Calculation of the triton binding energy in relativistic quantum mechanics

F. M. Lev

Institute of Applied Mathematics, Khabarovsk Branch, Far East Branch of the USSR Academy of Sciences, Khabarovsk

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The use of relativistic quantum mechanics to calculate the triton binding energy is reviewed. An equation is derived which can be used to calculate the relativistic correction to the binding energy in first order in v^2/c^2 (where v is the nucleon speed in the triton and c is the speed of light) if the relativistic binding energy and the partial-wave expansion of the Faddeev components of the nonrelativistic wave function are known. A partial-wave expansion is made of the relativistic Faddeev equations, the solution of which determines the triton binding energy and wave function without expanding in powers of v^2/c^2.

INTRODUCTION

The problem of the deficiency in the triton binding energy

One of the fundamental problems in the physics of few-nucleon systems is the problem of the deficiency in the binding energy of the lightest nuclei ^{3}H, ^{3}He, and ^{4}He. This problem is essentially the following: when only realistic nucleon–nucleon pair interactions are taken into account in nonrelativistic quantum mechanics, the calculated binding energy is smaller in absolute value than the corresponding experimental value (moreover, there is a discrepancy between theory and experiment also for such low-energy parameters as the charge radii of the nuclei, the doublet nucleon–deuteron scattering length, and so on). Since in the case of the helium nucleus the problem is complicated by the need to include the Coulomb interaction, it is clear that the most accurate calculations can be carried out for the triton. By now, the results obtained on this problem can be considered practically exhaustive. For example, in Refs. 1 and 2 by groups at Los Alamos and Sandia, calculations have been carried out in the 34-channel approximation for the Faddeev equations, and for various realistic nucleon–nucleon potentials the result for the absolute value of the triton binding energy has turned out to be smaller than the experimental value of 8.48 MeV by 0.8–1.1 MeV. The 34-channel approximation corresponds to the inclusion of only those elements of the basis $\{LSJl_j\}$ (see below) for which $J\leqslant 4$. An estimate of the contribution of elements of this basis with $5\leqslant J\leqslant 8$ has shown[3] that this contribution is no larger than 10 keV. The 34-channel calculation has also been carried out for the Bonn potential,[4] and the result of 8.35 MeV is close to the experimental value. The success of the Bonn potential is also discussed in Ref. 5, but in other studies (see, for example, Refs. 6–8) it is suggested that the Bonn potential cannot really be considered realistic. In particular, the larger binding energy compared with the results for other potentials is related mainly to the smaller tensor forces in the Bonn potential, but this contradicts the experimental data on the deuteron magnetic form factor, and, in addition, the parameters of the Bonn potential were fitted using only *pn* scattering data, neglecting the data on *pp* scattering. A recent measurement of the asymmetry coefficient in *pn* scattering at 10.03 MeV (Ref. 9) showed that the results are described well by the Paris potential, somewhat worse by the Nijmegen potential, and poorly by the Bonn potential. Therefore, the problem of the deficiency in the triton binding energy remains unsolved. Recently, Mukhtarova[10] carried out calculations of the triton binding energy using the method of hyperspherical harmonics with accuracy at least as good as in the 34-channel Faddeev calculations, and the results of Refs. 1 and 2 were confirmed for the known realistic potentials.

In order to explain this discrepancy between theory and experiment, various authors have most often cited the contribution of three-nucleon interactions due to two-pion exchange. There are also studies in which the problem of the binding-energy deficiency is attributed to the need to include quark and other non-nucleon degrees of freedom in nuclei, and, finally, there are a very few studies in which relativistic effects are included.

In recent 34-channel calculations of the triton binding energy taking into account three-nucleon interactions[2,11] it was shown that these interactions can lead to an additional attraction of order 2 MeV, and therefore the triton is even overbound. However, the result is strongly dependent on the choice of cutoff parameter of the pion–nucleon form factor. By changing this parameter one can even obtain complete agreement between theory and experiment; however, as yet there are no reliable theoretical grounds for preferring any particular value of this parameter. For example, the author of Ref. 12 thinks that the contribution of three-nucleon interactions does not exceed 0.3 MeV. The studies on three-nucleon interactions carried out from 1978 to 1988 were reviewed in Ref. 13. As far as the contribution of quark degrees of freedom is concerned, recently a group at ITEP carried out a calculation using the compound quark bag (CQB) model and obtained the value 8.1 MeV (Ref. 14), but only five channels were included. The use of the CQB method to calculate other low-energy parameters of few-nucleon systems was discussed in Refs. 15–18 and references cited therein. Yet another variant of the realistic nucleon–nucleon potential which effectively includes the contribution of quark degrees of freedom has been developed at the Nuclear Physics Institute of Moscow State University.[19–22] In calculations of various low-energy parameters using this model the contribution of large relative momenta plays a much more important role than in other realistic models, so that the contribution of higher partial waves and of relativistic effects is more important.[23,24] Up to now the triton binding energy has been calculated in the latter model only in the 5-

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channel approximation with simplifying assumptions,[25] which is clearly insufficient. We also note that on the basis of an analysis of the nucleon–nucleon data up to energies of 1 GeV, the authors of Ref. 26 conclude that quark degrees of freedom have not yet manifested themselves in nucleon–nucleon interactions.

On the isolation of the contribution of relativistic effects to three-nucleon observables

The studies in which the relativistic correction (RC) to the triton binding energy is calculated will be discussed at the end of this article; for now, we note that the very possibility of separately taking into account relativistic effects must be justified, since in quantum field theory the inclusion of relativity leads simultaneously to the need to include the production of new particles. It seems to us, however, that in the case of few-nucleon systems at low energies, and particularly three-nucleon systems, the contribution of effects from relativistic kinematics can be reliably isolated in view of the following considerations.

First of all, it is well known that in reasonable relativistic classical and quantum theories such as classical and quantum electrodynamics, the general theory of relativity (GTR), a quantum theory with scalar and pseudoscalar coupling, quantum chromodynamics (QCD), and so on, it is possible to restrict oneself to a finite number of degrees of freedom not only in zeroth, but also in first order on $1/c^2$ (and in the GTR even in order $1/c^4$) (Refs. 27–31), so that one can attempt to first calculate the RC to the triton binding energy only in first order in $1/c^2$. It is still not possible to explicitly calculate the nucleon–nucleon potential in order $1/c^2$ from the QCD Lagrangian, but there is no reason to doubt that this potential exists. Another argument is that at energies below the π-meson production threshold all nonnucleon degrees of freedom can perhaps be taken into account by using certain effective two- and three-nucleon interactions. Arguments in favor of this were presented in Refs. 32–34, but there is still no rigorous proof. Finally, yet another argument in favor of the effective inclusion of only the nucleon degrees of freedom at low energies (see Refs. 35–37) is that inelasticities in the lowest partial waves for nucleon–nucleon scattering begin to appear only at energies of 1–1.5 GeV, i.e., way above the π-meson production threshold. From the viewpoint of quantum field theory this situation is unusual and implies that at low energies the effects of relativistic kinematics begin to appear earlier than inelastic effects.

More detailed arguments are given in Refs. 32–38, but it seems to us that the above are sufficient to convince the reader that at low energies the natural apparatus for calculating relativistic effects in few-nucleon systems is relativistic quantum mechanics (RQM), i.e., an approach in which, as in ordinary quantum mechanics, the number of particles is conserved, but the invariance group is the Poincaré group rather than the Galilean group. A detailed review of RQM can be found in, for example, the lectures of the author,[38] where there is also a discussion of the connection between RQM and studies of how the relativistic quantum-mechanical description can be obtained from quantum field theory. Since the main goal of the present study is to derive explicit expressions for calculating the triton binding energy in RQM, here we shall not discuss the other approaches and shall present a condensed version of RQM sufficient for our purposes.

As usual, by the relativistic invariance of a quantum system we shall mean that the wave function of this system transforms according to a unitary representation of the Poincaré group in some Hilbert space. This representation can be specified by means of ten generators, i.e., self-adjoint operators satisfying the commutation relations of the Lie algebra of the Poincaré group. The main goal of RQM is the derivation of explicit equations for all ten generators such that, together with the needed commutation relations, the cluster separability condition is also satisfied for each generator. In the special case of there particles this condition implies that if, for example, the entire interaction involving particle 3 is switched off, then each generator becomes the sum of the corresponding generators for two systems: the system in which particles 1 and 2 interact, and the system consisting of the single free particle 3. References 39 and 40 can be consulted for more details about cluster separability. We note that the separability of all ten generators is, in general, a much stronger requirement than only separability of the S matrix. It can be argued (see Ref. 38, for example) that the relativistic three-particle equations derived in Refs. 41–43 and in previous studies on the basis of the diagrammatic and other approaches are inconsistent with separability in all ten generators.

To calculate the triton binding energy it is sufficient to have only an explicit equation for the mass operator of the three-nucleon system. Various expressions for the three-particle mass operator have been derived in RQM by Coester,[44,45] Sokolov,[40] Bakker, Kondratyuk, and Terent'ev[46] (who, in particular, improved on the results of Berestetskiĭ and Terent'ev[47]), Gudavadze, Kopaleishvili, and Machavariani[48] (who studied only the spinless case), and the present author.[49] Although various forms of the dynamics and different choices of the momentum and spin variables were used in these studies, in our opinion a very important "experimental" fact is that (neglecting the error in Ref. 44) all the operators in question are unitarily equivalent (which will be shown in a separate review by Kondratyuk and the present author), so that they lead to identical physical results. These three-particle mass operators are also obtained naturally using the general approach based on the technique of Sokolov packing operators,[40,50] which prescribes how the generators of a representation can be constructed for a system of any given number of particles,[51–55] and in first order in $1/c^2$ this result is consistent with the approaches of Foldy and Krajcik[56] and Gaĭda[57] (see Refs. 38, 58, and 59). As was shown in Refs. 38, 54, and 55, the solutions found in Refs. 40, 44–49, and 51–55 are consistent with quantum field theory in the sense that the choice of packing operators in them is the same as in quantum field theory, and the linear addition of interactions in the three-particle mass operator corresponds to the representation of the full interaction Lagrangian in the form of a sum of the Lagrangians of the various interactions. Finally, the solution found for the three-particle mass operator is obtained naturally if in addition to Poincaré invariance there is also invariance under supersymmetry transformations.[60]

In the case of four or more particles the general solution for the representation generators contains additional arbitrary functions,[50–52] and a unique solution exists only in first

order in $1/c^2$ (Ref. 59). Therefore, the three-nucleon case is special, since here the conditions for relativistic invariance, cluster separability, and correspondence to quantum field theory allow an exact solution to be found.

In view of the above discussion, it seems reasonable to calculate the triton binding energy using the exact equation for the three-particle mass operator derived in RQM. The main goal of the present study is to derive explicit expressions which, in principle, allow this calculation to be carried out in the Faddeev approach. The derivation of these equations is quite awkward, but wherever calculations are omitted we give everything needed for the interested reader to carry out the calculations independently. We discuss the complexities resulting from the inclusion of relativity in detail.

The relativistic calculation of the triton binding energy also seems to us to be very important because the standard nonrelativistic approach, strictly speaking, is logically inconsistent. In fact, in the standard approach the mass operator of a three-particle system is written as (the index *nr* stands for "nonrelativistic")

$$\hat{M}^{nr} = T^{nr} + \sum_{\alpha} v_{\alpha}^{nr}, \tag{1}$$

where T^{nr} is the nonrelativistic kinetic-energy operator of the three-particle system in its c.m. frame, α takes the values 12, 31, and 23, and the interaction operators v_{α}^{nr} are chosen so as to correctly describe the corresponding two-particle data. Here v_{α}^{nr} includes terms of relativistic origin—the spin–orbit and spin–spin terms (and, in the case of the Bonn potential, also a spinless term quadratic in the momentum). However, then the theory becomes partially relativistic, and the simple addition of interactions as in (1) contradicts the relativistic commutation relations already in first order in $1/c^2$. This can be verified using the explicit form of the three-particle mass operator in this approximation (see Refs. 38 and 58 and Sec. 2 of the present study).

Above, we discussed the possibility of expanding in powers of $1/c^2$, and in view of what we have just said this possibility requires clarification. The point is that if we want to use in the relativistic calculation results already obtained in the standard approach, we must express the relativistic three-particle mass operator in terms of the same operators v_{α}^{nr} as above [in this case the relation is more complicated than in (1)]. After this, all functions of $\mathbf{q}$ and $\mathbf{Q}$ giving rise to this relation can be expanded in powers of q/mc and Q/mc, where $\mathbf{q}$ and $\mathbf{Q}$ are the relative momenta of the nucleons [see Eqs. (14) and (31)], $q = |\mathbf{q}|$, $Q = |\mathbf{Q}|$, and m is the nucleon mass. Here we leave the operators v_{α}^{nr} unchanged. Of course, in this procedure only some of the terms of order $1/c^2$ are explicitly isolated (since such terms are also contained in the operators v_{α}^{nr}), and the expansion in powers of $1/c^2$ is here explicitly realized as an expansion in powers of $1/m^2$. Therefore, we are considering an expansion in powers of $1/m^2$ such that the result of the standard approach is obtained as the zeroth approximation. This makes sense if the triton wave function (WF) obtained in the standard approach falls off sufficiently rapidly for large q and Q.

In this study we use the system of units with $\hbar = c = 1$, and the metric tensor in Minkowski space has the nonzero components $g_{00} = -g_{11} = -g_{22} = -g_{33} = 1$. The commutation relations for the generators of representations of the Poincaré group are realized in the form

$$\left.\begin{aligned}&[P_\mu,\ P_\nu] = 0; \quad [M_{\mu\nu},\ P_\rho] = -\mathrm{i}\,(g_{\mu\rho}P_\nu - g_{\nu\rho}P_\mu);\\ &[M_{\mu\nu},\ M_{\rho\sigma}] = -\mathrm{i}\,(g_{\mu\rho}M_{\nu\sigma} + g_{\nu\sigma}M_{\mu\rho} - g_{\mu\sigma}M_{\nu\rho} - g_{\nu\rho}M_{\mu\sigma}),\end{aligned}\right\} \tag{2}$$

where $\mu,\nu = 0,1,2,3$; P_μ are the 4-momentum operators, and $M_{\mu\nu}$ are the 4-dimensional angular-momentum operators.

1. A SYSTEM OF TWO INTERACTING PARTICLES IN RQM AND THE PRINCIPLE OF MINIMAL RELATIVITY

Realization of a representation of the Poincaré group for a system of two free particles

As usual, we shall assume that an elementary particle of mass m is described by a unitary irreducible representation of the Poincaré group. The methods of realizing such a representation are well known (see, for example, Ref. 61). We choose the realization in the space of functions $\varphi(\mathbf{p},\sigma)$ such that

$$\sum_{\sigma}\int |\,\varphi\,(\mathbf{p},\ \sigma)\,|^2\,d^3\mathbf{p} < \infty, \tag{3}$$

where $\mathbf{p}$ is the particle momentum and σ is the spin projection. We consider only the case of spin-$\frac{1}{2}$ particles, when σ takes the values $\pm\frac{1}{2}$. Here the generators of the representation, which satisfy the commutation relations (2), can be realized in the form (see, for example, Ref. 61)

$$\mathbf{P} = \mathbf{p}, \quad E = (m^2 + \mathbf{p}^2)^{1/2}, \quad \mathbf{M} = \mathbf{l}\,(\mathbf{p}) + \mathbf{s},$$
$$\mathbf{N} = -\mathrm{i}\,(m^2+\mathbf{p}^2)^{1/4}\frac{\partial}{\partial\mathbf{p}}(m^2+\mathbf{p}^2)^{1/4} + \frac{\mathbf{s}\times\mathbf{p}}{m+(m^2+\mathbf{p}^2)^{1/2}}, \tag{4}$$

where $\mathbf{P}$ is the ordinary momentum operator, $\mathbf{p}$ is the operator of multiplication by $\mathbf{p}$, E is the energy operator, $\mathbf{l}(\mathbf{p}) = -i\mathbf{p}\times\partial/\partial\mathbf{p}$ is the orbital angular-momentum operator, $\mathbf{s}$ is the spin operator, and

$$\mathbf{M} = \{M^{23},\ M^{31},\ M^{12}\}, \quad \mathbf{N} = \{M^{01},\ M^{02},\ M^{03}\}.$$

If we take the volume element in momentum space to be not $d^3\mathbf{p}$, but the relativistically invariant measure $d\nu(\mathbf{p}) = d^3\mathbf{p}/2\omega(\mathbf{p})$, with $\omega(\mathbf{p}) = (m^2+\mathbf{p}^2)^{1/2}$, i.e., instead of (3) we consider the space of functions $\varphi(\mathbf{p},\sigma)$ such that

$$\sum_{\sigma}\int |\,\varphi\,(\mathbf{p},\ \sigma)\,|^2\,d\nu\,(\mathbf{p}) < \infty, \tag{3'}$$

then, as is easily seen, the form of the operators $\mathbf{P}$, E, and $\mathbf{M}$ remains unchanged, but the boost operator will have the form

$$\mathbf{N} = -\mathrm{i}\omega\,(\mathbf{p})\frac{\partial}{\partial\mathbf{p}} + \frac{\mathbf{s}\times\mathbf{p}}{m+\omega\,(\mathbf{p})}. \tag{4'}$$

Let us now consider a system of two free particles. Since we shall neglect the proton–neutron mass difference, it is sufficient to restrict ourselves to the case of particles of identical mass m. It can be assumed that, by definition, the particles do not interact with each other if their common WF transforms like the tensor product of the corresponding one-particle representations. Here, if the particles are identical, the WF is assumed to be antisymmetrized. Therefore, the representation describing the system of two free particles 1 and 2 can be realized, for example, in the space of functions $\varphi(\mathbf{p}_1,\sigma_1,\mathbf{p}_2,\sigma_2)$ such that

$$\sum_{\sigma_1\sigma_2}|\,\varphi\,(\mathbf{p}_1,\ \sigma_1,\ \mathbf{p}_2,\ \sigma_2)\,|^2\,d\nu\,(\mathbf{p}_1)\,d\nu\,(\mathbf{p}_2) < \infty, \tag{5}$$

and each of the representation generators is equal to the sum of the corresponding one-particle generators of the form (4) (for $\mathbf{P}$, E, $\mathbf{M}$) and (4′) (for $\mathbf{N}$).

