 \longrightarrow \uparrow\\
E_7 \to E_6 \to D_5 \to A_4 \to A_3 \to A_2 \to A_1;\\
\qquad\qquad \lfloor\!\!\longrightarrow D_4 \longrightarrow \uparrow\\
E_6 \to D_5 \to A_4 \to A_3 \to A_2 \to A_1.\\
\qquad \lfloor\!\!\longrightarrow D_4 \longrightarrow \uparrow
\end{array}\right\} \tag{170}$$

We agree to label the simple roots of the algebra in such a way that in each successive subalgebra along the chain the singular root has the maximal number. As a result, we arrive at the decomposition

$$\mathscr{L}_r = \bigoplus_{s=1}^{r} (\mathscr{E}_{-s} \oplus H_s \oplus \mathscr{E}_{+s}). \tag{171a}$$

Introducing the notation

$$\mathscr{E}_0 \equiv \bigoplus_{s=1}^{r} H_s, \tag{172}$$

we can rewrite Eq. (170) in a different way:

$$\mathscr{L}_r = \bigoplus_{s=-r}^{r} \mathscr{E}_s. \tag{171b}$$

The commutation relations in $\mathscr{L}_r$ can now be represented in the form

$$[\mathscr{E}_q, \mathscr{E}_p] = \mathscr{E}_p, \quad 0 < |q| < |p|; \tag{173a}$$

$$[\mathscr{E}_q, \mathscr{E}_{-q}] = \bigoplus_{s=-q}^{q} \mathscr{E}_s, \quad 0 < |q|. \tag{173b}$$

To write out fully these commutation relations, we denote the generators of the subalgebras $\mathscr{E}_{\pm s}$ by $E_{\pm s}$. Then we have

$$\left.\begin{array}{l}
[E_q, E_p] = -\hat{E}_q(\mathscr{E}_p) E_p\\
[H_q, E_p] = -\hat{H}_q(\mathscr{E}_p) E_p
\end{array}\right\} 0 < |q| < |p|; \tag{174a}$$

$$[H_q, E_{\pm q}] = \pm Q_q E_{\pm q}; \tag{174b}$$

$$\begin{aligned}
[E_{\pm p} \otimes E_{\mp p}] = {} & \pm Q_p H_p \hat{1}_p - \sum_{q=1}^{p-1} H_q \hat{H}_q(\mathscr{E}_{\pm p})\\
& - \sum_{q=1}^{p-1} E_{-q} \hat{E}_{+q}(\mathscr{E}_{\pm p})\\
& - \sum_{q=1}^{p-1} E_{+q} \hat{E}_{-q}(\mathscr{E}_{\pm p}).
\end{aligned} \tag{174c}$$

The elements $E_{\pm s}$ and H_s satisfy the normalization conditions

$$\langle H_q, H_p \rangle = \delta_{qp}, \tag{175a}$$

$$\langle E_q \otimes E_{-q} \rangle = \hat{1}_q. \tag{175b}$$

The sets of matrices $\hat{H}_q(\mathscr{E}_p)$, $\hat{E}_{\pm q}(\mathscr{E}_p)$, $q = 1, .., p-1$, realize representations of the algebra $\mathscr{L}_{p-1}$ of dimension dim $\mathscr{E}_p$ with highest weights $\Lambda_{0s} = \gamma_{sp}$, $s \in N_{p-1}$.

We introduce the matrix S_q^i, which relates H_q to the basis elements I_i:

$$H_q = \sum_{i=1}^{r} S_q^i I_i. \tag{176}$$

It is readily seen that by virtue of (151a) the matrix S_q^i is triangular:

$$S_q^i = 0, \ i > q. \tag{177}$$

Substituting (176) in (175), we obtain the helpful formula

$$\sum_{i,k \in N_r} S_p^i S_q^k \gamma_{ik} = \delta_{pq}, \tag{178}$$

which in conjunction with the condition (177) enables us to determine the matrix S_q^i uniquely (by means of the standard Gram–Schmidt orthogonalization procedure). Many properties of the considered basis can be expressed in terms of the matrix S_q^i. For example, we have

$$\begin{aligned}
&\hat{E}_q(\mathscr{E}_p)|0\rangle = 0, \quad q \in N_{p-1};\\
&\hat{H}_q(\mathscr{E}_p)|0\langle = S_{qp}|0\rangle, \quad q \in N_{p-1}.
\end{aligned} \tag{179}$$

It is also easy to show that

$$Q_q = S_{qq} \tag{180}$$

and

$$\sum_{q=1}^{r} S_q^i R_q = \nu^i. \tag{181}$$

The last formula can be deduced by comparing (92) with the other expression for the Casimir operator

$$K_r = \sum_{q=1}^{r} H_q^2 + \sum_{q=1}^{r} R_q H_q + 2 \sum_{q=1}^{r} E_{-q} E_{+q}, \tag{182}$$

which follows from (169).

We now turn to the construction of differential realizations of the representations of the singular simple Lie algebras based on the decomposition (172). In accordance with this decomposition, any element of the groups can be parametrized as follows:

$$g(x) = \prod_{q=-r}^{r} \exp(x_q E_q). \tag{183}$$

We shall read the product in (183) from the left to the right. Note that any partial product $\exp(x_s E_s) \dots \exp(x_r E_r)$ forms a subgroup. For this reason, Eq. (183) can be split into $2r+1$ formulas of the form

$$\begin{aligned}
&\left(\prod_{q=s}^{r} \exp(x_q E_q) \right) \exp(\varepsilon_s E_s)\\
&\quad = \prod_{q=s}^{r} \exp[x_q E_q + \varepsilon_s (E_s | x_q) E_q].
\end{aligned} \tag{184}$$

After simple manipulations, each of these formulas can be rewritten in one of the following forms:

$$\left(\prod_{q=p}^{r}\exp(x_qE_q)\right)E_s\left(\prod_{q=p}^{r}\exp(x_qE_q)\right)^{-1}=(E_s|x_p)E_p+\sum_{n=p+1}^{r}(E_s|x_n)\left(\prod_{q=p}^{n-1}\exp(x_qE_q)\right)E_n\left(\prod_{q=p}^{n-1}\exp(x_qE_q)\right)^{-1}$$

$$+\sum_{n=s}^{p-1}(E_s|x_n)\left(\prod_{q=n}^{p-1}\exp(x_qE_q)\right)^{-1}E_n\left(\prod_{q=n}^{p-1}\exp(x_qE_q)\right),$$

$$s=-r,\dots,+r. \tag{185}$$

Multiplying both sides of (185) by E_{-p}, we find

$$(E_s|x_p)=\left\langle E_s\otimes\left(\prod_{q=p}^{r}\exp(-\operatorname{ad} x_qE_q)\right)E_{-p}\right\rangle. \tag{186}$$

From this it is easy to obtain differential representations for the operators

$$E_s=\sum_{p=1}^{r}\left\langle E_s\otimes\left(\prod_{q=p}^{r}\exp(-\operatorname{ad} x_qE_q)\right)E_{-p}\right\rangle\frac{\partial}{\partial x_p}. \tag{187}$$

To simplify these expressions, we project the operators (187) onto the class of functions

$$\psi=\exp(x_0h_0)\psi(x_1,\dots,x_r), \tag{188}$$

as a result of which they take the form

$$E_{+s}=\frac{\partial}{\partial x_s}-\sum_{a=s+1}^{r}\langle E_s\otimes\operatorname{ad}(x_aE_a)E_{-a}\rangle\frac{\partial}{\partial x_a},$$

$$H_s=h_s-\sum_{a=s}^{r}\langle H_s\otimes\operatorname{ad}(x_aE_a)E_{-a}\rangle\frac{\partial}{\partial x_a},$$

$$E_{-s}=\sum_{a=1}^{s}\left\langle E_{-s}\otimes\prod_{q=a}^{s}\exp(-\operatorname{ad} x_qE_q)H_a\right\rangle h_a$$

$$+\sum_{a=1}^{s}\left\langle E_{-s}\otimes\prod_{q=a}^{s}\exp(-\operatorname{ad} x_qE_q)E_{-a}\right\rangle\frac{\partial}{\partial x_a}$$

$$+\sum_{a=s+1}^{r}\langle E_{-s}\otimes\exp(-\operatorname{ad} x_aE_a)E_{-a}\rangle\frac{\partial}{\partial x_a}. \tag{189}$$

These are the required differential realizations of the representations of the Lie algebras with highest weights Λ_0. Here, h_a are the eigenvalues of the operators H_a associated with the highest weights by means of the matrix $\hat{S}$:

$$h_a=\sum_{i\in N_r}S_a^i\Lambda_{0i}. \tag{190}$$

4. GENERALIZED GAUDIN MODEL BASED ON ARBITRARY LIE ALGEBRAS AND ITS EXACT SOLUTIONS

Definition of the Gaudin model in the general case

With the simple Lie algebra $\mathscr{L}_r$ we associate the infinite-dimensional Gaudin algebra $G(\mathscr{L}_r)$, whose generators $I_a(\lambda)$, $a\in\Omega_r$, depend on the complex parameter λ and satisfy the commutation relations

$$[I_a(\lambda),I_b(\mu)]=\sum_{c\in\Omega_r}\Gamma_{ab}^{c}\frac{I_c(\lambda)-I_c(\mu)}{\lambda-\mu}. \tag{191}$$

