Spectral problems of mathematical physics related to nonlinear equations of the KdV type

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The spectral theory of expansions in products of solutions of two regular Sturm–Liouville problems and the associated theory of Λ operators are described. It is shown that this can be used as a general basis for obtaining uniqueness theorems and constructive methods of solving the corresponding inverse problems, and also a Hamiltonian formalism for the Korteweg–de Vries equation in the periodic case. A scheme for generalizing these results to the case of two regular Dirac operators is sketched.

INTRODUCTION

The development of contemporary mathematical physics has been profoundly influenced by a new method, usually referred to as the "inverse scattering method" (ISM): the integration of nonlinear evolution equations which can be written in abstract form as the Cauchy problem

$$v_t=K(v),\quad 0<t<\infty,\quad v(0)=v_0, \tag{1}$$

where the explicit expression for $K(v)$ is determined by the actual physical problem. Here a fundamental role is played by the Korteweg–de Vries (KdV) equation

$$r_t=6rr_x-r_{xxx},\quad r(x,0)=r_0(x)\in\varphi. \tag{2}$$

The essence of the ISM, proposed in Ref. 1, is the construction of transformations which linearize this problem in a suitable space of spectral data uniquely determining the Sturm–Liouville equation

$$l(r)y\equiv\left(-\frac{d^2}{dx^2}+r(x,t)\right)y=\lambda y, \tag{3}$$

where henceforth we use the conventional term "potential" to refer to the function $r(x)$. This leads to the corresponding inverse problem (IP), which for the operator (3) reduces to the by now thoroughly studied Gel'fand–Levitan–Marchenko equation.[2] After the fundamental studies of Lax,[3] Zakharov and Shabat,[4] and Zakharov and Faddeev[5] came the study of Ref. 6, which developed a very general approach giving the linearizing transformation in the form of a generalized Fourier transform, which for the KdV case reduces to the theory of expansion of the solutions of (3) in quadratics $Y=y^2(x,\lambda)$ satisfying the equation

$$LY'\equiv\frac{1}{4}\left\{-\frac{d^2}{dx^2}+4r(x)+r_x(x)\left(\int_{-\infty}^{x}-\int_{x}^{\infty}\right)\right\}Y'$$
$$=\lambda Y'\left('=\frac{d}{dx}\right). \tag{4}$$

The KdV equation itself can be written in the form $r_t=4Lr_x$. As was shown in Ref. 7, the construction of a general form of the Bäcklund transformation leads to an operator for which the eigenfunctions are products $Y=y_1y_2$ of the solutions of the two equations $l(r_j)y_j=\lambda y_j$, $j=1,2$. According to established tradition, such operators are usually referred to as Λ operators. As is apparent from, for example, the monographs of Refs. 7–9, the properties of these operators play a very important role in the ISM. As a rule, they have been studied on the entire axis for the class of potentials which fall off for $|x|\to\infty$. The question of the completeness of the product Y of solutions of two Sturm–Liouville problems on a finite interval has already been studied in the classical work of Borg,[10] where the first uniqueness theorems for the IP were proved. The constructive reconstruction of the Sturm–Liouville operator from the spectral function was solved by Gel'fand and Levitan in Ref. 11, where the inverse problem was reduced to a Fredholm equation of the second kind. A similar result for the scattering IP on the semiaxis was obtained by Marchenko in Ref. 12. It was later shown that in some cases the Gel'fand–Levitan equation admits explicit solutions, which formed the basis of the soliton solutions for the KdV equation. The Gel'fand–Levitan equation was also used as the basis for obtaining the *a priori* characteristics of the class of functions which are the spectral functions for the Sturm–Liouville operator with potential from a given space (see, for example, Sec. 2 below). This played an important role in the further development of IP methods, mainly in connection with extending the application of the ISM to various soliton equations. The theory of the equations to which the ISM is applicable naturally also affected IP investigations. The similarity between the methods of solving the inverse scattering problem for the radial Schrödinger operator and the properties of the KdV equation was noted, in particular, in Ref. 13, where the solution of the IP was obtained on the basis of the continuous analog of the Newton method (CANM),[14] which is an equation of the form (1) (see Sec. 2 for more details). The theory of Λ operators on the semiaxis was developed for its constructive realization.[15]

This review can be viewed as a continuation of these investigations. Here a fundamental role is played by the theory of expansions in products of the solutions of two self-adjoint Sturm–Liouville problems on a finite interval and their application to the IP and to the ISM for the KdV equation.

In Sec. 1 we consider the two problems

$$l(q_j)y=\lambda y,\quad y'(0)-h_jy(0)=0,$$

$$y'(\pi)+H_jy(\pi)=0,\quad 0\leqslant x\leqslant\pi,\quad j=1,2, \tag{5}$$

where $q_j \in L_2(0,\pi)$, h_j, H_j are finite numbers. Owing to the presence of boundary conditions in (5), the product expansion formulas here are somewhat more complicated than in the case of the full axis,[16] as was noted already in Ref. 17. As elementary applications we obtain simple proofs of the corresponding uniqueness theorems for the IP which harmonize with their more constructive treatment.[18,19] The idea of using the CANM to solve the IP, which is discussed in Sec. 1, originated in Refs. 17, 20, and 21. This approach is similar to the constructions of Refs. 22 and 23. A characteristic feature of the equations constructed in Sec. 2 is the possibility of solving them explicitly by equations which are the analogs of the known soliton solutions for the KdV equations. In Sec. 4 we study a similar set of questions for the problems

$$l(r_j)y=\lambda y, \quad y(0)=y(\pi)=0, \quad r_j \in L_2(0,\pi), \quad j=1,2. \tag{6}$$

In Sec. 3 we first obtain equations for product expansions of the solutions of these problems, following the construction of Sec. 1, in a form convenient for studying inverse periodic Sturm–Liouville problems. We use them to prove uniqueness theorems and also a representation for finite-zone potentials which is important for the later discussion. Then we show how, starting from the familiar Crum–Kreĭn transformation taking the problem (5) into (6), we can construct a transformation of the expansion formulas obtained in Sec. 2 for $q(x)=q(\pi-x)$, h, $H=h$ into the corresponding equations for $r(x)=r(\pi-x)$, bypassing the constructions of Sec. 1. The new feature in Sec. 4 is the formulation of the problem of the product expansion of the solution (6) in the form of a boundary-value eigenvalue problem for an integro-differential operator which is a generalization of the L operator mentioned above to the case of a finite interval.[24,25] In Sec. 5 we show that, starting from the symplectic expansion equations which are the expansion of unity for the operator L on a finite interval, it is possible to construct the Hamiltonian formalism for the KdV equation in the periodic case, as was first suggested in Ref. 26. The equations obtained here in the canonical variables of Ref. 26 give, in the case of N-zone potentials, the familiar Dubrovin equations,[27] while in the limit $N\to\infty$ they give equations similar to those in Ref. 28. The approach to the KdV equation discussed in Sec. 5 is a generalization of the method of Refs. 6 and 8 to the periodic case.[25] As an aside, we discuss the relation between these constructions and those of Ref. 23. In Sec. 6 we give a scheme for constructing the spectral theory of Λ operators related to the regular Dirac operator on a finite interval, which is used as the basis for developing a Hamiltonian theory similar to that in Refs. 25 and 26 for the nonlinear Schrödinger equation in the periodic case.

Let us make a few remarks about the organization of this review. We note that even though throughout the text we consider only regular, self-adjoint problems, the basic constructions of the expansion formulas are presented in a form convenient for studying also non-self-adjoint problems, and also for generalization to the case of singular Sturm–Liouville operators. In formulating the fundamental statements we stick to the level of mathematical rigor which is usual in discussions of problems of this type. The desire to keep the constructions of Secs. 1–3 within the Hilbert space $L_2(0,\pi)$ is the reason why some of our constructions may appear rather far-fetched. As a rule, the proofs are just sketched, and references are given to the studies in which they are presented in detail. The information on Sturm–Liouville operators needed for this study will be given without references and can be found in Refs. 29 and 30. The citations in the review do not pretend to be complete. A detailed exposition of the methods and history of inverse problems can be found in, for example, Ref. 31. The applications of spectral theory discussed here mainly reflect the scientific interests of the authors and their own work. This review does not cover a number of interesting problems in theoretical physics in which the completeness of the products plays an important role, such as the stability of the IP,[32] the Cauchy problem for the linearized KdV equation,[33,34] symplectic structures and the Poisson-bracket hierarchy,[35] the problem of perturbation of the quantum-mechanical oscillator,[36] and the theory of Λ operators for more complicated spectral problems, including their multidimensional generalizations.[37,38]

1. FORMULAS FOR THE EXPANSION IN PRODUCTS OF THE SOLUTIONS OF TWO STURM–LIOUVILLE PROBLEMS

The Expansion Equations

Let us consider two self-adjoint Sturm–Liouville problems determined by the differential equations

$$l(q_j)y_j \equiv \left(-\frac{d^2}{dx^2}+q_j(x)\right)y_j=\lambda y_j, \quad 0\leqslant x\leqslant\pi, \quad j=1,2, \tag{7}$$

and the boundary conditions

$$y_j'(0)-h_jy_j(0)=0, \quad y_j'(\pi)+H_jy_j(\pi)=0, \tag{8}$$

where the real functions $q_j \in L_2(0,\pi) = L_2$ and the numbers h_j, H_j are finite. We use $\varphi_j(x,\lambda)$ and $\psi_j(x,\lambda)$ to denote the solutions of Eqs. (7) for which

$$\varphi_j(0,\lambda)=1, \quad \varphi_j'(0,\lambda)=h_j, \quad \psi_j(\pi,\lambda)=1, \quad \psi_j'(\pi,\lambda)=H_j.$$

Then the spectrum $\sigma_j=\sigma\{\hat{q}_j=[q_j(x),h_j,H_j]\}$ of the problem (7), (8) is defined as the set of zeros $\lambda_n^{(j)} = \lambda_n(\hat{q}_j)$, $n\geqslant 0$, and their characteristic functions

$$\begin{aligned}\omega_j(\lambda)=W(\psi_j,\varphi_j)&=\varphi_j'(\pi)+H_j\varphi_j(\pi)\\&=h_j\psi_j(0)-\psi_j'(0),\end{aligned} \tag{9}$$

with the Wronskian $W(f,g)=fg'-f'g$. For brevity we set $\lambda_n^{(j)} = \lambda_{2n+j}$, $n=0,1,\ldots$, $j=1,2$, and over the spectra σ_j we construct the sets $\sigma = \sigma_1 \cup \sigma_2$, $\sigma'' = \sigma_1 \cap \sigma_2$, $\sigma' = \sigma\setminus\sigma''$. Since for $n\to\infty$ we have the asymptote

$$\lambda_{2n+j}=n^2+\frac{2}{\pi}a_j+o(1);$$

$$a_j=h_j+H_j+\frac{1}{2}\int_0^\pi q_j(x)dx, \tag{10}$$

we shall assume without loss of generality that if $\sigma''\neq\phi$, then from $\lambda_{2n+1}=\lambda_{2m+2}$ it follows that $n=m$. We recall that the eigenfunctions are

$$\varphi_n(\hat{q}_j;x)=C_n(\hat{q}_j)\psi_n(\hat{q}_j;x),$$

$$C_{2n+j}=\varphi_j(\pi,\lambda_n(\hat{q}_j))=\psi_j^{-1}(0,\lambda_n(\hat{q}_j)), \tag{11}$$

and their norms are

$$\alpha_{2n+j}=\|\varphi_n(\hat{q}_j)\|_{L_2}^{-2}$$

$$=-(C_{2n+j}\dot{\omega}_j(\lambda_{2n+j}))^{-1}(\dot{}=\partial/\partial\lambda); \tag{12}$$

$$\beta_{2n+j}=\|\psi_n(\hat{q}_j)\|_{L_2}^{-2}=-C_{2n+j}\dot{\omega}_j^{-1}(\lambda_{2n+j}). \tag{13}$$

From the representation

$$\omega_j(\lambda)=\pi(\lambda_0^{(j)}-\lambda)\prod_{n\geqslant 1}n^{-2}(\lambda_n^{(j)}-\lambda) \tag{14}$$

it follows that

$$C_n(\hat{q}_j)=(-1)^n|C_n(\hat{q}_j)|. \tag{15}$$

We introduce the Hilbert spaces $\mathfrak{R}_2=L_2\times\mathbb{R}^2\ni\hat{f}=(f(x),\alpha,\beta)$ and $\mathfrak{R}_1=L_2\times\mathbb{R}\ni\tilde{f}=(f(x),\alpha)$ with the scalar products

$$(\hat{f}_1,\hat{f}_2)_2=\int_0^\pi f_1(x)f_2(x)dx+\alpha_1\alpha_2+\beta_1\beta_2$$

and

$$(\hat{f}_1,\hat{f}_1)_1=\int_0^\pi f_1(x)f_2(x)dx+\alpha_1\alpha_2,$$

respectively. Let

$$\mathfrak{M}=\left\{\hat{f}\in\mathfrak{R}_2\,\middle|\,\frac{1}{2}\int_0^\pi f(x)dx+\alpha+\beta=0\right\} \tag{16}$$

be the subspace $\mathfrak{R}_2$ of elements $\hat{f}$ orthogonal to the element $\hat{f}_0=(\frac{1}{2},1,1)$. As usual, we shall denote the scalar product in $L_2(0,\pi)$ by $(f,g)=\int_0^\pi f(x)g(x)dx$. Let

$$\Phi(x,\lambda)=\varphi_1(x,\lambda)\varphi_2(x,\lambda),$$

$$\hat{\Phi}(\lambda)=[\overline{\Phi}(x,\lambda),1,\Psi(\pi,\lambda)],$$

$$\Psi(x,\lambda)=\psi_1(x,\lambda)\psi_2(x,\lambda),$$

$$\hat{\Psi}(\lambda)=[\Psi(x,\lambda),\Psi(0,\lambda),1],$$

where $\hat{\Phi}$ and $\hat{\Psi}$ are treated as elements of $\mathfrak{R}_2$. We introduce the operator

$$J=\left(2\frac{d}{dx},\begin{pmatrix}1&0\\0&-1\end{pmatrix}\right),\quad J\hat{f}=(2f'(x),\alpha,-\beta). \tag{17}$$

We note that on a linear manifold

$$\mathfrak{R}_2^{(c)}\ni\hat{f}_c=(f(x)\in C_1(0,\pi),\ \alpha=f(0),\ \beta=f(\pi))$$

the operator J is skew-symmetric,

$$[\hat{f}_c,\hat{g}_c]_2=(\hat{f}_c,J\hat{g}_c)_2=-[\hat{g}_c,\hat{f}_c]_2,\quad J\hat{f}_c\in\mathfrak{M}. \tag{18}$$

In $\mathfrak{R}_1$ we construct the functions

$$\tilde{\Phi}(\lambda)=(\Phi(x,\lambda)-\tfrac{1}{2},\Phi(\pi,\lambda)-1),$$

$$\tilde{\Psi}(\lambda)=(2\Psi'(x,\lambda),-1); \tag{19}$$

$$\tilde{\Psi}^+(\lambda)=(\Psi(x,\lambda)-\tfrac{1}{2},\Psi(0,\lambda)-1),$$

$$\tilde{\Phi}^+(\lambda)=(-2\Phi'(x,\lambda),-1). \tag{20}$$

The following equations are satisfied:

$$\left.\begin{array}{ll}[\hat{f},\hat{\Psi}(\lambda)]_2=[\tilde{f},\tilde{\Psi}(\lambda)]_1; & \hat{f}\in\mathfrak{M},\ \tilde{f}=[f(x),\beta];\\ [\hat{f},\hat{\Psi}(\lambda)]_2=[\tilde{f},\tilde{\Psi}^+(\lambda)]_1; & \hat{f}\in\mathfrak{M},\ \tilde{f}=[f(x),\alpha].\end{array}\right\} \tag{21}$$

The well-known identity[5]

$$W[Y(x,\lambda)Z(x,\mu)]$$

$$=\frac{1}{\lambda-\mu}\frac{d}{dx}\prod_{j=1,2}W[y_j(x,\lambda),z_j(x,\mu)], \tag{22}$$

where $Y=y_1y_2$, $Z=z_1z_2$, y_j and z_j are solutions of Eqs. (7) for λ and μ, respectively, leads, owing to (9), to the relations

$$[\tilde{\Phi}(\lambda),\tilde{\Psi}(\mu)]_1=[\tilde{\Phi}^+(\lambda),\tilde{\Psi}^+(\mu)]_1$$

$$=[\hat{\Phi}(\lambda),\hat{\Psi}(\mu)]_2$$

$$=(\lambda-\mu)^{-1}[\Omega(\lambda)-\Omega(\mu)],$$

$$\Omega(\lambda)=\omega_1(\lambda)\omega_2(\lambda). \tag{23}$$

We define in $\mathfrak{R}_1$ the systems $\{\tilde{U}_n=(U_n(x),U_n^{(0)})\}_{n=1}^\infty$ and

$$\{\tilde{V}_n=(V_n(x),V_n^{(0)})\}_{n=1}^\infty$$

in the following manner: for $\lambda_{2n+j}\in\sigma'$ we set

$$\tilde{U}_{2n+j}=\dot{\Omega}^{-1}(\lambda_{2n+j})\tilde{\Phi}(\lambda_{2n+j}),\quad \tilde{V}_{2n+j}=\tilde{\Psi}(\lambda_{2n+j}), \tag{24}$$

and for $\lambda=\lambda_{2n+j}\in\sigma''$ we set

$$\tilde{U}_{2n+1}=2\ddot{\Omega}^{-1}(\lambda)\tilde{\Phi}(\lambda),\quad \tilde{U}_{2n+2}=2\ddot{\Omega}^{-1}(\lambda)\dot{\tilde{\Phi}}(\lambda),$$

$$\tilde{V}_{2n+1}=\dot{\tilde{\Psi}}(\lambda)-\dddot{\Omega}(\lambda)[3\ddot{\Omega}(\lambda)]^{-1}\tilde{\Psi}(\lambda),$$

$$\tilde{V}_{2n+2}=\tilde{\Psi}(\lambda). \tag{25}$$

The identities (23) lead to:

Lemma 1.1. The following biorthogonality relations are valid:

$$(\tilde{V}_n,\tilde{U}_m)_1=(\hat{V}_n,\hat{U}_m)_2=\delta_{nm};$$

$$(\tilde{V}_n^+,\tilde{U}_m^+)_1=(\hat{V}_n^+,\hat{U}_m^+)_2=\delta_{nm},\quad n,m\geqslant 1,$$

where the functions $\hat{U}_n=[U_n(x),U_n^{(0)},U_n^{(1)}]$ and $\hat{V}_n=[V_n(x),V_n^{(0)},V_n^{(1)}]$ are obtained from Eqs. (24) and (25) by replacing $\tilde{\Phi}$ by $\hat{\Phi}$ and $\tilde{\Psi}$ by $\hat{\Psi}$, the function $\tilde{U}_n^+$ is obtained by replacing $\tilde{\Phi}$ by $\tilde{\Psi}^+$, $\tilde{V}_n^+$ is obtained by replacing $\tilde{\Psi}$ by $\tilde{\Phi}^+$, and $\hat{U}_n^+$ and $\hat{V}_n^+$ are constructed from $\tilde{U}_n^+$ and $\tilde{V}_n^+$ just as $\hat{U}_n$ and $\hat{V}_n$ are constructed from $\tilde{U}_n$ and $\tilde{V}_n$. With each function $\hat{f}\in\mathfrak{M}$ we associate partial sums of the series

$$S_N(\hat{f}\in\mathfrak{M})=\sum_{n=1}^{2N}\hat{V}_n(\hat{f},\hat{U}_n)_2=\sum_{n=1}^{2N}\tilde{V}_n(\tilde{f},\tilde{U}_n)_1, \tag{26}$$

$$S_N^+(\hat{f}\in\mathfrak{M})=\sum_{n=1}^{2N}\hat{V}_n^+(\hat{f},\hat{U}_n^+)_2=\sum_{n=1}^{2N}\tilde{V}_n^+(\tilde{f},\tilde{U}_n^+)_1, \quad (27)$$

where in (26) $\tilde{f}=(f(x),\beta)$, and in (27) $\tilde{f}=(f(x),\alpha)$.

Theorem 1.1. For any $\tilde{f}=(f(x),\beta)\in\mathfrak{R}_1$ the sums $S_N(\tilde{f})$ (26) converge in the sense of the norm of $\mathfrak{R}_1$ to f, i.e.,

$$\lim_{n\to\infty}\left\|f(x)-\sum_{n=1}^{2N}V_n(x)(\tilde{f},\tilde{U}_n)_1\right\|_{L_2}=0, \quad (28)$$

$$\lim_{N\to\infty}\left|\beta-\sum_{n=1}^{2N}V_n^{(0)}(\tilde{f},\tilde{U}_n)_1\right|=0, \quad (29)$$

and the sums $\tilde{S}_N^+$ converge to $\tilde{f}$, i.e.,

$$\lim_{N\to\infty}\|f-S_N^+(\tilde{f}=(f(x),\alpha))\|_{\mathfrak{R}_1}=0. \quad (30)$$

This theorem is related to:

Theorem 1.2. For any $f(x)\in L_1(0,\pi)$, α, $\beta\in\mathbb{C}$ such that the condition (16) is satisfied uniformly on $0\leqslant x\leqslant\pi$ we have

$$-\int_x^\pi f(s)ds-2\beta$$

$$\equiv\int_0^x f(s)ds+2\alpha=\lim_{N\to\infty}\sum_{n=1}^{2N}W_n(x)(\hat{f},\hat{U}_n)_2, \quad (31)$$

$$-\int_x^\pi f(s)ds-2\beta$$

$$\equiv\int_0^x f(s)ds+2\alpha=\lim_{N\to\infty}\sum_{n=1}^{2N}W_n^+(x)(\hat{f},\hat{U}_n^+)_2, \quad (32)$$

where for $\lambda_n\in\sigma'$, $\lambda_{2n+2}\in\sigma''$, $W_n(x)=2\Psi(x,\lambda_n)$, $W_n^+(x)=-2\Phi(x,\lambda_n)$, and for $\lambda=\lambda_{2n+1}\in\sigma''$, $W_{2n+1}(x)=2\dot{\Psi}(x,\lambda)-2\ddot{\Omega}(\lambda)\,[3\dddot{\Omega}(\lambda)]^{-1}\Psi(x,\lambda)$, $W_{2n+1}^+(x)=-2\dot{\Phi}(x,\lambda)+2\ddot{\Omega}(\lambda)\,[3\dddot{\Omega}(\lambda)]^{-1}\Phi(x,\lambda)$.

Corollary 1. The expansion formulas (28)–(30) can be written as

$$\hat{f}=\sum_{n=1}^{\infty}\hat{V}_n(\hat{f},\hat{U}_n)_2,\quad \hat{f}=\sum_{n=1}^{\infty}\hat{V}_n^+(\hat{f},\hat{U}_n)_2,\quad \hat{f}\in\mathfrak{M}.$$

Remark 1. Substituting $x=\pi$ into (31), we obtain (29), and for $x=0$ we have $\alpha=\Sigma_{n=1}^\infty V_n^{(1)}(\hat{f},\hat{U}_n)_2$. Equation (28) is formally obtained from (31) by differentiating the latter with respect to x.

Remark 2. We define in the space $\mathfrak{R}_1$ the system of orthogonal projectors P_n by the equations

$$P_n\tilde{f}=\tilde{f}_n=\tilde{V}_{2n-1}(\tilde{f},\tilde{U}_{2n-1})_1+\tilde{V}_{2n}(\tilde{f},\tilde{U}_{2n})_1,\quad n\geqslant1.$$

Then Theorem 1.1 states that the sequence of two-dimensional subspaces $\mathfrak{R}_1^{(n)}=P_n(\mathfrak{R}_1)$ is a basis in $\mathfrak{R}_1$, i.e., any function $\tilde{f}\in\mathfrak{R}_1$ can be expanded uniquely in a series of the form $\tilde{f}=\Sigma_{n=1}^\infty f_n$, where $\tilde{f}_n\in\mathfrak{R}_1^{(n)}$. We see from the equations (57) obtained below that owing to the known asymptotic form $\beta_n=2/\pi+o(n^{-1})$, the system $\{\tilde{V}_n\}_{n=1}^\infty$ itself is in general not a basis in $\mathfrak{R}_1$.

