Three-particle quantum scattering theory for fixed total orbital angular momentum

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Fiz. Elem. Chastits At. Yadra **21**, 1301–1359 (November–December 1990)

The quantum scattering problem for three-particle systems with fixed total orbital angular momentum is considered. Faddeev's equations for the components of the T matrix and the wave functions are obtained, and the structure of the wave operators and the S matrix and the asymptotic behavior of the wave functions are investigated. A series of spectral identities (Levinson formulas) is proved for systems with a rapidly decreasing interaction.

INTRODUCTION

Quantum scattering theory for three-particle systems was originally formulated in the momentum space.[1] The solution of problems with long-range potentials made it necessary to investigate the properties of the wave functions and kernel of the resolvent in the coordinate representation as well.[2] As a result, not only Faddeev's equations for the components of the T matrix but also the equations for the components of the wave functions in the configuration space have been widely used.

These equations provided the basis for effective methods of calculating the properties of specific systems. They have been most strongly developed in nuclear physics, mainly to describe three-nucleon systems (see, for example, the reviews of Refs. 3 and 4). These methods are based on expansions of the components of the wave functions with respect to particular angular bases (the most widely used expansions are those with respect to bispherical[3] and hyperspherical[4] harmonics).

The point is that in their original form the Faddeev equations contain operators in a six-dimensional space. The direct solution of such equations is beyond the scope of modern computers. In contrast, expansions with respect to basis functions lead to infinite systems of two-dimensional or one-dimensional equations. For their numerical solution, a finite number of equations that make the main contribution to the solution are retained. In the case of systems with short-range potentials, their number is usually small, so that such methods are very effective. This is why significant successes have been achieved by applying Faddeev's equations in nuclear-physics problems in which the so-called s states make the main contribution.

A much more complicated situation arises in the case of systems in which the Coulomb interaction is dominant. In calculations of the properties of such systems, it is, as a rule, necessary to take into account a very large number of basis functions. For this reason, the Faddeev equations have not been widely used in problems of atomic and μ-mesic molecular physics.

The main difficulties in the application of the traditional methods of partial-wave expansions are due to the fact that the basis functions do not take into account the symmetry properties of the system's Hamiltonian. Therefore, the representations of the Hamiltonian in such bases are given by infinite filled matrices. In the case of Coulomb systems, the elements of these matrices decrease slowly with increasing values of the quantum numbers that label them. As a result, the corresponding expansions for the wave functions also converge slowly.

This difficulty disappears if the elements of invariant subspaces of the Hamiltonian are taken as basis functions. In this case, the energy operator decomposes into a direct sum of operators acting on spaces of finite dimension. Therefore, the solution of the scattering problem reduces to the solution of a system of equations of finite rank.

For three-particle systems with central interactions, the corresponding basis consists of Wigner functions—the eigenfunctions of the operators of the total orbital angular momentum and one of its projections. The Wigner functions form invariant subspaces of the Hamiltonian that correspond to states with fixed total orbital angular momentum. Restriction of the Hamiltonian to these subspaces makes it possible to separate three angular variables that describe the rotation of the system as a whole. The technique of the corresponding partial-wave analysis was already developed at the dawn of quantum mechanics[5] and was then generalized to systems with an arbitrary number of particles.[6]

As a result of the separation of the rotational degrees of freedom, the Schrödinger equation reduces to independent finite systems of equations in the three-dimensional space of "intrinsic" coordinates. These equations are well known. They have been widely used in calculations of bound states of atomic and molecular systems. However, there have been practically no investigations of the problems of the scattering theory associated with such equations. These include study of the structure of the wave operators and S matrix, analysis of the asymptotic behavior of the continuum wave functions, formulation of Faddeev equations for their components, etc. The aim of the present paper is to solve these problems, i.e., to construct the scattering theory for three-particle systems with fixed total orbital angular momentum. This theory can serve as the mathematical basis for the creation of new powerful computational methods based on the corresponding Faddeev equations in the three-dimensional space of the intrinsic coordinates. Such equations admit direct solution on modern supercomputers without any intermediate approximations and simplifying assumptions about the structure of the interaction.

New results are presented in the main body of the review. Some of them were announced in the papers of Refs. 7–10.

The scattering theory for three-particle systems with fixed total orbital angular momentum contains all the main difficulties and specific features of the original scattering

 0090-4759/90/060553-26$04.00

problem in the complete six-dimensional configuration space. For brevity, we shall in what follows call this last problem the complete three-body problem.

We make wide use of the general methodology of the scattering theory developed in the complete problem.[1,2] When we refer to the corresponding results without specifying the source, we shall have in mind the monograph of Ref. 11.

1. CONFIGURATION SPACE AND HAMILTONIAN OF A THREE-PARTICLE SYSTEM WITH FIXED TOTAL ORBITAL ANGULAR MOMENTUM

In this section, we reduce the Hamiltonian of the complete three-body problem to the subspace of states corresponding to a fixed total orbital angular momentum of the system. The formal side of the matter reduces to separating the rotational degrees of freedom describing the orientation in space of the triangle formed by the particles. The three remaining degrees of freedom determine the shape of this triangle. We shall call them the intrinsic coordinates of the system, and the collection of these coordinates will be called the intrinsic space. This space is the configuration space of the system for the fixed total orbital angular momentum.

The intrinsic space has a nontrivial geometrical structure—it is a three-dimensional Riemannian manifold with non-Euclidean metric. A systematic study of the geometry of the intrinsic space was made in Ref. 12, and it provides the basis of our exposition of the corresponding questions. It uses the standard formalism and terminology of differential geometry.[13]

Configuration space of the complete three-body problem

We consider a system of three spinless particles in the space R^3. The particles are labeled by the index $\alpha = 1, 2, 3$. By the pair α we shall mean the system of the two particles with labels $\beta \neq \alpha$. Let $\mathbf{r}_\alpha$ and m_α be the radius vectors and masses of the particles. In the center-of-mass system, the configuration of the system is specified by the set of reduced Jacobi coordinates $\{\mathbf{x}_\alpha, \mathbf{y}_\alpha\}$. For $\alpha = 1$, these vectors are defined by the equations

$$\mathbf{x}_1 = \left(\frac{2m_2m_3}{m_2+m_3}\right)^{1/2} (\mathbf{r}_3 - \mathbf{r}_2),$$
$$\mathbf{y}_1 = \left[\frac{2(m_2+m_3)\,m_1}{m_1+m_2+m_3}\right]^{1/2} \left(\mathbf{r}_1 - \frac{m_2\mathbf{r}_2+m_3\mathbf{r}_3}{m_2+m_3}\right). \quad (1)$$

The expressions for the Jacobi vectors with $\alpha = 2, 3$ are obtained from (1) by cyclic permutation of the indices. The vectors with different indices are related by an orthogonal transformation,

$$\begin{pmatrix}\mathbf{x}_\alpha\\ \mathbf{y}_\alpha\end{pmatrix} = \begin{pmatrix} c_{\alpha\beta} & s_{\alpha\beta}\\ -s_{\alpha\beta} & c_{\alpha\beta}\end{pmatrix}\begin{pmatrix}\mathbf{x}_\beta\\ \mathbf{y}_\beta\end{pmatrix}, \quad c_{\alpha\beta}^2 + s_{\alpha\beta}^2 = 1, \quad (2)$$

whose coefficients depend on the masses of the particles:

$$c_{\alpha\beta} = -\left[\frac{m_\alpha m_\beta}{(M-m_\alpha)(M-m_\beta)}\right]^{1/2}, \quad M = \sum_\alpha m_\alpha,$$
$$s_{\alpha\beta} = (-1)^{\beta-\alpha}\,\mathrm{sign}\,(\beta-\alpha)\,(1-c_{\alpha\beta}^2)^{1/2}.$$

Thus, in the center-of-mass system the configuration space of the three particles is a Euclidean space $Q \simeq R^6 \simeq R^3 \otimes R^3$ with elements $X = \{\mathbf{x}_\alpha, \mathbf{y}_\alpha\}$. The scalar product, metric, and volume element in Q are defined in the standard manner:

$$(X, X') = (\mathbf{x}_\alpha, \mathbf{x}'_\alpha) + (\mathbf{y}_\alpha, \mathbf{y}'_\alpha),$$
$$K_X = dX^2 = d\mathbf{x}_\alpha^2 + d\mathbf{y}_\alpha^2, \quad dQ(X) = d\,\mathbf{x}_\alpha \wedge d\mathbf{y}_\alpha. \quad (3)$$

The Hamiltonian of the complete three-body problem in the center-of-mass system has the form

$$H = H_0 + V, \quad V = \sum_\alpha V_\alpha(|\mathbf{x}_\alpha|),$$
$$H_0 = -\Delta_X = -\Delta_{x\alpha} - \Delta_{y\alpha}, \quad (4)$$

where V_α are the central potentials of the two-body interactions.

Intrinsic space

The orientation in space of the plane spanned by the Jacobi vectors is specified by three Euler angles, which parametrize the elements g of the rotation group $SO(3)$. We consider the natural action of this group on the space Q:

$$X = \{\mathbf{x}_\alpha, \mathbf{y}_\alpha\} \to gX = \{g\mathbf{x}_\alpha, g\mathbf{y}_\alpha\}. \quad (5)$$

If the vectors $\mathbf{x}_\alpha$ and $\mathbf{y}_\alpha$ are linearly independent, then $gX = X$ implies $g = e$ [e is the identity element of $SO(3)$]. Therefore, the group $SO(3)$ acts freely on the set $\dot{Q} = Q - D$, where D corresponds to configurations in which all the particles lie on one straight line:

$$D = \{X: a\mathbf{x}_\alpha + b\mathbf{y}_\alpha = 0, \ a^2 + b^2 \neq 0\}.$$

Thus, $\dot{Q}$ is the space of the $SO(3)$ orbits of the factor manifold $M = \dot{Q}/SO(3)$. We call the manifold M the intrinsic space of the three-particle system.

This structure, which relates $\dot{Q}$, the intrinsic space M, and the group $SO(3)$, is the principal fiber bundle $\dot{Q}(M, SO(3), \pi)$ with fiber space $\dot{Q}$, base M, and structure group $SO(3)$. The canonical projection π associates the points of $\dot{Q}$ with elements of the base M, $\pi\dot{Q} = M$.

It is obvious that M is a three-dimensional manifold. We shall show that it is topologically equivalent to the space

$$R_+^3 = \{r = (z_\alpha^1, z_\alpha^2, z_\alpha^3); \ z_\alpha^1, z_\alpha^2 \in (-\infty, \infty), \ z_\alpha^3 \in (0, \infty)\}, \quad (6)$$

which consists of three-dimensional vectors with components

$$z_\alpha^1 = (\mathbf{x}_\alpha^2 - \mathbf{y}_\alpha^2)/\rho, \quad z_\alpha^2 = 2(\mathbf{x}_\alpha, \mathbf{y}_\alpha)/\rho, \quad z_\alpha^3 = 2\,|\mathbf{x}_\alpha \times \mathbf{y}_\alpha|/\rho, \quad (7)$$

where ρ is the hyperradius of the system:

$$\rho = (\mathbf{x}_\alpha^2 + \mathbf{y}_\alpha^2)^{1/2} = \left[\sum_i (z_\alpha^i)^2\right]^{1/2}.$$

The mapping $\{\mathbf{x}_\alpha, \mathbf{y}_\alpha\} \xrightarrow{\pi} r$ defines the canonical projection of the fiber bundle $\dot{Q}(M, SO(3), \pi)$.

Indeed, z_α^i are invariant with respect to the action of $SO(3)$. The preimage $\pi^{-1}(r)$ is formed by the vectors $\mathbf{x}_\alpha$ and $\mathbf{y}_\alpha$ that satisfy the system of equations (7). They determine the lengths of the Jacobi vectors and the angle between them, i.e., they specify a triangle formed by certain fixed vectors $\mathring{\mathbf{x}}_\alpha(r)$, $\mathring{\mathbf{y}}_\alpha(r)$ in R^3. Therefore, the preimage $\pi^{-1}(r)$ is the $SO(3)$ orbit of the vector $\mathring{X} = \{\mathring{\mathbf{x}}_\alpha(r), \mathring{\mathbf{y}}_\alpha(r)\}$. Therefore $M = \mathring{Q}/SO(3) \simeq R_+^3$.

Since the base $M \simeq R_+^3$ can be contracted to a point, the fiber bundle $\dot{Q}(M, SO(3), \pi)$ is trivial, i.e.,

$\dot{Q} \simeq SO(3) \otimes M$. In other words, every element $X \in \dot{Q}$ can be represented in the form

$$X = \{g,\ r\} \in SO(3) \otimes M;\ g : X = g\overset{\circ}{X}(r);\ r = \pi X. \tag{8}$$

The intrinsic space can be equipped with the natural structure of the vector space R^3_+ with the scalar product

$$(r,\ \tilde{r})_M = \sum_i z^i_\alpha \tilde{z}^i_\alpha. \tag{9}$$

The connection between the parametrizations (6) and the different indices α is specified by a rotation transformation in R^3_+, which is generated by the analogous transformation (2) of the Jacobi vectors:

$$\begin{pmatrix} z^1_\alpha \\ z^2_\alpha \\ z^3_\alpha \end{pmatrix} = \begin{pmatrix} \cos\omega_{\alpha\beta} & \sin\omega_{\alpha\beta} & 0 \\ -\sin\omega_{\alpha\beta} & \cos\omega_{\alpha\beta} & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} z^1_\beta \\ z^2_\beta \\ z^3_\beta \end{pmatrix}, \tag{10}$$
$$\cos\omega_{\alpha\beta} = c^2_{\alpha\beta} - s^2_{\alpha\beta},\quad \sin\omega_{\alpha\beta} = 2c_{\alpha\beta}s_{\alpha\beta}.$$

We shall call the coordinates $(\xi^1\xi^2\xi^3)$, which parametrize the intrinsic space, the intrinsic coordinates of the three-particle system. Besides the Cartesian coordinates (7), we shall also use three sets of intrinsic coordinates:

1. Dragt's coordinates[14] $(\xi^1\xi^2\xi^3) = (\rho\psi\varphi_\alpha)$:

$$r = (z^1_\alpha,\ z^2_\alpha,\ z^3_\alpha) = (\rho\cos\psi\cos\varphi_\alpha,\ \rho\cos\psi\sin\varphi_\alpha,\ \rho\sin\psi),$$
$$\psi \in (0,\ \pi/2),\ \varphi_\alpha \in [0,\ 2\pi]. \tag{11}$$

2. The hyperspherical coordinates $(\xi^1\xi^2\xi^3) = (\rho\chi_\alpha\theta_\alpha)$:

$$r = (z^1_\alpha,\ z^2_\alpha,\ z^3_\alpha) = (\rho\cos\chi_\alpha,\ \rho\sin\chi_\alpha\cos\theta_\alpha,\ \rho\sin\chi_\alpha\sin\theta_\alpha),$$
$$\chi_\alpha \in [0,\ \pi],\ \theta_\alpha \in (0,\ \pi). \tag{12}$$

3. The Jacobi coordinates $(\xi^1\xi^2\xi^3) = (x_\alpha y_\alpha \theta_\alpha)$:

$$x_\alpha = |\mathbf{x}_\alpha| = \rho\cos(\chi_\alpha/2),\quad \cos\theta_\alpha = (\hat{\mathbf{x}}_\alpha,\ \hat{\mathbf{y}}_\alpha),\quad y_\alpha = |\mathbf{y}_\alpha| = \rho\sin(\chi_\alpha/2), \tag{13}$$

In (12) and (10), the angles θ_α are the same. The first two sets of intrinsic coordinates are ordinary spherical coordinates in R^3_+ with the polar angles $\pi/2 - \psi$ and φ_α defined relative to different axes. The connection between the intrinsic coordinates with different indices is specified by the transformation (10). It follows from it, in particular, that on replacement of an index the angles φ_α are shifted by a constant,

$$\varphi_\beta = \varphi_\alpha + \omega_{\alpha\beta},$$

while the coordinates ρ and ψ are invariant with respect to the index α.

This invariance has a simple physical origin—the coordinates ρ and ψ determine the principal moments of inertia of the triangle formed by the three particles. To see this, let the configuration of the system be described by a vector $X \in \dot{Q}$. The corresponding inertia tensor $A(X)$ has the form

$$[A(X)]_{ij} = \delta_{ij}\rho^2 - \mathbf{x}^i_\alpha\mathbf{x}^j_\alpha - \mathbf{y}^i_\alpha\mathbf{y}^j_\alpha, \tag{14}$$

where the indices $i, j = 1, 2, 3$ label the Cartesian coordinates of the Jacobi vectors. It is readily verified that the principal axes of the tensor $A(X)$ form the vectors e_1, e_2, and $e_3 = e_1 \times e_2$, which are related to the Jacobi vectors by

$$\begin{pmatrix} \mathbf{x}_\alpha \\ \mathbf{y}_\alpha \end{pmatrix} = \rho R(\varphi_\alpha/2) \begin{pmatrix} \cos(\psi/2)\, e_1 \\ \sin(\psi/2)\, e_2 \end{pmatrix},\quad R(\varphi) = \begin{pmatrix} \cos\varphi & -\sin\varphi \\ \sin\varphi & \cos\varphi \end{pmatrix}.$$

The principal moments of inertia $I_i = [e_i, A(X)e_i]$ are

$$I_1 = \rho^2\sin^2(\psi/2),\ I_2 = \rho^2\cos^2(\psi/2),\ I_3 = \rho^2. \tag{15}$$

Finally, we fix the parametrization of the structure group $SO(3)$. Let the rotation $g \in SO(3)$ specify the orientation of the principal axies of inertia e_i with respect to the system of fixed unit vectors $\overset{\circ}{e}_i$. We denote by $(\phi^1\phi^2\phi^3)$ the set of corresponding Euler angles:

$$e_i = g\overset{\circ}{e}_i,$$
$$g = \begin{pmatrix} \mathcal{R}_{11} & \mathcal{R}_{12} & \sin\phi^2\cos\phi^1 \\ \mathcal{R}_{21} & \mathcal{R}_{22} & \sin\phi^2\sin\phi^1 \\ -\cos\phi_3\sin\phi^2 & \sin\phi^3\sin\phi^2 & \cos\phi^2 \end{pmatrix}, \tag{16}$$

where $\mathcal{R}_{11} = \cos\phi^1\cos\phi^2\cos\phi^3 - \sin\phi^1\sin\phi^3$; $\mathcal{R}_{12} = -\cos\phi^1\cos\phi^2\sin\phi^3 - \sin\phi^1\cos\phi^3$; $\mathcal{R}_{21} = \sin\phi^1\cos\phi^2\sin\phi^3 + \cos\phi^1\sin\phi^3$; $\mathcal{R}_{22} = -\sin\phi^1\cos\phi^2\sin\phi^3 + \cos\phi^1\cos\phi^3$. By virtue of (8), the following parametrization of $SO(3)$ in $\dot{Q}$ corresponds to such a set of Euler angles:

$$X = g\overset{\circ}{X},\ \overset{\circ}{X} = \{\overset{\circ}{\mathbf{x}}_\alpha,\ \overset{\circ}{\mathbf{y}}_\alpha\},\ \begin{pmatrix} \overset{\circ}{\mathbf{x}}_\alpha \\ \overset{\circ}{\mathbf{y}}_\alpha \end{pmatrix} = \rho R(\varphi_\alpha/2) \begin{pmatrix} \cos(\psi/2)\,\overset{\circ}{e}_1 \\ \sin(\psi/2)\,\overset{\circ}{e}_2 \end{pmatrix}. \tag{17}$$

Metric and volume element of the intrinsic space

The Euclidean metric (3) of the space Q induces a metric structure of the fiber bundle $\dot{Q}(M, SO(3), \pi)$. In particular, it determines the metric of the base M. In this subsection, we calculate the metric tensor and the volume element of M.

We proceed as follows. The metric K_X generates the scalar product K^*_X in the cotangent space $T^*_X\dot{Q}$ to the manifold $\dot{Q}$ at the point X. We separate in it two subspaces: $(T^*_X\dot{Q})_{\rm int}$ and $(T^*_X\dot{Q})_{\rm rot}$, which correspond to the intrinsic (int) and rotational (rot) degrees of freedom. These subspaces are orthogonal in the scalar product K^*_X. Therefore, the metric tensor of the intrinsic space is determined by the restriction of K^*_X to the subspace $(T^*_X\dot{Q})_{\rm int}$.

We shall describe the structure of spaces $T^*_X\dot{Q}$, $(T^*_X\dot{Q})_{\rm int}$, and $(T^*_X\dot{Q})_{\rm rot}$. The first of them is the standard linear space of the 1-forms on $\dot{Q}$ with basis $d\mathbf{x}^i_\alpha$, $d\mathbf{y}^j_\alpha$ or $d\xi^i$, $d\phi^j$ $(i, j = 1, 2, 3)$, where ξ^i is an arbitrary set of intrinsic coordinates, and ϕ^j are the Euler angles in (16). As the subspace $(T^*_X\dot{Q})_{\rm int}$, we take the linear hull of the basis 1-forms $d\xi^i$ corresponding to the internal coordinates. Finally, $(T^*_X\dot{Q})_{\rm rot}$ consists of 1-forms associated with rotations of the three-particle system. These are described by the angular-velocity vector

$$\omega = [A(X)]^{-1}(\mathbf{x}_\alpha \times d\mathbf{x}_\alpha + \mathbf{y}_\alpha \times d\mathbf{y}_\alpha), \tag{18}$$

where the matrix $[A(X)]^{-1}$ is the inverse of the matrix of the inertia tensor (14). As a basis in $(T^*_X\dot{Q})_{\rm rot}$ we choose the components of the vector 1-form ω with respect to the principal axes of inertia e_i. They are calculated on the basis of the representation (17) and a relation that follows from (16),

$$\begin{pmatrix} de_1 \\ de_2 \\ de_3 \end{pmatrix} = \begin{pmatrix} 0 & \sigma^3 & -\sigma^2 \\ -\sigma^3 & 0 & \sigma^1 \\ \sigma^2 & -\sigma^1 & 0 \end{pmatrix} \begin{pmatrix} e_1 \\ e_2 \\ e_3 \end{pmatrix}, \tag{19}$$

where σ^i are 1-forms on the group $SO(3)$:

$$\begin{pmatrix} \sigma_1 \\ \sigma_2 \\ \sigma_3 \end{pmatrix} = \begin{pmatrix} -\sin\phi^2\cos\phi^3 & \sin\phi^3 & 0 \\ \sin\phi^2\sin\phi^3 & \cos\phi^3 & 0 \\ \cos\phi^2 & 0 & 1 \end{pmatrix} \begin{pmatrix} d\phi^1 \\ d\phi^2 \\ d\phi^3 \end{pmatrix}.$$

As a result, we obtain the decomposition

$$\omega = \sum_i \omega^i e_i,\ \omega^1 = -\sigma^1,\ \omega^2 = -\sigma^2,$$

$$\omega^3 = -\sigma^3 + \frac{1}{2}\sin\psi\, d\varphi_\alpha. \tag{20}$$

Thus, $(T^*_X\dot{Q})_{\rm rot}$ is the linear hull of the 1-forms ω^i.