Using (191), we associate the Cartan–Weyl decomposition (81) in the algebra $\mathscr{L}_r$ with the analogous decomposition in the algebra $G(\mathscr{L}_r)$:

$$G(\mathscr{L}_r)=G(\mathscr{L}_r^-)\oplus G(\mathscr{L}_r^0)\oplus G(\mathscr{L}_r^+). \tag{192}$$

We denote the elements of the subalgebras $G(\mathscr{L}_r^{\pm})$ and $G(\mathscr{L}_r^0)$ by $I_\alpha(\lambda)$, $\alpha\in\Delta_r^{\pm}$, and $I_i(\lambda)$, $i\in N_r$, respectively. We define a representation of the Gaudin algebra by means of the formulas

$$\left.\begin{aligned}&I_\alpha(\lambda)|0\rangle=0,\quad \alpha\in\Delta_r^+;\\&I_i(\lambda)|0\rangle=F_i(\lambda)|0\rangle,\quad i\in N_r.\end{aligned}\right\} \tag{193}$$

Here, $|0\rangle$ is the highest vector of the representation, and the functions $F_i(\lambda)$, $i\in N_r$, play the part of the components of the highest weight. Let $M=\{M^i\}$, $i\in N_r$, be sets of nonnegative integers. With each such set, we associate the space $|M\rangle$, which is formed from all possible linear combinations of vectors of the form

$$I_{\alpha_1}(\lambda_1)\cdot\ldots\cdot I_{\alpha_K}(\lambda_K)|0\rangle, \tag{194}$$

where $\lambda_1, \ldots, \lambda_K$ are arbitrary complex numbers, and $\alpha_1, \ldots, \alpha_K \in \Delta_r^-$ are different sets of non-negative roots of the algebra $\mathscr{L}_r$ that satisfy the restriction

$$\alpha_1 + \alpha_2 + \ldots + \alpha_K = -\sum_{i \in N_r} M^i \pi_i. \tag{195}$$

Subject to this restriction, the number of roots in the set can be arbitrary. If the operators $I_a(\lambda)$ in (194) satisfy the conditions (193), then the space $W_r\{F(\lambda)\}$ of the representation of the Gaudin algebra realized by them is defined as

$$W_r\{F(\lambda)\} = \bigoplus_{M \geqslant 0} |M\rangle. \tag{196}$$

Following Ref. 3, we introduce the operators

$$K_r(\lambda) = \sum_{a,b \in \Psi_r} g^{ab} I_a(\lambda) I_b(\lambda), \tag{197}$$

which depend on λ and in their structure recall the Casimir operators (89) in the algebra $\mathscr{L}_r$. However, in reality (197) are not Casimir operators for the Gaudin algebra $G(\mathscr{L}_r)$, since they do not commute with all of its elements. They possess a different remarkable property:

$$[K_r(\lambda), K_r(\mu)] = 0, \tag{198}$$

i.e., they form a commutative family. For this reason, the spectral problem

$$K_r(\lambda)\phi_r = E_r(\lambda)\phi_r, \quad \phi_r \in W_r\{F(\lambda)\}, \tag{199}$$

is completely integrable.[3] In the following section, we shall show that it can be solved exactly for all singular simple Lie algebras in the framework of the algebraic Bethe ansatz. In the exposition, we shall follow the studies of Ref. 18. For other methods of solving the Gaudin problem for the simple classical Lie algebras, see, for example, Ref. 20.

Gaudin algebra in the special basis

Beginning with this section, we restrict the treatment to only the singular simple Lie algebras, i.e., the algebras $A_r (r \geqslant 1)$, $B_r (r \geqslant 2)$, $C_r (r \geqslant 3)$, $D_r (r \geqslant 4)$, E_6, and E_7. By virtue of the correspondence noted in the previous section between these algebras and the Gaudin algebras, the latter have decompositions analogous to the decompositions (152):

$$G(\mathscr{L}_r) = \{G(\mathscr{E}_{-r}) \oplus G(H_r) \oplus (\mathscr{E}_{+r})\} \oplus G(\mathscr{L}_{r-1}). \tag{200}$$

We denote the elements of the subalgebras $G(\mathscr{E}_{\pm r})$, $G(H_r)$, and $G(\mathscr{L}_{r-1})$ by $E_{\pm r}(\lambda)$, $H_r(\lambda)$, and $I_a(\lambda)$, where $a \in \Psi_{r-1}$. Then in accordance with (191) and the results of the previous sections, the following commutation relations hold for them:

$$[I_a(\lambda), I_b(\xi)] = \sum_{c \in \Omega_{r-1}} \Gamma_{ab}^c \frac{I_c(\lambda) - I_c(\xi)}{\lambda - \xi}, \quad a,b \in \Omega_{r-1}; \tag{201}$$

$$[I_a(\lambda), E_{\pm r}(\xi)] = -\hat{I}_a(\mathscr{E}_{\pm r}) \frac{E_{\pm r}(\lambda) - E_{\pm r}(\xi)}{\lambda - \xi}, \quad a \in \Omega_{r-1}; \tag{202}$$

$$[I_a(\lambda), H_r(\xi)] = 0, \quad a \in \Omega_{r-1}; \tag{203}$$

$$[H_r(\lambda), E_{\pm r}(\xi)] = \pm Q_r \frac{E_{\pm r}(\lambda) - E_{\pm r}(\xi)}{\lambda - \xi}; \tag{204}$$

$$[H_r(\lambda), H_r(\xi)] = 0; \tag{205}$$

$$[E_{\pm r}(\lambda) \otimes E_{\pm r}(\xi)] = 0, \tag{206}$$

$$[E_{\pm r}(\lambda) \otimes E_{\mp r}(\xi)] = \pm Q_r \frac{H_r(\lambda) - H_r(\xi)}{\lambda - \xi} \hat{1}_r - \sum_{a \in \Omega_{r-1}} \frac{I_a(\lambda) - I_a(\xi)}{\lambda - \xi} \hat{I}^a(\mathscr{E}_{\pm r}). \tag{207}$$

The superscript a in (206) is understood in the sense of conjugation with respect to the bilinear form of the algebra $\mathscr{L}_{r-1}$. We have the formula

$$[E_{\pm r}(\lambda) \cdot E_{\mp r}(\lambda)] = \pm R_r H_r'(\lambda), \tag{208}$$

which is the Gaudin analog of (167). For the operators $K_r(\lambda)$, we have

$$K_r(\lambda) = H_r^2(\lambda) + R_r H_r'(\lambda) + 2E_{-r}(\lambda)E_{+r}(\lambda) + K_{r-1}(\lambda). \tag{209}$$

We now determine the vacuum subspace $W_{r-1}\{F(\lambda)\}$ of the representation space $W_r\{F(\lambda)\}$ as a linear hull of the vectors

$$|0\rangle, \ I_{a_1}(\lambda_1)|0\rangle, \ I_{a_1}(\lambda_1) I_{a_2}(\lambda_2)|0\rangle \ldots, \tag{210}$$

in which the numbers $\lambda_1, \lambda_2, \ldots$ are arbitrary, and $a_1, a_2, \ldots \in \Omega_{r-1}$. It is readily verified that the elements ϕ_{r-1} of the vacuum subspace possess the properties

$$\left.\begin{aligned} &E_{+r}(\lambda)\phi_{r-1} = 0; \\ &H_r(\lambda)\phi_{r-1} = h_r(\lambda)\phi_{r-1}. \end{aligned}\right\} \tag{211}$$

Here, $h_r(\lambda)$ are the eigenvalues of the operator $H_r(\lambda)$ on $|0\rangle$, and they are equal to

$$h_r(\lambda) = \sum_{i \in N_r} S_r^i F_i(\lambda), \tag{212}$$

and S_r^i is the matrix introduced earlier [see Eq. (176)].

Bethe solution of the generalized Gaudin problem

We now turn to the construction of exact solutions of the spectral problem (199). Let M^r be a fixed natural number, and $\xi_{r,i}$, $i = 1, \ldots, M^r$, be as yet unknown numerical parameters. We shall seek an eigenvector of the Casimir–Gaudin operator $K_r(\lambda)$ in the form

$$\phi_r = E_{-r}(\xi_{r,1}) \otimes \ldots \otimes E_{-r}(\xi_{r,M^r})\phi_{r-1}, \tag{213}$$

where ϕ_{r-1} is a tensor of rank M^r contracted with the vectors of the generators E_{-r} of the algebra $G(\mathscr{E}_{-r})$ with respect to all indices. We shall regard the components of this tensor as elements of the vacuum subspace $W_{r-1}\{F(\lambda)\}$. Using Eqs. (211), we can write

$$K_r(\lambda)\phi_{r-1}=[h_r^2(\lambda)+R_rh_r'(\lambda)]\phi_{r-1}+K_{r-1}(\lambda)\phi_{r-1}. \quad (214)$$

We now calculate the result of applying the operator $K_r(\lambda)$ to the vector ϕ_r. For this, we require the commutation relations

$$[K_r(\lambda),E_{-r}(\xi)]=-\frac{Q}{(\lambda-\xi)}\{E_{-r}(\lambda)H_r(\xi)-E_{-r}(\xi)H_r(\lambda)\}-\frac{1}{\lambda-\xi}\sum_{a\in\Omega_{r-1}}\hat{I}^a(\mathscr{C}_{-r})\{E_{-r}(\lambda)I_a(\xi)-E_{-r}(\xi)I_a(\lambda)\}; \quad (215)$$

$$[I_a(\lambda),E_{-r}(\xi)]=\frac{1}{\lambda-\xi}\hat{I}_a(\mathscr{C}_{-r})\{E_{-r}(\lambda)-E_{-r}(\xi)\}; \quad (216)$$

$$[H_r(\lambda),E_{-r}(\xi)]=-\frac{Q_r}{\lambda-\xi}\{E_{-r}(\lambda)-E_{-r}(\xi)\}, \quad (217)$$

which can be readily deduced from the relations (191) and the definition (197).