As already noted, the operator J determines the symplectic structure in $\mathfrak{R}_2$ via the bilinear form $[\hat{f}_c,\hat{g}_c]_2$. The simplest symplectic expansion formula following directly from Theorems 1.1 and 1.2 is given by:

Theorem 1.3. Let the boundary-value problems (7) and (8) be isospectral, i.e., $\sigma'=\varnothing$, $\lambda_{2n+1}=\lambda_{2n+2}=\lambda_n$, $n\geqslant0$. With each λ_n we associate a pair of functions

$$\hat{P}_n=\beta_{2n+1}\hat{\Psi}(\lambda_n),$$

$$\hat{Q}_n=\tfrac{1}{2}\,[\beta_{2n+2}\dot{\hat{\Psi}}(\lambda_n)-\alpha_{2n+1}\dot{\hat{\Phi}}(\lambda_n)]. \quad (33)$$

Then: (I) The system $\{\hat{P}_n,\hat{Q}_n\}_{n=0}^\infty$ is a symplectic basis in $\mathfrak{M}$, i.e., for any $\hat{f}\in\mathfrak{M}$ we have the expansion formula

$$\hat{f}=\lim_{N\to\infty}\sum_{n=0}^{2N}\{J\hat{Q}_n(\hat{f},\hat{P}_n)_2-JJ\hat{P}_n(\hat{f},\hat{Q}_n)_2\}, \quad (34)$$

where $J\hat{P}_n$, $J\hat{Q}_n\in\mathfrak{M}$ and

$$[\hat{P}_n,\hat{Q}_m]_2=\delta_{nm},\quad [\hat{P}_n,\hat{P}_m]_2=[\hat{Q}_n,\hat{Q}_m]_2=0. \quad (35)$$

(II) If $\hat{f}$ satisfies the conditions of Theorem 1.2, we have uniformly in $0\leqslant x\leqslant\pi$ the expansion

$$\frac{1}{2}\left(\int_0^x-\int_x^\pi\right)f(s)ds+\alpha-\beta$$

$$=\sum_{n=0}^{\infty}2\{Q_n(x)(\hat{f},\hat{P}_n)_2-P_n(x)(\hat{f},\hat{Q}_n)_2\}, \quad (36)$$

where $P_n=\beta_{2n+1}\Psi(x,\lambda_n)$, $Q_n=1/2[\beta_{2n+2}\dot{\Psi}(x,\lambda_n)-\alpha_{2n+1}\dot{\Phi}(x,\lambda_n)]$. This theorem can be viewed as a special case of Theorem 1.4. We have singled it out mainly because its proof, in contrast to that of Theorem 1.4, is a direct consequence of Theorems 1.1 and 1.2. In fact, since for $\lambda_{2n+j}\in\sigma''$ from (II) we have $\hat{\Phi}(\lambda_n)=C_n^{(1)}C_n^{(2)}\hat{\Psi}(\lambda_n)$, then

$$\frac{1}{2}\sum_{j=1,2}\{W_{2n+j}(x)(\hat{f},\hat{U}_{2n+j})_2+W_{2n+j}^+(x)(\hat{f},\hat{U}_{2n+j})_2\}$$

$$=2Q_n(x)(\hat{f},\hat{P}_n)_2-2P_n(x)(\hat{f},\hat{Q}_n)_2.$$

Therefore, Eq. (36) is obtained by combining the expansions (31) and (32). The relations (35) are verified using the identity (22).

Generalization of Theorem 1.3 to the case where $\sigma'\neq\varnothing$ gives:

Theorem 1.4. Let us construct, using the boundary-value problems (7), (8), the systems $\{\hat{P}_n^{(j)},\hat{Q}_n^{(j)}\}_{n=0}^\infty$, $j=1$, 2 in the following manner:

$$\hat{P}_n^{(j)}=\dot{\Omega}^{-1}(\lambda_{2n+j})\hat{\Phi}(\lambda_{2n+j}),$$

$$\hat{Q}_n^{(j)}=\hat{\Psi}(\lambda_{2n+j}),\quad \lambda_{2n+j}\in\sigma'; \quad (37)$$

$$P_n=2\ddot{\Omega}^{-1}(\lambda_{(n)})\Phi(\lambda_{(n)}),$$

$$\hat{Q}_n^{(j)}=\widehat{\psi_j(x,\lambda_{(n)})\chi_{3-j}(x,\lambda_{(n)})},\quad \lambda_{(n)}\in\sigma'', \quad (38)$$

where $\chi_j(x,\lambda_{(n)})=\psi_j(x,\lambda_{(n)})-C_{2n+j}^{-1}\varphi_j(x,\lambda_{(n)})$ is the solution of Eq. (7) for $\lambda=\lambda_{(n)}$, $W(\chi_j,\varphi_j)=\dot{\omega}_j(\lambda_{(n)})\neq0$. Then for any $\hat{f}\in\mathfrak{M}$ we have the expansion

$$\hat{f}=\sum_{n=0}^{\infty}\{J\hat{Q}_n^{(j)}(\hat{f},\hat{P}_n^{(j)})_2-J\hat{P}_n^{(j)}(\hat{f},\hat{Q}_n^{(j)})_2\},\qquad(39)$$

where

$$[\hat{P}_n^{(j)},\hat{Q}_m^{(j)}]_2=\delta_{nm},\quad[\hat{P}_n^{(j)},\hat{P}_m^{(j)}]_2=[\hat{Q}_n^{(j)},\hat{Q}_m^{(j)}]_2=0.\quad(40)$$

Here, if $\hat{f}$ satisfies the conditions of Theorem 1.2, then

$$\frac{1}{2}\left(\int_0^x-\int_x^\pi\right)f(s)ds+\alpha-\beta$$

$$=\sum_{n=0}^{\infty}2\{Q_n^{(j)}(x)(\hat{f},\hat{P}_n^{(j)})_2-P_n^{(j)}(x)(\hat{f},\hat{Q}_n^{(j)})_2\},\qquad(41)$$

where the convergence of the series in (39) is understood, as in Theorem 1.1, in the sense of the norm of $\mathfrak{R}_2$, and in (41) it is uniform in $0\leqslant x\leqslant\pi$.

Remark 1. The expansion formula (41) remains valid for any $\hat{f}\in\mathfrak{R}_2$ for $x\in\Delta\subset(0,\pi)$. The condition $\hat{f}\in\mathfrak{M}$ ensures uniform convergence in $0\leqslant x\leqslant\pi$, and also the possibility of term-by-term differentiation with respect to x in L_2, i.e., the validity of (39). More precisely, if we use $S_N^{(j)}(\hat{f}\in\mathfrak{M})$ to denote the partial sum of the series (39) and

$$\sigma_N(f;x)=\frac{1}{\pi}\int_0^\pi f(y)dy+\sum_{n=1}^{N}\left\{\frac{2}{\pi}\cos 2nx\int_0^\pi f(y)\cos 2nydy+\frac{2}{\pi}\sin 2nx\int_0^\pi f(y)\sin 2nydy\right\}$$

to denote the partial sum of the Fourier series for $f\in L_2$, then

$$\lim_{N\to\infty}\|S_N^{(j)}(f;x)-\sigma_N(f;x)\|_{L_2}=0.$$

Let us sketch the proofs of Theorems 1.2 and 1.1, following Ref. 17. We introduce the function

$$G(x,y,\lambda)=\frac{2}{\Omega(\lambda)}\begin{cases}\Psi(x,\lambda)\Phi(y,\lambda), & 0\leqslant y\leqslant x;\\ \sum_{j=1,2}U_j(x,\lambda)U_{3-j}(y,\lambda) & \\ \quad-\Phi(x,\lambda)\Psi(y,\lambda), & x\leqslant y\leqslant\pi,\end{cases}\qquad(42)$$

where $U_j(x,\lambda)=\varphi_j(x,\lambda)\psi_{3-j}(x,\lambda)$, and then construct the functions

$$R(x,y,\lambda)=G(x,y,\lambda)-\tfrac{1}{2}G(x,0,\lambda),$$

$$S(x,\lambda)=R(x,\pi,\lambda)-R(x,0,\lambda).$$

The functions G, R, and S are entire functions of λ for any $x,y\in[0,\pi]$ and have poles of no higher than the second order for $\lambda_n\in\sigma$. Let $\lambda=k^2$ and $\tilde{c}_N$ be a circle in the k plane with radius $N-1/2$, with c_N being its image in the λ plane. With $f(x)\in L_1(0,\pi)$ and $\beta\in\mathbb{C}$ we consider the contour integral

$$I_N(x)=\frac{1}{2\pi i}\oint_{\tilde{c}_N}\left\{\int_0^\pi R(x,y,\lambda)f(y)dy+\beta S(x,\lambda)\right\}d\lambda$$

$$=\frac{1}{2\pi i}\oint_{\tilde{c}_N}\left\{\int_0^\pi R(x,y,k^2)f(y)dy+\beta S(x,k^2)\right\}kdk=I_{N,1}(x)+\beta I_{N,2}(x),$$

where the circles c_N and $\tilde{c}_N$ go around anticlockwise one time. From the residue theorem, using (11), we find

$$I_N(x)=\sum_{n=1}^{2N}W_n(x)(\tilde{f},\tilde{U}_n)_1.\qquad(43)$$

Let us now compute $I_N(x)$ directly using the contour $\tilde{c}_N$ for $N\to\infty$. Let $\Gamma(x,y,k^2)$ be the function obtained from R for $q_j\equiv0$, $h_j=H_j=0$. Since uniformly in $0\leqslant x\leqslant\pi$ for $|k|\to\infty$, $k=\sigma+i\tau$,

$$\varphi(x,k)=\cos kx+O\left(\frac{1}{k}e^{|\tau|x}\right),$$

$$\psi(x,k)=\cos k(\pi-x)+O\left(\frac{1}{k}e^{|\tau|(\pi-x)}\right),$$

where $\omega^{-1}(k^2)=(-k\sin k\pi)^{-1}[1+O(k^{-1})]$, $k\in\tilde{c}_N$, then uniformly in $0\leqslant x,y\leqslant\pi$ for $|k|\to\infty$, $k\in\tilde{c}_N$ we have the asymptotes $R(x,y,k^2)=\Gamma(x,y,k^2)+O(k^{-3})$, $S(x,k^2)=-2k^{-2}+O(k^{-3})$. From this we find, as usual (see, for example, Ref. 30), that uniformly in $x\in[0,\pi]$

$$\lim_{N\to\infty}I_N(x)=\lim_{N\to\infty}\frac{1}{2\pi i}\oint_{c_N}\left\{\int_0^\pi\Gamma(x,y,k^2)f(y)dy-\frac{2\beta}{k^2}\right\}kdk=-\int_x^\pi f(y)dy-2\beta.$$

Comparing this with (43), we obtain the expansion (31). The proof of Theorem 1.1 is based on the following two lemmas.

Lemma 1.2 (uniform convergence). For any function $f(x)\in L_1(0,\pi)$ we have

$$\lim_{N\to\infty}\left\|\sigma_N(f;x)-\sum_{n=1}^{2N}V_n(x)(f,U_n)\right\|_{L_2}=0,\qquad(44)$$

where

$$\sigma_N=(f;x)=\sum_{n=0}^{N-2}v_n(x)(f,u_n),\quad u_0(x)=1,$$

$$u_{2n-1}(k)=\cos 2nx,\quad u_{2n}(x)=x\sin 2nx(n\geqslant1),$$

$$v_0(x)=2\pi^{-2}(\pi-x),$$

$$v_{2n-1}(x)=4\pi^{-2}(\pi-x)\cos 2nx,$$

$$v_{2n}(x)=4\pi^{-2}\sin 2nx(n\geqslant1).$$

Proof. This is proved using the contour integral

$$I'_{N,1}(x)=\frac{1}{2\pi i}\oint_{c_N}\{R'_x(x,y,\lambda)f(y)dy\}d\lambda.$$

According to the residue theorem, $I'_{N,1}(x) = \Sigma_{n=1}^{2N}V_n(x) \times(f,U_n)$ and

$$J_N(x)=\frac{1}{2\pi i}\oint_{c_N}\left\{\int_0^\pi (x,y,\lambda)f(y)dy\right\}d\lambda=\sigma_N(f;x).$$

Furthermore, for $|k|\to\infty$, $k\in\tilde{c}_N$ we have the estimate

$$\begin{aligned}R_x(x,y,k^2)-\Gamma_x(x,y,k^2)\\ &=O(k^{-2}\exp[2|\tau|(y-x)])\\ &\quad+O(k^{-2}\exp[2|\tau|(x-y)])\\ &\quad+k^2\exp[2|\tau|(|\pi-2x|-\pi)],\\ &0\leqslant y\leqslant x,\quad x\leqslant y\leqslant\pi.\end{aligned}$$

Now to obtain (44) we need only note that from this estimate we have uniformly in x in any interval $\Delta\subset(0,\pi)$

$$\lim_{N\to\infty}|J_N(x)-I'_{N,1}(x)|=0$$

and

$$\lim_{N\to\infty}\overline{\sup_{0<x<\pi}}|J_N(x)-I'_N(x)|<\infty.$$

Lemma 1.3. For any function $f(x)\in L_2(0,\pi)$ we have

$$\lim_{N\to\infty}\left\|f(x)-\sum_{n=1}^{2N}V_n(x)(f,U_n)\right\|_{L_2}=0.$$

Proof. Owing to Lemma 1.2, the proof reduces to establishing the convergence

$$\lim_{N\to\infty}\left\|f(x)-\sum_{n=1}^{N}v_n(x)(f,u_n)\right\|_{L_2}=0. \tag{45}$$

Let us consider the following non-self-adjoint boundary-value problem:

$$y''+4\lambda y=0\ (0\leqslant x\leqslant\pi),\quad y(\pi)=0,\quad y'(0)=y'(\pi). \tag{46}$$

It can be verified directly that the function $\Gamma_x(x,y,\lambda)$ determines the Green function of this problem; its eigenvalues are $\mu_n = k_n^2$ ($= 4\lambda_n$), where $k_0 = 0$, $k_{2n-1} = k_{2n} = 2n$ $(n\geqslant1)$, and to each μ_n $(n\geqslant1)$ there corresponds a single eigenfunction $v_{2n}(x)$ and one associated function $v_{2n-1}(x)$. It is easily shown that the system $u_n(x)$ is biorthogonal to v_n in $L_2(0,\pi)$ and gives the system of eigenfunctions $u_0(x)$, $u_{2n-1}(x)$, $n\geqslant1$, and associated functions $u_{2n}(x)$ of the boundary-value problem conjugate to (46): $y''+4\lambda y=0$, $y'(0)=0$, $y(0)=y(\pi)$. It is well known that any system of eigenfunctions and associated functions of the problem (46) is complete in $L_1(0,\pi)$. The property of forming a basis in L_2 [i.e., Eq. (45)], which is important for our purposes, follows from the necessary and sufficient condition, obtained by Il'in in Ref. 39, for a given complete and minimal system of eigenfunctions and associated functions of an ordinary differential operator to form a basis in L_2. In our case this condition is easily verified by the inequality $\|v_n\|_{L_2}\|u_n\|_{L_2}\leqslant$ const $(n\geqslant1)$.

To complete the proof of Theorem 1.1, we still need to note that from (43) we have

$$I'_{N,2}(x)=\frac{1}{2\pi i}\oint_{c_N}S_x(x,\lambda)d\lambda=\sum_{n=1}^{2N}V_n(x)U_n^{(0)}.$$

Furthermore, owing to the estimate

$$\begin{aligned}S'_x(x,k^2)=O(k^{-2}[&e^{-2|\tau|x}+e^{-2|\tau|(\pi-x)}\\ &+e^{-2|\tau|(\pi-|\pi-2x|)}])\end{aligned}$$

and Jordan's lemma, we obtain uniformly in $x\in\Delta\subset(0,\pi)$

$$\lim_{N\to\infty}I'_{N,2}(x)=0,\quad \lim_{N\to\infty}\overline{\sup_{0\leqslant x\leqslant\pi}}|I'_{N,2}(x)|\leqslant\text{const}.$$

Therefore, $\lim_{N\to\infty}\|\Sigma_{n=1}^{2N}V_n(x)U_n^{(0)}\|_{L_2}=0$, which together with Lemmas 1.2 and 1.3 proves the expansion (28).

The expansions given in Theorem 1.4 are obtained using the function

$$G^{(j)}(x,y,\lambda)=\frac{-2}{\Omega(\lambda)}\begin{cases}\Phi(x,\lambda)\Psi(y,\lambda)-U_{3-j}(x,\lambda)U_j(y,\lambda),\\ \quad x<y<\pi;\\ -\Psi(x,\lambda)\Phi(y,\lambda)+U_j(x,\lambda)U_{3-j}(y,\lambda),\\ \quad 0<y<x,\end{cases} \tag{47}$$

as a result of computing, as in the proof of Theorems 1.1 and 1.2, the contour integral

$$I_N^{(j)}(f;x)=\frac{1}{2\pi i}\oint_{c_N}\left\{\int_0^x G^{(j)}(x,y,\lambda)f(y)dy+\alpha G^{(j)}(x,0,\lambda)+\beta G^{(j)}(x,\pi,\lambda)\right\}d\lambda.$$

We note that direct calculation of the integral $I_N^{(j)}(f,x)$ for $N\to\infty$ gives the uniform convergence, noted in Remark 1 to Theorem 1.4, with an ordinary Fourier series, which greatly simplifies the demonstration of the convergence in $L_2(0,\pi)$ $(\mathfrak{R}_2)$ for the series of partial sums in (39) compared with the proof of Theorem 1.1.

Theorem 1.2 leads directly to:

Lemma 1.4. The system $\{\tilde{U}_n\}_{n=1}^{\infty}$ is complete and minimal in the space $\mathfrak{R}_1$, i.e., if $(\tilde{f},\tilde{U}_n)_1 = 0$, $n\geqslant1$, then $f(x)=0$ and $\beta=0$. The minimality follows from the biorthogonality relations of Lemma 1.1. This leads to:

Lemma 1.5. The system $\{\hat{U}_n\}_{n=1}^{\infty}$ is complete and minimal in $\mathfrak{R}_2$, i.e., if $f(x)\in L_2(0,\pi)$, α, $\beta\in\mathbb{C}$,

$$\int_0^\pi f(x)\Phi(x,\lambda_n)dx+\alpha+\beta\Phi(\pi,\lambda_n)=0,\quad \lambda_n\in\sigma;$$

$$\int_0^\pi f(x)\dot{\Phi}(x,\lambda_n)dx+\beta\dot{\Phi}(\pi,\lambda_n)=0,\quad \lambda_n\in\sigma'';$$

then $\alpha=\beta=0$ and $f(x)=0$ almost everywhere.

Remark 1. The completeness and minimality in $\mathfrak{R}_2$ of any of the systems $\{\hat{U}_n^+\}$, $\{\hat{P}_n,\hat{Q}_n\}$, and $\{\hat{P}_n^{(j)},\hat{Q}_n^{(j)}\}$ are established in a completely analogous way, starting from the expansions given in Theorems 1.2–1.4.

Let us now consider in somewhat more detail the case of the boundary-value problems

$$y'' + (\lambda - q_j(x))y=0, \quad q_j(x)=q_j(\pi - x)\in L_2^{(s)}, \qquad (48)$$

$$y'(0) - h_j y(0)=0, \quad y'(\pi) + h_j y(\pi)=0, \qquad (49)$$

which are obtained from (7), (8) for $h_j = H_j$, $q_j\in L_2^{(s)} = \{f\in L_2 | f(x) = f(\pi - x)\}$, i.e., $\hat{q}_j = \hat{q}_j^{(s)}\in \mathfrak{R}_2^{(s)} = \{\hat{f}\in \mathfrak{R}_2 | f\in L_2^{(s)}, \alpha,\beta = \alpha\}$. It is well known (see, for example, Ref. 19) that the eigenfunctions in this case satisfy the equations

$$\varphi_j(x,\lambda_{2n+j})=(-1)^n\psi_j(x,\lambda_{2n+j}), \quad n\geqslant 0, \qquad (50)$$

and, therefore, their norms are

$$(\alpha_{2n+j}^{(s)})^{-1}=(\beta_{2n+j}^{(s)})^{-1}=(-1)^{n+1}\dot{\omega}_j(\lambda_{2n+j}).$$

The expansion formulas given above simplify considerably in this case. We shall illustrate this for the example of Theorem 1.4, with further applications in mind. Suppose that for some $N<\infty$ we have

$$\lambda_n^{(1)}\equiv\lambda_{2n+1}\neq\lambda_n^{(2)}\equiv\lambda_{2n+2}, \quad n=0,1,\dots,N,$$

$$\lambda_{2n+1}=\lambda_{2n+2}=\lambda_{(n)}, \quad n>N. \qquad (51)$$

For $\lambda_{2n+j}\in\sigma'$ we set

$$P_{n,j}^{(s)}(x)=\dot{\Omega}^{-1}(\lambda_n^{(j)})(\Phi(x,\lambda_n^{(j)}) - \tfrac{1}{4}[1+\Phi(\pi,\lambda_n^{(j)})]),$$

$$Q_{n,j}^{(s)}(x)=\Psi(x,\lambda_n^{(j)}) - \Phi(x,\lambda_n^{(j)}),$$

and for $\lambda_n^{(j)} = \lambda_{(n)}\in\sigma''$ we set

$$P_n^{(s)}(x)=2\ddot{\Omega}^{-1}(\lambda_{(n)})(\Phi(x,\lambda_{(n)}) - \tfrac{1}{2}), \qquad (52)$$

$$Q_{n,j}^{(s)}(x)=\chi_{3-j}(x,\lambda_{(n)})\psi_j(x,\lambda_{(n)}),$$

$$\chi_j(x)=(-1)^n\dot{\varphi}_j(x,\lambda_{(n)}) + \dot{\psi}(x,\lambda_{(n)}). \qquad (53)$$

Theorem 1.5. For any function $f\in L_2^{(s)}$ the following expansion formulas are valid:

$$f(x)=\lim_{N\to\infty}\sum_{n=0}^{N} 2Q_{n,j}^{(s)'}(x)(f,P_{n,j}^{(s)}),$$

$$\frac{1}{2}\left(\int_0^x - \int_x^\pi\right)f(y)dy=\sum_{n=0}^{\infty} 2Q_{n,j}^{(s)}(x)(f,P_{n,j}^{(s)}),$$

$$j=1,2, \qquad (54)$$

where

$$2(Q_{n,j}^{(s)'},P_{m,j}^{(s)})=\delta_{nm}, \quad j=1,2, \quad n,m\geqslant 0. \qquad (55)$$

Proof. These expansions are a direct consequence of Theorem 1.4 if we set

$$\mathfrak{M}^{(s)}=\left\{\hat{f}\in\mathfrak{R}_2^{(s)}\left|\frac{1}{2}\int_0^\pi f(x)dx + 2\alpha=0\right.\right\},$$

and we shall treat $f(x)$ as an element of $\mathfrak{M}^{(s)}$. Then from $\hat{q}^{(s)}\in\mathfrak{R}_2^{(s)}$ we have $Q_j^{(s)}(x,\lambda_{(n)}) = -Q_j^{(s)}(\pi - x,\lambda_{(n)})$ for $\lambda_{(n)}\in\sigma''$ and, therefore, $(\hat{f}^{(s)},\hat{Q}_{n,j}^{(s)}) = 0$, and for $\lambda_{2n+j}\in\sigma'$ we have $[\hat{f}^{(s)},\hat{\Phi}(\lambda_{2n+j})]_2 = [f^{(s)},\hat{\Phi}(\lambda_{2n+j})]_2$, since $[f,\Phi(\lambda)] = [f,\Psi(\lambda)]$, $\forall f\in L_2^{(s)}$. Equations (55) follow directly from (40).

Uniqueness theorems in inverse problems

The above expansion formulas can be used to obtain an elementary proof of the fundamental uniqueness theorems in inverse problems for a regular Sturm–Liouville operator. For this we first recall that from Eqs. (7) we have the identity

$$\frac{d}{dx}W[y_2(x,\lambda),y_1(x,\lambda)]=\Delta q(x)y_1(x,\lambda)y_2(x,\lambda),$$

$$\Delta q=q_1 - q_2. \qquad (56)$$

From this it follows that for the coefficients of the expansion of the function $\Delta\hat{q} = [q_1(x) - q_2(x), h_1 - h_2, H_1 - H_2]$ in the system $\{\hat{U}_n\}$ the following representations are valid:

$$\left.\begin{aligned}(\Delta\hat{q},\hat{U}_{2n+j})_2&=(-1)^{j-1}\beta_{2n+j}, \quad \lambda_{2n+j}\in\sigma',\\ (\Delta\hat{q},\hat{U}_{2n+1})_2&=0,\\ (\Delta\hat{q},\hat{U}_{2n+2})_2&=\beta_{2n+1} - \beta_{2n+2}, \quad \lambda_{2n+j}\in\sigma''.\end{aligned}\right\} \qquad (57)$$

Equation (57) and Lemma 1.5 lead directly to:

Theorem 1.6 (Marchenko uniqueness[40]). If for the boundary-value problems (7), (8) we have $\lambda_{2n+1} = \lambda_{2n+2}$, $\beta_{2n+1} = \beta_{2n+2}$, $n\geqslant 0$, then almost everywhere $q_1(x) = q_2(x)$, $h_1 = h_2$, $H_1 = H_2$.

Because of Eqs. (57), Theorem 1.2 makes it possible to find the explicit form of $\Delta\hat{q}$. More precisely, we have:

Theorem 1.7. For any two Sturm–Liouville problems (7), (8), where $a(\hat{q}_1) = a(\hat{q}_2)$, $a(\hat{q})$ are determined from (10), we have

$$\int_x^\pi (q_2(s) - q_1(s))ds + 2(H_2 - H_1)$$

$$= \sum_{\lambda_n\in\sigma''}(\beta_{2n+1} - \beta_{2n+2})W_{2n+2}(x)$$

$$+ \sum_{\lambda_n\in\sigma'}\sum_{j=1,2}(-1)^{j-1}\beta_{2n+j}W_{2n+j}(x).$$

Here, if $\lambda_n(\hat{q}_1) = \lambda_n(\hat{q}_2)$, $n\geqslant 0$, i.e., the problems (7), (8) are isospectral, then

$$\int_x^\pi (q_2(s) - q_1(s))ds + 2(H_2 - H_1)$$

$$= \sum_{n=0}^{\infty}(-1)^n(\exp[l_n(\hat{q}_1)]$$

$$- \exp[l_n(\hat{q}_2)])\dot{\omega}^{-1}(\lambda_n)W_{2n+2}(x),$$

where

$$l_n(\hat{q})=\ln|C_n|, \quad C_n(\hat{q})=(-1)^n|C_n|. \qquad (58)$$

Corollary 1. If $\lambda_n(\hat{q}_1) = \lambda_n(\hat{q}_2)$, $l_n(\hat{q}_1) = l_m(\hat{q}_2)$, $n\geqslant 0$, then $\hat{q}_1 = \hat{q}_2$.