It is obvious that the 1-forms $d\xi^i$ and ω^j form a basis in $T^*_X\dot{Q}$. Note that the part $\frac{1}{2}\sin\psi d\varphi_\alpha$ of the 1-form ω^3 couples the intrinsic and rotational degrees of freedom of the system and represents the Coriolis interaction.

Using the expressions (17), (19), and (20), we now rewrite the metric (3) in terms of the basis 1-forms of the subspaces $(T^*_X\dot{Q})_{\rm int}$ and $(T^*_X\dot{Q})_{\rm rot}$:

$$K_X = K_r + \sum_i I_i\,(\omega^i_{\cdot})^2, \tag{21}$$

where I_i are the principal moments of inertia (15). The first term corresponds to the intrinsic coordinates:

$$K_r = \sum_{ij} b_{ij}\,d\xi^i\,d\xi^j = d\rho^2 + \frac{\rho^2}{4}(d\psi^2 + \cos^2\psi\, d\varphi_\alpha^2), \tag{22}$$

where $r = \pi X$ is a point of the base M in the fiber bundle $\dot{Q}(M, SO(3), \pi)$. The metric (21) defines a scalar product K^*_X in $T^*_X\dot{Q}$:

$$K^*_X(d\xi^i,\ d\xi^j) = b^{ij},\ K^*_X(\omega^i,\ \omega^j) = \delta_{ij}I_i^{-1},$$
$$K^*_X(\omega^i,\ d\xi^j) = 0, \tag{23}$$

where $(b^{ij}) = (b_{ij})^{-1}$. In this scalar product, the subspaces $(T^*_X\dot{Q})_{\rm int}$ and $(T^*_X\dot{Q})_{\rm rot}$ are orthogonal. The restriction of K^*_X to $(T^*_X\dot{Q})_{\rm int}$ is determined by the metric (22). Thus, the expression (22) defines a metric tensor of the intrinsic space. It induces a scalar product in the cotangent space T^*_rM to the manifold M at the point r:

$$K^*_r(d\xi^i,\ d\xi^j) = b^{ij}. \tag{24}$$

We now calculate the volume element of the intrinsic space. To this end, we express the measure (3) of the space Q in terms of the 1-forms $d\xi^i$ and ω^j by means of the representation (21) for the metric of Q:

$$dQ(X) = \sigma(r) b^{1/2} (\bigwedge_i \omega^i) \wedge (\bigwedge_i d\xi^i) = 8\pi^2 dg \wedge dM(r), \tag{25}$$

where $b = \det(b_{ij})$, $\sigma = (I_1 I_2 I_3)^{1/2}$, dg is the normalized invariant measure on the group $SO(3)$,

$$dg = \frac{1}{8\pi^2}\sin\phi^2 \bigwedge_i d\phi^i, \tag{26}$$

and dM is the volume element in the intrinsic space:

$$dM(r) = \sigma(r)\, b^{1/2} \bigwedge_i d\xi^i. \tag{27}$$

We give a list of formulas for $\sigma(r)$, the metric, and the volume element of the intrinsic space in the various coordinates:

1. Dragt's coordinates (11):

$$K_r = d\rho^2 + \frac{\rho^2}{4}(d\psi^2 + \cos^2\psi\, d\varphi_\alpha^2),\quad \sigma(r) = \frac{\rho^3}{2}\sin\psi,$$
$$dM(r) = \frac{\rho^5}{16}\sin 2\psi\, d\rho \wedge d\psi \wedge d\varphi_\alpha. \tag{28}$$

2. The hyperspherical coordinates (12):

$$K_r = d\rho^2 + \frac{\rho^2}{4}(d\chi_\alpha^2 + \sin^2\chi_\alpha\, d\theta_\alpha^2),\ \sigma(r) = \frac{\rho^3}{2}\sin\chi_\alpha \sin\theta_\alpha,$$
$$dM(r) = \frac{\rho^5}{8}\sin^2\chi_\alpha \sin\theta_\alpha\, d\rho \wedge d\chi_\alpha \wedge d\theta_\alpha. \tag{29}$$

3. The Jacobi coordinates (13):

$$K_r = dx_\alpha^2 + dy_\alpha^2 + \frac{x_\alpha^2 y_\alpha^2}{x_\alpha^2 + y_\alpha^2}d\theta_\alpha^2,\ \sigma(r) = x_\alpha y_\alpha \sqrt{x_\alpha^2 + y_\alpha^2}\sin\theta_\alpha,$$
$$dM(r) = x_\alpha^2 y_\alpha^2 \sin\theta_\alpha\, dx_\alpha \wedge dy_\alpha \wedge d\theta_\alpha. \tag{30}$$

Contribution of intrinsic and rotational degrees of freedom to the kinetic-energy operator

The kinetic-energy operator of the complete three-body problem is given by the Laplacian (4) on the space Q. We now express it in terms of vector fields on Q corresponding to the intrinsic and rotational degrees of freedom.

The decomposition of the cotangent space $T^*_X\dot{Q}$ constructed above into intrinsic and rotational subspaces induces an analogous decomposition of the tangent space $T_X\dot{Q}$ to the manifold $\dot{Q}$ at the point X into subspaces $(T_X\dot{Q})_{\rm int}$ and $(T_X\dot{Q})_{\rm rot}$. The latter are the linear hulls of the vectors dual to the basis 1-forms $d\xi^i$ and ω^j. We denote these vectors by $(\partial/\partial\xi^i)^*$ and L_j. They satisfy the duality relations

$$d\xi^i((\partial/\partial\xi^j)^*) = \delta_{ij},\quad \omega^i(L_j) = \delta_{ij},$$
$$\omega^i((\partial/\partial\xi^j)^*) = 0,\quad d\xi^i(L_j) = 0. \tag{31}$$

By the definition of the subspaces, $(T_X\dot{Q})_{\rm int}$ and $(T_X\dot{Q})_{\rm rot}$ are orthogonal in the scalar product (21), since

$$K_X((\partial/\partial\xi^i)^*,\ (\partial/\partial\xi^j)^*) = b_{ij},$$
$$K_X(L_i,\ L_j) = \delta_{ij}I_i,\ K_X(L_j,\ (\partial/\partial\xi^i)^*) = 0.$$

We shall describe the explicit form of the vector fields $(\partial/\partial\xi^i)^*$ and L_i. In accordance with the definition (18), (19) of the 1-forms ω^i, the vectors L_i dual to them are the components of the operator

$$L = \mathbf{x}_\alpha \times \frac{\partial}{\partial\mathbf{x}_\alpha} + \mathbf{y}_\alpha \times \frac{\partial}{\partial\mathbf{y}_\alpha} \tag{32}$$

with respect to the principal axes of inertia. They can be calculated on the basis of (17) and (19) and are the generators of the right action of the group $SO(3)$ on $\dot{Q}$:

$$L = -\sum_i e_i L_i,$$

$$\begin{pmatrix} L_1 \\ L_2 \\ L_3 \end{pmatrix} = \begin{pmatrix} \cos\phi^3\sin\phi^2 & -\sin\phi^3 & -\cos\phi^3\operatorname{ctg}\phi^2 \\ -\sin\phi^3/\sin\phi^2 & -\cos\phi^3 & \sin\phi^3\operatorname{ctg}\phi^2 \\ 0 & 0 & -1 \end{pmatrix} \times \begin{pmatrix} \partial/\partial\phi^1 \\ \partial/\partial\phi^2 \\ \partial/\partial\phi^3 \end{pmatrix}.$$

The vector fields $(\partial/\partial\xi^i)^*$ are fixed by the duality conditions (31). For example, in Dragt's coordinates they have the form

$$\left(\frac{\partial}{\partial\rho}\right)^*=\frac{\partial}{\partial\rho},\quad \left(\frac{\partial}{\partial\psi}\right)^*=\frac{\partial}{\partial\psi},$$

$$\left(\frac{\partial}{\partial\varphi_\alpha}\right)^*=\frac{\partial}{\partial\varphi_\alpha}-\frac{1}{2}\sin\psi L_3. \tag{33}$$

Note that, apart from a factor, the operator (32) is identical to the operator $\hat{L}$ of the total orbital angular momentum of the three-particle system:

$$\hat{L}=-iL=-\sum_k e_k\hat{L}_k, \qquad \hat{L}_k=-iL_k. \tag{34}$$

The vector 1-form of the angular velocity (18) dual to this operator can be interpreted as a connection on the principal fiber bundle $\dot{Q}(M, SO(3), \pi)$. The vector fields $(\partial/\partial\xi^i)^*$ are the horizontal lifts of the fields $\partial/\partial\xi^i$ with respect to the connection ω.

We now represent the Laplacian (4) on the space Q in terms of the vector fields $(\partial/\partial\xi^i)^*$ and L_i. To this end, we consider the kinetic-energy functional corresponding to it:

$$\int_Q\left|\frac{\partial}{\partial\mathbf{x}_\alpha}f\right|^2+\left|\frac{\partial}{\partial\mathbf{y}_\alpha}f\right|^2 dQ(X)=\int_Q K_X^*(\overline{df},\ df)\,dQ(X), \tag{35}$$

where df is the gradient 1-form:

$$df=\sum_i\left(\frac{\partial f}{\partial\mathbf{x}_\alpha^i}d\mathbf{x}_\alpha^i+\frac{\partial f}{\partial\mathbf{y}_\alpha^i}d\mathbf{y}_\alpha^i\right).$$

We decompose it into components lying in the subspaces $(T_X^*\dot{Q})_{\rm int}$ and $(T_X^*\dot{Q})_{\rm rot}$:

$$df=\sum_i\left[\left(\frac{\partial}{\partial\xi^i}\right)^*f\right]d\xi^i+\sum_i(L_if)\,\omega^i.$$

Using this representation and the definition (23) of the scalar product K_X^*, we arrive at the following expression for the kinetic-energy Lagrangian:

$$K_X^*(\overline{df},\ df)=\sum_{ij}b^{ij}\left[\left(\frac{\partial}{\partial\xi^i}\right)^*f\right]\left[\left(\frac{\partial}{\partial\xi^j}\right)^*f\right]+\sum_i I_i^{-1}(L_if)^2.$$

We substitute it in the functional (35) and integrate by parts, using the factorization (25) of the measure dQ. As a result, we obtain the required representation for H_0:

$$H_0=-\frac{1}{\sigma b^{1/2}}\sum_{i,j}\left(\frac{\partial}{\partial\xi^i}\right)^*\sigma b^{1/2}b^{ij}\left(\frac{\partial}{\partial\xi^j}\right)^*+\sum_i\frac{\hat{L}_i^2}{I_i}, \tag{36}$$

where $\hat{L}_i$ are the components of the operator of the total orbital angular momentum (34).

In this expression, the horizontal lifts of the vector fields $\partial/\partial\xi^i$ contain terms that couple the intrinsic and rotational degrees of freedom. For example in Dragt's coordinates, the coupling is generated by the last term of the field $(\partial/\partial\varphi_\alpha)^*$ in (33). It describes the Coriolis interaction in the three-particle system.

Hamiltonian of the three-particle system with fixed total orbital angular momentum

We now define the rotational degrees of freedom and reduce the Hamiltonian of the complete three-body problem to the subspaces of states corresponding to fixed total orbital angular momentum.

We consider the Hilbert space $\mathscr{H}=L^2(\dot{Q})$, in which the Hamiltonian (4) acts. We recall that $\dot{Q}\simeq SO(3)\otimes M$: to the vector $X\in\dot{Q}$ there corresponds the point (8), $\{g, r\}\in SO(3)\otimes M$. The action (5) of the rotation group on $\dot{Q}$ induces the representation $T(g)$ of it in the space $\mathscr{H}=L^2(SO(3)\otimes M)$:

$$T(g)\Psi(g',\ r)=\Psi(gg',\ r). \tag{37}$$

We shall decompose this representation into irreducible representations and show that the spaces of the irreducible representations are invariant subspaces of the Hamiltonian H. The restriction of H to these subspaces determines the Hamiltonian of the three-particle system with fixed total orbital angular momentum.

As is well known,[15] the irreducible representations of $SO(3)$ are classified by the eigenvalues $l(l+1)$ ($l=0,1,...$) of the Laplace–Beltrami operator on the group $SO(3)$. It is identical to the square of the orbital angular momentum (34), $\hat{L}^2=\hat{L}_1^2+\hat{L}_2^2+\hat{L}_3^2$. To each l there correspond $2l+1$ equivalent representations $T^{lm}(g)$, $m=-l, -l+1,...,l$. They are labeled by the eigenvalues of the projection $\hat{L}_z$ of the angular-momentum operator $\hat{L}$ onto the fixed axis $\mathring{e}_3$ in (16):

$$\hat{L}_z=(\mathring{e}_3,\ \hat{L})=-i\frac{\partial}{\partial\phi^1}. \tag{38}$$

The irreducible representations T^{lm} are specified by matrices $D^l(g)=[D^l_{mn}(g)]$ ($m,\ n=-l,...,l$) formed from the Wigner functions. These are the common eigenfunctions of the operators $\hat{L}^2$ and $\hat{L}_z$. The definition of the Wigner functions and some of their properties are described in the Appendix.

We now realize the irreducible representations T^{lm} in the space $\mathscr{H}$. Let $\mathscr{H}^{lm}$ be the invariant subspaces in $\mathscr{H}$ of the operators $\hat{L}^2$ and $\hat{L}_z$. These are the linear hulls of the functions $D^l_{mn}(g)$ with fixed l and m:

$$\mathscr{H}^{lm}=\left\{\Psi^{lm}(g,\ r)=\sum_{n=-l}^{l}D^l_{mn}(g)\,\Psi_n^{lm}(r)\right\}. \tag{39}$$

We define the action of T^{lm} in $\mathscr{H}^{lm}$ in accordance with (37):

$$T^{lm}(g)\,\Psi^{lm}(g',\ r)=\sum_{n=-l}^{l}D^l_{mn}(gg')\,\Psi_n^{lm}(r).$$

Further, we consider the kinetic-energy operator (36). On the basis of the canonical commutation relations for the components of the angular-momentum operator $\hat{L}$, we can readily show that H_0 commutes with $\hat{L}^2$ and $\hat{L}_z$. This is also true for the total Hamiltonian H, since the potentials depend only on the intrinsic coordinates. Therefore, $\mathscr{H}^{lm}$ are invariant subspaces of H.

It is obvious that $\mathscr{H}^{lm}$ are isomorphic to the Hilbert spaces $\mathscr{H}^l=L^2(M,\mathbb{C}^{2l+1},dM)$. The elements of $\mathscr{H}^l$ are complex $(2l+1)$-vector functions $f(r)$ on the intrinsic space formed from the coefficients of the expansion (39): $f(r) = [\Psi_n^{lm}(r), n = -l,...,l]$. Their components are labeled by the values of the projection $\hat{L}_3 = (e_3, \hat{L})$ of the angular momentum onto the principal axis e_3 of the inertia tensor (14). The scalar product in $\mathscr{H}^l$, made consistent with the standard scalar product in $L^2(Q)$, is determined by the measure (27) of the intrinsic space:

$$(f, h)_{\mathscr{H}^l} = \frac{8\pi^2}{2l+1} \sum_{n=-l}^{l} \int_M f_n(r) \bar{h}_n(r)\, dM(r). \qquad (40)$$

We shall call $\mathscr{H}^l$ the state space of the three-particle system with fixed total orbital angular momentum l.

The restriction of the Hamiltonian H to $\mathscr{H}^{lm}$ generates the matrix operator H^l, which acts in the state space $\mathscr{H}^l$:

$$H^l = D^l(g^{-1}) H D^l(g). \qquad (41)$$

From the representation (36) we obtain for this operator the expression

$$H^l = H_0^l + V \otimes I^l, \qquad (42)$$

where I^l is the unit matrix of rank $2l+1$, and

$$H_0^l = -\frac{1}{\sigma b^{1/2}} \sum_{i,j} (\partial/\partial\xi^i)_l^* \sigma b^{1/2} b^{ij} (\partial/\partial\xi^j)_l^* + \sum_i I_i^{-1} (\hat{L}_i)_l^2. \qquad (43)$$

Here, $(\partial/\partial\xi^i)_l^*$ and $(\hat{L}_i)_l$ are matrix operators that arise as a result of the reduction (41) of the vector fields $(\partial/\partial\xi^i)^*$ and $\hat{L}_i$. The operators $(\hat{L}_i)_l$ are specified by the constant matrices described in the Appendix. Explicit expressions for $(\partial/\partial\xi^i)_l^*$ in Dragt's coordinates follow from (33):

$$\left(\frac{\partial}{\partial\rho}\right)_l^* = \frac{\partial}{\partial\rho} \otimes I^l, \quad \left(\frac{\partial}{\partial\psi}\right)_l^* = \frac{\partial}{\partial\psi} \otimes I^l,$$
$$\left(\frac{\partial}{\partial\varphi_\alpha}\right)_l^* = \frac{\partial}{\partial\varphi_\alpha} \otimes I^l - \frac{i}{2} \sin\psi\, (\hat{L}_3)_l.$$

Thus, in Dragt's coordinates the kinetic part of the Hamiltonian H^l has the form

$$H_0^l = -\left\{\rho^{-5}\partial_\rho \rho^5 \partial_\rho + \frac{4}{\rho^2 \sin 2\psi} \partial_\psi \sin 2\psi \partial_\psi\right\} \otimes I^l$$
$$-\frac{4}{\rho^2 \cos^2\psi} \left[\left(\frac{\partial}{\partial\varphi_\alpha}\right)_l^*\right]^2$$
$$+\frac{1}{\rho^2}\left\{\frac{(\hat{L}_1)_l^2}{\sin^2(\psi/2)} + \frac{(\hat{L}_2)_l^2}{\cos^2(\psi/2)} + (\hat{L}_3)_l^2\right\}, \qquad (44)$$

where $\partial_{\xi} i = \partial/\partial\xi^i$. The Coriolis term $(i/2)\sin\psi(\hat{L}_3)_l$ of the operator $(\partial/\partial\varphi_\alpha)_l^*$ is a matrix gauge field in the intrinsic space, coupling the intrinsic and rotational degrees of freedom.

The intrinsic space M, in which the Hamiltonian H^l is defined, has a boundary ∂M. It consists of the points with zero coordinate z_α^3 in (6), i.e.,

$$\partial M = \{r = (\rho\psi\varphi_\alpha) : \psi = 0\} = \{r = (\rho\chi_\alpha\theta_\alpha) : \sin\chi_\alpha \sin\theta_\alpha = 0\}. \qquad (45)$$

For $r \in \partial M$ the centrifugal term $(\hat{L}_1)_l^2/\sin^2(\psi/2)$ of the operator (44) is singular. Therefore, H^l is defined on the dense, in $\mathscr{H}^l$, set of smooth vector functions that satisfy the condition $(\hat{L}_1)_l f(r)|_{r\in\partial M} = 0$. It follows from the expression (A.2) for the matrix $(\hat{L}_1)_l$ that the solutions of this equation have the form

$$f(r)\,|_{r\in\partial M} = c(\rho, \varphi_\alpha) Y^l, \qquad (46)$$

where c is an arbitrary function, and Y^l is a constant $(2l+1)$-vector. Its coordinates are equal to the values of the spherical functions $Y_{lm}(\pi/2, 0)$ $(m = -l,...,l)$. Equation (46) gives the boundary condition for the eigenfunctions of the Hamiltonian H^l.

2. SCATTERING THEORY FOR THE HAMILTONIAN H^l. RAPIDLY DECREASING POTENTIALS

In this section, we develop a general scheme of scattering theory for the Hamiltonian (42) with rapidly decreasing two-body potentials

$$V_\alpha(x) \underset{x\to\infty}{=} O(x^{-1-\varepsilon}), \quad \varepsilon > 0. \qquad (47)$$

We introduce the T matrix of the Hamiltonian H^l, construct the Faddeev equations for its components, describe the singularities of their kernels, and calculate the kernels of the wave operators and the S matrix.

These constructions are based on a special representation for the Hamiltonian H^l. It is specified by the unitary transformation that diagonalizes its kinetic part H_0^l. In it, H_0^l is a diagonal matrix operator of multiplication, while the potentials are integral matrix operators. Thus, in this representation the operator H^l has essentially the same structure as the Hamiltonian of the complete three-particle problem in the momentum representation. This makes it possible to investigate the Faddeev equations for H^l by the methods of the theory of integral equations developed in the complete problem. The necessary modification of these methods due to the matrix structure of the Hamiltonian H^l, the difference between the measures of the spaces, etc., are fairly trivial. We shall not dwell on a detailed description of them and shall give many results without proofs, since they largely repeat the existing constructions of the scattering theory for the complete three-body problem.

Diagonal representation for the kinetic-energy operator

As is well known, the kinetic-energy operator (4) of the complete three-body problem can be diagonalized in the momentum representation. It is found by a Fourier transformation $\mathscr{F}$ in $L^2(Q)$:

$$(\mathscr{F}\Psi)(X) = (2\pi)^{-3} \int_Q \exp\{i(X, P)\}\, \Psi(P)\, dQ(P). \qquad (48)$$

Here, the vector $P = \{\mathbf{k}_\alpha, \mathbf{p}_\alpha\}$ is composed of the momenta conjugate to the Jacobi vectors $\mathbf{x}_\alpha$ and $\mathbf{y}_\alpha$. The transition to the diagonal representation for H_0 is realized by the unitary transformation $H_0 \to \tilde{H}_0 = \mathscr{F}^* H_0 \mathscr{F}$, which carries it into an operator of multiplication: $H_0\Psi(P) = P^2\Psi(P)$, $P^2 = \mathbf{k}_\alpha^2 + \mathbf{p}_\alpha^2$.

We now recall that the kinetic part of the Hamiltonian (42) is obtained by the restriction (41) of the operator H_0 to the subspace $\mathscr{H}^{lm}$ in (39). Therefore, the diagonal representation for H^l_0 must be determined by the restriction of the Fourier transform (48) to $\mathscr{H}^{lm}$.

Indeed, let $\{g(P), q\}$ and $\{g(X), r\}$ be the points of the space $SO(3)\times M$ that correspond to the vectors P and X in (48) in the trivial fiber bundle $\dot{Q}(M, SO(3), \pi)$:

$$g = \pi P, \quad r = \pi X,$$
$$P = g(P)\,\mathring{P}(q), \quad X = g(X)\,\mathring{X}(r),$$

where the vectors $\mathring{P}$ and $\mathring{X}$ are defined by equations of the type (17). We expand the kernel $\mathscr{F}$ in a Fourier series on the group $SO(3)$:[15]

$$\exp\{i(X, P)\} = \exp\{i(g^{-1}(P)\,g(X)\,\mathring{X},\ \mathring{P})\} \tag{49}$$
$$= \pi \sum_{l=0}^{\infty} (2l+1)\,\mathrm{Sp}\,[\mathscr{F}^l(r, q)\,D^l(g^{-1}(P)\,g(X))],$$

where $\mathscr{F}^l$ is the $(2l+1)\times(2l+1)$ matrix

$$\mathscr{F}^l(r, q) = \frac{1}{\pi}\int\limits_{SO(3)} \exp\{i(\mathring{X}, g\mathring{P})\}\,D^l(g)\,dg. \tag{50}$$

(Here and everywhere below, the integral of a matrix is understood as the matrix of the integrals of the matrix elements.)