In the expression

$$K_r(\lambda)\phi_r=K_r(\lambda)E_{-r}(\xi_{r,1})\otimes\ldots\otimes E_{-r}(\xi_{r,M^r})\phi_{r-1} \quad (218)$$

we move the operator $K_r(\lambda)$ to the right and, using the formulas (214) and (215), we obtain

$$K_r(\lambda)\phi_r=[h_r^2(\lambda)+R_rh_r'(\lambda)]\phi_r+E_{-r}(\xi_{r,1})\otimes\ldots\otimes E_{-r}(\xi_{r,M^r})K_{r-1}(\lambda)\phi_{r-1}$$
$$-2Q_r\sum_{m=1}^{M^r}\frac{1}{\lambda-\xi_{r,m}}E_{-r}(\xi_{r,1})\otimes\ldots\otimes\{E_{-r}(\lambda)H_r(\xi_{r,m})-E_{-r}(\xi_{r,m})H_r(\lambda)\}\otimes\ldots\otimes E_{-r}(\xi_{r,M^r})\phi_{r-1}$$
$$+2\sum_{a\in\Omega_{r-1}}\sum_{m=1}^{M^r}\frac{1}{\lambda-\xi_{r,m}}E_{-r}(\xi_{r,1})\otimes\ldots\otimes\{E_{-r}(\lambda)I_a(\xi_{r,m})$$
$$-E_{-r}(\xi_{r,m})I_a(\lambda)\}\otimes\ldots\otimes E_{-r}(\xi_{r,M^r})\hat{I}_a(\mathscr{C}_r)\phi_{r-1}. \quad (219)$$

Further, shifting the operators $H_r(\lambda)$, $H_r(\xi_{r,m})$, $I_a(\lambda)$, $I_a(\xi_{r,m})$, $a\in\Omega_{r-1}$, to the right by means of the commutation relations (216) and (217), and using (211), we find that the expression $K_r(\lambda)\phi_r$ can be represented as a sum of terms of two types:

$$K_r(\lambda)\phi_r=\{K_r(\lambda)\phi_r\}_1+\{K_r(\lambda)\phi_r\}_2. \quad (220)$$

The terms identified by the index 1 are proportional to the vector ϕ_r and, therefore, determine an eigenvalue $E_r(\lambda)$ of the operator $K_r(\lambda)$. The remaining terms, identified by the index 2, are not proportional to the vector ϕ_r and they must be annihilated. We begin by considering the first group of terms. They have the form

$$\{K_r(\lambda)\phi_r\}_1=\left\{\left(h_r(\lambda)+Q_r\sum_{m=1}^{M_r}\frac{1}{\lambda-\xi_{r,m}}\right]^2+R_r\left[h_r(\lambda)+Q_r\sum_{m=1}^{M^r}\frac{1}{\lambda-\xi_{r,m}}\right]'\right\}\phi_r+E_{-r}(\xi_{r,1})\otimes\ldots\otimes E_{-r}(\xi_{r,M^r})\tilde{K}_{r-1}(\lambda)\phi_{r-1}, \quad (221)$$

where

$$\tilde{K}_{r-1}(\lambda)=\sum_{a,b\in\Omega_{r-1}}g^{ab}\tilde{I}_a(\lambda)\tilde{I}_b(\lambda). \quad (222)$$

It is readily verified that the operators

$$\tilde{I}_a(\lambda)=I_a(\lambda)-\sum_{m=1}^{M^r}\frac{\tilde{I}_a(\mathscr{C}_r)}{\lambda-\xi_{r,m}},\quad a\in Q_{r-1}, \quad (223)$$

satisfy the same commutation relations as the operators $I_a(\lambda)$. Therefore, $\tilde{K}_{r-1}(\lambda)$, defined by (222), is a Gaudin operator for the algebra $G(\mathscr{L}_{r-1})$. The set of vectors ϕ_{r-1} can be considered as the space $W_{r-1}\{\tilde{F}(\lambda)\}$ of a representation of the Gaudin algebra characterized by highest weight with components

$$\tilde{F}_i(\lambda)=F_i(\lambda)-\sum_{m-1}^{M^r}\frac{\gamma_{ri}}{\lambda-\xi_{r,m}},\quad i\in N_{r-1}. \quad (224)$$

Therefore, the spectral problem

$$\tilde{K}_{r-1}(\lambda)\phi_{r-1}=\tilde{E}_{r-1}(\lambda)\phi_{r-1},\quad \phi_{r-1}\in W_{r-1}\{F(\lambda)\}, \quad (225)$$

is the Gaudin problem for the algebra $G(\mathscr{L}_{r-1})$. If it has solutions, then we can write

$$K_r(\lambda)\phi_r=\left\{\left[h_r(\lambda)+Q_r\sum_{m=1}^{M^r}\frac{1}{\lambda-\xi_{r,m}}\right]^2+R_r\left[h_r(\lambda)+Q_r\sum_{M=1}^{M^r}\frac{1}{\lambda-\xi_{r,m}}\right]'+\tilde{E}_{r-1}(\lambda)\right\}\phi_r, \quad (226)$$

provided, of course, the terms of the second type are equal to zero. We write out these terms separately. They have the form

$$\{K_r(\lambda)\phi_r\}_2=2\sum_{m>n}\frac{1}{(\lambda-\xi_{r,n})(\xi_{r,n}-\xi_{r,m})}E_{-r}(\xi_{r,1})\otimes...\otimes \underset{\underset{n}{\uparrow}}{E_{-r}(\lambda)}\otimes...\otimes \underset{\underset{m}{\uparrow}}{E_{-r}(\xi_{r,n})}\otimes...\otimes E_{-r}(\xi_{r,M^r})\Bigg\{\Bigg[Q_r^2\hat{1}_r^{(n)}\otimes\hat{1}_r^{(m)}$$

$$+\sum_{a,b,\in\Omega_{r-1}}g^{ab}\hat{I}_a(\mathscr{E}_r^{(n)})\hat{I}_b(\mathscr{E}_r^{(m)})\Bigg]-\Bigg[Q_r^2\hat{1}_r^{(m)}\otimes\hat{1}_r^{(n)}+\sum_{a,b\in\Omega_{r-1}}g^{ab}\hat{I}_a(\mathscr{E}_r^{(m)})\hat{I}_b(\mathscr{E}_r^{(n)})\Bigg]\Bigg\}\phi_{r-1}$$

$$-\sum_{n=1}^{M^r}\frac{1}{\lambda-\xi_{r,n}}E_{-r}(\xi_{r,1})\otimes...\otimes \underset{\underset{n}{\uparrow}}{E_{-r}(\lambda)}\otimes...\otimes E_{-r}(\xi_{r,M^r})\Bigg\{2Q_rh_r(\xi_{r,n})+2Q_r^2\sum_{m=1}^{M^r}{}'\frac{1}{\xi_{r,n}-\xi_{r,m}}$$

$$+2\sum_{a,b\in\Omega_{r-1}}\hat{I}^a(\mathscr{E}_r)I_a(\xi_{r,n})-2\sum_{m=1}^{M^r}{}'\frac{1}{\xi_{r,n}-\xi_{r,m}}\sum_{a,b\in\Omega_{r-1}}\hat{I}^a(\mathscr{E}_r)\hat{I}_a(\mathscr{E}_r)\Bigg\}\phi_{r-1}. \tag{227}$$

The numbers with the arrows indicate the serial number of the operator in the product. Using the formulas (167) and (222), we find

$$\{K_r(\lambda)\phi_r\}_2=\sum_{n=1}^{M^r}\frac{1}{\lambda-\xi_{r,n}}E_{-r}(\xi_{r,1})\otimes...\otimes E_{-r}(\lambda)\otimes...\otimes E_{-r}(\xi_{r,M^r})\Bigg\{2Q_rh_r(\xi_{r,n})+2Q_r^2\sum_{m=1}^{m^2}{}'\frac{1}{\xi_{r,n}-\xi_{r,m}}$$

$$-\operatorname{Res}_{\xi_{r,n}}\widetilde{K}_{r-1}(\lambda)\Bigg\}\phi_{r-1}. \tag{228}$$

Taking into account (225), we finally find that the condition of vanishing of the terms (228) has the form

$$2Q_rh_r(\xi_{r,n})+2Q_r^2\sum_{m=1}^{M^r}{}'\frac{1}{\xi_{r,n}-\xi_{r,m}}=\operatorname{Res}_{\xi_{r,n}}\widetilde{E}_{r-1}(\lambda),$$

$$n=1,\ldots,M^r, \tag{229}$$

where by $\operatorname{Res}_\xi\widetilde{E}_{r-1}(\lambda)$ we have denoted the residue of the function $\widetilde{E}_{r-1}(\lambda)$ at the simple pole situated at the point ξ.

