Few-baryon systems in the chiral-field soliton model

V. A. Nikolaev and O. G. Tkachev

Joint Institute for Nuclear Research, Dubna

Fiz. Elem. Chastits At. Yadra **21**, 1499–1538 (November–December 1990)

The main results obtained in recent years on the construction of a theory of nuclear states using the Skyrme model are reviewed. In this approach nuclei are associated with the quantum solitons of a chiral field and have nontrivial topology. The results of calculating the effective Hamiltonian, the mass formulas, the static electromagnetic characteristics, and the form factors of the lightest nuclei and nucleons within a unified approach are given. The possible role of the gauge bosons of the hidden symmetry of the Skyrme model and also the role of the dilaton–quarkonium in the structure of few-baryon systems are discussed.

INTRODUCTION

The chiral soliton model as a model of baryons and baryonic systems was born in the 1960s in the fundamental studies of Skyrme.[1] The main statement in these papers is essentially that the topological solitons of a chiral field (solitons with nonzero topological charge) can represent all the properties of baryons and their interactions if their topological charge is identified with the baryon number. The solitons in question are those of the $SU(2)\times SU(2)$ chirally symmetric model of a pseudoscalar field. The Skyrme model was the simplest generalization of the nonlinear σ model,[2] which has stable soliton solutions with integer topological charge. The nonlinear σ model is a theory of a self-interacting pion field in which the interaction is introduced via a chirally symmetric constraint involving an auxiliary scalar meson field. In quantum chromodynamics (QCD), which later was developed as a gauge theory of strong interactions, the low-energy properties of baryons and baryonic systems are associated with the "nonperturbative" regime. The nontrivial and essentially nonlinear properties of the QCD vacuum are very important in this energy range.[3,4]

Meanwhile, intense development began, and continues, of a phenomenological approach to the description of the structure of baryons and their interactions on the basis of effective chiral meson fields.[5-8] Great efforts have been made to justify the approach using an effective chiral Lagrangian for describing strongly interacting systems.[9-14] Some success has been achieved in this area in recent years. We shall not dwell on these studies, but only mention some of the main points. The quark and gluon degrees of freedom in the QCD generating functional are replaced by effective bosonic degrees of freedom, leading to a new nonlocal generating functional for the bosonic fields. By expansion in the derivatives of the fields, a local Lagrangian is obtained which dictates the equations of motion for the bosonic fields. In the approximation where the fields vary slowly in space, only the lowest derivatives need be kept in the Lagrangian, and this leads to the Skyrme model or generalizations of it.

In 1983 Witten[15,16] showed that the Skyrme idea can be extended to numbers of flavors $N_f\geqslant 3$. He succeeded in showing that solitons with odd topological charge are identified with fermions if the number of colors N_c involved in the gauge theory of the strong interactions is odd. In QCD with $N_c = 3$ solitons with odd topological charge are fermions, and solitons with even topological charge are bosons. Therefore, the topological charge must be identified with the baryon number, since there are no other fermions except baryons involved in the strong interaction. The fundamental hypothesis of the Skyrme model was thereby justified.

Although, in general, bosonized QCD (the bosonic representation of QCD) must include an infinite number of meson fields, it can be expected that the lightest mesons will be important in the sense of the quantitative magnitude of their contributions to the low-energy properties of baryons and nuclei. This is why, keeping only the lightest mesons, in studying the low-energy properties of baryons it is possible to begin with the Skyrme model and when necessary resort to certain generalizations of it. Among the possible direct generalizations are models which reproduce the conformal anomaly of QCD,[17-21] including the dilaton field, and also models including vector mesons, the presence of which reflects a hidden chiral gauge symmetry of the Skyrme model.[22-24] They involve a small number of parameters, which in principle can be derived from QCD, and are treated as fitted to some data. The first calculations[25] using the Skyrme model have already shown that the observed static properties of baryons are reproduced with an error of no more than 30%. The use of the generalized variant of the model including vector-meson fields makes it possible to satisfactorily describe not only the static baryon observables, but also the baryon form factors.[26]

The application of the method of topological chiral solitons to a system of two interacting nucleons has proved to be extremely fruitful. The most characteristic features of the nucleon–nucleon interaction are already predicted by the original Skyrme model, for example, the one-pion exchange interaction,[1] the short-range repulsion,[27] the isospin-dependent spin–orbit interaction,[28] and the long-range part of the isovector electromagnetic exchange current operators.[39] In the topological chiral soliton model the operators of the isoscalar electromagnetic currents of baryons and nuclei are proportional to the model-independent anomalous baryon-current operator. The isoscalar exchange current operators are being investigated on this basis.[30]

The successes of the Skyrme model[1] in describing nucleons as the quantum states of chiral solitons make it natural to apply the model to nuclei. In traditional nuclear physics nuclei are viewed as bound states of nucleons. The first description of nuclei using the Skyrme model was based on the potential approximation.[27,31] In this approach it is necessary to first compute the interaction potential between Skyrmions for all separations R. Nuclei can be viewed as bound states of Skyrmions in this potential. There are several difficulties encountered in this construction of nuclear states which should be noted. First, the Skyrmion separation

 0090-4759/90/060643-18$04.00

is defined absolutely arbitrarily, and the interaction potentials are meaningful only at large distances, where they must be identified with the one-boson exchange potential. In order to satisfy this condition, the so-called product ansatz is used for chiral-field configurations. Here we encounter the problem of finding the attractive part of the central potential at intermediate separations, which is responsible for nuclear binding. This problem has been under intense discussion recently and will probably be solved soon. It has also been noted that attraction in the intermediate region, which is independent of the state of the central part of the potential, can arise also from quantum effects, such as pion fluctuations or soliton-deformation effects.[32-34]

The potential approximation is not the only possible one in describing nuclear states, and it is also not natural to the Skyrme model. In this model there is a more direct method of constructing systems with arbitrary baryon charge. Here we first seek solitons of the classical fields with the corresponding topological charge and then quantize the soliton degrees of freedom so as to obtain an object with the nuclear quantum numbers. From studies of systems with baryon number 2, 3, and 4, neglecting vibrations[35-39] and including the breathing mode,[40,41] it can be concluded that the problem of the absence of attraction at intermediate distances is probably a technical rather than a fundamental one, or simply an artifact of the potential approximation. This can be seen from the fact that even the first calculations carried out using the variational method and the more refined numerical calculations lead to a large binding energy for systems with baryon number $B=2$. Naturally, in this approach the distance R does not appear anywhere, but if necessary it can serve as a variational parameter determining the field configuration. In general, in this approach to the theory of nuclear states it must be assumed that the nucleons themselves can be created only near the nuclear surface and, strictly speaking, do not exist inside nuclei. In fact, in order to get the nucleon quantum numbers, the Skyrmion must be able to rotate freely in coordinate and isotopic space. However, there are no such conditions on it inside the nucleus, since the Skyrmion interaction potential depends on the relative orientation of the Skyrmions in the two spaces. As a result, only the nucleus as a whole can have good quantum numbers.[42]

The most striking results of the calculations of nuclear states have been obtained by direct numerical methods for configurations with minimum energy and topological charge $B=2$ (Refs. 36 and 43). Recently, a new variational ansatz was proposed independently in Refs. 44 and 45. This ansatz satisfies the symmetry conditions formulated in Refs. 37 and 38, and, since its structure is very simple, it makes it possible to advance another step in the analytic study of the problem, that is, to include some vibrational modes in the consideration. In this way, among the new solutions are ones which can be interpreted as compound-nucleus states, whose structure includes antinucleons. Such states can appear in reactions of light nuclei with stopped antinucleons. The discovery of such states would be evidence in favor of the model of baryons and nuclei as the solitons of a chiral field.

In this review we present a scheme for calculating the effective quantum Hamiltonian describing the rotational and some vibrational degrees of freedom in the collective-variable method. We also give the main results known at present on the calculation of the spectrum of this Hamiltonian and the properties of states with the quantum numbers of the lightest nuclei.

The reviews of the Skyrme model which have appeared in recent years have mainly been devoted to the one-baryon sector.[46-49] Here we attempt to systematize the results obtained for few-baryon states in the Skyrme model and some of its generalizations.

1. PROPERTIES OF STATIC FIELD CONFIGURATIONS OF FEW-BARYON SYSTEMS

The $SU(2)$ Skyrme model is defined by the Lagrangian density

$$\mathscr{L}=\frac{F_\pi^2}{16}\,\mathrm{Tr}\,L_\mu L_\mu+\frac{1}{32e^2}\,\mathrm{Tr}\,[L_\mu L_\nu]^2, \tag{1}$$

constructed using the left-invariant Cartan forms

$$L_\mu=U^+\partial_\mu U, \tag{2}$$

where the $SU(2)$ matrices $U(x)$ in the exponential parametrization are

$$U=\exp\left\{\mathrm{i}\,\frac{2}{F_\pi}\,\boldsymbol{\tau}\cdot\boldsymbol{\pi}\right\} \tag{3}$$

and are given by the triplet of pion fields $\boldsymbol{\pi}$. In the last equation τ_i are the Pauli matrices. The Lagrangian defines a model characterized by spontaneous breakdown of chiral $SU(2)_L\times SU(2)_R$ symmetry. The solitons of this model carry an exactly conserved topological charge corresponding to the trivially conserved topological current

$$J_\mu^B=-\frac{1}{24\pi^2}\,\varepsilon_{\mu\nu\rho\sigma}\,\mathrm{Tr}\,(L_\nu L_\rho L_\sigma). \tag{4}$$

The solitons are extended, strongly interacting objects. They are very massive in comparison with the fields appearing in the Lagrangian. Because of these properties, the quantum states of the solitons can be identified with baryons and baryonic systems (nuclei).

Generalized ansatz for the static solutions

It is well known that historically the first stationary solution of the equations of the Skyrme model in the sector with topological charge n was the hedgehog-type configuration or the Skyrme–Witten solution:

$$U_{\mathrm{SW}}(\mathbf{r})=\cos F(r)+\mathrm{i}\,(\boldsymbol{\tau}\cdot\mathbf{N})\sin F(r). \tag{5}$$

Here $\mathbf{N}$ specifies a direction in isotopic space, and, strictly speaking, the hedgehog configuration is specified by the vector $\mathbf{N}=\mathbf{r}/|\mathbf{r}|$. In (5), $F(r)$ is the chiral angle describing the absolute value of the pion field. The function $F(r)$ satisfies the following boundary conditions: $F(0)=n\pi$, $F(\infty)=0$. These conditions ensure the finiteness of the energy for solitons with topological charge n equal to the baryon number B. As has been shown in Ref. 50, the hedgehog configuration ensures the absolute minimum of the energy of the soliton with $n=1$. However, for other sectors solutions of this type do not necessarily ensure even a local minimum.

Let us discuss the sector with baryon number $B=2$ and try to answer the question of whether or not the deuteron can be identified with the quantum state of a soliton of topological charge equal to two. We shall dwell in somewhat more detail on the problem of seeking configurations with minimum energy in this sector. As already noted, the initial efforts were concentrated on the calculation of the internucleon potential in the Skyrme model. These studies were based on Skyrme's observation that the $B=2$ solution describing well separated Skyrmions can be approximated by the product ansatz. In the reference frame in which the centers of the Skyrmions are symmetrically located relative to origin on the x axis the product ansatz is

$$U(\mathbf{r},\ s,\ A)=U_1(\mathbf{r}+s\hat{x})\,AU_1(\mathbf{r}-s\hat{x})\,A^+, \qquad (6)$$

where U_1 is the hedgehog solution minimizing the energy in the sector with $B=1$. In Eq. (5) the $SU(2)$ matrix A describes the relative isospin orientation of the two Skyrmions. The potential $V(s,A)$ as a function of the relative separation $2s$ in this approach is determined by the difference of the energy E of the configuration (6) and the two masses M_1 of Skyrmions with $B=1$:

$$V(s,\ A)=E\,[U\mathbf{r},\ s,\ A]-2M_1. \qquad (7)$$

The internucleon potential is then obtained by projecting $V(s,A)$ (as a function of A) on states of definite spin and isospin (Refs. 27, 31, and 51–53). In Ref. 54, $V(s,A)$ was estimated as a function of s for various values of A. According to the results obtained in Ref. 54, the interaction energy is minimized if the isospin axes of one of the Skyrmions are rotated 180° relative to some direction perpendicular to the line on which the Skyrmions are localized. This corresponds to the matrix $A=\exp\{i\pi(\mathbf{n}\cdot\boldsymbol{\tau}/2\}$, where $\mathbf{n}$ is a unit vector perpendicular to the x axis. For definiteness we can take $n=\hat{z}$, which corresponds to $A=i\tau_3$. Therefore, we arrive at the configuration

$$U_2(x,\ y,\ z)=U_1(x+s,\ y,\ z)\,\tau_3\cdot U_1(x-s,\ y,\ z)\,\tau_3. \qquad (8)$$

This configuration reaches its minimum at a distance $s=s_0=2.8/eF_\pi$, which corresponds to the classical binding energy $-V(s_0,i\tau_3)=1.06F_\pi/e$ for the following values of the Skyrme-model constants selected by the authors: $F_\pi=108$ MeV, $e=4.84$, and $m_\pi/eF_\pi=0.263$.