Remark. We recall that the quantities $\{\lambda_n(\hat{q}),\beta_n(\hat{q}),n\geqslant 0\}$ determine the spectral function

$\rho(\hat{q};\lambda)$ of the problem (7), (8). From the representations (13) for $\beta_n(\hat{q})$ and (14) for $\omega(\hat{q};\lambda)$ it follows that the function $\rho(\hat{q};\lambda)$ is determined uniquely from the quantities $\{\lambda_n(\hat{q}),l_n(\hat{q}),n \geqslant 0\}$. In Ref. 22 it was suggested that, in addition to $\lambda_n(\hat{q})$, the quantities $l_n(\hat{q})$ instead of $\beta_n(\hat{q})$ be treated as the spectral characteristics for the problems (7) and (8). This makes it somewhat easier to study the corresponding IP, which is done in detail in Sec. 2 below.

It is more complicated to establish:

Theorem 1.8 (Ref. 18). Let there be given the boundary-value problems $\hat{q}_1 = [q_1(x),h_1,H_1]$ and $\tilde{q}_1 = [q_1(x),h_1,\tilde{H}_1]$, $H_1 \neq \tilde{H}_1$, for which

$$\sigma(\hat{q}_1)=\{\lambda_n(\hat{q}_1)\}_{n=0}^{\infty}, \quad \sigma(\hat{\tilde{q}}_1)=\{\lambda_n(\hat{\tilde{q}}_1)\}_{n=0}^{\infty},$$

and together with them the boundary-value problems $\hat{q}_2 = [q_2(x),h_2,H_2]$ and $\hat{\tilde{q}}_2 = [q_2(x),h_2,\tilde{H}_2]$, $H_2 \neq \tilde{H}_2$, for which

$$\lambda_{2n+1} \equiv \lambda_n(\hat{q}_1)=\lambda_n(\tilde{q}_2), \quad n=0,1,... \tag{59}$$

$$\lambda_n(\hat{\tilde{q}}_1)=\lambda_n(\hat{q}_2) \equiv \lambda_{2n+2}, \quad n=N,N+1,...,$$

$$\lambda_n(\hat{\tilde{q}}_1) \neq \lambda_n(\hat{q}_2), \quad n=0,1,...,N-1. \tag{60}$$

Then

$$\int_x^{\pi} [q_2(s)-q_1(s)]ds+\tilde{H}_2-H_1$$

$$=-\sum_{n=0}^{N-1} \frac{\omega(\tilde{q}_1,\lambda_{2n+2})}{\omega(\hat{q}_1,\lambda_{2n+2})} \beta_{2n+2} W_{2n+2}(x), \tag{61}$$

where $W_{2n+2}(x)$ is defined as in Theorem 1.2. [Here for $\lambda_n(\hat{\tilde{q}}_1) \neq \lambda_n(\hat{q}_2)$, $\omega(\hat{\tilde{q}}_1,\lambda_{2n+2}) \neq 0$, and from $H_2 \neq \tilde{H}_2$ it follows that $\sigma''=0$ and, therefore, $\omega(\hat{q}_1;\lambda_{2n+2}) \neq 0$.]

Proof. We first note that (59) and (60) lead to the equations

$$h_1-h_2+H_1-\tilde{H}_2+\frac{1}{2}\int_0^{\pi} (q_1(x)-q_2(x))dx=0,$$

$$H_1-\tilde{H}_2=\tilde{H}_1-H_2.$$

Now let the boundary-value problems (7) and (8) be used to construct the system $\{\tilde{U}_n\}$ ($\sigma''=\varnothing$). Since the conditions (59) and (60) are equivalent to the equations $\omega_1(\lambda_{2n+1}) = \tilde{\omega}_2(\lambda_{2n+2}) = \omega_2(\lambda_{2n+2}) = 0$, $n \geqslant 0$, $\tilde{\omega}_1(\lambda_{2n+2}) = 0$, $n \geqslant N$, then from the identity (56) we obtain $(\Delta\tilde{q},\tilde{U}_{2n+1})_1=0, n \geqslant 0$,

$$(\Delta\tilde{q},\tilde{U}_{2n+2})_1=0, \quad n \geqslant N,$$

$$[\Delta\tilde{q},\Phi(\lambda_{2n+2})]_1=\varphi_2(\pi,\lambda_{2n+2})\omega_1(\lambda_{2n+2}),$$

$$n=0,1,...,N-1.$$

where $\Delta\tilde{q} = [q_1(x) - q_2(x),H_1 - \tilde{H}_2]$, which, owing to the expansion (31), gives the representation (61).

Corollary 1 (the Borg uniqueness theorem[10]). From Eqs. (59), (60) with $N=0$, $H_2 \neq \tilde{H}_2$ it follows that $q_1(x)=q_2(x), h_1=h_2, H_1=\tilde{H}_2, \tilde{H}_1=H_2$.

Remark 1. Bearing in mind the Borg uniqueness theorem formulated above, Theorem 1.7 can be viewed as the analog of Theorem 1.6 for the case of an N-dimensional perturbation of the initial spectral characteristics $\lambda_n(\hat{q}_1)$, $\lambda_n(\hat{\tilde{q}}_1)$, $n \geqslant 0$, which uniquely determine q_1, h_1, H_1, $\tilde{H}_1$. It follows from the biorthogonality relations of Lemma 1.1 that the corresponding perturbation $\Delta\tilde{q}$ is also N-dimensional, which corresponds to the generalized degeneracy of the Gel'fand–Levitan equation established in Ref. 19, where a different proof of (61) is given. The analog of Theorem 1.8 for the boundary-value problems (48), (49) is:

Theorem 1.9. Let the boundary-value problems (48), (49), where the spectra satisfy the condition (51), be used to construct the function $\Delta\tilde{q}^{(s)} = (\Delta q = q_1 - q_2,\Delta h = h_1 - h_2)$. Then

$$\frac{1}{2}\int_0^x \Delta q^{(s)}(y)dy+\Delta h$$

$$=\sum_{r=0}^{N} \alpha_{2n+j}(\hat{q}_j^{(s)})Q_{n,j}^{(s)}(x), \quad j=1,2. \tag{62}$$

Corollary 1 (the Borg uniqueness theorem[10]). If $\lambda_n(\hat{q}_1^{(s)})=\lambda_n(\hat{q}_2^{(s)})$, $n \geqslant 0$, then $\hat{q}_1^{(s)}=\hat{q}_2^{(s)}$.

Remark 1. Equations of the type (62) were first obtained in Ref. 18. Owing to the biorthogonality relations (55), the dimensionality of the representation (62) is equal to that of the perturbation of the spectrum in (51). In contrast to Theorem 1.7, which can also be obtained directly from the corresponding Gel'fand–Levitan equation in the IP, Theorems 1.8 and 1.9 are not a direct consequence of this equation. In Ref. 19, where the generalized degeneracy of the Gel'fand–Levitan equation was proved under the conditions of Theorem 1.9, the following refinement of the corresponding equation from Ref. 18 was obtained:

$$\frac{1}{2}\int_0^x \Delta q(y)dy+\Delta h$$

$$=\sum_{n=0}^{N} -\alpha_n^{(1)}\{[\varphi_2(\pi,\lambda_n^{(1)})-(-1)^n]c_2(x,\lambda_n^{(1)})$$

$$+[\varphi_2'(\pi,\lambda_n^{(1)})+h_1\varphi_2(\pi,\lambda_n^{(1)})$$

$$+(-1)^n h_2]s_2(x,\lambda_n^{(1)})\}\varphi_1(x,\lambda_n^{(1)}), \tag{63}$$

where $s_2(x,\lambda)$, $c_2(x,\lambda)$ are the solutions of Eqs. (48) with $j=2$ for which $s_2(\pi,\lambda) = c_2'(\pi,\lambda) = 0$, $s_2'(\pi,\lambda) = c_2(\pi,\lambda) = 1$. It can be verified directly that

$$Q_{2n+1}^{(s)}(x)=-\varphi_1(x,\lambda_n^{(1)})\{[\varphi_2(\pi,\lambda_n^{(1)})$$

$$-(-1)^n]c_2(x,\lambda_n^{(1)})+[\varphi_2'(\pi,\lambda_n^{(1)})$$

$$+(-1)^n h_2]s_2(x,\lambda_n^{(1)})\}.$$

Therefore, Eq. (62) differs from (63) by the sum

$$\sum_{n=0}^{N} -h_1\alpha_n^{(1)}\omega_2^{-1}(\lambda_n^{(1)})N_{2n+1}(x),$$

$$N_{2n+1}(x)=\Phi(x,\lambda_{2n+1})$$

$$-(-1)^n\varphi_2(\pi,\lambda_{2n+1})\Psi(x,\lambda_{2n+1}),$$

which must be equal to zero.

2. THE CONTINUOUS ANALOG OF THE NEWTON METHOD IN STURM–LIOUVILLE INVERSE SCATTERING PROBLEMS

In this section we briefly describe the methods of constructing the continuous analog of the Newton method (CANM) for the solution of the IP in the formulations of Sec. 1. These methods are based on the general idea of Gavurin[14] about the solution of the operator equation

$$f(v)=y_* \tag{64}$$

by means of the Cauchy problem

$$v_t=-[f'(v)]^{-1}(f(v)-y_*),v(0)=v_0,\quad 0\leqslant t<\infty. \tag{65}$$

where $f: X\to Y$ is a nonlinear operator, X, Y are B spaces, $f'(v)$ is the Fréchet derivative, $[f'(v)]^{-1}$ is its inverse operator, and y_* is a given element of Y. The desired solution v_* of Eq. (64) is obtained as the limit $\|v(t,v_0)-v_*\|=0$, $t\to\infty$, of the solution $v(t,v_0)$ of Eq. (65), usually referred to as the CANM. An important feature of (65) is the fact that $v(t,v_0)$ is the inverse image of the segment:

$$f[v(t)]=f(v_0)\exp(-t)+y_*[1-\exp(-t)],$$
$$0\leqslant t<\infty. \tag{66}$$

Let us formulate the IP in the form of the operator equation (64). With the Sturm–Liouville problems (7), (8), i.e., the element $\hat{q}=(q(x),h,H)\in\mathfrak{R}_2$, we associate $\tilde{q}=(q(x),H)\in\mathfrak{R}_1$ and consider the operator

$$f_{\rm I}(\tilde{q})=\left\{\lambda_n(\tilde{q}),\beta_n(\tilde{q}),h+H+\frac{1}{2}\int_0^\pi q(x)dx=a_*,\ n\geqslant 0\right\}, \tag{67}$$

where a_* is a fixed number, $\lambda_n=\lambda_n(\hat{q})$ are the eigenvalues, and $\beta_n=\beta_n(\hat{q})$ are normalization constants (13). It is well known that for any $\tilde{q}\in\mathfrak{R}_1$

$$\lambda_m(\tilde{q})<\lambda_n(\tilde{q})(m<n),\quad \sum_{n=0}^\infty\left(\lambda_n(\tilde{q})-n^2-\frac{2}{\pi}a_*\right)^2<\infty, \tag{68}$$

$$\beta_n(\tilde{q})>0(n\geqslant 0),\quad \sum_{n=1}^\infty n^2\left(\beta_n(\tilde{q})-\frac{2}{\pi}\right)^2<\infty. \tag{69}$$

The converse is also true.

Theorem 2.1 (Ref. 11). If the conditions (68), (69) are satisfied for the two sequences of numbers λ_n^*, β_n^*, $n\geqslant 0$, there exists the single element $\tilde{q}_*=(q_*(x),H_*)\in\mathfrak{R}_1$ ($h_*=a_*-H_*-1/2\int_0^\pi q_*(x)dx$), for which $\lambda_n^*=\lambda_n(\tilde{q}_*)$, $\beta_n^*=\beta_n(\tilde{q}_*)$.

This inverse problem, referred to henceforth as IP I, is related to the problem of determining the Sturm–Liouville operator from the two spectra, hereafter referred to as IP II. Let us consider the two boundary-value problems (7), (8), where $\hat{q}_j=(q(x),h,H_j)\in\mathfrak{R}_2$ ($H_1<H_2$).

Using $\hat{q}_j$, where with certain fixed a_j^* ($a_1^*<a_2^*$) we have $h+H_j+1/2\int_0^\pi q(x)dx=a_j^*$, $j=1,2$, we construct the element $\tilde{q}=[q(x),H_1]\in\mathfrak{R}_1$, and denote the corresponding eigenvalues by $\lambda_{2n+j}(\tilde{q})$, $n\geqslant 0$, $j=1,2$. It is well known that

$$\lambda_{2n+1}<\lambda_{2n+2}<\lambda_{2(n+1)+1}<(n\geqslant 0),$$

$$\sum_{n=0}^\infty\left\{\left(\lambda_{2n+j}(\tilde{q})-n^2-\frac{2}{\pi}a_j^*\right)^2+n^2\left(\lambda_{2n+2}-\lambda_{2n+1}-\frac{2}{\pi}(a_2^*-a_1^*)\right)^2\right\}<\infty \tag{70}$$

and the converse is also true.

Theorem 2.2 (Ref. 41). If the sequences of numbers λ_{2n+j}^*, $n\geqslant 0$, $j=1,2$, satisfy the conditions (70), there exists the single element $\tilde{q}_*=(q_*(x),H_1^*)\in\mathfrak{R}_1$, for which $\lambda_{2n+j}^*=\lambda_{2n+j}^*(\tilde{q}_*)$, $n\geqslant 0$, $j=1,2$, where $h_*=a_1^*-H_1^*-1/2\int_0^\pi q_*(x)dx$, $H_2^*=H_1^*+(a_2^*-a_1^*)$.

The corresponding IP II operator is

$$f_{\rm II}(\tilde{q})=\{\lambda_{2n+j}(\tilde{q})a_j(\tilde{q})=a_j^*,\ \tilde{q}=(q(x),H_1)\}_{n=0}^\infty. \tag{71}$$

It follows from Theorems 2.1 and 2.2 that the equations

$$f_{\rm I}(\tilde{q})=y_{\rm I}^*=\{\lambda_n^*,\beta_n^*\}_{n=0}^\infty,$$

$$f_{\rm II}(\tilde{q})=y_{\rm II}^*=\{\lambda_{2n+j}^*,j=1,2\}_{n=0}^\infty \tag{72}$$

have unique solutions in the space $\mathfrak{R}_1$. The next two theorems give the explicit form of Eq. (65) for solving Eqs. (72).

Theorem 2.3. Let the following set be constructed according to (72):

$$M_N(y_{\rm I}^*)=\{\tilde{q}=(q(x),H)\in\mathfrak{R}_1\,|\,\lambda_n(\tilde{q})=\lambda_n^*,\beta_n(\tilde{q})=\beta_n^*,$$
$$n\geqslant N+1\}, \tag{73}$$

where N is a finite number. Then for any initial value $\tilde{q}_0=[q_0(x),H_0]\in M_N(y_{\rm I}^*)$ the Cauchy problem

$$\tilde{q}_t(t)=\sum_{n=0}^N\{\beta_n[\tilde{q}(t)]\{\lambda_n^*-\lambda_n[\tilde{q}(t)]\tilde{V}_{2n+1}[\tilde{q}(t)]\}+\{\beta_n^*-\beta_n[\tilde{q}(t)]\}\tilde{V}_{2n+2}[\tilde{q}(t)]\},$$

$$\tilde{q}(t)=[q(x,t),H(t)],\quad \tilde{q}(0)=\tilde{q}_0, \tag{74}$$

where $\tilde{V}_{2n+1}=\dot{\Psi}(\lambda_n)$, $\tilde{V}_{2n+2}=\Psi(\lambda_n)$, $\Psi(\lambda)$ is determined by Eq. (19) with $\hat{q}_1=\hat{q}_2=\hat{q}(t)$, $h(t)=a^*-H(t)-1/2\int_0^\pi q(x,t)dx$, has the unique solution $\tilde{q}(t)\in M_N(y_{\rm I}^*)$, for which

$$\lim_{t\to\infty}\|\tilde{q}(t)-\tilde{q}_*\|_{\mathfrak{R}_1}=0(f(\tilde{q}_*)=y_{\rm I}^*). \tag{75}$$

Theorem 2.4. Let $y_{\rm II}^*$ (72) be used to construct the set

$$M_N(y_{\rm II}^*)=\{\tilde{q}=[q(x),H_1]\in\mathfrak{R}_1\,|\,\lambda_{2n+j}(\tilde{q})=\lambda_{2n+j}^*,$$
$$n\geqslant N+1,j=1,2\}. \tag{76}$$

Then for any initial value $\tilde{q}_0=[q_0(x),H_{1,0}]\in M_N(y_{\rm II}^*)$ the Cauchy problem is

$$\tilde{q}_t(t)=\frac{1}{a_2^*-a_1^*}\sum_{n=0}^N\sum_{j=1,2}(-1)^j\{\lambda_{2n+j}^*$$

$$-\lambda_{2n+j}[\tilde{q}(t)]\}V_{2n+j}[\tilde{q}(t)],$$

$$q(t)=[q(x,t),H_1(t)],$$

$$\tilde{q}(0)=\tilde{q}_0 \in M_N(y^*_{\rm II}),\quad 0<t<\infty, \qquad (77)$$

where the functions $\tilde{V}_{2n+j}[\tilde{q}(t)]$ are determined by Eqs. (24),

$$\hat{q}_j(t)=(q(x,t),h(t),H_j(t)),h(t)+H_j(t)$$

$$+\frac{1}{2}\int_0^\pi q(x,t)dx=a_j^*,$$

has a unique solution $\tilde{q}(t)\in M_N(y^*_{\rm II})$, for which the limit $\lim_{t\to\infty}\tilde{q}(t)=\tilde{q}_*=(q_*,H_{1,*}),f(\tilde{q}^*)=y^*_{\rm II}$ exists.

Before proving these theorems we give two lemmas which are of independent interest.

Lemma 2.1. At any point $\hat{q}\in\mathfrak{R}_2$ there exist the differentials

$$\frac{d}{d\varepsilon}\lambda_n(\hat{q}+\varepsilon\hat{f})\Big|_{\varepsilon=0,\hat{f}\in\mathfrak{R}_2}$$

$$=\left(\frac{\partial\lambda_n}{\partial\hat{q}},\hat{f}\right)_2,\frac{d}{d\varepsilon}\beta_n(\hat{q}+\varepsilon\hat{f})\Big|_{\varepsilon=0}=\left(\frac{\partial\beta_n}{\partial\hat{q}},\hat{f}\right)_2, \qquad (78)$$

where the gradients are

$$\frac{\partial\lambda_n}{\partial\hat{q}}=\left(\frac{\partial\lambda_n}{\partial q(x)},\frac{\partial\lambda_n}{\partial h},\frac{\partial\lambda_n}{\partial H}\right)=\alpha_n\hat{\Phi}(\lambda_n)=\beta_n\hat{\Psi}(\lambda_n), \qquad (79)$$

$$\frac{\partial\beta_n}{\partial\hat{q}}=\left(\frac{\partial\beta_n}{\partial q(x)},\frac{\partial\beta_n}{\partial},\frac{\partial\beta_n}{\partial H}\right)$$

$$=\frac{1}{\dot{\omega}^2(\lambda_n)}\left\{\dot{\hat{\Phi}}(\lambda_n)-\frac{\ddot{\omega}(\lambda_n)}{\dot{\omega}(\lambda_n)}\hat{\Phi}(\lambda_n)\right\}, \qquad (80)$$

where $\hat{\Phi}(\lambda_n)=[\Phi(x,\lambda_n)=\varphi^2(x,\lambda_n),1,\Phi(\pi,\lambda_n)]$, α_n and β_n are determined by (12) and (13), and $\omega(\lambda)$ is given by (9).

Proof. This is generally well known.[17,20,22,42] Following, for example, the proof of Lemma 5.1 given below, we find for $f(x)\in L_2(0,\pi)$ the equations

$$\frac{d}{d\varepsilon}\lambda_n(q+\varepsilon f)\Big|_{\varepsilon=0}=\alpha_n[f,\Phi(\lambda_n)],$$

$$\frac{\partial}{\partial\varepsilon}\beta_n(q+\varepsilon f)\Big|_{\varepsilon=0}=\dot{\omega}^{-2}(\lambda_n)[f,\dot{\Phi}(\lambda_n)$$

$$-\ddot{\omega}(\lambda_n)\dot{\omega}^{-1}(\lambda_n)\Phi(\lambda_n)],$$

determining, for fixed h and H, the part of the increase of $\lambda_n(\varepsilon)$ and $\beta_n(\varepsilon)$ linear in ε when $q(x)$ is replaced by $q(x)+\varepsilon f(x)$. Furthermore, since $\partial\psi(x,\lambda)/\partial h=0$, $\partial\varphi(x,\lambda)/\partial H=0$, by differentiating the equations $\omega[\lambda_n(h)]\equiv h\psi[0,\lambda_n(h)]-\psi'[0,\lambda_n(h)]=0$ and $\omega[\lambda_n(H)]\equiv\varphi'[\pi,\lambda_n(H)]+H\varphi[\pi,\lambda_n(H)]=0$, with respect to h and H, respectively, we find the partial derivatives $\partial\lambda_n/\partial h=\alpha_n$, $\partial\lambda_n/\partial H=\beta_n$. From this, using for β_n the expressions, following from (9) and (11),

$$\beta_n(h)=-\psi^{-1}[0,\lambda_n(h)](h\dot{\psi}[0,\lambda_n,(h)]$$

$$-\dot{\psi}'[0,\lambda_n(h)])^{-1}$$

and

$$\beta_n(H)=-\varphi[\pi,\lambda_n(H)](\dot{\varphi}'[\pi,\lambda_n(H)]$$

$$+H\dot{\varphi}[\pi,\lambda_n(H)])^{-1},$$

we obtain

$$\frac{\partial\beta_n}{\partial h}=-\frac{\ddot{\omega}(\lambda_n)}{\dot{\omega}(\lambda_n)},$$

$$\frac{\partial\beta_n}{\partial H}=\frac{1}{\dot{\omega}^2(\lambda_n)}\left\{\dot{\Phi}(\pi,\lambda_n)-\frac{\ddot{\omega}(\lambda_n)}{\dot{\omega}(\lambda_n)}\Phi(\pi,\lambda_n)\right\},$$

which is what we wanted to prove.

Noting that for $\hat{q}_j=[q(x),h,H_j]$ we have

$$\dot{\Omega}(\lambda_{2n+j})=(-1)^j(H_2-H_1)\alpha_{2n+j}^{-1}$$

$$=(-1)^j\frac{2}{\pi}(a_2-a_1)\alpha_{2n+j}^{-1},$$

we obtain:

Corollary 1. For $H_1\neq H_2$ the gradients are

$$\frac{\partial\lambda_{2n+j}}{\partial\hat{q}_j}=\left(\frac{\partial\lambda_n^{(j)}}{\partial q(x)},\frac{\partial\lambda_n^{(j)}}{\partial h},\frac{\partial\lambda_n^{(j)}}{\partial H_j}\right)$$

$$=\frac{(-1)^j\dot{\Omega}(\lambda_n^{(j)})}{H_2-H_1}\hat{\Phi}(\lambda_{2n+j}).$$

Furthermore, since the differential of the functional $a(\hat{q})$ (10) is

$$da(\hat{q})\hat{f}=\frac{d}{d\varepsilon}a(\hat{q}+\varepsilon\hat{f})\Big|_{\varepsilon=0}=\frac{1}{2}\int_0^\pi f(x)dx+\alpha+\beta,$$

we find, taking into account the equations in (21), the following:

Corollary 2. At any point $\tilde{q}=[q(x),H]\in\mathfrak{R}_1$ the differential of the operator $f_{\rm I}(\tilde{q})$ (67) has the form

$$df_{\rm I}(\tilde{q})\tilde{f}=\{\beta_n^{-1}(\tilde{f},\tilde{U}_{2n+1})_1,(\tilde{f},\tilde{U}_{2n+2})_1,\tilde{f}$$

$$=[f(x),\beta]\in\mathfrak{R}_1,\}\quad n\geqslant 0,$$

where the functions $\tilde{U}_{2n+j}$ are determined by (79), (80) with the subsequent restriction (21), and the differential of the operator $f_{\rm II}(\tilde{q})$ (71) has the form

$$df_{\rm II}(\tilde{q})\tilde{f}=\{(-1)^j(H_2-H_1)^{-1}(\tilde{f},\tilde{U}_{2n+j})_1,$$

$$j=1,2,n\geqslant 0\},$$

where $\tilde{U}_{2n+j}$ are determined by (24) with $\hat{q}_j=(q(x),h,H_j)$. From this, reducing the expansion formula (28), (29) obtained in Theorem 1.1 to the case $\hat{q}_1=\hat{q}_2=\hat{q}$ and $\hat{q}_j=[q(x),h,H_j]$, $H_1\neq H_2$, we immediately find the inverses of the operators $df_{\rm I}(\dot{q})\tilde{f}$ and $df_{\rm II}(\tilde{q})\tilde{f}$, respectively. More precisely, we have:

Lemma 2.2. The inverse of the operator $df_{\rm I}(\tilde{q})\tilde{f}$ is given by the inversion formula

$$\tilde{f}=\sum_{n=0}^{\infty}\beta_n\tilde{V}_{2n+1}(\tilde{f},\tilde{U}_{2n+1})_1$$

$$+\widetilde{V}_{2n+2}(\widetilde{f},\widetilde{U}_{2n+2})_1,\quad \widetilde{f}\in\mathfrak{R}_1,$$

where $\widetilde{V}_{2n+j}$ are defined as in Theorem 2.3, and the inverse of the operator $df_{\rm II}(\widetilde{q})\widetilde{f}$ is given by

$$\widetilde{f}=\sum_{n=0}^{\infty}\sum_{j=1,2}(H_1-H_2)^{-1}\widetilde{V}_{2n+j}(\widetilde{f},\widetilde{U}_{2n+j})_1,\quad \widetilde{f}\in\mathfrak{R}_1,$$

where $\widetilde{V}_{2n+j}$ are defined as in Theorem 2.4.