We consider further the action of $\mathscr{F}$ in the subspace $\mathscr{H}^{lm}$. From its definition (39), the expansion (49), and the factorization (25) of the measure dQ it follows that $\mathscr{H}^{lm}$ is an invariant subspace of $\mathscr{F}$. The restriction of $\mathscr{F}$ to $\mathscr{H}^{lm}$ generates the matrix integral operator $\mathscr{F}^l$ with kernel (50) acting in the state space $\mathscr{H}^l$ of the Hamiltonian H^l:

$$(\mathscr{F}^l f)(r) = \int\limits_M \mathscr{F}^l(r, q)\,f(q)\,dM(q), \quad f \in \mathscr{H}^l.$$

It is readily verified that this representation is unitary in the scalar product (40) of the space $\mathscr{H}^l$ and defines a representation in which the Hamiltonian H^l_0 is an operator of multiplication by a diagonal matrix:

$$H^l_0 \to \widetilde{H}^l_0 = [\mathscr{F}^l]^*\,H^l_0\mathscr{F}^l = q^2 \otimes I^l, \tag{51}$$

where $q^2 = (q,q)_M$ is the square of the Euclidean length (9) of the vector q in the realization of M as the vector space R^3_+.

The potential part of the Hamiltonian (42) in the diagonal representation for H^l_0 is equal to a sum of matrix integral operators

$$V^l_\alpha = [\mathscr{F}^l]^*\,(V_\alpha \otimes I^l)\,\mathscr{F}^l. \tag{52}$$

To calculate their kernel, we obtain an explicit expression for the kernel of the transformation $\mathscr{F}^l$.

In the integral (50), we parametrize the points r and q by the intrinsic Jacobi coordinates (13),

$$r = (x_\alpha y_\alpha \theta_\alpha), \quad q = (k_\alpha p_\alpha \theta_\alpha), \tag{53}$$

and we rewrite the definition (17) of the vectors $\mathring{X}$ and $\mathring{P}$ as follows:

$$\left.\begin{aligned} &\mathring{X} = g_\alpha(r)\,\{x_\alpha \mathring{e}_3,\ y_\alpha e(\theta\alpha)\},\\ &\qquad\qquad e(\theta) = \sin\theta\,\mathring{e}_1 + \cos\theta\,\mathring{e}_2\\ &\mathring{P} = \tilde{g}_\alpha(q)\,\{k_\alpha e(\theta'_\alpha),\ p_\alpha \mathring{e}_\alpha\}. \end{aligned}\right\} \tag{54}$$

The rotations g_α and $\tilde{g}_\alpha$ can be calculated by comparing the representations (17) and (54). They are defined by the Euler angles

$$g_\alpha(r) = g(\phi_\alpha,\ \pi/2,\ \pi/2), \quad \tilde{g}_\alpha(q) = g(\tilde{\phi}_\alpha,\ \pi/2,\ 3\pi/2),$$

which can be expressed in terms of the Dragt coordinates $(\psi\varphi_\alpha)$ and $\psi'\varphi'_\alpha$ of the points r and q:

$$\phi_\alpha \in [\pi, 2\pi] : \mathrm{tg}\,\phi_\alpha = -\mathrm{tg}\,(\varphi_\alpha/2)\,\mathrm{tg}\,(\psi/2),$$
$$\tilde{\phi}_\alpha \in [0, \pi/2] \cup [3\pi/2, 2\pi] : \mathrm{tg}\,\tilde{\phi}_\alpha = \mathrm{ctg}\,(\varphi'_\alpha/2)\,\mathrm{tg}\,(\psi'/2).$$

We substitute (54) in the integral (50) and go over to integration with respect to the new variable: $g \to g_\alpha^{-1}(r)\,g\tilde{g}_\alpha(q)$. As a result, for the kernel $\mathscr{F}^l$ we obtain the representation

$$\mathscr{F}^l(r, q) = D^l(g_\alpha(r))\,\mathscr{F}^l_\alpha(r, q)\,D^l(\tilde{g}_\alpha^{-1}(r)), \tag{55}$$

in which the matrix $\mathscr{F}^l_\alpha$ has the form

$$\mathscr{F}^l_\alpha(r, q) = \pi^{-1}\int\limits_{SO(3)} \exp\{ik_\alpha x_\alpha(g^{-1}\mathring{e}_3,\ e(\theta'_\alpha))\}$$
$$\times \exp\{ip_\alpha y_\alpha(e(\theta'_\alpha),\ g\mathring{e}_3)\}\,D^l(g)\,dg. \tag{56}$$

In this integral, we expand each of the exponentials in a series in spherical functions:

$$\exp\{iz(\hat{n}_1, \hat{n}_2)\} = 4\pi\sum_{\lambda=0} i^\lambda j_\lambda(z) \sum_{m=-\lambda}^{\lambda} Y_{\lambda m}(\hat{n}_1)\,Y^*_{\lambda m}(\hat{n}_2),$$

where j_λ are spherical Bessel functions.[16] Further, we express the spherical functions $Y_{\lambda m}$ of the arguments $g^{-1}\mathring{e}_3$ and $g\mathring{e}_3$ in terms of the Wigner functions by means of Eq. (A.5). Then in (56) we obtain an integral of a product of three Wigner functions, and it can be expressed in terms of Clebsch–Gordan coefficients by Eq. (A.4). The upshot is the following representation for the matrix $\mathscr{F}^l_\alpha$:

$$\mathscr{F}^l_\alpha(r, q) = \frac{2}{\pi}\sum_{l_1 l_2=0}^{\infty} j_{l_1}(p_\alpha y_\alpha)\,j_{l_2}(k_\alpha x_\alpha)$$
$$\times Q^l_{l_1}(\cos\theta_\alpha)\,C^l_{l_1 l_2}\,Q^l_{l_2}(\cos\theta'_\alpha), \tag{57}$$

where $C^l_{l_1 l_2}$ are the matrices of numbers

$$[C^l_{l_1 l_2}]_{mn} = (-1)^{m+n}\,(2l+1)^{-1} i^{l_1+l_2}$$
$$\times [(2l_1+1)(2l_2+1)]^{1/2}$$
$$\times \langle l_2\, 0\, l_1\, m \mid lm\rangle\ \langle l_2 n l_1\, 0 \mid ln\rangle, \tag{58}$$

and Q^l_λ are diagonal matrices of normalized associated Legendre polynomials:[16]

$$[Q^l_\lambda(t)]_{mn} = \delta_{mn}\left[\frac{(2\lambda+1)(\lambda-|n|)!}{2(\lambda+|n|)!}\right]^{1/2} P^{|n|}_\lambda(t). \tag{59}$$

Note that these matrices satisfy the orthogonality relations

$$\int\limits_{-1}^{1} [Q^l_{l_1}(t)]^*\,Q^l_{l_2}(t)\,dt = \delta_{l_1 l_2} I^l_{l_1 2} \tag{60}$$

$$\sum_{l_1} [C^l_{l_1 l_2}]^* I^l_{l_1} C^l_{l_1 l'_2} = \delta_{l_2 l'_2} I^l_{l_2}, \quad (61)$$

where I^l_λ are diagonal unit matrices for which only part of the diagonal is nonzero for $\lambda < l$:

$$[I^l_\lambda]_{mn} = \begin{cases} \delta_{mn}, & \lambda \geqslant l, \\ \delta_{mn}, & \lambda < l, \ |m| \leqslant \lambda, \\ 0, & \lambda < l, \ |m| > \lambda. \end{cases} \quad (62)$$

Equation (60) is trivial, and (61) follows from the orthogonality properties of the Clebsch–Gordan coefficients (A.6).

Thus, the representations (55) and (57) describe the explicit form of the kernel of the diagonalizing transformation $\mathscr{F}^l$. It is now easy to calculate the kernels of the operators of the two-body potentials (52), using the well-known orthogonality conditions for the spherical Bessel functions and the relations (60) and (61):

$$V^l_\alpha(q, q') = D^l(\widetilde{g}_\alpha(q)) \widetilde{V}^l_\alpha(q, q') D^l(\widetilde{g}^{-1}_\alpha(q')),$$

$$\widetilde{V}^l_\alpha(q, q') = \frac{\delta(p_\alpha - p'_\alpha)}{p^2_\alpha} \times \sum_{\lambda=0}^{\infty} v^{(\lambda)}_\alpha(k_\alpha, k'_\alpha) Q^l_\lambda(\cos\theta_\alpha) Q^l_\lambda(\cos\theta'_\alpha). \quad (63)$$

The last equation is expressed in terms of the Jacobi coordinates of the points q and q':

$$q = (k_\alpha p_\alpha \theta_\alpha), \quad q' = (k'_\alpha p'_\alpha \theta'_\alpha). \quad (64)$$

The coefficients $v^{\prime}_\alpha$ are the ordinary partial potentials of the two-body interactions in the momentum representation:

$$v^{(\lambda)}_\alpha(k, k') = \frac{2}{\pi} \int_0^\infty j_\lambda(kx) V_\alpha(x) j_\lambda(k'x) x^2 dx.$$

Thus, the Hamiltonian of the three-particle system with fixed total orbital angular momentum in the diagonal representation for H^l_0 has the form

$$H^l = [\mathscr{F}^l]^* H^l \mathscr{F}^l = \widetilde{H}^l_0 + V^l; \quad V^l = \sum_\alpha V^l_\alpha. \quad (65)$$

In its structure, it is very similar to the Hamiltonian of the complete three-body problem in the momentum representation—the kinetic part H^l_0 is the operator of multiplication (51), while the two-body potentials are integral operators whose kernels (63) contain δ-function singularities.

The T matrix and Faddeev's equations

On the basis of the representation (65), we introduce the T matrix of the Hamiltonian H^l and formulate Faddeev equations for its components. At the algebraic level, this can be done in the same way as in the complete three-body problem.

We denote by $R^l(z)$ and $R^l_0(z)$ the resolvents of the Hamiltonian (65) and of its kinetic part:

$$R^l(z) = (\widetilde{H}^l - z)^{-1}, \quad R^l_0(z) = (\widetilde{H}^l_0 - z)^{-1}. \quad (66)$$

These are matrix integral operators in $\mathscr{H}^l$ where R^l_0 is a diagonal operator of multiplication, $[R^l_0(z) f](q) = (q^2 - z)^{-1} f(q)$. We define the T matrix of the Hamiltonian (65) and its Faddeev components $M_{\alpha\beta}$ $(\alpha,\beta = 1,2,3)$ by the standard equations

$$T^l(z) = V^l - V^l R^l(z) V^l,$$
$$M^l_{\alpha\beta}(z) = \delta_{\alpha\beta} V^l_\alpha - V^l_\alpha R^l(z) V^l_\beta.$$

It is obvious that the T matrix is equal to the sum of its components:

$$T^l(z) = \sum_{\alpha, \beta} M^l_{\alpha\beta}(z). \quad (67)$$

Repeating the calculations made in the complete three-particle problem, we obtain the Faddeev equations for the components $M^l_{\alpha\beta}$:

$$M^l_{\alpha\beta}(z) = \delta_{\alpha\beta} T^l_\alpha(z) - T^l_\alpha(z) R^l_0(z) \sum_{\gamma \neq \alpha} M^l_{\gamma\beta}(z). \quad (68)$$

Here, the operators T^l_α are the T matrices of the Hamiltonians

$$\widetilde{H}^l_\alpha = \widetilde{H}^l_0 + V^l_\alpha. \quad (69)$$

We shall describe the structure of the kernels of the operators T^l_α. They satisfy the Lippmann–Schwinger equations

$$T^l_\alpha(z) = V^l_\alpha - V^l_\alpha R^l_\alpha(z) T^l_\alpha(z), \quad R^l_\alpha(z) = (\widetilde{H}^l_\alpha - z)^{-1}.$$

By virtue of the representations (63), the variables in these equations separate in the Jacobi coordinates (64). For the kernels of the operators T^l_α we obtain explicit expressions of the type (63):

$$T^l_\alpha(q, q'; z) = D^l(\widetilde{g}_\alpha(q)) \widetilde{T}^l_\alpha(q, q'; z) D^l(g^{-1}_\alpha(q')),$$

$$\widetilde{T}^l_\alpha(q, q'; z) = \frac{\delta(p_\alpha - p'_\alpha)}{p^2_\alpha} \times \sum_{\lambda=0}^{\infty} t^{(\lambda)}_\alpha(k_\alpha, k'_\alpha; z - p^2_\alpha) Q^l_\lambda(\cos\theta_\alpha) Q^l_\lambda(\cos\theta'_\alpha), \quad (70)$$

where $t^{(\lambda)}_\alpha$ is the T matrix of the radial Schrödinger operator

$$h^{(\lambda)}_\alpha = -x^{-2}\partial^2_x x^2 \partial^2_x + \lambda(\lambda+1) x^{-2} + V_\alpha(x). \quad (71)$$

Thus, on the diagonal $(q = q')$ the kernels T_α contain δ-function singularities. They also have pole singularities with respect to the spectral parameter, which correspond to the discrete spectra of the two-body potentials (71). To describe these singularities, we introduce some notation for the characteristics of the bound states of these Hamiltonians.

We classify the discrete spectrum of the operators $h^{(\lambda)}_\alpha$ by the multiple index $A_\alpha = \{\lambda, i\}$, where i is the number of bound states of the Hamiltonian $h^{(\lambda)}_\alpha$ with fixed α and λ. We combine A_α and the pair number α in the symbol $A = \{\alpha; A_\alpha\}$. Let $-\varkappa^2_A$ be the binding energies, and $\varphi_A(k)$ be the form factors[11] of the bound states $h^{(\lambda)}_\alpha$. We shall characterize the bound states of pair β by an analogous index $B = \{\beta, B_\beta\}$.

As is well known,[11] at the points of the discrete spectrum the two-body T matrices have singularities of the form

$$t^{(\lambda)}_\alpha(k, k'; z) = \sum_{A_\alpha} \frac{\varphi_A(k) \overline{\varphi_A(k')}}{z + \varkappa^2_A} + \dots,$$

which generate analogous singularities of the kernels T^l_α:

$$T^l_\alpha(q,\ q';\ z)=\frac{\delta(p_\alpha-p'_\alpha)}{p^2_\alpha}\left\{\sum_{A_\alpha}\frac{\Phi^l_A(q)\,[\Phi^l_A(q')]^*}{z-p^2_\alpha+\varkappa^2_A}+\ldots\right\}. \tag{72}$$

Here, $\Phi^{(\lambda)}_A$ are $(2l+1)\times(2l+1)$ matrices formed from the form factors of the bound states of pair α:

$$\Phi^l_A(q)=\varphi_A(k_\alpha)\,D^l(\widetilde{g}_\alpha(q))\,Q^l_\lambda(\cos\theta_\alpha). \tag{73}$$

Note that for $\lambda<l$ this matrix has $2(l-\lambda)$ zero columns. They correspond to vanishing elements of the diagonal of the matrix Q^l_λ in (59):

$$[\Phi^{(l)}_A]_{mn}=0 \text{ for } \lambda<l,\ |m|>\lambda\ (A_\alpha=\{\lambda,\ i\}). \tag{74}$$

Therefore, for the matrix form factors we have a normalization condition of the type (60):

$$\left.\begin{aligned}&\int\frac{[\Phi^l_{A'}(q)]^*\,\Phi^l_A(q)}{(k^2_\alpha+\varkappa^2_{A'})\,(k^2_\alpha+\varkappa^2_A)}\,k^2_\alpha\,dk_\alpha\,\sin\theta_\alpha\,d\theta_\alpha=\delta_{AA'}I^l_A,\\&I^l_A\equiv I^l_\lambda(A_\alpha=\{\lambda,\ i\}).\end{aligned}\right\} \tag{75}$$

Thus, we have described the kernels of the inhomogeneous terms of the Faddeev equations (68). On the basis of these results, we can investigate (68) using the same scheme as in the complete three-body problem, namely, it is necessary to consider the iterations of (68) and to study the singularities of their kernels. It is easy to show that the singularities of the kernels T^l_α of δ-function type disappear already in the first iteration. At the same time, the kernel of each iteration contains two-body pole singularities generated by the singularities (72) of the inhomogeneous term. As a result, for the components $M_{\alpha\beta}$ we can prove the representation

$$M^l_{\alpha\beta}(z)=\delta_{\alpha\beta}T^l_\alpha(z)+W^l_{\alpha\beta}(z), \tag{76}$$

in which the structure of the kernels $W^l_{\alpha\beta}$ is described by the expansion

$$\begin{aligned}W^l_{\alpha\beta}(q,\ q';\ z)&=\mathscr{F}^l_{\alpha\beta}(q,\ q';\ z)\\&+\sum_{A_\alpha}\frac{\Phi^l_A(q)}{p^2_\alpha-\varkappa^2_A-z}\,\mathscr{J}^l_{A\beta}(p_\alpha,\ q';\ z)\\&+\sum_{B_\beta}\mathscr{G}^l_{\alpha B}(q,\ p'_\beta;\ z)\,\frac{[\Phi^l_B(q')]^*}{p'^2_\beta-\varkappa^2_B-z}+\\&+\sum_{A_\alpha,\ B_\beta}\frac{\Phi^l_A(q)\,\mathscr{H}^l_{AB}(p_\alpha,\ p'_\beta;\ z)\,[\Phi^l_B(q')]^*}{(p^2_\alpha-\varkappa^2_A-z)\,(p'^2_\beta-\varkappa^2_B-z)}.\end{aligned} \tag{77}$$

In it, we have separated all the two-body pole singularities of these kernels. All the matrix coefficients of the type $\mathscr{F}$, $\mathscr{J}$, $\mathscr{G}$, $\mathscr{H}$ are smooth functions of their variables.

In the complete three-body problem, the components of the T matrix are also described by representations analogous to (76) and (77). However, there is a very important difference between them. The kernels of the Faddeev components in the complete problem have, in addition to the two-body singularities, additional so-called three-particle singularities. They are contained in the term of the type $\mathscr{F}$ of an expansion analogous to (77). In our case, this is not so—the kernel $\mathscr{F}^l_{\alpha\beta}$ does not contain any singularities.

We shall clarify the reasons for this important result. In the complete problem, the highest three-particle singularity arises in the first iteration of the Faddeev equations. In our case, the first iteration has the form

$$Q^l_{\alpha\beta}(z)=-T^l_\alpha(z)\,R^l_0(z)\,T^l_\beta(z)\,(1-\delta_{\alpha\beta}). \tag{78}$$

It is clear that for real $z>0$ the kernel $(q^2-z)^{-1}\otimes I^l$ of the operator R^l_0 becomes singular, and its singularities do not intersect the two-body singularities (72) of the kernels T^l_α. In the complete three-body problem, the analogous singularities of the kernel of the free resolvent also generate the three-particle singularities of the Faddeev components. In our case, the contribution of the singularities of the kernel R^l_0 to (78) is given by the integral

$$-\int_M dM(\hat{q})\,\frac{\delta(p_\alpha-\widetilde{p}_\alpha)\,\delta(p_\beta-\widetilde{p}_\beta)}{\widetilde{q}^2-z}\,T_{\alpha\beta}(q,\ q';\ \widetilde{q};\ z), \tag{79}$$

in which $T_{\alpha\beta}$ is a smooth function of all of its arguments [since on the surface $\tilde{q}^2=z$ the kernels $T^l_\alpha(q,\tilde{q};z)$ and $T^l_\beta(\hat{q},q';z)$ do not have two-body singularities]. For $\alpha\neq\beta$, we can take as the variables of integration in (79) the parameters $\tilde{p}_\alpha$, $\tilde{p}_\beta$, and $\tilde{q}^2$. The integrals over $\tilde{p}_\alpha$ and $\tilde{p}_\beta$ can be eliminated on account of the δ functions. The remaining integral over $\tilde{q}^2$ contains the singular denominator $(\tilde{q}^2-z)$. In the limit $z\to\lambda\pm i0$ $(\lambda>0)$, this is an integral of Cauchy type.[17] Such integrals are smooth functions of λ. Therefore, the integral of the first iteration has no other singularities apart from the two-body singularities. It is clear that this is also true for any iteration of (68).

It is obvious from our arguments that the geometrical reason for the absence of three-particle singularities in our problem is the low codimensionality of the manifolds $\tilde{p}_\alpha=p_\alpha$ and $\tilde{p}_\beta=p'_\beta$ in the intrinsic space M. These are two-dimensional surfaces in M that intersect along a one-dimensional curve. It is the integral along this curve in (79) that eliminates the singularity of the denominator. In the complete problem, the analogous manifolds $\tilde{\mathbf{p}}_\alpha=\mathbf{p}_\alpha$ and $\tilde{\mathbf{p}}_\beta=\mathbf{p}'_\beta$ are three-dimensional hyperplanes in Q that do not intersect at a single point.

Thus, we have described the structure of the T matrix of the Hamiltonian H^l. We now turn to the construction of the wave operators and the S matrix of the three-particle system with fixed total orbital angular momentum.

Wave operators and *S* matrix

We first describe the spaces of the reaction channels of the three-particle system with fixed total orbital angular momentum. They correspond to the possible asymptotic states of the system. These can be divided into two classes: three-particle and two-particle states. The three-particle states correspond to three noninteracting particles and are described by the Hamiltonian H^l_0. We identify the space of the three-particle channel with the Hilbert space $\mathscr{H}_0\equiv\mathscr{H}^l$ of states of the Hamiltonian H^l. In the two-body asymptotic states, there is a bound pair and a free third particle. The dynamics of these states is determined by the operator H^l_α in (69). The spaces of the two-body channels are the subspaces of $\mathscr{H}^l$ spanned by the eigenfunctions of the operators $\widetilde{H}^l_\alpha$. By virtue of the representation (63) for the potential V^l_α, these eigenfunctions are the columns of the matrices

$$L_A^l(q,\ p_\alpha') = \frac{\delta(p_\alpha - p_\alpha')}{p_\alpha^2}\,\Psi_A^l(q), \tag{80}$$

where Ψ_A^l is the matrix of wave functions of the two-body Hamiltonians (71). It is proportional to the matrix form factor (73):

$$\Psi_A^l(q) = (k_\alpha^2 + \varkappa_A^2)^{-1}\,\Phi_A^l(q). \tag{81}$$

Note that this matrix has zero columns under the condition

$$A = \{\alpha,\ A_\alpha\} : \lambda < l\ (A_\alpha = \{\lambda,\ i\}), \tag{82}$$

since the matrix form factor possesses the same property (74). We denote by $n(A)$ the number of nonzero columns of the matrix Ψ_A^l:

$$n(A) = \begin{cases} 2l+1, & \lambda \geqslant l, \\ 2\lambda+1, & \lambda < l. \end{cases}$$

Let $\tilde{\mathscr{H}}_A$ be the subspace of $\mathscr{H}^l$, spanned by the nonzero columns of the matrix (80). It is clear that it is isomorphic to the subspace $\mathscr{H}_A = L^2(R_+, \mathbb{C}_A^{2l+1}; p_\alpha^2 dp_\alpha)$ of the $(2l+1)$-vector functions $f(p_\alpha)$ for which $n(A)$ components are nonzero:

$$f(p_\alpha) = (\underbrace{0,\ \ldots,\ 0}_{l-\lambda},\ f_{-\lambda}(p_\alpha),$$

$$f_{-\lambda+1}(p_\alpha),\ \ldots,\ f_\lambda(p_\alpha),\ \underbrace{0,\ \ldots,\ 0}_{l-\lambda}) \tag{83}$$

(for $\lambda \geqslant l$, all the components of f are nontrivial).