Quantization of the rotational degrees of freedom using the method of Ref. 25 led to two closely spaced quantum states with the quantum numbers of the deuteron: spin $s=1$, isospin $T=0$, and parity $P=+1$. The calculated values of the rms charge radius and the magnetic and quadrupole moments characterizing these states agree with the deuteron observables with an error of 30%. However, in Refs. 36 and 37 it was shown by numerical methods that configurations with $B=2$ which are symmetric under spatial reflections have considerably lower energy than configurations corresponding to the product ansatz. Moreover, it was shown that the configuration of minimum energy is also axially symmetric and has a toroidal baryon-number density distribution. Meanwhile, a search was carried out for the analytic form of such a solution. It was proposed that an approximate solution with reflection and axial symmetries be used, introducing torsion into the field—the "$k\varphi$" ansatz:

$$\mathbf{N}=\{\cos(k\varphi)\sin\theta;\ \sin(k\varphi)\sin\theta;\ \cos\theta\}, \qquad (9)$$

where (θ,φ) are the angles of the vector $\mathbf{r}$ in spherical coordinates. In (9), k is an integer, which also determines the topological charge. Some interesting properties of the states generated by these solutions have been studied in Refs. 35, 40, 55, and 56. In the sector with baryon number $B=2$ this form of solution leads to masses of the quantum states (with the deuteron quantum numbers) of the order of two nucleon masses. The quantization procedure leads to a rich spectrum of rotational bands.[40,55] It should be stressed that for this ansatz the energy of the lowest classical state (the soliton) is somewhat higher than the two-soliton threshold. However, it can be shown that this solution can be improved so that configurations of considerably lower energy are obtained.

The symmetries of the solutions in the $B=2$ sector

In order to understand the problem, let us briefly discuss the symmetries of the solutions in the $B=2$ sector. As far as the product ansatz is concerned, such solutions satisfy the nontrivial condition

$$U_2(x,\ -y,\ -z)=\tau_1U_2(x,\ y,\ z)\,\tau_1, \qquad (10)$$

and also have simple transformation properties under parity transformations $PU(\mathbf{r})\Rightarrow U^+(-\mathbf{r})$:

$$PU_2(\mathbf{r})\Rightarrow U_2(-\mathbf{r})=\tau_3U_2(\mathbf{r})\tau_3. \qquad (11)$$

Equation (10) implies that an arbitrary spatial rotation by an angle π about the axis on which the Skyrmions are localized is equivalent to an isospin rotation by an angle π about the same axis. As was shown in Ref. 54, this symmetry is very important for obtaining the correct spectrum of quantum numbers. Symmetries of this type are related to quantum theory using the Finkelstein–Rubenstein criterion[57] which physically allowed states of the Hilbert space must satisfy. This criterion ensures that the Pauli principle is satisfied. As far as the symmetry (10) is concerned, the Finkelstein–Rubenstein criterion applied to such solutions eliminates from the spectrum of states with $B=2$ the states with the quantum numbers $T=S=0$, thereby leaving the states with the deuteron quantum numbers as the lowest states in the spectrum. It follows from the studies carried out in Ref. 58 that the following are also exact symmetries of configurations of minimum energy in the $B=2$ sector:

$$U_2(-x,\ y,\ z)=\tau_1U_2(x,\ y,\ z)\,\tau_1; \qquad (12)$$

$$U_2(-x,\ -y,\ z)=U_2(x,\ y,\ z). \qquad (13)$$

Therefore, two new symmetries in addition to the nontrivial symmetry (10) typical of the product ansatz are valid for the exact solution.

In Ref. 38 it was shown (and then confirmed by direct numerical calculations in Ref. 37) that U_2 also possesses a continuous cylindrical symmetry. For example, if the z axis is chosen to be the symmetry axis, the cylindrical symmetry of the solution takes the form

$$U_2(\rho,\ \varphi+\alpha,\ z)=e^{-i\alpha\tau_3}U_2(\rho,\ \varphi,\ z)\,e^{i\alpha\tau_3}. \qquad (14)$$

This continuous symmetry includes the last of the discrete symmetries (13) discussed above for $\alpha=\pi$.

In the following section we use the ansatz[41,44,45] realizing these symmetries in sectors with $B\geqslant 2$:

$$\mathbf{N}=\{\cos\Phi(\varphi)\sin T(\theta),\ \sin\Phi(\varphi)\cdot\sin T(\theta),\ \cos T(\theta)\}, \qquad (15)$$

where $\Phi(\varphi)$ and $T(\theta)$ are arbitrary functions of the angles

φ, θ of the spherical coordinates determined by the local orientation of the isotopic vector **N**. We shall show that this ansatz is a generalization of the Skyrme–Witten ansatz and the $k\varphi$ ansatz, and in a certain sense explains the origin and approximate nature of the latter. As will become clear in what follows, this ansatz leads to a series of new solutions in both the baryonic and the topologically trivial sectors. Some of the new states are classically stable.

The mass functional and solutions of the static equations

We write the Lagrangian density of the $SU(2)$ Skyrme model for stationary solutions as

$$\mathscr{L}=\frac{F_\pi^2}{16}\,\mathrm{Tr}\,(L_kL_k)+\frac{1}{32e^2}\,\mathrm{Tr}\,[L_k,\ L_l]^2. \quad (16)$$

Here $L_k(x)=U^+(x)\partial_k U(x)$ are the spatial components of the left-handed currents ($k=1,2,3$). After tedious algebra, using the assumption about the form of the solution (15), the Lagrangian L can be written as

$$L=\int(\mathscr{L}_2+\mathscr{L}_4)\,dV, \quad (17)$$

where

$$\mathscr{L}_2=-\frac{F_\pi^2}{8}\left\{(F')^2+\left[\frac{\sin^2T}{\sin^2\theta}(\Phi')^2+(T')^2\right]\frac{\sin^2F}{r^2}\right\}; \quad (18)$$

$$\mathscr{L}_4=-\frac{1}{2e^2}\,\frac{\sin^2F}{r^2}\left\{\frac{\sin^2T}{\sin^2\theta}(T')^2(\Phi')^2\frac{\sin^2F}{r^2}+\left[\frac{\sin^2T}{\sin^2\theta}(\Phi')^2+(T')^2\right](F')^2\right\}. \quad (19)$$

In the last two expressions the prime denotes differentiation with respect to the argument:

$$\Phi'=d\Phi/d\varphi,\quad T'=dT/d\theta,\quad F'=dF/dr. \quad (20)$$

Variation of (17) with respect to Φ gives

$$\Phi''=0, \quad (21)$$

i.e.,

$$\Phi(\varphi)=k\varphi+\mathrm{const}. \quad (22)$$

In what follows we shall study only solutions with const $=0$. In order to ensure that the full solution $U(\mathbf{r})$ is single-valued in all space, the quantity k must be an integer.

Now for the soliton mass we easily find the expressions

$$M=M_2+M_4; \quad (23)$$

$$M_2=\frac{\gamma}{4}\int_0^\infty dx\cdot x^2\int_0^\pi d\theta\sin\theta\times\left\{(F')^2+\left[\frac{\sin^2T}{\sin\theta}k^2+(T')^2\right]\frac{\sin^2F}{x^2}\right\}; \quad (24)$$

$$M_4=\gamma\int_0^\infty dx\,x^2\int_0^\pi d\theta\sin\theta\left\{\left[\frac{\sin^2T}{\sin^2\theta}k^2+(T')^2\right](F')^2+\frac{\sin^2F}{x^2}\,\frac{\sin^2T}{\sin^2\theta}k^2(T')^2\right\}\frac{\sin^2F}{x^2}, \quad (25)$$

where $\gamma=\pi F_\pi/e$ and $x=F_\pi er$.

From the minimization conditions for the mass functional

$$\delta M/\delta T=0,\ \delta M/\delta F=0 \quad (26)$$

we immediately obtain a system of coupled equations determining the functions $F(x)$ and $T(\theta)$:

$$[x^2+2a\sin^2F]F''+2xF'+\left[a(F')^2-\frac{a}{4}-2b\frac{\sin^2F}{x^2}\right]\sin(2F)=0; \quad (27)$$

$$2\left[A+k^2B\frac{\sin^2T}{\sin^2\theta}\right]T''-k^2A\frac{\sin^2(2T)}{\sin^2\theta}+k^2B\frac{\sin(2T)}{\sin^2\theta}(T')^2+2T'\operatorname{ctg}\theta\left[A-k^2B\frac{\sin^2T}{\sin^2\theta}\right]=0. \quad (28)$$

The coefficients a, b and A, B are given by the following functionals of the solutions:

$$a=\int_0^\pi\left[k^2\frac{\sin^2T}{\sin^2\theta}+(T')^2\right]\sin\theta\,d\theta; \quad (29)$$

$$b=k^2\int_0^\pi\frac{\sin^2T}{\sin^2\theta}(T')^2\sin\theta\,d\theta; \quad (30)$$

$$A=\int_0^\infty\sin^2F\left[\frac{1}{4}+(F')^2\right]dx; \quad (31)$$

$$B=\int_0^\infty\frac{\sin^4F}{x^2}\,dx. \quad (32)$$

From (24), (25), and (29), (30) we can conclude that the value of the function $T(\theta)$ must be a multiple of π at the points $\theta=0$ and $\theta=\pi$. By considering only configurations with finite mass we restrict ourselves to functions $F(x)$ satisfying the conditions $F(0)=\pi\cdot n$ with integer n. Without loss of generality we can set $F(\infty)=0$. It is easy to check that the asymptotic behavior of $F(x)$ is that of a polynomial (for $m_\pi=0$):

$$F|_{x\to\infty}\sim 1/x^{p+1},\ p=\frac{\sqrt{1+2a}-1}{2}. \quad (33)$$

Near the origin

$$F|_{x\to 0}\sim\pi n-\alpha x^p, \quad (34)$$

where α is a numerical factor. It is also obvious that $T(\theta)$ has the following behavior near the boundary of the region in which it is defined:

$$T(\theta)\sim\theta^k\ \text{for}\ \theta\to 0 \quad (35)$$

and

$$T(\theta)\sim\pi l-(\pi-\theta)^k\ \text{for}\ \theta\to\pi, \quad (36)$$

where l is an integer. Therefore, all solutions U are characterized by a set of integers (n,k,l).

The solution $T(\theta)$ can be approximated by the following series:[45]

$$T(\theta)=\theta+\sum_{p=1}^m g_p\sin(2p\theta), \quad (37)$$

where g_p are variational parameters.

The form of this solution satisfies the conditions of reflection symmetry[37] on the field:

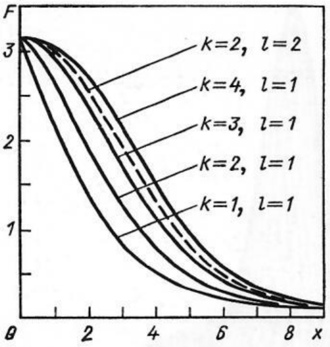

FIG. 1. Solution $F(x)$ of the system of Eqs. (27)–(32) for several values of l and k.

$$\mathbf{N}(x,\ y,\ -z)=\begin{bmatrix}1&0&0\\0&1&0\\0&0&-1\end{bmatrix}\mathbf{N}(x,\ y,\ z); \tag{38}$$

$$\mathbf{N}(x,\ -y,\ z)=\begin{bmatrix}1&0&0\\0&-1&0\\0&0&1\end{bmatrix}\mathbf{N}(x,\ y,\ z); \tag{39}$$

$$\mathbf{N}(-x,\ y,\ z)=\begin{bmatrix}-1&0&0\\0&1&0\\0&0&1\end{bmatrix}\mathbf{N}(x,\ y,\ z). \tag{40}$$

The variational calculation requires the inclusion of at least two terms in (37) in order to accurately reproduce the solution of the equation obtained in Ref. 41 by direct numerical integration of the equation for $T(\theta)$.

The result of numerically integrating the system of Eqs. (27)–(32) is shown graphically in Figs. 1 and 2 for several values of k and l.

Baryon-number distribution and soliton structure

Let us consider the soliton structure in more detail. For this we calculate the baryon-number distribution using Eq. (4). Direct calculation leads to the following expression:

$$J_0^B(r,\ \theta)=-\frac{1}{2\pi^2}\frac{\sin^2 F}{r^2}\frac{dF}{dr}\frac{\sin T}{\sin\theta}\frac{dT}{d\theta}\frac{d\Phi}{d\varphi}. \tag{41}$$

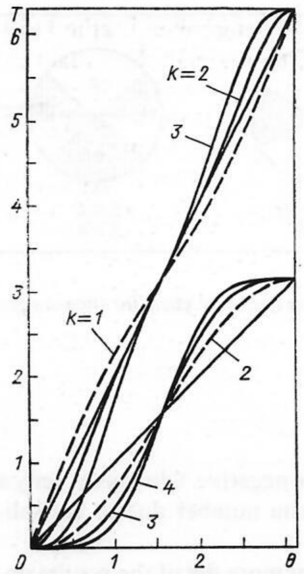

FIG. 2. Solution $T(\theta)$ of the system of Eqs. (27)–(32) for several values of l and k.