Proof. The proofs of Theorems 2.3 and 2.4 are quite similar, and we need only give that of Theorem 2.3. We consider the functions

$$\lambda_n(t)=\lambda_n(\widetilde{q}_0)e^{-t}+\lambda_n^*(1-e^{-t}),$$

$$\beta_n(t)=\beta_n(\widetilde{q}_0)e^{-t}+\beta_n^*(1-e^{-t})+\beta_n^*(1-e^{-t}), \tag{81}$$

where $n\geqslant 0$, $0\leqslant t<\infty$, $\widetilde{q}_0\in M_N(y_I^*)$ (73). It follows from Theorem 2.1 and the definition of $M_N(y_I^*)$ that these functions uniquely determine a one-parameter (in t) family of boundary-value problems $\hat{q}(t)=[q(x,t),\ h(t),\ H(t)]$, where $h(t)+H(t)+1/2\int_0^\pi q(x,t)dx=a_*$, for which $\lambda_n(t)=\lambda_n[\hat{q}(t)]$, $\beta_n(t)=\beta_n[\hat{q}(t)]$, and a_* is found using y_I^* from the asymptote (10). Therefore, differentiating (81) with respect to t and taking into account Lemma 2.1, we obtain the system of equations

$$\{\widetilde{q}_t,\widetilde{U}_{2n+1}[\widetilde{q}(t)]\}_1=\beta_m[\widetilde{q}(t)]\{\lambda_n^*-\lambda_n[\widetilde{q}(t)]\};$$

$$(\widetilde{q}_t,\widetilde{U}_{2n+2}[\widetilde{q}(t)])_1=\beta_n^*-\beta_n[\widetilde{q}(t)],$$

$$\widetilde{q}(t)=[q(x,t),H(t)]. \tag{82}$$

From this and owing to Lemma 2.2 it follows that $\widetilde{q}(t)$ is the unique solution of Eq. (74). [We note that, owing to the biorthogonality relations of Lemma 1.2, Eq. (74) leads to (82), which, owing to Lemma 2.1, can be written as $\lambda_{n,t}=[\lambda_n^*-\lambda_n(\widetilde{q}_0)]e^{-t}$, $\beta_{n,t}=[\beta_n^*-\beta_n(\widetilde{q}_0)]e^{-t}$. Integrating over t, we arrive at the system (81), which is equivalent to the first integral of (66) for Eq. (74).]

Remark 1. The restrictions (73) and (76) on the regions of the initial values for the Cauchy problems (74) and (77) play an important role in the construction of numerical methods for solving the IP I and IP II, since they ensure the stability of the corresponding algorithms.[20] Using the estimates from Ref. 29, it can be shown that for $N=\infty$ in (73) and (76), $M_N(y_I^*)=M_N(y_{II}^*)=\mathfrak{R}_1$. This assumption is naturally consistent with the unique solvability of the IP I and IP II in the entire $\mathfrak{R}_1$ and with the convexity of the regions of values of the operators $f_{\rm I}(\widetilde{q})$ and $f_{\rm II}(\widetilde{q})$. The conditions $a[\hat{q}(0)]=a^*$ and $a[q_j(0)]=a_j^*$ give those first integrals of Eqs. (74), (77) for which the series for $N\to\infty$ converge in the sense of the norm of $\mathfrak{R}_1$, because from (81) it follows that, owing to the asymptote (10), $a[\hat{q}(t)]=a[\hat{q}(0)]=a_*$, $0\leqslant t<\infty$. Theorem 2.4 can be viewed as a nontrivial generalization of the iteration process proposed in Ref. 43 for solving the corresponding inverse problem.

Remark 2. The solutions of Eqs. (74) and (77) can be written out explicitly. The simplest example here is obtained if in (74) we set $\lambda_n(\widetilde{q}_0)=\lambda_n^*$ $(n\geqslant 0)$, $\beta_n(\widetilde{q}_0)=\beta_n^*$ $(n\neq m,\ n\geqslant 0)$, where $\beta_m^*>0$ is an arbitrary number. Then, using the Gel'fand–Levitan equation, we find that the solution to (74) is given by the equations

$$q(x,t)=q_0(x)-2\frac{d^2}{dx^2}\ln\left[1+(\beta_m(\widetilde{q}_0)-\beta_m^*)\times(1-e^{-t})\int_x^\pi \psi_m^2(\widetilde{q}_0;s)ds\right];$$

$$H(t)=H_0+(\beta_m(\widetilde{q}_0)-\beta_m^*)(1-e^{-t}).$$

V. Daskalov has shown that an example of a solution of Eq. (77) can be obtained by setting $N=1$ in Theorem 1.8, i.e., $\lambda_m(\hat{\ }q_1)\neq\lambda_m(\widetilde{q}_2)$. Then

$$q_2(x)=q_1(x)-2\frac{d^2}{dx^2}\ln W\{\varphi[q_1;x,\lambda_m(\hat{\widetilde{q}}_1)],\ \psi[q_1;x,\lambda_m(\hat{q}_2)]\},$$

$$H_2(H_1)=H_1(\widetilde{H}_1)+[\lambda_m(\hat{q}_2)-\lambda_m(\hat{\widetilde{q}}_1)]\times(\widetilde{H}_1-H_1)^{-1}.$$

Let us now consider the differential properties of the operator

$$g(\hat{q})=\{\lambda_n(\hat{q}),l_n(\hat{q}),\ n\geqslant 0,\ \hat{q}\in\mathfrak{R}_2\}, \tag{83}$$

where $l_n(\hat{q})$ are determined by (58). The operator g, as was pointed out in Ref. 22, is a more sensitive indicator of the behavior of the spectral characteristics according to which $\hat{q}$ is reconstructed, in the sense that from (50) we have only $\alpha_n(\hat{q}^{(s)})=\beta_n(\hat{q}^{(s)})$, whereas $l_n(\hat{q}^{(s)})=0$, $n\geqslant 0$.

Theorem 2.5. We differentiate the operator g at any point $\hat{q}\in\mathfrak{R}_2$:

$$dg(\hat{q})\hat{f}=\left\{\left(\frac{\partial\lambda_n}{\partial\hat{q}},\hat{f}\right)_2,\left(\frac{\partial l_n}{\partial\hat{q}},\hat{f}\right)_2,\quad n\geqslant 0,\hat{f}\in\mathfrak{R}_2\right\}, \tag{84}$$

and if $\hat{q}\in\mathfrak{R}_2^{(s)}$, i.e., if we have the problem (48), (49), the differential is

$$dg(\hat{q}^{(s)})\hat{f}^{(s)}=\left\{\left(\frac{\partial\lambda_n}{\partial\hat{q}^{(s)}},\hat{f}^{(s)}\right)_2,\quad n\geqslant 0,\hat{f}^{(s)}\in\mathfrak{R}_2^{(s)}\right\}, \tag{85}$$

where $\partial\lambda_n/\partial\hat{q}=\hat{P}_n$, $\partial l_n/\partial\hat{q}=\hat{Q}_n$, and $\hat{P}_n$ and $\hat{Q}_n$ are determined by (33) for $\hat{q}_1=\hat{q}_2=\hat{q}$.

Here for any $\hat{f}\in\mathfrak{M}$ (16) the inverse of the operator (84) is given by

$$\hat{f}=\sum_{n=0}^{\infty}\left\{J\frac{\partial\lambda_n}{\partial\hat{q}}\left(\hat{f},\frac{\partial l_n}{\partial\hat{q}}\right)_2-J\frac{\partial l_n}{\partial\hat{q}}\left(\hat{f},\frac{\partial\lambda_n}{\partial\hat{q}}\right)_2\right\}, \tag{86}$$

where J is determined by (17), and the inverse of the operator (85) for $\hat{f}^{(s)}\in\mathfrak{R}^{(0)}{}_2^{(s)}$ is given by

$$f(x)=\sum_{n=0}^{\infty}Q_n^{(s)\prime}(x)(f,P_n^{(s)}),\quad f\in L_2^{(s)}, \tag{87}$$

where the functions $P_n^{(s)}$, $Q_n^{(s)}$ are given by (52) and (53) for $\hat{q}_1^{(s)}=\hat{q}_2^{(s)}=\hat{q}^{(s)}$.

Proof. The gradient $\partial l_n/\partial\hat{q}$ is obtained from Eq. (80) and the analogous expression

$$\partial\alpha_n/\partial\hat{q}=\dot{\omega}^{-2}(\lambda_n)\{\dot{\hat{\Phi}}(\lambda_n)-\ddot{\omega}(\lambda_n)\dot{\omega}^{-1}(\lambda_n)\Phi(\lambda_n)\},$$

since $l_n(\hat{q})=2^{-1}\ln(\beta_n\alpha_n^{-1})$. The equations $(\partial l_n/\partial\hat{q}^{(s)},\hat{f}^{(s)})_2=0$ follow from (50), as already stated in the proof of Theorem 1.5. Equations (86) and (87) are a direct consequence of Theorems 1.3 and 1.5.

Remark. If, following Ref. 22, we introduce into $\mathfrak{R}_2$ the isospectral manifold $M(\hat{q}^*)=\{\hat{q}\in\mathfrak{R}_2|\lambda_n(\hat{q})=\lambda_n(\hat{q}^*), n\geqslant 0\}$, then from (86) we obtain the explicit form of the tangent space $T_{\hat{q}}M(\hat{q}^*)$ at the point $\hat{q}\in M(\hat{q}^*)$. The condition $f\in\mathfrak{M}$ (16) is satisfied, owing to the asymptote (10). The properties of the fluxes $\hat{q}_t=J\partial\lambda_n/\partial\hat{q}$, given in Ref. 22, follow from the relations (35). The explicit form of $\hat{q}(t)$ (Ref. 22) essentially coincides with the solution of Eq. (74) indicated in Remark 2. Substituting $M_{N=\infty}(y_1^*)=M(\hat{q}^*)$ into (73) and using the familiar technique for solving the Gel'fand–Levitan equation, we can obtain an explicit solution for $N\geqslant 1$ of Eq. (74) with $\hat{q}\in M(\hat{q}^*)$; a form convenient for taking the limit $N\to\infty$ is given in Ref. 22.

3. THE INVERSE PERIODIC STURM–LIOUVILLE PROBLEM. DIFFERENTIAL PROPERTIES OF THE CRUM–KREIN TRANSFORMATION AND APPLICATIONS

1. The Inverse Periodic Sturm–Liouville Problem

Let us consider the two Sturm–Liouville problems

$$l(r_j)y_j\equiv\left(-\frac{d^2}{dx^2}+r_j(x)\right)y_j=\lambda y_j,\quad 0\leqslant x\leqslant\pi;\tag{88}$$

$$y_j(0)=y_j(\pi)=0,\quad j=1,2,\tag{89}$$

with real potentials $r_j(x)\in L_2(0,\pi)$. Here we give the theorem analogous to Theorem 1.4 for the boundary-value problems (88), (89). On the basis of it we shall show how the fundamental uniqueness theorem can be proved in the inverse periodic Sturm–Liouville problem, and we shall also obtain a representation for finite-zone potentials which will be important for the later constructions (see Sec. 5).

We use $g_j(x,\lambda)$ and $h_j(x,\lambda)$ to denote the solutions of (88) for which

$$g_j(0,\lambda)=h_j(\pi,\lambda)=0,\quad g_j'(0,\lambda)=h_j'(\pi,\lambda)=1,\tag{90}$$

and let

$$\chi_j(\lambda)=g_j(\pi,\lambda)=-h_j(0,\lambda)\equiv W(g_j,h_j)\tag{91}$$

be the characteristic functions of the problems (88) and (89). Their spectra are $\sigma_j=\sigma(r_j)=\{\nu_n^{(j)}=\nu_{2n+j}|\chi_j(\nu_{2n+j})=0, n\geqslant 1\}$, where the eigenvalues are

$$\nu_n^{(j)}=n^2+\frac{1}{\pi}\int_0^\pi r_j(x)dx+o(1),\quad n\to\infty.\tag{92}$$

As before, let $\sigma=\sigma_1\cup\sigma_2$, $\sigma''=\sigma_1\cap\sigma_2$, $\sigma'=\sigma\backslash\sigma''$, where if $\nu_{2n+1}=\nu_{2m+2}$ we shall assume that $n=m$. We also recall that

$$g_j(x,\nu_n^{(j)})=S_{2n+j}h_j(x,\nu_n^{(j)}),$$

$$S_{2n+j}=g_j'(\pi,\nu_n^{(j)})=(h_j'(0,\nu_n^{(j)}))^{-1};\tag{93}$$

$$\gamma_{2n+j}=\|g_{2n+j}\|_{L_2}^{-2}=\frac{1}{S_{2n+j}\dot{\chi}_j(\nu_{2n+j})},$$

$$\delta_{2n+j}=\|h_{2n+j}\|_{L_2}^{-2}=S_{2n+j}/\dot{\chi}_j(\nu_{2n+j}),\tag{94}$$

$$\chi_j(\lambda)=\pi\prod_{n\geqslant 1}n^{-2}(\nu_n^{(j)}-\lambda)\tag{95}$$

and, therefore, $S_n^{(j)}=(-1)^n|S_n^{(j)}|$, $n\geqslant 1$. Let

$$G(x,\lambda)=g_1(x,\lambda)g_2(x,\lambda),\quad H(x,\lambda)=h_1(x,\lambda)h_2(x,\lambda),$$

$$X(\lambda)=\chi_1\chi_2.\tag{96}$$

We introduce the systems of functions $\{P_n^{(j)},Q_n^{(j)}\}$ as follows: for $\nu_{2n+j}\in\sigma'$ we set

$$P_n^{(j)}(x)=2X^{-1}(\nu_{2n+j})G(x,\nu_{2n+j}),$$

$$Q_n^{(j)}(x)=H(x,\nu_{2n+j})\tag{97}$$

and with each $\nu_{2n+1}=\nu_{2n+2}=\nu_{(n)}$ we associate the functions

$$P_n(x)=4\ddot{X}^{-1}(\nu_{(n)})G(x,\nu_{(n)}),$$

$$Q_n^{(j)}(x)=h_j(x,\nu_{(n)})z_{3-j}(x,\nu_{(n)}),\tag{98}$$

where $z_j(x,\nu_{(n)})=\dot{h}_j(x,\nu_{(n)})-S_{2n+j}^{-1}\dot{g}_j(x,\nu_{(n)})$ is the solution of the equation $lz_j=\nu_{(n)}z_j$, $W(z_j,h_j)=\chi_j(\nu_n)\neq 0$.

Theorem 3.1. Let the systems $\{P_n^{(j)},Q_n^{(j)}\}$, $j=1,2$, be constructed in the manner described above from the boundary-value problems (88), (89). Then for any, possibly complex-valued, function $f\in L_1$ we have the expansion formulas

$$\frac{1}{2}\left(\int_0^x-\int_x^\pi\right)f(s)ds$$

$$=\sum_{\nu_{2n+j}\in\sigma_j}\{Q_n^{(j)}(x)(f,P_n^{(j)})-P_n^{(j)}(x)(f,Q_n^{(j)})\}\tag{99}$$

and if

$$f\in L_2^{(0)}=\left\{f\in L_2\Big|\int_0^\pi f(x)dx=0\right\},\tag{100}$$

these formulas admit term-by-term differentiation with respect to x, i.e.,

$$f(x)=\lim\sum_{\nu_{2n+j}\in\sigma_j}\{Q_h^{(j)\prime}(x)(f,P_n^{(j)})-P_n^{(j)\prime}(x)(f,Q_n^{(j)})\},\tag{101}$$

where the convergence in (99) is uniform in $x\in\Delta\subset(0,\pi)$, and in (101) it is uniform in the sense of the norm of L_2. Here

$$[P_n^{(j)},Q_m^{(j)}]=\delta_{nm},\quad[P_n^{(j)},P_m^{(j)}]=[Q_n^{(j)},Q_m^{(j)}]=0,\tag{102}$$

where

$$[f,g]=(f,Dg),\quad D=d/dx.\tag{103}$$

Proof. The proof of this theorem is completely analogous to that of Theorem 1.4, and we omit it.

The identity (56) and the id ıtity

$$Y(x,\lambda)[2\lambda - s(x)] + 2y_1'(\ldots,\lambda)\,{}_{2}(x,\lambda) + W[y_2(x,\lambda),y_1(x,\lambda)] \int_{x_0}^{x} \Delta(t)dt - Y(x_0,\lambda)(2\lambda - s(x_0)) - 2y_1'(x_0,\lambda)y_2'(x_0,\lambda) = \int_{x_0}^{x} w(t)Y(t,\lambda)dt,$$

where

$$w(x) = -s_x(x) + \Delta(x)\int_0^x \Delta(t)dt,$$

$$s = r_1 + r_2,\quad \Delta = r_1 - r_2,$$

together with Eqs. (91), (93), and (94) lead to:

Theorem 3.2. Suppose that we have functions $\Delta(x)$, $w(x)\in L_2^{(0)}$. Then we have the expansions

$$\Delta(x) = \sum_{\nu_{2n+j}\in\sigma_j'} 2(-1)^{3-j}\{\delta_{2n+j}H'(x,\nu_{2n+j}) - \gamma_{2n+j}G'(x,\nu_{2n+j})\} + \sum_{\nu_{(n)}\in\sigma''} 2(S_{2n+1} - S_{2n+2})\dot{\chi}_j^{-1}(\nu_{(n)})H'(x,\nu_{(n)}), \tag{104}$$

$$w(x) = \sum_{\nu_{2n+1}\in\sigma'} \{4\delta_{2n+1}\chi_2^{-1}(\nu_{2n+1})[g_2'(\pi,\nu_{2n+1}) - S_{2n+1}^{-1}]H'(x,\nu_{2n+1}) - 4\gamma_{2n+1}\chi_2^{-1}(\nu_{2n+1})[S_{2n+1} - h_2'(0,\nu_{2n+1})]G'(x,\nu_{2n+1})\} + \sum_{\nu_{(n)}\in\sigma''} \{4\dot{\chi}_1^{-1}(\nu_{(n)})\dot{\chi}_2^{-1}(\nu_{(n)})(S_{2n+1}S_{2n+2} - 1)Q_n^{(1)'}(x) + 2[S_{2n+2}^{-1}2\dot{g}_2'(\pi,\nu_{(n)}) - S_{2n+1}^{-1}h_2'(0,\nu_{(n)})]P_n'(x)\}. \tag{105}$$

From (104) we immediately obtain:

Corollary 1 (the Borg uniqueness theorem[10]). The eigenvalues $\nu_n(r)$ and normalization constants $\gamma_n(r)$ or $\delta_n(r)$ uniquely determine the boundary-value problems (88), (89) with $r(x)\in L_2$.

Let us now consider, along with the boundary-value problems (88), (89), the following boundary-value problems with periodic and antiperiodic boundary conditions:

$$y'' + [\lambda - r_j(x)]y = 0,\quad y(0) = y(\pi),\quad y'(0) = y'(\pi), \tag{106}$$

$$y'' + [\lambda - r_j(x)]y = 0,\quad y(0) = -y(\pi),\quad y'(0) = -y'(\pi), \tag{107}$$

where the potentials $r_j(x)$ are assumed to be smooth periodic (with period π) functions, i.e.,

$$r\in\widetilde{C}_n = \{f(x)\in C_n(0,\pi)\,|\,f^{(k)}(0) = f^{(k)}(\pi),\ 0\leqslant k\leqslant n\}. \tag{108}$$

We use $\Delta_j(\lambda) = g_j'(\pi,\lambda) + c_j(\pi,\lambda)$ to denote their Hill discriminants, where $c_j(x,\lambda)$ is the solution of Eq. (88) for which $c_j(0,\lambda) = 1$, $c_j'(0,\lambda) = 0$. It is well known (see, for example, Ref. 29) that the spectrum of the boundary-value problem (106) is determined by the zeros of the equation $\Delta_j(\lambda) = 2$: $\mu_0^{(j)}$, $\mu_{2n+1}^{(j)}$, $\mu_{2n+2}^{(j)}$, $n = 1,3\ldots$, and the spectrum of the problem (107) is determined by the zeros of $\Delta_j(\lambda) = -2$: $\mu_{2n+1}^{(j)}$, $\mu_{2n+2}^{(j)}$, $n = 0,2\ldots$. The eigenvalues $\nu_n^{(j)}$, $\mu_n^{(j)}$ are ordered as follows:

$$\mu_0^{(j)} < \mu_1^{(j)} \leqslant \nu_1^{(j)} \leqslant \mu_2^{(j)} < \mu_3^{(j)} \leqslant \nu_2^{(j)} \leqslant \mu_4^{(j)} < \ldots.$$

The potential $r_j(x)$ is termed an N-zone potential if the equations

$$\mu_{2n-1}^{(j)} = \nu_n^{(j)} = \mu_{2n}^{(j)},\quad n = N+1,\ N+2,\ldots$$

are satisfied, which is equivalent to the conditions

$$\dot{\Delta}_j(\nu_n^{(j)}) = 0,\quad S_n^{(j)} = (-1)^n,\quad n\geqslant N+1. \tag{109}$$

We now note that the equation

$$h(x,\lambda) = c(\pi,\lambda)g(x,\lambda) - g(\pi,\lambda)c(x,\lambda) \tag{110}$$

leads to $h'(0,\lambda) = c(\pi,\lambda)$, which gives the following representation for the Hill discriminant:

$$\Delta_j(\lambda) = g_j'(\pi,\lambda) + h_j'(0,\lambda). \tag{111}$$

Theorem 3.3. Let the potentials $r_j(x)$ be N-zone potentials with the Hill discriminants

$$\Delta_1(\lambda) = \Delta_2(\lambda) = \Delta(\lambda). \tag{112}$$

Then: (I) If $\nu_{2n+1}\neq\nu_{2n+2}$ for $n = 1,2,\ldots,N$, then

$$\Delta(x) = \sum_{n=1}^{N} 2(-1)^{3-j}\{\delta_{2n+j}H'(x,\nu_{2n+j}) - \gamma_{2n+j}G'(x,\nu_{2n+j})\}, \tag{113}$$

$$w(x) = \sum_{n=1}^{N} \{4\delta_{2n+1}\chi_2^{-1}(\nu_{2n+1})[g_2'(\pi,\nu_{2n+1}) - S_{2n+1}^{-1}]H'(x,\nu_{2n+1}) - 4\gamma_{2n+1}\chi_2^{-1}(\nu_{2n+1})\times[S_{2n+1} - h_2'(0,\nu_{2n+1})]G'(x,\nu_{2n+1})\}.$$

(II) For $\nu_{2n+1} = \nu_{2n+2} = \nu_{(n)}$, $n = 1,2,\ldots,N$,

$$\Delta(x) = \sum_{n=1}^{N} \{2\dot{\chi}^{-1}(\nu_{(n)})(S_{2n+1} - S_{2n+2})H'(x,\nu_{(n)})\}; \tag{114}$$

$$w(x) = \sum_{n=1}^{N} \{4\dot{\chi}^{-2}(\nu_{(n)})(S_{2n+1}S_{2n+2} - 1)Q_n^{(1)'}(x) + 2[S_{2n+2}^{-1}\dot{g}_2'(\pi,\nu_{(n)}) - S_{2n+1}^{-1}\dot{h}_2'(0,\nu_{(n)})]P_n'(x)\}. \tag{115}$$

Proof. This follows directly from Theorem 3.2 if we take into account the fact that (109) and (111) imply that all the terms in (104) and (105) for $n>N$ are equal to zero.

Remark. We use $\tilde{c}(x,\lambda)$ to denote the solution of (88) for which $\tilde{c}(\pi,\lambda)=1$, $\tilde{c}'(\pi,\lambda)=0$. Then we have $g_2(x,\lambda) = g_2'(\pi,\lambda)h_2(x,\lambda) + g_2(\pi,\lambda)\tilde{c}_2(x,\lambda)$. Substituting this into (113) for $j=2$, we obtain the equation derived by Levitan.[29] Equation (114) can be viewed as following from (113) in the limit $\nu_{2n+1} = \nu_{2n+2}$, $n=1,2,...,N$. The representations (113)–(115) remain valid in the case of infinite-zone potentials, where, as is clear from the proof given above, the terms on the right-hand sides for which (109) holds (for some n) are equal to zero.