The isomorphism is realized by an integral operator from $\mathscr{H}_A$ to $\tilde{\mathscr{H}}_A$:

$$(L_A^l f)(q) = \Psi_A^l(q)\, f(p_\alpha).$$

We call the space $\mathscr{H}_A$ the space of channel A. The Hamiltonian of channel A is determined by the restriction of the operator $\tilde{H}_\alpha^l$ to the subspace $\tilde{\mathscr{H}}_A$, which generates a matrix operator of multiplication in $\mathscr{H}_A$:

$$H_A^l = (p_\alpha^2 - \varkappa_A^2) \otimes I_A^l,$$

where the matrix I_A^l is determined by Eqs. (62) and (75).

We now determine the wave operators of the three-body system with fixed total orbital angular momentum. They act from the spaces of the three-body and two-body channels to the state space $\mathscr{H}^l$ of the Hamiltonian H^l and are determined by the canonical limits of the corresponding evolution operators:

$$\left.\begin{aligned} U_0^{(\pm)} &= s-\lim_{t\to\mp\infty} \exp\{itH^l\}\exp\{-itH_0^l\}\,(\mathscr{H}_0 \to \mathscr{H}^l), \\ U_A^{(\pm)} &= s-\lim_{t\to\mp\infty} \exp\{itH^l\}\,L_A^l \exp\{-itH_A^l\}\,(\mathscr{H}_A \to \mathscr{H}^l). \end{aligned}\right\} \tag{84}$$

The existence of these limits for the rapidly decreasing potentials (47) can be verified in the same way as in the complete three-body problem.

The wave operators (84) possess the set of standard properties:

1. Intertwining properties:

$$H^l U_0^{(\pm)} = U_0^{(\pm)} H_0^l, \quad H^l U_A^{(\pm)} = U_A^{(\pm)} H_A^l. \tag{85}$$

2. Partial isometry:

$$[U_A^{(\pm)}]^* U_B^{(\pm)} = \delta_{AB}\,\mathrm{id}_{\mathscr{H}_A}, \tag{86}$$

where $A = 0$ or $\{\alpha, A_\alpha\}$; $B = 0$ or $\{\beta, B_\beta\}$; id are the identity operators in the spaces of the corresponding channels.

3. Asymptotic completeness:

$$U_0^{(\pm)}[U_0^{(\pm)}]^* + \sum_A U_A^{(\pm)}[U_A^{(\pm)}]^* = \mathrm{id}_{\mathscr{H}^l} - P_d^l,$$

where P_d^l is the projection operator onto the subspace of the discrete spectrum of the Hamiltonian H^l.

The intertwining property and the partial isometry of the wave operators follow directly from the definitions (80), (81), and (84) and the orthogonality (75) of the matrix form factors. The asymptotic completeness can be proved by the formalism of time-independent scattering theory. The corresponding constructions effectively repeat the proof of asymptotic completeness in the complete three-body problem.

We now express the kernels of the wave operators in terms of the T matrix of the Hamiltonian H^l. To this end, it is convenient to go over from the dynamical definitions (84) of these operators to the equivalent representations for their kernels in terms of the resolvents of the operator H^l. The transition is made by means of the elementary transformation given in Ref. 11 (p. 45):

$$U_0^{(\pm)}(q,\ q') = \lim_{\varepsilon\downarrow 0}(\mp i\varepsilon)\,R^l(q,\ q';\ q'^2 \pm i\varepsilon),$$

$$U_A^{(\pm)}(q,\ p_\alpha') = \lim_{\varepsilon\downarrow 0}(\mp i\varepsilon)\int_M R^l(q,\ \tilde{q};\ p_\alpha'^2 - \varkappa_A^2 \pm i\varepsilon)$$

$$\times\, L_A^l(\tilde{q},\ p_\alpha')\,dM(\tilde{q}).$$

It is clear that the nontrivial terms of these limits are generated by the singularities of the kernel $R^l(z)$ for real z. These singularities are described by the standard expression for the resolvent in terms of the T matrix,

$$R^l(z) = R_0^l(z) - R_0^l(z)\,T^l(z)\,R_0^l(z), \tag{87}$$

and by the representations (67), (76), and (77) for the components of the T matrix. It is easy to show that the kernels $U_0^{(\pm)}$ are determined by the residues of the kernel $R^l(q,q',z)$ at the poles of the type $(q'^2 - z)$ corresponding to the kernels of the resolvents R_0^l in (87). The kernels of the wave operators $U_A^{(\pm)}$ are generated by the residues of the kernel $R^l(q,q';z)$ at the two-body singularities $(z - p_\beta'^2 + \varkappa_B^2)^{-1}$, which are contained in the components of the T matrix and are described by the representation (77). Simple calculations lead to the following expressions for the kernels of the wave operators:

$$U_0^{(\pm)}(q,\ q') = \delta(q - q') \otimes I^l - \frac{T^l(q,\ q';\ q'^2 \pm i0)}{q^2 - q'^2 \pm i0},$$

$$U_A^{(\pm)}(q,\ p_\alpha') = L_A^l(q,\ p_\alpha') - \frac{T_A^l(q,\ p_\alpha';\ p_\alpha'^2 - \varkappa_A^2 \pm i0)\,I_A^l}{q^2 - p_\alpha'^2 + \varkappa_A^2 \mp i0}, \tag{88}$$

where the matrix T_A^l is related to the coefficients of the type $\mathscr{G}$ and $\mathscr{H}$ in the expansion (77):

$$T_A^l=\sum_\beta T_{\beta A}^l,$$

$$T_{\beta A}(q,\ p'_\alpha;\ z)=\mathcal{G}_{\beta A}(q,\ p'_\alpha;\ z)+\sum_{B_\beta}\frac{\Phi_B^l(q)\,\mathcal{H}_{BA}^l(p_\beta,\ p'_\alpha;\ z)}{p_\beta^2-\varkappa_B^2-z}. \quad (89)$$

Note that under the condition (82) the matrices $U_A^{(\pm)}$ have $2l+1-n(A)$ zero columns. They are generated by the zero columns of the matrix L_A^l and the zero elements of the diagonal of the matrix I_A^l.

The *S* matrix

We now describe the structure of the S matrix of the Hamiltonian H^l. It consists of elements of four types: S_{AB}, S_{A0}, S_{0A}, and S_{00}. These operators correspond to reactions with different clustering of the particles in the initial and final states. In accordance with the terminology of the complete problem, we shall call them the S matrices of the scattering processes $(2\to2)$, $(3\to2)$, $(2\to3)$, and $(3\to3)$. These S matrices couple the spaces of the reaction channels and can be expressed canonically in terms of the wave operators:

$$S_{BA}=[U_B^{(-)}]^*\,U_A^{(+)}\,(\mathcal{H}_A\to\mathcal{H}_B),$$

where the indices A and B take the same values as in (86).

On the basis of the representations (88), we can obtain expressions for the kernels of these S matrices in terms of the components of the T matrix:

$$S_{BA}(p_\beta,\ p'_\alpha)=\delta_{AB}\,\frac{\delta(p_\beta-p'_\alpha)}{p_\beta^2}\otimes I_A^l$$

$$-2\pi i\delta(p_\beta^2-\varkappa_B^2-p_\alpha'^2+\varkappa_A^2)$$

$$\times I_B^l\mathcal{H}_{BA}^l(p_\beta,\ p'_\alpha;\ p_\alpha'^2-\varkappa_A^2+i0)\,I_A^l,$$

$$S_{A0}(p_\alpha,\ q')=-2\pi i\delta(p_\alpha^2-\varkappa_A^2-q'^2)$$

$$\times I_A^l\widetilde{T}_A^l(p_\alpha,\ q',\ q'^2+i0),$$

$$S_{0A}(q,\ p'_\alpha)=-2\pi i\delta(q^2-p_\alpha'^2+\varkappa_A^2)$$

$$\times T_A^l(q,\ p'_\alpha;\ p_\alpha'^2-\varkappa_A^2+i0)\,I_A^l,$$

$$S_{00}(q,\ q')=\delta(q-q')\otimes I^l-2\pi i\delta(q^2-q'^2)$$

$$\times T^l(q,\ q';\ q'^2+i0).$$

Here, the matrices T_A^l are defined in (89), and $\widetilde{T}_A^l$ are related to the coefficients of the type $\mathcal{J}$ and $\mathcal{H}$ in the expansion (77):

$$\widetilde{T}_A^l=\sum_\beta\widetilde{T}_{A\beta}^l,$$

$$\widetilde{T}_{A\beta}^l(p_\alpha,\ q';\ z)=\mathcal{Y}_{A\beta}^l(p_\alpha,\ q';\ z)\cdot$$

$$+\sum_{B_\beta}\frac{\mathcal{H}_{AB}^l(p_\alpha,\ p_\beta;\ z)}{p_\beta'^2-\varkappa_B^2-z}\,[\Phi_B^l(q')]^*.$$

The presence of the δ functions in the kernels of the S matrices reflects the energy conservation law. The coefficients of these δ functions on the energy shell determine the S matrix of the system at fixed energy. The construction of this S matrix is realized by an expansion into direct integrals[18] of the spaces of the reaction channels:

$$\mathcal{H}_0=\int_0^\infty\oplus L^2(\hat{M},\mathbb{C}^{2l+1};\ d\hat{M})\,\rho_0(E)\,dE,$$
$$\mathcal{H}_A=\int_{-\varkappa_A^2}^\infty\oplus\,\mathbb{C}_A^{2l+1}\rho_A(E)\,dE \quad (90)$$

with weights

$$\rho_0(E)=E^2/16,\ \rho_A(E)=\sqrt{E_A}/2;\ E_A=E+\varkappa_A^2. \quad (91)$$

We recall that the space $\mathbb{C}_A^{2l+1}$ consists of vectors of the type (83). In the first expansion, $\hat{M}$ is the unit hemisphere in the intrinsic space M, which is regarded as the vector space R_+^3 with the scalar product (9). The elements of $\hat{M}$ are unit vectors $\hat{q}=q/|q|$, $q\in M$. The measure $d\hat{M}$ is equal to the angular part of the measure (27) of the intrinsic space:

$$d\hat{M}(\hat{q})=\frac{1}{2}\sin 2\psi\,d\psi\wedge d\varphi_\alpha=\sin^2\chi_\alpha\sin\theta_\alpha\,d\chi_\alpha\wedge d\theta_\alpha. \quad (92)$$

In accordance with the representations (90), the elements of the S matrix can be expanded into direct integrals of the S matrices for fixed energy:

$$S_{AB}=\int_{\lambda_{AB}}^\infty\oplus\,S_{AB}(E)\,dE,$$

where the indices A and B are understood in the wider sense as in (86). For the S matrices of the $(2\to2)$ processes $(A=\{\alpha,A_\alpha\},\ B=\{\beta,B_\beta\})$, $\lambda_{AB}=\max(-\varkappa_A^2,-\varkappa_B^2)$. In other cases, $\lambda_{AB}=0$.

The S matrices at fixed energy couple the fiber spaces of the expansions (90) and are determined by the kernels

$$S_{BA}(E)=\delta_{AB}I_A^l-2\pi i\rho_A(E)$$
$$\times I_B^l\mathcal{H}_{BA}^l(E_B^{1/2},\ E_A^{1/2};\ E+i0)\,I_A^l, \quad (93)$$

$$S_{0A}(\hat{q};\ E)=-2\pi i\rho_A(E)\,T_A^l(E^{1/2}\hat{q},\ E_A^{1/2};\ E+i0)\,I_A^l, \quad (94)$$

$$S_{A0}(E;\ \hat{q})=-2\pi i\rho_0(E)\,I_A^l\widetilde{T}_A^l(E_A^{1/2},\ E^{1/2}\hat{q};\ E+i0), \quad (95)$$

$$S_{00}(\hat{q},\ \hat{q}';\ E)=\delta(\hat{q}-\hat{q}')$$
$$\otimes I^l-2\pi i\rho_0(E)\,T^l(E^{1/2}\hat{q},\ E^{1/2}\hat{q}';\ E+i0). \quad (96)$$

The kernel S_{00} is a $(2l+1)\times(2l+1)$ matrix. Its elements are labeled by the values of the projection of the total orbital angular momentum. The kernels of the other S matrices are also $(2l+1)\times(2l+1)$ matrices. If the indices of the two-body channels satisfy the condition (82), they contain zero blocks, which are generated by the zero elements of the diagonals of the matrices I_A^l. The presence of such blocks has a simple physical explanation—the modulus of the projection of the angular momentum λ of pair α in channel A cannot be greater than λ, whereas the projection of the total angular momentum varies from $-l$ to l. Thus, physical meaning is associated with only the nonvanishing elements of the matrices S_{BA}, S_{0A}, and S_{A0}, which form rectangular blocks measuring $n(B)\times n(A)$, $(2l+1)\times n(A)$, and $n(A)\times(2l+1)$, respectively.

Note that the S matrices S_{BA} and S_{0A} are operators of multiplication by a matrix in the vector space $\mathbb{C}_A^{2l+1}$ with values in the spaces $\mathbb{C}_B^{2l+1}$ and $\widehat{\mathscr{H}}=L^2(\widehat{M},\mathbb{C}^{2l+1};d\widehat{M})$. The $(3\to 2)$ and $(3\to 3)$ processes are described by the matrix integral operators $S_{A0}:\mathscr{H}\to\mathbb{C}_A^{2l+1}$ and $S_{00}:\widehat{\mathscr{H}}\to\widehat{\mathscr{H}}$.

The expressions obtained above for the wave operators and S matrices have much in common with the corresponding expressions of the complete three-body problem. However, there are important differences. They are due to the fact that these operators act, in our case and in the complete problem, in spaces with very different structures. For example, in our case the S matrices of the $(2\to 2)$ and $(2\to 3)$ processes at fixed energy are operators of multiplication in a vector space, while in the complete problem they are integral operators in Hilbert spaces.

3. SCATTERING PROBLEM IN THE CONFIGURATION SPACE. RAPIDLY DECREASING POTENTIALS

In this section, we describe the formulation of the scattering problem for the Hamiltonian H^l in the configuration representation (42). We construct the continuum wave functions of this Hamiltonian, investigate their asymptotic behavior, and formulate differential Faddeev equations for the components of the wave functions.

The main attention is devoted to studying the asymptotic behaviors of the wave functions. We show that they include standard spherical waves describing $(2\to 2, 3)$ and $(3\to 2, 3)$ scattering processes. The amplitudes of these waves are equal to the kernels of the S matrices for fixed energy. The wave functions of the $(3\to 2, 3)$ processes also contain so-called single-rescattering terms, which also arise in the complete three-body problem.

In the complete problem, the asymptotic behavior of the $(3\to 2, 3)$ wave functions also contains double-rescattering terms. They are generated by the three-particle singularities of the kernel of the T matrix. As was shown in the previous section, in our problem the T matrix does not have three-particle singularities. Therefore, the wave functions do not contain double-rescattering terms, and this greatly simplifies their asymptotic structure.

Wave functions of (2→2, 3) processes

The wave functions of the Hamiltonian H^l are related to the kernels of the wave operators by the transformation (51), which diagonalizes the kinetic-energy operator H_0^l. In this subsection, we study the wave functions corresponding to the wave operators $U_A^{(+)}$:

$$\Psi_A(r,\ p_\alpha)=\int_M \mathscr{F}^l(r,\ q')\,U_A^{(+)}(q',\ p_\alpha)\,dM(q'). \qquad (97)$$

Like the kernels $U_A^{(+)}$, the functions Ψ_A are $(2l+1)\times(2l+1)$ matrices. They satisfy a matrix Schrödinger equation that follows from the intertwining property of the wave operators (85):

$$(H^l-E\otimes I^l)\,\Psi_A(r,\ p_\alpha)=0,\ E=p_\alpha^2-\varkappa_A^2. \qquad (98)$$

Each column of the matrix Ψ_A is also a solution of this equation and is one of the continuum eigenfunctions of the Hamiltonian H^l. They describe scattering of the third particle with energy p_α^2 by the bound state A of pair α and are labeled by the values of the projection of the orbital angular momentum of pair α in the initial state. Note that under the condition (82) the matrix Ψ_A has $2l+1-n(A)$ zero columns.

We turn to the study of the asymptotic behavior of the functions Ψ_A. Using the expression (88) for the kernel $U_A^{(+)}$, we represent them as the sum of components

$$\Psi_A=\chi_A+\sum_\beta \Psi_{\beta A}. \qquad (99)$$

Here, χ_A is generated by the first term of (88), $\chi_A=\mathscr{F}^l L_A^l$, and the terms $\Psi_{\beta A}$ express the contribution of the components of the T matrix (89):

$$\Psi_{\beta A}(r,\ p_\alpha)=-\int_M \mathscr{F}^l(r,\ q')\,\frac{T_{\beta A}(q',\ p_\alpha;\ E+i0)\,I_A^l}{q'^2-E-i0}\,dM(q'). \qquad (100)$$

By definition, the matrix χ_A satisfies the Schrödinger equation

$$(H_0^l+V_\alpha\otimes I^l-E\otimes I^l)\,\chi_A=0.$$

Its columns are the wave functions of the initial state of the system. The matrix χ_A can be readily calculated explicitly. To this end, it is necessary to go over in (97) to an integration over the Jacobi coordinates of the point q' and to use the representations (55), (57), and (80) for the kernels $\mathscr{F}^l$, L_A^l and the orthogonality relation (60). As a result, we obtain the following expression for χ_A in terms of the Jacobi coordinates of the point r:

$$\chi_A(r,\ p_\alpha)=\psi_A(x_\alpha)\sum_{k=|l-\lambda|}^{l+\lambda} j_k(p_\alpha y_\alpha)\,\mathscr{Y}_A^{(k)}(\hat{r})\ (A_\alpha=\{\lambda,\ i\}), \qquad (101)$$

where ψ_A is the normalized radial wave function of the two-body Hamiltonian (71), and $\mathscr{Y}_A^{(k)}$ are the $(2l+1)\times(2l+1)$ matrices

$$\mathscr{Y}_A^{(k)}(\hat{r})=(2/\pi)^{3/2}\,D^l(g_\alpha(r))\,Q_k^l(\cos\theta_\alpha)\,C_{k\lambda}^l I_A^l. \qquad (102)$$

We now consider the integral (100). We recall that the kernel $T_{\beta A}^l$ contains a smooth part and two-body singularities, which are reflected in the representation (89). These singularities do not intersect the singularities of the denominator in (100). Thus, each singularity of the integrand makes an additive contribution to the asymptotic behavior of the component $\Psi_{\beta A}$.

The contribution of the two-body singularities is readily calculated. To this end, we must express the integral (100) in terms of the Jacobi coordinates $(k'_\beta p'_\beta \theta'_\beta)$ of the point q' and replace the smooth coefficients of the two-body singularities by their values on the surface $p'^2_\beta=E+\varkappa_B^2$. Then the integral over θ'_β can be calculated by virtue of the orthogonality condition (60), and the integral over k'_β gives the radial wave function of the bound pair β. The asymptotic behavior of the remaining integral over p'_β is determined by the representation

$$\int_0^\infty j_k(p'_\beta y_\beta)\,\frac{f(p'_\beta)\,p'^2_\beta\,dp'_\beta}{p'^2_\beta-a^2-i0}\underset{y_\beta\to\infty}{\sim}(-i)^k\,\frac{\pi}{2}\,\frac{e^{i|a|y_\beta}}{y_\beta}\,f(|a|). \qquad (103)$$

We now describe the calculation of the contribution of the singular denominator $(q'^2 - E - i0)^{-1}$. In the intrinsic space, we introduce the vector structure (6) and denote by $\hat{q}' = q'/|q'|$ the element of the unit hemisphere $\widehat{M} \subset M$ [see the representation (90)]. Since the singularity $(q'^2 - E - i0)^{-1}$ does not depend on $\hat{q}'$, its contribution to the integral (100) can be expressed in the form

$$\frac{1}{8}\int_0^\infty \frac{|q'|^5\, d\,|q'|}{q'^2 - E - \mathrm{i}0} U(r,\ |q'|), \tag{104}$$

where U is expressed by an integral over $\widehat{M}$ with the measure (92):

$$U(r,\ t) = -\int_{\widehat{M}} d\widehat{M}(\hat{q}')\,\mathcal{F}^l(r,\ t\hat{q}')\,W(t\hat{q}'). \tag{105}$$

Here $W(q') = T^l_{\beta A}(q', p_\alpha; E + i0)$. We shall show that the asymptotic behavior of the last integral is given by the representation

$$U(r,\ t) \underset{\rho\to\infty}{\sim} \pi^{-1/2}(2/t\rho)^{5/2}\sum_{\pm}(\pm 1)$$

$$\times \exp\{\pm \mathrm{i}t\rho \mp \mathrm{i}\pi/4\}\, I^l_\pm W(t\hat{r}), \tag{106}$$

where $\rho = |r|$, and $I^l_\pm$ are diagonal $(2l+1)\times(2l+1)$ matrices:

$$(I^l_\pm)_{mn} = (\pm 1)^{m+1}\delta_{mn}. \tag{107}$$

In (105) we substitute the integral representation (50) for the kernel $\mathcal{F}^l$. As a result, we obtain an integral over the manifold $SO(3)\times\widehat{M} \simeq S^5$, where $S^{(5)}$ is the sphere in $\mathring{Q}$. Its elements are the unit vectors $\widehat{P} = g\widehat{\mathring{P}}$. By virtue of (25), the standard measure $d\widehat{P}$ on $S^{(5)}$ is equal to the product of measures $\pi^2 dg \wedge d\widehat{M}$. Thus, we can express (105) as an integral over the sphere $S^{(5)}$ with kernel $\exp\{it(\mathring{X}, g\widehat{\mathring{P}})\}$. As is well known,[11] for $|\mathring{X}| = \rho \to \infty$ such an exponential is a generalized function in $L^2(S^{(5)}; d\widehat{P})$:

$$\exp\{\mathrm{i}t(\mathring{X},\ \hat{P})\} \sim (2\pi/t\rho)^{5/2}\sum_{\pm}(\mp \mathrm{i})^{5/2}\, e^{\pm \mathrm{i}t\rho}\,\delta(\hat{P} \mp \hat{\mathring{X}}).$$

Thus, the asymptotic behavior of U is determined by the values of the matrix $D^l(g)\,W(t\hat{q})$ at the points $g\widehat{\mathring{P}} = \pm\widehat{\mathring{X}}$. In accordance with the definition (17) of the vectors $\widehat{\mathring{P}}$ and $\widehat{\mathring{X}}$, at these points $\hat{q} = \hat{r}$, $g = g_\pm = g(\pi/2 \mp \pi/2, 0, 0)$. Using the identity $D^l(g_\pm) = \pm I^l_\pm$, we obtain the asymptotic behavior (106).

We now substitute the representation (106) in the integral (104). In the limit $\rho\to\infty$, the term with the minus sign gives an exponentially small contribution, while the contribution of the term with the plus sign is equal to half the residue of the integrand at the point $|q'| = \sqrt{E}$. It determines the term of the asymptotic behavior of $\Psi_{\beta A}$ generated by the singularity $(q'^2 - E - i0)^{-1}$.