Instead of the variables $\mathbf{p}_1$ and $\mathbf{p}_2$ describing the separate particle momenta, we introduce the variables $\mathbf{P}$ and $\mathbf{k}$:

$$\mathbf{P}=\mathbf{p}_1+\mathbf{p}_2, \quad \mathbf{k}=\mathbf{p}_1-\frac{\mathbf{P}}{M}\omega(\mathbf{p}_1)+\frac{(\mathbf{P}\mathbf{p}_1)\mathbf{P}}{M(M+E)}, \tag{6}$$

where $E=\omega(\mathbf{p}_1)+\omega(\mathbf{p}_2)$ and $M=(E^2-\mathbf{P}^2)^{1/2}=2\omega(\mathbf{k})$. Then, instead of the realization of the representation in the space of functions satisfying the condition (5), we obtain the realization in the space H—in the space of functions $\varphi(\mathbf{P},\mathbf{k},\sigma_1,\sigma_2)$ such that

$$\sum_{\sigma_1\sigma_2}\int|\varphi(\mathbf{P},\mathbf{k},\sigma_1,\sigma_2)|^2\frac{d^3\mathbf{k}}{M^2}\frac{d^3\mathbf{P}}{\left(1+\frac{\mathbf{P}}{M^2}\right)^{1/2}}<\infty. \tag{7}$$

We introduce the operator

$$\mathcal{U}=\left(1+\frac{\mathbf{P}^2}{M^2}\right)^{1/4}\gamma(\mathbf{P},\sigma_1,\mathbf{k})\,\gamma(\mathbf{P},\sigma_2,-\mathbf{k}), \tag{8}$$

where

$$\gamma(\mathbf{P},\boldsymbol{\sigma},\mathbf{k})=\frac{(E+M)(\omega(\mathbf{k})+m)+\mathbf{P}\mathbf{k}+i\boldsymbol{\sigma}(\mathbf{P}\times\mathbf{k})}{\{2(E+M)(\omega(\mathbf{k})+m)(E\omega(\mathbf{k})+Mm+\mathbf{P}\mathbf{k})\}^{1/2}}; \tag{9}$$

$M=2\omega(\mathbf{k})$, $E=(M^2+\mathbf{P}^2)^{1/2}$, and $\boldsymbol{\sigma}$ are the Pauli matrices. Here it is assumed that in Eq. (8) the operator $\boldsymbol{\sigma}_i$ $(i=1,2)$ acts only on the corresponding variable σ_i.

Let $\tilde{H}$ be the space of functions $\tilde{\varphi}(\mathbf{P},\mathbf{k},\sigma_1,\sigma_2)$ such that

$$\sum_{\sigma_1\sigma_2}\int|\tilde{\varphi}(\mathbf{P},\mathbf{k},\sigma_1,\sigma_2)|^2\frac{d^3\mathbf{k}}{M^2}d^3\mathbf{P}<\infty. \tag{10}$$

Then it is easy to see that $\mathcal{U}^{-1}$ is a unitary operator from H to $\tilde{H}$, and by direct calculation it can be verified that the action of the generators of this two-particle representation in $\tilde{H}$ has the form

$$\mathbf{P}=\mathbf{P}, \quad E=(M^2+\mathbf{P}^2)^{1/2}, \quad \mathbf{M}=\mathbf{l}(\mathbf{P})+\mathbf{S},$$
$$\mathbf{N}=-i(M^2+\mathbf{P}^2)^{1/4}\frac{\partial}{\partial\mathbf{P}}(M^2+\mathbf{P}^2)^{1/4}+\frac{\mathbf{S}\times\mathbf{P}}{M+(M^2+\mathbf{P}^2)^{1/2}}, \tag{11}$$

where $\mathbf{S}=\mathbf{l}(\mathbf{k})+\mathbf{s}_1+\mathbf{s}_2$, and $\mathbf{s}_i$ $(i=1,2)$ is the spin operator of the corresponding particle. This calculation is rather complicated, and a simpler (and more general) method of deriving (11) is to use the direct-integral formalism (see, for example, Refs. 38, 54, and 55).

Comparing Eqs. (3) and (4) on the one hand, and Eqs. (10) and (11) on the other, we can conclude that the transformation to the space $\tilde{H}$ in the two-particle case can be interpreted as a splitting of the variables into external and internal ones: the role of the external variable is played by $\mathbf{P}$, and that of the internal variables is played by $\mathbf{k}$, σ_1, and σ_2. Here M is the mass operator of the compound system, and $\mathbf{S}$ is its spin operator. These operators act only in H_{int}, the space of functions $\chi(\mathbf{k},\sigma_1,\sigma_2)$ such that

$$\sum_{\sigma_1\sigma_2}\int|\chi(\mathbf{k},\sigma_1,\sigma_2)|^2\frac{d^3\mathbf{k}}{4\omega(\mathbf{k})^2}<\infty. \tag{12}$$

Returning to the space H, we find that the action of the representation generators Γ^i $(i=1,2,\ldots,10)$ in this space has the form

$$\Gamma^i=\mathcal{U}\tilde{\Gamma}^i\mathcal{U}^{-1}, \tag{13}$$

where $\tilde{\Gamma}^i$ are the generators (11) in $\tilde{H}$.

In the nonrelativistic limit the operator $\mathcal{U}$ obviously becomes unity, and the vector $\mathbf{k}$ becomes the vector $\mathbf{q}$, which is defined by the usual expression

$$\mathbf{q}=\tfrac{1}{2}(\mathbf{p}_1-\mathbf{p}_2). \tag{14}$$

Below, we shall need the expression relating $\mathbf{k}$ and $\mathbf{q}$ in first order in $1/m^2$:

$$\mathbf{k}=\mathbf{q}-\frac{(\mathbf{P}\mathbf{q})\mathbf{P}}{8m^2}. \tag{15}$$

Introduction of the interaction into the two-particle system and the principle of minimal relativity

It is easily found from (11) that if in these equations the operator M is replaced by some operator $\hat{M}$ in H_{int} which commutes with $\mathbf{S}$, the resulting new set of generators $\hat{\tilde{\Gamma}}^i$ will also satisfy the necessary commutation relations. Therefore, as is clear from (13), the simplest way of introducing the interaction into the two-particle system is to replace the operators Γ^i by the operators

$$\hat{\Gamma}^i=\mathcal{U}\hat{\tilde{\Gamma}}^i\mathcal{U}^{-1}. \tag{16}$$

This technique was first used by Bakamjian and Thomas.[62] In the general case the operators $\mathcal{U}$ also must be replaced by certain operators $\hat{\mathcal{U}}$ including the interaction. However, it can be shown[58,38] that for particles of equal mass Eq. (16) is consistent with quantum field theory in order $1/m^2$, and therefore, at least in this approximation, the introduction of the interaction as in (16) is valid.

The separability conditions (see above) require that when the interaction is switched on the operator $\hat{M}$ becomes M. In addition, from physical considerations it is necessary that the operator $\hat{M}$ have the same continuous spectrum as M, and that it have the corresponding point of the discrete spectrum in the channel with the deuteron quantum numbers. These conditions can be satisfied in many ways. It will be convenient for us to assume that $\hat{M}^2=M^2+w$, where w is an integral operator in H_{int}:

$$w\chi(\mathbf{k})=\int w(\mathbf{k},\mathbf{k}')\,\chi(\mathbf{k}')\frac{d^3\mathbf{k}'}{4\omega(\mathbf{k}')^2}. \tag{17}$$

In this equation it is assumed that the function $\chi(\mathbf{k})$ is a spinor in the variables σ_1 and σ_2, and that the kernel $w(\mathbf{k},\mathbf{k}')$ is an operator in these variables. The eigenvalue equation for the operator $\hat{M}^2$ in H_{int} has the form $\hat{M}^2\chi=\mu^2\chi$, and if we use the notation $\mu^2=4(m^2+\varkappa^2)$, this equation can be written as

$$\left(\frac{\mathbf{k}^2}{m}+\frac{w}{4m}\right)\chi=\frac{\varkappa^2}{m}\chi. \tag{18}$$

If instead of χ we introduce the function $\tilde{\chi}(\mathbf{k})=\chi(\mathbf{k})/2\omega(\mathbf{k})$ such that $\tilde{\chi}(\mathbf{k})$ belongs to the space of functions with nonrelativistic normalization $\int|\tilde{\chi}(\mathbf{k})|^2d^3\mathbf{k}<\infty$ (rather, the correspondingly equipped space), then for $\tilde{\chi}(\mathbf{k})$ we have the equation

$$\left(\frac{\mathbf{k}^2}{m}+v\right)\tilde{\chi}=\frac{\varkappa^2}{m}\tilde{\chi}, \tag{19}$$

where the kernels of the operators w and v are related as

$$w(\mathbf{k},\mathbf{k}')=16m\omega(\mathbf{k})\,\omega(\mathbf{k}')\,v(\mathbf{k},\mathbf{k}'). \tag{20}$$

If we consider a two-nucleon system in its c.m. frame

(i.e., for $\mathbf{P}=0$), the vectors $\mathbf{k}$ and $\mathbf{q}$ coincide and Eq. (19) has the form of the ordinary Schrödinger equation. The fact that the eigenvalue equation for the mass operator reduces to the ordinary Schrödinger equation when, for particles of equal mass, the interaction is introduced by adding it to the square of the mass operator is widely known as the principle of minimal relativity (see, for example, Refs. 63–66).

We shall see that in the relativistic case the two-particle interaction operator can be chosen on the basis of the description of the experimental data using the ordinary Schrödinger equation. Strictly speaking, in an experiment one determines the S matrix, which obeys the equation $S=W_+^* W_-$, where

$$W_\pm = s\text{-}\lim_{t\to\pm\infty} e^{i\hat{M}t}e^{-iMt} \tag{21}$$

(s-lim denotes the strong limit) and, using Eq. (19), one finds the operator $\widetilde{S}=\widetilde{W}_+^*\widetilde{W}_-$, where

$$\widetilde{W}_\pm = s\text{-}\lim_{t\to\pm\infty} e^{i\hat{M}^2t}e^{-iM^2t}. \tag{21'}$$

However, using the approach proposed by Faddeev,[67,68] it can easily be verified that if the kernel of the operator v is sufficiently smooth and falls off rapidly at large momenta, the invariance principle $W_\pm = \widetilde{W}_\pm$ is satisfied.

The quantity $\varkappa$ on the right-hand side of (19) is interpreted as the absolute value of the nucleon momentum in the nucleon c.m. frame. Meanwhile, in the standard approach the two-particle interaction operator v^{nr} must be chosen on the basis of the Schrödinger equation with $\mu-2m$ on the right-hand side instead of $\varkappa^2/m$. The relation between these quantities is obviously

$$\frac{\varkappa^2}{m}=(\mu-2m)\left(1+\frac{\mu-2m}{4m}\right). \tag{22}$$

It would therefore seem that the operator v cannot be identified with any known realistic operator v^{nr} (the Reid potential, the Paris potential, and so on). However, Coester made the following important observation.[69] Since the scattering phase shifts are known experimentally as functions of the laboratory energy E_{lab}, in the standard approach the quantity $\mu-2m$ is expressed in terms of E_{lab} according to the nonrelativistic expression $\mu-2m=E_{\text{lab}}/2$, and Eq. (19) is solved with $E_{\text{lab}}/2$ on the right-hand side. However, $\varkappa^2/m=E_{\text{lab}}/2$ is the exact relativistic expression. Therefore, no relativistic corrections arise in the scattering region. A correction arises at the single point of the discrete spectrum corresponding to the deuteron (here $\varkappa$ is imaginary). Substituting the experimental value of the deuteron binding energy -2.2246 MeV into (22) instead of $\mu-2m$, we find that at this point $\varkappa^2/m=-2.2233$ MeV. Therefore, the only change introduced by relativity in the two-particle problem is that the nucleon–nucleon potentials should be fitted not to the actual deuteron binding energy of -2.2246 MeV, but to the "effective" value, equal to -2.2233 MeV. It is clear that this difference is practically insignificant, and henceforth we shall assume that $v=v^{nr}$.

To conclude this section, we note that it follows from (11) and (16) and from the definition of $\widehat{\widetilde{\Gamma}}^i$ that the energy operator of the two-particle system can be written as $\widetilde{E}=E+V$, where E is the energy operator for the system of noninteracting particles, and the operator V is given by the expression

$$V=(4m^2+4\mathbf{k}^2+\mathcal{U}w\mathcal{U}^{-1}+\mathbf{P}^2)^{1/2}-(4m^2+4\mathbf{k}^2+\mathbf{P}^2)^{1/2}. \tag{23}$$

This is clear from the fact that the operator $\mathcal{U}$ commutes with the operators of multiplication by $\mathbf{k}$ and $\mathbf{P}$.

1. A SYSTEM OF THREE INTERACTING PARTICLES IN RQM AND THE RELATIVISTIC CORRECTION TO THE TRITON BINDING ENERGY IN THE $1/m^2$ APPROXIMATION

A representation of the Poincaré group describing a system of three particles 1, 2, and 3 with identical masses m can be realized in the space of functions $\varphi(p_1,\sigma_1,\mathbf{p}_2,\sigma_2,\mathbf{p}_3,\sigma_3)$ such that

$$\sum_{\sigma_1\sigma_2\sigma_3}\int |\varphi(\mathbf{p}_1,\sigma_1,\mathbf{p}_2,\sigma_2,\mathbf{p}_3,\sigma_3)|^2 \times dv(\mathbf{p}_1)\,dv(\mathbf{p}_2)\,dv(\mathbf{p}_3)<\infty. \tag{24}$$

Now we use $\mathbf{P}=\mathbf{p}_1+\mathbf{p}_2+\mathbf{p}_3$ to denote the total momentum of the three-particle system and $\mathbf{k}_i$ $(i=1,2,3)$ to denote the particle momenta in the c.m. frame, i.e., in the reference frame where $\mathbf{P}=0$. By analogy with Eq. (6) we set

$$\mathbf{k}_i=\mathbf{p}_i-\frac{\mathbf{P}}{M}\omega(\mathbf{p}_i)+\frac{(\mathbf{P}\mathbf{p}_i)\,\mathbf{P}}{M(E+M)}, \tag{25}$$

where now M denotes the free mass operator of the three-particle system and E is the corresponding energy operator: $E=\omega(\mathbf{p}_1)+\omega(\mathbf{p}_2)+\omega(\mathbf{p}_3)$, $M=(E^2-P^2)^{1/2}$. From (25) it follows that $\mathbf{k}_1+\mathbf{k}_2+\mathbf{k}_3=0$, as expected. In the variables $\mathbf{P}$, $\mathbf{k}_i$ $(i=1,2,3)$ the representation space is realized in the form of the space of functions $\varphi(\mathbf{P},\mathbf{k}_1,\mathbf{k}_2,\mathbf{k}_3,\sigma_1,\sigma_2,\sigma_3)$, such that

$$\sum_{\sigma_1\sigma_2\sigma_3}\int |\varphi(\mathbf{P},\mathbf{k}_1,\mathbf{k}_2,\mathbf{k}_3,\sigma_1,\sigma_2,\sigma_3)|^2 \times dv(\text{int})\,\frac{d^3\mathbf{P}}{\left(1+\frac{\mathbf{P}^2}{M^2}\right)^{1/2}}, \tag{26}$$

where

$$dv(\text{int})=dv(\mathbf{k}_1)\,dv(\mathbf{k}_2)\,dv(\mathbf{k}_3)\,\delta^{(3)}(\mathbf{k}_1+\mathbf{k}_2+\mathbf{k}_3). \tag{27}$$