Here we give two corollaries of Theorem 3.3 which are important for what follows.

Corollary 1 (the uniqueness theorem for the inverse periodic Sturm–Liouville problem; see, for example, Refs. 28 and 31). If the condition (112) is satisfied, then $r_1(x) = r_2(x)$ if and only if $\nu_n^{(1)} = \nu_n^{(2)} = \nu_n$, $S_n^{(1)} = S_n^{(2)} = S_n$, $n\geqslant 1$, with

$$S_n - S_n^{-1} = \sqrt{\Delta^2(\nu_n) - 4}, \quad S_n + S_n^{-1} = \Delta(\nu_n), \tag{116}$$

where the sign in front of the radical $(+,-)$ is determined from the equation

$$\sqrt{\Delta^2(\nu_n)} - 4 = 2S_n - \Delta(\nu_n)\,(S_n = (-1)^n|S_n|). \tag{117}$$

Remark. Following Ref. 26, as the spectral characteristics determining $r(x)$ we shall take the quantities

$$\nu_n(r), f_n(r) = -2\ln|S_n|, (S_n = (-1)^n|S_n|), \tag{118}$$

which, owing to the Borg uniqueness theorem (see Corollary 1 of Theorem 3.2) and the representation (95), uniquely determine $r(x)\in L_2(0,\pi)$ and also the Hill discriminant $\Delta(\lambda)$ (see, for example, Ref. 28). The conditions of periodicity and smoothness of $r(x)$ impose definite additional constraints on ν_n and f_n. A general description of the infinite-zone potentials $r(x)\in C_n$ $(n\geqslant 0)$ has been obtained in the well-known study of Ref. 44 in terms of the asymptotic expansions of the spectra μ_n $(n\to\infty)$, i.e., the zeros $\Delta(\lambda) = \pm 2$. For finite-zone potentials this problem is easily solved by:

Corollary 2. For the potential $r(x)\in\tilde{C}_1$ to admit the representation

$$r_x(x) = \sum_{n=1}^{N} \{\dot{\chi}^{-2}(\nu_n)(S_n^2 - 1)Q_n'(x) - S_n\dot{\Delta}(\nu_n)P_n'(x)\}, \tag{119}$$

where P_n and Q_n are determined by (98) (for $r_1 = r_2$), it is necessary and sufficient that $r(x)$ be an N-zone potential. Here $r(x)\in\tilde{C}_\infty(0,\pi)$.

Proof. If $r(x)$ is an N-zone potential, the representation (119) is obtained directly from (115) for $r_1 = r_2$. Conversely, from (105) (for $r_1 = r_2$), owing to (102), we obtain

$$(r_x,P_n) = \dot{\chi}^{-2}(\nu_n)(S_n^2 - 1), \quad (r_x,Q_n) = S_n\dot{\Delta}(\nu_n), \tag{120}$$

and therefore the representation (119) is valid if and only if $S_n^2 - 1 = \dot{\Delta}(\nu_n) = 0$ for $n\geqslant N+1$, which, owing to (109), is equivalent to the condition that $r(x)$ is an N-zone potential. The condition $r(x)\in\tilde{C}_\infty(0,\pi)$, which is well known,[44] can be obtained from $r(x)\in\tilde{C}_1(0,\pi)$ owing to Theorem 4.8.

Remark. After the pioneering study of Novikov,[45] where, in particular, the well-known finite-zone criterion was given, Its and Matveev[46] obtained an explicit expression for N-zone potentials. A different representation for N-zone potentials, which is somewhat closer to the methods of solving the IP described in Sec. 2, can be found in Ref. 47. Further applications of the representation (119) are given in Sec. 5.

2. Differential properties of the Crum–Kreĭn transformation and applications

In addition to the problem

$$y'' + [\lambda - q(x)]y = 0,$$

$$y'(0) - hy(0) = y'(\pi) + Hy(\pi) = 0$$

let us consider the problem

$$y'' + [\lambda - r(x)]y = 0, \quad y(0) = y(\pi) = 0,$$

where the potentials $\hat{q} = [q(x),h,H]$ and r are related by the equation

$$\mathscr{F}(\hat{q}) \overset{\text{def}}{=} r(x) = q(x) - 2\frac{d^2}{dx^2}\ln\varphi_0(x),$$

$$\varphi_0(x) = \varphi(\hat{q};x,\lambda_0), \tag{121}$$

usually referred to as the Crum–Kreĭn transformation. Then, as is well known,[48] the characteristic function is

$$\hat{\omega}(\hat{q};\lambda) = (\lambda_0 - \lambda)\chi(r,\lambda), \tag{122}$$

where ω and χ are determined by (9) and (91), respectively. Here the solutions $g(x,\lambda)$ and $h(x,\lambda)$ are expressed in terms of the solutions $\varphi(x,\lambda)$ and $\psi(x,\lambda)$ as

$$g(x,\lambda) = \frac{W[\varphi_0(x),\varphi(x,\lambda)]}{(\lambda_0 - \lambda)\varphi_0(x)},$$

$$h(x,\lambda) = \frac{W[\varphi_0(x),\psi(x,\lambda)]}{(\lambda_0 - \lambda)\varphi_0(x)}. \tag{123}$$

From this it follows that

$$\lambda_n(\hat{q}) = \nu_n(r), \quad l_n(\hat{q}) = -\frac{1}{2}f_n(r), \quad n\geqslant 1, \tag{124}$$

where $l_n(\hat{q})$ and $f_n(r)$ are determined by (58) and (118).

The main purpose of this section is to use Eqs. (123) as the basis for constructing a representation for the expansion formulas (87) corresponding to the problem (88), (89) with $r(x) = r(\pi - x)$ and vice versa. For this[49] we first study the differential properties of the operator (123), which are of independent interest.

Theorem 3.4. The transformation (121) as an operator $\mathscr{F}(\hat{q}):\mathfrak{R}_2 \to L_2$ is differentiable at any point $\hat{q}\in\mathfrak{R}_2$, and its differential is

$$\frac{d}{d\varepsilon}\mathscr{F}(\hat{q}+\varepsilon\hat{f})\Big|_{\varepsilon=0} =\frac{\partial \mathscr{F}}{\partial \hat{q}}\hat{f}=\frac{\partial \mathscr{F}}{\partial q(x)}f(x)+\frac{\partial \mathscr{F}}{\partial h}\alpha+\frac{\partial \mathscr{F}}{\partial H}\beta, \qquad (125)$$

where

$$\frac{\partial \mathscr{F}}{\partial q}f(x)=-f(x)-\left(\frac{d}{dx}\Phi_0^{-1}(x)\right)\left(\int_0^x-\int_x^\pi\right)\times\Phi_0(s)f(s)ds +\frac{d}{dx}\left\{\Phi_0^{-1}(x)\left(\int_0^x-\int_x^\pi\right)\times\Phi_0(s)ds\right\}\left(\frac{\partial\lambda_0}{\partial q},f\right), \qquad (126)$$

$$\Phi_0=\varphi^2(x,\lambda_0),$$

$$\frac{\partial \mathscr{F}}{\partial h}(x)=-2\frac{\partial}{\partial x}\left(\Phi_0^{-1}(x)\int_x^\pi\Phi_0(s)ds\right)\frac{\partial\lambda_0}{\partial h}, \qquad (127)$$

$$\frac{\partial \mathscr{F}}{\partial H}(x)=2\frac{d}{dx}\left(\Phi_0^{-1}(x)\int_0^x\Phi_0(s)ds\right)\frac{\partial\lambda_0}{\partial H}. \qquad (128)$$

Proof. Since $\partial\varphi/\partial H=\partial\psi/\partial h=0$, from $\varphi_0(x)=C_0\psi_0(x)$ we find the equations

$$\frac{\partial \mathscr{F}}{\partial h}=-\frac{d^2}{dx^2}\frac{\dot{\varphi}(x,\lambda_0)}{\psi(x,\lambda_0)}\frac{\partial\lambda_0}{\partial h},$$

$$\frac{\partial \mathscr{F}}{\partial H}=-\frac{d^2}{dx^2}\frac{\dot{\varphi}(x,\lambda_0)}{\varphi(x,\lambda_0)}\frac{\partial\lambda_0}{\partial H}. \qquad (129)$$

The identity $ly=\lambda y$ leads to the identity

$$\frac{d}{dx}W[\dot{y}(x,\lambda),y(x,\lambda)]=y^2(x,\lambda).$$

Assuming that here $y=\varphi(x,\lambda_0)$, we find

$$\frac{d}{dx}\frac{\dot{\varphi}(x,\lambda_0)}{\varphi(x,\lambda_0)}=-\Phi_0^{-1}(x)\int_0^x\Phi_0(s)ds, \qquad (130)$$

which together with (129) gives the desired equation (128). Equation (127) is established in a similar manner. Furthermore, again taking into account (130), we have

$$\frac{\partial \mathscr{F}}{\partial q}f(x) =f(x)-2\frac{d^2}{dx^2}\left\{\varphi^{-1}(q;x,\lambda_0)\frac{\partial}{\partial q}\varphi(q;x,\lambda_0)f(x)\right\} +2\frac{d}{dx}\left\{\Phi_0^{-1}(x)\int_0^x\Phi_0(s)ds\right\}\left(\frac{\partial\lambda_0}{\partial q},f\right). \qquad (131)$$

In order to calculate the derivative $\partial\varphi(q;x,\lambda)/\partial q$, we first note that for $\lambda=\lambda_0$ the solution $\varphi_0(q+f;x)$ of the equation $\varphi_0''+[\lambda_0-q(x)]\varphi_0=f(x)\varphi_0$ for which $\varphi_0(q+f;0)=1$, $\varphi_0'(q+f;0)=h$ satisfies the integral equation

$$\varphi_0(q+f;x)=\varphi_0(q;x)+\int_0^x\{\chi_0(x)\varphi_0(s)-\chi_0(s)\varphi_0(x)\}f(s)(\varphi_0(q+f;s)ds,$$

where

$$\chi_0(x)=\chi_0(x,\lambda_0):\chi_0(0)=-h(1+h^2)^{-1},$$

$$\chi_0'(0)=(1+h^2)^{-1}.$$

Since $W(\varphi_0,\chi_0)=1$, we have $\chi_0(x)=\varphi_0(x)\int_0^x\varphi_0^{-2}(s)ds-h(1+h^2)^{-1}\varphi_0(x)$ and, therefore, $\chi_0(x)\varphi_0(x)-\chi_0(s)\varphi_0(x)=\varphi_0(x)\varphi_0(s)\int_s^x\Phi_0^{-1}(\xi)d\xi$. From this we find the equation

$$\varphi_0(q+f;x)=\varphi_0(q;x)+\varphi_0(q;x)\int_0^x\Phi_0^{-1}(s)\times\left(\int_0^s f(\xi)\varphi_0(q;\xi)\varphi_0(q+f;\xi)d\xi\right)ds,$$

which serves as the basis for finding the expression for the differential:

$$\frac{\partial}{\partial q}\varphi_0(q;x)f(x) =\varphi_0(x)\int_0^x\Phi_0^{-1}(s)\left(\int_0^s\Phi(\xi)f(\xi)d\xi\right)ds.$$

Substituting this into the right-hand side of (131), we obtain Eq. (126), since $\mathscr{F}(\hat{q})=q(x)-2d^2/dx^2\ln\psi(q;x,\lambda_0)$. We now write Eqs. (124) in the form $\nu_n(\mathscr{F}(\hat{q}+\varepsilon\hat{f}))=\lambda_n(\hat{q}+\varepsilon\hat{f})$, $n\geqslant 1$, $\hat{f}\in\mathfrak{R}_2$, $r=\mathscr{F}(\hat{q})\in L_2$. Differentiating this with respect to ε and then setting $\varepsilon=0$, owing to Theorem 3.4 we obtain the following lemma.

Lemma 3.1. Let $\mu_n(r)$ be the eigenvalues of the problem (88), (89), $r=\mathscr{F}(\hat{q})$. Then

$$\left(\frac{\partial\lambda_n}{\partial\hat{q}},\hat{f}\right)_2=\left(\frac{\partial\mu_n}{\partial\mathscr{F}},\frac{\partial\mathscr{F}}{\partial\hat{q}}\hat{f}\right),\quad n\geqslant 1,\quad \hat{f}\in\mathfrak{R}_2, \qquad (132)$$

where $\partial\lambda_n/\partial\hat{q}=\alpha_n\hat{\Phi}(\lambda_n)$, $\partial\nu_n/\partial r=\gamma_n g^2(x,\nu_n)$.

Using the proof of Theorem 1.5, we find that Theorem 3.1 leads to:

Theorem 3.5. Let $\sigma(r)=\{\nu_n,n\geqslant 1\}$ be the spectrum of the boundary-value problem (88), (89) with $r(x)\in L_2^{(s)}[r(x)=r(\pi-x)]$. Then for any function $f\in L_2^{0(s)}$ we have the expansion

$$f(x)=\sum_{n=1}^{\infty}\dot{\chi}^{-2}(\nu_n)L_n'(x)[f,G(\nu_n)], \qquad (133)$$

where $L_n(x)=\dot{H}(x,\nu_n)-\dot{G}(x,\nu_n)$. The analog of this theorem for the problem (48), (49) with $\hat{q}^{(s)}=(q\in L_2^{(s)},h,H=h)\in\mathfrak{R}_2^{(s)}$ is the expansion (87), i.e., we have:

Theorem 3.6. Let the boundary-value problem (48), (49) with $\hat{q}=\hat{q}^{(s)}$ be used to construct the functions

$$W_n(x)=\dot{\Psi}(x,\lambda_n)-\dot{\Phi}(x,\lambda_n),$$

$$\Phi_n(x)=\Phi(x,\lambda_n),\quad n\geqslant 0.$$

Then for any function $h\in L_2^{(s)}$ we have the expansion

$$h(x)=\sum_{n=0}^{\infty}\dot{\omega}^{-2}(\lambda_n)W_n'(x)\left(h,\Phi_n-\frac{1}{2}\right). \qquad (134)$$

Let us give the derivation of Theorem 3.6 from Theorem 3.5 in more detail. We shall break up the proof into several lemmas. First we note that Lemma 3.1 leads directly to:

Lemma 3.2. Let $\widehat{f}^{(s)}\in\mathfrak{R}_2^{(s)}=\{\widehat{f}\in\mathfrak{R}_2|\,f(x)=f(\pi-x),\alpha,\beta=\alpha\}$. Then if $\widehat{q}^s\in\mathfrak{R}_2^{(s)}$, for any $f\in\mathfrak{R}_2^{0(s)}=\{\widehat{f}\in\mathfrak{R}_2^{(s)}|\,1/2\int_0^\pi f(x)dx+2\alpha=0\}$ we have

$$\frac{\partial\mathscr{F}}{\partial\widehat{q}^{(s)}}f\equiv h(f;\lambda_0,x)$$

$$=-f(x)-\left(\frac{d}{dx}\Phi_0^{-1}(x)\right)\left(\int_0^x-\int_x^\pi\right)$$

$$\times\Phi_0(s)f(s)ds+\alpha_0\left(f,\Phi_0-\frac{1}{2}\right)\frac{d}{dx}$$

$$\times\left\{\Phi_0^{-1}(x)\left(\int_0^x-\int_x^\pi\right)\Phi_0(s)ds\right\}\in L_2^{0(s)}. \qquad (135)$$

Remark. The equation $\partial\mathscr{F}(\widehat{q}^{(s)})/\partial h=\partial\mathscr{F}(\widehat{q}^{(s)})/\partial H$, $H=h$, should be understood in the sense that $(\partial\mathscr{F}/\partial h,f)=(\partial\mathscr{F}/\partial H,f)$, $f\in L_2^{(s)}$.

Lemma 3.3. Let the function $f\in L_2^{(s)}$ be used to construct the function $h(f;x,\lambda_0)$, where $r=\mathscr{F}(\widehat{q}^{(s)})$. Then for $n\geqslant 1$ we have

$$\gamma_n(r)\int_0^\pi h(f;x,\lambda_0)G_n(r,x)dx$$

$$=\alpha_n(\widehat{q}^{(s)})\int_0^\pi f(x)\left(\Phi_n(\widehat{q}^{(s)};x)-\frac{1}{2}\right)dx, \qquad (136)$$

where $\alpha_n=(-1)^{n+1}\dot{\omega}^{-1}$ $(\lambda_n=\nu_n)$, $\gamma_n=(-1)^n\dot{\chi}^{-1}(\nu_n)$.

Proof. The proof of (136) is a direct consequence of Lemmas 3.1 and 3.2. From this and Theorem 3.5 we find that if $r=\mathscr{F}(\widehat{q}^{(s)},x)$, then for any $f\in L_2^{(s)}$ we have the expansion

$$-h(f;\lambda_0,x)=\sum_{n=1}^{\infty}\frac{L_n'(r;x)}{\dot{\chi}(r;\nu_n)\dot{\omega}(\widehat{q}^{(s)};\nu_n)}$$

$$\times\left(f,\Phi_n(\widehat{q}^{(s)})-\frac{1}{2}\right), \qquad (137)$$

where $h(f;\lambda_0,x)$ is determined by (135).

Let us now use the function $\Phi_0(x)$ to construct the operators

$$Af=f(x)+\left(\frac{d}{dx}\Phi_0^{-1}(x)\right)$$

$$\times\left(\int_0^x-\int_x^\pi\right)\Phi_0(s)f(s)ds,$$

$$Bf=f(x)+\Phi_0'(x)\left(\int_0^x-\int_x^\pi\right)\Phi_0^{-1}(s)f(s)ds.$$

Lemma 3.4. In the space $L_2^{(s)}$, $A=B^{-1}$, i.e., for any $h\in L_2^{(s)}$ the equation $Af(x)=h(x)$ has the unique solution $f=Bh\in L_2^{(s)}$ and vice versa. Here

$$2\int_0^\pi h(x)\left(\Phi_0(x)-\frac{1}{2}\right)dx=\int_0^\pi f(x)dx,$$

$$h(x)=Bf(x). \qquad (138)$$

Proof. The proof of this lemma is completely analogous to that of Lemma 6.6 of Ref. 50, and we omit it. It is more difficult to establish the following:

Lemma 3.5. Let the functions $L_n(x)$ and $W_n(x)$ $(r=\mathscr{F}(\widehat{q}^{(s)}))$ be constructed as in Theorems 3.5 and 3.6, respectively. Then for $n\geqslant 1$ we have

$$(\lambda_0-\lambda_n)^{-1}W_n'(x)=BL_n'(x) \qquad (139)$$

and

$$\frac{d}{dx}\left\{\Phi_0(x)\left(\int_0^x\int_x^\pi\right)\Phi_0^{-1}(s)ds\right\}=\alpha_0^{-1}BW_0'x. \qquad (140)$$

Proof. It is well known (see, for example, Ref. 48) that the inverses of the transformations (123) have the form

$$\varphi(x,\lambda)=\frac{W[z_0(x),g(x,\lambda)]}{z_0(x)},$$

$$\psi(x,\lambda)=\frac{W[z_0(x),h(x,\lambda)]}{z_0(x)},\quad z_0(x)=\varphi_0^{-1}(x).$$

From this and the identity (22) we find

$$(\lambda_0-\lambda)^{-1}\frac{d}{dx}[Z_0(x)\Phi(x,\lambda)]=W[Z_0(x),G(x,\lambda)],$$

$$(\lambda_0-\lambda)^{-1}\frac{d}{dx}[Z_0(x)\Psi(x,\lambda)]=W[Z_0(x),H(x,\lambda)],$$

$$Z_0=\Phi_0^{-1},$$

where $G=g^2$, $H=h^2$, $\Phi=\varphi^2$, and $\Psi=\psi^2$, from which it follows that

$$(\lambda_0-\lambda)^{-2}Z_0(x)(\Psi(x,\lambda)-\Phi(x,\lambda))$$

$$+(\lambda_0-\lambda)^{-1}Z_0(x)(\dot{\Psi}(x,\lambda)-\dot{\Phi}(x,\lambda))$$

$$=\frac{1}{2}\left(\int_0^x-\int_x^\pi\right)W(Z_0(s),\dot{H}(s,\lambda)-\dot{G}(s,\lambda))ds,$$

since $\int_0^\pi W(Z_0(s),\dot{H}(s,\lambda)+\dot{G}(s,\lambda))ds=0$. Now to obtain (139) we must set $\lambda=\nu_n$ and take into account the fact that, owing to (50), $\Psi(x,\nu_n)=\Phi(x,\nu_n)$, $\Psi(0,\lambda_0)=\Phi(\pi,\lambda_0)=1$. A simple calculation gives

$$B\frac{d}{dx}\left(\Phi_0(x)\left(\int_0^x-\int_x^\pi\right)\Phi_0^{-1}(s)ds\right)$$

$$=\frac{d}{dx}\left(\Phi_0(x)\left(\int_0^x-\int_x^\pi\right)\Phi_0^{-1}(s)ds\right).$$

Then Eq. (140) follows from the biorthogonality relations (55) and the derivation of the expansion formula (134) given below.

Now let us apply the operator B to both sides of Eq. (137). Owing to Lemmas 3.4 and 3.5, we obtain the expansion formula

$$f(x)-\alpha_0\left(f,\Phi_0-\frac{1}{2}\right)\times\frac{d}{dx}\left(\Phi_0(x)\left(\int_0^x-\int_x^\pi\right)\Phi_0^{-1}(s)ds\right)=\sum_{n=1}^{\infty}\dot{\omega}^{-2}(\lambda_n)W_n'(x)\left(f,\Phi_n-\frac{1}{2}\right).$$

In this we set $f=W_0'(x)$. Then owing to (55) we obtain $W_0'(x)=(d/dx)(\Phi_0(x)(\int_0^x-\int_x^\pi)\Phi_0^{-1}(s)ds)$, which is what we wanted to prove.

The expansion (133) is obtained from (134) by a similar scheme. From the representations (123) and the identity (22) we can show that for any $f\in L_2^{0(s)}$ we have

$$\int_0^\pi f(x)G(x,\nu_n)dx=(\lambda_0-\nu_n)^{-1}\int_0^\pi h(x)\left(\Phi(x,\nu_n)-\frac{1}{2}\right)dx,\quad n\geqslant 1,$$

where $h=Bf$. From this, setting $h=Bf$ in the expansion (134), we obtain

$$h(x)=\sum_{n=1}^{\infty}(\lambda_0-\nu_n)\dot{\omega}^{-2}(\nu_n)W_n'(x)[f,G(\nu_n)],\tag{141}$$

since, owing to (138), the condition $f\in L_2^{0(s)}$ gives $(h,\Phi_0-1/2)=0$. Lemma 3.4 leads to the equations $(\lambda_0-\nu_n)L_n'(x)=AW_n'(x)$, $n\geqslant 1$. Furthermore, applying the operator A to both sides of Eq. (141), we obtain the desired expansion (133), using the fact that $A=B^{-1}$.

4. THE SPECTRAL THEORY OF Λ OPERATORS

In this section we obtain expansions analogous to Theorems 1.1 and 1.3 for the boundary-value problems (88), (89), using a scheme which is traditional in spectral theory. The essence of this scheme is its formulation as an eigenvalue problem for a suitable integro-differential operator with subsequent construction of the resolvent operator, from which the expansion of unity is obtained by contour integration as in Sec. 1. Our constructions of Λ operators is based on the following theorem.