The calculations described above lead to an asymptotic representation for the components $\Psi_{\beta A}$ in the form

$$\Psi_{\beta A}(r,\ p_\alpha) \underset{\rho\to\infty}{\sim} \sum_{B_\beta} \psi_B(x_\beta)\,\frac{\exp(\mathrm{i}E_B^{1/2}y_\beta)}{y_\beta}\,\mathcal{Y}^l_B(\hat{r})\, f^l_{BA}(E)$$

$$+\ \rho^{-5/2}\exp(\mathrm{i}E^{1/2}\rho)\,A^l_{\beta A}(\hat{r},\ E), \tag{108}$$

where the energies E_B are determined in (91), and the matrix $\mathcal{Y}^l_B$ is equal to the sum of the matrices (102):

$$\mathcal{Y}^l_B(\hat{r}) = -\frac{\pi}{2}\sum_h(-\mathrm{i})^h\,\mathcal{Y}^{(h)}_B(\hat{r}).$$

Contributions to the sum over B_β are made by the open channels of the $(2\to 2)$ reactions with $E_B > 0$. The matrix f^l_{BA} consists of the target-rearrangement amplitudes, which determine the S matrix (93) of the $(2\to 2)$ processes:

$$f^l_{BA}(E) = I^l_B \mathcal{H}^l_{BA}(E_B^{1/2},\ E_A^{1/2};\ E + \mathrm{i}0)\, I^l_A.$$

The final term of the asymptotic behavior (108) corresponds to the $(2\to 3)$ scattering process. In it, the coefficient of the spherical wave is the part of the disintegration amplitude that is contained in the given component of the wave function:

$$A^l_{\beta A}(\hat{r},\ E) = (\pi \mathrm{i}/2)^{1/2} E^{3/4} T^l_{\beta A}(E^{1/2}\hat{r},\ p_\alpha;\ E + \mathrm{i}0)\, I^l_A.$$

The sum of these amplitudes over all components is proportional to the kernel of the corresponding S matrix (94).

Thus, the asymptotic behavior of the wave functions is described by the representations (99), (101), and (108).

Wave functions of the (3→2, 3) processes

These wave functions correspond to the wave operators $U_0^{(+)}$:

$$\Psi_0(r,\ q) = \int_M \mathcal{F}^l(r,\ q')\, U_0^{(+)}(q',\ q)\, dM(q').$$

They satisfy a matrix Schrödinger equation equivalent to the intertwining property (85):

$$(H^l - E \otimes I^l)\,\Psi_0(r,\ q) = 0,\quad E = q^2. \tag{109}$$

Thus, the columns of the matrix Ψ_0 are the wave functions of the branch $(0,\infty)$ of the continuous spectrum of the Hamiltonian H^l. They are labeled by the values of the projection of the total orbital angular momentum l and describe scattering processes with three free particles in the initial state (the point $q\in M$ specifies the relative momenta of the particles).

In accordance with the representations (88) and (67) for the kernel $U_0^{(+)}$, the wave function Ψ_0 is equal to the sum

$$\Psi_0 = \mathcal{F}^l + \sum_\alpha U_\alpha + \sum_\alpha \Phi_\alpha, \tag{110}$$

where the terms $\mathcal{F}^l$ and U_α are generated by the first terms of Eqs. (88) and (76),

$$U_\alpha(r,\ q) = -\int_M \frac{\mathcal{F}^l(r,\ q')\, T^l_\alpha(q',\ q;\ E + \mathrm{i}0)}{q'^2 - E - \mathrm{i}0}\, dM(q'), \tag{111}$$

while Φ_α express the contribution of the components of the T matrix $W_{\alpha\beta}$:

$$\Phi_\alpha(r,\ q) = -\sum_\beta\int_M \frac{\mathcal{F}^l(r,\ q')\, W_{\alpha\beta}(q',\ q;\ E + \mathrm{i}0)}{q'^2 - E - \mathrm{i}0}\, dM(q'). \tag{112}$$

In accordance with the definition (51), the kernel $\mathcal{F}^l$ satisfies the free Schrödinger equation $(H^l_0 - E\otimes I^l)\mathcal{F}^l = 0$. Therefore, the columns of the matrix $\mathcal{F}^l$ are the wave functions of the system of three noninteracting particles with fixed total orbital angular momentum. By analogy with the usual terminology of scattering theory, we shall in what follows refer to $\mathscr{F}^l$ as a plane wave in the intrinsic space. We recall that its explicit form is given by the expressions (55) and (57).

The terms U_α in (110) take into account the δ-function singularities of the T matrix that are contained in the kernels T_α [see (70)]. In the complete three-body problem, the $(3\to 2, 3)$ wave functions include terms of similar structure. They describe the contribution to the asymptotic behavior of the processes of single two-particle collisions. In our case, the matrices U_α express the contribution of the single two-particle collisions at fixed total orbital angular momentum of the system.

We now turn to the analysis of the asymptotic behavior of the wave function Ψ_0. We first consider its component Φ_α. In the integral (112), the kernels $W_{\alpha\beta}$ have two-body singularities, which are reflected in the representation (77). Thus, the integrand has essentially the same structure as in the integral (100), whose asymptotic behavior we have studied. Repeating the calculations made then, we obtain for Φ_α an asymptotic representation of the type (108):

$$\Phi_\alpha(r, q) \underset{\rho\to\infty}{\sim} \sum_\alpha \psi_A(x_\alpha) \frac{\exp\{iE_A^{1/2} y_\alpha\}}{y_\alpha} \times \mathscr{Y}_A^l(\hat{r}) f_{A0}^l(q) + \frac{\exp\{iE^{1/2}\rho\}}{\rho^{5/2}} \mathscr{A}_{\alpha 0}^l(\hat{r}, \hat{q}). \tag{113}$$

Here, the matrices f_{A0}^l are the amplitudes of the $(3\to 2)$ processes with formation of bound pair α. They are proportional to the kernels of the corresponding S matrices (95):

$$f_{A0}^l(q) = I_A^l \widetilde{T}_A^l(E_A^{1/2}, q; E+i0).$$

The last term in (113) describes the $(3\to 3)$ processes. The amplitudes $\mathscr{A}_{\alpha 0}^l$ can be expressed in terms of the components of the T matrix $W_{\alpha\beta}^l$:

$$\mathscr{A}_{\alpha 0}^l(\hat{r}, q) = (\pi i/2)^{1/2} E^{3/4} \sum_\beta W_{\alpha\beta}^l(E^{1/2}\hat{r}, q; E+i0).$$

Contribution of single two-particle collisions

The explicit form of the matrix U_α is given by the following representation, which is a consequence of the definition (111) of the matrix and Eqs. (55), (57), and (70) for the kernels $\mathscr{F}^l$ and T_α:

$$U_\alpha(r, q) = \frac{2}{\pi} \sum_{l_1, l_2=0}^{\infty} j_{l_1}(p_\alpha y_\alpha) u_\alpha^{(l_2)}(x_\alpha, k_\alpha) \times D^l(g_\alpha(r)) Q_{l_1}^l(\cos\theta_\alpha) C_{l_1 l_2}^l Q_{l_2}^{l\,\dagger}(\cos\theta'_\alpha) D^l(\widetilde{g}_\alpha^{-1}(q)). \tag{114}$$

It is expressed in terms of the Jacobi coordinates (53) of the points r and q. The functions $u_\alpha^{(\lambda)}$ can be expressed in terms of the T matrices of the partial two-body Hamiltonians (71):

$$u_\alpha^{(\lambda)}(x, k) = -\int_0^\infty j_\lambda(k'x) \frac{t_\alpha^{(\lambda)}(k', k; k^2+i0)}{k'^2-k^2-i0} k'^2\, dk'.$$

Thus, the sum $\psi_\alpha^{(\lambda)}(x,k) = j_\lambda(kx) + u_\alpha^{(\lambda)}(x,k)$ is a continuum wave function of the Hamiltonian $h_\alpha^{(\lambda)}$. In accordance with (103), the asymptotic behavior of the function $u_\alpha^{(\lambda)}$ can be expressed in terms of the phase shift $\delta_\alpha^{(\lambda)}(k)$ associated with this Hamiltonian:

$$u_\alpha^{(\lambda)}(x, k) \underset{x\to\infty}{\sim} \frac{\exp[2i\delta_\alpha^{(\lambda)}(k)]-1}{2ik} \frac{\exp(ikx - i\pi\lambda/2)}{x}. \tag{115}$$

We now describe the asymptotic behavior of the matrix U_α. In the limit $y_\alpha\to\infty$, we can replace the spherical Bessel functions in (114) by their asymptotic behavior.[16] We then obtain a sum over l_1 of products of associated Legendre polynomials (59) and Clebsch–Gordan coefficients (58). This sum can be calculated by means of Eqs. (A.5) and (A.3). As a result, we obtain a representation that describes the asymptotic behavior of U_α in the region $x_\alpha \ll y_\alpha \to \infty$:

$$U_\alpha(r, q) \underset{y_\alpha\to\infty}{\sim} (\pi i p_\alpha y_\alpha)^{-1} \sum_\pm e^{\pm i p_\alpha y_\alpha} \sum_{\lambda=0}^{\infty} u_\alpha^{(\lambda)}(x_\alpha, k_\alpha) \times i^\lambda D^l(g_\alpha(r)) I_\pm^l d^l(\pm\cos\theta_\alpha) Q_\lambda^l(\pm\cos\theta_\alpha) \times Q_\lambda^l(\cos\theta'_\alpha) D^l(\widetilde{g}_\alpha^{-1}(q)), \tag{116}$$

where the matrices $I_\pm^l$ and d^l are defined by Eqs. (107) and (A.1).

In the limit $x_\alpha\to\infty$ we can replace the functions $u_\alpha^{(\lambda)}$ in (116) by their asymptotic behavior (115):

$$U_\alpha = \sum_\pm U_\alpha^{(\pm)}, \qquad U_\alpha^{(\pm)}(r, q) \sim A_\alpha^{(\pm)}(r, q) \exp\{i\sqrt{E} Z_\alpha^{(\pm)}\}. \tag{117}$$

This representation describes the asymptotic behavior of the matrix U_α in the region $x_\alpha \sim y_\alpha \to \infty$. The amplitudes $A_\alpha^{(\pm)}$ have the form

$$A_\alpha^{(\pm)}(r, q) = (\pi i p_\alpha x_\alpha y_\alpha)^{-1} D^l(g_\alpha(r)) I_\pm^l d^l(\pm\cos\theta_\alpha) \times f_\alpha^{(\pm)}(k_\alpha, \theta_\alpha, \theta'_\alpha) D^l(\widetilde{g}_\alpha^{-1}(q)), \tag{118}$$

where $f_\alpha^{(\pm)}$ are diagonal matrices that can be expressed in terms of the two particle phase shifts:

$$f_\alpha^{(\pm)}(k, \theta, \theta') = (2ik)^{-1} \sum_{\lambda=0}^{\infty} \{\exp[2i\delta_\alpha^{(\lambda)}(k)]-1\} \times Q_\lambda^l(\pm\cos\theta) Q_\lambda^l(\cos\theta'). \tag{119}$$

The arguments of the exponentials in (117) are

$$Z_\alpha^{(\pm)}(r, q) = E^{-1/2}(k_\alpha x_\alpha \pm p_\alpha y_\alpha). \tag{120}$$

In the following section, we shall show that the functions Z_α satisfy an eikonal equation in the intrinsic space. Thus, they are the eikonals of single two-particle collisions in systems with fixed total orbital angular momentum.

Note that the amplitudes (119) are smooth functions of all their variables for the rapidly decreasing two-body potentials (47) with $\varepsilon > 2$. For $\varepsilon \leqslant 2$, the amplitudes $f_\alpha^{(\pm)}$ have low-energy singularities as $k\to 0$ and angle singularities at $\theta' = \theta(f_\alpha^{(+)})$ and $\theta = \pi - \theta'(f'_\alpha)$. The singularities of these amplitudes can be investigated by the methods developed in the studies of Ref. 19.

Thus, we have described the asymptotic behavior of the terms of the wave function Ψ_0 that correspond to $(3\to2, 3)$ scattering processes and single two-particle collisions. We now study the asymptotic behavior of the plane wave $\mathscr{F}^l$. This problem is nontrivial, since the explicit expressions (55) and (57) for the plane wave do not reflect its $\rho\to\infty$ behavior. The problem is that in the sum (57) it is not possible to go to the limit $\rho\to\infty$ term by term, since the series which then arises diverges.

Asymptotic behavior of the plane wave

We study this asymptotic behavior on the basis of the integral representation (56) for the kernel $\mathscr{F}^l$, which is related to the plane wave by Eq. (55). We express the integral (56) in terms of the Euler angles $(\phi^1\phi^2\phi^3)$, which give the parametrization (16) of the elements of the group $SO(3)$. The integrals over the angles ϕ^1 and ϕ^3 that then arise can be expressed in terms of Bessel functions. The remaining integral over $\phi\equiv\phi^2$ has the form

$$\mathscr{F}^l_\alpha(r, q) = -\frac{1}{2\pi}\int_0^\pi d\phi \sin\phi \exp\{\mathrm{i}\cos\phi\,(k_\alpha x_\alpha\cos\theta'_\alpha + p_\alpha y_\alpha\cos\theta_\alpha)\} \times J^l(p_\alpha y_\alpha\sin\theta_\alpha\sin\phi)\, d^l(\cos\phi)\, I^l_- J^l(k_\alpha x_\alpha\sin\theta'_\alpha\sin\phi), \quad (121)$$

where the matrices I^l_- and d^l are defined in (107) and (A.1), and J^l are diagonal matrices of Bessel functions: $[J^l(x)]_{mn} = \delta_{mn} i^n J_n(x)$.

Suppose that $\sin\theta_\alpha\neq0$ and $\sin\theta'_\alpha\neq0$, i.e., the points r and q are separated from the boundary (45) of the intrinsic space M. Then in (121) it is possible to replace the Bessel functions by their asymptotic behaviors as $y_\alpha\to\infty$ and $x_\alpha\to\infty$. At the same time, the kernel $\mathscr{F}^l_\alpha$ can be decomposed into a sum of four integrals of rapidly oscillating exponentials. We represent the argument of each of them as the product of a large ϕ-independent parameter and the cosine of a certain angle:

$$\mathscr{F}^l_\alpha(r, q) \sim (16\pi^4 x_\alpha y_\alpha \sin\theta_\alpha k_\alpha p_\alpha \sin\theta'_\alpha)^{-1/2}\sum_{k=1}^{2}\sum_{\pm}(\pm \mathrm{i})^{k-2} \times \int_0^\pi d\phi \exp\{\mathrm{i}\sqrt{E}\, Z_k\cos(\phi\pm\Delta_k)\} \times (\delta_{k1} I^l_\mp + \delta_{k2} I^l_\pm)\, d^l(\cos\phi)\, I^l_\pm. \quad (122)$$

Here

$$\left.\begin{aligned} Z_k(r, q) &= E^{-1/2}[k_\alpha^2 x_\alpha^2 + p_\alpha^2 y_\alpha^2 + 2k_\alpha p_\alpha x_\alpha y_\alpha\cos(\theta_\alpha + (-1)^k\theta'_\alpha)]^{1/2},\\ \cos\Delta_k &= (k_\alpha x_\alpha\cos\theta'_\alpha + p_\alpha y_\alpha\cos\theta_\alpha)/(\sqrt{E}\, Z_k),\\ \sin\Delta_k &= (k_\alpha x_\alpha\sin\theta'_\alpha - (-1)^k p_\alpha y_\alpha\sin\theta_\alpha)/\sqrt{E}\, Z_k. \end{aligned}\right\} \quad (123)$$

We now calculate the points of stationary phase $\phi_k^{(\pm)}$ of the integrals (122). We consider the case $k=1$. It follows from (123) that $\Delta_1\in[0,\pi]$, and therefore $\phi_1^{(+)} = \pi - \Delta_1$, $\phi_1^{(-)} = \Delta_1$. For $k=2$, the positions of the points of stationary phase $\phi_2^{(\pm)}$ depend on the sign of the parameter $\sin\Delta_2$:

$$\sin\Delta_2 > 0:\ \phi_2^{(+)} = \pi - \Delta_2,\ \phi_2^{(-)} = \Delta^2, \quad (124a)$$

$$\sin\Delta_2 < 0:\ \phi_2^{(+)} = 2\pi - \Delta_2,\ \phi_2^{(-)} = \Delta_2 - \pi. \quad (124b)$$

Thus, the conditions $\mathrm{sign}(\sin\Delta_2) = \mp1$ divide the intrinsic space M into two regions Γ_α and M/Γ_α, in which the asymptotic behavior of the integrals (122) with $k=2$ has different forms. We find the region Γ_α, which is distinguished by the inequality

$$k_\alpha x_\alpha\sin\theta'_\alpha - p_\alpha y_\alpha\sin\theta_\alpha < 0. \quad (125)$$

To this end, we realize M as the vector space (6). Multiplying the inequality (125) by x_α, we can express it in terms of the scalar product (9):

$$(\hat{r}, \hat{n}_\alpha(q))_M > \cos\omega_\alpha(q),$$

where $\cos\omega_\alpha = k_\alpha\sin\theta'_\alpha(p_\alpha^2 + k_\alpha^2\sin^2\theta'_\alpha)^{-1/2}$, and $\hat{n}_\alpha(q)$ is the unit vector with coordinates $(-\cos\omega_\alpha, 0, \sin\omega_\alpha)$. Therefore, Γ_α is the interior of the cone with symmetry axis $\hat{n}_\alpha$ and opening angle $2\omega_\alpha$ (Fig. 1).

Thus, in the limit $r\to\infty$ in Γ_α the points of stationary phase of the integrals (122) with $k=2$ are the angles (124b), while outside Γ_α they are the angles (124a).

The further calculation of the asymptotic behavior of the integrals (122) is performed by the standard formulas of the method of stationary phase[19] and leads to the following representation for the plane wave:

$$\left.\begin{aligned} \mathscr{F}^l &= \sum_{k=1}^{2}\sum_{\pm}\mathscr{F}_k^{(\pm)},\\ \mathscr{F}_k^{(\pm)}(r, q) &\underset{r\to\infty}{\sim} [8\pi^3\sigma(r)\sigma(q)\cos(\Omega_k/2)]^{-1/2} \times A_k^{(\pm)}(\hat{r}, \hat{q})\exp\{\pm\mathrm{i}E^{1/2}Z_k \pm (-1)^k \mathrm{i}\pi/4\}. \end{aligned}\right\} \quad (126)$$

We recall that the function σ is defined in (28)–(30). It specifies the determinant of the inertia tensor (14). The angles $\Omega_k\in[0,\pi]$ depend on the directions of the vectors r and q in the realization of M as the vector space (6):

$$\cos\Omega_1 = (\hat{r}, \hat{q}_M), \quad \cos\Omega_2 = -(\hat{r}, \hat{q}_*)_M,$$

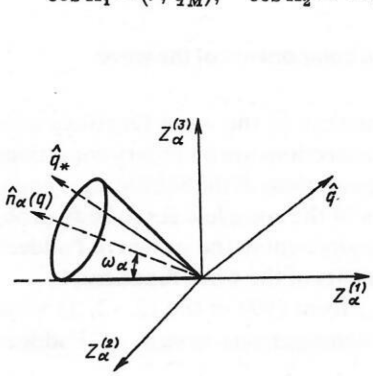

FIG. 1. The cone Γ_α in R^3_+ within which the inequality (92) holds. It is tangent to the plane $(Z_\alpha^{(1)}, Z_\alpha^{(2)})$ along the ray $(-1,0,0)$. The surface of Γ_α contains the ray $\hat{q}_*$, the mirror reflection of the ray $\hat{q}$ from the plane $(Z_\alpha^{(1)}, Z_\alpha^{(2)})$.

where $\hat{q}_*$ is the reflection of the vector $\hat{q}$ from the boundary R^3_+ (see Fig. 1), i.e., the hyperspherical coordinates $(\chi^*_\alpha \theta^*_\alpha)$ and $(\chi_\alpha \theta_\alpha)$ of the vectors $\hat{q}_*$ and $\hat{q}$ are related by the equations $\chi^*_\alpha = \pi - \chi_\alpha$ and $\theta^*_\alpha = \pi - \theta_\alpha$.

The expressions (123) for the functions Z_k can be rewritten in an elegant form in terms of the angles Ω_k:

$$Z_k(r, q) = \rho \cos(\Omega_k/2). \qquad (127)$$

In the following section, we show that the functions Z_k satisfy the eikonal equation in the intrinsic space. Thus, they are the eikonals of the plane wave for systems with fixed orbital angular momentum.

The amplitudes $A_k^{(\pm)}$ depend on the angles (123) and are given by the product of the matrices (105) and (A.1):

$$A_k^{(\pm)}(\hat{r}, \hat{q}) = D^l(g_\alpha(r)) B_k^{(\pm)}(\Delta_k) D^l(\tilde{g}_\alpha^{-1}(q)), \qquad (128)$$

where

$$B_1^{(\pm)}(\Delta_1) = I^l_\pm d^l(\pm\cos\Delta_1) I^l_\mp,$$

$$B_2^{(\pm)}(\Delta_2) = \begin{cases} I^l_\pm d^l(\pm\cos\Delta_2) I^l_\pm, & r \in \Gamma_\alpha, \\ I^l_\mp d^l(\pm\cos\Delta_2) I^l_\mp, & r \in M/\Gamma_\alpha. \end{cases}$$

The amplitudes $B_2^{(\pm)}$ have a discontinuity on the boundary Γ_α [on it, $\sin\Delta_2 = 0$ and the points of stationary phase (124) merge with the ends of the interval of integration in (122)]. The transition regime of these amplitudes in the neighborhood of the boundary Γ_α is described by the Fresnel integrals $\Phi_\pm(t) = (\mp i/\pi)^{1/2} \int_t^\infty dx \exp(\pm ix^2)$:

$$= (\mp i/\pi)^{1/2} \int_t^\infty dx \exp(\pm ix^2):$$

$$B_2^{(+)} = \sum_\pm \Phi_-(\pm\xi) I^l_\pm d^l(\cos\Delta_2) I^l_\pm,$$

$$B_2^{(-)} = \sum_\pm \Phi_+(\pm\xi) I^l_\mp d^l(-\cos\Delta_2) I^l_\mp,$$

where $\xi = (\sqrt{E} Z_2/2)^{1/2} \sin\Delta_2$.

Note that the asymptotic behavior (126) is incorrect in the singular direction $\hat{r} = \hat{q}_*$, which lies on the surface of the cone Γ_α (see Fig. 1). The point is that $Z_2(r,q) = 0$ for $\hat{r} = \hat{q}_*$, and therefore in the integrals (122) with $k = 2$ it is not possible to go to the limit $Z_2 \to \infty$ in the neighborhood of this direction. As a result, for $\hat{r} = \hat{q}_*$ the terms of the plane wave $\mathscr{F}_2^{(\pm)}$ decrease more slowly than in other directions of the intrinsic space. We shall investigate this singular case in the following section.

Faddeev equations for the components of the wave functions

The asymptotic behaviors of the wave functions constructed in the previous subsections are boundary conditions that fix the corresponding solutions of the Schrödinger equations (98) and (109). As in the complete three-body problem, these equations are equivalent to the system of Faddeev equations for the components of the wave functions.