Thus, we see that the Gaudin problem for the algebra $G(\mathscr{L}_r)$ has been reduced to the analogous problem for the algebra $G(\mathscr{L}_{r-1})$. In its structure, the latter completely repeats the former, and therefore it, in its turn, can be reduced to the Gaudin problem for the algebra $G(\mathscr{L}_{r-2})$, etc., and this goes on until we reach the algebra $G(\mathscr{L}_1)=G(A_1)$, for which the solution of this problem is known (see Sec. 1). This enables us to write the solution of the Gaudin problem in the explicit form

$$\phi_r=\phi_r(M,\xi)$$

$$=\Bigg\{\bigotimes_{i=1}^{M^r}E_{-r}(\xi_{r,i})\Bigg\}\Bigg\{\bigotimes_{i=1}^{M^{r-1}}\Bigg[E_{-(r-1)}(\xi_{r-1,i})$$

$$-\sum_{m=1}^{M^r}\frac{\hat{E}_{-(r-1)}(\mathscr{E}_r)}{\xi_{r-1,i}-\xi_{r,m}}\Bigg]\Bigg\}$$

$$\cdot...\cdot\Bigg\{\bigotimes_{i=1}^{M^1}\Bigg[E_{-1}(\xi_{1,i})-\sum_{q=2}^{r}\sum_{m=1}^{M^q}\frac{E_{-1}(\mathscr{E}_q)}{\xi_{1,i}-\xi_{q,m}}\Bigg]\Bigg\}|0\rangle, \tag{230}$$

$$E_r(\lambda)=E_r(M,\xi;\lambda)$$

$$=\sum_{s=1}^{r}\{h_s^2(\lambda)+R_sh_s'(\lambda)\}$$

$$+2\sum_{p=1}^{r}\sum_{m=1}^{M^p}S_{sp}\frac{h_s(\lambda)-h_s(\xi_{p,m})}{\lambda-\xi_{p,m}}, \tag{231}$$

where $\xi_{p,m}$, $m=1,\ldots,M^p$, $p=1,\ldots,r$ are numbers that satisfy the system of equations

$$\sum_{p=1}^{r}\sum_{m=1}^{m^p}{}'\frac{\gamma_{sp}}{\xi_{s,n}-\xi_{p,m}}+\sum_{q=1}^{r}S_{qs}h_q(\xi_{s,n})=0. \tag{232}$$

Using the properties of the matrix S_{pq} and the results of Sec. 3, we can rewrite the last two expressions in terms of the highest weights $F(\lambda)$ of the representation of the Gaudin algebra:

$$E_r(M,\xi;\lambda)=\sum_{i\in N_r}\Bigg[F^i(\lambda)+\nu^i\frac{\partial}{\partial\lambda}\Bigg]F_i(\lambda)$$

$$+2\sum_{p=1}^{r}\sum_{i=1}^{M^p}\frac{F_p(\lambda)-F_p(\xi_{p,i})}{\lambda-\xi_{p,i}}; \tag{233}$$

$$\sum_{k=1}^{r}\sum_{m=1}^{M^k}{}' \frac{(\pi_i,\pi_k)}{\xi_{i,n}-\xi_{k,m}}+F_i(\xi_{i,n})=0,$$

$$n=1,\ldots,M^i, i=1,\ldots,r. \tag{234}$$

5. FROM GENERALIZED GAUDIN MODELS TO QUASI-EXACTLY SOLVABLE EQUATIONS

Symmetry of the Gaudin model

We assume that the functions $F_i(\lambda)$, $i\in N_r$, are such that the limits

$$F_i=-\lim_{\lambda\to\infty}\lambda F_i(\lambda), \quad i\in N_r, \tag{235}$$

exist. Then by virtue of (193) and (191) there must also exist the operator limits

$$I_a=-\lim_{\lambda\to\infty}\lambda I_a(\lambda), \quad a\in\Omega_r. \tag{236}$$

Using Eqs. (191), (197), and (236), we find that the operators (236) form the algebra $\mathscr{L}_r$:

$$[I_a,I_b]=\sum_{c\in\Omega_r}\Gamma^c_{ab}I_c, \quad a,b\in\Omega_r, \tag{237}$$

and commute with the family of operators $K_r(\lambda)$:

$$[K_r(\lambda),I_a]=0, \quad a\in\Omega_r. \tag{238}$$

This means that the Gaudin model possesses the global symmetry algebra $\mathscr{L}_r$.

We now consider what representations of the symmetry algebra can be realized in the space $W_r\{F(\lambda)\}$. For this, we apply to the Bethe solutions (230) the operators I_α, $\alpha\in\Delta_r^+$, and I_i, $i\in N_r$. Shifting them to the right by means of the commutation relations

$$[I_a,I_b(\lambda)]=\sum_{c\in\Omega_r}\Gamma^c_{ab}I_c(\lambda), \quad a,b\in\Omega_r, \tag{239}$$

and using the restrictions (232) on the parameters $\xi_{p,m}$, we obtain

$$I_i\phi_r(M,\xi)=(F_i-M_i)\phi_r(M,\xi), \quad i\in N_r; \tag{240a}$$

$$I_\alpha\phi_r(M,\xi)=0, \quad \alpha\in\Delta_r^+. \tag{240b}$$

It can be seen from Eqs. (240a) and (240b) that the Bethe solutions play the role of highest vectors for representations of the symmetry algebra $\mathscr{L}_r$. The numbers F_i-M_i, $i\in N_r$, are the components of the corresponding highest weights. The representations described by Eqs. (240a) and (240b) are infinite-dimensional. Indeed, applying successively to $\phi_r(M,\xi)$ the operators I_α, $\alpha\in\Delta_r^-$, we will obtain more and more new solutions of the Gaudin model with the same eigenvalues $E_r(M,\xi;\lambda)$. This means that the Bethe solutions found in the previous section are only some of all the solutions of the Gaudin problem.

We denote by $\Phi_M\{F\}$ the set of all solutions of the system:

$$I_i\phi=(F_i-M_i)\phi, \quad i\in N_r; \tag{241a}$$

$$I_\alpha\phi=0, \quad \alpha\in\Delta_r^+; \tag{241b}$$

$$\phi\in W_r\{F(\lambda)\}. \tag{241c}$$

By virtue of (238), the space $\Phi_M\{F\}$ is invariant with respect to the action of the operators $K_r(\lambda)$. Therefore, the spectral problems

$$K_r(\lambda)\phi_r=E_r(\lambda)\phi_r, \quad \phi_r\in\Phi_M\{F\}, \tag{242}$$

are defined for all $M=\{M^i\}$, $i\in N_r$. It can be shown that the linear hull of the functions (230) is identical to the space $\Phi_M\{F\}$, and for this reason the solutions of any of the problems (242) are completely described by the explicit Bethe formulas (230), (233), and (234).

An important property of the spaces $\Phi_M\{F\}$ is that they are all finite-dimensional if $F(\lambda)$ is a rational function. This has the consequence that in the rational case Eqs. (242) have only a finite number of exact solutions and therefore can be used as starting points in the construction of quasi-exactly solvable problems.

Differential form of the Gaudin equations and transition to quasi-exactly solvable equations

The finite dimensionality of the invariant spaces $\Phi_M\{F\}$ is most readily proved in the case when the rational functions $F_i(\lambda)$ are nondegenerate, i.e., they are described by the formulas

$$F_i(\lambda)=-\sum_{A=1}^{N}\frac{f_{Ai}}{\lambda-\sigma_A}, \quad i\in N_r. \tag{243}$$

If (191) and (193) are to agree, the generators of the Gaudin algebra must have an analogous form:

$$I_a(\lambda)=-\sum_{A=1}^{N}\frac{I_{Aa}}{\lambda-\sigma_A}, \quad a\in\Omega_r, \tag{244}$$

where I_{Aa} are certain operators. Substituting (244) in the commutation relations (191), we find that they satisfy the relations

$$[I_{Aa},I_{Ab}]=\sum_{c\in\Omega_r}\Gamma^c_{ab}I_{AC}, \quad a,b\in\Omega_r;$$

$$[I_{Aa},I_{Bb}]=0, \quad a,b\in\Omega_r, \quad A\neq B, \tag{245}$$

i.e., they form the algebra $\mathscr{L}_r\oplus\ldots\oplus\mathscr{L}_r$ (N times). In this case, the operators $K_r(\lambda)$ take the form

$$K_r(\lambda)=\sum_{A,B=1}^{N}\frac{\sum_{a,b\in\Omega_r}g^{ab}I_{Aa}I_{Bb}}{(\lambda-\sigma_A)(\lambda-\sigma_B)}, \tag{246}$$

i.e., they are transformed into the Hamiltonians of N-site models of a magnet based on the algebra $\mathscr{L}_r\oplus\ldots\oplus\mathscr{L}_r$ (N times). It is obvious that at site A there acts the representation of the algebra $\mathscr{L}_r$ with highest weight $f_A=\{f_{Ai}\}$, $i\in N_r$. For the numbers F_i, we have in this case

$$F_i=\sum_{A=1}^{N}f_{Ai}, \quad i\in N_r. \tag{247}$$

The expressions for the symmetry operators become extremely simple:

$$I_a = \sum_{A=1}^{N} I_{Aa}, \quad a\in\Omega_r. \tag{248}$$

We are now ready to use the differential realizations of the representations of the algebra $\mathscr{L}_r$ obtained in Sec. 2:

$$I_{Aa} = \hat{t}_a^+(x_A)\frac{\partial}{\partial x_A} + \hat{t}_a^0(x_A)f_A. \tag{249}$$