Theorem 4.1 (Ref. 51). Let $s(x)=r_1(x)+r_2(x)$, $\Delta(x)=r_1(x)-r_2(x)$, where $r_j(x)\in C_1[0,\pi]$. Then the product $Y=y_1y_2$ of the solutions of the equations $l(r_j)y_j=\lambda y_j$, $j=1,2$, satisfies the equation

$$\widetilde{\Lambda}_{x_0}Y(x,\lambda)=\lambda Y(x,\lambda)+B(x_0,x,\lambda),\quad x_0\in[0,\pi],\tag{142}$$

where

$$\widetilde{\Lambda}_{x_0}=\frac{1}{4}\left\{-D^2+2s(x)-\int_{x_0}^x s_y(y)dy-\int_{x_0}^x\Delta(y)dy\int_{x_0}^y\Delta(z)dz\right\};$$

$$B(x_0,x,\lambda)=-\frac{1}{4}\left\{W[y_1(x_0,\lambda),y_2(x_0,\lambda)]\int_{x_0}^x\Delta(y)dy+2[y_1'(x_0,\lambda)y_2'(x_0,\lambda)+2Y(x_0,\lambda)]-s(x_0)Y(x_0,\lambda)\right\}\left(D=\frac{d}{dx}\right).$$

Differentiating (142) with respect to x gives:

Corollary 1. The function $Y(x,\lambda)$ is the solution of the equation

$$\Lambda_{x_0}Y(x,\lambda)=\lambda DY(x,\lambda)-\tfrac{1}{4}\Delta(x)W[y_1(x_0,\lambda),y_2(x_0,\lambda)],\tag{143}$$

where

$$\Lambda_{x_0}=D\widetilde{\Lambda}_{x_0}=\frac{1}{4}\left\{-D^3+2s(x)D+s_x(x)-\Delta(x)\int_{x_0}^x\Delta(y)dy\right\}.$$

For $\Delta(x)=0$ this leads to the well-known fact that the product $Y(x,\lambda)$ of any two solutions of the equation $l(r)y=\lambda y$ is a solution of the equation

$$LY=\tfrac{1}{4}\{-D^3+4r(x)D+2r_x(x)\}Y=\lambda DY.$$

Corollary 2. From the identity (56) and the equation $W(y_1,y_2)=Y'-2Z$, where $Z=y_1'(x,\lambda)y_2(x,\lambda)$, it follows that Eq. (143) is equivalent to the system of differential equations[36]

$$-Y'''+s_x(x)Y+(2s(x)+\Delta(x))Y'-2\Delta(x)Z=4\lambda Y',$$

$$-Z''+r_{1,x}(x)Y+2r_1(x)Y'-\Delta(x)Z=2\lambda Y'.\tag{144}$$

Taking into account the initial conditions (90), we can verify that Theorem 4.1 gives:

Theorem 4.2. The functions $G(x,\lambda)$ and $H(x,\lambda)$ determined by (96) satisfy the equations

$$\widetilde{\Lambda}_0G(x,\lambda)=\lambda G(x,\lambda)-\tfrac{1}{2},\quad \widetilde{\Lambda}_\pi H(x,\lambda)=\lambda H(x,\lambda)-\tfrac{1}{2},$$

where $\widetilde{\Lambda}_{0(\pi)}=\widetilde{\Lambda}_{x_0=0(\pi)}$, and also the equations

$$\Lambda_0G\equiv D\widetilde{\Lambda}_0G=\lambda DG,\quad \Lambda_\pi H\equiv D\widetilde{\Lambda}_\pi H=\lambda DH,\tag{145}$$

where $\Lambda_{0(\pi)}=\Lambda_{x_0=0(\pi)}$.

Corollary 1. If we introduce the operators

$$\widetilde{\Lambda}^*_{\pi(0)}=\frac{1}{4}\left\{-D^2+2s(x)+s_x(x)\int_{0(\pi)}^x dy-\Delta x\int_{0(\pi)}^x\Delta(y)dy\int_{0(\pi)}^y dz\right\},$$

then Eqs. (145) can be written as

$$\Lambda_0G\equiv\widetilde{\Lambda}^*_\pi DG=\lambda DG,\quad \widetilde{\Lambda}^*_0DH=\lambda DH,$$

since $G(x,\lambda)=\int_0^xG'(s,\lambda)ds$, $H(x,\lambda)=-\int_x^\pi H'(s,\lambda)ds$.

Let us now consider the boundary-value problems

$$\Lambda_0 Y(x,\lambda) \equiv \tilde{\Lambda}_\pi^* D Y(x,\lambda) = \lambda D Y(x,\lambda),$$

$$Y(0) = Y'(0) = Y(\pi) = 0, \tag{146}$$

$$\Lambda_\pi Y(x,\lambda) \equiv \tilde{\Lambda}_0^* D Y(x,\lambda) = \lambda D Y(x,\lambda),$$

$$Y(0) = Y(\pi) = Y'(\pi) = 0. \tag{147}$$

Since the conditions $Y(0) = Y'(0) = 0$, $Y''(0) = 2$ uniquely determine the function $G(x,\lambda)$, the eigenvalues of the problem (146) are determined by the condition $G(\pi,\lambda) = 0$, i.e., the set $\sigma = \sigma_1 \cup \sigma_2$ of zeros of the function $X(\lambda) = G(\pi,\lambda) = H(0,\lambda)$. This equation together with the conditions $H(\pi) = H'(\pi) = 0$, $H''(\pi) = 2$ shows that the spectra of the problems (146) and (147) coincide. Obviously, the problems (146), (147) are equivalent to the problems

$$\tilde{\Lambda}_\pi^* Z(x,\lambda) = \lambda Z(x,\lambda), \quad \int_0^\pi Z(x,\lambda)dx = 0,$$

$$Z(0,\lambda) = 0, \tag{148}$$

$$\tilde{\Lambda}_0^* Z(x,\lambda) = \lambda Z(x,\lambda), \quad \int_0^\pi Z(x,\lambda)dx = 0,$$

$$Z(\pi,\lambda) = 0, \tag{149}$$

respectively, in the sense that if for some λ the function $Y(x,\lambda)$ is a solution of the problem (146), then the function $Z(x,\lambda) = Y'(x,\lambda)$ is a solution of the problem (148); conversely, if $Z(x,\lambda)$ is a solution of the problem (148), then $Y(x,\lambda) = \int_0^x Z(s,\lambda)ds$ is a solution of the problem (146).

We shall study the operators Λ_0, Λ_π in the space L_2 with the domains of definition

$$\mathscr{D}(\Lambda_{0(\pi)}) = \{ f(x) \in C_3, | f(0) = f'(0) = f(\pi) = 0 (f(0) = f(\pi) = f'(\pi) = 0)\},$$

and the operators $\tilde{\Lambda}_\pi^*$ and $\tilde{\Lambda}_0^*$ in the space

$$\mathscr{D}(\tilde{\Lambda}_{\pi(0)}^*) = \left\{ f(x) \in C_2 \;\middle|\; \int_0^\pi f(x)dx = 0, f(0) = 0 [f(\pi) = 0] \right\}.$$

From $f \in \mathscr{D}(\tilde{\Lambda}_\pi^*) (\mathscr{D}(\tilde{\Lambda}_0^*))$ we have $F(x) = \int_0^x f(s)ds \in \mathscr{D}(\Lambda_0)$ $(F(x) = -\int_x^\pi f(s)ds \in \mathscr{D}(\Lambda_\pi))$, and if $f \in \mathscr{D}(\Lambda_{\pi(0)})$, then $f' \in \mathscr{D}(\tilde{\Lambda}_{0(\pi)}^*)$. Here

$$\left.\begin{aligned} &\Lambda_{0(\pi)} f \equiv D\tilde{\Lambda}_{0(\pi)} f = \tilde{\Lambda}_{\pi(0)}^* Df, \quad \forall f \in C_3, \\ &f(0)(f(\pi)) = 0, \quad (\Lambda_0 f, g) = -(f, \Lambda_\pi g), \\ &[\tilde{\Lambda}_0 f, g] = [f, \tilde{\Lambda}_\pi g] f \in \mathscr{D}(\Lambda_0), \quad g \in \mathscr{D}(\Lambda_\pi) \end{aligned}\right\} \tag{150}$$

and

$$(\tilde{\Lambda}_{0(\pi)} f, g) = (f, \tilde{\Lambda}_{0(\pi)}^* g) f \in \mathscr{D}(\Lambda_{0(\pi)}), \quad g \in \mathscr{D}(\tilde{\Lambda}_{0(\pi)}), \tag{151}$$

where the skew-scalar product [,] is given by (103).

We introduce the function

$$R_0(x,y,\lambda) = \frac{2}{X(\lambda)} \begin{cases} G(x,\lambda)H(y,\lambda), & x < y < \pi, \\ \sum_{j=1,2} U_j(x,\lambda)U_{3-j}(y,\lambda) \\ \quad - H(x,\lambda)G(y,\lambda), & 0 < y < x, \end{cases}$$

where $U_j(x,\lambda) = g_j(x,\lambda)h_{3-j}(x,\lambda)$.

Theorem 4.3. The integral operator

$$R_0(f;\lambda)(x) = \int_0^\pi R_0(x,y,\lambda) f(y)dy$$

is, for any $\lambda \in \rho(\Lambda_0) = \mathbb{C} \setminus \cup_{j=1,2} \sigma_j$, the resolvent of the operator $\Lambda_0 - \lambda D$, i.e.,

$$R_0(f;\lambda)(x) \in \mathscr{D}(\Lambda_0), \; (\Lambda_0 - \Lambda D) R_0(f;\lambda)(x) = f(x),$$

$$f \in L_1, \quad R_0(\lambda, (\Lambda_0 - \lambda D) f)(x) = f(x), \quad f \in \mathscr{D}(\Lambda_0).$$

Proof. The proof reduces to direct verification using the identity (56) and the equation following from (88):

$$Y''(x,\lambda) = [s(x) - 2\lambda] Y(x,\lambda) + 2y_1'(x,\lambda)y_2'(x,\lambda).$$

This calculation is described in detail in Ref. 24.

Corollary 1. The resolvent operator $\tilde{R}_\pi^*(\lambda)$ of the boundary-value problem (148) is determined in terms of $R_0(\lambda)$ by the equation

$$\tilde{R}_\pi^*(\lambda) = DR_0(\lambda) = (\tilde{\Lambda}_\pi^* - \lambda I)^{-1}, \quad \lambda \in \rho(\Lambda_0).$$

Taking into account (150), we obtain:

Corollary 2. For any $\lambda \in \rho(\Lambda_0)$ the operator

$$R_\pi(\lambda, f)(x) = \int_0^\pi R_\pi(x,y,\lambda) f(y)dy,$$

$$R_\pi(x,y,\lambda) = - R_0(y,x,\lambda)$$

determines the resolvent of the boundary-value problem (147). Here the operator

$$\tilde{R}_0^*(\lambda) = (\tilde{\Lambda}_0^* - \lambda I)^{-1} = DR_\pi(\lambda)$$

is the resolvent for the boundary-value problem (149).

Now we introduce the operators

$$\Lambda = \tfrac{1}{2}(\Delta_0 + \Lambda_\pi),$$

$$\mathscr{D}(\Lambda) = \{ f(x) \in C_3 | f(0) = f(\pi), f'(0) = f'(\pi)\},$$

$$\tilde{\Lambda}^* = \tfrac{1}{2}(\tilde{\Lambda}_0^* + \Lambda_\pi^*),$$

$$\mathscr{D}(\tilde{\Lambda}_*) = \left\{ f(x) \in C_2 \;\middle|\; \int_0^\pi f(x)dx = 0, f(0) = f(\pi) \right\}.$$

This leads to the equations

$$\Lambda f = \tilde{\Lambda}^* Df, \quad f \in C_3, \quad f(0) = f(\pi) = 0,$$

$$(\Lambda f, g) = -(f, \Lambda g), \quad f, g \in \mathscr{D}(\Lambda), \tag{152}$$

the last of which shows that Λ is a skew-symmetric operator. We also note the relations

$$[\tilde{\Lambda} f, g] = [f, \tilde{\Lambda} g], \quad \tilde{\Lambda} = \tfrac{1}{2}(\tilde{\Lambda}_0 + \tilde{\Lambda}_\pi), \quad f, g \in \mathscr{D}(\Lambda),$$

$$(\tilde{\Lambda}^* f, g) = (f, \tilde{\Lambda} g), \quad f \in \mathscr{D}(\tilde{\Lambda}^*), \quad g \in \mathscr{D}(\Lambda).$$

Let us consider the boundary-value problem

$$\Lambda Y(x,\lambda)=\lambda DY(x,\lambda),$$

$$Y(0)=Y(\pi)=0,\quad Y'(0)=Y'(\pi), \tag{153}$$

which, owing to (152), is equivalent to the problem

$$\widetilde{\Lambda}^* Z(x,\lambda)=\lambda Z(x,\lambda),$$

$$\int_0^\pi Z(x,\lambda)dx=0,\quad Z(0)=Z(\pi).$$

The analog of Theorem 4.3 is:

Theorem 4.4. If the boundary-value problems (88), (89) are isospectral, i.e., $\sigma_1=\sigma_2$, then the operator

$$R(\lambda)=\tfrac{1}{2}[R_0(\lambda)+R_\pi(\lambda)]=(\Lambda-\lambda D)^{-1},$$

$$\lambda\in\rho(\Lambda)=\mathbb{C}\setminus\sigma$$

is the resolvent of the problem (153), and we have the operator

$$\widetilde{R}^*(\lambda)=DR(\lambda)=(\widetilde{\Lambda}^*-\lambda I)^{-1}.$$

We introduce the systems of functions $\{U_n\}$ and $\{V_n\}$ as follows: for $\nu_n\in\sigma'$ we set

$$U_n(x)=\dot{X}^{-1}(\nu_n)G(x,\nu_n),\quad V_n(x)=2H(x,\nu_n),$$

and for $\nu_{2n+1}=\nu_{2n+2}=\nu_{(n)}\in\sigma''$ we set

$$U_{2n+1}(x)=2\ddot{X}^{-1}(\nu_{(n)})G(x,\nu_{(n)}),$$

$$U_{2n+2}=2\ddot{X}^{-1}(\nu_{(n)})\dot{G}(x,\nu_{(n)}),$$

$$V_{2n+1}=2\dot{H}(x,\nu_{(n)})-2\dddot{X}(\nu_{(n)})\ddot{X}^{-1}(\nu_{(n)})H(x,\nu_{(n)}),$$

$$V_{2n+2}=2H(x,\nu_{(n)}).$$

Using the identity (22), we can verify the following lemma.

Lemma 4.1. The system $\{U_n\}$ is biorthogonal to the system $\{V_n\}$ under the skew-scalar product (103):

$$[U_n,V_m]=-[V_m,U_n]=\delta_{nm}.$$

Furthermore, from Eqs. (146), taking into account the boundary conditions (90) and Eq. (93), we find:

Lemma 4.2. $U_n\in\mathscr{D}(\Lambda_0)$, $V_n\in\mathscr{D}(\Lambda_\pi)$, $U_n'\in\mathscr{D}(\widetilde{\Lambda}_\pi^*)$, $V_n'\in\mathscr{D}(\widetilde{\Lambda}_0^*)$. Here, if $\nu_n\in\sigma'$, then

$$\Lambda_0U_n\equiv\widetilde{\Lambda}_\pi^*U_n'=\nu_nU_n',\quad \Lambda_\pi V_n\equiv\widetilde{\Lambda}_0^*V_n'=\nu_nV_n',$$

and if $\nu_{(n)}\in\sigma''$, then

$$\Lambda_0U_{2n+1}\equiv\widetilde{\Lambda}_\pi^*U_{2n+1}'=\nu_{(n)}U_{2n+1}',$$

$$\Lambda_0U_{2n+2}=\nu_{(n)}U_{2n+2}'+U_{2n+1}',$$

$$\Lambda_\pi V_{2n+2}=\nu_{(n)}V_{2n+2}',$$

$$\Lambda V_{2n+1}=\nu_{(n)}V_{2n+1}'+V_{2n+2}'.$$

The fundamental question of whether or not the systems $\{U_n\}$ and $\{V_n\}$ form a basis and are complete is resolved by:

Theorem 4.5. This concerns the expansion of unity for the operators $\Lambda_{0(\pi)}$ and $\widetilde{\Lambda}_{\pi(0)}^*$. For any, possibly complex-valued, function $f\in L_1$ we have the expansion formulas

$$-\int_x^\pi f(s)ds=\lim_{N\to\infty}\sum_{n=3}^{2N}V_n(x)(f,U_n), \tag{154}$$

$$\int_0^x f(s)ds=\lim_{N\to\infty}\sum_{n=3}^{\infty}U_n(x)(f,V_n), \tag{155}$$

where the convergence in (154) is uniform in $x\in\Delta\subseteq(0,\pi]$, and that in (155) is uniform for $x\in\Delta\subseteq(0,\pi]$. Here, if $f\in L_2^{(0)}$, then in the sense of the norm of L_2 we have

$$f(x)=\lim_{N\to\infty}\sum_{n=3}^{2N}V_n'(x)(f,U_n), \tag{156}$$

$$f(x)=\lim_{N\to\infty}\sum_{n=3}^{2N}U_n'(x)(f,V_n). \tag{157}$$

Proof. This is proved by calculation, as in the proof of Theorems 1.1 and 1.2, of the contour integrals

$$\frac{1}{2\pi i}\oint_{c_N}R_{0(\pi)}(\lambda,f)(x)d\lambda,$$

$$\frac{1}{2\pi i}\oint_{c_N}\widetilde{R}_{0(\pi)}^*(\lambda,f)(x)d\lambda$$

with subsequent application of the criterion for forming a basis.[39]

It follows from Lemma 4.2, Eq. (150), and Theorem 4.2 that if

$$f\in\mathscr{D}(\Lambda_\pi)\ \text{and}\ \nu_{2n+j}\in\sigma',\ \nu_{2n+1}\in\sigma'',$$

then

$$(\Lambda_\pi f,U_{2n+j})=-(f,\Lambda_0U_{2n+j})$$
$$=-\nu_{2n+j}[f,U_{2n+j}],$$

while for $\nu_{2n+2}\in\sigma''$ we have

$$(\Lambda_\pi f,U_{2n+2})=-\nu_{2n+2}[f,U_{2n+2}]-[f,U_{2n+1}].$$

Furthermore, taking into account (151), we find that if $f\in\mathscr{D}(\widetilde{\Lambda}_0^*)$, then

$$(\widetilde{\Lambda}_0^*f,U_{2n+j})=(f,\widetilde{\Lambda}_0U_{2n+j})=\nu_{2n+j}(f,U_{2n+j});$$

$$(\widetilde{\Lambda}_0^*f,U_{2n+2})=\nu_{2n+2}(f,U_{2n+2})+(f,U_{2n+1}).$$

From this, after replacing f by $\Lambda_\pi f$ in (154) and f by $\widetilde{\Lambda}_0^*f$ in (156), we find:

Theorem 4.6. This concerns the spectral expansion of the operators Λ_π and $\widetilde{\Lambda}_0^*$. For any $f\in\mathscr{D}(\Lambda_\pi)$ we have the expansion

$$-\int_x^\pi\Lambda_\pi f(s)ds$$

$$=\sum_{n=1}^{\infty}\left\{\sum_{j=1,2}\nu_{2n+j}V_{2n+j}(x)[f,U_{2n+j}]\right.$$
$$\left.+\delta_nV_{2n+2}(x)[f,U_{2n+1}]\right\}, \tag{158}$$

and if $f\in\mathscr{D}(\widetilde{\Lambda}_0^*)$, $\widetilde{\Lambda}_0^*f\in L_2^{(0)}$, then

$$\widetilde{\Lambda}_0^* f(x) = \lim_{N\to\infty} \sum_{n=1}^{2N} \left\{ \sum_{j=1,2} \nu_{2n+j} V'_{2n+j}(x)(f,U_{2n+j}) + \delta_n V'_{2n+2}(x)(f,U_{2n+1}) \right\}, \qquad (159)$$

where $\delta_n = 0$ for $\nu_{2n+1} \in \sigma'$, and $\delta_n = 1$ for $\nu_{2n+1} \in \sigma''$.

The expansions for the operators Λ_0 and $\widetilde{\Lambda}_\pi^*$, which are obtained from (155) and (157), respectively, are completely analogous to (158) and (159) given above and are not written out here.

Corollary 1. The spectrum $\sigma(\Lambda_0) = \mathbb{C} \setminus \rho(\Lambda_0)$ of the boundary-value problem (146), (148) coincides with the spectrum of the problem (147), (149), and $\sigma(\Lambda_0) = \sigma(\Lambda_\pi) = \sigma_1 \cup \sigma_2, \rho(\Lambda_0) = \rho(\Lambda_\pi)$ is the region in which the operators $R_0(\lambda)[\widetilde{R}_\pi^*(\lambda)]$ and $R_\pi(\lambda)[\widetilde{R}_0^*(\lambda)]$ are regular. The numbers $\nu_n \in \sigma'$ are simple eigenvalues for which $U_n(U'_n)$ and $V_n(V'_n)$ are the corresponding eigenfunctions. The numbers $\nu_{(n)} \in \sigma''$ are double eigenvalues for which $U_{2n+1}(U'_{2n+1})$, $V_{2n+2}(V'_{2n+2})$ and $U_{2n+2}(U'_{2n+2})$, $V_{2n+1}(V'_{2n+1})$ are the eigenfunctions and associated functions of the boundary-value problems (146), (148) and (147), (149), respectively.

Remark 1. It follows from the equations for the functions V'_n given in Lemma 4.2 that Eq. (159) can be viewed as resulting from the term-by-term application of the operator $\widetilde{\Lambda}_0^*$ to the expansion (159), i.e.,

$$\widetilde{\Lambda}_0^* f(x) = \sum_{n=3}^{\infty} \widetilde{\Lambda}_0^* V'_n(x)(f,U_n),$$
$$f \in \mathscr{D}(\widetilde{\Lambda}_0^*), \quad \widetilde{\Lambda}_0^* f \in L_2^0.$$

A similar statement can be made for (158) if together with $f \in \mathscr{D}(\Lambda_\pi)$ we require that $\Lambda_\pi f \in L_2^0$.

Remark 2. Computing, as in the proof of Theorem 1.1, the contour integral

$$\frac{1}{2\pi i} \oint \widetilde{R}_0^*(z,f)(x)(z-\lambda)^{-1} dz, \quad \lambda \in \rho(\widetilde{\Lambda}_0^*),$$

we obtain the following spectral representation for the resolvent:

$$\widetilde{R}_0^*(\lambda,f)(x) = \sum_{n=1}^{\infty} \sum_{j=1,2} \left\{ \frac{V'_{2n+j}(x)(f,U_{2n+j})}{\nu_{2n+j} - \lambda} - \frac{\delta_n V'_{2n+2}(x)}{(\nu_{(n)} - \lambda)^2}(f,U_{2n+1}) \right\},$$

which is valid for any $f \in L_1$. From this, owing to Corollary 1, Theorem 4.2, and Lemma 4.3, we obtain the expansion formula (156) for any $f \in \mathscr{D}(\widetilde{\Lambda}_0^*)$. In this aspect Theorem 4.5 enlarges the set of allowed f for which the given expansions hold.

Let us now consider the spectral problem (153). Here the analog of Lemma 4.2 is:

Lemma 4.3. The functions

$$P_n(x) = 2\ddot{X}^{-1}(\nu_n) H(x,\nu_n),$$
$$Q_n(x) = S_{2n+1} S_{2n+2} \dot{H}(x,\nu_n) - \dot{G}(x,\nu_n) \qquad (160)$$

are the eigenfunctions of the boundary-value problem (153) corresponding to eigenvalue ν_n:

$$\Lambda P_n \equiv \widetilde{\Lambda}^* P'_n = \nu_n P'_n, \quad \Lambda Q_n \equiv \widetilde{\Lambda}^* Q'_n = \nu_n Q'_n, \qquad (161)$$

where $P_n, Q_n \in \mathscr{D}(\Lambda)$. Here

$$[P_n, Q_m] = \delta_{nm}, \; [P_n, P_m] = [Q_n, Q_m] = 0. \qquad (162)$$

The following two theorems give the expansions of unity and the spectral expansions for the operators Λ and $\widetilde{\Lambda}^*$, respectively.

Theorem 4.7. For every function $f \in L_2^0$ there is the expansion

$$f(x) = \lim_{N\to\infty} \sum_{n=1}^{N} \{Q'_n(x)(f,P_n) - P'_n(x)(f,Q_n)\}, \qquad (163)$$

and if $f \in L_1$ we have

$$\frac{1}{2}\left(\int_0^x - \int_x^\pi \right) f(s) ds = \sum_{n=1}^{\infty} \{Q_n(x)(f,P_n) - P_n(x)(f,Q_n)\}, \qquad (164)$$

where in (163) the series converges in the sense of the norm of L_2, and in (164) the convergence is uniform in $x \in \Delta \subset (0,\pi)$.

The proof of these expansions is obtained by calculating the contour integrals

$$\frac{1}{2\pi i} \oint \widetilde{R}^*(\lambda,f)(x) d\lambda, \quad \frac{1}{2\pi i} \oint R(\lambda,f)(x) d\lambda$$

by the method discussed in the proof of Theorem 4.5. We only note that if we use $S_N(f;x)$ to denote the partial sum of the series (163) and

$$\sigma_N(f;x) = \sum_{n=1}^{N} \left\{ \frac{2}{\pi} \cos 2nx \int_0^\pi f(y) \cos 2ny \, dy + \frac{2}{\pi} \sin 2nx \int_0^\pi f(y) \sin 2ny \, dy \right\}$$

for the partial sum of the Fourier series for $f \in L_2^0$, then $\lim_{N\to\infty} \|S_N(f;x) - \sigma_N(f;x)\|_{L_2} = 0$, which is equivalent to Eq. (163).