The components $\Psi_{\beta A}$ from (99) of the $(2\to2, 3)$ wave functions satisfy the inhomogeneous system of Faddeev equations

$$(H_0^l + V_\beta \otimes I^l - E \otimes I^l)\Psi_{\beta A} = -V_\beta\{\delta_{\alpha\beta}\chi_A + \sum_{\gamma\neq\beta} \Psi_{\gamma A}\}, \qquad (129)$$

where the matrices χ_A are determined by the representation (101). They describe the wave functions of the initial state. The asymptotic boundary conditions (108) for the components $\Psi_{\beta A}$ uniquely fix the solutions of (129).

The Faddeev equations for the components $\Psi_\alpha = U_\alpha + \Phi_\alpha$ of the $(3\to2, 3)$ wave functions have a similar form [see (110)]:

$$(H_0^l + V_\alpha \otimes I^l - E \otimes I^l)\Psi_\alpha = -V_\alpha\{\delta_{\alpha 1}\mathscr{F}^l + \sum_{\beta\neq\alpha} \Psi_\beta\},$$

where $\mathscr{F}^l$ is the plane wave (55). The boundary conditions are specified by the asymptotic behaviors (113) and (114) of the functions U_α and Φ_α.

Note that besides these asymptotic boundary conditions the components of the wave functions must also satisfy the boundary condition (46) on the boundary of the intrinsic space.

4. SYSTEMS WITH COULOMB INTERACTION

In this section, we investigate the scattering problem for a Hamiltonian H^l in which the two-body potentials contain a long-range Coulomb part,

$$V_\alpha(x_\alpha) = n_\alpha/x_\alpha + v_\alpha(x_\alpha), \qquad (130)$$

while the rapidly decreasing corrections v_α satisfy the condition (59). We describe the asymptotic behavior of the wave functions and formulate boundary conditions that determine them on the basis of the Schrödinger equation and Faddeev equations. The main attention is devoted to a study of the asymptotic behavior of the wave function Ψ_0 corresponding to $(3\to2, 3)$ processes. The investigation of the functions Ψ_A is much simpler in the technical respect. As in the complete three-body problem, the Coulomb interaction leads to fairly obvious modifications in the asymptotic behavior of Ψ_A in relation to the case of rapidly decreasing potentials. We describe these modifications without going into trivial derivations.

We construct the asymptotic behavior of the functions Ψ_0 by the eikonal method, which was also used in the complete problem. In the framework of the method, the asymptotic behaviors of the wave functions of Coulomb systems are determined by the same set of asymptotic waves as in the case of rapidly decreasing potentials. The Coulomb effects are taken into account by additional phases, which distort these waves. The phases can be calculated by solving the eikonal equation associated with the kinetic-energy operator. In our case, this is the eikonal equation on the Riemannian manifold M with metric (22).

Asymptotic behavior of the $(2\to2, 3)$ wave functions and Faddeev equations for their components

Asymptotic behaviors

For Coulomb systems, the wave functions of $(2\to2, 3)$ processes can be represented as sums of components analogous to (99):

$$\Psi_A(r, p_\alpha) = \chi_A^{(c)}(r, p_\alpha) + \sum_\beta \Psi_{\beta A}^{(c)}(r, p_\alpha),$$

where the matrix $\chi_A^{(c)}$ is the incident Coulomb wave. It describes the motion of the third particle in the Coulomb field of the bound pair α and satisfies a Schrödinger equation with the effective Hamiltonian

$$H_\alpha^{(c)} = H_0^l + V_\alpha \otimes I^l + \frac{n_{\alpha\alpha}}{y_\alpha} \otimes I^l,$$

where $n_{\alpha\alpha} = \Sigma_{\beta\neq\alpha} n_\beta / |s_{\beta\alpha}|$ [the kinematic coefficients $s_{\beta\alpha}$ are defined in (2)]. The operator $H_\alpha^{(c)}$ admits separation of the variables, and therefore the matrix $\chi_A^{(c)}$ can be calculated explicitly:

$$\chi_A^{(c)}(r, p_\alpha) = \psi_A(x_\alpha) \sum_{k=|l-\lambda|}^{l+\lambda} F_k(\eta_A, p_\alpha y_\alpha) \mathcal{Y}_A^{(k)}(\hat{r}). \tag{131}$$

This representation is analogous to (101) and goes over into it for $\eta_{\alpha\alpha} = 0$. In it, η_A are standard Coulomb parameters,

$$\eta_A = n_{\alpha\alpha}(2E_A^{1/2})^{-1},$$

and F_k are regular Coulomb wave functions,

$$F_k(\eta, z) = [2^k e^{-\pi\eta/2}\Gamma(k+1+i\eta)/(2k+1)!]$$
$$\times z^k e^{-iz}\Phi(k+1-i\eta, 2k+2, 2iz),$$

where Φ is the confluent hypergeometric function.[16]

The asymptotic behavior of the components $\Psi_{\beta A}^{(c)}$ contains spherical waves from (108):

$$\Psi_{\beta A}^{(c)}(r, p_\alpha) \underset{r\to\infty}{\sim} \sum_{B_\beta} \psi_B(x_\beta) \frac{\exp\{iE_B^{1/2} y_\beta + iW_B(y_\beta)\}}{y_\beta} \mathcal{Y}_B^l(\hat{r}) f_{BA}^l(E)$$
$$+ \frac{\exp\{iE^{1/2}\rho + iW_0(r)\}}{\rho^{5/2}} \mathcal{A}_{\beta A}^l(\hat{r}, E), \tag{132}$$

which are distorted by the Coulomb phases:

$$W_B(y_\beta) = -\eta_B \ln(2\sqrt{E_B}\, y_\beta),$$
$$W_0(r) = -\frac{\rho}{2\sqrt{E}} \sum_\gamma \frac{n_\gamma}{x_\gamma} \ln(2\sqrt{E}\,\rho). \tag{133}$$

The final term in (132) describes the $(2\to3)$ breakup processes, and the matrices f_{BA}^l determine the amplitudes of the inelastic $(2\to2)$ processes. The elastic scattering $(2\to2)$ amplitudes are equal to the sum

$$f_A^l(E) = f_{AA}^l(E) + f_{A,c}^l(E),$$

where $f_{A,c}^l$ is the matrix of Coulomb amplitudes generated by the function $\chi_A^{(c)}$:

$$[f_{A,c}^l(E)]_{mn} = (-1)^{m+n+1}[i\pi\sqrt{E_A}(2l+1)]^{-1}$$
$$\times \sum_{k=|l-\lambda|}^{l+\lambda} (2k+1)$$
$$\times \exp[2i\delta_k^{(c)}(\eta_A)] \langle\lambda mk0 \mid lm\rangle \langle\lambda nk0 \mid ln\rangle.$$

Here, $\delta_k^{(c)}(\eta)$ are the standard Coulomb phases: $\delta_k^{(c)}(\eta) = \arg\Gamma(k+1+i\eta)$.

Faddeev equations

The Faddeev equations for the components $\Psi_{\beta A}^{(c)}$ have the same form as the corresponding equations in the complete problem of three charged particles:

$$\{H_0^l + V_\beta \otimes I^l + \sum_{\gamma\neq\beta} V_\gamma^{(0)} \otimes I^l - E \otimes I^l\} \Psi_{\beta A}$$
$$= F_{\beta A} - \hat{V}_\beta \sum_{\gamma\neq\beta} \Psi_{\gamma A}.$$

The functions $V_\alpha^{(0)}$ and $\hat{V}_\alpha$ determine the standard[11] splitting of the two-body potentials (130) into long- and short-range parts: $V_\alpha = \hat{V}_\alpha + V_\alpha^{(0)}$. The inhomogeneous terms of these equations can be expressed in terms of the wave function (131) of the initial state:

$$F_{\beta A} = \left\{\delta_{\alpha\beta}\left(\frac{n_{\alpha\alpha}}{y_\alpha} - \sum_{\gamma\neq\alpha} V_\gamma^{(0)}\right) + (\delta_{\alpha\beta} - 1)\hat{V}_\beta\right\} \chi_A^{(c)}.$$

Eikonal method

We describe the scheme of construction for the asymptotic behavior of the wave function Ψ_0 based on the eikonal equation in the intrinsic space.

We recall that the kinetic part of the Hamiltonian H^l is specified by the differential operator (43) on the Riemannian manifold M. We shall seek a solution of the Schrödinger equation

$$(H_0^l + V \otimes I^l - E \otimes I^l)\Psi = 0 \tag{134}$$

in the form

$$\Psi = A e^{iS}, \tag{135}$$

where S is a real function, and A is a $(2l+1)\times(2l+1)$ matrix. Substituting this representation in (134), we obtain the system of equations

$$(K_r^*(dS, dS) + V - E) \otimes I^l = -A^{-1}H_0^l A, \tag{136}$$

$$2\sum_{i,j} b^{ij}\left(\frac{\partial}{\partial\xi^i}S\right)\left(\frac{\partial}{\partial\xi^j}\right)_l^* A - A\Delta_M S = 0. \tag{137}$$

Here, $dS \in T_r^*M$ is the gradient 1-form, and K_r^* is the scalar product (24) in the cotangent space T_r^*M. The matrix operators $(\partial/\partial\xi^j)_l^*$ are defined in (43), and Δ_M is the differential operator

$$\Delta_M = -[\sigma b^{1/2}]^{-1} \sum_{i,j} (\partial/\partial\xi^i)\, \sigma b^{1/2}\, b^{ij}\, (\partial/\partial\xi^j).$$

Note that it is identical to the kinetic-energy operator (43) for $l = 0$.

We now assume that at large distances the amplitude A varies fairly slowly:

$$\| A^{-1}H_0^l A \| \ll |V(r)|,$$

where $\|\cdot\|$ is the ordinary matrix norm. Then in Eq. (136) the right-hand side can be ignored. As a result, it takes the form of the Hamilton–Jacobi equation on the Riemannian manifold M:

$$K_r^*(dS, dS) = E - V(r). \tag{138}$$

Equation (137) is the corresponding continuity equation.

In the limit $r\to\infty$, the potential in the Hamilton–Jacobi equation (138) is a small perturbation, and therefore asymptotically its solution is equal to the sum

$$S(r)=\sqrt{E}\,Z(r)+W(r), \tag{139}$$

where the function Z satisfies the eikonal equation

$$K_r^*(dZ,\ dZ)=1, \tag{140}$$

and in the leading order the phase W is determined by the equation

$$K_r^*(dZ,\ dW)=-(2\sqrt{E})^{-1}V(r). \tag{141}$$

It is readily integrated. On the manifold M, we introduce the vector field $\partial/\partial Z$. Let $\gamma_r(t)$ be the integral curve of this field that for $t=Z$ passes through the given point $r\in M$. Then (141) can be rewritten in the form of an ordinary differential equation on the curve γ_r:

$$\frac{\partial W(\gamma_r(t))}{\partial t}=-(2\sqrt{E})^{-1}V(\gamma_r(t)), \tag{142}$$

whose solution has the form

$$W(r)=-(2\sqrt{E})^{-1}\int^{Z}V(\gamma_r(t))\,dt+C(\xi_Z^2,\ \xi_Z^3). \tag{143}$$

Here, C is a constant of integration. It may depend on the coordinates ξ^i_Z orthogonal to the eikonal Z:

$$K_r^*(dZ,\ d\xi_Z^i)=0,\quad K_r^*(d\xi_Z^i,\ d\xi_Z^j)=\delta_{ij}K_r^*(d\xi_Z^i,\ d\xi_Z^i). \tag{144}$$

It is clear that in the limit $Z\to\infty$ only the slowly decreasing terms of the potential contribute to the phase (143).

Thus, to each solution of the eikonal equation (140) there corresponds an asymptotic solution (135) of the Schrödinger equation. Its amplitude is determined by the continuity equation (137), while the phase is equal to the sum (139) of the eikonal and the additional phase (143). We now turn to the construction of asymptotic solutions associated with the wave function Ψ_0 of the Hamiltonian H'.

Asymptotic behavior of the (3→2, 3) wave functions

In the previous section, we showed that in the case of rapidly decreasing potentials the asymptotic behavior of the wave functions Ψ_0 is determined by the following set of eikonals: a) the plane-wave eikonals $Z=\pm Z_k$ $(k=1,2)$; b) the eikonals of single two-particle collisions, $Z=Z_\alpha^{(\pm)}$; c) the spherical eikonal $Z=\rho$. By direct calculation it can be verified that all these eikonals satisfy the eikonal equation (140), while the amplitudes corresponding to them in the representations (113), (117), and (126) satisfy the continuity equation (137). The problem thus reduces to the calculation of the phase corrections (143) to these eikonals that are generated by the Coulomb potential

$$V_c(r)=\sum_\alpha \frac{n_\alpha}{x_\alpha}. \tag{145}$$

Spherical eikonal $Z=\rho$

As the coordinates orthogonal to this eikonal, we can take the hyperspherical coordinates $(\chi_\alpha,\theta_\alpha)$ of the point r. The integral curves of the vector field $\partial/\partial\rho$ are the rays $\{\chi_\alpha=\text{const},\ \theta_\alpha=\text{const}\}$. The phase (143) is readily calculated and is described by (133). The corresponding asymptotic solution is identical to the final term of the asymptotic behavior (132) of the wave functions Ψ_A.

Plane-wave eikonals $Z=\pm Z_k$ $(k=1,2)$

We first define intrinsic coordinates orthogonal to the eikonals Z_k in the sense of (144). Note that for these eikonals the expression (127) is very similar to the definition (13) of the Jacobi coordinate x_α. They differ only in that the polar angles of the point r are measured from different directions. Let $(\Omega_k\Phi_k^{(\alpha)})$ be the spherical coordinates of the vector $\hat r$ relative to the axis $\hat q_k$ $(\hat q_1=\hat q,\ \hat q_2=-\hat q_*)$. It is clear that $(Z_k,\ u_k=\rho\sin(\Omega_k/2),\ \phi_k^{(\alpha)})$ are the Jacobi coordinates of the point r constructed relative to the axis $\hat q_k$. Since the metric (22) of the intrinsic space is invariant with respect to rotations, the coordinates $(Z_k u_k\phi_k^{(\alpha)})$ are orthogonal,

$$K_r=dZ_k^2+du_k^2+\frac{Z_k^2u_k^2}{Z_k^2+u_k^2}[d\phi_k^{(\alpha)}]^2. \tag{146}$$

At the same time, the angles $\Omega_k,\ \phi_k^{(\alpha)}$ are related to the hyperspherical coordinates $(\chi_\alpha\theta_\alpha)$ by a rotation transformation in R^3_+:

$$(\cos\chi_\alpha,\ \sin\chi_\alpha,\ \cos\theta_\alpha,\ \sin\chi_\alpha\sin\theta_\alpha)^T=$$
$$=g_k(\hat g)(\cos\Omega_k,\ \sin\Omega_k\cos\phi_k^{(\alpha)},\ \sin\Omega_k\sin\phi_k^{(\alpha)})^T, \tag{147}$$

which carries the axis $\hat e_1=(1,0,0)$ to $\hat q_k$. It is determined by the matrix

$$g_k(\hat q)=\begin{pmatrix}\cos\chi'_\alpha & -\sin\chi'_\alpha & 0\\ \sin\chi'_\alpha\cos\theta'_\alpha & \cos\chi'_\alpha\cos\theta'_\alpha & (-1)^k\sin\theta'_\alpha\\ (-1)^{k+1}\sin\chi'_\alpha\sin\theta'_\alpha & (-1)^{k+1}\cos\chi'_\alpha\sin\theta'_\alpha & \cos\theta'_\alpha\end{pmatrix}, \tag{148}$$

where $(\chi'_\alpha\theta'_\alpha)$ are the hyperspherical coordinates of the point $\hat q$.

Note that the angles $\phi_k^{(\alpha)}$ with different α differ by a constant:

$$\phi_k^{(\beta)}=\phi_k^{(\alpha)}-\delta_{\beta\alpha},$$
$$\sin\delta_{\beta\alpha}=\sin\omega_{\beta\alpha}\sin\theta'_\alpha/\sin\chi'_\beta,$$
$$\cos\delta_{\beta\alpha}=(\cos\omega_{\beta\alpha}\sin\chi'_\alpha-\sin\omega_{\beta\alpha}\cos\chi'_\alpha\cos\theta'_\alpha)/\sin\chi'_\beta, \tag{149}$$

where the angles $\omega_{\beta\alpha}$ are defined in (10).

Using the relation (148), we now express the Jacobi coordinates in terms of the orthogonal coordinates $Z_k,\ u_k,\ \phi_k^{(\alpha)}$:

$$x_\alpha=\Big[Z_k^2\cos^2\frac{\chi'_\alpha}{2}+u_k^2\sin^2\frac{\chi'_\alpha}{2}-Z_ku_k\cos\phi_k^{(\alpha)}\sin\chi'_\alpha\Big]^{1/2},$$
$$y_\alpha=\Big[Z_k^2\sin^2\frac{\chi'_\alpha}{2}+u_k^2\cos^2\frac{\chi'_\alpha}{2}+Z_ku_k\cos\phi_k^{(\alpha)}\sin\chi'_\alpha\Big]^{1/2},$$
$$x_\alpha y_\alpha\cos\theta_\alpha=\frac{1}{2}(Z_k^2-u_k^2)\sin\chi'_\alpha\cos\theta'_\alpha$$
$$+Z_ku_k(\cos\chi'_\alpha\cos\theta'_\alpha\cos\phi_k^{(\alpha)}+(-1)^k\sin\theta'_\alpha\sin\phi_k^{(\alpha)}). \tag{150}$$

For fixed u_k, $\phi_k^{(\alpha)}$ and $Z_k = t\in[0,\infty)$, these formulas parametrize the integral curves of the vector fields $\partial/\partial Z_k$. Equation (150) determines the Coulomb potential (145) on these curves. Calculation of the integral (143) leads to the following expressions for the phases corresponding to the eikonals $\pm Z_k$:

$$W_k^{(\pm)}(r, q) = \sum_\alpha \eta_\alpha \ln[2k_\alpha x_\alpha (1 \mp \zeta_\alpha^{(k)}(\hat{r}, \hat{q}))], \qquad (151)$$

where

$$\eta_\alpha = n_\alpha/(2k_\alpha),$$

$$\zeta_\alpha^{(k)} = \frac{k_\alpha Z_k - p_\alpha u_k \cos\phi_k^{(\alpha)}}{\sqrt{E}\, x_\alpha} = \frac{k_\alpha x_\alpha + p_\alpha y_\alpha \cos(\theta_\alpha + (-1)^k \theta'_\alpha)}{\sqrt{E}\, Z_k}.$$

Such phases correspond to the choice of vanishing constants of integration in (143), this being consistent with the analogous choice in the complete problem of three charged particles.

The corresponding asymptotic solution of the Schrödinger equation is described by the representation (126), in which the eikonals are distorted by the Coulomb phases:

$$\mathcal{F}_c^l = \sum_{k=1}^{2} \sum_\pm \mathcal{F}_{k,c}^{(\pm)},$$

$$\mathcal{F}_{k,c}^{(\pm)}(r, q) \underset{r\to\infty}{\sim} [8\pi^3 \sigma(r)\sigma(q)\cos(\Omega_k/2)]^{-1/2}$$
$$\times A_k^{(\pm)}(\hat{r}, \hat{q}) \exp\{\pm i\sqrt{E}\, Z_k + iW_k^{(\pm)} \pm (-1)^k i\pi/4\}. \qquad (152)$$

This asymptotic behavior is incorrect in the neighborhood of two singular directions:

$$\text{i) } \hat{r} = \hat{q}\ (\Omega_1 = 0); \quad \text{ii) } \hat{r} = \hat{q}_*\ (\Omega_2 = \pi). \qquad (153)$$

The first of them is the direction of forward scattering. In this direction, $\zeta_\alpha^{(1)} = 1$ and $W_1^{(+)} = \infty$. A similar effect is also well known in the complete three-body problem—the Coulomb phases that distort the plane wave are singular in the direction of forward scattering. The presence of the singular direction (ii) is not related to the long range of the Coulomb potential. In the previous section, we saw that in this direction the original asymptotic behavior (126) of the plane wave is incorrect. For the time being, we postpone the analysis of the asymptotic behavior of the function $\mathcal{F}_c^l$ in the singular directions (153) and turn to the construction of the asymptotic solutions generated by the eikonals $Z_\alpha^{(\pm)}$.