We use $\widetilde{H}$ to denote the space of functions $\tilde{\varphi}(\mathbf{P},\mathbf{k}_1,\mathbf{k}_2,\mathbf{k}_3,\sigma_1,\sigma_2,\sigma_3)$ such that

$$\sum_{\sigma_1\sigma_2\sigma_3}\int |\tilde{\varphi}(\mathbf{P},\mathbf{k}_1,\mathbf{k}_2,\mathbf{k}_3,\sigma_1,\sigma_2,\sigma_3)|^2\,dv(\text{int})\,d^3\mathbf{P}<\infty, \tag{28}$$

and H_{int} to denote the space of functions $\chi(\mathbf{k}_1,\mathbf{k}_2,\mathbf{k}_3,\sigma_1,\sigma_2,\sigma_3)$, such that

$$\sum_{\sigma_1\sigma_2\sigma_3}|\chi(\mathbf{k}_1,\mathbf{k}_2,\mathbf{k}_3,\sigma_1,\sigma_2,\sigma_3)|^2\,dv(\text{int})<\infty. \tag{29}$$

Let us consider the case where particles 1 and 2 interact and particle 3 is free. then the representation of the Poincaré group is the tensor product of the representation describing the interacting system (12) and the representation describing the free particle 3. The direct-integral formalism[54,55] can be used to find a unitary operator $\mathcal{U}_{12,3}$ from H to $\widetilde{H}$ such that the generators of the representation in $\widetilde{H}$ have the "canonical" form [cf. (4) and (11)]:

$$\mathbf{P}_{12,3}=\mathbf{P},\quad E_{12,3}=(M_{12,3}^2+\mathbf{P}^2)^{1/2},$$

$$\mathbf{M}_{12,3}=\mathbf{M}=\mathbf{l}(\mathbf{P})+\mathbf{S},$$

$$\mathbf{N}_{12,3}=-\mathrm{i}(M_{12,3}^2+\mathbf{P}^2)^{1/4}\frac{\partial}{\partial\mathbf{P}}(M_{12,3}^2+\mathbf{P}^2)^{1/4}+\frac{\mathbf{S}\times\mathbf{P}}{M_{12,3}+(M_{12,3}^2+\mathbf{P}^2)^{1/2}}. \tag{30}$$

Here the spin operator of the three-particle system acts in H_{int} and has the standard form $\mathbf{S}=\mathbf{l}(\mathbf{k}_1)+\mathbf{l}(\mathbf{k}_2)+\mathbf{s}_1+\mathbf{s}_2+\mathbf{s}_3$ (of the three momenta $\mathbf{k}_i$ only two are independent), and the mass operator $M_{12,3}$ also acts only in H_{int} and is written as $M_{12,3}=M+V_{12}$, where V_{12} stands for the action of the operator (23), restricted to H_{int}. Let us give a more detailed description of how this restriction is defined.

By analogy with (6) we introduce the quantities

$$\mathbf{Q}_{12}=\mathbf{k}_1+\mathbf{k}_2,\quad \mathbf{k}_{12}=\mathbf{k}_1-\frac{\mathbf{Q}_{12}}{M_{12}}\omega(\mathbf{k}_1)+\frac{(\mathbf{Q}_{12}\mathbf{k}_1)\mathbf{Q}_{12}}{M_{12}(E_{12}+M_{12})}, \tag{31}$$

where $E_{12}=\omega(\mathbf{k}_1)+\omega(\mathbf{k}_2)$, $M_{12}=(E_{12}^2-\mathbf{Q}_{12}^2)^{1/2}$. Therefore, $\mathbf{Q}_{12}$ and $\mathbf{k}_{12}$ are, respectively, the total and relative momenta for the system $\{1,2\}$ in the c.m. frame of the three-particle system. In these variables the space H_{int} is realized as the space of functions $\chi(\mathbf{Q}_{12},\mathbf{k}_{12},\sigma_1,\sigma_2,\sigma_3)$ such that

$$\sum_{\sigma_1\sigma_2\sigma_3}\int|\chi(\mathbf{Q}_{12},\mathbf{k}_{12},\sigma_1,\sigma_2,\sigma_3)|^2\times\rho(Q_{12},k_{12})\,d^3\mathbf{Q}_{12}\,d^3\mathbf{k}_{12}<\infty, \tag{32}$$

where $Q_{12}=|\mathbf{Q}_{12}|$, $k_{12}=|\mathbf{k}_{12}|$, and $\rho(Q_{12},k_{12})=[4\omega(k_{12})\omega(Q_{12})(4m^2+4k_{12}^2+Q_{12}^2)^{1/2}]^{-1}$. We also use $\mathscr{U}_{12}$ to denote the operator $\mathscr{U}$ defined in the preceding section with $\mathbf{P}$ and $\mathbf{k}$ in it replaced by $\mathbf{Q}_{12}$ and $\mathbf{k}_{12}$, respectively, and w_{12} to denote the operator which acts through the variables $\mathbf{k}_1$, $\mathbf{k}_2$, σ_1, σ_2 in the same way that the operator w acts through $\mathbf{p}_1$, $\mathbf{p}_2$, σ_1, σ_2. In other words, the operator w_{12} does not act through the variables $\mathbf{Q}_{12}$, σ_3 and its action on functions of $\mathbf{k}_{12}$, σ_1, σ_2 is given by [cf. (17)]

$$w_{12}\chi(\mathbf{k}_{12})=\int w_{12}(\mathbf{k}_{12},\mathbf{k}'_{12})\chi(\mathbf{k}'_{12})\frac{d^3\mathbf{k}'_{12}}{4\omega(k'_{12})^2}, \tag{33}$$

where $w_{12}(\mathbf{k}_{12},\mathbf{k}'_{12})=w(\mathbf{k}_{12},\mathbf{k}'_{12})$. If we use $v_{12}(\mathbf{k},\mathbf{k}')$ to denote the kernel of the operator v determined from the experimental data for the system $\{1,2\}$ using Eq. (19), then instead of (20) we will have

$$w_{12}(\mathbf{k}_{12},\mathbf{k}'_{12})=16m\omega(k_{12})\,\omega(k'_{12})\,v_{12}(\mathbf{k}_{12},\mathbf{k}'_{12}). \tag{34}$$

Using these definitions, we can now write the operator V_{12} in the form

$$V_{12}=(4m^2+4k_{12}^2+\mathscr{U}_{12}w_{12}\mathscr{U}_{12}^{-1}+Q_{12}^2)^{1/2}-(4m^2+4k_{12}^2+Q_{12}^2)^{1/2}. \tag{35}$$

Now we use $\widetilde{\Gamma}^i(M_{12,3})$ $(i=1,2,\ldots,10)$ to denote the generators (30) in $\widetilde{H}$ as functions of $M_{12,3}$. Then the generators of this representation in H have the form

$$\Gamma_{12,3}^i=\mathscr{U}_{12,3}\widetilde{\Gamma}^i(M_{12,3})\mathscr{U}_{12,3}^{-1}. \tag{36}$$

The following question arises. How do we construct the generators of a representation for the case where all the particles have pairwise interactions with each other? Using the method of Sokolov packing operators,[40] this can be done as follows.[54,55] We use the notation

$$\hat{M}=M+\sum_\alpha V_\alpha, \tag{37}$$

where V_α for $\alpha=31$ and $\alpha=23$ are defined in complete analogy with V_{12}. Then we assume that from the operators $\mathscr{U}_{12,3}$, $\mathscr{U}_{31,2}$, and $\mathscr{U}_{23,1}$, defined for cases in which only one pair of particles interacts, we have constructed a unitary operator $\hat{\mathscr{U}}$ such that if only particles 1 and 2 interact with each other, $\hat{\mathscr{U}}$ becomes $\mathscr{U}_{12,3}$, and so on. Then the generators of the desired representation, which satisfy the separability conditions and the necessary commutation relations, can be written as

$$\hat{\Gamma}^i=\hat{\mathscr{U}}\hat{\widetilde{\Gamma}}^i(\hat{M})\hat{\mathscr{U}}^{-1}. \tag{38}$$

The operator (37) is thereby the three-particle mass operator discussed in the Introduction. It is important that the separability in all ten generators leads to the appearance of the operators v_α in the expression for $\hat{M}$ in the square root. This does not happen for the variants of the relativistic three-particle equations which have been derived up to now using the diagrammatic approach (see Refs. 41–43 and the literature cited therein). Of course, in the nonrelativistic limit the operators V_α become v_α. As far as the explicit form of the operators $\mathscr{U}_{12,3}$, $\hat{\mathscr{U}}$, and so on, is concerned, we shall need it only if we treat our three-particle system as a component of a more complex system (for example, in calculating the binding energy of an N-particle system for $N\geqslant 4$; Ref. 59). However, if we are interested in the three-body problem, the only thing needed to include only pairwise interactions is knowledge of the operator $\hat{M}$ as a function of the operators v_α. Here the problem of finding the triton binding energy reduces to the eigenvalue problem for the operator $\hat{M}$ in H_{int}:

$$\hat{M}\chi=(3m+\varepsilon)\chi, \tag{39}$$

where ε is the binding energy ($\varepsilon<0$).

Let us introduce the notation [cf. (8) and (9)]

$$W_{12}=\left(1+\frac{Q_{12}^2}{M_{12}^2}\right)^{1/4}w_{12}\left(1+\frac{Q_{12}^2}{M_{12}^2}\right)^{-1/4}; \tag{40}$$

$$U_{12}=\gamma(\mathbf{Q}_{12},\sigma_1,\mathbf{k}_{12})\,\gamma(\mathbf{Q}_{12},\sigma_2,-\mathbf{k}_{12}). \tag{41}$$

Then the operator (35) can be written as

$$V_{12}=(4m^2+4k_{12}^2+U_{12}W_{12}U_{12}+Q_{12}^2)^{1/2}-(4m^2+4k_{12}^2+Q_{12}^2)^{1/2} \tag{42}$$

and, as follows from (33) and (34), the action of the operator W_{12} can be written as

$$W_{12}\chi(\mathbf{Q}_{12},\mathbf{k}_{12})=4m\int\left(\frac{4m^2+4k_{12}^2+Q_{12}^2}{4m^2+4k_{12}'^2+Q_{12}^2}\right)^{1/4}\times\left[\frac{\omega(k_{12})}{\omega(k'_{12})}\right]^{1/2}v_{12}(\mathbf{k}_{12},\mathbf{k}'_{12})\chi(\mathbf{Q}_{12},\mathbf{k}'_{12})\,d^3\mathbf{k}'_{12}, \tag{43}$$

where it is assumed that the function $\chi(\mathbf{Q}_{12},\mathbf{k}_{12})$ is a spinor in the variables σ_1, σ_2, and σ_3. Of course, the operators U_α and W_α, which have the same form in the corresponding variables $\mathbf{Q}_\alpha$ and $\mathbf{k}_\alpha$, can be defined in a similar manner.

The mass operator of the three-nucleon system in the $1/m^2$ approximation

Since we want to calculate the order-$(1/m^2)$ correction to the mass operator (1) in the standard approach, we must first, as in the nonrelativistic case, choose as the two inde-

pendent momenta not $\{\mathbf{k}_\alpha, \mathbf{Q}_\alpha\}$ for some α, but $\{\mathbf{q}_\alpha, \mathbf{Q}_\alpha\}$, where [cf. (14)] $\mathbf{q}_{12} = (\mathbf{k}_1 - \mathbf{k}_2)/2$ and $\mathbf{q}_\alpha$ are defined similarly for other α. Then we must go from the space of functions satisfying the condition (32) to the space H_{int}^{nr} of functions $\Psi(\mathbf{q}_\alpha, \mathbf{Q}_\alpha, \sigma_1, \sigma_2, \sigma_3)$ such that

$$\sum_{\sigma_1\sigma_2\sigma_3} \int |\Psi(\mathbf{q}_\alpha, \mathbf{Q}_\alpha, \sigma_1, \sigma_2, \sigma_3)|^2 d^3\mathbf{Q}_\alpha d^3\mathbf{q}_\alpha. \quad (44)$$

The explicit form of all the operators in H_{int}^{nr} is easily obtained from the expressions [cf. (15)]

$$\mathbf{k}_\alpha = \mathbf{q}_\alpha - \frac{(\mathbf{Q}_\alpha\mathbf{q}_\alpha)\,\mathbf{Q}_\alpha}{8m^2}, \quad \left|\frac{\partial^3\mathbf{k}_\alpha}{\partial^3\mathbf{q}_\alpha}\right| = 1 - \frac{Q_\alpha^2}{8m^2}, \quad (45)$$

which are valid in first order in $1/m^2$. In particular, it is easy to show that in this approximation the action of the operator W_α (rather, of the corresponding unitarily transformed operator) in H_{int}^{nr} is given by the expression

$$W_\alpha = 4m\left(v_\alpha - \left[\frac{(\mathbf{Q}_\alpha\mathbf{q}_\alpha)}{8m^2}\left(\mathbf{Q}_\alpha\frac{\partial}{\partial\mathbf{q}_\alpha}\right), v_\alpha\right]\right), \quad (46)$$

where here and below, as usual, we use the symbol [...] for the commutator and {...} for the anticommutator. It is easy to see that in this approximation

$$U_\alpha = 1 - \frac{i}{8m^2}\boldsymbol{\sigma}_\alpha(\mathbf{Q}_\alpha \times \mathbf{q}_\alpha), \quad (47)$$

where $\boldsymbol{\sigma}_{12} = \boldsymbol{\sigma}_1 - \boldsymbol{\sigma}_2$ and the other $\boldsymbol{\sigma}_\alpha$ are defined similarly. Therefore, taking into account (46) and (47), the expansion of the square root in (42) leads to the following expression for the action of V_α in H_{int}^{nr} (in the $1/m^2$ approximation):

$$V_\alpha = v_\alpha + V'_\alpha, \quad V'_\alpha = -\frac{1}{4m}v_\alpha^2 - \frac{1}{16m^2} \times \{v_\alpha, (4q_\alpha^2 + Q_\alpha^2)\} - \frac{1}{8m^2}\left[(\mathbf{Q}_\alpha\mathbf{q}_\alpha)\left(\mathbf{Q}_\alpha\frac{\partial}{\partial\mathbf{q}_\alpha}\right), v_\alpha\right] + \frac{i}{8m^2}[\boldsymbol{\sigma}_\alpha(\mathbf{Q}_\alpha \times \mathbf{q}_\alpha), v_\alpha]. \quad (48)$$

It is easy to see that in the variables $\{k_\alpha, Q_\alpha\}$ we can write

$$M = (4m^2 + 4k_\alpha^2 + Q_\alpha^2)^{1/2} + \omega(Q_\alpha). \quad (49)$$

From this it follows that in our approximation

$$M = 3m + T^{nr} + T', \quad T^{nr} = \frac{1}{m}\left(q_\alpha^2 + \frac{3}{4}Q_\alpha^2\right),$$
$$T' = -\frac{1}{8m^3}\left[2q_\alpha^4 + q_\alpha^2Q_\alpha^2 + 2(\mathbf{q}_\alpha\mathbf{Q}_\alpha)^2 + \frac{9}{8}Q_\alpha^4\right]. \quad (50)$$

Therefore, taking into account (1), (37), (48), and (50) and the fact that $v_\alpha = v_\alpha^{nr}$ (see Sec. 2), we have

$$\hat{M} = 3m + \hat{M}^{nr} + T' + \sum_\alpha V'_\alpha, \quad (51)$$

and v_α can be replaced by v_α^{nr} in V'_α.

We see that the order-$(1/m^2)$ correction to $\hat{M}^{nr}$ arising from inclusion of the relativistic commutation relations is, in general, of the same order as the relativistic terms included in $\hat{M}^{nr}$, so that the standard approach is, in general, not fully justified (see the discussion in the Introduction). The order-$(1/m^2)$ corrections to $\hat{M}$ were first studied by Shirokov,[70] but not all the terms in (51) were calculated in his study. Of course, there are no problems in the calculation of T', and the question involves only the terms entering into V'_α. This problem is discussed in detail in Refs. 38 and 57, for example.