The expansions (163) and (164) can be viewed as resulting from the addition of the expansions (156) and (157) and (154), (155), respectively, with the subsequent transformation of the corresponding terms similar to that used in the derivation of Theorem 1.3 from Theorems 1.1 and 1.2. Following this scheme, Theorem 4.6 and Lemma 4.3 lead to:

Theorem 4.8. Equation (163) is the expansion of unity for the operator $\widetilde{\Lambda}^*$, i.e., for any $f \in \mathscr{D}(\widetilde{\Lambda}^*)$, $\widetilde{\Lambda}^* f \in L_0^{(2)}$ we have

$$\widetilde{\Lambda}^* f(x) = \sum_{n=1}^{\infty} \{\nu_n Q'_n(x)(f,P_n) - \nu_n P'_n(x)(f,Q_n)\}, \qquad (165)$$

and Eq. (164) is the expansion of unity for the operator Λ, i.e., for any $f \in \mathscr{D}(\Lambda)$ we have

$$-\frac{1}{2}\left(\int_0^x-\int_x^\pi\right)\Lambda f(s)ds$$

$$=\sum_{n=1}^{\infty}\{\nu_nP_n(x)[f,Q_n]-\nu_nQ_n(x)[f,P_n]\}.$$

Remark 1. In contrast to the function Q_n (160), the functions $Q_n^{(j)}$ (98) are not solutions of Eq. (161). More precisely, we have the equation

$$\Lambda Q_n^{(j)}(x)=\nu_nQ_n^{(j)}(x)+(-1)^j4^{-1}\Delta(x)\dot\chi(\nu_n)\times(S_{2n+1}^{-1}+S_{2n+2}^{-1}),\quad j=1,2,$$

from which Eq. (161) is obtained, owing to the representation

$$Q_n(x)=-S_{2n+1}S_{2n+2}[Q_n^{(1)}(x)+Q_n^{(2)}(x)](\sigma'=0).$$

Remark 2. It follows from the representation (110) that for $0<y<x$

$$R_0(x,y,\lambda)=2X^{-1}(\lambda)G(x,\lambda)H(y,\lambda)-2\prod_{j=1,2}[g_j(x,\lambda)\times c_j(y,\lambda)-c_j(x,\lambda)g_j(y,\lambda)].$$

From this, since $W(c_j,g_j)=1$, it follows that the function R_0 is continuous together with the first derivative with respect to x for $0<y<\pi$. The second derivative with respect to x for $x\neq y$ exists, is continuous, and

$$\left.\frac{\partial^2R(x,y,\lambda)}{\partial x^2}\right|_{y=x+0}-\left.\frac{\partial^2R(x,y,\lambda)}{\partial x^2}\right|_{y=x-0}=4.$$

On the basis of these requirements, the authors of Ref. 36 obtained the Green function for the system (144) on the entire axis with arbitrary potentials $r_j(x)\in L_{\rm loc}$. A formula similar to that of Ref. 36 was proposed earlier in Ref. 51. The presence of boundary conditions somewhat complicates the formulation of the problem of expanding in products of the solutions of two Sturm–Liouville problems in the form of the spectral problem for the Λ operator, as is clear, for example, from the constructions of Ref. 52.

5. ON THE HAMILTONIAN OF THE THEORY FOR THE KdV EQUATION IN THE PERIODIC CASE

In this section we use the symplectic expansions obtained in Sec. 4 and the related operator spectral theory to construct for the KdV equation a canonical change of variables, which corresponds to the well-known study of Flaschka and McLaughlin.[26] In addition, we show how these expansions can be used to obtain the equations proposed by Trubowitz in Ref. 28 for solving the periodic inverse Sturm–Liouville problem. When not stated otherwise, everywhere below we use the notation of Secs. 3 and 4, assuming that $r_1(x)=r_2(x)=r(x)$.

We introduce the functions

$$\tilde P_n(x)=\delta_nH(x,\nu_n)=\gamma_nG(x,\nu_n),$$
$$\tilde Q_n(x)=\delta_n\dot H(x,\nu_m)-\gamma_n\dot G(x,\nu_n). \qquad (166)$$

Since $\tilde P_n(x)\tilde Q_n(y)=P_n(x)Q_n(y)$, where P_n and Q_n are defined by (160), the expansion formula (163) can be written as

$$f(x)=\sum_{n=1}^{\infty}\{\tilde Q_n'(x)(f,\tilde P_n)-\tilde P_n'(x)(f,\tilde Q_n)\},\quad f\in L_2^{(0)}, \qquad (167)$$

and Eqs. (162) give

$$[\tilde P_n,\tilde Q_m]=\delta_{nm},\quad [\tilde P_n,\tilde P_m]=[\tilde Q_n,\tilde Q_m]=0,\quad n,m\geqslant1. \qquad (168)$$

Lemma 5.1. At any point $r\in L_2$ the eigenvalues $\nu_n(r)$ and the normalization factors $\gamma_n(r)$, $\delta_n(r)$ (94) and $f_n(r)$ (110) are differentiable, i.e., if $f\in L_2$, the following derivatives exist:

$$\left.\frac{d}{d\varepsilon}\nu_n(r+\varepsilon f)\right|_{\varepsilon=0}=\left(\frac{\partial\nu_n}{\partial r},f\right),\quad\frac{\partial\nu_n}{\partial r}=\tilde P_n(x), \qquad (169)$$

$$\left.\frac{d}{d\varepsilon}\gamma_n(r+\varepsilon f)\right|_{\varepsilon=0}=\left(\frac{\partial\gamma_n}{\partial r},f\right),$$

$$\frac{\partial\gamma_n}{\partial r}=\frac{1}{\dot\chi^2(\nu_n)}\left\{\dot H_n(x)-\frac{\ddot\chi(\nu_n)}{\dot\chi(\nu_n)}H_n(x)\right\}, \qquad (170)$$

$$\left.\frac{d}{d\varepsilon}\delta_n(r+\varepsilon f)\right|_{\varepsilon=0}=\left(\frac{\partial\delta_n}{\partial r},f\right),$$

$$\frac{\partial\delta_n}{\partial r}=\frac{1}{\dot\chi^2(\nu_n)}\left\{\dot G_n(x)-\frac{\ddot\chi(\nu_n)}{\dot\chi(\nu_n)}G_n(x)\right\}, \qquad (171)$$

$$\left.\frac{d}{d\varepsilon}f_n(r+\varepsilon f)\right|_{\varepsilon=0}=\left(\frac{\partial f_n}{\partial r},f\right),\quad\frac{\partial f_n}{\partial r}=\tilde Q_n(x). \qquad (172)$$

Proof. The proof of Eqs. (169) is well known (see, for example, Ref. 23). Since $S_n^2(r)=\delta_n(r)\gamma_n^{-1}(r)$, Eq. (172) follows directly from (170) and (171), which are derived in a similar manner. Let us prove (171). From the integral equation

$$g(r+h;x,\lambda)\equiv g(x,\lambda)+\int_0^x\{g(x,\lambda)c(y,\lambda)-c(x,\lambda)g(y,\lambda)\}h(y)g(r+h;y,\lambda)dy,$$

where g and c are solutions of the equation $y''+[\lambda-r(x)]y=0$ and $g(r+h;x,\lambda)$ is the solution of the equation $y''+[\lambda-r(x)]y=h(x)y$, we find that the functionals $g'(\pi,\lambda)=g'(r;\pi,\lambda)$, $\dot g(\pi,\lambda)=\dot g(r;\pi,\lambda):L_2\to\mathbb{C}$ are differentiable at any point $r\in L_2$, $\lambda\in\mathbb{C}$:

$$\partial g'(\pi,\lambda)/\partial r(x)=g'(\pi,\lambda)c(x,\lambda)g(x,\lambda)-c'(\pi,\lambda)g^2(x,\lambda),$$

$$\partial\dot g(\pi,\lambda)/\partial r(x)=\dot g(\pi,\lambda)c(x,\lambda)g(x,\lambda)-\dot c(\pi,\lambda)g^2(x,\lambda)-2c(\pi,\lambda)g(x,\lambda)\dot g(x,\lambda)+g(\pi,\lambda)\times[\dot c(x,\lambda)g(x,\lambda)+c(x,\lambda)\dot g(x,\lambda)]. \qquad (173)$$

From this, taking into account the representation (94) and Eq. (22), it follows that $\delta_n(r):L_2 \to \mathbb{R}$ is a differentiable functional, where

$$\frac{d}{d\varepsilon}\delta_n(r+\varepsilon f)\Big|_{\varepsilon=0}$$

$$=\frac{d}{d\varepsilon}\frac{g'(r+\varepsilon f;\pi,\nu_m(r+\varepsilon f))}{\dot g(r+\varepsilon f;\pi,\nu_m(r+\varepsilon f))}\Big|_{\varepsilon=0}$$

$$=\dot g^{-1}(\pi,\nu_m)\left(\frac{\partial\varphi'(\pi,\nu_m)}{\partial r},f\right)$$

$$-g'(\pi,\nu_m)\dot g^{-1}(\pi,\nu_m)\left(\frac{\partial\dot g(\pi,\nu_n)}{\partial r},f\right)$$

$$+\{\dot g'(\pi,\nu_n)\dot g^{-1}(\pi,\nu_n)$$

$$-g'(\pi,\nu_n)\ddot g(\pi,\nu_n)\dot g^{-1}(\pi,\nu_n)\}\left(\frac{\partial\nu_n}{\partial r},f\right).$$

Now to obtain Eq. (24) it is necessary to substitute into the right-hand side of this equation the expressions found for $\partial\nu_n/\partial r$, $\partial g'(\pi,\nu_n)/\partial r$, and $\partial\dot g(\pi,\nu_n)/\partial r$ with $g(\pi,\nu_n) = 0$ and then use the equations $g'(\pi,\nu_n) = c^{-1}(\pi,\nu_n)$, $\dot g(\pi,\nu_n)c(\pi,\nu_n) + g'(\pi,\nu_n)\dot c(\pi,\nu_n) - \dot g(\pi,\nu_n)c'(\pi,\nu_n) = 0$, which follow from $W(c,g) = 1$, $g(\pi,\nu_n) = 0$.

Remark 1. Equation (172) can also be obtained directly using (169) and (173). In Refs. 23 and 26 it was shown that

$$\frac{\partial f_n}{\partial r}(x) = -2g(x,\nu_n)c(x,\nu_n)$$

$$+2\dot g^{-1}(\pi,\nu_n)\dot c(\pi,\nu_n)g^2(x,\nu_n).$$

From this we obtain (172) after noting that, owing to (110), we have for $\lambda = \nu_n$:

$$\dot h(x,\nu_n) - c(\pi,\nu_n)\dot g(x,\nu_n)$$

$$=\dot c(\pi,\nu_n)g(x,\nu_n) - \dot g(\pi,\nu_n)c(x,\nu_n),$$

where

$$\int_0^\pi g(x,\nu_n)c(x,\nu_n)dx = \dot g^{-1}(\pi,\nu_n)\dot c(\pi,\nu_n)$$

and, therefore, $\int_0^\pi \partial f_n/\partial r(x)dx = 0$.

Remark 2. Equation (167) plus the equations $\int_0^\pi \widetilde P_n(x)dx = 1$, $\int_0^\pi \widetilde Q_n(x)dx = 0$, lead directly to the expansion

$$f(x) = \frac{1}{\pi}\int_0^\pi f(y)dy + \sum_{n=1}^\infty \widetilde Q_n'(x)\left(f,\widetilde P_n - \frac{1}{\pi}\right)$$

$$-\widetilde P_n'(x)(f,\widetilde Q_n), \tag{174}$$

which holds for any $f\in L_2$, since $f^{(0)} = f(x) - (1/\pi)\int_0^\pi f(x)dx \in L_2^{(0)}$. As can be seen from the asymptote (92), the functions $\widetilde P_n - 1/\pi$ are the gradients of the functionals $\widetilde\nu_n(r) = \nu_n(r) - n^2 - (1/\pi)\int_0^\pi r(x)dx$, as was pointed out in Ref. 23. Equation (174) was obtained in a different manner in Ref. 23 and along with Eqs. (168) plays a fundamental role in the scheme for solving the IP proposed in that study, which is similar to those constructed in Sec. 2.

Let us now consider the one-parameter family of boundary-value problems

$$y'' + [\lambda - r(x,t)]y = 0, \quad y(0) = y(\pi) = 0, \quad t\in\mathbb{R}. \tag{175}$$

From Eqs. (169), (172), and (120) plus the expansion formula (167) we find the following important theorem.

Theorem 5.1. In Eq. (175) suppose that we have $r(x,\cdot)\in\widetilde C_1(0,\pi)$ (108) and the derivative $r_t(x,\cdot)\in L_2^{(0)}$. Then the following expansions are valid:

$$r_t(x,t) = \sum_{n=1}^\infty \{\lambda_{n,t}\widetilde Q_n'(x,t) - f_{n,t}\widetilde P_n'(x,t)\}, \tag{176}$$

$$r_x(x,t) = \sum_{n=1}^\infty \left\{\frac{S_n^{-1}(t) - S_n(t)}{\dot\chi(\nu_n)}\widetilde Q_n'(x,t)\right.$$

$$\left. -2\frac{\dot\Delta(\nu_n)}{\dot\chi(\nu_n)}\widetilde P_n'(x,t)\right\}, \tag{177}$$

where the functions $\widetilde P(x,t)$ and $\widetilde Q_n(x,t)$ are constructed from Eqs. (166) with $r = r(x,t)$, $\nu_n(t)$ and $f_n(t)$ are the quantities (118) corresponding to the boundary-value problem (175), $\nu_{n,t} = (r_t,\widetilde P_n)$, $f_{n,t} = (r_t,\widetilde Q_n)$, and $\Delta(\lambda)$ is the Hill discriminant (111).

Let us consider the Cauchy problem

$$r_t(x,t) - r_x(x,t) = 0, \quad 0\leqslant x\leqslant\pi, \quad t\in\mathbb{R}, \quad r(x,0) = \widetilde C_1. \tag{178}$$

From Theorem 5.1 we obtain:

Corollary 1. The potential $r(x,t)$ satisfies Eq. (178) if and only if

$$\frac{d}{dt}\nu_n(t) = \frac{S_n^{-1}(t) - S_n(t)}{\dot\chi[\nu_n(t)]}, \quad \frac{d}{dt}f_n(t) = \frac{2\dot\Delta[\nu_n(t)]}{\dot\chi[\nu_n(t)]}. \tag{179}$$

The solution of the problem (178) is obviously the function $r(x+t)$ and, therefore, Eqs. (179) give the t evolution of the spectral data $\nu_n(t)$, $f_n(t)$, $n\geqslant 1$, of the boundary-value problem

$$y'' + [\lambda - r(x+t)]y = 0, \quad y(0) = y(\pi) = 0. \tag{180}$$

Let us now take into account the fact that the Hill discriminant $\Delta(\lambda)$ of Eq. (180) is independent of t. From this it follows that the spectra of the periodic and aperiodic problems (106) and (107) are independent of t; here the inequalities $\mu_{2n-1} \leqslant \nu_n(t) \leqslant \mu_{2n}$, $n\geqslant 1$, are satisfied. From (116) and the representation (95) it follows that the first of the equations in (179) can be written as

$$\frac{d}{dt}\nu_n(t) = \sqrt{\Delta^2(\nu_n) - 4}\left(\frac{\partial}{\partial\lambda}\prod_{l\geqslant 1} l^{-2}(\nu_l - \lambda)\Big|_{\lambda=\nu_n}\right)^{-1} \tag{181}$$

The second equation can be solved, assuming that the solution of the system (181) is known. The solution is given by

$$\sqrt{\Delta^2[\nu_n(t)] - 4} = 2(-1)^{n+1}\exp(-\tfrac{1}{2}f_n(t))$$

$$-\Delta[\nu_n(t)],$$

which can be viewed as determining the sign in front of the radical.

Remark 1. In connection with the solution of the inverse problem for periodic potentials, the system of equations (181), where the sign in front of $\sqrt{\Delta^2(\nu_n)-4}$ is determined as described above, was obtained by a different method in Ref. 28. There it was shown that the solution is unique and given by

$$r(t)=\mu_0+\sum_{n=1}^{\infty}\mu_{2n-1}+\mu_{2n}-2\nu_n(t), \qquad (182)$$

which is the well-known trace identity.

Remark 2. Owing to the representation

$$\Delta^2(\lambda)-4=4\pi^2(\mu_0-\lambda)\prod_{n>1}n^{-4}(\mu_{2n-1}-\lambda)(\mu_{2n}-\lambda)$$

in the case of finite-zone potentials the system (181) reduces to the equations

$$\frac{d\nu_n}{dt}=2\sqrt{R_N(\nu_n(t))}\left(\prod_{l\neq n}(\nu_l(t)-\nu_n(t))\right)^{-1},$$

$$n=1,2,...,N,$$

where

$$R_N(\lambda)=(\mu_0-\lambda)\prod_{n>1}(\mu_{2n-1}-\lambda)(\mu_{2n}-\lambda)$$

were first obtained by Dubrovin.[27]

Let us consider the periodic Cauchy problem for the KdV equation:

$$r_t=6rr_x-r_{xxx},\quad 0\leqslant x\leqslant\pi,\quad r(0,t)=r(\pi,t),\quad r(x,0)\in\widetilde{C}_\infty \qquad (183)$$

and together with it the Sturm–Liouville problem (175). It is well known[1] that if we introduce into $\widetilde{C}_\infty$ the Poisson bracket

$$\{F,G\}=\left[\frac{\partial F}{\partial r},\frac{\partial G}{\partial r}\right]=\left(\frac{\partial F}{\partial r},D\frac{\partial G}{\partial r}\right), \qquad (184)$$

where F and G are differentiable functionals of r, r_x, and so on, then (183) has the Hamiltonian structure

$$z_t=\frac{\partial}{\partial x}\frac{\partial H}{\partial r},\quad H(r)=\int_0^{\pi}\left(r^3(x)-\frac{1}{2}r_x^2(x)\right)dx. \qquad (185)$$

Now, following Ref. 26, we introduce the new variables $\nu_n(t)$, $f_n(t)$, which, as noted in Corollary 1 of Theorem 3.3, uniquely determine $r(x,t)$ in (183). With respect to the Poisson brackets (184), ν_n and f_n are canonical variables,[53] which follows from Eqs. (168), (169), and (172). Equation (185) takes the form

$$\frac{d}{dt}\nu_n=\frac{\partial H}{\partial f_n},\quad \frac{d}{dt}f_n=-\frac{\partial H}{\partial \nu_n}.$$

The following theorem gives the explicit form of the right-hand sides of these equations.

Theorem 5.2. For the function $r(x,t)$ to be a solution of the Cauchy problem (183), it is necessary and sufficient that $\nu_n(t)$ and $f_n(t)$ satisfy the system of equations

$$\frac{d}{dt}\nu_n=\frac{2\sqrt{\Delta^2(\nu_n(t))-4}}{\dot{\chi}(\nu_n(t))}\left\{2\nu_n(t)+\mu_0+\sum_{l=1}^{\infty}[\mu_{2l-1}+\mu_{2l}-2\nu_l(t)]\right\}, \qquad (186)$$

$$\frac{d}{dt}f_n=\frac{4\dot{\Delta}[\nu_n(t)]}{\dot{\chi}[\nu_n(t)]}\left\{2\nu_n(t)+\mu_0+\sum_{l=1}^{\infty}[\mu_{2l-1}+\mu_{2l}-2\nu_l(t)]\right\}, \qquad (187)$$

where μ_n are the zeros of the equations $\Delta^2(\lambda)-4=0$, which are determined through $r(x,0)$ and are the first integrals of the KdV equation.

Proof. We use $\widetilde{L}^*$ to denote the operator $\widetilde{\Lambda}^*$ for $r_1=r_2=r(x,t)$:

$$\widetilde{L}^*=\frac{1}{4}\left\{D^2+4r(x,t)+r_x(x,t)\left(\int_0^x-\int_x^{\pi}\right)dy\right\}. \qquad (188)$$

Applying the operator $4\widetilde{L}^*$ to both sides of (177), where $r_x=r_x(x,t), S_n^{-1}-S_n=\sqrt{\Delta^2(\nu_n)-4}$, owing to the spectral expansion (165) we obtain

$$4\widetilde{L}^*r_x=6r(x,t)r_x(x,t)-r_{xxx}(x,t)-2r_x(x,t)r(0,t)$$
$$=\sum_{n=1}^{\infty}\left\{4\nu_n(t)\frac{\sqrt{\Delta^2(\nu_n(t))-4}}{\dot{\chi}(\nu_n(t))}\widetilde{Q}_n'(x,t)-8\nu_n(t)\frac{\dot{\Delta}(\nu_n(t))}{\dot{\chi}(\nu_n(t))}\widetilde{P}_n'(x,t)\right\}.$$

From this, taking into account (176) and (177), we find that

$$r_t-6rr_x+r_{xxx}$$
$$=\sum_{n=1}^{\infty}\left\{\left[\nu_{n,t}-\frac{\sqrt{\Delta^2(\nu_n)-4}}{\dot{\chi}(\nu_n)}(4\nu_n-2r(0,t))\right]Q_n'(x)-\left[f_{n,t}-2\frac{\dot{\Delta}(\nu_n)}{\dot{\chi}(\nu_n)}[4\nu_n-2r(0,t)]\right]P_n'(x)\right\}.$$

Now in order to obtain Eqs. (186) and (187) we need to use the trace identity (182) with $r=r(0,t)$, where it is necessary to take into account the well-known fact that for the problem (183) $\mu_n(t)=\mu_n(0)$.

Remark 1. Equations (187), as was shown above, determine the sign in front of the radical in (186). In the case of the N-zone potential $r(x,0)$, Eq. (186) reduces to the finite system

$$\frac{d\nu_n}{dt}=\frac{4\sqrt{R_N[\nu_n(t)]}}{\prod_{l\neq n}[\nu_l(t)-\nu_n(t)]}\left\{\mu_0+\sum_{l=1,l\neq n}^{N}[\mu_{2l-1}+\mu_{2l}-2\nu_l(t)]\right\}, \qquad (189)$$

in which the polynomial $R_N(\lambda)$ is defined as in (181). It is well known that (189), in addition to the system (181) $(t\to x)$, can be solved explicitly using the Riemann θ function.[46] Solving the systems (189) and (181) $(t\to x)$ together, we find, as was shown in Refs. 55 and 56, the N-zone KdV solution in the form

$$r(x,t)=\mu_0+\sum_{l=1}^{N}[\mu_{2l-1}+\mu_{2l}-2\nu_l(x,t)].$$

Remark 2. The scheme described above for constructing the Hamiltonian equations in the variables ν_n, f_n is easily generalized to higher KdV equations. For example, for the Hamiltonian following from $H(r)$ (185)

$$H_2(r)=\frac{1}{2}\int_0^{\pi}\{r^2_{xx}(x)-5r^2(x)r_{xx}(x)+5r^4(x)\}dx$$

we have

$$\frac{d}{dx}\frac{\partial H_2}{\partial r}=16\widetilde{L}^{*2}r_x(x)+8r(0)\widetilde{L}^{*}r_x(x)+8r_x(x)\times[-r_{xx}(0)+3r^2(0)].$$

From this, writing out $r(0,t)$, $-r_{xx}(0,t)+3r^2(0,t)$, using the well-known trace identities (see, for example, Ref. 54), we derive, as in the proof of Theorem 5.2, the corresponding system of equations similar to the system (186), (187), the explicit form of which we omit, owing to its awkwardness. It is obvious that the system (178) is also a Hamiltonian system with $H_0=1/2\int_0^{\pi}r^2(x)dx$. Further substantive study of the properties of Hamiltonian systems of the KdV type in the periodic case requires the use of the techniques of algebraic geometry,[54] the discussion of which lies outside the scope of this study. We note that representations of the form (119) for finite-zone potentials are of interest in connection with finite-dimensional dynamical systems.[56]

6. THE SPECTRAL THEORY OF Λ OPERATORS FOR A DIRAC SYSTEM ON A FINITE INTERVAL

The theory of Λ operators related to a Dirac (Zakharov–Shabat) system on the entire axis in the class of potentials falling off for $|x|\to\infty$ played a very important role in the development of the ISM (Ref. 8). Several difficulties arise in generalizing this theory to the case of a finite interval for applications to periodic problems for equations like the nonlinear Schrödinger equation. We note that if one considers only the problem of constructing the formulas for expansion in products of the solutions of two regular Dirac operators, the analogy with the constructions for Sturm–Liouville problems is almost completely preserved.[17,57] Some differences arise in the construction of Λ operators, for which these expansions are expansions of unity.[58,59]

