Dispersion approach to the problem of including meson and quark–gluon degrees of freedom in hadron–hadron interactions at intermediate energies

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Hadron–hadron scattering is studied in the S-matrix formalism, taking into account particle (hadron) exchanges in the t and u channels and the interaction mechanism related to the formation of a compound system (a quark–gluon bag) in the intermediate state in the s channel. A general representation is obtained for the partial-wave scattering amplitude which is consistent with unitarity, analyticity, relativistic invariance, and the confinement condition. Its relation to the P-matrix formalism of Jaffe and Low is determined. In this approach the solution of the dispersion relations for the partial-wave amplitudes contains the CDD-type poles of some "renormalized" amplitude, which are the relativistic analogs of the P-matrix poles and correspond to the eigenstates of the compound system. These poles carry information about the interaction at short distances, and the dynamical cuts of the partial-wave amplitude arising from the exchange mechanisms in the t and u channels determine the peripheral part of the interaction. The contributions of distant singularities are strongly suppressed by the quark–gluon bag-formation mechanism. This resolves a fundamental problem in meson theories of nuclear forces. The effects of coupling of the compound quark bag to the scattering channels are studied. The method is used to analyze NN, $\pi\pi$, πK, πN, and KN scattering at low and intermediate energies.

INTRODUCTION

For almost half a century the meson picture of strong interactions has served (and to a considerable degree still does serve) as the basis for various phenomenological or semiphenomenological approaches used to describe hadron scattering processes at low and intermediate energies (see, for example, Refs. 1–3). Meson theories of nuclear forces naturally describe the peripheral interaction region, but encounter serious difficulties at short distances. For example, when realistic values of the ω-meson–nucleon coupling constant are used in a meson theory it is difficult to understand the nature of the repulsive core in NN scattering.[1,4]

In spite of the significant progress made in understanding the features of strong interactions using quantum chromodynamics (QCD) (Refs. 5–13), the construction of a hadron scattering theory including the quark–gluon structure of hadrons at low and intermediate energies, where nonperturbative effects play an important role, is a very complicated and as yet unsolved problem. Therefore, at the present stage of theoretical studies the various approximation methods which to some degree allow the inclusion of the features of QCD are very important. Quark–gluon systems have been studied using QCD sum rules,[14,15] a method based on the $1/N_c$ expansion,[16,17] which leads to topological chiral models,[18–20] and lattice gauge theories.[21–23] A number of more phenomenological approaches have also been proposed.[24–53] It should be noted that these methods have been used most successfully to describe the static properties of hadrons.

Along with potential models of constituent quarks,[24–30] beginning in the late 1960s an approach was developed to modeling the properties of QCD based on relativistic bag models.[31–51] The Dubna quark-bag model first proposed in 1967 (Refs. 31 and 36) was the first to realize the fundamental idea of the approach in which hadronic states are formed owing to localization (confinement) of the fields of valence quarks and gluons in restricted spatial cavities determining the hadron sizes. This approach was developed further in the MIT bag model,[32–34] in which effects related to cavity formation in the physical QCD vacuum were taken into account and the bag equilibrium condition was formulated.

Bag models led not only to the successful description of the static properties of hadrons,[31–36] but also predicted the existence of a number of multiquark systems,[37–44] many of which, however, have not been seen in the hadron mass spectrum. The calculations carried out using the MIT bag model[32–34] predicted, for example, states of the Q^6 system with the quantum numbers of two nucleons in different channels and masses in the range 2.16–2.4 GeV (Refs. 40 and 43), and a state of the $Q^2\overline{Q}^2$ system with mass ~ 0.65 GeV (Refs. 37 and 43) and the quantum numbers of two π mesons, $I = J = 0$ (I is the isospin and J is the angular momentum), in addition to a number of other states.

The masses of these multiquark systems are calculated using the boundary conditions of the MIT bag model, which ensure that the confinement condition is satisfied in all channels, including color-singlet ones where it need not be. Therefore, the boundary conditions of the MIT bag model in color-singlet channels are clearly artificial, and the question arises of the status of multiquark bag states. Jaffe and Low[54] suggested that these states (referred to by them as "primitives") be associated not with poles of the S matrix, as for ordinary hadronic states, but with poles of the P matrix, which is closely related to the Wigner–Eisenbud R matrix.[55] In the one-channel case the P matrix is the logarithmic derivative of the wave function describing the relative motion of the colliding particles at some value[1)] $r = b$ (r is the relative separation of the hadron centers of mass) linearly related to the radius R of the compound quark bag formed in the fusion of hadronic quark bags into a single common bag, i.e.,

$$b = cR, \tag{1}$$

where c is a coefficient for which there are simple estimates.[43,54] For example, in the case of massless quarks[43]

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$$c \cong 1{,}37\ (n/n_1n_2)^{1/2}, \quad (2)$$

where n_1 and n_2 are the numbers of valence quarks and antiquarks in the colliding hadrons and $n = n_1 + n_2$.

In analyzing the available experimental data on meson–meson ($\pi\pi$ and πK) scattering using the P-matrix formalism, Jaffe and Low[54] discovered the agreement between the locations of the P-matrix poles closest to threshold and the calculations of the masses of the lowest-lying states of the $Q^2\overline{Q}^2$ system with the quantum numbers of the corresponding scattering channels carried out using the MIT bag model. After the work of Ref. 54, P-matrix analyses of other hadron–hadron scattering processes were also carried out, and definitively showed that at least the P-matrix poles closest to threshold correspond to the lowest levels of compound quark bags.[56–64] In particular, the model of rotating quark bags[44] was used to interpret the data on dibaryon resonances.[65]

The approach of Jaffe and Low[54] was developed further in the quantum-mechanical compound quark bag (CQB) model proposed by Simonov,[66,67] in which the full wave function of the system is represented as the sum $\Psi_{\rm Cl} + \Psi_Q$, where $\Psi_{\rm Cl}$ describes the relative motion of the hadrons as a quark clusters and Ψ_Q is the CQB wave function. In the original version of the CQB model[66] it was assumed that the quark and hadron channels are orthogonal to each other and coupled only in an infinitesimally narrow region on the surface of the quark bag. In the later version of the CQB model[67,68] it was assumed that the hadron and quark channels are not, in general, orthogonal. In this variant of the model the effective hadron interaction at short distances is described by a nonlocal, energy-dependent potential of the form $V_Q(r,r';E) = \Sigma_\nu f_\nu(r,E) f_\nu(r',E)(E-E_\nu)^{-1}$, where the form factors $f_\nu(r,E)$ are the sum of two terms, one determining the surface interaction of a hadron channel with the compound quark bag, and the other determining the volume interaction arising because of the nonorthogonality and containing a linear energy dependence. The CQB model has been used to analyze NN (Refs. 69–77), πN (Ref. 78), and KN (Ref. 79) scattering and certain other hadronic systems.[76,77,80,81]

To describe hadron–hadron scattering, de Tar proposed[82,83] the two-center MIT bag model. The procedure used to calculate the effective hadron–hadron potential in the Schrödinger equation in the de Tar method is based on use of the adiabatic approximation, which can hardly be considered valid for describing hadron–hadron scattering processes at intermediate energies. We also note the theory of extension of nonrelativistic Hamiltonians with internal degrees of freedom lying in an auxiliary space[84] and a number of other models,[4,85–87] the detailed discussion of which, however, lies outside the scope of the present study.