Substitution of (249) in (246) transforms $K_r(\lambda)$ into a single-parameter family of differential operators of second order:

$$K_r(\lambda) = \sum_{A,B=1}^{N} \frac{\sum_{a,b\in\Omega_r} g^{ab}\left[\hat{t}_a^+(x_A)\frac{\partial}{\partial x_A} + \hat{t}_a^0(x_A)f_A\right]\left[\hat{t}_b^+(x_B)\frac{\partial}{\partial x_B} + \hat{t}_B^0(x_B)f_B\right]}{(\lambda-\sigma_A)(\lambda-\sigma_B)}. \tag{250}$$

The invariant space $\Phi_M\{F\}$ on which the operators (250) act is transformed accordingly, into the space of functions (polynomials) in the variables x_A, $A=1, \dots, N$. To describe the structure of this function space, we use the differential realizations of the symmetry operators I_α, $\alpha\in\Delta_r^+$, and I_i, $i\in N_r$, the explicit form of which can be obtained from (248) and (249) with allowance for the formulas (120) of Sec. 2:

$$I_\alpha = \sum_{A=1}^{N} \hat{t}_\alpha^+(x_A)\frac{\partial}{\partial x_A}, \quad \alpha\in\Delta_r^+;$$

$$I_i = \sum_{A=1}^{N} \hat{t}_i^+(x_A)\frac{\partial}{\partial x_A} + F_i, \quad i\in N_r. \tag{251}$$

Then it follows from the definition (241) of the space $\Phi_M\{F\}$ that its elements are functions that satisfy the system of first-order differential equations

$$\left\{\sum_{A=1}^{N} \hat{t}_\alpha^+(x_A)\frac{\partial}{\partial x_A}\right\}\phi = 0, \quad \alpha\in\Delta_r^+; \tag{252a}$$

$$\left\{\sum_{A=1}^{N} \hat{t}_i^+(x_A)\frac{\partial}{\partial x_A}\right\}\phi = -M_i\phi, \quad i\in N_r. \tag{252b}$$

The subsidiary restriction (241c) reduces to the requirement that the solutions of the system (252) be sought in the class of polynomials in x_A, $A=1, \dots, N$.

In accordance with the results of Sec. 2, the general solution of the subsystem (252a) has the form of a function that depends on $N-1$ vector variables:

$$\xi_A = s_A \dot{-} x_N, \quad A=1, \dots N-1. \tag{253}$$

If this function is to satisfy the second subsystem (252b) and simultaneously be a polynomial, it must have the form of a linear combination of monomials,

$$\prod_{\alpha\in\Delta_r^+} (\xi_{1\alpha})^{K_{1\alpha}}(\xi_{2\alpha})^{K_{2\alpha}}\dots(\xi_{N-1,\alpha})^{K_{N-1,\alpha}}, \tag{254}$$

in which the non-negative integers $K_{A\alpha}$, $A=0, \dots, N-1$, $\alpha\in\Delta_r^+$, satisfy the system of equations

$$\sum_{\alpha\in\Delta_r^+} \sum_{A=1}^{N-1} \alpha K_{A\alpha} = \sum_{i\in N_r} M^i\pi_i. \tag{255}$$

The number of solutions of the system (255) for $K_{A\alpha}$ for given M^i determines the dimension of the space $\Phi_M\{F\}$. We see that in all cases it is finite.

Now that we have at our disposal the explicit form of the operators $K_r(\lambda)$ and the function spaces $\Phi_M\{F\}$, we can construct differential analogs of the spectral equations (242). For this, it is sufficient to project $K_r(\lambda)$ onto the spaces $\Phi_M\{F\}$. We go over to the new variables in accordance with the formulas

$$\xi_A = x_A \dot{-} x_N, \quad A=1, \dots, N-1; \quad \xi_N = x_N; \tag{256}$$

$$\frac{\partial}{\partial x_A} = \frac{\partial(x_A \dot{-} x_N)}{\partial x_A}\frac{\partial}{\partial \xi_A}, \quad A=1, \dots, N-1; \tag{257}$$

$$\frac{\partial}{\partial x_N} = \frac{\partial}{\partial \xi_N} + \sum_{A=1}^{N-1} \frac{\partial(x_A \dot{-} x_N)}{\partial x_N}\frac{\partial}{\partial \xi_A}.$$

By virtue of the "translational" invariance of the operator $K_r(\lambda)$ and the functions of the space $\Phi_M\{F\}$, they do not depend explicitly on ξ_N. Therefore, in Eqs. (256) and (257) we can set $\xi_N = 0$, simultaneously omitting the derivatives with respect to ξ_N. This gives, in place of (256) and (257),

$$x_A = \xi_A, \quad A=1,\dots,N-1; \quad x_N = 0; \tag{258}$$

$$\frac{\partial}{\partial x_A} = \frac{\partial}{\partial \xi_A}, \quad A=1,\dots,N-1;$$

$$\frac{\partial}{\partial x_N} = -\sum_{A=1}^{N-1} t_+^+(\xi_A)\frac{\partial}{\partial \xi_A} \tag{259}$$

[in obtaining the last formula, we have used the definition of the matrix $\hat{t}(\xi)$ given in Sec. 2]. In addition, we have the equations

$$\hat{t}_+^+(0) = \hat{1}, \quad \hat{t}_0^+(0) = 0, \quad \hat{t}_-^+(0) = 0,$$
$$\hat{t}_+^0(0) = 0, \quad \hat{t}_0^0(0) = \hat{1}, \quad \hat{t}_-^0(0) = 0. \tag{260}$$

Using the relations (258)–(260), we obtain for $K_r(\lambda)$:

$$K_r(\lambda)=\sum_{A,B=1}^{N-1}\frac{\sum_{a,b\in\Omega_r} g^{ab}I_{Aa}I_{Bb}}{(\lambda-\sigma_A)(\lambda-\sigma_B)}$$

$$+2\sum_{A=1}^{N-1}\frac{\sum_{a,b\in\Omega_r} g^{ab}I_{Aa}J_b}{(\lambda-\sigma_A)(\lambda-\sigma_N)}+\frac{\sum_{a,b\in\Omega_r} g^{ab}J_aJ_b}{(\lambda-\sigma_N)^2}. \tag{261}$$

Here

$$I_{Aa}=\sum_{\alpha\in\Delta_r^+} t_a^\alpha(\xi_A)\frac{\partial}{\partial\xi_A^\alpha}+\sum_{i\in N_r} t_A^i(\xi_A)f_{Ai},\quad a\in\Omega_r; \tag{262a}$$

$$J_\alpha=-\sum_{B=1}^{N-1}\sum_{\beta\in\Delta_r^+} t_\alpha^\beta(\xi_B)\frac{\partial}{\partial\xi_B^\beta},\quad \alpha\in\Delta_r^+;$$

$$J_i=f_{Ni},\quad i\in N_r;$$

$$J_\alpha=0,\quad \alpha\in\Delta_r^-. \tag{262b}$$

The obtained operators (261) act on the space of homogeneous polynomials in ξ_A, $A=1,\ldots,N-1$, with basis (254)–(255). In accordance with the results of Sec. 2, the elements of this space can be represented in the form

$$\phi=\prod_{i\in N_r}(\xi_{N-1}^{\pi_i})^{M_i}\psi(\eta,\nu), \tag{263}$$

where η is the vector of dimension $(N-2)(d_r-r)/2$ with components

$$\eta_A^\alpha=\frac{\xi_A^\alpha}{\prod_{i\in N_r}(\xi_{N-1}^{\pi_i})^{(\alpha,\pi^i)}},\quad A=1,\ldots,N-2,\quad \alpha\in\Delta_r^+, \tag{264}$$

and ν is the $(d_r-r)/2$-dimensional vector with the components

$$\nu^\alpha=\frac{\xi_{N-1}^\alpha}{\prod_{i\in N_r}(\xi_{N-1}^{\pi_i})^{(\alpha,\pi^i)}},\quad \alpha\in\Delta_r^+. \tag{265}$$

For this vector, the only trivial components are the $(d_r-3r)/2$ components with $\alpha\in\Delta_r^+-\Pi_r^+$, where Π_r^+ is the set of simple roots of the algebra $\mathscr{L}_r$. The remaining r components with $\alpha\in\Pi_r^+$ are equal to unity. The functions $\psi(\eta,\nu)$ in (263) are polynomials in the $[(N-2)(d_r-r)+(d_r-3r)]/2$ variables η and ν. The form of these polynomials can be determined from the condition that the elements (263) belong to the spaces $\Phi_M\{F\}$. We denote the set of allowed polynomials $\psi(\eta,\nu)$ by $\Psi_M\{F\}$.