The relativistic correction to the triton binding energy in leading order in $1/m^2$

Let φ be the triton wave function calculated in the standard approach:

$$\hat{M}^{nr}\varphi = \varepsilon_0\varphi, \quad (52)$$

where ε_0 is the nonrelativistic value of the binding energy ($\varepsilon_0 < 0$). Then from (39) and (51) it follows that the RC to the binding energy in leading order in $1/m^2$ is given by the expression

$$\Delta\varepsilon = \langle\varphi | T' + \sum_\alpha V'_\alpha | \varphi\rangle. \quad (53)$$

Let us consider the case where the function φ is found by solving the system of Faddeev equations

$$(T^{nr} - \varepsilon_0)\varphi_\alpha = -v_\alpha^{nr}\varphi, \quad \varphi = \sum_\alpha \varphi_\alpha, \quad \alpha = 12, 31, 23. \quad (54)$$

Then from (48), (53), and (54) we easily find that

$$\Delta\varepsilon = \langle\varphi | T' | \varphi\rangle + \frac{1}{4m^2}\,\mathrm{Re} \times \sum_\alpha \left\{\langle\varphi | \left[2q_\alpha^2 + Q_\alpha^2 + (\mathbf{Q}_\alpha\mathbf{q}_\alpha)\left(\mathbf{Q}_\alpha\frac{\partial}{\partial\mathbf{q}_\alpha}\right) - i\boldsymbol{\sigma}_\alpha(\mathbf{Q}_\alpha \times \mathbf{q}_\alpha)\right](T^{nr} - \varepsilon_0) | \varphi_\alpha\rangle - m\,\|(T^{nr} - \varepsilon_0)\varphi_\alpha\|^2\right\}. \quad (55)$$

We now use the fact that if additional isospin variables are introduced, in the case of the triton all the φ_α are identical functions of "their" variables and can therefore be obtained from each other by cyclic permutation of the nucleons. Moreover, all the φ_α are antisymmetric under interchange of the corresponding pair of particles (for example, $\varphi_{21} = -\varphi_{12}$) and, as is easily seen, $\mathbf{Q}_\alpha \times \mathbf{q}_\alpha$ is the same for all $\alpha = 12, 31, 23$. Therefore, from (50) and (55) we find that

$$\Delta\varepsilon = -\frac{3}{4m^3}\langle\varphi_{12} | \frac{3}{4}Q^4 + Q^2m | \varepsilon_0 | + m^2\varepsilon_0^2 | \varphi_{12}\rangle + \frac{3}{4m^3}\,\mathrm{Re}\,\langle\varphi_{31} + \varphi_{23} | q^4 + 2q^2Q^2 + (\mathbf{qQ})^2 + \frac{3}{16}Q^4 + (2q^2 + Q^2)m | \varepsilon_0 | + m\left[(\mathbf{Qq})\left(\mathbf{Q}\frac{\partial}{\partial\mathbf{q}}\right) - i\boldsymbol{\sigma}_{12}(\mathbf{Q} \times \mathbf{q})\right] \times (T^{nr} + | \varepsilon_0 |) | \varphi_{12}\rangle, \quad (56)$$

where for brevity we write $\mathbf{q} = \mathbf{q}_{12}$, $\mathbf{Q} = \mathbf{Q}_{12}$.

We see that the RC to the binding energy can be calculated by direct integration if the Faddeev components of the nonrelativistic triton WF (more precisely, the Faddeev components of the triton WF calculated in the standard approach) and the nonrelativistic binding energy are known. Therefore, in this approximation it is not necessary to solve the relativistic Faddeev equations. However, as was noted in the Introduction, the question of the accuracy of the first order in $1/m^2$ can be answered only by comparing $\Delta\varepsilon$ with the result of the exact calculation. As far as the technical part of the calculation of $\Delta\varepsilon$ is concerned, clearly here it is desirable to have an equation directly expressing this quantity in terms of the partial-wave expansion of the Faddeev components of the nonrelativistic problem. This equation will be derived below.

3. PARTIAL-WAVE EXPANSION OF THE RELATIVISTIC FADDEEV EQUATIONS

Formulation of the relativistic three-body problem in terms of Faddeev components

By analogy with the usual case, instead of Eq. (39) with the operator $\hat{M}$ in the form (37), we can solve the system of equations for the Faddeev components χ_α:

$$(M+V_\alpha-3m-\varepsilon)\,\chi_\alpha=-V_\alpha\sum_{\beta\neq\alpha}\chi_\beta, \tag{57}$$

where α and β take the values 12, 31, and 23. We shall assume that the χ_α are also functions of the isospin variables, which we shall denote by $\tau_i (i=1,2,3)$, and for any fixed values of these variables the χ_α belong to the "relativistic" space H_{int} [see (32)]. Then in the triton case all the χ_α are obtained from each other by cyclic permutations of the particles, and instead of the three equations (57) it is sufficient to solve one of them, for example,

$$(M+V-s)\,\chi=-V\hat{P}\chi, \tag{58}$$

where $s=3m+\varepsilon$; here and below, $\chi\equiv\chi_{12}$, $V=V_{12}$, and the cyclic-permutation operator $\hat{P}$ operates according to the rule $\hat{P}\chi_{12}=\chi_{13}+\chi_{23}$. We change from Eq. (58) to the equation

$$\chi=-R(s)\,V\hat{P}\chi, \tag{59}$$

where $R(s)=(M+V-s)^{-1}$ is the resolvent of the operator $M+V$, or to the equation

$$\chi=-R_0(s)\,T(s)\,\hat{P}\chi, \tag{60}$$

where $T(s)=V-VR(s)V$, and $R_0(s)$ is the resolvent of the operator M. It is important that the T matrix for the operator $M+V$ enters into the three-particle problem, rather than the two-particle t matrix as in the standard approach (this was first pointed out in Ref. 46).

For the explicit solution of Eqs. (58)–(60) we need to expand the function χ in some basis. In the standard approach the basis is usually characterized by the quantum numbers $\{LSJlj\}$ defined as follows: L is the orbital angular momentum of particles 1 and 2, S is the spin of the system $\{1,2\}$, J is the total angular momentum of the system $\{1,2\}$ obtained by adding the momenta L and S, l is the orbital angular momentum of particle 3 relative to the system $\{1,2\}$, and j is the total angular momentum of particle 3 obtained by adding l to the spin of particle 3, which is equal to $\frac{1}{2}$. Here the action of the two-particle interaction operator is taken into account only in states with a finite number of quantities $\{LSJ\}$.

In the relativistic case the operator determining the interaction of two particles in their c.m. frame enters into the three-particle problem, being unitarily transformed using the operator U_{12}. It is easy to see that U_{12} is a Wigner rotation describing the transformation of the spin variables in taking the system $\{1,2\}$ from its c.m. frame to the system in which the total momentum of the particles $\{1,2\}$ is equal to $\mathbf{Q}_{12}$. It is therefore natural to introduce the function $\tilde{\chi}$ such that $\chi=U_{12}\tilde{\chi}$, and to expand not χ but $\tilde{\chi}$ in the basis $\{LSJlj\}$.

Since χ (and, therefore, $\tilde{\chi}$) must be antisymmetric under the interchange of particles 1 and 2, the quantum numbers L and S determine the isotopic spin of the system $\{1,2\}$, which when added to the isotopic spin of particle 3 (equal to $\frac{1}{2}$) must give the value $\frac{1}{2}$ for the isotopic spin of the triton.

We shall use $C^{jm}_{j_1m_1j_2m_2}$ to denote the Clebsch–Gordan coefficient arising when the momenta j_1 and j_2 are added to form the total angular momentum j. We shall assume for definiteness that the three-nucleon system under consideration is in a state in which the z projection of the total angular momentum and the total isotopic spin are equal to $\frac{1}{2}$. In view of the above discussion and the fact that the triton spin is $\frac{1}{2}$, we can now write the function χ as

$$\chi=U_{12}\sum_n C^{1/2\,1/2}_{J\mu j\nu}C^{J\mu}_{L\rho S\sigma}Y_{L\rho}(\boldsymbol{\nu}_{12})\,C^{S\sigma}_{1/2\,\xi\,1/2\,\eta}\chi_\xi(\sigma_1)\,\chi_\eta(\sigma_2)\,C^{j\nu}_{l\lambda\,1/2\,\varepsilon}\times Y_{l\lambda}(-\boldsymbol{\nu}_3)\,\chi_\varepsilon(\sigma_3)\,\chi_{LS}(\tau_1,\tau_2,\tau_3)\,\chi_n(k,Q). \tag{61}$$

In this equation it is assumed that n is the number of the channel with quantum numbers $\{LSJlj\}$, $\mathbf{k}=\mathbf{k}_{12}$, $k=|\mathbf{k}|$, $\mathbf{Q}=\mathbf{Q}_{12}$, $Q=|\mathbf{Q}|$, $\boldsymbol{\nu}_{12}=\mathbf{k}/k$, $\boldsymbol{\nu}_3=\mathbf{Q}/Q$, there is a sum over repeated indices of the momentum projections, $\chi_\xi(\sigma_i)$ is the standard eigenfunction of the operator $(\mathbf{s}_i)_z$ with eigenvalues ξ, $Y_{L\rho}(\boldsymbol{\nu}_{12})$ is the standard spherical function [see Eq. (71)], and the explicit dependence on the isospin variables τ_i is determined by the quantum numbers L and S. It is easy to see that

$$\chi_{LS}(\tau_1,\tau_2,\tau_3)=\frac{1}{\sqrt{6}}[2\chi_+(\tau_1)\,\chi_+(\tau_2)\,\chi_-(\tau_3)-\chi_+(\tau_1)\times\chi_-(\tau_2)\,\chi_+(\tau_3)-\chi_-(\tau_1)\,\chi_+(\tau_2)\,\chi_+(\tau_3)] \tag{62}$$

if $L+S$ is even, and

$$\chi_{LS}(\tau_1,\tau_2,\tau_3)=\frac{1}{\sqrt{2}}[\chi_+(\tau_1)\,\chi_-(\tau_2)-\chi_-(\tau_1)\,\chi_+(\tau_2)]\,\chi_+(\tau_3) \tag{63}$$

if $L+S$ is odd, where for brevity we have written the subscripts as $\pm$ instead of $\pm 1/2$. The minus sign in the argument of the spherical function $Y_{l\lambda}$ is related to the fact that the momentum of particle 3 in the c.m. frame of the three-particle system is $-\mathbf{Q}$. Of course, we could omit this sign and introduce the factor $(-1)^l$ in $\chi_n(k,Q)$.

From (61)–(63) it follows that χ is determined by a set of functions of two variables $\{\chi_n(k,Q)\}$, where the normalization integral for χ has the form

$$\|\chi\|^2=\sum_n\int_0^\infty\int_0^\infty\rho(Q,k)\,\chi_n(k,Q)^2\,k^2Q^2\,dk\,dQ, \tag{64}$$

since, as will become clear below, the functions $\chi_n(k,Q)$ can be chosen to be real. The meaning of the expansion (61) is that if the two-particle interaction operator acts only in channels with numbers $n\leqslant N$, Eqs. (58)–(60) reduce to a system of equations for N functions $\chi_1(k,Q),\ldots,\chi_N(k,Q)$, i.e., we can set $\chi_n=0$ for $n>N$. In order to transform to this system of equations, we must understand the following for the operators A entering into (58)–(60). If some function Ψ is determined using the expansion (61) by the set $\{\Psi_1(k,Q),\ldots,\Psi_N(k,Q)\}$, then what is the set $\{\Psi'_1(k,Q),\ldots,\Psi'_N(k,Q)\}$ determining the function $\Psi'=A\Psi$? We shall see below that the greatest complication relative to the standard case arises in the case of the operator $\hat{P}$.

Partial-wave expansion of the operator $\hat{P}$

If the function χ is written as (61), then it is obvious that $\hat{P}\chi = \chi_{31} + \chi_{23}$, where

$$\chi_{31} = U_{31} \sum_{n'} C^{1/2\,1/2}_{J'\mu'j'\nu'} C_{L'\rho'S'\sigma'} Y_{L'\rho'} \times \left(\frac{\mathbf{k}_{31}}{k_{31}}\right) C^{S'\sigma'}_{1/2\xi'1/2\eta'} \chi_{\xi'}(\sigma_3)\chi_{\eta'}(\sigma_1) C^{j'\nu'}_{l'\lambda'1/2\varepsilon'} Y_{l'\lambda'} \times \left(-\frac{\mathbf{Q}_{31}}{Q_{31}}\right) \chi_{\varepsilon'}(\sigma_2)\chi_{L'S'}(\tau_3,\tau_1,\tau_2)\chi_{n'}(k_{31}, Q_{31}), \quad (65)$$

and χ_{23} is defined similarly. Therefore, if the function $\hat{P}\chi = \varphi$ is determined by the set $\{\varphi_1(k,Q),\ldots,\varphi_N(k,Q)\}$, then from (61)–(63) it follows that

$$\varphi_n(k,Q) = \sum_{n'} C^{1/2\,1/2}_{J\mu j\nu} C^{J\mu}_{L\rho S\sigma} C^{S\sigma}_{1/2\xi 1/2\eta} \times C^{j\nu}_{l\lambda 1/2\varepsilon} C^{1/2\,1/2}_{J'\mu'j'\nu'} C_{L'\rho'S'\sigma'} C^{S'\sigma'}_{1/2\xi' 1/2\eta'} \times C^{j'\nu'}_{l'\lambda' 1/2\varepsilon'} \int d^2\nu_{12} \int d^2\nu_3 Y_{L\rho}(\mathbf{v}_{12})^* \times Y_{l\lambda}(-\mathbf{v}_3)^* Y_{L'\rho'}\left(\frac{\mathbf{k}_{31}}{k_{31}}\right) Y_{l'\lambda'}\left(-\frac{\mathbf{Q}_{31}}{Q_{31}}\right) \times f_{LSL'S'}(\gamma_{12}^{-1}\gamma_{13})_{\xi\eta'}(\gamma_{21}^{-1})_{\eta\varepsilon'}(\gamma_{31})_{\varepsilon\xi'}\chi_{n'}(k_{31}, Q_{31}) + (\ldots), \quad (66)$$

where we have used the following notation [cf. (9) and (41)]:

$$\gamma_\alpha = \frac{(E_\alpha + M_\alpha)(\omega(k_\alpha)+m) + Q_\alpha k_\alpha + i\sigma_\alpha(\mathbf{Q}_\alpha \times \mathbf{k}_\alpha)}{[2(E_\alpha+M_\alpha)(\omega(k_\alpha)+m)(E_\alpha\omega(k_\alpha)+M_\alpha m + Q_\alpha k_\alpha)]^{1/2}}, \quad (67)$$

M_α and E_α are the mass and energy of the pair α in the three-particle c.m. frame, and α can take the values 12, 21, 31, 13. Below, $(\gamma_\alpha)_{\xi\eta'}$ will denote the element of the matrix γ_α between the states with spin eigenvalues ξ and η', i.e., $(\gamma_\alpha)_{\xi\eta'} = \langle \xi | \gamma_\alpha | \eta' \rangle$. It is assumed that all the vectors in (66) are expressed in terms of $\mathbf{k}$ and $\mathbf{Q}$, and then after integration over $\mathbf{v}_{12}$ and $\mathbf{v}_3$ only the dependence on k and Q remains. It is easy to see that $f_{LS'S'} = \langle \chi_{LS}(\tau_3,\tau_1,\tau_2) | \chi_{L'S'}(\tau_1,\tau_2,\tau_3)\rangle$ is equal to $-\frac{1}{2}$ if $L+S+L'+S'$ is even, and equal to $(\sqrt{3}/2)\times(-1)^{L+S}$ otherwise. Finally, (...) in Eq. (66) stands for the contribution of χ_{23}, which is calculated in an analogous manner. If we write this contribution explicitly and make the change of variable $\mathbf{v}_{12} \to -\mathbf{v}_{12}$ in the integral (here $\mathbf{k}_{23}$ becomes $\mathbf{k}_{13} = -\mathbf{k}_{31}$, and so on), then it can be checked that (...) is equal to the contribution from χ_{31} written explicitly in (66).

It is clear from physical considerations that the result for $\varphi_n(k,Q)$ is not changed if we take z, the triton spin projection, to be $-\frac{1}{2}$ rather than $\frac{1}{2}$. This can easily be checked as follows. After the calculations below it will become clear that the structure of the integral in (66).

It is clear from physical considerations that the result for $\varphi_n(k,Q)$ is not changed if we take z, the triton spin projection, to be $-\frac{1}{2}$ rather than $\frac{1}{2}$. This can easily be checked as follows. After the calculations below it will become clear that the structure of the integral in (66) is such that if the $\chi_{n'}(k,Q)$ are real, then the $\varphi_n(k,Q)$ are also real. We take the complex conjugate in (66) and use the fact that $(\gamma^*)_{\xi\eta} = (-1)^{\xi+\eta+1}\gamma_{-\xi,-\eta}$, $Y^*_{L\rho} = (-1)^{L-\rho}Y_{L,-\rho}$. We then change the signs of the momentum projections in all the Clebsch–Gordan coefficients and use the fact that $C^{j,-m}_{j_1,-m_1,j_2,-m_2} = (-1)^{j_1+j_2+j+2m} C^{jm}_{j_1m_1j_2m_2}$, after which this property is easily verified. Therefore, Eq. (66) can be transformed by discarding (...) and replacing $C^{1/2\,1/2}_{J_{\mu j\nu}} C^{1/2\,1/2}_{J'\mu'j'\nu'}$ by $C^{1/2\varepsilon}_{J\mu j\nu} C^{1/2\varepsilon}_{J'\mu'j'\nu'}$.