Eikonals of single two-particle collisions, $Z = Z_\alpha^{(\pm)}$

In this case, the set of suitable orthogonal coordinates is formed by the variables $(Z_\alpha^{(\pm)} u_\alpha^{(\pm)} \theta_\alpha)$, where

$$u_\alpha^{(\pm)} = (k_\alpha y_\alpha \mp p_\alpha x_\alpha) E^{-1/2}.$$

Indeed, in accordance with the definition (120) of the eikonals $Z_\alpha^{(\pm)}$ the transition from the Jacobi coordinates (x_α, y_α) to $(Z_\alpha^{(\pm)}, u_\alpha^{(\pm)})$ is realized by a rotation in the x_α, y_α plane. Since the radial part of the metric (30) is invariant with respect to such rotations, the coordinates that we have introduced satisfy the orthogonality criterion (144). The integral curves of the vector fields $\partial/\partial Z_\alpha^{(\pm)}$ are determined by the inverse rotation:

$$x_\alpha = (k_\alpha Z_\alpha^{(\pm)} \mp p_\alpha u_\alpha^{(\pm)}) E^{-1/2},$$
$$y_\alpha = (k_\alpha u_\alpha^{(\pm)} \pm p_\alpha Z_\alpha^{\pm}) E^{-1/2}$$

and by the conditions $u_\alpha^{(\pm)} = \text{const}$, $\theta_\alpha = \text{const}$. At the same time, it follows from the connection (10) between hyperspherical coordinates with different indices that on these integral curves the coordinates x_β with $\beta \neq \alpha$ are

$$x_\beta^2 = [Z_\alpha^{(\pm)}]^2 \cos^2\tau_{\beta\alpha}^{(\pm)} + [u_\alpha^{(\pm)}]^2 \sin^2\tau_{\beta\alpha}^{(\pm)} \pm Z_\alpha^{(\pm)} u_\alpha^{(\pm)} \cos\sigma_{\beta\alpha}^{\pm},$$

where

$$\cos 2\tau_{\beta\alpha}^{(\pm)} = \cos\omega_{\beta\alpha} \cos\chi'_\alpha \pm \sin\omega_{\beta\alpha} \sin\chi'_\alpha \cos\theta_\alpha;$$
$$\cos\sigma_{\beta\alpha}^{(\pm)} = -\cos\omega_{\beta\alpha} \sin\chi'_\alpha \pm \sin\omega_{\beta\alpha} \cos\chi'_\alpha \cos\theta_\alpha.$$

Calculating the integral (143) by means of these representations, we obtain the following expression for the corresponding phases:

$$W_\alpha^{(\pm)} = -\eta_\alpha \ln(2k_\alpha x_\alpha) - \sum_{\beta\neq\alpha} \eta_{\beta\alpha}^{(\pm)} \ln 2\{k_{\beta\alpha}^{(\pm)} x_\beta + \xi_{\beta\alpha}^{(\pm)}\}$$
$$+ C_\alpha^{(\pm)}(u_\alpha^{\pm}, \theta_\alpha), \qquad (154)$$

where

$$k_{\beta\alpha}^{(\pm)} = \sqrt{E} \cos\tau_{\beta\alpha}^{(\pm)}, \quad \eta_{\beta\alpha}^{(\pm)} = n_\beta/2k_{\beta\alpha}^{(\pm)};$$
$$\xi_{\beta\alpha}^{(\pm)} = \sqrt{E}(\cos^2\tau_{\beta\alpha}^{(\pm)} Z_\alpha^{(\pm)} \pm \cos\sigma_{\beta\alpha}^{(\pm)} u_\alpha^{(\pm)}),$$

and $C_\alpha^{(\pm)}$ are constants of integration. They can be fixed by the comparison-equation method. (It is necessary to match the eikonal approximation to the asymptotic solution of the Schrödinger equation in the region $x_\alpha \ll y_\alpha \to \infty$; this construction can be carried out explicitly by separation of the variables.) The corresponding technique is described in detail in Ref. 11 for the example of the complete three-body problem. Therefore, we give only the result:

$$C_\alpha^{(\pm)} = \sum_{\beta\neq\alpha} \{\eta_\beta \ln(\sqrt{E}\,|u_\alpha^{(\pm)}|\,\nu_{\beta\alpha}^{(\pm)})$$
$$+ \eta_{\beta\alpha}^{(\pm)} \ln(\sqrt{E}\,|u_\alpha^{(\pm)}|\,\mu_{\beta\alpha}^{(\pm)})\}, \qquad (155)$$

where the parameters ν and μ can be expressed in terms of the coefficients of the transformation (2):

$$\nu_{\beta\alpha}^{(\pm)} = k_\alpha^{-1}[k_\beta\,|s_{\beta\alpha}| \mp s_{\beta\alpha}(s_{\beta\alpha} p_\alpha + c_{\beta\alpha} k_\alpha \cos\theta'_\alpha)],$$
$$\mu_{\beta\alpha}^{(\pm)} = k_\alpha^{-1}[k_{\beta\alpha}^{(\pm)}\,|s_{\beta\alpha}| + s_{\beta\alpha}(\pm s_{\beta\alpha} p_\alpha + c_{\beta\alpha} k_\alpha \cos\theta_\alpha)].$$

The asymptotic solution of the Schrödinger equation corresponding to the eikonals $Z_\alpha^{(\pm)}$ is described by the representation (117), in which the eikonals are distorted by the Coulomb phases (154):

$$U_\alpha^{(c)} = \sum_\pm U_{\alpha,c}^{(\pm)},$$
$$U_{\alpha,c}^{(\pm)} \sim A_{\alpha,c}^{(\pm)} \exp\{i\sqrt{E}\, Z_\alpha^{(\pm)} + iW_\alpha^{(\pm)}\}. \qquad (156)$$

The matrices $A_{\alpha,c}^{(\pm)}$ are determined by Eq. (118), in which the amplitudes $f_\alpha^{(\pm)}$ are redefined:

$$f_\alpha^{(\pm)}(k,\ \theta,\ \theta') = (2ik)^{-1}\sum_{\lambda=0}^{\infty}\exp\{2i\delta_\alpha^{(\lambda)}(k)\}$$

$$\times Q_\lambda^l(\pm\cos\theta)\, Q_\lambda^l(\cos\theta'), \qquad (157)$$

where $\delta_\alpha^{(\lambda)}$ are the phase shifts for scattering by the potential (130). They are equal to the sum of the Coulomb phases $\delta_\lambda^{(c)}(\eta_\alpha)$ and the phases $\delta_{\alpha\lambda}^{(s)}$ generated by the short-range part of the potential (130).

The amplitudes $f_\alpha^{(\pm)}$ have singularities. They are contained in their purely Coulomb part $f_{\alpha,c}^{(\pm)}$, which is given by the series (157) with the Coulomb phases $\delta_\lambda^{(c)}$. This series can be summed by means of the multiplication formula for the associated Legendre polynomials:[15]

$$f_{\alpha,\,c}^{(\pm)} = -\frac{2^{i\eta_\alpha}}{4\pi k}\,\eta_\alpha\exp\{2i\delta_0^{(c)}(\eta_\alpha)\}$$

$$\times\int_0^{2\pi}[1+\cos\theta\cos\theta' - \sin\theta\sin\theta'\cos\varphi]^{-1-i\eta_\alpha}E^l(\varphi)\,d\varphi,$$

where $[E^l(\varphi)]_{mn} = \delta_{mn}e^{in\varphi}$. It follows from this representation that the amplitude $f_{\alpha,c}^{(+)}$ is singular at $\theta=\theta'$, while $f_{\alpha,c}^{(-)}$ is singular at $\theta = \pi - \theta'$:

$$f_{\alpha,\,c}^{(\pm)}(k,\ \theta,\ \theta') = -\frac{\eta_\alpha\Gamma(1/2+i\eta_\alpha)}{2k\Gamma(1-i\eta_\alpha)}$$

$$\times(2\pi\sin\theta\sin\theta')^{-1/2}\left[\frac{1\mp\cos(\theta\mp\theta')}{2}\right]^{-1/2-i\eta_\alpha}I^l+\dots \qquad (158)$$

The constructed asymptotic solution (156) becomes meaningless at $u_\alpha^{(+)}=0$, since at such points the phase $W_\alpha^{(+)}$ is singular [see (155)]. The set of these points forms the conical surface $\Xi_\alpha \subset M$ shown in Fig. 2:

$$\Xi_\alpha = \{r = (\rho\chi_\alpha\theta_\alpha) : \chi_\alpha = \chi_\alpha^*\}. \qquad (159)$$

We now summarize some of our results.

Asymptotic behavior of the wave functions Ψ_0

The asymptotic behavior of the wave functions Ψ_0 in systems with Coulomb interaction is given by the sum of the eikonal approximations constructed above:

$$\Psi_0 \sim \mathscr{F}_s^l + \sum_\alpha U_\alpha^{(c)} + \sum_\alpha \Phi_\alpha^{(c)}.$$

The first term is the distorted plane wave (152). The contribution of the processes of single two-particle collisions is described by the representation (156). The functions $\Phi_\alpha^{(c)}$ contain the distorted spherical waves corresponding to the $(3\to2,3)$ processes. For them a representation analogous to (132) is valid.

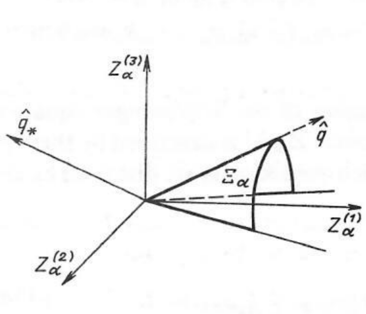

FIG. 2. Singular directions in R_+^3 of the asymptotic behavior of the $(3\to2,3)$ wave function of Coulomb systems. The rays $\hat{q}_*$ and $\hat{q}$ are the same as in Fig. 1. The surfaces Ξ_α are formed by rotating the ray $\hat{q}$ around the axes $\Xi_\alpha^{(1)}$ $(\alpha=1,2,3)$. One of them is shown in the figure.

Figure 2 shows the set of singular points at which the resulting asymptotic behaviors of the functions $\mathscr{F}_c^l$ and $U_\alpha^{(c)}$ become meaningless. We now turn to the investigation of the structure of these functions in the neighborhoods of these singularities.

Singular directions

Forward-scattering direction ($xr = xq$)

In this direction the asymptotic behavior (152) is not valid for the function $\mathscr{F}_{c,1}^{(+)}$, since $W_1^{(+)}=\infty$ for $\hat{r}=\hat{q}$ (the remaining three terms of the distorted plane wave do not have singularities for $\hat{r}=\hat{q}$). We construct an asymptotic solution of the Schrödinger equation that remains smooth for $\hat{r}=\hat{q}$ and outside this direction is identical to the eikonal approximation $\mathscr{F}_{1c}^{(+)}$. We shall find that the singularity in the asymptotic behavior of $\mathscr{F}_{1c}^{(+)}$ generates a singularity of the $(3\to3)$ scattering amplitude in the given direction.

To avoid distraction by secondary details, we describe all the constructions for the example of systems with zero total orbital angular momentum. In this case, $A_k^{(\pm)}=1$ in the representation (152). In accordance with this representation, we shall seek the required solution in the form

$$\mathscr{F}_{1,\,c}^{(+)} = [8\pi^3\sigma(r)\,\sigma(q)\cos(\Omega_1/2)]^{-1/2}U. \qquad (160)$$

As a consequence, away from the singular direction the asymptotic behavior of U is described by the eikonal approximation

$$U(r,\ q) \sim \exp\{i\sqrt{E}\,Z_1 + iW_1^{(+)} - i\pi/4\}. \qquad (161)$$

We shall obtain an equation for U. To this end, it is convenient to express the Schrödinger equation (134) for $\mathscr{F}^{(+)}$ in the spherical coordinates $(\rho\Omega_1\phi_1^{(\alpha)})$ defined in (147). The metric of the intrinsic space in these coordinates is analogous to (29):

$$K_r = d\rho^2 + \frac{\rho^2}{4}[d\Omega_1^2 + \sin^2\Omega_1(d\phi_1^{(\alpha)})^2],$$

and the kinetic-energy operator can be determined from it by means of Eq. (43), in which $(\partial/\partial\xi^i)_l^* = \partial/\partial\xi^i$ for $l=0$. As a result, we arrive at the equation

$$(-\Delta_1 + V_c + \delta V - E)\,U = 0, \qquad (162)$$

where V_c is the total Coulomb potential (145), and the correction δV is

$$\delta V = -\rho^{-2}\left(1+\frac{1}{\sin^2\psi}+\frac{1}{4\cos^2(\Omega_1/2)}\right)$$

(ψ is the Dragt coordinate of the point r). The differential operator Δ_1 has the form

$$\Delta_1 = \left\{\partial_\rho^2 + \frac{2}{\rho}\partial_\rho + \frac{1}{\rho^2}\left[\frac{1}{\sin^2(\Omega_1/2)}\partial^2_{\phi_1^{(\alpha)}} + \frac{4}{\sin(\Omega_1/2)}\partial_{\Omega_1}\sin(\Omega_1/2)\,\partial_{\Omega_1}\right]\right\} + \frac{1}{\rho^2\cos^2(\Omega_1/2)}\partial^2_{\phi_1^{(\alpha)}}.$$

Note that the expression in the curly brackets is identical to the Laplacian in Euclidean space:

$$R^3=\left\{(x^1,\ x^2,\ x^3)\right.$$

$$\left.=\left(\rho\cos\frac{\Omega_1}{2},\ \rho\sin\frac{\Omega_1}{2}\cos\phi_1^{(\alpha)},\ \rho\sin\frac{\Omega_1}{2}\sin\phi_1^{(\alpha)}\right)\right\}. \quad (163)$$

We now construct an asymptotic solution of Eq. (162) in the neighborhood of the ray $\hat{r}=\hat{q}$ by Fock's parabolic-equation method. Suitable coordinates in our case are the standard parabolic coordinates $(\zeta\xi\phi_1^{(\alpha)})$ in the space (163):

$$\zeta=\rho\left(1+\cos\frac{\Omega_1}{2}\right)=\rho+Z_1,$$

$$\xi=\rho\left(1-\cos\frac{\Omega_1}{2}\right)=\rho-Z_1.$$

In them, the operator Δ_1 is given by the expression

$$\Delta_1=\frac{4}{\zeta+\xi}\left(\partial_\zeta\zeta\partial_\zeta+\partial_\xi\xi\partial_\xi\right)+\frac{1}{\zeta\xi}\left(\frac{\zeta+\xi}{\zeta+\xi}\right)^2\partial^2_{\phi_1^{(\alpha)}}.$$

As $r\to\infty$ in the singular direction $\zeta\to 0$, $\xi=0$. Therefore, up to terms $\sim\xi/\zeta^2$, we can replace the Coulomb potential in (162) by the expression

$$V_c(r)\sim\frac{2n_0}{\zeta+\xi},\quad n_0=\sqrt{E}\sum_\alpha\frac{n_\alpha}{k_\alpha}$$

and ignore the correction δV [this last step is valid if the ray is separated from the boundary (45) of the intrinsic space]. After these approximations, Eq. (162) admits separation of the variables in the parabolic coordinates. We shall seek a solution of it in the form

$$U=\exp\left\{i\sqrt{E}(\zeta-\xi)/2\right\}\sum_{m=-\infty}^{+\infty}\frac{\exp\{im\phi_1^{(\alpha)}\}}{\sqrt{2\pi}}F_m(\xi,\ \zeta). \quad (164)$$

With accuracy up to terms $\sim\xi/\zeta^2$, the functions F_m satisfy the equations

$$\left\{\xi\partial_\xi^2+(1-i\sqrt{E}\xi)\partial_\xi-n_0/2-\frac{m^2}{4\xi}\right\}F_m$$
$$+\left\{\zeta\partial_\zeta^2+(1-i\sqrt{E}\zeta)\partial_\zeta-\frac{5m^2}{4\zeta}\right\}F_m=0. \quad (165)$$

The asymptotic behavior of the required solutions to these equations is determined by the condition of matching of the expansion (164) to the eikonal approximation (161) in the limit $\xi\to\infty$. It follows from the expression (151) for the phase $W_1^{(+)}$ that in the leading order as $\xi/\zeta\to 0$ it is equal to

$$W_1^{(+)}\sim\sum_\beta\eta_\beta\ln\left(\frac{p_\beta^2}{2\sqrt{E}}\sin^2\phi_1^{(\beta)}\xi\right),\quad \eta_\beta=\frac{n_\beta}{2k_\beta}.$$

Therefore, the asymptotic behavior of F_m is given by the expansion of the exponential $\exp(iW_1^{(+)})$ in the Fourier series (164):

$$F_m(\xi,\ \zeta)\underset{\xi\to\infty}{\sim}c_m(\sqrt{E}\xi)^{i\eta_0},\quad \eta_0=\frac{n_0}{2\sqrt{E}}, \quad (166)$$

with coefficients

$$c_m=\left(-\frac{i}{2\pi}\right)^{1/2}\prod_\beta(p_\beta^2/2E)^{i\eta_\beta}$$
$$\times\int_0^{2\pi}d\phi_1^{(\alpha)}\exp\{-im\phi_1^{(\alpha)}\}\prod_\beta(\sin^2\phi_1^{(\beta)})^{i\eta_\beta}.$$

In this integral, the angles $\phi_1^{(\beta)}$ with different indices are related by the shift (149).

In the leading order as $\zeta\to\infty$, the solutions of Eq. (165) with the asymptotic behavior (166) do not depend on ζ and can be expressed in terms of the regular confluent hypergeometric function:

$$F_m(\xi)=c_m\frac{\Gamma(|m|/2+1+i\eta_0)}{\Gamma(|m|+1)}e^{-\pi\eta_0/2}\left(-i\sqrt{E}\xi\right)^{|m|/2}$$
$$\times\Phi\left(|m|/2-i\eta_0,\ |m|+1,\ i\sqrt{E}\xi\right). \quad (167)$$

Thus, the asymptotic behavior of the function $\mathscr{F}_{1,c}^{(+)}$ in the neighborhood of the singular direction is described by the representations (160), (164), and (167). In the limit $\xi\to\infty$, it decomposes into a sum of the eikonal approximation (152) and a distorted spherical wave with amplitude singular in the forward scattering direction:

$$A_s(\hat{r},\ \hat{q})=a\left[\sin\left(\Omega_1(\hat{r},\ \hat{q})/2\right)\right]^{-4-2i\eta_0},$$

$$a=-\frac{\sqrt{E}}{16\pi^2\sigma(q)}\sum_{m=-\infty}^{+\infty}c_m(-i)^m$$

$$\times\frac{\Gamma(|m|/2+1+i\eta_0)}{\Gamma(|m|/2-i\eta_0)}\exp\{im\phi_1^{(\alpha)}\}. \quad (168)$$

This representation describes the main angular singularity of the $(3\to3)$ scattering amplitude in Coulomb systems. This singularity is weaker than the analogous singularity in the complete three-body problem.

We considered above the case of zero total orbital angular momentum. For $l\geqslant1$ all the expressions obtained for $\mathscr{F}_{1,c}^{(+)}$ must be multiplied by the matrix amplitudes $A_1^{(+)}$ from (152).

The singular direction $\hat{r}=\hat{q}_*$

In this direction, the asymptotic behavior (152) for the function $\mathscr{F}_{c,2}^{(+)}$ is not valid, since $Z_2=0$ for $\hat{r}=\hat{q}_*$. We now construct an asymptotic solution of the Schrödinger equation that away from the ray $\hat{r}=\hat{q}_*$ is equal to the sum of the eikonal approximations $\mathscr{F}_{c,2}^{(+)}+\mathscr{F}_{c,2}^{(-)}$ and remains smooth for $\hat{r}=\hat{q}_*$.

In accordance with the representation (152), we shall seek such a solution in the form

$$\mathscr{F}_2=\mathscr{F}_{2,c}^{(+)}+\mathscr{F}_{2,c}^{(-)}=\left[8\pi^3\rho^{-1}\sigma(r)\sigma(q)\right]^{-1/2}\Phi. \quad (169)$$

Away from the singular direction, the asymptotic behavior of Φ is described by the eikonal approximation

$$\Phi\sim\sum_\pm A_2^{(\pm)}\exp\left\{\pm i\sqrt{E}Z_2+iW_2^{(\pm)}\pm i\pi/4\right\}/\sqrt{Z_2}, \quad (170)$$

where $A_2^{(\pm)}$ are the matrix amplitudes from (152). It is clear from the functional form of the asymptotic behavior (170) that it is convenient to construct the function Φ in cylindrical coordinates (Z_2, $u_2 = \rho \sin(\Omega_2/2)$, $\phi_2^{(\alpha)}$). In them, the metric of the intrinsic space is given by the expression (146), and the Hamiltonian H^l can be determined from this metric by Eq. (43). Substituting (169) in the Schrödinger equation (134), we obtain for Φ a matrix equation of the type (162). With accuracy up to terms $\sim Z_2/u_2^2$, it has the form

$$\left(-\Delta_2 \otimes I^l + \frac{\tilde{n}_0}{u_2} \otimes I^l - E \otimes I^l\right)\Phi = 0, \qquad (171)$$

where $\tilde{n}_0 = \Sigma_\alpha n_\alpha \sqrt{E}/p_\alpha$ and

$$\Delta_2 = \partial^2_{Z_2} + \frac{1}{Z_2}\partial_{Z_2} + \partial^2_{u_2} + \frac{1}{u_2}\partial_{u_2} + (Z_2^{-2} + u_2^{-2})\,\partial^2_{\phi_2^{(\alpha)}}.$$

The variables in this equation separate. We shall establish the asymptotic boundary conditions that fix the required solution. To this end, we calculate the leading term of the eikonal asymptotic behavior (170) in the neighborhood of the singular direction.

In accordance with the definition (151), the phases $W_2^{(\pm)}$ in the leading order as $Z_2/u_2 \to 0$ are

$$W_2^{(\pm)} \sim \sum_\beta \eta_\beta \ln\left\{\frac{\sqrt{E}\,u_2}{2}\sin\chi'_\beta\,(1 \pm \cos\phi_2^{(\beta)})\right\}.$$

We consider further the amplitudes $\mathscr{A}_2^{(\pm)}$. From the expressions (123) for the angle Δ_2 and the definition (124) of the angles $\phi_2^{(\alpha)}$ we obtain the relation

$$\cos\Delta_2 = -\cos(\phi_2^{(\alpha)} + \theta'_\alpha) + O(Z_2/u_2).$$

At the same time, the position of the point r relative to the cone Γ_α in Fig. 1 in the neighborhood of the singular direction is determined by the angle $\phi_2^{(\alpha)}$:

$$\begin{aligned} &r \in \Gamma_\alpha : \phi_2^{(\alpha)} \in (0,\ \pi - \theta'_\alpha) \cup (2\pi - \theta'_\alpha,\ 2\pi),\\ &r \in M/\Gamma_\alpha : \phi_2^{(\alpha)} \in (\pi - \theta'_\alpha,\ 2\pi - \theta'_\alpha). \end{aligned} \qquad (172)$$

Then near the ray $\hat{r} = \hat{q}_*$ the representation (128) for the amplitude $A_2^{(\pm)}$ takes the form

$$A_2^{(\pm)}(\phi_2^{(\alpha)}) = D^l(g_\alpha(q_*))\,B_2^{(\pm)}(\phi_2^{(\alpha)})\,D^l(\tilde{g}_\alpha^{-1}(q)),$$

$$B_2^{(\pm)}(\phi_2^{(\alpha)}) = \begin{cases} I^l_\pm d^l(\mp\cos(\phi_2^{(\alpha)} + \theta'_\alpha))\,I^l_\pm, & r \in \Gamma_\alpha,\\ I^l_\mp d^l(\mp\cos(\phi_2^{(\alpha)} + \theta'_\alpha))\,I^l_\mp, & r \in M/\Gamma_\alpha. \end{cases}$$

Therefore, the solution of Eq. (171) must have the following asymptotic behavior for $u_2, Z_2 \to \infty$:

$$\Phi \sim c(q)\,u_2^{i\eta_0} \sum_\pm Z_2^{-1/2} \exp\{\pm i\sqrt{E}\,Z_2 \pm i\pi/4\}\,\tilde{A}^{(\pm)}(\phi_2^{(\alpha)}),$$

where η_0 is determined in (166), and

$$\begin{aligned} &c(q) = \prod_\beta \left(\frac{\sqrt{E}}{2}\sin\chi'_\beta\right)^{i\eta_\beta},\\ &\tilde{A}^{(\pm)}(\phi_2^{(\alpha)}) = \prod_\beta (1 \pm \cos\phi_2^{(\beta)})^{i\eta_\beta} A_2^{(\pm)}(\phi_2^{(\alpha)}). \end{aligned} \qquad (173)$$

Such a solution is given by the series

$$\Phi = c(q)\,u_2^{i\eta_0} \sum_{m=-\infty}^{\infty} A_m J_m(\sqrt{E}\,Z_2)\,\frac{\exp\{im\phi_2^{(\alpha)}\}}{\sqrt{2\pi}}, \qquad (174)$$

where J_m is a Bessel function,[16] and A_m are constant matrices of rank $2l+1$. They are the coefficients of the Fourier expansion of the matrix amplitudes (173):

$$A_m = \frac{(\pm i)^{m/2}}{\sqrt{2\pi}} \int_0^{2\pi} d\phi_2^{(\alpha)}\,\tilde{A}^{(\pm)}(\phi_2^{(\alpha)})\exp\{im\phi_2^{(\alpha)}\}.$$

The right-hand side of this equation takes the same values for both signs ($\pm$). This follows from the symmetry properties of the integrand with respect to the shift $\phi_2^{(\alpha)} \to \phi_2^{(\alpha)} + \pi$. [Under such a shift, the neighborhoods of the singular direction inside and outside the cone Γ_α are interchanged [see (172)], and the angles $\phi_2^{(\beta)}$ with $\beta \neq \alpha$ are also shifted by π by virtue of (149).]

Thus, the representations (169) and (174) describe the asymptotic structure of the distorted plane wave in the neighborhood of the singular direction $\hat{r} = \hat{q}_*$. They are also valid for systems with rapidly decreasing potentials (in this case, it is necessary to set $\eta_\beta = \eta_0 = 0$ in them).