It follows from the invariance of the spaces $\Phi_M\{F\}$ that the result of applying (261) to functions of the form (263) must have the same form. This enables us to write

$$K_r(\lambda)\prod_{i\in N_r}(\xi_{N-1}^{\pi_i})^{M^i}\psi(\eta,\nu)$$

$$=\prod_{i\in N_r}(\xi_{N-1}^{\pi_i})^{M^i}K_r(M,\lambda;\eta,\nu)\psi(\eta,\nu), \tag{266}$$

where $K_r(M,\lambda;\eta,\nu)$ is a differential operator that acts on the space $\Psi_M\{F\}$ and depends on the non-negative integers M^i. Accordingly, Eqs. (242) can be written in the form

$$K_r(M,\lambda;\eta,\nu)\psi(\eta,\nu)=E_r(M,\lambda)\psi(\eta,\nu),\quad \psi(\eta,\nu)\in\Psi_M\{F\}. \tag{267}$$

If we denote by Ψ the set of all analytic functions of the variables η and ν, then the equations

$$K_r(M,\lambda;\eta,\nu)\psi(\eta,\nu)=E_r(M,\lambda)\psi(\eta,\nu),\quad \psi(\eta,\nu)\in\Psi, \tag{268}$$

will obviously have in Ψ only a finite number of exact solutions, which are determined by the Bethe functions (233) and (234) and lie in the class of polynomials in the variables η and ν of the form $\Psi_M\{F\}$.

Thus, we arrive at an infinite series of quasi-exactly solvable equations associated with Gaudin models on the representations of the algebras $\mathscr{L}_r$ with highest weights $F(\lambda)$.

It now remains to find the explicit form of the operators $K_r(M,\lambda;\eta,\nu)$. For this, we must go over in (261) to new variables in accordance with the formulas

$$\left.\begin{aligned}&\xi_{N-1}^{\pi_i}=\xi^i,\quad i\in N_r;\\&\xi_A^\alpha=\prod_{i\in N_r}(\xi^i)^{(\alpha,\pi^i)}\eta_A^\alpha,\quad A=1,\ldots,N-2;\quad \alpha\in\Delta_r^+;\\&\xi_{N-1}^\alpha=\prod_{i\in N_r}(\xi^i)^{(\alpha,\pi^i)}\nu^\alpha,\quad \alpha\in\Delta_r^+-\Pi_r^+;\end{aligned}\right\} \tag{269}$$

$$\frac{\partial}{\partial\xi_{N-1}^{\pi_i}}=\frac{\partial}{\partial\xi^i}-\sum_{A=1}^{N-2}\sum_{\alpha\in\Delta_r^+}\frac{(\alpha,\pi^i)}{\xi^i}\eta_A^\alpha\frac{\partial}{\partial\eta_A^\alpha}$$

$$-\sum_{\alpha\in\Delta_r^+-\Pi_r^+}\frac{(\alpha,\pi^i)}{\xi^i}\nu^\alpha\frac{\partial}{\partial\nu^\alpha},\quad i\in N_r;$$

$$\frac{\partial}{\partial\xi_A^\alpha}=\prod_{i\in N_r}(\xi^i)^{-(\alpha,\pi^i)}\frac{\partial}{\partial\eta_A^\alpha},\quad A=1,\ldots,N-2,\quad \alpha\in\Delta_r^+;$$

$$\frac{\partial}{\partial\xi_{N-1}^\alpha}=\prod_{i\in N_r}(\xi^i)^{-(\alpha,\pi^i)}\frac{\partial}{\partial\nu^\alpha},\quad \alpha\in\Delta_r^+-\Pi_r^+. \tag{270}$$

Since the operator (261) is "scale-invariant," it can contain a dependence on ξ^i and $\partial/\partial\xi^i$ only in the form of the combinations $\xi^i(\partial/\partial\xi^i)$, $i\in N_r$. As a result of the projection of (261) onto the class of functions of the form (263), these combinations are replaced by numbers M^i. In practice, this is done as follows. We use the scaling property of the coefficient matrices $t_a^\alpha(\xi)$ and $t_a^i(\xi)$:

$$t^{\alpha}_{a}(\xi_A)=\prod_{i\in N_r}(\xi^i)^{(\alpha-a,\pi^i)}t^{\alpha}_{a}(\eta_A);$$

$$t^{i}_{a}(\xi_A)=\prod_{i\in N_r}(\xi^i)^{-(a,\pi^i)}t^{i}_{a}(\eta_A);$$

$$t^{\alpha}_{a}(\xi_{N-1})=\prod_{i\in N_r}(\xi^i)^{(\alpha-a,\pi^i)}t^{\alpha}_{a}(\nu);$$

$$t^{i}_{a}(\xi_{N-1})=\prod_{i\in N_r}(\xi^i)^{-(a,\pi^i)}t^{i}_{a}(\nu). \tag{271}$$

Here $A=1,\dots,N-2$, $\alpha\in\Delta_r^+$, $i\in N_r$, $a\in\Omega_r$.[2)] Substituting the formulas (269), (270), and (271) in (261) and (262), and replacing the operators $\xi^i(\partial/\partial\xi^i)$ by the numbers M^i, we find

$$K_r(M,\lambda;\eta,\nu)=\sum_{A,B=1}^{N-2}\frac{\sum_{a,b\in\Omega_r}g^{ab}I_{Aa}I_{Bb}}{(\lambda-\sigma_A)(\lambda-\sigma_B)}+\sum_{A=1}^{N-2}\frac{\sum_{a,b\in\Omega_r}g^{ab}[\tilde K_bI_{Aa}+I_{Ab}K_a]}{(\lambda-\sigma_A)(\lambda-\sigma_{N-1})}+\frac{\sum_{a,b\in\Omega_r}g^{ab}\tilde K_aK_b}{(\lambda-\sigma_{N-1})^2}$$
$$+2\sum_{A=1}^{N-2}\frac{\sum_{a,b\in\Omega_r}g^{ab}I_{Aa}L_b}{(\lambda-\sigma_A)(\lambda-\sigma_N)}+2\frac{\sum_{a,b\in\Omega_r}g^{ab}\tilde K_aL_b}{(\lambda-\sigma_{N-1})(\lambda-\sigma_N)}+\frac{\sum_{a,b\in\Omega_r}g^{ab}L_aL_b}{(\lambda-\sigma_N)^2}, \tag{272}$$

where

$$I_{Aa}=\sum_{\alpha\in\Delta_r^+}t^{\alpha}_{a}(\eta_A)\frac{\partial}{\partial\eta^{\alpha}_{A}}+\sum_{i\in N_r}t^{i}_{a}(\eta_A)f_{Ai},\quad a\in\Omega_r,$$
$$A=1,\dots,N-2; \tag{273a}$$

$$K_a\sum_{i\in N_r}t^{\pi_i}_{a}(\nu)\Big[M^i-\sum_{B=1}^{N-2}\sum_{\beta\in\Delta_r^+}(\beta,\pi^i)\eta^{\beta}_{B}\frac{\partial}{\partial\eta^{\beta}_{B}}$$
$$-\sum_{\beta\in\Delta_r^+-\Pi_r^+}(\beta,\pi^i)\nu^{\beta}\frac{\partial}{\partial\nu^{\beta}}\Big]+\sum_{\alpha\in\Delta_r^+-\Pi_r^+}t^{\alpha}_{a}(\nu)\frac{\partial}{\partial\nu^{\alpha}}$$
$$+\sum_{i\in N_r}t^{i}_{a}(\nu)f_{N-1,i},\quad a\in\Omega_r; \tag{273b}$$

$$\tilde K_a=K_a-\sum_{i\in N_r}t^{\pi_i}_{a}(\nu)(a,\pi^i),\quad a\in\Omega_r; \tag{273c}$$

$$L_\alpha=-\sum_{B=1}^{N-2}\sum_{\beta\in\Delta_r^+}t^{\beta}_{\alpha}(\eta_B)\frac{\partial}{\partial\eta^{\beta}_{B}}-\sum_{i\in N_r}t^{\pi_i}_{\alpha}(\nu)\Big[M^i$$
$$-\sum_{B=1}^{N-2}\sum_{\beta\in\Delta_r^+}(\beta,\pi^i)\eta^{\beta}_{B}\frac{\partial}{\partial\eta^{\beta}_{B}}$$
$$-\sum_{\beta\in\Delta_r^+-\Pi_r^+}(\beta,\pi^i)\nu^{\beta}\frac{\partial}{\partial\nu^{\beta}}\Big]$$
$$-\sum_{\beta\in\Delta_r^+-\Pi_r^+}t^{\beta}_{\alpha}(\nu)\frac{\partial}{\partial\nu_{\beta}},\quad\alpha\in\Delta_r^+;$$
$$L_i=f_{N_i},\quad i\in N_r;\quad L_\alpha=0;\quad\alpha\in\Delta_r^-. \tag{273d}$$

With this, we complete the construction of the multidimensional quasi-exactly solvable differential equations of second order associated with the completely integrable Gaudin models in the case when the functions $F_i(\lambda)$, $i\in N_r$, which play the part of highest weights of the representations of the Gaudin algebra, are rational and nondegenerate. The transition to the degenerate rational case can be made as follows.