Now we change to new variables in order to explicitly perform three of the four integrations entering into (66). We note that by a rotation it is always possible to put the vectors $\mathbf{k}$ and $\mathbf{Q}$ in the zx plane, with the vector $\mathbf{k}_1$ directed along the z axis, the angle θ between $\mathbf{k}$ and $\mathbf{Q}$ lying in the range $0 \leqslant \theta \leqslant 180°$, and the rotation taking $\mathbf{v}_{12}$ into $\mathbf{v}_3$ done by counterclockwise rotation by the angle θ. If we use θ' to denote the angle between $\mathbf{k}$ and $\mathbf{k}_1$, and θ'' to denote the angle between $\mathbf{Q}$ and $\mathbf{k}_1$, then $\theta' + \theta'' = \theta$, and the vectors $\mathbf{v}_{12}$ and $\mathbf{v}_3$ in this reference frame (we denote them by $\mathbf{v}^0_{12}$ and $\mathbf{v}^0_3$) have the components $(\mathbf{v}^{(0)}_{12})_x = -\sin\theta'$, $(\mathbf{v}^{(0)}_{12})_z = \cos\theta''$, $(\mathbf{v}^{(0)}_3)_x = \sin\theta''$, $(\mathbf{v}^0_3)_z = \cos\theta''$. We use g to denote an element of the rotation group such that

$$\mathbf{v}_{12} = g\mathbf{v}^{(0)}_{12}, \quad \mathbf{v}_3 = g\mathbf{v}^0_3. \quad (68)$$

This element is parametrized by three Euler angles. A technically simple but rather awkward calculation leads to the result

$$d^2\mathbf{v}_{12}d^2\mathbf{v}_3 = 8\pi^2 dg \sin\theta\, d\theta, \quad (69)$$

where dg is the invariant volume element of the group $SO(3)$ such that the volume of the entire group is unity.

Below, we shall use the expressions

$$Y_{L\rho}(g\mathbf{v}_{12}) = \sum_{\rho'} \mathscr{D}^L_{\rho\rho'}(g^*) Y_{L\rho'}(\mathbf{v}_{12}); \quad (70)$$

$$Y_{L\rho}(\mathbf{v}) = (-1)^{[(\rho+|\rho|)/2]} i^L \times \left[\frac{2L+1}{4\pi}\frac{(L-|\rho|)!}{(L+|\rho|)!}\right]^{1/2} P^{|\rho|}_L(\cos\theta) e^{i\rho\varphi}, \quad (71)$$

where $\mathscr{D}^j_{mm'}(g)$ is the matrix of the irreducible representation of $SO(3)$ labeled by the spin j; θ and φ are the polar angles characterizing the unit vector $\mathbf{v}$, and $P^{|\rho|}_L$ is the associated Legendre polynomial. If we also use the familiar orthogonality relations for the $\mathscr{D}$ functions and the equation for expanding products of them in $\mathscr{D}$ functions, then in the nonrelativistic case these expressions are sufficient, but in the relativistic case serious difficulties arise because (66) involves γ matrices including Wigner rotations. To transform them we first use the expression

$$(i\sigma\mathbf{v})_{\xi\eta} = -(4\pi)^{1/2} C^{1/2\eta}_{1\delta 1/2\xi} Y_{1\delta}(\mathbf{v}). \quad (72)$$

Then from (66), (67), and (69)–(71), and the known expressions for the $\mathscr{D}$ functions it follows that the functions $\varphi_n(k,Q)$ are real if $\chi_n(k,Q)$ are real, since the Clebsch–Gordan coefficients are real and the triton WF is described by only states in which $L+l$ is even. Therefore, Eq. (66) can actually be transformed as shown above. Using (70) and the fact that the vector $\mathbf{v}^{(0)}_3 \times \mathbf{v}^{(0)}_{12}$ obviously has components, $(0,-1,0)$, we obtain

$$\{i\sigma(\mathbf{Q}\times\mathbf{k})\}_{\xi\eta} = -\sqrt{\frac{3}{2}}|\mathbf{Q}\times\mathbf{k}| \sum_{\delta'=\pm1} C^{1/2\eta}_{1\delta 1/2\xi} \mathscr{D}^1_{\delta 0'}(g) \quad (73)$$

(we recall that there is a sum over repeated indices of the momentum projections). Therefore, using (67), the terms

arising from the inclusion of Wigner rotations can be represented as

$$(U_{12}^{-1}U_{31})_{\xi\eta\varepsilon\xi'\eta'\varepsilon'}=(\gamma_{12}^{-1}\gamma_{13})_{\xi\eta'}(\gamma_{21}^{-1})_{\eta\varepsilon'}(\gamma_{31})_{\varepsilon\xi'}$$
$$=\frac{1}{c}\Big[(a_1a_4-b^2)\,\delta_{\xi\eta'}-\Big(\frac{3}{2}\Big)^{1/2}b\,(a_1+a_4)\,C^{1/2\eta'}_{1\delta_1 1/2\xi}$$
$$\times\sum_{\delta_1'}\mathscr{D}^{1}_{\delta_1\delta_1'}(g)^*\Big]$$
$$\times\Big[a_2\delta_{\eta\varepsilon'}+\Big(\frac{3}{2}\Big)^{1/2}bC^{1/2\varepsilon'}_{1\delta_2 1/2\eta}\sum_{\delta_2'}\mathscr{D}^{1}_{\delta_2\delta_2'}(g)^*\Big]$$
$$\times\Big[a_3\delta_{\varepsilon\xi'}+\Big(\frac{3}{2}\Big)^{1/2}bC^{1/2\xi'}_{1\delta_3 1/2\varepsilon}\times\sum_{\delta_3'}\mathscr{D}^{1}_{\delta_3\delta_3'}(g)^*\Big],\qquad(74)$$

where the δ_i' $(i=1,2,3)$ take only the values ± 1, a_i $(i=1,2,3,4)$; b and c are functions only of k, Q, and θ; $b=|\mathbf{Q}\times\mathbf{k}|=Qk\sin\theta$, and the explicit form of the other functions will be given below [see Eq. (89)].

Let us also introduce the notation $\mathbf{k}_{13}=-\mathbf{k}_{31}$ and write θ'_{13} for the angle between the vectors $\mathbf{k}_{13}$ and $\mathbf{k}_1$, θ''_{13} for the angle between $\mathbf{Q}_{13}=\mathbf{Q}_{31}$ and $\mathbf{k}_1$, and $d_{LSL'S'}=(-1)^{L'+S+1}f_{LSL'S'D15}$. It is easily seen that

$$d_{LSL'S'}=\begin{cases}\sqrt{3}/2, & \text{if } L+S+L'+S' \text{ odd};\\ 1/2, & \text{if } L+S\,u\,L'+S' \text{ even};\\ -1/2, & \text{if } L+S\,u\,L'+S' \text{ odd}.\end{cases}\qquad(75)$$

For the integration over g in (66) we first use (70), and transform all the products of $\mathscr{D}$ functions into products of two $\mathscr{D}$ functions, after which the integration over g is carried out using the orthogonality relations for the $\mathscr{D}$ functions and Eqs. (72)–(74). Then we first use the orthogonality relation for the Clebsch–Gordan coefficients and the equations (see, for example, Refs. 71 and 72)

$$C^{1/2\varepsilon}_{J\mu j\nu}C^{J\mu}_{L\rho S\sigma}C^{j\nu}_{l\lambda 1/2\varepsilon}C^{K\varkappa}_{L\rho l\lambda}C^{ym}_{S\sigma 1/2\varepsilon}$$
$$=[(2J+1)(2j+1)(2K+1)(2y+1)]^{1/2}$$
$$\times C^{1/2\varepsilon}_{K\varkappa ym}\begin{Bmatrix}L & S & J\\ l & 1/2 & j\\ K & y & 1/2\end{Bmatrix};\qquad(76)$$

$$C^{1/2\varepsilon}_{K\varkappa ym}C^{1/2\varepsilon}_{K'\varkappa'y'm'}C^{K\varkappa}_{K'\varkappa'M\delta}=2(-1)^{K'+M+y+\frac{1}{2}}\Big(\frac{2K+1}{2y'+1}\Big)^{1/2}$$
$$\times\begin{Bmatrix}y & y' & M\\ K' & K & 1/2\end{Bmatrix}C^{y'm'}_{M\delta ym}.\qquad(77)$$

All the $3nj$ symbols can be expressed in terms of products of $6j$, $9j$, and $3(n-3)j$ symbols using these equations. The rest of the calculation can be done using the "diagram cutting" technique for the $3nj$ symbols.[71,72] This amounts to the appropriate insertion into the corresponding expressions of selected orthogonality relations for the Clebsch–Gordan coefficients in such a way that when the order of summation is changed the products of these coefficients are systematically expressed in terms of $6j$ and $9j$ symbols. Here, along with (76) and (77), it is convenient to use the expressions

$$C^{jm}_{j_2m_2j_1m_1}=(-1)^{j-j_1-j_2}C^{jm}_{j_1m_1j_2m_2};\qquad(78)$$

$$C^{j_3m_3}_{j_4m_4j_5m_5}C^{j_6m_6}_{j_1m_1j_5m_5}C^{j_2m_2}_{j_6m_6j_4m_4}=(-1)^{2j_2}[(2j_3+1)(2j_6+1)]^{1/2}$$
$$\times\begin{Bmatrix}j_1 & j_2 & j_3\\ j_4 & j_5 & j_6\end{Bmatrix}C^{j_2m_2}_{j_3m_3j_1m_1},\qquad(79)$$

which can be verified on the basis of the relation between the Clebsch–Gordan and Wigner coefficients and the symmetry properties of the Wigner coefficients ($3j$ symbols).

The final result can be written in the following form. We introduce the notation

$$F_1=(-1)^{\frac{1}{2}(L+L'-l-l')}d_{LSL'S'}$$
$$\times(2K+1)(2y+1)(2K'+1)(2y'+1)$$
$$\times\Big[(2L+1)(2l+1)(2J+1)(2j+1)(2S+1)$$
$$\times(2L'+1)(2l'+1)(2J'+1)(2j'+1)(2S'+1)$$
$$\times\frac{(L-|\rho|)!\,(l-|\lambda|)!\,(L'-|\rho'|)!\,(l'-|\lambda'|)!}{(L+|\rho|)!\,(l+|\lambda|)!\,(L'+|\rho'|)!\,(l'+|\lambda'|)!}\Big]^{1/2}$$
$$\times\begin{pmatrix}L & l & K\\ \rho & \lambda & -\varkappa\end{pmatrix}\begin{pmatrix}L' & l' & K'\\ \rho' & \lambda' & -\varkappa'\end{pmatrix}\times\begin{Bmatrix}L & S & J\\ l & 1/2 & j\\ K & y & 1/2\end{Bmatrix}\begin{Bmatrix}L' & S' & J'\\ l' & 1/2 & j'\\ K' & y' & 1/2\end{Bmatrix}$$
$$\times P_L^{|\rho|}(\cos\theta')\,P_l^{|\lambda|}(\cos\theta'')\,P_{L'}^{|\rho'|}(\cos\theta'_{13})\,P_{l'}^{|\lambda'|}(\cos\theta''_{13});\qquad(80)$$

$$F_2=\frac{\delta_{KK'}\delta_{yy'}\delta_{\varkappa\varkappa'}}{(2K+1)(2y+1)}(-1)^{\varkappa+\frac{1}{2}(|\rho|+|\lambda|+|\rho'|+|\lambda'|)}$$
$$\times\Bigg[(-1)^{S+S'+1}a_2a_3(a_1a_4-b^2)\begin{Bmatrix}1/2 & y & S'\\ 1/2 & 1/2 & S\end{Bmatrix}-2b^2(a_1a_4-b^2)$$
$$\times\begin{Bmatrix}1/2 & 1/2 & S\\ 1/2 & 1 & 1/2\\ S' & 1/2 & y\end{Bmatrix}+b^2(a_1+a_4)\begin{Bmatrix}1/2 & y & S'\\ 1/2 & 1/2 & S\end{Bmatrix}$$
$$\times\Big(\frac{a_2(-1)^S}{2S'+1}+\frac{a_3(-1)^{S'}}{2S+1}\Big)\Bigg]+\sqrt{3}b(-1)^{\frac{1}{2}(1+|\rho|+|\lambda|+|\rho'|+|\lambda'|)}$$
$$\times\Big[\begin{pmatrix}K' & 1 & K\\ \varkappa' & 1 & -\varkappa\end{pmatrix}-\begin{pmatrix}K & 1 & K'\\ \varkappa & 1 & -\varkappa'\end{pmatrix}\Big]\begin{Bmatrix}y & y' & 1\\ K' & K & 1/2\end{Bmatrix}$$
$$\times\Bigg\{(a_1a_4-b^2)\,a_2(-1)^{S'}\begin{Bmatrix}1/2 & y' & S\\ 1/2 & 1/2 & S'\end{Bmatrix}\begin{Bmatrix}1 & y' & y\\ S & 1/2 & 1/2\end{Bmatrix}$$
$$+(a_1a_4-b^2)\,a_3(-1)^{S}\begin{Bmatrix}1/2 & y & S'\\ 1/2 & 1/2 & S\end{Bmatrix}\begin{Bmatrix}1 & y & y'\\ S' & 1/2 & 1/2\end{Bmatrix}$$
$$-a_2a_3(a_1+a_4)\begin{Bmatrix}1 & y' & S'\\ 1/2 & 1 & 1/2\\ S & y & 1/2\end{Bmatrix}+b^2(a_1+a_4)$$
$$\times\Bigg[\begin{Bmatrix}1/2 & y & 1\\ 1/2 & 1/2 & S\end{Bmatrix}\begin{Bmatrix}1/2 & y' & 1\\ 1/2 & 1/2 & S'\end{Bmatrix}\Bigg(\begin{Bmatrix}1 & y' & y\\ 1 & 1/2 & 1/2\end{Bmatrix}$$
$$-12(-1)^{y+\frac{1}{2}}\begin{Bmatrix}1 & 1/2 & y\\ 1 & 1/2 & y'\\ 2 & 1 & 1\end{Bmatrix}\Bigg)-\frac{(-1)^{S+S'}}{8}\delta_{y1/2}\delta_{y'1/2}\Bigg]\Bigg\}$$
$$+\sqrt{15}b^2(-1)^{\frac{1}{2}(|\rho|+|\lambda|+|\rho'|+|\lambda'|)}\begin{Bmatrix}y & y' & 2\\ K' & K & 1/2\end{Bmatrix}$$
$$\times\Big[\sqrt{2}\begin{pmatrix}K' & 2 & K\\ \varkappa' & 0 & -\varkappa\end{pmatrix}-\sqrt{3}\begin{pmatrix}K' & 2 & K\\ \varkappa' & 2 & -\varkappa\end{pmatrix}$$
$$-\sqrt{3}\begin{pmatrix}K & 2 & K'\\ \varkappa & 2 & -\varkappa\end{pmatrix}\Big]\Big[(-1)^{y+y'}(a_1a_4-b^2)$$
$$\times\begin{Bmatrix}1/2 & y & 1\\ 1/2 & 1/2 & S\end{Bmatrix}\begin{Bmatrix}1/2 & y' & 1\\ 1/2 & 1/2 & S'\end{Bmatrix}\begin{Bmatrix}y & y' & 2\\ 1 & 1 & 1/2\end{Bmatrix}$$
$$+\frac{2}{3\sqrt{3}}(a_1+a_4)\Bigg(a_2\delta_{S'_1}\begin{Bmatrix}3/2 & y' & S\\ 1/2 & 1/2 & 1\end{Bmatrix}$$
$$\times\begin{Bmatrix}2 & y' & y\\ S & 1/2 & 3/2\end{Bmatrix}+a_3\delta_{S1}\begin{Bmatrix}3/2 & y & S'\\ 1/2 & 1/2 & 1\end{Bmatrix}\begin{Bmatrix}2 & y & y'\\ S' & 1/2 & 3/2\end{Bmatrix}\Bigg)\Big]$$

$$-\frac{\sqrt{7}}{12}b^3(a_1+a_4)(-1)^{\frac{1}{2}(1+|\rho|+|\lambda|+|\rho'|+|\lambda'|)}$$
$$\times\begin{Bmatrix}3/2 & 3/2 & 3\\ K' & K & 1/2\end{Bmatrix}\delta_{y3/2}\delta_{y'3/2}\delta_{S1}\delta_{S'1}\left[\sqrt{\frac{3}{5}}\left(\begin{pmatrix}K' & 3 & K\\ \varkappa' & 1 & -\varkappa\end{pmatrix}-\begin{pmatrix}K & 3 & K'\\ \varkappa & 1 & -\varkappa'\end{pmatrix}\right)-\left(\begin{pmatrix}K' & 3 & K\\ \varkappa' & 3 & -\varkappa\end{pmatrix}-\begin{pmatrix}K & 3 & K'\\ \varkappa & 3 & -\varkappa'\end{pmatrix}\right)\right]; \tag{81}$$

$$F_{nn'}=\frac{1}{c}\sum_{KyK'y'}\sum_{\rho\lambda\varkappa\rho'\lambda'\varkappa'}F_1F_2=F_{nn'}(k,Q,x),\quad x=\cos\theta. \tag{82}$$