A relativistically invariant approach to the problem of the inclusion of the meson and quark degrees of freedom in hadron–hadron interactions based on the use of unitarity and the analyticity properties of the partial-wave scattering amplitudes was first proposed in Ref. 88 and then used to analyze specific processes in Refs. 89–102. This approach is based on analytic S-matrix theory, which allows the *ab initio* inclusion of the most fundamental physical principles assumed as the starting point in constructing the formalism: unitarity, analyticity, relativistic invariance, and the requirement of confinement.

In Secs. 2–6 we give the theoretical fundamentals of this method, and in Secs. 7 and 8 we study its application for describing NN, $\pi\pi$, πK, πN, and KN scattering processes at low and intermediate energies.

1. CONFINEMENT AND THE ANALYTIC STRUCTURE OF THE SCATTERING AMPLITUDE

The confinement requirement means that only colorless complexes of quarks and gluons, i.e., hadrons, can propagate freely in the physical vacuum of QCD. The QCD vacuum is an opaque medium with respect to the fundamental constituents of hadrons—quarks and gluons, and also colored clusters. It should be noted that a rigorous mathematical formulation of confinement in field theory is difficult,[13] and the study of the proposed confinement criteria[21,52] requires the development of effective methods which are not based on perturbation theory. However, it is possible to formulate the confinement condition directly in the language of analytic S-matrix theory.

Actually, any particle scattering experiment fundamentally amounts to the following. The initial state of two or more noninteracting particles at infinite separation (for simplicity, we assume that the Coulomb and gravitational interactions are switched off) is fixed. This can be associated with the state vector of the system $|\alpha,{\rm in}\rangle$ an infinite time in the past ($t = -\infty$). After the particles approach and interact with each other, the final state is fixed. It is described by the state vector $|\beta,{\rm out}\rangle$ at an infinite time in the future ($t=\infty$). We stress that in the present context the term "particle" is used for any object, including composite ones (hadrons, nuclei), appearing in the asymptotic in- and out-states and separated by infinitely large space-time intervals. Here the hyperplanes on which the in- and out-states are defined are separated by an infinitely large timelike interval.

Composite objects are understood as objects consisting of elementary constituents localized in restricted regions of configuration space. The particles entering into the composition of the asymptotic states are real, i.e., they are on-shell ($p_i^2 = m_i^2$, where p_i is the 4-momentum of particle i and m_i is its mass). The confinement condition states that neither quarks nor gluons themselves, nor any complexes of them which are not color singlets, can be realized in nature as "particles" entering into the composition of the asymptotic in- and out-states.

As our basic principles for constructing the theory we take the following postulates fundamental to analytic S-matrix theory (see, for example, Ref. 103): 1) the superposition principle; 2) completeness and orthogonality of the in- and out-states; 3) existence of the unitary S operator relating sets of in- and out-states; 4) Lorentz invariance of the S operator; 5) cluster separability,[104] which is a consequence of the finiteness of the range of the strong interaction; 6) maximal analyticity of the first degree, according to which the only singularities of the S matrix can be poles corresponding to one-particle states and also cuts generated by these poles owing to unitarity.

As is well known, unitarity of the S matrix physically implies that the sum of the probabilities for transitions into all channels in a scattering process is equal to unity, and analyticity is a consequence of one of the most fundamental physical principles determining interaction dynamics—the principle of microcausality.[105,106] The requirements of unitarity and analyticity necessarily lead to the consequence that the S matrix must have branch points in the invariant variables, which are determined by the thresholds of the scattering channels. The sixth postulate requires that on the physical sheet the S matrix has, in addition to poles corresponding to one-particle states, only those singularities needed to satisfy the unitarity conditions in all possible scattering channels.

It therefore follows from the principle of analyticity that only the on-shell particles involved in the asymptotic in- and out-states can generate singularities on the physical sheet in the scattering amplitudes. Therefore, owing to the confinement requirement, in strong interactions the only source of such singularities can be colorless complexes, i.e., hadronic states. The residues of the scattering amplitudes at the poles and the discontinuities on the cuts are determined by the renormalized coupling constants and by the amplitudes for processes involving on-shell particles (the Cutkosky rule[107]).

Therefore, the manifestations of hadronic degrees of freedom are clearly defined in the language of S-matrix theory: They are effects due to the poles and cuts of the scattering amplitude on the physical sheet in the invariant variables. Meanwhile, it is well known that the specification of the discontinuity of the partial-wave amplitude on a dynamical cut and the unitarity condition do not uniquely determine the solution of the scattering problem.[103,108,109] As long ago as the mid-1950s it was suggested that this ambiguity, which is related to the presence of Castillejo–Dalitz–Dyson (CDD) poles, can be a consequence of internal structure of the particles (Refs. 108 and 110).[2)]

Of course, it should be borne in mind that it is in practice impossible to calculate the contributions of dynamical cuts far from the physical region, which correspond to hadronic exchange mechanisms at short distances. In the light of new ideas about the nature of strong interactions, however, it is clear that at short distances a decisive role must be played by the quark–gluon structure of hadrons. Here "asymptotic freedom" (Refs. 7 and 8) is of fundamental importance. Therefore, the problem of constructing a general representation for the S matrix which is compatible with the fundamental physical principles discussed above is solved in the approach described below[88–102] in such a way that the short-distance (on the order of the confinement radius) interaction in the system can be treated in terms of quark–gluon states. As far as the interaction in the peripheral region is concerned, it is determined by the closest dynamical cuts and is described in the language of hadron exchanges. Therefore, this approach permits the meson picture of nuclear forces at large distances to be joined to the quark–gluon picture at short distances.