The expression obtained for the baryon-number density becomes the expression for the hedgehog configuration if we set $k=1$ and $T(\theta)=\theta$, as it should. Expression (41) also leads directly to its integral, i.e., the baryon number B:

$$B=\frac{kn}{2}[1-\cos(\pi l)]. \tag{42}$$

Even values of l will obviously correspond to configurations from the vacuum sector $B=0$, and odd l will correspond to ones from the baryon sector. In Fig. 3 we show schematically the baryon-number distribution in the (x,z) plane for solitons characterized by the numbers n, k, l. In Table I we represent symbolically the soliton structure, the total baryon number, and the values of the rms radii for several values of k and l. For example, we show the structure $2S$–$2\bar{S}$ for the solution with $k=2$, $l=2$, where the baryon-number distribution divides the entire space into two axially symmetric regions. Two units of baryon number are concentrated in one region, and two units of antibaryon number (negative charge) are concentrated in the other. The calculated rms radii of the baryon-number distributions reveal a large difference in the shape of the stationary configurations ob-

TABLE I. Structure of the states ($n=1$, k, l) and rms radii of the baryon-number distributions.

k	$l=1$	$l=2$	$l=3$
1	$B=1\ (S)$ $r_x^2-r_y^2=r_z^2=1{,}49$	$B=0\ (S-\bar{S})$ $r_x=r_y=0$ $r_z=0$	$B=1\ (S-\bar{S}-S)$ $r_x=r_y=-4{,}1$ $r_z=22{,}8$
2	$B=2\ (2S)$ $r_x^2=r_y^2=6{,}5$ $r_z=2{,}9$	$B=0\ (2S-\overline{2S})$ $r_x=r_y=0$ $r_z=0$	$B=2\ (2S-\overline{2S}-2S)$ $r_x=r_y=-8{,}7$ $r_z=62$
3	$B=3\ (3S)$ $r_x^2=r_y^2=16{,}2$ $r_z=4{,}2$	$B=0\ (3S-\overline{3S})$ $r_x=r_y=0$ $r_z=0$	$B=2\ (3S-\overline{3S}-3S)$ $r_x=r_y=-9{,}9$ $r_z=114{,}3$

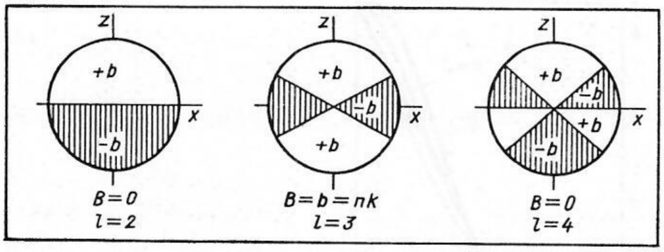

FIG. 3. Baryon-number distribution in the (x,z) plane for the solution specified by the numbers n, l, k.

tained. Some of the rms radii are negative. Obviously, only a predominance of negative baryon number due to antisolitons can lead to such values.

In Figs. 4 and 5 we show in more detail the results obtained in Ref. 59 for the baryon-number density distribution in the (x,z) plane and a "three-dimensional" picture of the same distribution for the dibaryon. According to Fig. 4, the peak in the baryon-density distribution lies near $\rho = \sqrt{x^2+y^2} \approx 1.5/F_\pi e$, $z=0$. The solution with $k=2$, $l=1$ has a toroidal structure, as was pointed out in Ref. 36.

In Figs. 6 and 7 we show lines corresponding to constant values of the baryon-number density and a "three-dimensional" picture of the baryon-number density for S–$\bar{S}$–S Skyrmions, which gives a clearer picture of this considerably more complicated object. This soliton obviously no longer has a simple toroidal structure. Only one of the Skyrmions of the substructure of this object is a diffuse torus (with $B=-1$), and the other two are not like this at all. It can also be concluded (see Fig. 3) that $B=0$, $l=4$ solitons consist of one toroidal Skyrmion, one toroidal anti-Skyrmion, and a Skyrmion–anti-Skyrmion pair of nontoroidal shape localized near the z axis.

Here it is relevant to note that quantum states of the type S–$\bar{S}$–S could be observed experimentally as compound-nucleus states in interactions of stopped antiprotons with deuterons. It therefore becomes possible to theoretically include antinucleons in the structure of compound states on an equal footing with nucleons.

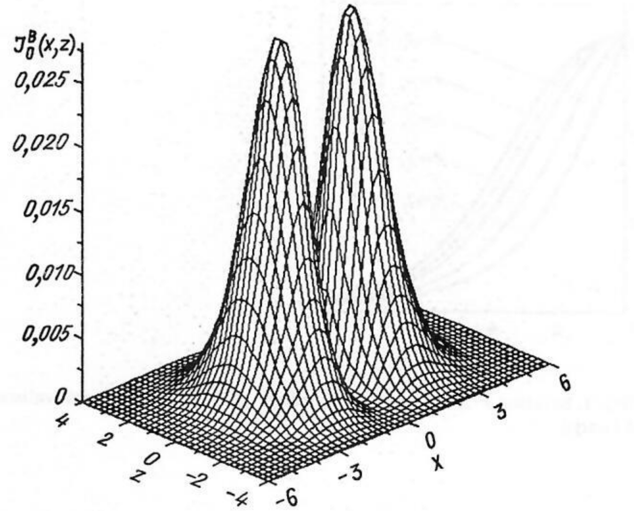

FIG. 5. "Three-dimensional" picture of the baryon-number density distribution of the two-Skyrmion configuration.

The classical soliton masses

Here we note that when we study multi-Skyrmion configurations we, in general, are dealing not only with stable classical configurations (the breakup of which into two or more Skyrmions is energetically forbidden), but also with classically unstable ones. This is related to the obvious fact that classically unstable solitons can become stable objects when quantum fluctuations are included. (A similar situation arises in the theory of the hydrogen atom.)

The numbers given in Table II correspond to our calculations in the chirally symmetric limit (the pion mass is $m_\pi = 0$). The variational calculation taking into account the chiral-symmetry violating term

$$\mathcal{L}_\pi = -\frac{F_\pi^2 m_\pi^2}{8} \operatorname{Tr}(1-U), \tag{43}$$

which includes the nonzero mass of the π meson ($m_\pi = 139$ MeV), allows us to compare these results with those obtained by direct numerical calculation[60] (the method of "caps"); see also Table III. Comparison shows that the values obtained in Refs. 41 and 61 closely coincide. For example, the variational calculation using the generalized ansatz (15) with the constants $F_\pi = 108$ MeV and $e = 4.84$ gives

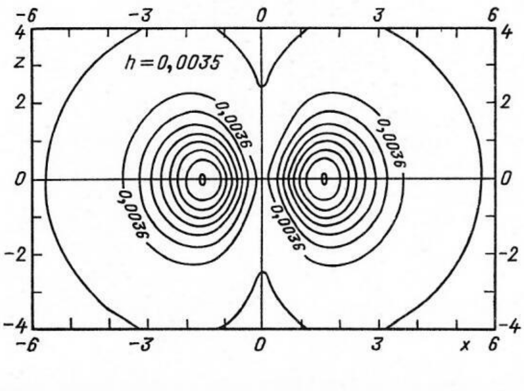

FIG. 4. Lines of equal values of the baryon-number distribution density in the (x,z) plane for the two-Skyrmion configuration. The interval between the lines is $h = 0.0035$.

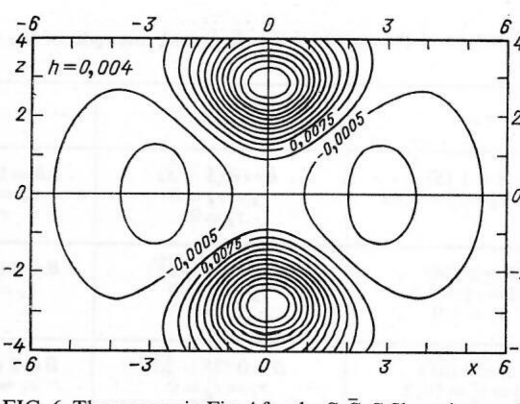

FIG. 6. The same as in Fig. 4 for the S–$\bar{S}$–S Skyrmion configuration. The interval between the lines is $h = 0.004$.

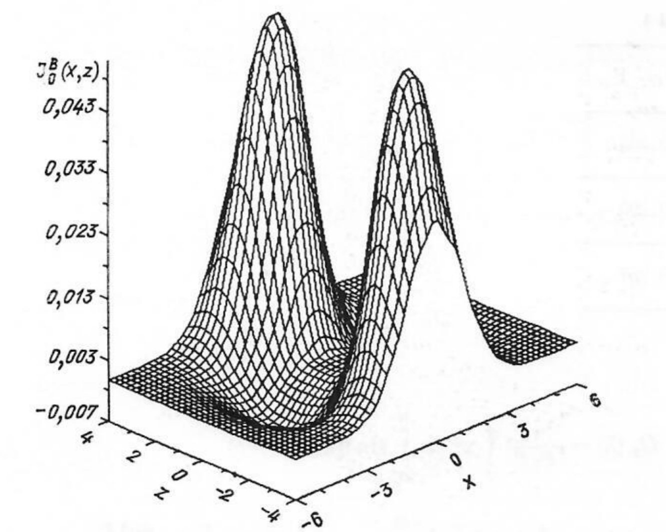

FIG. 7. The same as in Fig. 5 for the S–$\bar{S}$–S Skyrmion configuration.

1670 MeV for the di-Skyrmion mass, which should be compared with 1660 MeV from Ref. 61. The values of the masses for other Skyrmions taken from Refs. 44 and 61 are also in satisfactory agreement with each other: the 3-Skyrmion mass $M = 2580$ and 2530 MeV, the 4-Skyrmion mass $M_4 = 3572$ and 3452 MeV, and the 5-Skyrmion mass $M_5 = 4635$ and 4420 MeV. The slight discrepancy in the calculated masses for multi-Skyrmions allows us to estimate the errors probably introduced by the variational ansatz. The errors are 5% for $B \leqslant 5$.

To estimate and control the error in the numerical calculations we can use a variant of the virial theorem. According to this theorem, for the problem in question the quantity

$$\Delta = \gamma \{(A_2 - A_1)\, a + bB - C\} \qquad (44)$$

should be zero. Here A_1, A_2, and C are the values of the following integrals:

$$A_1 = \frac{1}{4}\int_0^\infty \sin^2 F\, dx,\ A_2 = \int_0^\infty (F')^2 \sin^2 F\, dx, \qquad (45)$$

$$C = \frac{1}{2}\int_0^\infty (xF')^2\, dx. \qquad (46)$$

For Δ we obtained the value $|\Delta| = 1$ MeV for the most "difficult" case ($k = 4$, $l = 3$).[59]

Let us conclude this section by reviewing the main results in Tables I and II. We have extended the spectrum of classical n-multibaryons to the spectrum of nk-multi-Skyrmion configurations. For example, the $k = 3$, $n = 1$, $l = 1$ term in Table II with binding energy of about 5.4 MeV per baryon corresponds to the tribaryon state. In addition, a new series of meson-like solutions appears ($nkl/2$-baryon–$nkl/2$-antibaryon configurations for even l). (See, for example, the $k = 2$, $n = 1$, $l = 2$ case, which corresponds to a two-baryon–two-antibaryon configuration with a mass of about 3192 MeV.) Some of the configurations thus obtained are classically stable objects, as can be seen from Table II (these are given in boldface). The masses of such objects are smaller than the sums of the masses of their baryon constituents. The classical binding energy of these states can easily be obtained using Table II for arbitrary F_π and e.

In Table II it is easy to see that there is a nearly linear dependence of the classical mass on the baryon number. This dependence differs dramatically from the dependence $M \sim B(B+1)$ which is found[62] for the Skyrme–Witten ansatz.

TABLE II. Masses of classical solitons (Ref. 59) in units of $\pi F_\pi/e$.

k	$l=1$	$l=2$	$l=3$	$l=4$
1	**11,605**	26,358	46,332	71,169
2	**22,458**	**45,536**	73,533	106,61
3	**34,585**	**66,701**	**103,08**	144,32
4	47,675	**89,310**	**134,45**	
5	61,569	**113,12**		

2. THE SPECTRUM OF QUANTUM STATES OF MULTI-SKYRMIONS

Our goal is to obtain the quantum-mechanical effective Hamiltonian using the collective-coordinate method. For this we introduce breather and rotational degrees of freedom[41] as collective coordinates and calculate the binding energies of the lowest quantum states.

The effective Hamiltonian in terms of collective variables

Let us describe the most important steps necessary for obtaining the effective Hamiltonian. We now choose the time-dependent chiral field in the form

$$U(\mathbf{r},\, t) = \exp\{i\tau^i I^{ij}(t)\, N^j\, (R^{-1}_{mk} x_k)\, F(xe^{-\lambda})\}, \qquad (47)$$

where $R(t)$ and $I(t)$ are 3×3 spatial and isospin rotation matrices, respectively, and $\lambda(t)$ is the time-dependent pa-

TABLE III. Properties of classical configurations (Ref. 60) with $B=2$, 3, and 4.