We note that all rational functions $F_i(\lambda)$, $i\in N_r$, admit representation in the form

$$F_i(\lambda)=\sum_A\tilde f_{Ai}\omega^A(\lambda), \tag{274}$$

where $\omega^A(\lambda)$ are elementary rational functions of the form $(\lambda-\sigma)^{-n}$, $\sigma\in C$, $n\in N$. The index A labeling them is in fact a multiple index $A=(\sigma,n)$. The sum in (279) is assumed to be finite. In the decomposition (274) we have used only decreasing elementary rational functions because the components of the highest weights $F_i(\lambda)$ must, by hypothesis, be regular at infinity.

For the functions $\omega^A(\lambda)$ we have the composition theorems

$$\frac{\omega^A(\lambda)-\omega^A(\mu)}{\lambda-\mu}=\sum_{B,C}C^A_{BC}\omega^B(\lambda)\omega^C(\mu), \tag{275}$$

$$\omega^B(\lambda)\omega^C(\lambda)=\sum_A D^{BC}_A\omega^A(\lambda), \tag{276}$$

in which C^A_{BC} and D^{BC}_A are certain structure constants. The sums over A, B, and C in (275) and (276) are also assumed to be finite.

In accordance with (193) and (191), the generators of the Gaudin algebra should be sought in an analogous form:

$$I_a(\lambda)=\sum_A\tilde I_{Aa}\omega^A(\lambda). \tag{277}$$

Substituting (277) in the commutation relations (191) and using (275), we obtain commutation relations directly for the coefficient operators $\tilde I_{Aa}$:

$$[\tilde I_{Aa},\tilde I_{BC}]=\sum_{c\in\Omega_r}\Gamma^c_{ab}\sum_C C^C_{AB}\tilde I_{C,c}. \tag{278}$$

By virtue of the finiteness of the sum over A in (277), the operators $\tilde I_{Aa}$ form a finite-dimensional Lie algebra. Substituting the expansion (277) in the expression (197) for the operators $K_r(\lambda)$, we can reduce them to the form

$$K_r(\lambda)=\sum_{a,b\in\Omega_r}\sum_{A,B,C}g^{ab}D_C^{AB}\omega^C(\lambda)\widetilde{I}_{Aa}\widetilde{I}_{Bb}. \qquad (279)$$

These are the Hamiltonians of magnets based on the finite-dimensional Lie algebra (278), which can be interpreted as a certain contraction of the algebra $\mathcal{L}_r\oplus\ldots\oplus\mathcal{L}_r$ (N times) if the functions $F_i(\lambda)$, $i\in N_r$, are obtained as a result of degeneracy of functions of the form (243).

To construct differential realizations of the operators $\widetilde{I}_{Aa}$, we first consider the procedure for going over from the nondegenerate functions $F_i(\lambda)$ to degenerate functions:

$$\sum_{A=1}^{N}\frac{f_{Ai}}{\lambda-\sigma_a}\to\sum_{A=1}^{N}\widetilde{f}_{Ai}\omega^A(\lambda). \qquad (280)$$

To realize this procedure, we must make a suitable linear substitution:

$$f_{Ai}=\sum_{B=1}^{N}C_A^B(\sigma_1,\ldots,\sigma_N)\widetilde{f}_{Bi},\quad i\in N_r, \qquad (281)$$

and we must then let the parameters $\sigma_1,\ldots,\sigma_N$ tend to their limiting values, merging all or some of the simple poles of the nondegenerate functions $F_i(\lambda)$. The explicit form of the matrix C_A^B is determined by the specific form of the degeneracy, i.e., by the requirement that the result be identical to the right-hand side of Eq. (280).

A similar procedure must be carried out for the operators $I_a(\lambda)$:

$$\sum_{A=1}^{N}\frac{I_{Aa}}{\lambda-\sigma_A}\to\sum_{A=1}^{N}\widetilde{I}_{Aa}\omega^A(\lambda). \qquad (282)$$

Here, it is most convenient to proceed from the nondegenerate operators $I_{A\alpha}$, $\alpha\in\Delta_r^+$, whose differential realizations contain in accordance with Eqs. (189) the operators $\partial/\partial x_A^\alpha$ as terms. Requiring that the degenerate operators contain as terms analogous operators of differentiation, but now with respect to the new variables, $\partial/\partial\widetilde{x}_A^\alpha$, we arrive at the need to consider the limiting process

$$\sum_{A=1}^{N}\frac{\partial/\partial x_A^\alpha}{\lambda-\sigma_A}\to\sum_{A=1}^{N}(\partial/\partial\widetilde{x}_A^\alpha)\omega^A(\lambda), \qquad (283)$$

the structure of which is completely analogous to (280). This enables us to write down the connection between the derivatives $\partial/\partial x_A^\alpha$ and $\partial/\partial\widetilde{x}_A^\alpha$:

$$\frac{\partial}{\partial x_A^\alpha}=\sum_{B=1}^{N}C_A^B(\sigma_1,\ldots,\sigma_N)\frac{\partial}{\partial\widetilde{x}_B^\alpha},\quad \alpha\in\Delta_r^+, \qquad (284)$$

and then express the old variables x_A^α in terms of the new ones $\widetilde{x}_A^\alpha$:

$$x_A^\alpha=\sum_{B=1}^{N}\widetilde{C}_A^B(\sigma_1,\ldots,\sigma_N)\widetilde{x}_B^\alpha,\quad \alpha\in\Delta_r^+. \qquad (285)$$

Here, $\widetilde{C}_A^B$ is the matrix that is the inverse of C_A^B. Substituting (281), (284), and (285) in the expressions for the remaining operators I_{Aa}, and making the necessary transition to the limiting values $\sigma_1,\ldots,\sigma_2$, we arrive at differential forms of the degenerate operators $\widetilde{I}_{Aa}$ that realize the representation of the contracted algebra $\mathcal{L}_r\oplus\ldots\oplus\mathcal{L}_r$ (N times).

Coulomb analogy

We return to the algebraic equations (253), from which we can find the spectra of the Gaudin magnets and the quasi-exactly solvable problems associated with them. The roots of these equations, i.e., the numbers $\xi_{i,q}$, $q=1,\ldots,M^I$, $i=1,\ldots,r$, are, in general, complex. Therefore, it is meaningful to introduce the two-dimensional vectors

$$\xi_{i,q}=(\mathrm{Re}\ \xi_{i,q},\mathrm{Im}\ \xi_{i,q}),\quad q=1,\ldots,M^I,\quad i=1,\ldots,r. \qquad (286)$$

If with them we introduce the notation

$$U_i(\xi)\equiv\mathrm{Re}\int F_i(\xi)d\xi,\quad i=1,\ldots,r, \qquad (287)$$

then the system (234) can be interpreted as the condition for an extremum of the function

$$U(\xi)=-\sum_{i,k=1}^{r}\sum_{q=1}^{M^i}\sum_{p=1}^{M^k}(\pi_i,\pi_k)\ln|\xi_{i,q}-\xi_{k,p}| -\sum_{i=1}^{r}\sum_{p=1}^{M^i}U_i(\xi_{i,p}). \qquad (288)$$

We now note that the function (288) is none other than the potential of a two-dimensional logarithmic many-particle Coulomb system in an external field. There are altogether r species of particles, labeled by the index $i=1,\ldots,r$. There are M^i of the particles of species i. The numbers $\xi_{i,p}$ denote coordinates of these particles, and the simple roots of the Lie algebra, π_i, play the role of their "vector" charges. Particles of the same species have the same vector charges, and therefore repel each other $[(\pi_i,\pi_i)>0]$, while particles of different species attract each other $[(\pi_i,\pi_k)\leqslant 0,\ i\neq k]$. In addition, there are r potentials $-U_i(\xi_i)$, each of which acts only on the particles of a definite species.

The Coulomb analogy is extremely helpful. We have already had the possibility of demonstrating this in the example of Ref. 12. The analogy makes it possible in a qualitative analysis of the solutions of quasi-exactly solvable equations to use our classical intuition, which is obviously much more developed than the quantum-mechanical intuition.

CONCLUSIONS. DEALGEBRAIZATION OF THE METHOD AND PROSPECTS

Thus, we have completed the exposition of our new approach to the problem of quasi-exact solvability in nonrelativistic quantum mechanics. The method developed in it is suitable for constructing both one-dimensional and multidimensional quasi-exactly solvable differential equations, which, using the procedure described in Ref. 12, can always be reduced to equations of Schrödinger type. The proposed approach, in contrast to the earlier ones (see, for example, the review of Ref. 12), is truly algebraic. The original objects in it are completely integrable Gaudin models based on the various simple Lie algebras, and they are exactly solvable in the framework of the algebraic Bethe ansatz. The global symmetry of these models makes it possible to carry out in them (or, rather, in the differential forms of the corresponding integral equations) a partial separation of the variables, after which these equations become quasi-exactly solvable. Using the Bethe-ansatz equations describing the spectra of the quasi-exactly solvable quantum-mechanical models obtained in this manner, one can show that these models are equivalent to classical models of two-dimensional Coulomb systems in an external field. This connection between three completely different, at the first glance, physical systems—models of magnets based on Lie algebras, quasi-exactly solvable quantum-mechanical models, and the classical many-particle Coulomb problem—was noted by us earlier, but we restricted ourselves to a discussion of the case of the algebra $sl(2)$. We now see that the connection also holds in the general case.

An interesting feature of the approach proposed here is that by its very essence it contains a possibility of further generalization. This assertion is important, and therefore it is worth dwelling on it in more detail.