2. SOLUTION OF THE DISPERSION RELATIONS FOR THE PARTIAL-WAVE AMPLITUDES BY THE GENERALIZED R-FUNCTION METHOD

For simplicity, let us consider the case of one-channel scattering of two spinless particles with equal masses m. The reduced partial-wave amplitude $A_l(s)$, which is related to the partial-wave S matrix as

$$S_l = 1 + 2i\rho_l(s)\,A_l(s), \tag{3}$$

where

$$\rho_l(s) = 2k^{2l+1}/\sqrt{s}, \tag{4}$$

s is the square of the energy in the c.m. frame, and $k = (s/4 - m^2)^{1/2}$ is the relative momentum of the colliding particles, has, in addition to the right-hand (unitarity) cut C_R ($s_R \leqslant s \infty$, $s_R = 4m^2$), a left-hand (dynamical) cut C_L on the real axis ($-\infty < s \leqslant s_L$), the discontinuity across which, Im $A_l(s)$, due to hadronic exchanges in the t and u channels is assumed to be known. The general solution of the scattering problem compatible with the initial requirements is found in two steps. In the first we construct a particular solution containing the dynamical cut of the amplitude $A_l(s)$ and depending on the external parameter b, which determines the splitting of the interaction into "external" and "internal" parts. Then in the second stage the problem reduces to the construction of the generalized R function[111] with given discontinuity on the right-hand cut.

We therefore write the partial-wave S matrix (1) as

$$S_l = \frac{D_l^{(-)}(b,\ s)}{D_l^{(+)}(b,\ s)}\,\widetilde{S}_l, \tag{5}$$

where the function $D_l^{(+)}(b,s)$ has only a right-hand cut C_R, and $D_l^{(-)}(b,s)$ is the analytic continuation of $D_l^{+}(b,s)$ onto the second sheet of the Riemann surface in the variable s relative to the cut C_R. Below, we shall formulate a system of equations to calculate these functions. We introduce the function $\widetilde{A}_l(b,s)$, henceforth referred to as the renormalized amplitude (RA) according to the definition

$$\widetilde{S}_l = 1 + 2i\widetilde{\rho}_l(b,\ s)\,\widetilde{A}_l(b,\ s), \tag{6}$$

where

$$\widetilde{\rho}_l(b,\ s) = \rho_l(s)\,[D_l^{(+)}(b,\ s)\,D_l^{(-)}(b,\ s)]^{-1}. \tag{7}$$

From (3)–(7) it follows that $A_l(s)$ and $\widetilde{A}_l(b,s)$ are related to each other as

$$A_l(s) = \bar{A}_l(b,\ s) + [D_l^{(+)}(b,\ s)]^{-2}\widetilde{A}_l(b,\ s), \tag{8}$$

where

$$\bar{A}_l(b,\ s) = [D_l^{(-)}(b,\ s)/D_l^{(+)}(b,\ s) - 1]/2i\rho_l(s). \tag{9}$$

We require that the RA $\widetilde{A}_l(b,s)$ have no left-hand cut C_L. Since disc $D_l^{(+)}(b,s) = 0$ on C_L, it follows from (8) that this requirement is equivalent to the following condition on the dynamical cut:

$$\text{disc}\,A_l(s) = \text{disc}\,\bar{A}_l(b,\ s). \tag{10}$$

The functions $D_l^{(\pm)}(b,s)$ are written as

$$D_l^{(\pm)}(b,\ s) = d_l^{(\pm)}(bk)\,\Delta_l^{(\pm)}(b,\ s), \tag{11}$$

where

$$d_l^{(+)}(x) = ix^l h_l^{(1)}(x)/(2l-1)\,!!,\quad d_l^{(-)}(x) = d_l^{(+)}(-x), \tag{12}$$

in which $h_l^{(1)}(x) = \sqrt{(\pi x/2)}\,H_{l+1/2}^{(1)}(x)$ is the Riccati–Hankel function of the first kind. The analytic properties of the functions $d_l^{(+)}(x)$ are studied in Appendix A.

The physical meaning of the representation of the S matrix in the form (5)–(12) is easiest to understand by considering the scattering in configuration space using exactly solvable models. In Sec. 5 we discuss this question in more detail.

We define the amplitude $T_l(b,s)$ related to $\overline{A}_l(b,s)$ (9) as

$$\overline{A}_l(b,\ s)=\frac{d_l^{(-)}(bk)/d_l^{(+)}(bk)-1}{2i\rho_l(s)}+[d_l^{(+)}(bk)]^{-2}T_l(b,\ s). \tag{13}$$

Taking into account (9), (11), and (13), $T_l(b,s)$ can be written as the ratio of two functions:

$$T_l(b,\ s)=n_l(b,\ s)/\Delta_l(b,\ s), \tag{14}$$

where

$$n_l(b,\ s)=[\Delta_l^{(-)}(b,\ s)-\Delta_l^{(+)}(b,\ s)]/2i\rho_l(s); \tag{15}$$

$$\Delta_l(b,\ s)=\Delta_l^{(+)}(b,\ s), \tag{16}$$

and $\Delta_l(b,s)$ has only a right-hand cut, while $n_l(b,s)$ has only a left-hand cut on the physical sheet. The unitarity condition for $T_l(b,s)$ has the form

$$\operatorname{Im} T_l^{-1}(b,\ s)=-\sigma_l(b,\ s)\,\theta(s-4m^2), \tag{17}$$

where

$$\sigma_l(b,\ s)=\rho_l(s)\,[d_l^{(+)}(bk)\,d_l^{(-)}(bk)]^{-1};\quad \theta(x)=1\ (0);\ x>0\ (x<0). \tag{18}$$

From (10) and (14), taking into account the Schwarz principle for the functions $A_l(s)$ and $\overline{A}_l(b,s)$ $[A_l(s^*)=A_l^*(s),\overline{A}_l(b,s^*)=\overline{A}_l^*(b,s)]$, it follows that the discointinuity of $T_l(b,s)$ is related to the discontinuity of $A_l(s)$ on the dynamical cut C_L as

$$\operatorname{Im} T_l(b,\ s)=\operatorname{Im} A_l(s)\,[R_l^{(+)}(s)]^2\exp\left(-b\sqrt{4m^2-s}\right)\times\theta(s_L-s)+\frac{(-1)^l}{2}\,|\rho_l(s)|^{-1}R_l^{(+)}(s)\,[R_l^{(-)}(s)-R_l^{(+)}(s)\exp\left(-b\sqrt{4m^2-s}\right)]\,\theta(-s), \tag{19}$$

where

$$R_l^{(\pm)}(s)=1+\sum_{n=1}^{l}a_n^{(l)}\left(\pm\frac{b}{2}\sqrt{4m^2-s}\right)^n, \tag{20}$$

and $a_n^{(l)}$ are positive coefficients (A2).