B	M, MeV	R_B^2, F^2	Q_B, F^2	$\langle z_M\rangle$, F	$\langle z_B\rangle$, F
2	1656,0	0,884	0,084	0,355	0,303
3	2523,5	1,356	0,187	0,356	0,295
4	3446,8	1,930	0,308	0,355	0,287

rameter of dilational transformations. Substituting (47) into the Lagrangian, in which the time components of the currents L_0 now play a fundamental role,

$$L=-M+\frac{F_\pi^2}{16}\int d^3r\,\mathrm{Tr}\,(L_0L_0)+\frac{1}{16e^2}\int \mathrm{Tr}\,[L_0,\ L_k]^2\,d^3r, \tag{48}$$

making a canonical transformation, and defining the conjugate variables

$$p=\frac{\partial L}{\partial\dot\lambda},\ T^i=\frac{\partial L}{\partial\omega^i},\ S_i=\frac{\partial L}{\partial\Omega_i}, \tag{49}$$

where the angular velocities Ω_i and ω^i for rotations and isospin rotations are given by

$$R_{ik}^{-1}\dot R_{kj}=\varepsilon_{ijk}\Omega_k,\ \dot I^{ik}(I^{-1})^{kj}=\varepsilon^{ijk}\omega^k, \tag{50}$$

we obtain the Hamiltonian for $k\neq 1$:

$$\hat H=M(\lambda)+\frac{\hat p^2}{2m(\lambda)}+\frac{\hat T^2}{2Q_T(\lambda)}+\frac{\hat S^2}{2Q_S(\lambda)}-\frac{1}{2}\left\{\frac{1}{Q_T(\lambda)}+\frac{k^2}{Q_S(\lambda)}-\frac{1}{Q(\lambda)}\right\}\hat T_3^2. \tag{51}$$

Here the symbols $\hat P$, $\hat T$, and $\hat S$ are now interpreted as the quantum momentum operators corresponding to vibrations, isospin, and spin, respectively. The vibrational potential $M(\lambda)$ is given by the expression

$$M(\lambda)=M_2e^{-\lambda}+M_4e^{\lambda}. \tag{52}$$

For the inertial quantities $m(\lambda)$, $Q_T(\lambda)$, $Q_S(\lambda)$, and $Q(\lambda)$ we have

$$m(\lambda)=\frac{2\pi}{F_\pi e^3}\int dx\,x^4(F')^2\times\left\{\frac{e^{-3\lambda}}{2}+e^{-\lambda}\frac{\sin^2F}{x^2}\int_0^\pi\left[k^2\frac{\sin^2T}{\sin^2\theta}+(T')^2\right]\sin\theta\,d\theta\right\}; \tag{53}$$

$$Q_T(\lambda)=\frac{\pi}{F_\pi e^3}\int_0^\infty dx\,x^2\int_0^\pi\sin\theta\,d\theta\times\left\{-e^{-\lambda}\frac{\sin^4F}{x^2}\left[k^2\frac{\sin^2T}{\sin^2\theta}\cos^2T+(T')^2\right]+\sin^2F\left[\frac{e^{-3\lambda}}{4}+e^{-\lambda}\left((F')^2+\left[k^2\frac{\sin^2T}{\sin\theta}+(T')^2\right]\frac{\sin^2F}{x^2}\right)\right]\times(1+\cos^2T)\right\}; \tag{54}$$

$$Q_s(\lambda)=\frac{\pi}{F_\pi e^3}\int_0^\infty x^2\,dx\int_0^\pi\sin\theta\,d\theta\times\left\{\sin^2F\left[\frac{e^{-3\lambda}}{4}+e^{-\lambda}\left((F')^2+\left[k^2\frac{\sin^2T}{\sin^2\theta}+(T')^2\right]\frac{\sin^2F}{x^2}\right)\right]\times\left(k^2\frac{\sin^2T}{\sin^2\theta}\cos^2\theta+(T')^2\right)-e^{-\lambda}\frac{\sin^4F}{x^2}\left(k^4\frac{\sin^4T}{\sin^4\theta}\cos^2\theta+(T')^4\right)\right\}; \tag{55}$$

$$Q(\lambda)=\frac{2\pi}{F_\pi e^3}\int_0^\infty dx\,x^2\int_0^\pi\sin\theta\,d\theta\times\left\{-e^{-\lambda}\frac{\sin^4F}{x^2}k^2\frac{\sin^4T}{\sin^2\theta}+\sin^2F\left[\frac{e^{-3\lambda}}{4}+e^{-\lambda}\left((F')^2+\left[k^2\frac{\sin^2T}{\sin^2\theta}+(T')^2\right]\frac{\sin^2F}{x^2}\right)\right]\sin^2T\right\}. \tag{56}$$

The Wess–Zumino action and statistics

In the quantization of systems of the type considered here the delicate question of the statistics of the system arises. Should the system be quantized as a bosonic or a fermionic system? In order to obtain an unambiguous answer to this question, we must get rid of the extra symmetries of the σ-model Lagrangian.[15]

For example, it is easy to check that the Lagrangian is invariant under the transformations $\mathbf{x}\to-\mathbf{x}$, $t\to t$, and also $U\to U^+$ separately. However, in strong interactions the combined symmetry P: $\mathbf{x}\to-\mathbf{x}$, $t\to t$, $U\to U^+$ is conserved. Other "extra" symmetries can also be found. Witten suggested the addition to the equations of motion of a term which is odd under $\mathbf{x}\to-\mathbf{x}$, $t\to-t$, thereby making the equations of motion

$$\frac{F_\pi^2}{8}\,\partial^\mu L_\mu+\lambda\varepsilon^{\mu\nu\rho\sigma}L_\mu L_\nu L_\rho L_\sigma=0 \tag{57}$$

themselves noninvariant under both the extra symmetries because the two terms of the equation have opposite parities under these symmetries. The equation of motion turns out to be symmetric only under the combined symmetry. The chirally symmetric action, the variation of which leads to the second term of Eq. (57), is written in the form of an integral over the boundary M of a five-dimensional space:

$$\Gamma=\lambda\int_M\varepsilon^{ijklm}\mathrm{Tr}\,(L_iL_jL_kL_lL_m)\,d^5x. \tag{58}$$

Witten[15] noticed that the coefficient λ in this expression must be an integer multiple n of the normalization factor $-i/(240\pi^2)$, which has a geometrical significance. The condition that the factor λ be quantized also has a geometrical origin. The action (58) itself was derived in Ref. 63 and reproduces the anomalies appearing in renormalizations of fermion loops related to external axial currents. The action is responsible, in particular, for the decay $\pi^0 \to 2\gamma$. It can be rigorously stated that the QCD anomalies are reproduced if the integer n is equal to the number of colors N_c. In the effective theory the number of colors is a vestige of the summation over quarks. The main result of Ref. 16 is that for odd N_c the solitons must be quantized as fermions, and for even N_c they must be quantized as bosons, in remarkable accordance with the fact that N_c quarks with baryon number $1/N_c$ form a baryon. This statement is usually proved by considering an adiabatic rotation of the soliton during a long time T. The anomalous action, which is linear in the time derivative, gives a finite contribution even for the infinite integral; this contribution is a multiple of π: $\Gamma = N_c\pi$. Therefore, the wave functional of the soliton rotated by the angle 2π differs by the phase $(-1)^{N_c}$ from the functional of the unrotated soliton, which proves the statement. For $SU(2)$ variants of the theory the anomalous action is identically equal to zero, and it is necessary to use other conditions[57] replacing the Witten conditions in the $SU(2)$ variant of the model.

The Finkelstein–Rubinstein criterion (the sector $B=2$)

We shall work in the coordinate representation $|I,R\rangle$ of the Hilbert space in which the state vectors $|\Psi\rangle$ are represented by wave functions $\Psi(J,R) = \langle I,R|\Psi\rangle$. A more convenient basis is given by the direct product

$$|TT_3,\ T_3^{\mathrm{b.f}}\rangle|\,SS_3,\ S_3^{\mathrm{b.f}}\rangle, \tag{59}$$

where the three projections of the isospin T in the laboratory frame T_3 and in the internal frame $T_3^{\mathrm{b.f.}}$ (the body frame), and also of the spin S, S_3 and $S_3^{\mathrm{b.f.}}$, satisfy the conditions

$$\left.\begin{array}{l} -T \leqslant T_3,\ T_3^{\mathrm{b.f}} \leqslant T; \\ -S \leqslant S_3,\ S_3^{\mathrm{b.f}} \leqslant S. \end{array}\right\} \tag{60}$$

It can be shown that our ansatz leads to the following relation between the internal projections:

$$kT_3^{\mathrm{b.f}} + S_3^{\mathrm{b.f}} = 0. \tag{61}$$

This condition should be viewed as a constraint on the choice of wave function of the quantized Skyrmion. This condition is satisfied by, for example, the function

$$\langle I,\ R|TT_3,\ SS_3L\rangle = \frac{\sqrt{(2T+1)(2S+1)}}{2\pi^2} D^T_{T_3L}(I)\, D^S_{S_3-kL}(R), \tag{62}$$

where the numbers L determine the three projections in the body coordinate frame.

In the basis (59) the (rotational) kinetic-energy operator is diagonal, and the condition (61) is easily satisfied by discarding as unphysical all states which do not satisfy this criterion. However, not all of the remaining states are physical. The Pauli principle must be satisfied. Another criterion can be formulated for this purpose.[57] This criterion must select, in particular, wave functions in the sector with $B=1$ which change sign upon rotation by 2π, so that an isolated Skyrmion is quantized as a fermion. In the context of soliton physics such a criterion was formulated in Refs. 57 and 64. It was noticed that a 2π rotation of an isolated Skyrmion is a closed path in the Hilbert space of configurations $|U(\mathbf{r})\rangle$ which cannot be shrunk to a point by continuous transformations. Moreover, according to the homotopy $\pi_4(SU(2)) = Z_2$, there are only two topologically inequivalent types of closed path. Paths which can be contracted to a point form the first class, and noncontractible paths form the second. Therefore, a general rule consistent with the fact that an isolated nucleon must be quantized as a fermion can be formulated as follows: the end points of a closed path in the space of states corresponds to relative phase $+1$ if the path is contractible, and -1 if it is incontractible. The realization of this criterion in the $B=2$ sector[43] leads to the following representation for the wave functions in the sector with baryon number $B=2$. The allowed states are determined by the product

$$1)\ \ |TT_30\rangle\,|SS_30\rangle \tag{63}$$

if $T+S$ is odd, and $L=0$;

$$2)\ \ \frac{1}{\sqrt{2}}\left[|TT_3L\rangle|SS_3-2L\rangle - (-1)^{T+S}|TT_3-L\rangle|SS_32L\rangle\right], \tag{64}$$

where

$$L = 1,\ \ldots,\ \min\{T,\ S/2\}.$$

Let us briefly discuss the parity operator P, which determines the parity of the allowed states. The operator P is defined by the transformation

$$PU(\mathbf{r},\ t)\,P^{-1} = U^+(-\mathbf{r},\ t). \tag{65}$$

However, as was shown in Ref. 58, a configuration with $B=2$ satisfies the condition

$$U_2^+(-\mathbf{r}) = \tau_3 U(\mathbf{r})\,\tau_3. \tag{66}$$

Therefore, P must transform the collective coordinates A, B:

$$PAP^{-1} = A\,(i\tau_3);\ PBP^{-1} = B. \tag{67}$$

In the last equation we have used the $SU(2)$ matrix representation of the spatial and isospin rotation groups, which are related to our coordinates I and R as

$$\left.\begin{array}{l} I_{ij} = \frac{1}{2i}\,\mathrm{Tr}\,(\tau_i A \tau_j A^+); \\ R_{ij} = \frac{1}{2i}\,\mathrm{Tr}\,(\tau_i B \tau_j B^+). \end{array}\right\} \tag{68}$$

It can be shown that

$$P = e^{i\pi|T_3^{\mathrm{b.f}}|} = e^{i\pi L}. \tag{69}$$

Therefore, the parity of the states is determined by the number L.

The masses of the lowest calculated soliton states with $B=2$ are given in Table IV.[59] The calculations were carried out in the harmonic approximation (for the vibration potential) with the constants $F_\pi = 108$ MeV and $e = 4.84$. These constants correspond to the nucleon mass $N_{\mathrm{nucl}} = 931$ MeV. In Table IV we give the results corresponding to zero pion mass.

TABLE IV. Calculated energies for $B = 2$ ($k = 2$, $l = 1$) states with isospin T, spin-parity S^P, and quantum number $n = 0$ corresponding to the vibrational mode.

T	0	0	1	1	1
S^P	0^+	1^+	0^+	1^+	2^-
$E-2M_{\rm nucl}$, MeV	−214	−172	−154	−118	−53

If we neglect the vibrational degrees of freedom and include only the rotational ones, the resulting expression for the mass spectrum with $B = 2$ ($k = 2$, $l = 1$) in the chirally symmetric limit takes the form[41]

$$E_{STT_3} \approx \frac{F_\pi}{e}\left\{70{,}55+\frac{e^4}{2}\left[\frac{S(S+1)}{272{,}4}+\frac{T(T+1)}{183{,}0}-\frac{T_3^2}{83{,}2}\right]\right\} \quad (70)$$

for arbitrary values of F_π and e. The numerical factors in the last expression are the dimensionless integrals, calculated using (53)–(56), determining the moments of inertia in the spin and isospin spaces. These moments of inertia, as we have shown, are functionals of the solutions of the classical equations. The inclusion of the explicit breaking of the chiral symmetry by the pion mass term leads to the mass formula[45]

$$E = [1674\ \ \text{MeV}] + [31\ \ \text{MeV}]\, S\,(S+1) + [47\ \ \text{MeV}]\, T\,(T+1) - [100\ \ \text{MeV}]\,(T_3^{\rm b.f.})^2 \quad (71)$$

for $F_\pi = 108$ MeV and $e = 4.84$, and to the binding energies given in Table V. The last equation predicts a series of two-nucleon states with isospin $T = 1$. The energies of these states are considerably higher than the energy of the deuteron state and can be interpreted as resonances.