Then

$$\varphi_n(k,Q)=\sum_{n'=1}^{N}\int_{-1}^{1}F_{nn'}(k,Q,x)\,\chi_{n'}(k_{13},Q_{13})\,dx. \tag{83}$$

Some kinematic relations

To completely define the operator $\hat{P}$ we need to express all the functions of the momentum variables entering into the integral (83) in terms of the independent variables k, Q, and x. Let us express all the vectors $\mathbf{k}_i$ $(i=1,2,3)$ in terms of $\mathbf{k}$ and $\mathbf{Q}$. It is easy to show that from (31) we have

$$\mathbf{k}_1=\mathbf{k}+\frac{1}{2}\mathbf{Q}+\frac{\mathbf{Q}(\mathbf{kQ})}{M_{12}(M_{12}+E_{12})},$$
$$\mathbf{k}_2=-\mathbf{k}+\frac{1}{2}\mathbf{Q}-\frac{\mathbf{Q}(\mathbf{kQ})}{M_{12}(E_{12}+M_{12})},\quad \mathbf{k}_3=-\mathbf{Q}, \tag{84}$$

where M_{12} and E_{12} are functions of only k and Q:

$$M_{12}=2\omega,\quad E_{12}=(M_{12}^2+Q_{12}^2)^{1/2},$$
$$\omega=\omega(k)=(m^2+k^2)^{1/2}. \tag{85}$$

From (84) it follows that $|\mathbf{k}_i|$ as functions of k, Q, and x have the form

$$\left.\begin{aligned}k_1&=|\mathbf{k}_1|=\left(k^2+(1+r)^{1/2}kQx+\frac{1}{4}Q^2+k^2rx^2\right)^{1/2};\\ k_2&=|\mathbf{k}_2|=\left(k^2-(1+r)^{1/2}kQx+\frac{1}{4}Q^2+k^2rx\right)^{1/2};\\ k_3&=|\mathbf{k}_3|=Q,\end{aligned}\right\} \tag{86}$$

where we have used the notation $r=r(k,Q)=Q^2/4\omega^2$. We also use $\omega_i=\omega_i(k_i)$ to denote the energies of particles 1, 2, and 3 in their common c.m. frame. Clearly,

$$\omega_i=(m^2+k_i^2)^{1/2}\quad(i=1,2,3), \tag{87}$$

where the k_i are determined from (86). We use w_{13} to denote the energy of particle 1 or 3 in their common c.m. frame (these quantities obviously coincide). It is easy to see that

$$\omega_{13}=\frac{1}{2}[(\omega_1+\omega_3)^2-k_2^2]^{1/2} \tag{88}$$

and therefore ω_{13} is also expressed in terms of k, Q, and x.

Now we can write down explicit expressions for the functions a_i $(i=1,2,3,4)$, b, and c:

$$\left.\begin{aligned}a_1&=m(\omega_1+\omega_3)+2\omega(\omega_1+\omega+m),\\ a_2&=m(\omega_1+\omega_2)+2\omega(\omega_2+\omega+m);\\ a_3&=m(\omega_1+\omega_3)+2\omega_{13}(\omega_3+\omega_{13}+m),\\ a_4&=m(\omega_1+\omega_3)+2\omega_{13}(\omega_1+\omega_{13}+m);\\ b&=kQ(1-x^2)^{1/2},\\ c&=16(\omega_1+\omega_2+2\omega)(\omega_1+\omega_3+2\omega_{13})\\ &\quad\times\omega\omega_{13}(\omega+m)(\omega_{13}+m)(\omega_1+m)\\ &\quad\times[(\omega_2+m)(\omega_3+m)]^{1/2}.\end{aligned}\right\} \tag{89}$$

Let us now turn to the arguments of the associated Legendre functions in Eq. (80). From (84) and (85) we find that

$$\cos\theta'=\frac{1}{k_1}\left[k+\frac{1}{2}Qx+kx^2((1+r)^{1/2}-1)\right],$$
$$\cos\theta''=\frac{1}{k_1}\left[\frac{1}{2}Q+(1+r)^{1/2}kx\right]. \tag{90}$$

Then, by analogy with Eq. (31) it can be shown that

$$\mathbf{k}_{13}=\frac{(\omega+\omega_{13})\mathbf{k}_1-(\omega_1+\omega_{13})\mathbf{k}_3}{\omega_1+\omega_3+2\omega_{13}}\equiv k_{13}=(\omega_{13}^2-m^2)^{1/2}. \tag{91}$$

Therefore, from (91) it follows that

$$\cos\theta'_{13}=\frac{(\omega_3+\omega_{13})k_1+(\omega_1+\omega_{13})Q\cos\theta'}{(\omega_1+\omega_3+2\omega_{13})k_{13}}. \tag{92}$$

Finally, since $\mathbf{Q}_{13}=-\mathbf{k}_2$, from (84) we find that

$$\cos\theta''_{13}=\frac{1}{k_1k_2}\left[k^2(1+rx^2)-\frac{1}{4}Q^2\right],\quad Q_{13}=k_2. \tag{93}$$

Conditions for Hermiticity of the operator $\hat{P}$

The operator $\hat{P}$ acts in the Hilbert space of the vector functions $\chi=\{\chi_1(k,Q),\ldots,\chi_N(k,Q)\}$, where the norm of χ is determined by Eq. (64). If $\Psi=\{\Psi_1(k,Q),\ldots,\Psi_N(k,Q)\}$, then

$$(\Psi,\hat{P}\chi)=\sum_{n,n'=1}^{N}\int_0^\infty\int_0^\infty dk\,dQ\,k^2Q^2\rho(Q,k)\int_{-1}^{1}dx\,\Psi_n(k,Q)^*\times F_{nn'}(k,Q,x)\,\chi_{n'}(k_{13},Q_{13}), \tag{94}$$

where k_{13} and Q_{13} are expressed in terms of k,Q, and x. We use x_{13} to denote the cosine of the angle between the vectors $\mathbf{k}_{13}$ and $\mathbf{Q}_{13}$. From the equations of the preceding subsection it is obvious that the sets $\{k,Q,x\}$ and $\{k_{13},Q_{13},x_{13}\}$ have a one-to-one relation to each other. The same is true of the relation between each of these sets and $\{k_{23},Q_{23},x_{23}\}$. We now use the fact that in the integration of functions depending only on $\{k,Q,x\}$ we have

$$\rho(Q,k)\,k^2Q^2\,dk\,dQ\,dx=\frac{1}{8\pi^2}\,d\nu\,(\text{int}), \tag{95}$$

as follows from (32). Since $d\nu(\text{int})$ does not change under any permutations of the particles, we can now conclude that the left-hand side of (95) also possesses this property; for example,

$$\rho(Q,k)\,k^2Q^2\,dk\,dQ\,dx=\rho(Q_{13},k_{13})\,k_{13}^2Q_{13}^2\,dk_{13}\,dQ_{13}\,dx_{13}. \tag{96}$$

It easily follows from the actual definition of the operator $\hat{P}$ that this operator is Hermitian, and from (94) and (96) it follows that a sufficient condition for Hermiticity is that

$$F_{nn'}(k,Q,x)=F_{n'n}(k_{13},Q_{13},x_{13}). \tag{97}$$

It should be borne in mind that $F_{nn'}$ also depends implicitly on the sign of the y component of the vector $\mathbf{v}_3^{(0)}\times\mathbf{v}_{12}^{(0)}$. It can be verified that the vector $\mathbf{Q}_\alpha\times\mathbf{k}_\alpha$ is the same for all $\alpha=12,31,23$ and therefore changes sign under the interchange of any pair of particles. Therefore, in verifying (97) we need to replace b by $-b$ on the right-hand side. At the same time, it follows from (89) that for the replacement $2\leftrightarrow3$ we have $a_1\leftrightarrow a_4$, $a_2\leftrightarrow a_3$, $c\leftrightarrow c$. After this it is easily verified that the condition (97) is satisfied, using Eqs. (80)–(82) and the symmetry properties of the $6j$ and $9j$ symbols.

Now it is also clear that the direction of the z axis was originally chosen to be the $\mathbf{k}_1$ direction so that the symmetry between the initial and final states was expressed by the reciprocal relation (97).

Expression for the relativistic correction to the triton binding energy in first order $1/m^2$ in terms of the partial-wave expansion of the Faddeev components

Since in Eq. (56) we have already isolated the terms containing the small parameter $1/m^2$ from the dominant contribution, the operator $\hat{P}$ entering into this equation can be taken in the purely nonrelativistic approximation. In this approximation all the terms in $\hat{P}$ containing the function $b(k,Q,x)$ can be dropped. Moreover, we can everywhere replace $\mathbf{k}$ by $\mathbf{q}$, the functions $a_i(k,Q,x)$ $(i=1,2,3,4)$ by $8m^2$, and the function $c(k,Q,x)$ by $2^{12}m^8$. In this subsection we assume that the nonrelativistic Faddeev component $\chi=\chi_{12}$ determines the functions $\chi_n(q,Q)$ via the expansion (61), where $U_{12}=1$ in the nonrelativistic limit, and for the real functions $\chi_n(q,Q)$ we take the nonrelativistic normalization:

$$\int_0^\infty\int_0^\infty \chi_n\ (q,\ Q)^2\ q^2Q^2\ dq\ dQ<\infty. \tag{98}$$

The if $\hat{P}_\chi$ is determined by the functions $\varphi_i(q,Q)$ $(i=1,2,...,N)$, it follows from (80)–(83) that

$$\begin{aligned}\varphi_n(q,Q)=&\sum_{n'=1}^{N}\sum_{Ky}\sum_{\rho\lambda\varkappa\rho'\lambda'}\\ &\times(-1)^{\frac{1}{2}(L+L'-l-l'+|\rho|+|\lambda|+|\rho'|+|\lambda'|)+\varkappa+S+S'+1}\\ &\times d_{LSL'S'}\ (2K+1)\,(2y+1)\left[(2L+1)\,(2l+1)\,(2J+1)\right.\\ &\times(2j+1)\,(2S+1)\,(2L'+1)\,(2l'+1)\\ &\times\left.(2J'+1)\,(2j'+1)\,(2S'+1)\right.\\ &\times\frac{(L-|\rho|)!\,(l-|\lambda|)!\,(L'-|\rho'|)!\,(l'-|\lambda'|)!}{(L+|\rho|)!\,(l+|\lambda|)!\,(L'+|\rho'|)!\,(l+|\lambda'|)!}\Big]^{1/2}\\ &\times\begin{pmatrix}L & l & K\\ \rho & \lambda & -\varkappa\end{pmatrix}\begin{pmatrix}L' & l' & K\\ \rho' & \lambda' & -\varkappa\end{pmatrix}\begin{Bmatrix}1/2 & y & S'\\ 1/2 & 1/2 & S\end{Bmatrix}\\ &\times\begin{Bmatrix}L & S & J\\ l & 1/2 & j\end{Bmatrix}\begin{Bmatrix}L' & S' & J'\\ l' & 1/2 & j'\end{Bmatrix}\begin{Bmatrix}K & y & 1/2\end{Bmatrix}\int_{-1}^{1}P_L^{|\rho|}(\cos\theta')\\ &\times P_l^{|\lambda|}(\cos\theta'')\,P_{L'}^{|\rho'|}(\cos\theta'_{13})\,P_{l'}^{|\lambda'|}(\cos\theta''_{13})\,\chi_{n'}(q_{13},\,Q_{13})\,dx,\end{aligned} \tag{99}$$

where $d_{LSL'S'}$ is determined by (75). The kinematic equations also simplify significantly in the nonrelativistic limit, and in the integration in (99) it follows from Eqs. (86)–(88) and (90)–(93) that

$$\left.\begin{aligned}q_1&=k_1=\left(q^2+qQx+\tfrac{1}{4}Q^2\right)^{1/2},\\ {}_2&=k_2=\left(q^2-qQx+\tfrac{1}{4}Q^2\right)^{1/2};\\ q_{13}&=\tfrac{1}{2}\left(q^2+3qQx+\tfrac{9}{4}Q^2\right)^{1/2},\\ Q_{13}&=\left(q^2-qQx+\tfrac{1}{4}Q^2\right)^{1/2};\\ \cos\theta'&=\frac{1}{q_1}\left(q+\tfrac{1}{2}Qx\right),\\ \cos\theta''&=\frac{1}{q_1}\left(qx+\tfrac{1}{2}Q\right);\\ \cos\theta'_{13}&=\frac{1}{2q_1q_{13}}\left(q^2+2qQx+\tfrac{3}{4}Q^2\right),\\ \cos\theta''_{13}&=\frac{1}{q_1q_2}\left(q^2-\tfrac{1}{4}Q^2\right).\end{aligned}\right\} \tag{100}$$

It follows from (56) that we also need to understand how the set of functions of q and Q describes the elements $(\mathbf{qQ})^2\chi$, $(\mathbf{Qq})/(\mathbf{Q}\partial/\partial\mathbf{q})\chi$, and $i\sigma_{12}(\mathbf{Q}\times\mathbf{q})\chi$. First we use $\{\Psi_{1n}(q,Q)\}$ $(n=1,2,...,N)$ to denote the set of functions describing $\Psi_1=i\sigma_{12}\times(\mathbf{Q}\times\mathbf{q})\chi$. It is easy to compute these expressions, starting from (61), transforming to the spherical components of the vectors, and using the equation for the integral of the product of three spherical functions and Eq. (79). The result is the following

$$\begin{aligned}\Psi_{1n}(q,\ Q)=&2\sqrt{6}\,qQ\sum_{n'}[1-(-1)^{S+S'}]\\ &\times(-1)^{\frac{1}{2}(L+l+L'+l')+j+j'}[(2L+1)\,(2l+1)\,(2j+1)\,(2L'+1)\\ &\times(2l'+1)\,(2j'+2)]^{1/2}\begin{pmatrix}L & L' & 1\\ 0 & 0 & 0\end{pmatrix}\begin{pmatrix}l & l' & 1\\ 0 & 0 & 0\end{pmatrix}\begin{Bmatrix}1 & j & j'\\ 1/2 & l' & l\end{Bmatrix}\\ &\times\begin{Bmatrix}1 & J & J'\\ 1/2 & j' & j\end{Bmatrix}\chi_{n'}(q,\ Q)\\ &\times\begin{cases}(-1)^{J'-l+1}(2J'+1)^{1/2}\begin{Bmatrix}1 & J' & L\\ L' & 1 & 1\end{Bmatrix},\\ \qquad\text{if}\quad S=0,\ S'=1;\\ (-1)^{J-l'}(2J+1)^{1/2}\begin{Bmatrix}1 & J & L'\\ L & 1 & 1\end{Bmatrix},\ \text{if}\quad S=1,\ S'=0.\end{cases}\end{aligned} \tag{101}$$

It is clear that (101) involves only terms with $|S-S'|=|L-L'|=|l-l'|=1$. Therefore, the operator $i\sigma_{12}(\mathbf{Q}\times\mathbf{q})$ gives a contribution only if orbital angular momenta of different parity are taken into account. It is also easy to see that this operator contributes only when the P wave is included. In fact, if the P wave is neglected, the operator $i\sigma_{12}(\mathbf{Q}\times\mathbf{q})$ can obviously have transitions only between D waves, since $\mathbf{Q}\times\mathbf{q}$ corresponds to orbital angular momentum equal to unity. However, the spin functions of D states are symmetric under all particle interchanges (since the D wave enters into the triton WF with spin 3/2), and the operator σ_{12}, being antisymmetric under the interchange $1\leftrightarrow2$, does not have transitions between such functions. Since the probability for the P wave in the triton is of order 0.1%,[73] the impression is that the contribution of the term with $i\sigma_{12}(\mathbf{Q}\times\mathbf{q})$ to (56) is insignificant. However, this is not completely certain, since the integral determining this contribution (see below) is concentrated mainly in the region of momenta considerably higher than those important in the normalization integral for the triton WF.

To describe the element $(\mathbf{qQ})^2\chi$ we decompose $(\mathbf{qQ})^2$ into its scalar and tensor parts:

$$(\mathbf{qQ})^2=\tfrac{1}{3}q^2Q^2+\left(q_iq_k-\tfrac{1}{3}q^2\delta_{ik}\right)\left(Q_iQ_k-\tfrac{1}{3}Q^2\delta_{ik}\right), \tag{102}$$

with a sum over $i,k=1,2,3$. We use $\{\Psi_{2n}(q,Q)\}$ to denote the set of functions defining the element $(q_iq_k-\frac{1}{3}\delta_{ik}q^2)(Q_iQ_k-\frac{1}{3}\delta_{ik}Q^2)\chi$. Using the same technique as in calculating the functions $\Psi_{1n}(q,Q)$, we obtain

$$\Psi_{2n}(q,\ Q)=\frac{2}{3}q^2Q^2\sum_{n'}(-1)^{\frac{1}{2}(L+l-L'-l')+S+j-j'}$$
$$\times[(2L+1)(2l+1)(2J+1)(2j+1)(2L'+1)$$
$$\times(2l'+1)(2J'+1)(2j'+1)]^{1/2}$$
$$\times\begin{pmatrix}L & L' & 2\\ 0 & 0 & 0\end{pmatrix}\begin{pmatrix}l & l' & 2\\ 0 & 0 & 0\end{pmatrix}\begin{Bmatrix}2 & J & J'\\ S & L' & L\end{Bmatrix}$$
$$\times\begin{Bmatrix}2 & j & j'\\ 1/2 & l' & l\end{Bmatrix}\begin{Bmatrix}1/2 & j & J\\ 2 & J' & j'\end{Bmatrix}\chi_{n'}(q,\ Q),\qquad(103)$$

where only terms with $S'=S$ contribute to the sum.