We note that the second term on the right-hand side of (19), arising from relativistic effects tends for $s\to-\infty$ to a positive constant. In order to avoid the technical complication due to the fact that $\operatorname{Im} T_l(b,s)$ (19) does not tend to zero for $s\to-\infty$, we modify the definition (3) of the amplitude by making the replacement $\rho_l(s)\to\rho_l(s)\sqrt{(s+s_a)/(4m^2+s_a)}$, where s_a is a large positive constant ($s_a\gg 4m^2$) playing the role of a cutoff parameter. In the region $|s|\ll s_a$ the modified definition of $A_l(s)$ differs from the standard one by an infinitesimal amount. However, in this case the second term in the expression for the discontinuity of $T_l(b,s)$ will contain an additional cutoff factor of the form $\theta(s+s_a)$.[95,101]

As a result, we obtain the following system of linear integral equations for calculating the functions $n_l(b,s)$ and $\Delta_l(b,s)$:

$$n_l(b,\ s)=\frac{1}{\pi}\int_{-s_a}^{s_L}\frac{\operatorname{Im} T_l(b,\ s')\,\Delta_l(b,\ s')}{s'-s}\,ds'; \tag{21}$$

$$\Delta_l(b,\ s)=1-\frac{s-s_0}{\pi}\int_{s_R}^{\infty}\frac{\sigma_l(b,\ s')\,n_l(b,\ s')}{(s'-s-i\eta)\,(s-s_0)}\,ds'. \tag{22}$$

Equations (21) and (22) appear to be similar to the usual equations of the N/D method.[103,109] However, this is only superficial. There are fundamental differences between them. The most important one is that in the expression for $\operatorname{Im} T_l(b,s)$ on the dynamical cut (19), the first term, proportional to the discontinuity of $A_l(s)$, contains an exponentially decreasing factor for $s\to-\infty$, which suppresses the contribution of dynamical singularities far from the physical region. Therefore, the solutions of Eqs. (21) and (22) are mainly determined by the singularities of the scattering amplitude in the invariants t and u closest to the physical region. Another difference is related to the form of the spectral function $\sigma_l(b,s)$ (18) in Eq. (22). For $s\to s_R$ it has the threshold behavior k^{2l+1} and goes as $\sqrt{s}$ for $s\to\infty$. For this reason the solution of the system of equations (21), (22) for arbitrary l does not involve the difficulties encountered in solving the usual equations of the N/D method[3)] (see for example, Ref. 112).

Therefore, the specification of $\operatorname{Im} A_l(s)$ on C_L and the parameter b with the use of Eqs. (21) and (22) and the relations (11) and (19) fixes the definition of the functions $D_l^{(\pm)}(b,s)$ in Eqs. (5), (7), and (9). We stress that from the formal point of view the specific method used to construct the functions $D_l^{(\pm)}(b,s)$ does not, of course, at all restrict the generality of the desired solution for the S matrix, since no restrictions on $\widetilde{S}_l$ in Eq. (5) we assumed. Now let us obtain the general solution for the renormalized amplitude $\widetilde{A}_l(b,s)$, which does not have a left-hand cut and satisfies the unitarity condition in the form

$$\operatorname{Im}\widetilde{A}_l(b,\ s)=\widetilde{\rho}_l(b,\ s)\,|\widetilde{A}_l(b,\ s)|^2\theta(s-4m^2). \tag{23}$$

We shall further assume that the function $\Delta_l(b,s)$ determined by solving the system (21), (22) does not have zeros on the first sheet in the s plane (i.e., when only the "external" part of the interaction is switched on, there are no bound states in the system). Otherwise, it is necessary to somewhat modify the following discussion to include such zeros.

If we assume that $A_l(s)\to 0$ for $|s|\to\infty$, then, as is easily verified, on the physical sheet $\widetilde{A}_l(b,s)$ also tends to zero for $|s|\to\infty$, so that we can write down the following dispersion relation for it:

$$\widetilde{A}_l(b,\ s)=-\sum_{n=1}^{N}\frac{\beta_{ln}}{s-\alpha_{ln}}+\frac{1}{\pi}\int_{s_R}^{\infty}\frac{\widetilde{\rho}_l(b,\ s')\,|\,\widetilde{A}_l(b,\ s')\,|^2}{s'-s-i\eta}\,ds', \tag{24}$$

where the poles at the points $s=\alpha_{ln}$ correspond to bound states of the system; the residues β_{ln} at these poles are real positive constants. The renormalized amplitude $\widetilde{A}_l(b,s)$ possesses the following properties: 1) it is a meromorphic function in the s plane with a cut on the real axis ($s_R<s<\infty$); 2) $\widetilde{A}_l^*(b,s)=\widetilde{A}_l(b,s^*)$; 3) for points of the complex s plane not belonging to the segment of the real axis $s>s_R$ we have

$$\mathrm{Im}\,\widetilde{A}_l(b, s) = \Lambda_l(s)\,\mathrm{Im}\, s, \tag{25}$$

where

$$\Lambda_l(s) = \sum_{n=1}^{N} \frac{\beta_{ln}}{|s-\alpha_{ln}|^2} + \frac{1}{\pi}\int_{s_R}^{\infty} \frac{\widetilde{\rho}_l(b, s')\,|\widetilde{A}_l(b, s')|^2}{|s'-s|^2}\,ds' \tag{26}$$

is a real positive-definite function.

If all the constants β_{ln} are equal to zero (i.e., the system does not have bound states), the function $\widetilde{A}_l(b,s)$ cannot have zeros anywhere except at points on the real axis for $s \geqslant s_R$. When there are N bound states in the system, it is easy to see from Eq. (24) that there can be one zero on each segment of the real axis between the threshold point $s = s_R$ and the first pole, and also between two adjacent poles of the function $\widetilde{A}_l(b,s)$. Therefore, for $s < s_R$, $\widetilde{A}_l(b,s)$ can have no more than N zeros. These features indicate that $\widetilde{A}_l(b,s)$ is a generalized R function with right-hand cut on the real axis ($s_R \leqslant s < \infty$). After studying the properties of the inverse function $\widetilde{A}_l^{-1}(b,s)$, we conclude that it is also a generalized R function with known value of the discontinuity on the right-hand cut. Using the Herglotz theorem,[108,111] we obtain for it a general solution of the form

$$\widetilde{A}_l^{-1}(b, s) = \widetilde{P}_l(b, s) - \frac{s-s_0}{\pi}\int_{s_R}^{\infty} \frac{\widetilde{\rho}_l(b, s')}{(s'-s-i\eta)(s'-s_0)}\,ds'; \tag{27}$$

$$\widetilde{P}_l(b, s) = a_l(b, s_0) - c_l(b)\,s + s\sum_i \frac{\gamma_{li}(b)}{s_{li}(b)\,[s-s_{li}(b)]}, \tag{28}$$

where $a_l(b,s_0)$, $c_l(b)$, $\gamma_{li}(b)$, and $s_{li}(b)$ are real constants depending parametrically on b, with $c_l(b), \gamma_{li}(b) \geqslant 0$; if the system does not have bound states, we have all the constants $s_{li}(b) > s_R$, and when there are N bound states, no more than N poles in expression (28) can be located below threshold, while expression (28) cannot have more than one pole between the threshold s_R and the point $s = \max\{\alpha_{ln}\}$, and also between two adjacent points $s = \alpha_{lm}$, α_{lm+1} ($m = 1,...,N-1$).