In Table VI we give the excitation energies of diproton states of natural parity. In the last line of the table we give the excitation energies measured in Ref. 65. The present interpretation of this series of states, which appear in pp elastic channels, is based on the idea of strong coupling to the $N\Delta$ state with aligned spin and orbital angular momentum.[66]

A somewhat more general representation of the form of the solution in the $B = 2$ sector was used in Ref. 43. The chiral angle $F = F(\rho,z)$ and the function $T(\rho,z)$ were chosen as functions of the independent variables (ρ,z) in cylindrical coordinates. The functions $T = T(\rho,z)$ found in Ref. 43 differ little from the ansatz $T = T$ (arctan z/ρ) used in Refs. 44 and 59. In Table VII we give the dibaryon mass spectrum in physical units obtained in Ref. 43 for $F_\pi = 108$ MeV, $e = 4.84$, and $m_\pi/F_\pi e = 0.263$. Vibrations were not taken into account. The results quoted in Ref. 60 for the spectrum of dibaryon states agree closely with the results from Ref. 43.

We should stress the fact that the state $|T = 2, S = 0\rangle$ has mass 1940 MeV. This should mean that it is stable with respect to strong interactions, since it cannot decay into two nucleons (isospin conservation) or into two nucleons and a pion (energy conservation). However, it must again be emphasized that this result is very sensitive to the selected values of the model parameters ($F_\pi = 108$ MeV, $e = 4.84$ without vibrations) fitting the nucleon and Δ-isobar masses.

It can be noted that the calculated mass spectrum predicts that the deuteron and the $^1S_0$ state are stable to decay into two nucleons, with binding energies of 158 and 123 MeV, respectively. (As is well known, only the first of these is bound.) The calculated value of the difference between the levels also differs from the experimental value by almost an order of magnitude. This discrepancy with experiment might be overcome by going beyond the semiclassical approximation. As was shown in Ref. 43, there is a remarkable correspondence between the lowest states with $T_3^{\rm b.f.} = 0$ ($L = 0$) and the states expected in the 50-dimensional irreducible representation of the $SU(4)$ spin–flavor group in the nonrelativistic quark model. This classification is shown in Table VII, where we also give the physical interpretation of such six-quark states. Similar states also appear in spherical quark bag models.

As far as higher states with $L > 0$ are concerned, there is no such correspondence. This difficulty is a consequence of the condition $T \geqslant |L|$, $S \geqslant |2L|$, since this feature is not characteristic of any $SU(2)$ representation. From this we can conclude that the spectrum of dibaryon resonances obtained in this model contains states which are absent in the quark bag model.

The spectroscopic ^{2S+1}l classification given in the table

TABLE V. The same as in Table IV for the chirally asymmetric case (Ref. 45).

T	0	0	1	1	1	0
S^P	0^+	1^+	0^+	1^+	2^-	2^+
$E-2M_{\rm nucl}$, MeV	−201,4	−140	−107,8	−46,4	−23,6	−15

TABLE VI. Calculated excitation energies for diproton states of natural parity. $\Delta E_{\rm exp}$ corresponds to application of a candidate to a resonance state according to Ref. 65.

State	$^1D_2$	$^3F_3$	$^1G_4$	$^3H_5$	$^1I_6$
$\Delta E_{\rm exc}$, MeV	218	304	650	860	1310
$\Delta E_{\rm exp}$, MeV	250	310	540	810	1010

is, of course, somewhat arbitrary, since in this model nucleons are not distinguished as constituents of nuclear states. Therefore, we cannot split the total spin into the internal spin of the nucleons and the relative orbital angular momentum.

Finally, in Table VIII we give the spectrum of lowest states in the $B = 3$ and $B = 4$ sectors obtained in Ref. 59 using the variational ansatz (47) taking into account the breather mode.

The contributions of the rotational and vibrational degrees of freedom to the tribaryon masses in this approach are given in more detail in Table IX.

It is easy to see that $\lambda = 0$ is not a stable point (a minimum of the effective potential) when Skyrmion rotations are taken into account. Therefore, $\lambda_{\min}$ corresponding to the minimum must be found before solving the Schrödinger equation. In the calculations of Ref. 59 it was shown that the role of this procedure is numerically small for low-lying states, except in the one-nucleon case. In Table IX we give the values of $\lambda_{\min}$, the rotational energy $E_{\rm rot}$ (including the value of the classical mass), and the vibrational photon energy $\hbar\omega_{\rm vibr}$ for $F_\pi = 108$ MeV and $e = 4.84$. The values of $\lambda_{\min} = 0$ correspond to calculations in which the effective potential was not minimized with respect to the scale parameter λ.

Therefore, all states which could correspond to states of light nuclei have very large binding energy. It is nevertheless possible that the nuclear-like states obtained need not correspond to nuclear states until all the important quantum corrections (or effects) are included in their structure.

We have also discovered the remarkable phenomenon that some states which are not stable at the classical level become stable at the quantum level. For example, the classically unstable state $k = 4$, $l = 1$ having energy $+88$ MeV above the decay threshold becomes stable after the inclusion of quantum corrections to the mass of the state (see Tables II and VIII).

Some remarks about the existence of nucleon–antinucleon states

According to (42), the case $l = 2$ corresponds to $B = 0$ states. Some of these states, after being quantized, can be viewed as nucleon–antinucleon bound states. The classical mass of such states is of the order of two nucleon masses ($k = 2$). It can therefore be expected that if such states are stable after the inclusion of quantum corrections, their energies will be smaller than two nucleon masses. The same must also be true for other states with $l = 2$.

According to the calculations of $T = S = 0$ states, configurations which were stable before the inclusion of quantum corrections remain stable after the inclusion of the breather mode. The state $k = 1$, $l = 2$, which was unstable before quantization,

$$[\Delta M_{\rm cl}\ (k=1,\ l=2) - 2M_{\rm cl}\ (k=1,\ l=1) = 220{,}7\ \text{MeV}],$$

remains unstable ($\Delta_{\rm qu} = 56.7$ MeV). The states with $k = 2$, 3, 4 and $l = 2$ are stable. Some of these states could appear as compound states in reactions involving stopped antinucleons. Here it should be noted that N–$\overline{N}$ states in the Skyrme model were first studied in Ref. 67 in a different approach.

Strange multi-Skyrmions

In connection with the development of accelerator technology and the construction of kaon and hadron factories, one can pose the theoretical question of the experimental observation of multibaryons with large strangeness. They can be created in reactions with strangeness exchange

TABLE VII. Mass spectrum in the $B = 2$ sector from Ref. 43.

Classification	T	S	L	Parity	Theory	Experiment
Deuteron ($^3S_1$)	0	1	0	+	1720	1876
NN ($^1S_0$)	1	0	0	+	1755	1880
? ($^3P_2$)	1	2	1	−	1838	
$N\Delta$ ($^5S_2$)	1	2	0	+	1938	
$N\Delta$ ($^3S_1$)	2	1	0	+	2007	
? ($^5P_3$)	1	3	1	−	2022	
$\Delta\Delta$ ($^7S_3$)	0	3	0	+	2026	
? ($^3P_2$)	2	2	1	−	2030	
? ($^5D_4$)	2	4	2	+	2159	
? ($^5P_3$)	2	3	1	−	2213	
$\Delta\Delta$ ($^1S_0$)	3	0	0	+	2233	

TABLE VIII. Spectrum of lowest multibaryon states with $B = 3$ and 4 ($m_\pi \neq 0$) (Ref. 59).

B	3	3	4	4	4
T	1/2	3/2	0	0	1
S	3/2	3/2	0	1	0
L	1/2	1/2	0	0	0
$E - B \cdot M_{\text{nucl}}$, MeV	−268,0	−210,5	−324,0	−312,7	−294,5

involving kaons and hyperons, and also in heavy-ion collisions. In relation to this, the authors of Ref. 68 suggested that solitons corresponding to the $SO(3)$ subgroup of $SU(3)$ be studied. In Ref. 69 it was established using this approach that there exists a tetralambda state with $B = -S = 4$ (S is the strangeness) which is stable to decay into two dilambda states. The authors of the latter study used the ansatz

$$U = \exp\left(\mathrm{i}\,\frac{2}{3}\,G\right) + \mathrm{i}\sin F \exp\left(-\mathrm{i}\,\frac{1}{3}\,G\right)\Lambda\cdot\mathbf{N} + \left[\cos F \exp\left(-\mathrm{i}\,\frac{1}{3}\,G\right) - \exp\left(\mathrm{i}\,\frac{2}{3}\,G\right)\right]\Lambda_i\Lambda_k N_i N_k, \tag{72}$$

where $\Lambda_x = \lambda_7$, $\Lambda_y = -\lambda_5$, $\Lambda_z = \lambda_2$ are the Gell-Mann $SU(3)$ matrices. The vector $\mathbf{N}$ is defined as in (9). For the multi-Skyrmions considered, the baryon number B is independent of the second profile function G.

The condition $1 - \cos F \cdot \cos G = 0$, which must be satisfied at the topological center of the Skyrmion, leads to definite values of both profile functions at this point. The Wess–Zumino term vanishes for such $SO(3)$ Skyrmions.[70] This leads to vanishing hypercharge $Y = 0$ and to strangeness $S = -B$, where

$$B = 2\,\frac{k}{\pi}\,[F(0) - F(\infty)]. \tag{73}$$

In the chirally symmetric limit the tetralambda with strangeness $S = -4$ is stable to decay into two dilambdas:

$$2M_2 - M_4 \approx 2{,}5\ F_\pi/e. \tag{74}$$

The inclusion of pseudoscalar mass terms explicitly breaking the chiral symmetry, in general, can lead to unbinding of the system. For example, for $F_\pi = 186$ MeV and $e = 6.08$ it has been found[68] that $M_2 \approx 2379$ MeV, $2M_2 - M_4 \approx 28$ MeV. For $F_\pi = 108$ MeV and $e = 4.84$, it is found that $2M_2 - M_4 \approx 0$. As was shown in Ref. 60, states with $k \geqslant 3$, $B \geqslant 6$ are already unstable to decay into dilambda states.

A more direct generalization of $SU(2)$ to $SU(3)$ multi-Skyrmions has also been considered. Reference 60 gives the results of studying dibaryons corresponding to the multiplets lowest in energy (10-, 27-, 35-, and 28-plets) for the following values of the Casimir operator $O^{(q)}_{(p)}$ of the representation:

$$G(SU(3)) = \frac{1}{3}\,[p^2 + q^2 + pq + 3(p+q)] = 6,\ 8,\ 12,\ 18. \tag{75}$$

The results for the tribaryon 35-plet and the tetrabaryon 28-plet are given in the same study. As a rule, the binding energy of strange few-baryon systems grows with increasing strangeness within a given multiplet.

Here we should note that the masses of the baryon octet and decuplet are not reproduced by the $SU(3)$ generalization under discussion, owing to the very large contribution of the mass term.[71] The value of F_π needed to reproduce the hyperon masses turns out to be very small. The description of the masses of the baryon octet and decuplet is more successful in the variant of the $SU(3)$ model in Ref. 72. The existence of strange dibaryons is discovered theoretically in this model.

3. ELECTROMAGNETIC FORM FACTORS

The states of the lightest nuclei described above are only very remotely reminiscent of the states with which we are accustomed to dealing in the traditional theoretical nuclear physics of nuclei as systems of interacting nucleons. For example, the quantum soliton with topological charge equal to

TABLE IX. Tribaryon spectrum.

$\lambda_{\min}$	T	S	T_3	E_{tot}	$\hbar\omega_{\text{vibr}}$
0	1/2	3/2	1/2	2461	
−0,027	1/2	3/2	1/2	2460	127
0	3/2	3/2	1/2	2526	
−0,071	3/2	3/2	1/2	2519	125
0	5/2	3/2	1/2	2633	
−0,133	5/2	3/2	1/2	2607	122

two with the deuteron quantum numbers has a toroidal structure, which is never considered in traditional representations of this lightest nucleus. The representation of light nuclei as the quantum solitons of a chiral field can be subjected to various tests. For example, the calculated electromagnetic characteristics must reproduce the experimental values, or be of the same order of magnitude as the corresponding quantities in the familiar models of these nuclei. The toroidal structure of the baryon-number distribution and the corresponding electric-charge distribution could be completely demolished by any disagreement with experiment. Therefore, the first studies on calculating the electromagnetic characteristics of the deuteron in this model have been carried out. Of course, here one cannot hope to numerically reproduce the electromagnetic characteristics with an error of less than 30%, as for the nucleon in the original model of Skyrme, but there should be qualitative agreement. The toroidal structure of the lightest nuclei could, in principle, be manifested in some unusual behavior of the nuclear form factors, which, of course, would not be evidence in favor of the chiral soliton model.