We begin with this question: What role in the approach is played by the complete integrability of the model of a Gaudin magnet? At the first glance, everything is founded upon it. However, on closer examination it becomes obvious that the role reduces merely to the possibility of representing the result in a closed Bethe form. This circumstance is undoubtedly helpful, since the Bethe form of expression is the most convenient for carrying out various limiting processes, for example, the passage to the infinite-dimensional $(N\to\infty)$ or the exactly solvable $(M\to\infty)$ cases. At the same time, the functional structure of the result remains the same, so that both the pre-limit and the limit models can be interpreted from the point of view of the Coulomb analogy. However, these facts have only a secondary nature. To the main question, that of whether integrability has fundamental significance for quasi-exact solvability (i.e., for the possibility of algebraization of the spectral problem), one can answer with confidence: no, it does not. To demonstrate this, we consider the model of a magnet based on the algebra $\mathscr{L}_r\oplus\ldots\oplus\mathscr{L}_r$ with Hamiltonian

$$H=\sum_{A,B=1}^{N}\sum_{a,b\in\Omega_r}C^{AB}g^{ab}I_{Aa}I_{Bb},\tag{289}$$

in which I_{Aa}, which act at site A, are the generators of the algebra $\mathscr{L}_r$ with highest weights $f_A=\{f_{Ai}\}$, $i\in N_r$, and C^{AB} are arbitrary numerical coefficients. The arbitrariness of C^{AB} means that we do not require integrability of the model (289). Despite this, it can also be associated with a certain quasi-exactly solvable model by means of the method discussed in the paper, which for this purpose is completely ready.

Indeed, the space W on which the operator H acts is the direct product of the spaces W_A of the representations of the algebra $\mathscr{L}_r$. They, in their turn, can be regarded as direct sums of the subspaces $|M_A\rangle$, defined as the sets of vectors of the form $I_{A\alpha_1}\cdot\ldots\cdot I_{A\alpha_K}|0\rangle$, provided that the roots $\alpha_1,\ldots,\alpha_K$ are negative and their sum is equal to $-\Sigma_{i\in N_r}M_A^i\pi_i$. This means that for W we have the decomposition

$$W=\bigoplus_{M\geqslant 0}\Phi_M,\tag{290}$$

where the spaces Φ_M are determined by the formulas

$$\Phi_M=\bigoplus_{\{M_1,\ldots,M_N\}}\{|M_1\rangle\otimes\ldots\otimes|M_N\rangle\},\tag{291}$$

subject to the condition that

$$\sum_{A=1}^{N}M_A^i=M^i,\quad i\in N_r.\tag{292}$$

A key property of the spaces Φ_M is that they are all finite-dimensional and invariant with respect to the action of the operator H.

Further, the operator H has the global symmetry group $\mathscr{L}_r$ realized by the operators

$$I_a=\sum_{A=1}^{N}I_{Aa},\quad a\in\Omega_r.\tag{293}$$

It is easy to show that the spaces Φ_M are eigenspaces with respect to the elements of the Cartan subalgebra of the symmetry algebra $\mathscr{L}_r$:

$$I_i\Phi_M=(F_i-M_i)\Phi_M,\tag{294}$$

where $F_i=\Sigma_{A=1}^N f_{Ai}$. This means that the sets of vectors satisfying the conditions

$$I_\alpha\phi=0,\quad \alpha\in\Delta_r^+;$$
$$I_i\phi=(F_i-M_i)\phi,\quad i\in N_r;\quad \phi\in W,\tag{295}$$

certainly belong to Φ_M and are therefore finite-dimensional. These sets, which we denote by Ψ_M, are also invariant with respect to H, and we therefore arrive at the infinite series of equations

$$H\phi=E\phi,\ \phi\in\Psi_M,\ M\geqslant 0,\tag{296}$$

each of which has only a finite number of solutions. The transition from the algebraic form of these equations to the differential form can be realized by the same method as in the integrable case. This transition can be made in two stages. In the first, allowance is made for the translational invariance of the operator (289), by virtue of which it is reduced to the form

$$H=\sum_{A,B=1}^{N-1}\sum_{\alpha,\beta\in\Delta_r^+}P_{AB}^{\alpha+\beta}(\xi)\frac{\partial^2}{\partial\xi_A^\alpha\partial\xi_B^\beta}+\sum_{A=1}^{N}\sum_{\alpha\in\Delta_r^+}Q_A^\alpha(\xi)\frac{\partial}{\partial\xi_A^\alpha}.\tag{297}$$

Here, $P_{AB}^{\alpha+\beta}$ and Q_A^α are homogeneous polynomials in the variables ξ_A^α consisting of monomials of the form

$$P^{\alpha+\beta}=\{\xi_{A_1}^{\alpha_1}...\xi_{A_K}^{\alpha_K}\},\quad \alpha_1+...+\alpha_K=\alpha+\beta;$$

$$Q^{\alpha}=\{\xi_{A_1}^{\alpha_1}...\xi_{A_K}^{\alpha_K}\},\quad \alpha_1+...+\alpha_K=\alpha. \qquad (298)$$

The spaces Φ_M on which the operator H acts are a linear combination of monomials:

$$\Phi_M=\{\xi_{A_1}^{\alpha_1}...\xi_{A_K}^{\alpha_K}\},\quad \alpha_1+...+\alpha_K=\sum_{i\in N_r}M^i\pi_i. \qquad (299)$$

In the second stage, we take into account the "scale" invariance (homogeneity) of the operator (289), which makes possible partial separation of the variables ξ in the spectral equation for H. At the same time, we use the ansatz (263), which transforms the spectral equation for (297) into a differential equation in a smaller number of variables η and ν. It depends explicitly on the non-negative integers M^i and is quasi-exactly solvable by construction.

It follows from the above derivation that the requirement of integrability of the original model (289) is indeed redundant. However, at the same time we are forced to recognize that this is also true of the entire algebraic structure of the model, i.e, actually the model itself. Indeed, we could with success start with the operator (297) acting on the space (299), taking as $P^{\alpha+\beta}$ and Q^{α} the most general polynomials of the form (298). After partial separation of the variables, we would again obtain a quasi-exactly solvable model. It could be objected here that the algebraic nature is implicitly present in Eq. (296), since in determining the spaces (299) and the coefficient functions in (297) we used the properties of the root system of the algebra $\mathscr{L}_r$. However, it can be shown that this last thread connecting equations of the type (296) to Lie algebras can also be readily broken. Indeed, let Δ_r^+ be a finite system of vectors of an r-dimensional space, including a basis of r vectors, Π_r^+, such that all the remaining vectors in Δ_r^+ (if there are any) can be decomposed with respect to Π_r^+ with non-negative integer coefficients. The system Δ_r^+ in general is not a root system. However, if in Eqs. (298) and (299) the vectors α_i are assumed to be elements of a root system, then all the arguments that reduce Eqs. (298) to quasi-exactly solvable form remain valid.

Thus, we arrive at a conclusion which at the first glance appears paradoxical: In the formulation of the theory of quasi-exact solvability one can get by perfectly well without a concept such as a Lie algebra. The basic principles of this phenomenon can be understood without going beyond the framework of the analytic approach. The abandonment of the language of symmetries not only simplifies the problem, but, as we have seen above, permits its formulation in a much more general form. Here, however, there may arise a natural question concerning the status of the Turbiner–Shifman method,[11] in the formulation of which a Lie algebra, or, rather, finite-dimensional representations of it, play a decisive role. To answer this question, we consider a typical Turbiner–Shifman Hamiltonian:

$$H=\sum_{a,b}P_{ab}S^aS^b+\sum_a Q_aS^a, \qquad (300)$$

in which S^a are the generators of a finite-dimensional representation of some Lie algebra, and P_{ab} and Q_a are arbitrary numerical coefficients. If our Hamiltonian (295) is to take the form (300), it must be possible to represent the operators S^a in the form

$$S^a=\sum_{\alpha,A}P_A^{a\alpha}(\xi)\frac{\partial}{\partial\xi_A^{\alpha}}, \qquad (301)$$

where $P_A^{a\alpha}(\xi)$ are homogeneous polynomials in ξ formed from monomials of the form (298). Only such operators close to make a finite-dimensional Lie algebra without taking us outside the space (299), i.e., realize on it finite-dimensional representations. However, it is readily seen that such a reduction of Hamiltonians of the type (297) to the rotator Hamiltonians (300) is by no means always possible. This could be prevented by the presence of terms of the form $\xi_C^{\alpha+\beta}\partial^2/(\partial\xi_A^{\alpha}\partial\xi_B^{\beta})$, which are not factorizable, i.e, cannot be represented as a product of two operators of the form (301). This means that the Turbiner–Shifman algebraic approach is not the most general, i.e., it does not exhaust all possible quasi-exactly solvable models.

Of course, it is as yet clearly premature to claim that the dealgebraized version of our approach discussed here lays claim to the greatest generality. Although we do have arguments for such a claim, this question can only be settled at the theorem level of rigor. Thus, there may be "surprises." In turn, this means that in the theory of quasi-exact solvability it is still early to put the final period.

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[1)]This definition differs from the standard one[24] by a factor.
[2)]The employed notation (a,π^i) is interpreted as follows: $(a,\pi^i)=(\alpha,\pi^i)$ if $a=\alpha\in\Delta_r^+$, and $(a,\pi^i)=0$ if $a=k\in N_r$.

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