The poles of the function $\widetilde{P}_l(b,s)$ are the zeros of the renormalized amplitude $\widetilde{A}_l(b,s)$, which justifies our calling them CDD(RA) poles. We use this terminology only owing to the mathematical analogy between the poles of (28) and the poles introduced in scattering theory by Castillejo, Dalitz, and Dyson. However, there is no direct relation between these two types of poles.

Let us show that the representation (27), (28) leads to certain restrictions on the constants β_{ln} in Eq. (24). In fact, from these equations we find

$$\beta_{ln} = \left\{c_l(b) + \sum_i \frac{\gamma_{li}(b)}{[s_{li}(b)-\alpha_{ln}]^2} + J_{ln}(b)\right\}^{-1}, \tag{29}$$

where

$$J_{ln}(b) = \frac{1}{\pi}\int_{s_R}^{\infty} \frac{\widetilde{\rho}_l(b, s')}{(s'-\alpha_{ln})^2}\,ds' > 0, \tag{30}$$

from which, taking into account the positivity of the constants $c_l(b)$ and $\gamma_{li}(b)$, we find the inequality

$$\beta_{ln} \leqslant J_{ln}^{-}(b). \tag{31}$$

As a result, for the vertex constant related to the residue of the partial-wave amplitude as

$$g_{ln}^2 = (-1)^{l+1} m^{-1} \lim_{s\to\alpha_{ln}} [(s-\alpha_{ln})\,k^{2l} A_l(s)], \tag{32}$$

from the inequality (31) we obtain the bound[101,113,114]

$$g_{ln}^2 \leqslant \varkappa_{ln}^{2l} m^{-1} J_{ln}^{-1}(b)\,[D_l^{(+)}(b, \alpha_{nl})]^{-2}, \tag{33}$$

where $\varkappa_{ln} = (m^2 - \alpha_{ln}/4)^{1/2.}$ It can be shown[114] that the inequality (33) remains valid also when coupling to inelastic channels is included.

If we assume that the partial-wave amplitude does not have a left-hand cut, then in the case of S-wave scattering in the nonrelativistic limit [here $\rho_0(s)$ must be replaced by $(s/4m^2-1)^{1/2}$] from (33) we obtain the inequality

$$g_{0n}^2 \leqslant 2\varkappa_{0n}\exp(2b\varkappa_{0n}). \tag{34}$$

This limit was obtained earlier by Ruderman and Gasiorowicz[115] by assuming that the potential determining the interaction of two nonrelativistic particles is nonzero in the region $r \leqslant b$. The inequality (33) can be viewed as the generalization of the results of Ruderman and Gasiorowicz[115] and of Gribov, Zel'dovich, and Perelomov[116] to the (relativistic) case where the partial-wave amplitude has a left-hand cut. The constraint on the vertex constant was obtained in Ref. 116 by the dispersion-relation method using the theory of R functions, assuming zero-range forces. We note that the question of constraints on the vertex constants in the axiomatic approach to quantum field theory has been studied in Ref. 117 on the basis of the Källén–Lehmann representation.

3. GENERALIZATION OF THE FORMALISM TO THE MULTICHANNEL CASE

Let us generalize the representation for the partial-wave amplitude of elastic scattering obtained in the preceding section to the multichannel case. We consider transitions between two-partial states $|\lambda\rangle$ characterized by sets of quantum numbers λ. The partial-wave amplitudes for a given total angular momentum of the system J form an $N\times N$ matrix $\hat{A}_J(s)$ (where N is the number of two-particle channels explicitly included), which is related to the S matrix as

$$\hat{S}_J = \hat{I} + 2i\hat{\rho}^{1/2}(s)\,\hat{A}_J(s)\,\hat{\rho}^{1/2}(s), \tag{35}$$

where $\hat{\rho}(s)$ is a diagonal matrix with elements[4)]

$$\langle\lambda'|\hat{\rho}(s)|\lambda\rangle = \delta_{\lambda'\lambda}\rho_\lambda(s)\,\theta(s-4m^2), \tag{36}$$

with $\rho_\lambda(s)$ determined by an expression of the form (4) in which k and l must be replaced by the relative momentum $k_\lambda = (s/4 - m_\lambda^2)^{1/2}$ and the orbital angular momentum l_λ in the channel λ, respectively.

The matrix of partial-wave amplitudes $\hat{A}_J(s)$ is written as

$$\hat{A}_J(s) = \hat{\widetilde{A}}_J(\mathbf{b}, s) + [\hat{D}_J^T(\mathbf{b}, s)]^{-1}\hat{\widetilde{A}}_J(\mathbf{b}, s)\,[\hat{D}_J(\mathbf{b}, s)]^{-1}, \tag{37}$$

where the symbol T denotes transposition, $\mathbf{b} = \{b_\lambda\}$ is the set of N external parameters,

$$\hat{\hat{A}}_J(\mathbf{b}, s) = \hat{N}_J(\mathbf{b}, s)\hat{D}_J^{-1}(\mathbf{b}, s), \quad (38)$$

and the matrix elements $\hat{D}_J(\mathbf{b},s)$ have only right-hand cuts, while the matrix elements $\hat{N}_J(\mathbf{b},s)$ have only left-hand cuts, and in the physical region we have

$$\operatorname{Im}\hat{D}_J(\mathbf{b}, s) = -\hat{\rho}(s)\hat{N}_J(\mathbf{b}, s). \quad (39)$$

Let us first of all formulate the equations for the matrices $\hat{D}_J(\mathbf{b},s)$ and $\hat{\bar{A}}_J(\mathbf{b},s)$. We write the symmetric matrix $\overline{A}Q_J(\mathbf{b},s)$ as

$$\hat{\bar{A}}_J(\mathbf{b}, s) = \hat{A}_0(\mathbf{b}, s) + \hat{d}^{-1}(\mathbf{b}, s)\hat{T}_J(\mathbf{b}, s)\hat{d}^{-1}(\mathbf{b}, s), \quad (40)$$

where $d(\mathbf{b},s)$ and $\hat{A}_0(\mathbf{b},s)$ are diagonal matrices with the elements

$$\langle\lambda'|\hat{d}(\mathbf{b}, s)|\lambda\rangle = \delta_{\lambda'\lambda}\, d_{l_\lambda}^{(+)}(b_\lambda k_\lambda); \quad (41)$$

$$\langle\lambda'|\hat{A}_0(\mathbf{b}, s)|\lambda\rangle = \delta_{\lambda'\lambda}[d_{l_\lambda}^{(-)}(b_\lambda k_\lambda)/d_{l_\lambda}^{(+)}(b_\lambda k_\lambda) - 1]/2i\rho_\lambda(s).$$