The explicit form of the electromagnetic current of the Skyrme model has already been tested in calculations of the electromagnetic characteristics of nucleon states.[74,75] The same operators are also used in sectors with $B\geqslant 1$. Substitution of the ansatz (47) into the expression

$$J_\mu(\mathbf{x})=\frac{1}{2}B_\mu(\mathbf{x})+J_\mu^{V_3}(\mathbf{x}), \quad (76)$$

where $B_\mu(\mathbf{x})/2$ is the isoscalar part of the electromagnetic current and $J_\mu^{V_3}(\mathbf{x})$ is the third component of the isovector current, determines the electromagnetic current-density operator $J_\mu(\mathbf{x})$. In turn, averaging of the resulting operator over the states described in the preceding sections determines all the electromagnetic observables of these states. For example, the Coulomb, $F_C(q^2)$, and quadrupole, $F_Q(q^2)$, form factors of the deuteron in the Breit frame are determined by the following matrix element:[76]

$$\langle dS'_3\mathbf{p}'\,|J^0(\mathbf{r}=0)\,dS_3\mathbf{p}\rangle=F_C(q^2)\,\delta_{S'_3S_3}$$
$$+\frac{1}{6M_d^2}F_Q(q^2)\,U_{S'_3a}(3q^aq^b-q^2\delta^{ab})\,U^+_{bS_3}. \quad (77)$$

Here the momentum transfer is $\mathbf{q}=\mathbf{p}'-\mathbf{p}$, $q=|\mathbf{q}|$, M_d is the deuteron mass, and U_{im} is a unitary matrix relating the Cartesian and spherical bases. The form factors are given by the integrals

$$F_C(q^2)=\frac{1}{2}\int d^3r j_0(qr)\,B_0(\mathbf{r}) \quad (78)$$

and

$$\frac{1}{M_d^2}F_Q(q^2)=\frac{3}{2}\,\frac{1}{q^2}\int d^3r j_2(qr)\,B_0(\mathbf{r})\,\frac{x^2+y^2-2z^2}{r^2}. \quad (79)$$

For small transfers $q^2\to 0$ the following behavior of the form factors is typical:

$$F_C(q^2)\to 1-\frac{1}{6}q^2\,\langle r^2\rangle_d \quad (80)$$

and

$$F_Q(0)=M_d^2Q, \quad (81)$$

where $\langle r^2\rangle_d$ and Q are the squared charge radius and the quadrupole moment:

$$\langle r^2\rangle_d=\frac{1}{2}\int d^3r r^2B^0(\mathbf{r}); \quad (82)$$

$$Q=\frac{1}{10}\int d^3r\,(x^2+y^2-2z^2)\,B^0(\mathbf{r}). \quad (83)$$

In the calculations carried out in Ref. 76 the following set of parameters was used: $F_\pi=108$ MeV, $e=4.84$, $m_\pi/F_\pi e=0.263$. This set optimizes the predictions of the static properties of the nucleon (neglecting vibrational degrees of freedom) and obviously does not correspond to the vacuum sector, where the pion decay constant is almost twice as large, $F_\pi=186$ MeV, in accordance with the normalization of the kinetic part of the Lagrangian. The static electromagnetic properties of the deuteron with these parameters are:[43]

$$\langle r^2\rangle_d^{1/2}=0{,}92\ \text{F};\ Q=0{,}082\ \text{F}^2;$$

$$\mu_d=0{,}74\mu_N;\ \mu_{d\to\pi p}=-4{,}4\ \mu_N.$$

The last two numbers correspond to the deuteron magnetic moment and the transition magnetic moment for the deuteron photodisintegration reaction in units of the nuclear magneton μ_N. These quantities are determined by the matrix elements of the spatial part of the electromagnetic current. The transition moment is determined by the isovector part of this current. The values obtained agree in magnitude and sign with the experimental values for the deuteron. It turned out that the toroidal structure of the soliton does not lead to any unusual behavior of the form factors. The qualitative behavior of these form factors is actually quite similar to the behavior in the traditional models of the deuteron, although the qualitative agreement with the experimental data is not very impressive.

In Fig. 8 we give the results of our calculations for the Coulomb, F_C, and quadrupole, F_Q, form factors of the state with the deuteron quantum numbers in the chirally symmetric limit ($m_\pi=0$, $F_\pi=129$ MeV, and $e=5.45$) and for the broken-symmetry case ($m_\pi=139$ MeV, $F_\pi=108$ MeV, and $e=4.84$). However, the predictions in this original variant of the Skyrme model are quite different from the experimental data. The model form factors fall off too slowly with increasing momentum transfer, and the behavior of the form factors is only reproduced qualitatively.

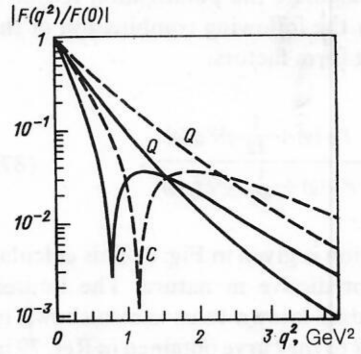

FIG. 8. Charge and quadrupole form factors of the state with the deuteron quantum numbers, normalized to unity at $q^2=0$: solid line—$F_\pi=108$ MeV, $e=4.84$, $m_\pi=139$ MeV; dashed line—$F_\pi=129$ MeV, $e=5.45$, $m_\pi=0$.

Here it should be noted that in the traditional approach the deuteron form factors are calculated in a completely different way. For example, in the impulse approximation the Coulomb form factor of the deuteron is written as

$$F_C(q^2) = [G_E^p(q^2) + G_E^n(q^2)] \int_0^\infty dr\, [u^2(r) + w^2(r)]\, j_0(qr/2). \quad (84)$$

In this expression G_E^p and G_E^n are the proton and neutron electromagnetic form factors, and $u(r)$ and $w(r)$ are the s and d components of the radial wave function of the deuteron. It should be noticed that in the latter expression the deuteron form factor is determined by the Fourier transform of both the internal wave function and the internucleon charge distributions. In the Skyrme model these two distributions are not distinguished, and the form factor is given by the Fourier transform of the baryon distribution.

The approach used in Ref. 77 is closer to the traditional one. The authors of that study use the product ansatz

$$A_1 U_1(\mathbf{r}-\mathbf{x}_1) A_1^+ A_2 U_1(\mathbf{r}-\mathbf{x}_2) A_2^+, \quad (85)$$

where U_1 is the solution in the $B=1$ sector, and the corresponding chiral-angle function $F(r)$ is fitted to the isoscalar nucleon form factor.

Substitution of this representation into the expression for the electromagnetic current leads to the decomposition

$$J^\mu(\mathbf{r}) = J_1^\mu(\mathbf{r}-\mathbf{x}_1, A_1) + J_2^\mu(\mathbf{r}-\mathbf{x}_2, A_2) + J_{ex}^\mu(\mathbf{r}, \mathbf{x}_1, \mathbf{x}_2, A_1, A_2). \quad (86)$$

The first two terms obviously correspond to the impulse approximation. The last term gives the contribution of meson exchange currents, as it is interpreted in the Skyrme model. In order to calculate this contribution to the electromagnetic form factors, we must average this term over the wave functions of $\mathbf{x}_1$, $\mathbf{x}_2$ of the potential approach (for example, over the wave functions in the Paris potential). The $SU(2)$ matrices for the soliton orientations in isospin space appearing in (86) must be averaged over the nucleon wave functions of the Skyrme model. Although in this "hybrid" approach reasonable results are obtained for the form factors, it is not completely systematic in the Skyrme model, at least at the present time. However, the possibility of rigorously justifying this approach in the future should not be excluded.

It is interesting to calculate the polarization tensor P (Ref. 78), determined by the following combination of the Coulomb and quadrupole form factors:

$$P(q) = \frac{\sqrt{2}}{3}\, q^2 F_Q(q)\, \frac{F_C(q) + \frac{1}{12} q^2 F_Q(q)}{F_C^2(q) + \frac{1}{18} q^4 F_Q^2(q)}, \quad (87)$$

The result of this calculation is given in Fig. 9. This calculation is qualitative, but predictive in nature. The limited amount of experimental data known to us is also shown in Fig. 9. Our curve is similar to the curve obtained in Ref. 79 in the hybrid model of the deuteron, including quark degrees of freedom and meson exchange currents.

The Coulomb form factor shown in Fig. 10 inspires somewhat more optimism.

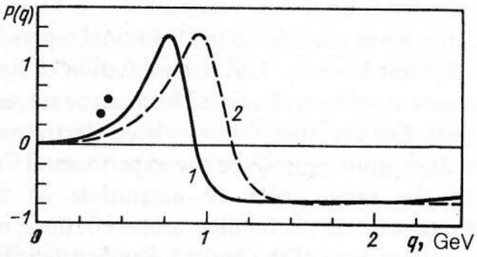

FIG. 9. The deuteron polarization tensor. Experimental data (points) from Ref. 78: 1) $F_\pi = 108$ MeV, $e = 4.84$, $m_\pi = 139$ MeV; 2) $F_\pi = 129$ MeV, $e = 5.45$, $m_\pi = 139$ MeV.

Vector mesons

It should be noted that the inclusion of vector mesons can improve the agreement between the calculated electromagnetic characteristics of few-baryon systems and experiment. One way of introducing vector meson fields is based on the use of the hidden $SU(2)$ symmetry of the nonlinear σ model.[22] This hidden symmetry is revealed if the soliton field is represented as the product of two $SU(2)$ left-handed and right-handed fields ξ_L and ξ_R:

$$U(\mathbf{x}) = \xi_L^+(\mathbf{x})\, \xi_R(\mathbf{x}). \quad (88)$$

Now it can be shown that the Lagrangian is invariant under local gauge transformations

$$\xi_{L,R} \Rightarrow h(\mathbf{x})\, \xi_{L,R}, \quad (89)$$

where $h(\mathbf{x})$ is a unitary scalar field.

The gauge bosons associated with this symmetry are identified with ρ mesons, which are associated with the field operators

$$\boldsymbol{\tau}\boldsymbol{\rho}_\mu = \frac{1}{ig}(\partial_\mu \xi_L \xi_L^+ + \partial_\mu \xi_R \xi_R^+), \quad (90)$$

as can be checked by replacing the derivatives ∂_μ by covariant derivatives:

$$D_\mu = \partial_\mu - \frac{g}{2}(\boldsymbol{\tau}\cdot\boldsymbol{\rho}_\mu + \omega_\mu). \quad (91)$$

In the last equation ω_μ corresponds to the ω-meson field. The resulting Lagrangian has the form of the Weinberg Lagrangian,[81] and the gauge constant can be identified with the constant of the decay $\rho \to 2\pi$.

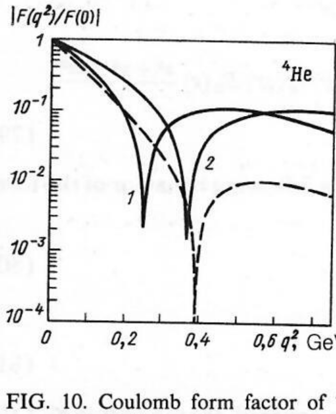

FIG. 10. Coulomb form factor of ^{4}He: 1) $F_\pi = 108$ MeV, $e = 4.87$, $m_\pi = 139$ MeV; 2) $F_\pi = 129$ MeV, $e = 5.45$, $m_\pi = 0$; the dashed line corresponds to the fit of the experimental data from Ref. 80.

The electromagnetic interaction is included by extending the covariant derivative in the effective Lagrangian:

$$D_\mu \xi = (\partial_\mu - iV_\mu)\, \xi + ieA_\mu \xi Q, \qquad (92)$$

where A_μ is the photon field and Q is the quark charge operator.

In this approach one of the variants leads to the model of vector-meson dominance (the direct coupling of photons to the pion field drops out). As a result, for the isoscalar current we find

$$J^\mu_{I=0} = -\frac{m^2_\omega}{3g}\, \omega_\mu(\mathbf{x}) \qquad (93)$$

(m_ω is the ω-meson mass), and for the isoscalar Coulomb form factor we obtain[82]

$$G^C_{I=0}(q^2) = -\frac{m^2_\omega}{3g}\int d^3r e^{i\mathbf{q}\cdot\mathbf{r}}\omega_0(r)$$

$$= \frac{1}{2}\frac{m^2_\omega}{m^2_\omega + q^2}\int d^3r B_0(\mathbf{r})\, j_0(qr). \qquad (94)$$

In the last integral it is not difficult to recognize the form factor, which we have already encountered in the original Skyrme model. From this for the charge radius we find

$$\langle r^2\rangle_{I=0} = -12\frac{dG^C_{I=0}(q^2)}{d(q^2)}\bigg|_{q^2=0} = \frac{6}{m^2_\omega} + \langle r^2_B\rangle. \qquad (95)$$

The isovector electromagnetic form factor has a similar factor $(q^2 + m^2_\rho)^{-1}$.