Finally, let us consider the contribution of the operator $(\mathbf{Qq})(\mathbf{Q}\partial/\partial\mathbf{q})$. It is obvious that $\partial\chi_n(q,Q)/\partial\mathbf{q}=\boldsymbol{\nu}\partial\chi_n(q,Q)/\partial q$, where $\boldsymbol{\nu}=\mathbf{q}/q$. Therefore, complications can arise only in operating with $\nabla_\nu=q\partial/\partial\mathbf{q}$ on the spherical function $Y_{l'\lambda'}(\boldsymbol{\nu})$. In this case only the tensor contribution remains, since $\boldsymbol{\nu}\nabla_\nu Y_{l'\lambda'}(\nu)=0$. This can be verified using the Wigner–Eckart theorem and the familiar expressions for the reduced matrix elements of the operators $\boldsymbol{\nu}$ and ∇_ν (see, for example, Sec. 107 in Ref. 74 and Sec. 7 in Ref. 75). Starting from this, it can also be shown that the functions $\{\Psi_{3n}(q,Q)\}$ determining the element $(\mathbf{Q}\boldsymbol{\nu})(\mathbf{Q}\nabla_\nu)\chi$ can be written as

$$\Psi_{3n}(q,\ Q)=\frac{2}{3}Q^2\sum_{n'}(-1)^{\frac{1}{2}(L+l-L'-l')+S+j-j'}$$
$$\times[(2L+1)(2l+1)(2J+1)(2j+1)$$
$$\times(2L'+1)(2l'+1)(2J'+1)(2j'+1)]^{1/2}$$
$$\times\begin{pmatrix}L & L' & 2\\ 0 & 0 & 0\end{pmatrix}\begin{pmatrix}l & l' & 2\\ 0 & 0 & 0\end{pmatrix}\begin{Bmatrix}1/2 & J & j\\ 2 & j' & J'\end{Bmatrix}\begin{Bmatrix}J & J' & 2\\ L' & L & S\end{Bmatrix}\begin{Bmatrix}j & j' & 2\\ l' & l & 1/2\end{Bmatrix}$$
$$\times\left[\frac{3}{2}+\frac{1}{4}(L+L'+1)(L'-L)\right]\chi_{n'}(q,\ Q),\qquad(104)$$

where the sum involves only terms with $S'=S$. As far as the contributions with $\partial\chi_n(q,Q)/\partial q$ are concerned, it is clear that by analogy with (103) the result for $\Delta\varepsilon$ will involve functions $\Psi_{4n}(q,Q)$ such that

$$\Psi_{4n}(q,\ Q)=\frac{2}{3}qQ^2\sum_{n'}(-1)^{\frac{1}{2}(L+l-L'-l')+S+j-j'}$$
$$\times[(2L+1)(2l+1)(2J+1)(2j+1)(2L'+1)(2l'+1)$$
$$\times(2J'+1)(2j'+1)]^{1/2}\begin{pmatrix}L & L' & 2\\ 0 & 0 & 0\end{pmatrix}\begin{pmatrix}l & l' & 2\\ 0 & 0 & 0\end{pmatrix}\begin{Bmatrix}2 & J & J'\\ S & L' & L\end{Bmatrix}$$
$$\times\begin{Bmatrix}2 & j & j'\\ 1/2 & l' & l\end{Bmatrix}\begin{Bmatrix}1/2 & J & j\\ 2 & j' & J'\end{Bmatrix}\frac{\partial\chi_{n'}(q,\ Q)}{\partial q}.\qquad(105)$$

Now from (56) we find that the final expression for $\Delta\varepsilon$ can be written as

$$\Delta\varepsilon=-\frac{3}{4m^3}\sum_{n=1}^{N}\int_0^\infty\int_0^\infty dq\,dQq^2Q^2\Big\{\Big(\frac{3}{4}Q^4+Q^2m\mid\varepsilon_0\mid+m^2\varepsilon_0^2\Big)$$
$$\times\chi_n(q,\ Q)^2-\Big[q^4+\frac{7}{3}q^2Q^2+\frac{3}{16}Q^4+(2q^2+Q^2)m\mid\varepsilon_0\mid\Big]$$
$$\times\varphi_n(q,\ Q)\chi_n(q,\ Q)-\varphi_n(q,\ Q)\Psi_{2n}(q,\ Q)$$
$$+\Big(q^2+\frac{3}{4}Q^2+m\mid\varepsilon_0\mid\Big)$$
$$\times\varphi_n(q,\ Q)\Big[\Psi_{1n}(q,\ Q)-\Psi_{3n}(q,\ Q)-\Psi_{4n}(q,\ Q)$$
$$-\frac{1}{3}Q^2q\frac{\partial\chi_n(q,\ Q)}{\partial q}\Big]\Big\}.\qquad(106)$$

Relativistic system of Faddeev equations for the triton wave function in the *N*-channel approximation

Now let us return to the purely relativistic problem, i.e., we shall not make use of the expansion in $1/m^2$. Our problem now is to write the relativistic system of equations in a form as close as possible to the ordinary system. Therefore, in the final formulation we shall work in the Hilbert space of elements $\chi=\{\chi_1(k,Q),\ldots,\chi_N(k,Q)\}$ such that the scalar product of the elements χ' and χ is defined, in complete analogy with the nonrelativistic case, by the expression [instead of (64)]

$$(\chi',\ \chi)=\sum_{n=1}^{N}\int_0^\infty\int_0^\infty\chi'_n(k,\ Q)^*\chi_n(k,\ Q)\,k^2Q^2\,dk\,dQ.\qquad(107)$$

Accordingly, instead of (83) the final expression for the operator $\hat{P}$ is the following: if $\chi=\{\chi_1(k,Q),\ldots,\chi_N(k,Q)\}$ and $\hat{P}\chi=\{\varphi_1(k,Q),\ldots,\varphi_N(k,Q)\}$, then

$$\varphi_n(k,\ Q)=\sum_{n'=1}^{N}\int_{-1}^{1}\Big[\frac{\omega_{13}\omega_2(\omega_1+\omega_3)}{\omega\omega_3(\omega_1+\omega_2)}\Big]^{1/2}$$
$$\times F_{nn'}(k,\ Q,\ x)\chi_{n'}(k_{13},\ Q_{13})\,dx,\qquad(108)$$

where all the functions of k, Q, and x are defined above.

Above, we expressed the operator V [see Eq. (42)] in terms of the operator v, which is determined from the two-particle data in almost complete analogy with the nonrelativistic case (see the discussion in Sec. 1). In the Hilbert space under consideration here the operator v, as in the usual case, is specified by the set of kernels $v_{nn'}(k,k')$ such that if $\chi=\{\chi_1(k,Q),\ldots,\chi_N(k,Q)\}$, then $v\chi$ is given by the set $\{\Psi_1(k,Q),\ldots,\Psi_N(k,Q)\}$ such that

$$\Psi_n(k,\ Q)=\sum_{n'}\int_0^\infty v_{nn'}(k,\ k')\chi_{n'}(k,\ Q)\,k^2\,dk.\qquad(109)$$

Actually, the sum (109) involves no more than two terms, since the operator v is diagonal in the quantum numbers $\{SJlj\}$. Since we are assuming that the operator v can be identified with v^{nr} (see the discussion in Sec. 1), we can take the $v_{nn'}(k,k')$ to be the known kernels for the standard phenomenological potentials.

From Eqs. (42) and (61) it follows that in the Hilbert space under consideration the operator V (or, rather, the corresponding unitarily transformed operator) is already written in a rather simple form:

$$V=(4m^2+4k^2+Q^2+4mv)^{1/2}-(4m^2+4k^2+Q^2)^{1/2}.\qquad(110)$$

Let us introduce the following notation: $B=4m^2+4k^2+Q^2+4mv$, $B_0=4m^2+4k^2+Q^2$, $G(s)=(B+s)^{-1}$, and $G_0(s)=(B_0+s)^{-1}$. Let us use the familiar expression for the square root of a positive operator:

$$B^{1/2}=\frac{1}{\pi}\int_0^\infty\frac{ds}{s^{1/2}}[1-sG(s)]\qquad(111)$$

(the integral is understood as the strong limit of the Riemann integral sums) and the analogous expression for the operator B_0. We also introduce the operator

$$t(s) = 4m\,[v - 4mvG(s)\,v]. \tag{112}$$

Then

$$G(s) - G_0(s) = -G_0(s)\,t(s)\,G_0(s). \tag{113}$$

Taking into account (110), (111), and (113), we obtain

$$V = \frac{1}{\pi}\int_0^\infty s^{1/2}\,G_0(s)\,t(s)\,G_0(s)\,ds. \tag{114}$$

The operator $t(s)$ is expressed in terms of v as

$$t(s) = 4m\left[1 + 4mv\frac{1}{4m^2+4k^2+Q^2+s}\right]^{-1} v. \tag{115}$$

In the Hilbert space under consideration this operator is specified by the set of kernels $t_{nn'}(k,k';Q^2+s)$, and if $\chi = \{\chi_1(k,Q),\ldots,\chi_N(k,Q)\}$, while $t(s)\chi$ is determined by the set $\{\Psi_1(k,Q),\ldots,\Psi_N(k,Q)\}$, then

$$\Psi_n(k,\,Q) = \sum_{n'}\int_0^\infty t_{nn'}(k,\,k';\,Q^2+s)\,\chi_{n'}(k',\,Q)\,k'^2\,dk'. \tag{116}$$

It therefore follows from (114) that the kernels determining the operator V can be calculated from the expression

$$V_{nn'}(k,\,k';\,Q) = \frac{1}{\pi}\int_0^\infty \frac{t_{nn'}(k,\,k';\,Q^2+s)\,s^{1/2}\,ds}{(4m^2+4k^2+Q^2+s)\,(4m^2+4k'^2+Q^2+s)} \tag{117}$$

and the action V has the form

$$(V\chi)_n(k,\,Q) = \sum_{n'}\int_0^\infty V_{nn'}(k,\,k';\,Q)\,\chi_{n'}(k',\,Q)\,k'^2\,dk'. \tag{118}$$

Equation (58) can now be represented as a system of equations for the functions $\chi_n(k,Q)$ $(n = 1,\ldots,N)$:

$$[(4m^2 + 4k^2 + Q^2)^{1/2} + (m^2 + Q^2)^{1/2} - s]\,\chi_n(k,\,Q) + \sum_{n'=1}^{N}\int_0^\infty V_{nn'}(k,\,k';\,Q) \times [\chi_{n'}(k',\,Q) + \varphi_{n'}(k',\,Q)]\,k'^2\,dk'^2 = 0, \tag{119}$$

where the functions $\{\varphi_n(k,Q)\}$ are expressed in terms of $\{\chi_n(k,Q)\}$ using (107).

The operator $T(s)$ in (60) is determined by the set of kernels $T_{nn'}(k,k';Q,s)$, so that

$$(T(s)\,\chi)_n(k,\,Q) = \sum_{n'}\int_0^\infty T_{nn'}(k,\,k';\,Q,\,s)\,\chi_{n'}(k',\,Q)\,k'^2\,dk'. \tag{120}$$

These kernels must be determined from the relation

$$T(s) = \left[1 + V\frac{1}{(4m^2+4k^2+Q^2)^{1/2}+(m^2+Q^2)^{1/2}-s}\right]^{-1} V. \tag{121}$$

Therefore, Eq. (60) can be represented as a system of equations for the functions $\chi_n(k,Q)$ $(n = 1,\ldots,N)$ in the form

$$\chi_n(k,\,Q) = -[(4m^2 + 4k^2 + Q^2)^{1/2} + (m^2 + Q^2)^{1/2} - s]^{-1} \times \sum_{n'=1}^{N}\int_0^\infty T_{nn'}(k,\,k';\,Q,\,s)\,\varphi_{n'}(k',\,Q)\,k'^2\,dk' \tag{122}$$

or in the form

$$\chi = -A(s)\,\chi; \tag{123}$$

$$(A(s)\,\chi)_n(k,\,Q) = \sum_{n'=1}^{N}\int_0^\infty dk'\,k'^2\int_{-1}^{1} dx\,A_{nn'}(k,\,Q;\,k',\,x;\,s) \times \chi_{n'}(k_{13}(k',\,Q,\,x),\,Q_{13}(k',\,Q,\,x)), \tag{124}$$

where, as a consequence of (108) and (122),

$$A_{nn'}(k,\,Q;\,k',\,x;\,s) = -[(4m^2+4k^2+Q^2)^{1/2} + (m'^2+Q'^2)^{1/2} - s]^{-1} \times \left[\frac{\omega'_{13}\omega'_2\,(\omega'_1+\omega'_3)}{\omega'\omega'_3\,(\omega'_1+\omega'_2)}\right]^{1/2} \times \sum_{n''=1}^{N} T_{nn''}(k,\,k';\,Q,\,s)\,F_{n''n'}(k',\,Q,\,x), \tag{125}$$

where in all the primed functions ω of k, Q, and x the argument k is replaced by k'.

In numerical calculations it usually becomes necessary to repeatedly calculate the action of the operator $A(s)$ (see Sec. 5), and it is therefore desirable to get rid of the operation of variable substitution in (124), i.e., it is desirable to write the action $A(s)$ as

$$(A(s)\,\chi)_n(k,\,Q) = \sum_{n'=1}^{N}\int_0^\infty\int_0^\infty A_{nn'}(k,\,Q;\,k_{13},\,Q_{13};\,s) \times \chi_{n''}(k_{13},\,Q_{13})\,dk_{13}\,dQ_{13}. \tag{126}$$

To calculate the kernel $A_{nn'}(k,Q;k_{13},Q_{13};s)$ it is necessary to transform from Q, k, x to the variables Q, k_{13}, Q_{13}.

Let us first transform the volume element in (124). We introduce into the integrand of (124) the quantity $\delta(Q'-Q)dQ'$, use (96), and then eliminate the δ function by integrating over x_{13}. From expressions analogous to (86), it can be shown that

$$x_{13} \equiv x_{13}(Q,\,k_{13},\,Q_{13}) = \frac{2\,(m^2+k_{13}^2)^{1/2}}{k_{13}Q_{13}} \times \left[\left(m^2+k_{13}^2+\frac{1}{4}Q_{13}^2\right)^{1/2} - (m^2+Q^2)^{1/2}\right]. \tag{127}$$

Therefore, the kernel $A_{nn'}(k,Q;k_{13},Q_{13};s)$ vanishes if the value of the function $x_{13}(Q,k_{13},Q_{13})$ does not lie in the interval $(-1,1)$. We introduce the notation

$$f(Q,\,k_{13},\,Q_{13}) = \begin{cases} 1, & \text{if } x_{13}(Q,\,k_{13},\,Q_{13}) \in (-1,\,1); \\ 0, & \text{if } x_{13}(Q,\,k_{13},\,Q_{13}) \notin (-1,\,1). \end{cases} \tag{128}$$

Then in view of the above discussion it can be shown that

$$A_{nn'}(k,\,Q;\,k_{13},\,Q_{13};\,s) = -2\frac{k_{13}Q_{13}f(Q,\,k_{13},\,Q_{13})}{Q\,[(4m^2+4k^2+Q^2)^{1/2}+(m^2+Q^2)^{1/2}-s]} \times \left[\frac{\omega_{13}\omega_{12}\,(\omega_1+\omega_2)}{\omega_3\omega_2\,(\omega_1+\omega_3)}\right]^{1/2} \times \sum_{n''=1}^{N} T_{nn''}(k,\,k_{12}(Q,\,k_{13},\,Q_{13});\,Q,\,s) \times F_{n''n'}(Q,\,k_{13},\,Q_{13}). \tag{129}$$

We distinguish k and k_{12} in this expression, assuming that k_{12} and all the functions under the square root depend on Q, k_{13}, and Q_{13}. The corresponding explicit expressions have the form

$$\omega_2 = (m^2 + Q_{13}^2)^{1/2}, \quad \omega_3 = (m^2 + Q^2)^{1/2};$$
$$\omega_1 = 2\left(m^2 + k_{13}^2 + \frac{1}{4}Q_{13}^2\right)^{1/2} - (m^2 + Q^2)^{1/2},$$
$$\omega_{13} = (m^2 + k_{13}^2)^{1/2};$$
$$\omega_{12} = \frac{1}{2}[(\omega_1 + \omega_2)^2 - Q^2]^{1/2}, \quad k_{12} = (\omega_{12}^2 - m^2)^{1/2}. \tag{130}$$

In the calculation of $F_{n''n'}(Q,k_{13},Q_{13})$ we must also express all the functions in terms of Q, k_{13}, and Q_{13}. Equations (89) remain valid for the functions a_i $(i = 1,2,3,4)$ and c if ω is replaced by ω_{12} and Eq. (130) is used. Moreover, it can be shown that

$$\left.\begin{aligned} &b = k_{13}Q_{13}(1 - x_{13}^2)^{1/2}, \quad k_1 = (\omega_1^2 - m^2)^{1/2};\\ &\cos\theta' = \frac{1}{k_1 k_{12}}\left[(\omega_1\omega_{12} - m^2) - \frac{2\omega_{12}(\omega_1\omega_2 - \omega_{12}^2)}{\omega_1 + \omega_2 + 2\omega_{12}}\right];\\ &\cos\theta'' = -\frac{1}{k_1 Q}(\omega_1\omega_3 - m^2 - 2k_{13}^2);\\ &\cos\theta'_{13} = \frac{k_{13}}{k_1} + \frac{(\omega_1 + \omega_{13})(\omega_1 - \omega_3)\omega_{13}}{k_1 k_{13}(\omega_1 + \omega_3 + 2\omega_{13})};\\ &\cos\theta''_{13} = \frac{1}{k_1 Q_{13}}(\omega_1\omega_2 - m^2 - 2k_{12}^2), \end{aligned}\right\} \tag{131}$$

and therefore the function $F_{n''n'}(Q,k_{13},Q_{13})$, as before, can be calculated using (80)–(82).