Here $d_l^{(\pm)}(x)$ are the functions (12), and $\hat{T}_J(\mathbf{b},s)$ is a symmetric matrix for which the unitarity condition has the form

$$\operatorname{Im}\hat{T}_J^{-1}(\mathbf{b}, s) = -\hat{\sigma}_J(\mathbf{b}, s), \quad (42)$$

where the elements of the matrix $\hat{\sigma}_J(\mathbf{b},s)$ are determined by the expression

$$\langle\lambda'|\hat{\sigma}_J(\mathbf{b}, s)|\lambda\rangle = \delta_{\lambda'\lambda}\rho_\lambda(s)[d_{l_\lambda}^{(+)}(b_\lambda k_\lambda)\, d_{l_\lambda}^{(-)}(b_\lambda k_\lambda)]^{-1}. \quad (43)$$

Taking

$$\hat{D}_J(\mathbf{b}, s) = \hat{d}(\mathbf{b}, s)\hat{\Delta}_J(\mathbf{b}, s), \quad (44)$$

for $\hat{T}_J(\mathbf{b},s)$ we obtain an expression of the form

$$\hat{T}_J(\mathbf{b}, s) = \hat{n}_J(\mathbf{b}, s)\hat{\Delta}_J^{-1}(\mathbf{b}, s), \quad (45)$$

where the matrix elements $\hat{\Delta}_J(\mathbf{b},s)$ have only right-hand cuts, and the matrix elements $\hat{n}_J(b,s)$ have only left-hand cuts. We require that the following relation be satisfied on the dynamical cuts:

$$\operatorname{Im}\hat{A}_J(s) = \operatorname{Im}\hat{\bar{A}}_J(\mathbf{b}, s), \quad (46)$$

and we normalize $\hat{\Delta}_J(\mathbf{b},s)$ to the unit matrix for $|s|\to\infty$. Then to calculate $\hat{n}_J(\mathbf{b},s)$ and $\hat{\Delta}_J(\mathbf{b},s)$ we can write down a system of integral equations of the form (21), (22) in which the functions $n_l(b,s)$, $\Delta_l(b,s)$, $T_l(b,s)$, and $\sigma_l(b,s)$ must be replaced by the matrices $\hat{n}_J(\mathbf{b},s)$, $\hat{\Delta}_J(\mathbf{b},s)$, $\hat{T}_J(\mathbf{b},s)$, and $\hat{\sigma}_J(\mathbf{b},s)$, respectively.

Using Eqs. (37) and (46), for the imaginary parts of the matrix elements $\hat{T}_J(\mathbf{b},s)$ on the left-hand cuts we can easily obtain an expression which is the analog of (19) for the one-channel case. As in the case of one-channel scattering, the contributions of dynamical singularities far from the physical region are suppressed by exponential factors.

After $\hat{\bar{A}}_J(\mathbf{b},s)$ and $\hat{D}_J(\mathbf{b},s)$ have been determined by solving these equations, the problem reduces to finding the matrix $\hat{\tilde{A}}_J(\mathbf{b},s)$. The unitarity condition for it has the form

$$\operatorname{Im}\hat{\tilde{A}}_J(\mathbf{b}, s) = \hat{\tilde{A}}_J^*(\mathbf{b}, s)\hat{\tilde{\rho}}_J(\mathbf{b}, s)\hat{\tilde{A}}_J(\mathbf{b}, s), \quad (47)$$

where

$$\hat{\tilde{\rho}}_J(\mathbf{b}, s) = [\hat{D}_J^*(\mathbf{b}, s)]^{-1}\hat{\rho}(s)[\hat{D}_J^T(\mathbf{b}, s)]^{-1} \quad (48)$$

is a symmetric Hermitian matrix. It follows from (37) and (46) that disc $\hat{\tilde{A}}_J(\mathbf{b},s) = 0$ on C_L, and, using (47) for the matrix $\tilde{A}_J(\mathbf{b},s)$, we can write the representation

$$\hat{\tilde{A}}_J(\mathbf{b}, s) = -\sum_{n=1}^{N}\frac{\hat{\beta}_{Jn}}{s-\alpha_{Jn}} + \frac{1}{\pi}\int_{s_R}^{\infty}\frac{\hat{\tilde{A}}_J^*(\mathbf{b}, s')\hat{\tilde{\rho}}_J(\mathbf{b}, s')\hat{\tilde{A}}_J(\mathbf{b}, s')}{s'-s-i\eta}ds', \quad (49)$$

where the residues $\hat{\beta}_{Jn}$ at the poles corresponding to bound states are real symmetric matrices with zero determinant. It can be shown that the diagonal elements $\hat{\beta}_{Jn}$ are non-negative constants.

From the representation (49) it follows that in the complex s plane except for points of the real axis for $s > s_{AR}$ the imaginary parts of the diagonal elements of the matrix $\hat{\tilde{A}}_J(\mathbf{b},s)$ are proportional to the imaginary parts of the argument s, i.e.,

$$\operatorname{Im}\langle\lambda\,|\,\hat{\tilde{A}}_J(\mathbf{b}, s)\,|\,\lambda\rangle = \Lambda_J^{(\lambda)}(s)\operatorname{Im}s, \quad (50)$$

where $\Lambda_J^{(\lambda)}(s)$ are positive-definite functions, which is a characteristic sign of a generalized R function. Analysis of the expression (49) leads to the conclusion that det $\tilde{A}_J(\mathbf{b},s)$ can vanish only on the real axis, where in the absence of bound states all the zeros must be located in the region $s > s_R$, and if there are bound states, they must be located at $s > \min\{|\alpha_{Jn}\}$. It follows from (47) that $\operatorname{Im}\hat{\tilde{A}}_J^{-1}(\mathbf{b},s)$ for $s > s_R$ is a known quantity. As a result, for $\hat{\tilde{A}}_J(\mathbf{b},s)$ we obtain a solution of the form (27), (28) in which $\tilde{A}_L(b,s)$, $\tilde{P}_l(b,s)$, $\tilde{\rho}_l(b,s)$, $a_l(b,s_0)$, $c_l(b)$, and $\gamma_{li}(b)$ must be replaced by the matrices $\hat{\tilde{A}}_J(\mathbf{b},s)$, $\hat{\tilde{P}}_J(\mathbf{b},s)$, $\hat{\tilde{\rho}}_J(\mathbf{b},s)$, $\hat{a}_J(\mathbf{b},s_0)$, $\hat{c}_J(\mathbf{b})$, and $\hat{\gamma}_{Ji}(\mathbf{b})$, respectively, and $s_{li}(b)$ must be replaced by $s_{Ji}(\mathbf{b})$; $\hat{a}_J(\mathbf{b},s_0)$, $\hat{c}_J(\mathbf{b})$, and $\hat{\gamma}_{Ji}(\mathbf{b})$ are real symmetric matrices. At the points $s = s_{Ji}(\mathbf{b})$, det $\hat{\tilde{A}}_J(\mathbf{b},s)$ vanishes. When bound states are present, the quantities $s_{Ji}(\mathbf{b})$ satisfy the condition $s_{Ji}(\mathbf{b}) > \min\{\alpha_{Jn}\}$, and in the absence of bound states $s_{Ji}(\mathbf{b}) > s_R$. The determinants of the matrices $\hat{\beta}_{Jn}$, $\hat{c}_F(\mathbf{b})$, and $\hat{\gamma}_{Ji}(\mathbf{b})$ are equal to zero, and the diagonal elements are non-negative.