The values of the static characteristics for nucleons obtained in this model are given in Table X. They suggest that the calculation of these quantities in the scheme with vector mesons for few-baryon systems also will lead to realistic values.

Other schemes for including vector mesons are discussed in Refs. 23, 24, and 83–87.

4. MULTIBARYONS AND SCALAR FIELDS

To complete this review we briefly discuss another possible variant for obtaining multibaryon states in soliton models. This variant involves use of an effective meson-field Lagrangian which is closer to the linear σ model, and an additional scalar field is introduced.

The introduction of the scalar field σ into the effective Lagrangian can have another motivation. For example, the authors of Ref. 88 studied the isospin-independent components of the spin–orbit interaction. In the Skyrme model they have the correct (negative) sign in the presence of terms of sixth order in derivatives of the type

$$\mathcal{L}_6 = -\varepsilon^2 B_\mu B^\mu, \qquad (96)$$

where B_μ is the anomalous baryon current of the model.

However, this is insufficient to ensure the required value of the negative part of the spin–orbit interaction. The needed value is obtained automatically if an additional scalar field is introduced according to the scheme proposed in Refs. 17 and 18, and also Refs. 19 and 20. The introduction of the scalar field makes it possible to reproduce yet another typical feature of the QCD Lagrangian. This is the conformal anomaly or the nonzero value of the trace of the energy–momentum tensor in QCD. This can be obtained within the framework of the effective-Lagrangian method at the cost of introducing the dilaton field. The introduction of the scalar also leads, as was shown in Refs. 84, 89, and 90, to the required strong attraction of the nucleon–nucleon potential at intermediate distances.

The trace of the energy–momentum tensor of QCD and, accordingly, the divergence of the dilatation current are not zero (see, for example, Refs. 91 and 92):

$$\partial_\mu j^\mu_D = \theta^\mu_\mu = \frac{\beta(g_s)}{g_s}\,\mathrm{Tr}\,(G^2_{\mu\nu}) + [1+\gamma(g_s)]\sum_{i=1}^{n_f} m_i\bar{q}_i q_i. \qquad (97)$$

Here j^μ_D is the dilatation current corresponding to scale transformations, θ^μ_μ is the canonical energy–momentum tensor, g_s is the running coupling constant, and $\beta(g_s)$ is the QCD β function. In the $SU(2)$ sector we can neglect the contribution of the quark mass terms and construct an effective theory representing the gluodynamics in terms of scalar fields.

In Ref. 93 it was suggested that there exists an order parameter: $H(\mathbf{x}) = \theta^\mu_\mu$, which can be identified with the scalar glueball.

Up to terms of second order in derivatives of H, the Lagrangian $\mathcal{L}_H$ is uniquely defined and is given by

$$\mathcal{L}_H = \frac{1}{2}\alpha H^{-3/2}(\partial_\mu H)^2 - \frac{1}{4}H\ln\left[\frac{H}{\Lambda^4}\right], \qquad (98)$$

where α is a dimensionless order parameter and Λ is the QCD scale. This potential has a minimum at $\langle H\rangle = \Lambda^4/e$, giving a negative vacuum energy density $\langle H\rangle/4$. A bubble in this vacuum would raise the total energy, but would be unstable. However, it turned out that such a bubble can be stabilized by a chiral soliton interacting with it. This is another possible way of constructing baryons and multibaryons.

The solutions obtained in the model of a dilaton–quarkonium field interacting with a chiral field possess a similar

TABLE X. Static electromagnetic properties of nucleons (Ref. 26).

Parameter	Model	Complete model	Experiment
$\langle r^2_E\rangle^{1/2}_p$, F	0,93	0,97	0,86±0,01
$\langle r^2_E\rangle^{1/2}_n$, F	−0,22	−0,25	−0,119±0,001
$\langle r^2_M\rangle^{1/2}_p$, F	0,84	0,94	0,86±0,06
$\langle r^2_M\rangle^{1/2}_n$, F	0,85	0,94	0,88±0,07
μ_p	3,36	2,77	2,79
μ_n	−2,57	−1,88	−1,91
$\|\mu_p/\mu_n\|$	1,31	1,51	1,46

structure. The Lagrangian for such a model was obtained by the simultaneous chiral and conformal bosonization of QCD in Ref. 19. The model represents a generalization of the Skyrme model and introduces a scalar field $\sigma(\mathbf{x})$ interacting with a chiral field $U(\mathbf{x})$ (Ref. 20):

$$\mathcal{L}_{\mathrm{eff}}(U,\sigma)=\frac{F_\pi^2}{4}e^{-2\sigma}\,\mathrm{Tr}\,\partial_\mu U\partial^\mu U^{+}+N_f\frac{F_\pi^2}{4}e^{-2\sigma}(\partial_\mu\sigma)^2+\frac{1}{128\pi^2}\mathrm{Tr}\,[\partial_\mu UU^{+},\ \partial_\mu UU^{+}]^2-C_g\frac{N_f}{48}\Big[e^{-4\sigma}-1+\frac{4}{\varepsilon}(1-e^{-\varepsilon\sigma})\Big],\qquad(99)$$

where F_π is the pion decay constant (93 MeV) and N_f is the number of flavors. The value of the gluon condensate $C_g=\langle G_{\mu\nu}^2\rangle$ lies in the range (300–400 MeV)^4. The first two terms are the kinetic terms of the chiral and scalar fields, and the third is the Skyrme self-interaction of the chiral field. The effective potential of the scalar fields, given by the last term in the Lagrangian, is an extrapolation of the low-energy potential[20] to high energies. The one-loop approximation for the Gell-Mann–Low β function of QCD was used for the extrapolation. The parameter ε is determined by the number of flavors, $\varepsilon=8N_f/(33-2N_f)$.

Turning to the baryon sector, let us make an assumption about the form of the static chiral and scalar fields. For example, for the chiral field we use the Skyrme–Witten ansatz $U(\mathbf{r})=\exp\{i\tau\cdot\mathbf{n}F(r)\}$, where $\mathbf{n}=\mathbf{r}/r$, and require that the dilaton field $\sigma(r)$ be spherically symmetric. Then for the mass functional M we have

$$M=M_2+M_4+V,\qquad(100)$$

in which

$$M_2=4\pi\frac{F_\pi}{e}\int_0^\infty dx\,\Big\{\frac{N_f}{4}x^2(\rho')^2+\rho^2\Big[\frac{x^2(F')^2}{2}+\sin^2F\Big]\Big\};\qquad(101)$$

$$M_4=4\pi\frac{F_\pi}{e}\int_0^\infty dx\,\Big\{\frac{1}{2}\frac{\sin^4F}{x^2}+\sin^2F\,(F')^2\Big\};\qquad(102)$$

$$V=4\pi\frac{F_\pi}{e}D_{\mathrm{eff}}\int_0^\infty dx\,x^2\Big[\rho^4-1+\frac{4}{\varepsilon}(1-\rho^\varepsilon)\Big];\qquad(103)$$

$$\rho(r)=\exp\{-\sigma(r)\}.\qquad(104)$$

In Eqs. (101)–(104) the dimensionless variable $x=F_\pi er$ is expressed in terms of the Skyrme parameter e, which according to (99) is equal to 2π. The contribution of the potential to the mass is determined by the coefficient $D_{\mathrm{eff}}=C_gN_f/(48e^2F_\pi^4)$. The mass functional (100) leads to the system of coupled equations (the primes denote derivatives with respect to x)

$$F''[\rho^2x^2+2\sin^2F]+2F'x[x\rho\rho'+\rho^2]+\sin(2F)(F')^2-\rho^2\sin(2F)-\frac{\sin^2F}{x^2}\sin(2F)=0;$$

$$\frac{N_f}{2}x[x\rho''+2\rho']-2\rho\Big[\frac{x^2(F')^2}{2}+\sin^2F\Big]-4D_{\mathrm{eff}}[\rho^3-\rho^{\varepsilon-1}]x^2=0.\qquad(105)$$

According to the virial theorem, the contributions of the individual terms of the functional on solutions of the system must satisfy the condition $M_4-M_2-3V=0$, which can be used to control the accuracy of the numerical solution of the system. Near the origin $F(x)\sim\pi N-\alpha x$ and $\rho(x)\sim\rho(0)+\beta x^2$. For sufficiently large x, we have $F(x)\sim a/x^2$, $\rho\sim1-b/x^6$. There are nontrivial relations between the numerical coefficients α and β, a and b. The boundary conditions chosen ensure that the mass functional is finite for a fixed value of the topological charge N. The rotational degrees of freedom of the soliton can be quantized using the collective-coordinate method. As a result, we arrive at the expression for the nucleon mass $M_N=M+(3/8)I$, in which now the rotational moment of inertia is[21]

$$I=\frac{8\pi}{3}(F_\pi e^3)^{-1}\int_0^\infty dx\,\sin^2F\,[\rho^2x^2+(F')^2x^2+\sin^2F].\qquad(106)$$

Some results of the numerical calculations are given in Table XI, in which we also give the rms radius of the baryon-number distribution $\langle r_B\rangle^{1/2}$. For comparison, in the last column of the table we give the results obtained in the original Skyrme model.

The partial restoration of the chiral invariance should be noted; it appears as a strong deviation of the value of ρ at the origin from the asymptotic value $\rho=1$. The considerably smaller classical component of the mass, and also the smaller value of the mass when rotational degrees of freedom are included than in the original Skyrme model, should also be noted.

TABLE XI. Some input and output values for the nucleon in the model with dilation–quarkonia (Ref. 21).

C_g	(300 MeV)^4	(300 MeV)^4	Skyrme
N_f	3	2	2
F_π, MeV	93	93	93
e	2π	2π	2π
$\rho(0)$	0,29	0,22	1
M, MeV	867	827	1087
$\langle r_B^2\rangle^{1/2}$, F	0,37	0,38	0,38
M_N, MeV	1072	1033	1260

TABLE XII. Results of calculations in the generalized model with a dilation field [$F_\pi = 93$ MeV, $e = 2\pi$, $C_g = (240\ \text{MeV})^4$, $N_f = 2$ for the nucleon and $N_f = 3$ for the dibaryon]. The results obtained in the original Skyrme model are given for comparison.

Parameter	Nucleon		Dibaryon ($S=1$)	
	Generalized model	Skyrme model	Generalized model	Skyrme model
M, MeV	758	1079	2012	3214
$\langle r^2\rangle^{1/2}$, F	0,42	0,36	0,75	0,70
M_B, MeV	962	1242	2165	3320
With minimization with respect to the scale parameter λ:				
$\langle r^2\rangle^{1/2}$, F	0,50	0,44	0,81	0,75
M_B, MeV	933	1204	2126	3320

The nucleon electromagnetic form factors calculated in this model qualitatively reproduce the experimental values, but again they fall off too slowly. Baryons are very compact objects. However, the significantly smaller values of the baryon masses than those given by the original Skyrme model provide impetus for seeking multibaryon solutions. In Table XII we give the results of calculations of the dibaryon mass with the Skyrme–Witten ansatz for the dibaryon. In spite of the considerably smaller value of the mass than for the solution in the original model with the same ansatz, there are no convincing arguments in favor of this solution corresponding to a minimum in the $B = 2$ sector. It is not impossible that the inclusion of the dilaton field does not spoil the toroidal solution of the type considered in the preceding sections.

CONCLUSION

We have studied the properties of few-baryon systems obtained in the Skyrme model and some generalizations of it. We have shown that, using very general assumptions about the form of the solutions of the equations of motion, in the original model we arrive at bound states in sectors with baryon number $B = 2$, 3, and 4 having a toroidal structure. We have also found bound states in the vacuum sector with $B = 0$, but they do not have a toroidal structure. We have also succeeded in obtaining more complicated states consisting of toroidal solitons and nontoroidal substructures. Some of these states might be manifested in reactions with stopped antinucleons as nuclear compound states containing antibaryons as their structural units. The search for such states is desirable in order to check the model of strongly interacting systems as solitons of a chiral field.

The calculations of the properties of states with the quantum numbers of light nuclei were mainly carried out using the collective-coordinate method. For this we first calculate the effective Hamiltonian taking into account the degrees of freedom corresponding to space and isospin rotations, and also some vibrational degrees of freedom. All the resulting candidates for states of light nuclei have binding energies which are larger than the experimental values. It is possible that the calculated states can still not be identified with nuclear states, since many quantum corrections have not been taken into account. The calculated electromagnetic form factors of states with the quantum numbers of the deuteron and ^{4}He qualitatively reproduce the experimental ones.

The calculations of the electromagnetic characteristics of nucleons with vector mesons corresponding to the hidden gauge symmetry of the model give reason to assume that the inclusion of these mesons in the effective Lagrangian will improve the agreement between the theoretical form factors of the lightest nuclei and the experimental ones at large momentum transfers.

Finally, we cannot exclude the possibility that the inclusion of the dilaton field, which is required to reproduce the conformal anomaly of QCD by the effective Lagrangian, does not spoil the toroidal structure of the lightest nuclei, but allows the observed binding energies to be reproduced for values of the coupling constants taken in the meson sector.

The authors are grateful to V. G. Kadyshevskiĭ, V. B. Belyaev, M. K. Volkov, G. V. Efimov, V. Yu. Novozhilov, A. A. Andrianov, V. A. Andrianov, V. B. Kopeliovich, M. Johnson, and V. Nerenberg for many useful discussions about the problems considered here.