Neighborhood of the cone Ξ_α

We shall construct an asymptotic solution of the Schrödinger equation that outside the cone (159) is identical to the eikonal approximation $U_{\alpha,c}^{(+)}$ from (156) and remains smooth for $r \in \Xi_\alpha$. We shall seek it in the form

$$U_{\alpha,c}^{(+)} = A_{\alpha,c}^{(+)} \exp\{i\sqrt{E}\,Z_\alpha^{(+)} + i\tilde{W}_\alpha\}\,F_\alpha, \qquad (175)$$

where F_α is an unknown function, the matrix amplitudes $A_{\alpha,c}^{(+)}$ are determined in (156), and $\tilde{W}_\alpha$ is the smooth part of the phase (154) in the limit $u_{\alpha,c}^{(+)} \to 0$:

$$\begin{aligned} &\tilde{W}_\alpha = W_\alpha^{(+)} - 2a_\alpha(\theta_\alpha)\ln(\sqrt{E}\,u_\alpha^{(+)}),\\ &a_\alpha(\theta_\alpha) = \frac{1}{2}\sum_{\beta\neq\alpha}\{\eta_\beta + \eta_{\beta\alpha}^{(+)}(\theta_\alpha)\}. \end{aligned}$$

The asymptotic behavior of F_α outside the cone Ξ_α is determined by the eikonal approximation (156):

$$F_\alpha \sim [\sqrt{E}\,u_\alpha^{(+)}]^{2ia_\alpha}. \qquad (176)$$

We shall obtain an equation for F_α. To this end, it is convenient first to write down the Schrödinger equation for $U_{\alpha,c}^{(+)}$ in the Jacobi coordinates (x_α, y_α, θ_α) and separate the factor $A_{\alpha,c}^{(+)}$. The radial part of the Hamiltonian H^l then takes the form of the Laplacian on a plane ($\partial^2_{x_\alpha} + \partial^2_{y_\alpha}$). The angular part of H^l, containing differentiation with respect to θ_α and centrifugal terms, generates corrections $\sim\rho^{-2}$, which can be ignored. One should then go over to the coordinates ($Z_\alpha^{(+)} u_\alpha^{(+)} \theta_\alpha$) and separate the exponential factor from (175). (In these coordinates, the Laplacian is $\partial^2_{Z_\alpha^{(+)}} + \partial^2_{u_\alpha^{(+)}}$.) In the resulting equation, the Coulomb potential cancels, since the phase satisfies the relation (142). With accuracy up to terms $\sim\rho$, it has the form

$$\Big\{\partial^2_{Z_\alpha^{(+)}} + \partial^2_{u_\alpha^{(+)}} + 2i\sqrt{E}\,\partial_{Z_\alpha^{(+)}} + 2i\left[\frac{\partial\tilde{W}_\alpha}{\partial Z_\alpha^{(+)}}\partial_{Z_\alpha^{(+)}} + \frac{\partial\tilde{W}_\alpha}{\partial u_\alpha^{(+)}}\partial_{u_\alpha^{(+)}}\right]\Big\}F_\alpha = 0. \qquad (177)$$

We now introduce the parabolic coordinates

$$\zeta_\alpha=\rho+Z_\alpha^{(+)},\quad \xi_\alpha=\rho-Z_\alpha^{(+)}$$

(as $r\to\infty$ on Ξ_α we have $\xi_\alpha=0$, $\zeta_\alpha\to\infty$). We rewrite Eq. (177) in these coordinates and retain in it only the leading terms as $\xi_\alpha/\zeta_\alpha\to 0$. They are generated by the first three terms in (177):

$$\left\{\xi_\alpha\partial^2_{\xi_\alpha}+\zeta_\alpha\partial^2_{\zeta_\alpha}+\mathrm{i}\sqrt{E}\,(\zeta_\alpha\partial_{\zeta_\alpha}-\xi_\alpha\partial_{\xi_\alpha})+\frac{1}{2}\,(\partial_{\xi_\alpha}+\partial_{\zeta_\alpha})\right\}F_\alpha=0.$$

In this equation, the variables separate, and the separation constant can be an arbitrary function of the angle θ_α. The required asymptotic solution with the boundary condition (176) as $\xi_\alpha\to\infty$ can be expressed in terms of the confluent hypergeometric function:

$$F_\alpha=\pi^{-1/2}e^{-\pi a_\alpha}\Gamma(1/2+\mathrm{i}a_\alpha)\times(\sqrt{E}\,\zeta_\alpha)^{\mathrm{i}a_\alpha}\,\Phi(-\mathrm{i}a_\alpha,\ 1/2,\ \mathrm{i}\sqrt{E}\,\xi_\alpha). \tag{178}$$

Thus, the asymptotic behavior of the function $U^{(+)}_{\alpha,c}$ in the neighborhood of the cone Ξ_α is described by the representations (175) and (178). In the limit $\xi_\alpha\to\infty$, the function $U^{(+)}_{\alpha,c}$ is transformed into the sum of the eikonal approximation (156) and the distorted spherical wave with amplitude

$$A^{(s)}_\alpha=-\mathrm{i}\,(\rho^2A^{(+)}_{\alpha,c})\,\frac{\Gamma(1/2+\mathrm{i}a_\alpha)}{\sqrt{2}\,\Gamma(-\mathrm{i}a_\alpha)}\times E^{-1/4}e^{\mathrm{i}b_\alpha}\left[\sin^2\left(\frac{\chi_\alpha-\chi'_\alpha}{4}\right)\right]^{-1/2-\mathrm{i}a_\alpha}, \tag{179}$$

where

$$b_\alpha=\eta_\alpha\ln\cos^2\left(\frac{\chi_\alpha}{2}\right)+\sum_{\beta\neq\alpha}\left\{\eta_\beta\ln\nu^{(+)}_{\beta\alpha}+\eta^{(+)}_{\beta\alpha}\ln\left[2\mu^{(+)}_{\beta\alpha}\cos^2\left(\frac{\chi_\beta}{2}\right)\right]\right\}.$$

This amplitude is singular as $r\to\Xi_\alpha$ ($\chi_\alpha\to\chi'_\alpha$). Therefore, the $(3\to3)$ scattering amplitude in Coulomb systems has angular singularities of the form (179) on the surfaces of the three cones Ξ_α ($\alpha=1,2,3$). These cones intersect along the ray $\hat{r}=\hat{q}$, which specifies the forward-scattering direction. On this ray, all three amplitudes $A^{(s)}_\alpha$ have additional singularities with respect to the angle θ_α. They are generated by the factor $\rho^2A^{(+)}_{\alpha,c}$ and are described by the representation (158). Thus, in the forward-scattering direction the $(3\to3)$ scattering amplitude contains, in addition to the main singularity (168), weaker singularities of the type

$$\sum_\alpha B_\alpha\left[\sin^2\left(\frac{\theta_\alpha-\theta'_\alpha}{2}\right)\right]^{-1/2-\mathrm{i}\eta_\alpha}\left[\sin^2\left(\frac{\chi_\alpha-\chi'_\alpha}{2}\right)\right]^{-1/2-\mathrm{i}a_\alpha},$$

which correspond to processes of single two-particle collisions.

5. LEVINSON'S FORMULA AND SPECTRAL IDENTITIES

In this section, we again consider rapidly decreasing potentials of the type (47) with $\varepsilon>2$. For a Hamiltonian H^l with such potentials, we obtain a complete series of spectral identities that relate the characteristics of the discrete spectrum and the S matrix. These results were proved for the first time in Refs. 9 and 10 for the example of systems with zero total orbital angular momentum. They are a direct generalization to the three-body problem of the trace formulas for the two-particle Schrödinger operator,[21] in particular, the classical Levinson's formula.[22]

The proof of the spectral identities is based on two formulas that also have independent interest: 1) a trace formula that expresses the trace of the connected part $R^l_c(z)$ of the resolvent of the Hamiltonian H^l in terms of the S matrix; 2) an asymptotic expansion of Sp $R^l_c(z)$ in the limit $|z|\to\infty$. Similar results have already been obtained in the complete three-body problem.[23,24] In the complete problem, the trace formula contains complicated regularizing corrections, which are generated by the three-particle singularities of the T matrix discussed in Sec. 2. The presence of such corrections makes it impossible to express the corresponding spectral identities[24] directly in terms of the S matrix. For systems with fixed total orbital angular momentum, this difficulty is absent, since the T matrix of the Hamiltonian H^l does not have three-particle singularities.

Trace formula

This formula relates the trace of the connected part

$$R^l_c(z)=R^l(z)-R^l_0(z)+\sum_\alpha R^l_0(z)\,T^l_\alpha(z)\,R^l_0(z) \tag{180}$$

of the resolvent of the Hamiltonian H^l to its S matrix. It can be derived in accordance with the usual scheme[11,23] by transformations of the singularities of the kernel of the T matrix. In our case, the T matrix has only two-body singularities, which are reflected in the representation (77) for its components. Therefore, in contrast to the complete three-body problem, there is no need to introduce complicated regularizations to take into account the three-particle singularities. Otherwise, the technique for deriving the trace formula for the operator (180) more or less repeats the corresponding constructions in the complete problem.[11,23] Therefore, we give only the result:

$$2\mathrm{i}\ \mathrm{Sp}\ \mathrm{Im}\ R^l_c(E+\mathrm{i}0)=\mathrm{Sp}\ (S^*\partial_E S)_c(E). \tag{181}$$

Here, the right-hand side is expressed in terms of the S matrix of the Hamiltonian H^l at fixed energy:

$$\begin{aligned}\mathrm{Sp}\,(S^*\partial_E S)_c(E)&\equiv\sum_{A,B}\theta(E+\varkappa_A^2)\,\theta(E+\varkappa_B^2)\,\mathrm{Sp}\,\{S^*_{AB}(E)\,\partial_E S_{AB}(E)\}\\&+\theta(E)\sum_A\{\mathrm{Sp}\,[S^*_{0A}(E)\,\partial_E S_{0A}(E)]+\mathrm{Sp}\,[S^*_{A0}(E)\,\partial_E S_{A0}(E)\}\\&+\theta(E)\,\mathrm{Sp}\left\{S^*_{00}(E)\,\partial_E S_{00}(E)-\sum_\alpha S^*_\alpha(E)\,\partial_E S_\alpha(E)\right\},\end{aligned} \tag{182}$$

where S_α are the S matrices of the Hamiltonians (69), and θ is the Heaviside function. The kernels of the remaining S matrices are determined in (93)–(96). We emphasize that in each term of Eqs. (181) and (182) the trace operation is understood in the sense of the function space of vector functions in which the corresponding matrix integral operator acts.

Asymptotic behavior of Sp $R_c^l(z)$ as $|z| \to \infty$

We first obtain the asymptotic behavior of the connected part of the T matrix: $T_c^l = T^l - \Sigma_\alpha T_\alpha^l$. To this end, we represent the operator $R_0^l(z)$ in the Lippmann–Schwinger equation $T^l(z) = V^l - V^l R_0^l(z) T^l(z)$ as a formal series in powers of z^{-1} [see (66)],

$$R_0^l(z) = -\sum_{k=0}^{\infty} (\widetilde{H}_0^l)^k z^{-k-1},$$

and seek the T matrix in a similar form:

$$T^l(z) = \sum_{n=0}^{\infty} T_n^l z^{-n}. \tag{183}$$

Equating the coefficients of z^{-n}, we obtain recursion relations for the operators T_n^l:

$$T_0^l = V^l, \quad T_n^l = V^l \sum_{k=0}^{n-1} (\widetilde{H}_0^l)^k T_{n-k-1}^l \; (n \geqslant 1). \tag{184}$$

For the operators T_α^l, the expansion (183) with coefficients $T_{\alpha,n}^l$ holds. The coefficients are determined by the expressions (184), in which V^l is to be replaced by V_α^l. As a result, we obtain an expansion for the connected part of the T matrix:

$$T_c^l(z) = \sum_{n=1}^{\infty} (T_n^l)_c z^{-n}, \quad (T_n^l)_c = T_n^l - \sum_\alpha T_{\alpha,n}^l. \tag{185}$$

Note that the coefficients of this expansion can be represented in the form

$$(T_n^l)_c = [\mathscr{F}^l]^* W_c^{(n)} \mathscr{F}^l, \tag{186}$$

where $\mathscr{F}^l$ is the diagonalizing transformation (51). The functions $W_c^{(n)}$ can be expressed in terms of the two-body potentials by the recursion relations (184) in the configuration representation,

$$W_c^{(n)} = W^{(n)} - \sum_\alpha W_\alpha^{(n)},$$

$$W^{(0)}(r) = V(r),$$

$$W^{(n)}(r) = V(r) \sum_{k=0}^{n-1} (H_0^l)^k W^{(n-k-1)}(r) \quad (n \geqslant 1),$$

while the $W_\alpha^{(n)}$ are determined by the same relations with the substitution $V \to V_\alpha$, so that

$$W_c^{(0)} = 0, \quad W_c^{(1)} = \sum_{\alpha \neq \beta} V_\alpha V_\beta,$$

$$W_c^{(2)} = \left(V^3 - \sum_\alpha V_\alpha^3\right) + \sum_{\alpha \neq \beta} V_\alpha H_0^l V_\beta$$

etc.

We now substitute the expansion (185) in the equation $R_c^l = -R_0^l T_c^l R_0^l$ and make a cyclic permutation of the operators under the trace symbol. Then, using the expression (186), we arrive at the representation

$$\mathrm{Sp}\, R_c^l(z) = -\sum_{n=1}^{\infty} f_n^{(l)}(z) z^{-n}. \tag{187}$$

The coefficients of this series are given by the integrals

$$f_n^l(z) = \int_0^\infty \frac{\lambda^5\, d\lambda}{(\lambda^2 - z)^2} Q_n^{(l)}(\lambda^2) \tag{188}$$

of the functions

$$Q_n^{(l)}(\lambda^2) = \frac{1}{8} \int_M dM(r)\, W_c^{(n)}(r)\, \mu_l(r, \lambda), \tag{189}$$

and the weight μ_l is expressed by an integral over the unit sphere in the intrinsic space with the measure (92):

$$\mu_l(r, \lambda) = \int_{\hat{M}} d\hat{M}(\hat{q})\, \mathrm{Sp}\, \{[\mathscr{F}^l(r, \lambda\hat{q})]^* \mathscr{F}^l(r, \lambda\hat{q})\}. \tag{190}$$

We calculate the asymptotic behavior of the coefficients in the series (187) as $|z| \to \infty$. By virtue of (188), it is determined by the behavior of the functions $Q_n^{(l)}(\lambda^2)$ as $\lambda \to \infty$. In this limit, we can replace the kernels $\mathscr{F}^l$ in the integral (190) by their asymptotic behaviors (126). We then obtain the representation

$$Q_n^{(l)}(\lambda^2) \underset{\lambda \to \infty}{\sim} \frac{2l+1}{2\pi\lambda^3} \int_M dM(r)\, W_c^{(n)}(r)\, \sigma^{-1}(r).$$

Making then in the integral (188) the scale transformation $\lambda \to |z|^{1/2}\lambda$, we can readily show that the asymptotic behavior of the coefficients $f_n^{(l)}$ has the form

$$f_n^l(z) \underset{|z| \to \infty}{\sim} \frac{i(2l+1)}{4\pi\sqrt{z}} \int_M dM(r)\, W_c^{(n)}(r)\, \sigma^{-1}(r). \tag{191}$$

Therefore, the series (187) is asymptotic as $|z| \to \infty$. It gives the required expansion of the function Sp $R_c^l(z)$.

Spectral identities

Suppose that the Hamiltonian H^l with rapidly decreasing two-body potentials has N_l bound states of multiplicity m_i with energies $E_i < 0$. We denote by ε_0 the smallest of the binding energies of all the two-body Hamiltonians (71): $\varepsilon_0 = \min_A(-\varkappa_A^2)$. If these Hamiltonians do not have bound states, we set $\varepsilon_0 = 0$.

We introduce the functions

$$h_n^l(z) = \mathrm{Sp}\, R_c^l(z) + (1 - \delta_{n0}) \sum_{k=1}^{n} f_n^{(l)}(z) z^{-n}, \quad n = 0, 1 \ldots$$

They are analytic in the z plane with a cut (ε_0, ∞) and have simple poles with residues $-m_i$ at the points $z = E_i$. As $|z| \to \infty$, we have, in accordance with (187) and (191), $h_n^{(l)}(z) \sim |z|^{-n-3/2}$. As follows from the trace formula (181) and the representation (188), the discontinuities of these functions across the cut are

$$h_n^{(l)}(z)\Big|_{E-i0}^{E+i0} = \mathrm{Sp}\,(S^* \partial_E S)_c(E) + (1 - \delta_{n0})\,\theta(E) \times \pi i \sum_{k=1}^{n} E^{-k} \partial_E [E^2 Q_k^{(l)}(E)], \quad E > \varepsilon_0. \tag{192}$$

We now consider the integral of the function $z^n h_n^{(l)}(z)$ around a contour that surrounds the cut (ε_0, ∞). On the one hand, the contour can be deformed to the cut, on which the integrand can be expressed in terms of the S matrix by Eq. (192). On the other hand, the contour can be closed by a circle of large radius, and the integral can be calculated from the residues at the points $z = E_i$. As a result, we obtain the required spectral identities:

1) $n = 0$ (Levinson's formula):

$$\pi = \sum_{k=1}^{N_l} m_k = \Omega_l(\varepsilon_0);$$

2) $n = 1, 2, \ldots$:

$$2\pi i \sum_{k=1}^{N_l} m_k E_k^n = -\int_{\varepsilon_0}^{\infty} dE \left\{ E^n \operatorname{Sp}(S^* \partial_E S)_c(E) + \pi i \theta(E) \sum_{k=1}^{n} E^{n-k} \partial_E [E^2 Q_k(E)] \right\},$$

where the function Ω_l can be expressed in terms of the S matrix:

$$\Omega_l(\varepsilon_0) = \frac{1}{2i} \int_{\infty}^{\varepsilon_0} \operatorname{Sp}(S^* \partial_E S)_c(E)\, dE.$$

The coefficients $Q_k^{(l)}$ are the integrals (189), which contain powers of the potentials and their derivatives. Note that in the case of zero total orbital angular momentum ($l = 0$) the weight factor (190) can be calculated explicitly. In the Dragt coordinates of the point r, it can be expressed in terms of Bessel functions and the hypergeometric function ${}_3F_2$:

$$\mu_l(r, \lambda)\Big|_{l=0} = \frac{2}{\pi} \left(\frac{2}{\lambda\rho}\right)^2 \sum_{n=0}^{\infty} J_{2n+2}^2(\lambda\rho)(n+1)^2 \times {}_3F_2(-n, n+2, 1/2; 3/2, 1; \sin^2\psi).$$

APPENDIX

We give here a list of formulas that describe the properties of the Wigner functions and some identities that we have used in the main text. A detailed exposition of the theory of Wigner functions can be found in Refs. 15 and 25.

Suppose that the rotations $g \in SO(3)$ are characterized by the Euler angles $(\phi^1\phi^2\phi^3)$. The Wigner functions have the form

$$D_{mn}^l(g) = e^{im\phi^1} d_{mn}^l(\cos\phi^2) e^{im\phi^3} \quad (m, n = -l, -l+1, \ldots, l),$$

where the factor d_{mn}^l can be expressed in terms of Jacobi polynomials:[16]

$$d_{mn}^l(x) = 2^{-m} \left[\frac{(l+m)!(l-m)!}{(l+n)!(l-n)!}\right]^{1/2} \times (1-x)^{\frac{m-n}{2}} (1+x)^{\frac{m+n}{2}} P_{l-m}^{(m-n, m+n)}(x). \tag{A.1}$$

They are the common eigenfunctions of the operators $\hat{L}^2$ and $\hat{L}_z$ in (34) and (38), normalized in accordance with the measure (26):

$$\int_{SO(3)} dg D_{mn}^l(g) \overline{D_{m'n'}^{l'}(g)} = (2l+1)^{-1} \delta_{ll'} \delta_{mm'} \delta_{nn'},$$

$$\hat{L}^2 D_{mn}^l = l(l+1) D_{mn}^l, \quad \hat{L}_z D_{mn}^l = m D_{mn}^l.$$

We denote by $D^l(g)$ the matrix formed from the Wigner functions $D_{mn}^l(g)$. For the components (34) of the operator of the total orbital angular momentum we have the relations $\hat{L}_i D^l(g) = D^l(g)(\hat{L}_i)_l$, where $(\hat{L}_i)_l$ are numerical matrices of rank $2l+1$ with elements

$$\begin{aligned} [(\hat{L}_1)_l]_{mn} &= +a(n)\delta_{m+1,n} - a(-n)\delta_{m-1,n}, \\ [(\hat{L}_2)_l]_{mn} &= -i(a(n)\delta_{m+1,n} - a(-n)\delta_{m-1,n}), \\ [(\hat{L}_3)_l]_{mn} &= m\delta_{mn}, \quad a(n) = \frac{1}{2}[(l+n)(l-n+1)]^{1/2}. \end{aligned} \tag{A.2}$$

The matrices $D^l(g)$ are unitary: $[D^l(g)]^{-1} = D^l(g^{-1}) = [D^l(g)]^*$. The following formula gives the decomposition of the tensor product $D^{l_1} \otimes D^{l_2}$:

$$D_{m_1n_1}^{l_1}(g) D_{m_2n_2}^{l_2}(g) = \sum_{l=|l_1-l_2|}^{l_1+l_2} \sum_{m,n=-l}^{l} \langle l_1m_1l_2m_2|lm\rangle \langle l_1n_1l_2n_2|ln\rangle D_{mn}^l(g). \tag{A.3}$$

From it we obtain an expression for the integral of a product of three Wigner functions:

$$\int_{SO(3)} dg D_{mn}^l(g) \overline{D_{m_1n_1}^{l_1}(g)} D_{m_2n_2}^{l_2}(g) = (2l+1)^{-1} \langle l_1m_1l_2m_2|lm\rangle \langle l_1n_1l_2n_2|ln\rangle. \tag{A.4}$$

The Wigner functions determine the transformation of the spherical functions under rotation:

$$Y_{lm}(g\hat{e}) = \sum_{n=-l}^{l} D_{mn}^l(g) Y_{ln}(\hat{e}).$$

In particular, for $\hat{e} = \mathring{e}_3 = (0,0,1)$ we obtain the relation

$$Y_{lm}(g\mathring{e}_3) = \left(\frac{2l+1}{4\pi}\right)^{1/2} D_{m0}^l(g). \tag{A.5}$$

Finally, we give two identities that were used in the proof of the expression (61):

$$\begin{aligned} &\sum_m \langle l_10l_2m|lm\rangle \langle l_1'0l_2m|lm\rangle = \frac{2l+1}{2l_1+1} \delta_{l_1l_1'}, \\ &\sum_{l_1} (-1)^{m+n} \frac{2l_1+1}{2l+1} \langle l_2ml_10|lm\rangle \langle l_2nl_10|ln\rangle = \delta_{mn}. \end{aligned} \tag{A.6}$$

These identities are not given directly in such a form in the textbooks. They follow from the standard orthogonality relations for the Clebsch–Gordan coefficients.

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Translated by Julian B. Barbour