4. THE METHOD OF INCLUDING INELASTIC EFFECTS

In analyzing hadron–hadron scattering processes at intermediate energies, in many cases it is important to include the contribution of inelastic channels. In practical calculations using the dispersion equations, usually only a small number (as a rule, one or two) or two-particle channels are taken into account explicitly.[5)] The others, including multiparticle channels, can be taken into account by the introduction of inelasticity parameters and the corresponding reformulation of the equations. There are two main methods of including inelastic channels in the ordinary one-channel N/D equations: the Froissart method[120] and the Frye and Warnock method,[121] in which it is assumed that the inelasticity parameter $\eta_l(s) = |S_l|$ is known in the entire energy range. The Frye–Warnock equations[121] have been generalized to the multichannel case by Warnock.[122] The inclusion of inelasticity in the Warnock method qualitatively changes the properties of the multichannel N/D equations (they become singular), and a rather awkward procedure is required to reduce them to the equivalent system of equations of the Fredholm type.

It should be noted that even such a practically important problem as the construction of the scattering matrix in the case of two coupled channels including inelasticity effects was not solved satisfactorily until 1982. This problem has been discussed in the literature (see, for example, Refs. 123–125) in connection with the description of the data on NN scattering at intermediate energies.[6)] Arndt and Roper[123] suggested a method of parametrizing the contribution of inelastic channels based on the K-matrix representation of the partial-wave S matrix. The real part of the K matrix in this method is expressed in terms of the Stapp parameter[126] using the same relations as in the absence of inelasticity, and the imaginary part determines the contribution of the explicitly excluded inelastic channels. This approach has a number of advantages over other methods (see the discussion in Refs. 123–125 for more details). In particular, the method of Arndt and Roper permits certain physical properties of the scattering amplitudes to be taken into account in a natural way. The parametrization under discussion has been used in new phase-shift analyses of NN (Refs. 127 and 128), πN (Ref. 129), and KN (Ref. 130) scattering at intermediate energies.

In Ref. 131 a new formulation of the multichannel N/D equations taking inelasticity into account was proposed. It is based on the K-matrix method of parametrizing the channels not explicitly included, which has advantages over the Warnock method.[122] Let us discuss the use of this approach[131] for including inelasticity effects in the equations formulated in Secs. 2 and 3.[98]

The absorptive part of the K matrix related to the matrix of partial-wave amplitudes $\hat{A}_J(s)$ as $\hat{A}_J^{-1}(s) = \hat{K}_J^{-1}(s) - i\hat{\rho}(s)$ above the threshold of the explicitly excluded channels s_I will be assumed to be known [along with $\mathrm{Im}\,\hat{A}_J(s)$ on the dynamical cuts]. We write $\hat{A}_J(s)$ as

$$\hat{A}_J(s) = \hat{A}_J^i(s) + [\hat{D}_J^{iT}(s)]^{-1}\hat{A}_J^e(s)\,[\hat{D}_J^i(s)]^{-1}, \tag{51}$$

where

$$\hat{A}_J^i(s) = \hat{N}_J^i(s)\,[\hat{D}_J^i(s)]^{-1}, \tag{52}$$

with the matrices $\hat{N}_J^i(s)$ and $\hat{D}_J^i(s)$ defined in such a way that in the physical region

$$\mathrm{Im}\,\hat{D}_J^i(s) = -\hat{\rho}(s)\,\mathrm{Re}\,\hat{N}_J^i(s), \tag{53}$$

and above the threshold of the channels not explicitly included

$$\mathrm{Im}\,\hat{N}_J^i(s) = \hat{\zeta}_J(s)\,\mathrm{Re}\,\hat{D}_J^i(s)\,\theta(s - s_I), \tag{54}$$

where

$$\hat{\zeta}_J(s) = \mathrm{Im}\,\hat{K}_J(s)\,[\hat{I} + \hat{\rho}(s)\,\mathrm{Im}\,\hat{K}_J(s)]^{-1} \tag{55}$$

is a quantity fully determined by the absorptive part of the K matrix.

To calculate the matrices $\mathrm{Re}\,\hat{N}_J^i(s)$ and $\mathrm{Re}\,\hat{D}_J^i(s)$ we write the system of equations

$$\mathrm{Re}\,\hat{N}_J^i(s) = \frac{1}{\pi}\int_{s_I}^{\infty}\frac{\hat{\zeta}_J(s')\,\mathrm{Re}\,\hat{D}_J^i(s')}{s' - s}\,ds'; \tag{56}$$

$$\mathrm{Re}\,\hat{D}_J^i(s) = \hat{I} - \frac{s - s_0}{\pi}\,\mathscr{P}\int_{s_R}^{\infty}\frac{\hat{\rho}(s')\,\mathrm{Re}\,\hat{N}_J^i(s')}{(s' - s)(s' - s_0)}\,ds', \tag{57}$$

where the symbol $\mathscr{P}$ means that the integral is understood in the sense of the principal value. It follows from these equations that on the dynamical cuts we have

$$\mathrm{disc}\,\hat{A}_J^i(s) = 0. \tag{58}$$

It can be shown that the unitarity condition for the matrix $\hat{A}_J^e(s)$ in Eq. (51) has the form

$$\mathrm{Im}\,\hat{A}_J^e(s) = \hat{A}_J^{e+}(s)\,\hat{\rho}_J^i(s)\,\hat{A}_J^e(s), \tag{59}$$

where

$$\hat{\rho}_J^i(s) = [\hat{D}_J^i(s)]^{-1}\hat{R}_J(s)\,[\hat{D}_J^{i+}(s)]^{-1} \tag{60}$$

is a symmetric Hermitian matrix and $\hat{R}_J(s)$ is determined by the expression

$$\hat{R}_J(s) = \hat{\rho}(s) + \hat{\rho}(s)\,\mathrm{Im}\,\hat{K}_J(s)\,\hat{\rho}(s). \tag{61}$$