[1] T. H. R. Skyrme, Proc. R. Soc. London, Ser. A **260**, 127 (1961); Nucl. Phys. **31**, 556 (1962).
[2] M. Gell-Mann and M. Levy, Nuovo Cimento **16**, 705 (1960).
[3] E. V. Shuryak, Nucl. Phys. **B319**, 511 (1989); Yu. A. Simonov, Nucl. Phys. **B307**, 512 (1988).
[4] D. I. Dyakonov and V. Yu. Petrov, Nucl. Phys. **B245**, 259 (1984); **B272**, 457 (1986).
[5] S. Weinberg, Phys. Rev. Lett. **18**, 188 (1967).
[6] M. K. Volkov, Fiz. Elem. Chastits At. Yadra **13**, 1070 (1982) [Sov. J. Part. Nucl. **13**, 446 (1982)]; D. Ebert and M. K. Volkov, Fortschr. Phys. **29**, 127 (1981).
[7] V. N. Pervushin, Teor. Mat. Fiz. **22**, 291 (1975) [Theor. Math. Phys. (USSR) **22**, 203 (1975)]; D. I. Kazakov, V. N. Pervoushin, and S. V. Pushkin, J. Phys. A **10**, 2093 (1978).
[8] A. A. Slavnov, Nucl. Phys. **B31**, 301 (1971).
[9] A. A. Andrianov and Yu. V. Novozhilov, Phys. Lett. **153B**, 422 (1985).
[10] A. A. Andrianov, V. A. Andrianov, and Yu. V. Novozhilov, Teor. Mat. Fiz. **70**, 63 (1987) [Theor. Math. Phys. (USSR) **70**, 43 (1987)].
[11] D. I. D'yakonov and M. I. Éĭdes, Pis'ma Zh. Eksp. Teor. Fiz. **38**, 358 (1983) [JETP Lett. **38**, 433 (1983)].
[12] D. Ebert and H. Reinhardt, Nucl. Phys. **B271**, 188 (1986).
[13] I. J. P. Aitchison and C. M. Fraser, Phys. Rev. D **31**, 2605 (1985).
[14] N. I. Karchev and A. A. Slavnov, Teor. Mat. Fiz. **65**, 192 (1985) [Theor. Math. Phys. (USSR) **65**, 1099 (1986)].
[15] E. Witten, Nucl. Phys. **B223**, 422 (1983).
[16] E. Witten, Nucl. Phys. **B223**, 433 (1983).
[17] H. Gomm, P. Jain, R. Johnson, and J. Schechter, Phys. Rev. D **33**, 801, 3476 (1986).
[18] P. Jain, R. Johnson, and J. Schechter, Phys. Rev. D **35**, 2230 (1987).
[19] A. A. Andrianov, V. A. Andrianov, V. Yu. Novozhilov, and Yu. V. Novozhilov, Phys. Lett. **186B**, 401 (1987).
[20] V. A. Andrianov and V. Yu. Novozhilov, Preprint IC/87/411, ICTP, Trieste (1987); Phys. Lett. **202B**, 580 (1988).
[21] V. A. Nikolaev, V. Yu. Novozhilov, and O. G. Tkachev, Preprint E4-88-536, JINR, Dubna (1988),
[22] M. Bando, T. Kugo, S. Uehara *et al.*, Phys. Rev. Lett. **54**, 1215 (1985).
[23] O. Kaymakcalan, S. Rajeev, and J. Schechter, Phys. Rev. D **30**, 594 (1984).

[24] O. Kaymakcalan and J. Schechter, Phys. Rev. D **31**, 1109 (1985).
[25] G. S. Adkins, C. R. Nappi, and E. Witten, Nucl. Phys. **B228**, 552 (1983).
[26] U. G. Meissner, N. Kaiser, and W. Weise, Nucl. Phys. **A466**, 685 (1987).
[27] A. Jackson, A. D. Jackson, and V. Pasquier, Nucl. Phys. **A432**, 567 (1985).
[28] E. M. Nyman and D. O. Riska, Phys. Lett. **183B**, 7 (1987).
[29] U. Blom and D. O. Riska, Nucl. Phys. **A476**, 603 (1988).
[30] E. M. Nyman and D. O. Riska, Phys. Rev. Lett. **57**, 3007 (1986); Nucl. Phys. **A454**, 498 (1986).
[31] R. Vinh Mau, M. Lacombe, B. Loiseau *et al.*, Phys. Lett. **150B**, 259 (1985).
[32] I. Zahed, A. Wirzba, and U.-G. Meissner, Phys. Rev. D **33**, 830 (1986).
[33] T. Otofuji, S. Saito, M. Yasino *et al.*, Prog. Theor. Phys. **78**, 527 (1987).
[34] A. De Pace, H. Muter, and A. Faessler, Phys. Lett. **188B**, 307 (1987); Z. Phys. A **326**, 33 (1987).
[35] H. Weigel, B. Schwesinger, and G. Holzwarth, Phys. Lett. **168B**, 321 (1986).
[36] V. B. Kopeliovich and B. E. Shtern, Pis'ma Zh. Eksp. Teor. Fiz. **45**, 165 (1987) [JETP Lett. **45**, 203 (1987)].
[37] J. J. M. Verbaarshot, Phys. Lett. **195B**, 235 (1987).
[38] N. S. Manton, Phys. Lett. **192B**, 177 (1987).
[39] I. R. Kozhevnikov, Yu. P. Rybakov, and M. B. Fomin, Teor. Mat. Fiz. **75**, 353 (1988) [Theor. Math. Phys. (USSR) **75**, 575 (1988)].
[40] V. A. Nikolaev and O. G. Tkachev, Communication R4-86-515, JINR, Dubna (1986).
[41] V. A. Nikolaev and O. G. Tkachev, Rapid Communication N1[34]-89, p. 28, JINR, Dubna (1986); Rapid Communication N4[37]-89, p. 18, JINR, Dubna (1986).
[42] D. I. D'yakonov, Yad. Fiz. **45**, 1592 (1987) [Sov. J. Nucl. Phys. **45**, 987 (1987)].
[43] E. Braaten and L. Carson, Phys. Rev. D **38**, 3525 (1988).
[44] V. A. Nikolaev and O. G. Tkachev, Preprint E4-89-56, JINR, Dubna (1989); in *Proc. of the Twelfth Few-Body Conf.*, TRIUMF, TRI-89-2, Vancouver, 1989, p. F25.
[45] T. Kurihara, H. Kanada, T. Otofuji, and S. Saito, in *Proc. of the Twelfth Few-Body Conf.*, TRIUMF, TRI-89-2, Vancouver, 1989, p. F31; Prog. Theor. Phys. **81**, 858 (1989).
[46] V. A. Nikolaev, Fiz. Elem. Chastits At. Yadra **20**, 401 (1989) [Sov. J. Part. Nucl. **20**, 173 (1989)].
[47] B. Schwesinger, H. Weigel, G. Holzwarth, and A. Hayashi, Phys. Rep. **173**, 1 (1989).
[48] D. O. Riska, Phys. Rep. **181**, 1 (1989).
[49] V. G. Makhan'kov, Yu. P. Rybakov, and V. I. Sanyuk, in *Lectures for Young Scientists*, R4-89-568, Dubna, 1989, No. 55 [in Russian].
[50] Yu. P. Rybakov and V. I. Sanyuk, Preprint NBI-HE-81-49, Niels Bohr Institute, Copenhagen (1981).
[51] U. B. Kaulfuss and U.-G. Meissner, Phys. Rev. C **30**, 2058 (1984); Phys. Rev. D **31**, 3024 (1985).
[52] V. Vento, Phys. Lett. **153B**, 198 (1985).
[53] A. Jackson, A. D. Jackson, A. S. Goldhaber, and G. E. Brown, Phys. Lett. **154B**, 101 (1985).
[54] E. Braaten and L. Carson, Phys. Rev. Lett. **56**, 1897 (1986).
[55] V. A. Nikolaev, in *Proc. of the Eighth Int. Seminar on High Energy Physics Problems*, D1.2-86-668, Dubna, 1986, Vol. 1, p. 78.
[56] V. A. Nikolaev, in *Proc. of the Ninth Int. Seminar on High Energy Physics Problems*, D1.2-88-652, Dubna, 1988, Vol. 1, p. 51.
[57] D. Finkelstein and J. Rubenstein, J. Math. Phys. **9**, 1762 (1968).
[58] J. J. M. Verbaarshot, T. S. Walhout, J. Wambach, and H. W. Wyld, Nucl. Phys. **A468**, 520 (1987).
[59] V. A. Nikolaev and O. G. Tkachev, Preprint E4-89-848, JINR, Dubna (1989).
[60] V. B. Kopeliovich and B. E. Stern, Preprint 89/34 N, Nordita, Copenhagen (1989).
[61] S. B. Zenkin, V. B. Kopeliovich, and B. E. Shtern, in *Proc. of the Symposium on Nucleon–Nucleon and Nucleon–Nucleus Interactions*, Leningrad Nuclear Physics Institute, Gatchina, 1986, p. 415 [in Russian].
[62] E. B. Bogomol'nyĭ and V. A. Fateev, Yad. Fiz. **37**, 228 (1983) [Sov. J. Nucl. Phys. **37**, 134 (1983)].
[63] J. Wess and B. Zumino, Phys. Lett. **37B**, 95 (1971).
[64] J. Williams, J. Math. Phys. **11**, 2611 (1983).
[65] I. P. Auer, C. Chang-Fang, E. Colton *et al.*, Phys. Rev. Lett. **48**, 1150 (1982).
[66] T. Otofuji, K. Sakai, H. Kanada *et al.*, Prog. Theor. Phys. **73**, 703 (1985).
[67] H. Walliser, A. Hayashi, and G. Holzwarth, Nucl. Phys. **A456**, 717 (1986).
[68] A. P. Balachandran, A. Barducci, F. Lizzi *et al.*, Phys. Rev. Lett. **52**, 887 (1984).
[69] A. I. Issinskiĭ, V. B. Kopeliovich, and B. E. Shtern, Yad. Fiz. **48**, 209 (1988) [Sov. J. Nucl. Phys. **48**, 133 (1988)].
[70] A. P. Balachandran, F. Lizzi, V. G. S. Rodgers, A. Stern *et al.*, Nucl. Phys. **B256**, 525 (1985).
[71] M. Praszalovich, Phys. Lett. **158B**, 264 (1985).
[72] C. G. Callan and I. Klebanov, Nucl. Phys. **B262**, 365 (1985).
[73] J. Kunz and P. J. Mulders, Phys. Lett. **215B**, 449 (1988).
[74] E. Braaten, S.-M. Tse, and C. Willcoh, Phys. Rev. D **34**, 1482 (1986).
[75] V. A. Nikolaev and O. G. Tkachev, Report R4-87-422, JINR, Dubna (1987) [in Russian].
[76] E. Braaten and L. Carson, Phys. Rev. D **39**, 838 (1989).
[77] E. M. Nyman and D. O. Riska, Nucl. Phys. **A468**, 473 (1987).
[78] M. E. Shulze, D. Beck, H. Farkhondeh *et al.*, Phys. Rev. Lett. **52**, 597 (1984).
[79] V. V. Burov and V. N. Dostovalov, Z. Phys. A **326**, 245 (1987).
[80] V. T. Voronchev, T. D. Eainov, V. M. Krasnopol'skiĭ *et al.*, Preprint 17-0472, Institute for Nuclear Research, USSR Academy of Sciences (1986) [in Russian].
[81] S. Weinberg, Phys. Rev. **166**, 1568 (1968).
[82] W. Weise, in: *Progress in Particle and Nuclear Physics*, edited by A. Faessler (Pergamon Press, Amsterdam, 1988), Vol. 20, p. 113.
[83] D. Ebert and M. K. Volkov, Z. Phys. C **16**, 205 (1983).
[84] M. Lacombe, B. Loiseau, R. Vinh Mau, W. N. Cottingham *et al.*, Preprint 87-28, IPN, Orsay (1987).
[85] D. Ebert and H. Reinhardt, Preprint E2-86-274, JINR, Dubna (1986).
[86] X. Q. Zhu, F. C. Khanna, and S. S. M. Wong, Nuovo Cimento A **101**, 563 (1989).
[87] V. Yu. Novozhilov, Preprint IC/89/95, ICTP, Trieste (1989).
[88] D. O. Riska and B. Schwesinger, Phys. Lett. **229B**, 339 (1989).
[89] H. Yabu, B. Schwesinger, and G. Holzwarth, Phys. Lett. **224B**, 25 (1989).
[90] M. Mashaal, T. N. Phom, and T. N. Truong, Phys. Rev. D **34**, 3484 (1986); M. Lacombe, B. Loiseau, R. Vinh Mau, and W. N. Cottingham, Phys. Rev. Lett. **57**, 170 (1989).
[91] J. C. Collins, A. Duncan, and S. D. Joglekar, Phys. Rev. D **16**, 438 (1977).
[92] N. K. Nielsen, Nucl. Phys. **B210**, 212 (1972).
[93] J. Schechter, Phys. Rev. D **21**, 3393 (1980).

Translated by Patricia Millard