Let us consider two self-adjoint boundary-value problems defined by the Dirac system

$$\left(B\frac{d}{dx}+Q_j(x)\right)y^{(j)}=\lambda y^{(j)},\quad 0\leqslant x\leqslant\pi,\quad j=1,2,\qquad(190)$$

and the boundary conditions

$$y_1^{(j)}(0,\lambda)=y_1^{(j)}(\pi,\lambda)=0.\qquad(191)$$

Here

$$B=\begin{pmatrix}0 & 1\\ -1 & 0\end{pmatrix},\quad Q_j=\begin{pmatrix}p_j(x) & q_j(x)\\ q_j(x) & -p_j(x)\end{pmatrix},$$

$$y^{(j)}=\begin{pmatrix}y_1^{(j)}\\ y_2^{(j)}\end{pmatrix},$$

with real functions $p_j(x)$, $q_j(x)\in C_1[0,\pi]$, and if $Q_1=Q_2$, we omit the index j. We use $\varphi^{(j)}(x,\lambda)$ and $\psi^{(j)}(x,\lambda)$ to denote the solutions of the system (190) for which $\varphi^{(j)}(0)=\psi^{j}(\pi)=(0,1)^T$, where T denotes transposition. Let

$$\omega_j(\lambda)=W(\varphi^{(j)},\psi^{(j)})=\varphi_1^{j}(\pi,\lambda)=-\psi_1^{(j)}(0,\lambda)\qquad(192)$$

be the characteristic functions of the boundary-value problems (190), (191), and the Wronskian is

$$W(f,g)=f^TBg=f_1g_2-f_2g_1,$$

$$f=(f_1,f_2),\quad g=(g_1,g_2).$$

It is well known (see, for example, Ref. 29) that the spectra of these problems consist of the simple eigenvalues

$$\sigma_j=\{\lambda_n^{(j)}=\lambda_{2n+j}|\omega_j(\lambda_{2n+j})=0\}_{n=-\infty}^{\infty},$$

$$\lambda_n^{(j)}\underset{n\to\pm\infty}{=}n+O(1),$$

where $\dot{\omega}_j(\lambda_{2n+j})\neq0$; $'=\partial/\partial\lambda$. The eigenfunctions are

$$\varphi^{(j)}(x,\lambda_n^{(j)})=C_{2n+j}\psi^{(j)}(x,\lambda_n^{(j)}),$$

$$C_{2n+j}=\varphi_2^{(j)}(\pi,\lambda_n^{(j)})=\psi_2^{(j)-1}(0,\lambda_n^{(j)})\qquad(193)$$

and their norms are

$$\alpha_{2n+j}=\|\varphi_{2n+j}\|_{L_2}^{-2}=-C_{2n+j}^{-1}\dot{\omega}_j^{-1}(\lambda_{2n+j}),$$

$$\beta_{2n+j}=\|\psi_{2n+j}\|_{L_2}^{-2}=-C_{2n+j}\dot{\omega}_j^{-1}(\lambda_{2n+j}),$$

where $(f,g)=\int_0^{\infty}(f_1(x)g_1(x)+f_2(x)g_2(x))dx$ is the scalar product in the space $L_2^{(2)}$ of the vector functions $f=(f_1,f_2)$ and $\|f\|=(f,f)^{1/2}$. We also introduce the skew-scalar product

$$[f,g]=(f,Bg)=-[g,f].\qquad(194)$$

We define, following Ref. 57, the product $Y(x,\lambda)$ of the solutions $y^{(1)}(x,\lambda)$ and $y^{(2)}(x,\lambda)$ of Eqs. (190) as

$$Y(x,\lambda)=y^{(1)}\circ y^{(2)}=(y_1^{(1)}y_1^{(2)}-y_2^{(1)}y_2^{(2)},y_1^{(1)}y_2^{(2)}+y_2^{(1)}y_1^{(2)})^T.\qquad(195)$$

The following operators play a fundamental role in our constructions:

$$\Lambda_{0(\pi)}=\left(L_{0(\pi)}=\frac{1}{2}B\frac{d}{dx}+\frac{1}{2}U(x)\int_{0(\pi)}^{x}[BU(y)]^Tdy,\ -\frac{1}{2}v^{(+)}(x)\right),\qquad(196)$$

where

$$U(x)=\begin{pmatrix} -q^{(-)} & p^{(+)} \\ p^{(-)} & q^{(+)} \end{pmatrix},$$

$$\begin{matrix} p^{(\pm)}=p_2\pm p_1 \\ q^{(\pm)}=q_2\pm q_1 \end{matrix} \quad v^{(+)}=\begin{pmatrix} p^{(+)} \\ q^{(+)} \end{pmatrix},$$

which operate according to the expressions

$$\Lambda_{0(\pi)}f^{(0(\pi))}(x)=L_{0(\pi)}f(x)-\tfrac{1}{2}v^{(+)}(x)f_1(0(\pi)),$$

where

$$f^{(0(\pi))}=\{f(x)=[f_1(x),f_2(x)]^T,f_1[0(\pi)]\}^T. \qquad (197)$$

The domain of definition is

$$\mathscr{D}(\Lambda_0)=\{f\in C_1^{(2)}\mid f_1(\pi)-f_1(0)-[v^{(+)},f]=0,f_2(0)=0\},$$

where $C_1^{(2)}=C_1^{(2)}[0,\pi]$ is the space of continuously differentiable vector functions $f(x)$. Since for any $f,g\in C_1^{(2)}$ we have the identity

$$\begin{aligned}&[\Lambda_0 f^{(0)},g]-[f,\Lambda_\pi f^{(\pi)}]\\ &=\tfrac{1}{2}\{f_1(0)(g_1(\pi)-g_1(0)-[v^{(+)},g])\\ &\quad+g_1(\pi)(f_1(\pi)-f_1(0)-[v^{(+)},f])\\ &\quad+f_2(\pi)g_2(\pi)-f_2(0)g_2(0)\},\end{aligned}$$

we have the following lemma.

Lemma 6.1. The operator Λ_π with the domain of definition

$$\mathscr{D}(\Lambda_\pi)=\{f\in C_1^{(2)}\mid f_1(\pi)-f_1(0)-[v^{(+)},f]=0,f_2(\pi)=0\}$$

is conjugate to the operator Λ_0 under the skew-scalar product (194), i.e.,

$$[\Lambda_0 f^{(0)},g]=[f,\Lambda_\pi f^{(\pi)}],\quad f\in\mathscr{D}(\Lambda_0),\quad g\in\mathscr{D}(\Lambda_\pi).$$

In addition to $Y(x,\lambda)$ we introduce the product

$$\begin{aligned}\hat{Y}(x,\lambda)&=y^{(1)}*y^{(2)}\\ &=(y_2^{(1)}y_1^{(2)}-y_1^{(1)}y_2^{(2)},y_1^{(1)}y_1^{(2)}+y_2^{(1)}y_2^{(2)}).\end{aligned}$$

The relation between $Y(x,\lambda)$ and $\hat{Y}(x,\lambda)$ is given by the identity

$$\frac{d}{dx}\hat{Y}(x,\lambda)=[BU(x)]^TY(x,\lambda), \qquad (198)$$

where

$$B\frac{d}{dx}Y(x,\lambda)+U(x)\hat{Y}(x,\lambda)=2\lambda Y(x,\lambda). \qquad (199)$$

From this it follows that $Y(x,\lambda)$ satisfies the equation

$$B\frac{d}{dx}Y(x,\lambda)+U(x)\int_{x_0}^{x}[BU(y)]^T Y(y,\lambda)dy+U(x)\hat{Y}(x_0,\lambda)=2\lambda Y(x,\lambda).$$

Here, setting $x_0=0,\pi$ and using the equations $\Phi(0,\lambda)=\Psi(\pi,\lambda)=(-1,0)$, we obtain:

Theorem 6.1. The vector functions $\Phi^{(0)}(x,\lambda)$ and $\Psi^{(\pi)}(x,\lambda)$ constructed using Eqs. (197), where

$$\Phi(x,\lambda)=\varphi^{(1)}\circ\varphi^{(2)},\quad \Psi(x,\lambda)=\psi^{(1)}\circ\psi^{(2)},$$

satisfy the equations

$$\Lambda_0\Phi^0(x,\lambda)\equiv L_0\Phi(x,\lambda)+\tfrac{1}{2}v^{(+)}(x)=\lambda\Phi(x,\lambda), \qquad (200)$$

$$\Lambda_\pi\Psi^{(\pi)}(x,\lambda)\equiv L_\pi\Psi(x,\lambda)+\tfrac{1}{2}v^{(+)}(x)=\lambda\Psi(x,\lambda). \qquad (201)$$

Here

$$\Phi_2(0,\lambda)=0,\Phi_1(\pi,\lambda)+1-[v^{(+)},\Phi(\lambda)]=2\Omega(\lambda), \qquad (202)$$

$$\Psi_2(\pi,\lambda)=0,1+\Psi_1(0,\lambda)+[v^{(+)},\Psi(\lambda)]=2\Omega(\lambda), \qquad (203)$$

where $\Omega(\lambda)=\omega_1(\lambda)\omega_2(\lambda)$. [The second equations in (202) and (203) follow from (198), owing to (192).]

Let us now consider the boundary-value problem

$$\Lambda_0 Y^{(0)}(x,\lambda)=\lambda Y(x,\lambda),$$

$$Y_2(0)=0,\quad Y_1(\pi)-Y_1(0)-[v^{(+)},Y]=0. \qquad (204)$$

We introduce the matrix

$$G_0(x,y,\lambda)=\frac{1}{\Omega(\lambda)}\begin{cases}\Phi(x,\lambda)\tilde{\Psi}(y,\lambda), & x<y<\pi,\\ \sum_{j=1,2}S^{(j)}(x,\lambda)\tilde{S}^{(3-j)}(y,\lambda)-\Psi(x,\lambda)\tilde{\Phi}(y,\lambda), & 0<y<x,\end{cases}$$

where $S^{(j)}=\psi^{(j)}\circ\varphi^{(3-j)}$, $\tilde{Y}=(BY)^T$. Then:

Theorem 6.2. The matrix G_0 is the kernel of the resolvent of the boundary-value problem (204). More precisely, for $\lambda\in\mathbb{C}\setminus\cup_{j=1,2}\sigma_j=\rho(\Lambda_0)$ the vector function

$$(\Lambda_0-\lambda I)^{-1}f(x)=R_0(f;x,\lambda)=\textstyle\int_0^\pi G_0(x,y,\lambda)f(y)dy,$$

where $I=\left(\begin{smallmatrix}1&0\\0&1\end{smallmatrix}\right)$, satisfies the equations

$$(\Lambda_0-\lambda I)R_0(f;x,\lambda)=f(x),R_0(f;x,\lambda)\in\mathscr{D}(\Lambda_0) \qquad (205)$$

and

$$R_0[(\Lambda_0-\lambda I)f;x,\lambda]=f(x),\quad f\in\mathscr{D}(\Lambda_0).$$

Proof. We set $H(x)=\Omega(\lambda)R_0(f;x,\lambda)$. Then Eq. (205) has the form

$$B\frac{d}{dx}H(x)+U(x)\int_0^x[BU(s)]^TH(s)ds-v^{(+)}(x)H_1(0)-2\lambda H(x)=2\Omega(\lambda)f(x). \qquad (206)$$

From (199), taking into account (192), we find

$$B\frac{d}{dx}H(x)=2\Omega(\lambda)f(x)+2\lambda H(x)-U(x)\times\left\{\hat{\Phi}(x)\int_x^\pi\tilde{\Psi}(y)f(y)dy+\sum_{j=1,2}\hat{\tilde{S}}^{(j)}(x)\int_0^x\tilde{S}^{(3-j)}(y)f(y)dy\right.$$

$$- \hat{\Psi}(x) \int_0^x \tilde{\Phi}(y) f(y) dy\} . \tag{207}$$

Furthermore, owing to (198),

$$U(x) \int_0^x [BU(y)]^T H(y) dy$$

$$= - U(x)\hat{\Phi}(0) \int_0^\pi \tilde{\Psi}(y) f(y) dy + \int_0^x \{\hat{\Phi}(y)\tilde{\Psi}(y)$$

$$- \sum_{j=1,2} \hat{S}^{(j)}(y)\tilde{S}^{(3-j)}(y) + \hat{\Psi}(y)\tilde{\Phi}(y)\} f(y) dy$$

$$+ U(x)\{...\},$$

where the brackets contain the same expression as in (207). Since the integrated matrix in the second term in this equation is equal to zero, in order to obtain (206) we need only note that $U(x)\hat{\Phi}(0) = v^{(+)}(x)$. The other statements of the theorem are proved in a similar manner.

Corollary. The matrix $G_\pi(x,y,\lambda) = - G_0(y,x,\lambda)$ determines the kernel of the resolvent of the boundary-value problem conjugate to (204):

$$\Lambda_\pi Y^{(\pi)}(x,\lambda) = \lambda Y(x,\lambda), \quad Y_2(\pi) = 0,$$

$$Y_1(\pi) - Y_1(0) - [v^{(+)}, Y] = 0.$$

Remark. The statements of Lemma 6.1 and Theorems 6.1–6.3 remain valid for any, possibly complex-valued, $p_j(x)$, $q_j(x)$.

Let us now show that, following the scheme of Sec. 4, we can construct the spectral expansions for the operators Λ_0 and Λ_π. Let the sets $\sigma = \sigma_1 \cup \sigma_2$, $\sigma'' = \sigma_1 \cap \sigma_2$, $\sigma' = \sigma \backslash \sigma''$ be determined from the spectra σ_j.

We introduce the systems $\{U_n\}$ and $\{V_n\}$, taking for $\lambda_{2n+j} \in \sigma'$

$$U_{2n+j} = \dot{\Omega}^{-1}(\lambda_{2n+j})\Phi(x,\lambda_{2n+j}),$$

$$V_{2n+j} = \Psi(x,\lambda_{2n+j})$$

and for $\lambda_{2n+1} = \lambda_{2n+2} = \lambda_n \in \sigma''$

$$U_{2n+1} = 2\ddot{\Omega}^{-1}(\lambda_{(n)})\Phi(x,\lambda_{(n)}),$$

$$U_{2n+2} = 2\ddot{\Omega}^{-1}(\lambda_{(n)})\dot{\Phi}(x,\lambda_{(n)}),$$

$$V_{2n+1} = \dot{\Psi}(x,\lambda_{(n)}) - \dddot{\Omega}(\lambda_{(n)})[3\ddot{\Omega}(\lambda_{(n)})]^{-1}\Psi(x,\lambda_{(n)}),$$

$$V_{2n+2} = \Psi(x,\lambda_{(n)})$$

Using the identity [following from Eqs. (190)]

$$[Y(\lambda), Z(\mu)] = (\mu - \lambda)^{-1} \prod_{j=1,2} W[y^{(j)}(x,\lambda), z^{(j)}(x,\mu)]\Big|_{x=0}^{\pi}, \tag{208}$$

we can establish the following biorthogonality relations:

$$[V_n, U_m] = \delta_{nm}, \quad n,m \in \mathbb{Z}.$$

Furthermore, Eqs. (200), (201) together with (202), (203) lead to:

Lemma 6.2. $U_n \in \mathscr{D}(\Lambda_0)$, $V_n \in \mathscr{D}(\Lambda_\pi)$. Here, if $\lambda_n \in \sigma'$, we have

$$\Lambda_0 U_n^{(0)}(x) = \lambda_n U_n(x), \quad \Lambda_\pi V_n^{(\pi)}(x) = \lambda_n V_n(x),$$

and for $\lambda_{(n)} \in \sigma''$ we have

$$\Lambda_0 U_{2n+1}^{(0)}(x) = \lambda_{(n)} U_{2n+1}(x),$$

$$\Lambda_0 U_{2n+2}^{(0)}(x) = \lambda_{(n)} U_{2n+2}(x) + U_{2n+1}(x),$$

$$\Lambda_\pi V_{2n+1}^{(\pi)}(x) = \lambda_{(n)} V_{2n+1}(x) + V_{2n+2}(x),$$

$$\Lambda_\pi V_{2n+2}^{(\pi)}(x) = \lambda_{(n)} V_{2n+2}(x).$$

The main theorem of this section is the following.

Theorem 6.2 (on the spectral expansions of the operators Λ_0 and Λ_π).

I. For every, possibly complex-valued, function $f \in L_2^{(2)}$ we have the expansion formulas

$$f(x) = \sum_{n=-\infty}^{n} - U_n(x)[f, V_n],$$

$$f(x) = \sum_{n=-\infty}^{\infty} V_n(x)[f, U_n], \tag{209}$$

where the series converge in the sense of the norm of $L_2^{(2)}$ as the $N \to \infty$ limit of the partial sums $S_\pi(N,f;x) = \Sigma_{n\in\Delta(N)} V_n(x)[f,U_n]$ and $S_0(N,f;x) = \Sigma_{n\in\Delta(N)} - U_n(x)[f,V_n]$, where $\Delta(N) = (-2N+1, -2N+2,...,2N+2)$.

II. For any $f \in \mathscr{D}(\Lambda_0)$ we have

$$\Lambda_0 f^{(0)}(x) = \sum_{\lambda_{2n+j} \in \sigma'} - \lambda_{2n+j} U_{2n+j}(x)[f, V_{2n+j}]$$

$$- \sum_{\lambda_{(n)} \in \sigma''} \{\lambda_{(n)} U_{2n+1}(x)[f, V_{2n+1}]$$

$$+ [\lambda_{(n)} U_{2n+2}(x) + U_{2n+1}(x)]$$

$$\times [f_1 V_{2n+2}]\} \tag{210}$$

and for any $f \in \mathscr{D}(\Lambda_\pi)$ we have

$$\Lambda_\pi f^{(\pi)}(x) = \sum_{\lambda_{2n+j} \in \sigma'} \lambda_{2n+j} V_{2n+j}(x)[f, U_{2n+j}]$$

$$+ \sum_{\lambda_{(n)} \in \sigma''} \{\lambda_{(n)} V_{2n+2}(x)[f, U_{2n+2}]$$

$$+ [\lambda_{(n)} V_{2n+1}(x) + V_{2n+2}(x)]$$

$$\times [f, U_{2n+1}]\},$$

where the series converge in the sense of the norm of $L_2^{(2)}$.

Proof. The expansion formulas (209) are proved just as in Ref. 57, and here we only outline the proof. We use $c_N = (N + \frac{1}{2}) \exp(i\varphi)$, $0 \leqslant \varphi < 2\pi$, to denote a circle in the λ plane and consider the contour integral

$$I_0(N, f; x) = \frac{1}{2\pi i} \oint_{c_N} \left\{ \int_0^\pi G_0(x,y,\lambda) f(y) dy \right\} d\lambda.$$

From the residue theorem, and taking into account (193), we find that for sufficiently large N we have $I_0(N,f;x) = S_0(N,f;x)$. Furthermore, owing to the known asymptotes

$$\varphi(x,\lambda)=\begin{pmatrix} -\sin\lambda x \\ \cos\lambda x\end{pmatrix}+O\left(\frac{1}{\lambda}e^{|\mathrm{Im}\,\lambda|x}\right),$$

$$\psi(x,\lambda)=\begin{pmatrix} -\sin\lambda(x-\pi) \\ \cos\lambda(x-\pi)\end{pmatrix}+O\left(\frac{1}{\lambda}e^{|\mathrm{Im}\,\lambda|(\pi-x)}\right)$$

we can find by the standard method using Jordan's lemma that $\lim_{N\to\infty}|S_0(N,f;x) - s_0(N,f;x)| = 0$ and $\lim_{N\to\infty}\sup_{0<x<\pi}|S_0(N,f;x) - s_0(N,f;x)| < \infty$ uniformly in x in any interval $\Delta\subset(0,\pi)$. Here $s_0(N,f;x)$ is the sum of the series obtained, as above, by calculation of the integral $(2\pi i)^{-1}\oint\{\int_0^\pi\Gamma_0(x,y,\lambda)f(y)dy\}d\lambda$, where Γ_0 is the matrix obtained from G_0 for $Q_j\equiv 0$. The explicit form of this series, which is analogous to the series (44), (45), is given in Ref. 57, and we omit it here. We only note that the convergence $\lim_{N\to\infty}s_0(N,f;x) = f(x)$ is established using the criterion for forming a basis.[39] In order to obtain the spectral expansion (210), in (209) we must replace the function $f(x)$ by $\Lambda_0 f^{(0)}(x)$, $f\in\mathscr{D}(\Lambda_0)$, and take into account Lemmas 6.1 and 6.2.

We conclude this section by considering the case where $\Delta=\varnothing$, i.e., the problems (190), (191) are isospectral. We introduce the operator

$$\Lambda=\tfrac{1}{2}(\Lambda_0+\Lambda_\pi)=(L=\tfrac{1}{2}(L_0+L_\pi),-\tfrac{1}{2}v^{(+)}(x)),\qquad(211)$$

acting on the functions

$$f^{(s)}=(f(x)=[f_1(x),f_2(x)]^T,\tfrac{1}{2}[f_1(0)+f_1(\pi)])^T$$

according to the expression

$$\Lambda f^{(s)}=Lf(x)-\tfrac{1}{4}[f_1(0)+f_1(\pi)]v^{(+)}(x).$$

Following the proof of Lemma 6.1, it is easy to prove:

Lemma 6.3. The operator Λ with domain of definition

$$\mathscr{D}(\Lambda)=\{f\in C_1^{(2)}\,|\,f_2(0)=f_2(\pi),f_1(\pi)-f_1(0)-[v^{(+)},f]=0\}$$

is self-adjoint under the skew-scalar product (194), i.e.,

$$[\Lambda f^{(s)},g]=[f,\Lambda g^{(s)}],\quad f,g\in\mathscr{D}(\Lambda).$$

Let us consider the boundary-value problem

$$\Lambda Y^{(s)}(x,\lambda)=\lambda Y(x,\lambda),\quad Y_2(0)=Y_2(\pi),$$
$$Y_1(\pi)-Y_1(0)-[v^{(+)},Y]=0.\qquad(212)$$

Following, for example, the proof of Theorem 1.3, Theorem 6.2 leads to:

Theorem 6.3. Let the following functions be constructed from the boundary-value problems (190), (191), where $\sigma_1=\sigma_2$:

$$P_n(x)=-\frac{C_{2n+1}}{\dot\omega(\lambda_n)}\Psi(x,\lambda_n),$$

$$Q_n(x)=\frac{1}{2\dot\omega(\lambda_n)}\{C_{2n+1}^{-1}\dot\Phi(x,\lambda_n)-C_{2n+2}\dot\Psi(x,\lambda_n)\},$$

where C_{2n+j} are given by (193). Then:

I. The functions P_n and Q_n are the eigenfunctions of the boundary-value problem (212), i.e.,

$$\Lambda P_n^{(s)}(x)=\lambda_nP_n(x),\quad \Lambda Q_n^{(s)}(x)=\lambda_nQ_n(x),\quad n\in\mathbb{Z}.$$

II. The system $\{P_n,Q_n\}$ is a symplectic basis in the space L_2, i.e., for any $f\in L_2^{(2)}$ we have the expansion

$$f(x)=\sum_{n=-\infty}^{n}\{Q_n(x)[f,P_n]-P_n(x)[f,Q_n]\},\qquad(213)$$

where

$$[Q_n,P_m]=\delta_{nm},\quad [P_n,P_m]=[Q_n,Q_m]=0.\qquad(214)$$

III. For any $f\in\mathscr{D}(\Lambda)$ we have

$$\Lambda f^{(s)}(x)=\sum_{n=-\infty}^{n}\{\lambda_nQ_n(x)[f,P_n]-\lambda_nP_n(x)[f,Q_n]\},$$

i.e., the expansion (213) is the expansion of unity for the operator Λ.

We now recall that if in the space of functionals of the periodic vector function $v = (p,q)\in C_\infty^{(2)}(0,\pi)[v(x) = v(x+\pi)]$ we introduce the Poisson bracket

$$\{F,G\}=\left[\frac{\partial F}{\partial v},\frac{\partial G}{\partial v}\right],\quad \frac{\partial F}{\partial v}=\left(\frac{\partial F}{\partial p},\frac{\partial F}{\partial q}\right),$$

where the skew-scalar product [,] is defined by (194), then the nonlinear Schrödinger equation (in the self-adjoint case) can be written as the Hamiltonian system

$$v_t=B\frac{\partial H_2}{\partial v},\quad H_2=\frac{1}{2}\int_0^\pi\{p_x^2+q_x^2+(p^2+q^2)^2\}dx,\qquad(215)$$

where the symplectic matrix B is defined as in (190). The results given above allow us, essentially following the construction of Sec. 5, to introduce as new variables the spectral quantities $\lambda_n(v)$ and $f_n(v) = -\ln|C_n|$, which uniquely determine the potential $v(x)$ in the operator (190). Since the functions P_n and Q_n determine the gradients $\partial\lambda_n/\partial v$ and $\partial f_n/\partial v$, respectively, it follows from (214) that these variables are canonical. In order to obtain the explicit form of the system (215) in the new variables λ_n, f_n, we must take into account the fact that this system can be written as $Bv_t = 4\Lambda^2v^{(s)}(x,t) + 4p(0,t)\Lambda v^{(s)}(x,t) + 2(q_x(0,t) + q^2(0,t) + q^2(0,t))v(x,t)$ and then use the representation

$$v(x,t)=\sum_{n=-\infty}^{\infty}\{(C_n-C_n^{-1})\dot\omega^{-1}(\lambda_n)Q_n(x,t)+\dot\Delta(\lambda_n)\dot\omega^{-1}(\lambda_n)P_m(x,t)\},\qquad(216)$$

where

$$C_n-C_n^{-1}=\pm\sqrt{\Delta^2(\lambda_n)-4},$$
$$\Delta(\lambda)=\psi_2(0,\lambda)+\varphi_2(\pi,\lambda)$$

is the Hill discriminant of the operator (190), and the summation runs over those λ_n which lie in nondegenerate lacunae of the periodic spectrum. We can also use this scheme to obtain the equations for the inverse periodic problem starting from the Hamiltonian $H_1 = \int_0^\pi pq_x dx$, which leads to the Cauchy problem

$$v_t = v_x \equiv B\partial H_1/\partial v, \quad v(t,x) = v(t,x+\pi). \tag{217}$$

From this, since $\partial H_1/\partial v = 2\Lambda v^{(s)}(x,t) + 2p(0,t)v(x,t)$, and owing to (216), we obtain the system

$$\lambda_{n,t} = (2\lambda_n + 2p(0,t)) \frac{\sqrt{\Delta^2(\lambda_n) - 4}}{2\dot{\omega}(\lambda_n)},$$

$$f_{n,t} = -(2\lambda_n + 2p(0,t))\dot{\Delta}(\lambda_n)/2\dot{\omega}(\lambda_n),$$

where $2p(0) = \Sigma_{n=-\infty}^{\infty}\{-2\lambda_n + \mu_n^- + \mu_n^+\}$, $\Delta(\mu_n^\pm) = \pm 2$. Equations for the periodic problem (217) similar to those given here have been obtained in Ref. 60.

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