4. SUMMARY OF THE RESULTS

In the first two sections of this study we reviewed the results which already exist in the literature, while the results of Sec. 3 are new. They pertain to two problems: 1) the derivation of an explicit expression for the RC to the triton binding energy when the nonrelativistic triton WF, calculated by solving the N-channel system of Faddeev equations, and the nonrelativistic binding energy are known; 2) the explicit derivation of the relativistic N-channel system of Faddeev equations, the solution of which determines the binding energy and the triton WF without expanding in powers of $1/m^2$. To aid the reader interested in carrying out numerical calculations, here we separately review the corresponding algorithms (of course, they are determined by the preceding discussion).

Calculation of the RC to the triton binding energy in first order in $1/m^2$

Let us assume that we know the set of functions $\{\chi_1(q,Q),\ldots,\chi_N(q,Q)\}$ obtained by solving the standard N-channel system of Faddeev equations for the triton wave function, where q and Q are the magnitudes of the vectors $\mathbf{q}$ and $\mathbf{Q}$ [see (14) and (31)] taken in the c.m. frame of the three particles. We assume that these functions are normalized in accordance with (98). Also, let ε_0 $(\varepsilon_0 < 0)$ be the nonrelativistic triton binding energy. Then the algorithm for calculating the RC to the triton binding energy in first order in $1/m^2$ is the following. First we use the relations (100) to introduce kinematic functions of q, Q, and x, and we determine the set of functions $\varphi_n(q,Q)$ using (99). Then we use Eqs. (101) and (103)–(105) to find the set of functions $\Psi_{in}(q,Q)$ $(i = 1,2,3,4;\ n = 1,2,\ldots,N)$. The desired relativistic correction to the triton binding energy is then determined by Eq. (106).

Algorithm for constructing the relativistic system of Faddeev equations for the triton wave function in the N-channel approximation

An algorithm for explicitly constructing Eqs. (58) or (60) is the following. First we choose N channels in which the states of the three-nucleon system are described by the quantum numbers $\{LSJlj\}$, and we consider the Hilbert space whose elements are the sets $\chi = \{\chi_1(k,Q),\ldots,\chi_N(k,Q)\}$, and the scalar product of two elements χ' and χ is given by (107). We assume that the kernels $v_{nn'}(k,k')$ of the operator v are chosen in such a way that the nucleon–nucleon data are correctly described by Eq. (19), where k is the magnitude of the relative momentum of the two nucleons in their c.m. frame [see (6)], and the functions $\widetilde{\chi}_n(k)$ describing the internal state of the two-nucleon system with quantum numbers $\{LSJ\}$ belongs to the Hilbert space of functions with the scalar product

$$(\chi',\ \chi) = \sum_n \int_0^\infty \chi'_n(k)^* \chi_n(k)\, k^2\, dk, \tag{132}$$

where the summation runs only over those sets $\{LSJ\}$ which are involved in the N-channel approximation. Above, we explained that $v_{nn'}(k,k')$ can be understood as the known kernels for phenomenological potentials when the normalization (132) is used (often the normalization for which the integration is done without the k^2 is chosen).

To determine the operator $\widehat{P}$ we first introduce functions k, Q, and x $(k,Q \in [0,\infty), x \in [-1,1])$ which are determined by Eqs. (85)–(93), and then functions F_1, F_2, and $F_{nn'}$ given by (80)–(82), after which the action of the operator $\widehat{P}$ is determined by Eq. (108).

To determine the operator V we first find the kernels $t_{nn'}(k,k';Q^2+s)$, which determine the operator (115), and then we use (117) to find the kernels $V_{nn'}(k,k';Q)$ and determine the action of the operator V using (118). After the operators $\widehat{P}$ and V have been found, Eq. (58) in the N-channel approximation reduces to the system of equations (119).

Imagine that now we want to represent Eq. (60), written in the form (123), in the N-channel approximation. Then, using (121), we also need to calculate the kernels $T_{nn'}(k,k';Q,s)$ of the operator $T(s)$, and if Eq. (123) is written in the form (124), the kernels of the operator $A(s)$ must be calculated using (125).

As already noted, for numerical calculations it is more convenient to write the operator $A(s)$ in the form (126). To calculate the kernel $A_{nn'}(k,Q;k_{13},Q_{13};s)$ we must in this case proceed as follows. First we introduce functions of Q, k_{13}, and Q_{13} which are determined by Eqs. (89), (127), (128), (130), and (131) (with $\omega = \omega_{12}$) and then use Eqs. (80)–(82) to calculate $F_{nn'}(Q,k_{13},Q_{13})$. Then in the kernels $T_{nn'}(k,k';Q,s)$ calculated earlier we replace k' by $k_{12}(Q,k_{13},Q_{13})$ and calculate the kernel $A_{nn'}(k,Q;k_{13},Q_{13};s)$ using Eq. (129), taking into account (127) and (130).

5. DISCUSSION OF THE RESULTS AND CONCLUSIONS

Since the system of equations in the form (119) or (123) has the same structure as in the standard approach, it

can be solved using the well known techniques used to find the triton binding energy in cases where the problem can be solved in the momentum representation. However, at present it is believed that it is most convenient to solve the Faddeev equations numerically in the coordinate representation (see the review of Ref. 76), and the most accurate calculations, which were mentioned in the Introduction, have also been carried out in this representation.

In the relativistic case the transformation to the coordinate representation is not completely clear, since, as is well known, here there is no operator possessing all the properties needed by the coordinate operator. If, however, we assume that the coordinate is determined by the Newton–Wigner operator or, as was suggested in Ref. 77, by the Shapiro transformation,[78] then the relativistic Faddeev equations in this coordinate representation become finite-difference rather than differential equations. Whereas in the two-body case some finite-difference analogs of the ordinary Schrödinger equation can even be solved analytically,[77] great technical difficulties arise in the three-body case. It therefore seems to us that it is advisable to solve the relativistic system of Faddeev equations in the momentum representation.

As noted by the authors of Ref. 2, the method of Horáček and Sasakawa[79] used in Ref. 2 is well suited to calculations in both the coordinate and the momentum representations. Let us briefly describe the main idea of this method.

We shall study the solution of the system of Faddeev equations written in the form (123)–(124) or (123), (126), (129). Assume that we have somehow made sure that the function F_0 is in some sense close to χ (in what sense will become clear in what follows). We use the notation $A_0(s) = A(s), F_{n+1} = -A_n(s)F_n, A_{n+1}(s) = A_n(s)P_n$, where P_i is the orthogonal projector perpendicular to F_i, i.e., the operator acting according to the rule

$$P_iF = F - \frac{(F_i, F)}{(F_i, F_i)} F_i. \tag{133}$$

We also define the functions $\Phi_{n-1} = [1 + A_n(s)]^{-1}F_n$. Then it is easy to show that

$$\Phi_{n-1} = F_n + \frac{(F_n, F_n)}{(F_n, F_n) - (F_n, \Phi_n)} \Phi_n \tag{134}$$

and the following is valid:

Proposition. $(F_n, \Phi_n) = (F_n, F_n)$ if and only if $\Phi_n = -A_n(s)\Phi_n$.

In particular, Φ_0 is the solution (123) (i.e., $\Phi_0 = \chi$) if and only if $(\Phi_0, F_0) = (F_0, F_0)$.

Suppose, for example, that we have accidentally guessed the solution and the binding energy, i.e., it has turned out that $F_0 = \chi$. Then it is obvious that $F_1 = F_0$, $F_2 = F_3 = \ldots = 0$, $\Phi_0 = F_0$. If F_0 is close to χ, and s is close to the true triton mass, then it is natural to expect that F_n rapidly falls off with increasing n, since F_n is obtained by $n!$ applications of the orthogonal projection perpendicular to F_0, $F_1, \ldots, F_{n-1}$. Therefore, for some $n = N_0$ we can take $\Phi_{N_0} = F_{N_0+1}$. Then it follows from (134) that

$$\Phi_{N_0-1} = F_{N_0} + \frac{(F_{N_0}, F_{N_0})}{(F_{N_0}, F_{N_0}) - (F_{N_0}, F_{N_0-1})}, \tag{135}$$

and we can carry out the iteration procedure $F_0 \to F_1 \to \ldots F_{N_0+1} = \Phi_{N_0}$. Then we calculate $(F_i, \Phi_{N_0}) = (F_i, F_{N_0+1})$ for $0 \leqslant i \leqslant N_0$ and start from the recursion equation following from (134):

$$(F_i, \Phi_{n-1}) = (F_i, F_n) + \frac{(F_i, \Phi_n)(F_n, F_n)}{(F_n, F_n) - (F_n, \Phi_n)} \quad (0 \leqslant i \leqslant n-1). \tag{136}$$

In the end, this equation can be used to calculate (F_0, Φ_0). Therefore, to find the triton binding energy we can proceed as follows. We fix some sufficiently large N_0 and for various s we go through the procedure of calculating (F_0, Φ_0). The true triton mass then corresponds to only that value of s for which $(F_0, \Phi_0) = (F_0, F_0)$. It is also clear that, starting from (134) and (135), we can recursively calculate also the function Φ_0 itself satisfying Eq. (123).

It follows from the above Proposition that this procedure is valid if the operators $A_n(s)$ do not have eigenvalues equal to -1 for $n \geqslant 1$ and s close to the true triton mass, and it is this condition that must be understood as F_0 being close to χ. Owing to the presence in $A_n(s)$ of orthogonal projectors perpendicular to $F_0, F_1, \ldots, F_{n-1}$, this condition is natural.

In an actual calculation the amount of machine time needed can be reduced by also using nonorthogonal projectors (see Ref. 2 for details), but our goal is only to present the idea behind the technique. One reason why this technique is convenient is, in particular, that the kernel of the operator $T(s)$ is used only for s located to the left of the leftmost singularity of this kernel for $s = m + m_d$, where m_d is the deuteron mass. In our case we can take for F_0, for example, simply the set $\{\tilde{\chi}_1(k,Q), \ldots, \tilde{\chi}_{\tilde{N}}(k,Q)\}$, where $\{\tilde{\chi}_1(q,Q), \ldots, \tilde{\chi}_{\tilde{N}}(q,Q)\}$ is the solution of the standard nonrelativistic problem, and, in principle, it is not necessary to take $\tilde{N} = N$, but we can take $\tilde{N} < N$, setting $\tilde{\chi}_n = 0$ for $\tilde{N} < n \leqslant N$.

Of course, we do not exclude the possibility that other computational techniques can also be effective. For example, in the standard nonrelativistic approach the so-called hybrid method suggested by Dzhibuti (see, for example, the review of Ref. 80 and also Ref. 81) appears to be very promising. Here, by using the Faddeev expansion in a hyperspherical basis, it is possible to reduce the problem to a system of one-dimensional integral equations in the momentum representation. Then one uses the fact that the kinetic energy T^{nr} is a quadratic form in the momenta $\mathbf{q}$ and $\mathbf{Q}$ and does not change under any particle interchanges. However, in the relativistic case the analog of T^{nr} is the free mass operator M, and no function of it is a quadratic form in the momenta $\mathbf{k}$ and $\mathbf{Q}$.

Until recently, calculations of the RC to the triton binding energy have been carried out in the s-channel approximation, neglecting spin effects and taking into account only the first terms in the expansion of powers of $1/m^2$ (Refs. 64–66 and 82–86). Here the contributions of the RC to the kinetic and potential energies are separately substantial, but owing to strong cancellation of these contributions the resulting effect in the triton binding energy is no larger than about 0.2 MeV (which has led some physicists to maintain that relativistic effects do not play an important role in the problem of the binding-energy deficiency of the lightest nuclei). The authors of Ref. 66 carried out the relativistic calculation of the triton binding energy in the s-channel approximation, but without expanding in powers of $1/m^2$. Here it turned out that the results greatly differed from those obtained when only the first term in the expansion in $1/m^2$ was taken into

account. As already noted, the first term in the expansion in $1/m^2$ leads to a reasonable result if the triton WF falls off sufficiently rapidly at large relative momenta, but in practice the rate of falloff can be insufficient, and this can be shown by a very simple example—that of the deuteron. For example, whereas in most realistic models the result of calculating the RC to the deuteron magnetic moment in first order in $1/m^2$ is characterized by an error of 10–20% (Refs. 87 and 88), for the model of the Nuclear Physics Institute at Moscow State University these results differ by almost a factor of 2 (Ref. 24), and in most cases there is a large difference for the RC to the quadrupole moment.[88] It can be concluded that the question of how close are the exact and approximate values of the energy can be settled only by comparing the corresponding numerical results.

In Ref. 66 the numerical value of the RC to the triton binding energy also turned out to be small (0.2 MeV), and it was suggested that the result of the earlier study of Ref. 89, in which the value -1.7 MeV was obtained, was incorrect. Values of the desired RC no larger than 0.25 MeV have also been obtained in approaches differing from RQM (for example, in the quasipotential and Bethe–Salpeter approaches), but in those cases calculations have been carried out only in the s-channel approximation.[90]

The RC to the triton binding energy was recently calculated in Ref. 91 using the five-channel approximation for the Reid soft-core potential, and the calculation was carried out in first order in $1/m^2$. The equations derived in Ref. 91 are a special case of Eqs. (99), (101), and (103)–(106) for $N = 5$ and neglecting the P-wave contribution [see the discussion of Eq. (101)]. The result -0.54 MeV was mainly due to the D wave, since when only two channels were included the RC was -0.10 MeV (in agreement with earlier calculations), and when three channels were included it was -0.50 MeV. This result might appear strange, since the D-wave probability in the triton is no greater than 10%. However, it should be remembered that the RC is mainly determined by the high-momentum part of the triton WF, where the D wave is much more important than the S wave. In the case of the RC to the deuteron magnetic moment, the dominant role of the D wave was first demonstrated in Ref. 87. In the case of the RC to the triton binding energy this effect must be even more pronounced, since this quantity is determined by the expectation values of the fourth power of the momenta [see (56)], rather than the second, as in the case of the RC to the deuteron magnetic moment.[87,88] It is also well known that in the calculation of the contribution of three-particle forces in the 34-channel approximation, the contribution of channels with high numbers is comparable with that of channels with small numbers.[2,11]

In view of the above discussion, it appears to be very important to carry out the relativistic calculation including a sufficiently large number of channels. As noted above, the need to include Wigner rotations and more complicated kinematic relations leads to great complications in the relativistic problem compared with the standard case. In our approach this led to a very complicated expression for the operator $\hat{P}$. Yet another complication arises from the fact that the operator v enters into V under the square root (the need for this was first pointed out in Ref. 44), so that to express V in terms of v it is necessary to first calculate the operator $t(s)$ [see (115)] and then carry out the integration in (114). Owing to these features, the relativistic calculation apparently requires a larger amount of machine time than the nonrelativistic one, but it is possible to carry out this calculation on powerful computers. The result of this calculation will definitively answer the question of what is the contribution of relativistic effects to the triton binding energy, after which the role of three-nucleon interactions and quark degrees of freedom will certainly become clearer. Then the relativistic multichannel calculation should be carried out also for other three-nucleon observables, since for many of them the agreement between theory and experiment is still unsatisfactory.

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