Equation (59) has the form of the unitarity relation for the matrix of partial-wave amplitudes $\hat{A}_J(s)$ with zero absorptive part of the K matrix and overdetermined matrix $\hat{\rho}(s)$ (36). It follows from Eqs. (51) and (58) that the discontinuities $\hat{A}_J^e(s)$ and $\hat{A}_J(s)$ on the dynamical cuts are related as

$$\mathrm{Im}\,\hat{A}_J^e(s) = \hat{D}_J^{iT}(s)\,\mathrm{Im}\,\hat{A}_J(s)\,\hat{D}_J^i(s). \tag{62}$$

Then we can construct the matrix $\hat{A}_J^e(s)$ in complete analogy with the method used in Secs. 2 and 3. Here we only need to replace the matrix $\hat{\rho}(s)$ (36) in the unitarity condition by the matrix $\hat{\rho}_J^i(s)$ (60) and take into account the renormalization (62) of $\mathrm{Im}\,\hat{A}_J^e(s)$ on the dynamical cuts. There is a slight formal difference in the construction of the matrices $\hat{A}_J^e(s)$ and $\hat{A}_J(s)$ in that the matrix $\hat{\rho}_J^i(s)$ (60), in contrast to $\hat{\rho}(s)$ (36), is not, in general, diagonal. However, this has no effect on the general properties of the representation under consideration. For example, $\hat{\hat{\rho}}_J(\mathbf{b},s)$ (48) is a symmetric Hermitian matrix, as before.

5. EXACTLY SOLVABLE MODELS

It is interesting to analyze various aspects of this method using exactly solvable models as examples. Of primary importance is the nonrelativistic potential model, which certainly satisfies all the requirements listed in Sec. 1.[132] In this important special case it is possible to rigorously demonstrate this representation starting directly from the dynamical equation. This exercise is also useful in that it allows us to get a clear physical picture of the various quantities.

We consider the solution of the Schrödinger partial-wave equation

$$\left(\frac{d^2}{dr^2} - \frac{l(l+1)}{r^2} - v(r) + k^2\right)u_l(r, k) = 0 \tag{63}$$

with boundary condition $u_l(r,k) \underset{r\to 0}{\cong} r^{l+1}$, assuming that the potential $v(r)$ is a superposition of Yukawa potentials. We define two linearly independent solutions of (63), $\varphi_l(r,k)$ and $\chi_l(r,k)$, with boundary conditions at the point $r = b$:

$$\varphi_l(b, k) = 1, \quad \varphi_l'(b, k) = 0, \tag{64}$$

$$\chi_l(b, k) = 0, \quad \chi_l'(b, k) = 1, \tag{65}$$

where the prime denotes differentiation with respect to the parameter b. Following Jaffe and Low,[54] we define the P matrix through

$$u_l(r, k) = a_l(b, k)\,[\varphi_l(r, k) + P_l(b, k)\,\chi_l(r, k)], \quad (66)$$

from which we see that $P_l(b,k)$ is the logarithmic derivative of the wave function $u_l(r,k)$ at $r = b$. Using the standard definition of the S matrix,[132] we find the relation between the S and P matrices in the form of (5), in which

$$D_l^{(\pm)}(b, s) \to D_l^{(\pm)}(b, \nu) = k^l h_l^{(\pm)}(b, k) \quad (67)$$

and

$$\widetilde{S}_l = \frac{P_l(b, k) - f_l^{(-)}(b, k)}{P_l(b, k) + f_l^{(+)}(b, k)}, \quad (68)$$

where $\nu = k^2$, and $h_l^{(\pm)}(r,k)$ are the solutions of (63) with the boundary conditions at infinity

$$h_l^{(\pm)}(r, k) \underset{r\to\infty}{\cong} \exp\left[\pm \mathrm{i}\left(kr - \frac{\pi l}{2}\right)\right]; \quad (69)$$

$$f_l^{(\pm)}(b, k) = d \ln h_l^{(\pm)}(b, k)/db. \quad (70)$$

The solution $u_l(r,k)$ is an entire even function of k for all r, since the boundary condition on it is independent of k (the Poincaré theorem[132]). Therefore, $P_l(b,k)$ is a meromorphic function in the entire complex ν plane, and its poles correspond to the zeros of $u_l(b,k)$. In the vicinity of the ith zero $u_l(b,k)$ has the form

$$u_l(b, k) \underset{\nu\to\nu_{li}(b)}{\cong} c_{li}(b)\,[\nu - \nu_{li}(b)] \quad (71)$$

and, therefore,

$$P_l(b, k) \underset{\nu\to\nu_{li}(b)}{\cong} \tau_{li}(b)\,[\nu - \nu_{li}(b)]^{-1}, \quad (72)$$

where

$$\tau_{li}(b) = -d\nu_{li}(b)/db. \quad (73)$$

As a result, using the fact that $P_l(b,k) \underset{k\to\infty}{\to} P_l^{(0)}(b,k)$, where $P_l^{(0)}(b,k)$ is a matrix corresponding to free motion of the particles for $P_l(b,k)$ we obtain an expression of the form (28), in which the replacements $s \to \nu$, $s_{li}(b) \leftarrow \nu_{li}(b)$, $\gamma_{li}(b) \to \tau_{li}(b)$ must be made and we must set $c_l(b) = 0$.

Using the properties of the solutions of (63), we can prove the validity (in the nonrelativistic limit) of all the relations of Sec. 2.[97] Here it can be verified that the function $\widetilde{P}_l(b,s)$ in (27) has a simple linear relation to $P_l(b,k)$ in the expression (66). Here we only present the scheme for proving the N/D representation (14) Eqs. (21) and (22). It is easy to see that the partial-wave S matrix for the potential $\tilde{v}(r) = v(r) = v_c(r)$, where

$$v_c(r) = \begin{cases} \infty & \text{for} \quad r \leqslant b; \\ 0 & \text{for} \quad r > b, \end{cases} \quad (74)$$

is determined by (5) with $\widetilde{S}_l = 1$ and the functions $D_l^{(\pm)}(b,s)$ of the form (67). The function $\overline{A}_l(b,s)$ in Eqs. (8), (9), and (13) is the reduced partial-wave scattering amplitude for the potential $\tilde{v}(r)$.[7)]

Since the potential $v_c(r)$ (74) is singular, let us consider the following limiting procedure. We define a potential $v_c^a(r)$ depending on the parameter a such that $\lim_{a\to\infty} v_c^a(r) = v_c(r)$. We assume that for a finite fixed value of a the scattering problem can be formulated for it in the momentum representation in the usual manner. The scattering operator $\hat{\tilde{t}}^{(a)}(z)$ for the sum of potentials $\tilde{v}^{(a)}(r) = v_c^{(a)}(r) + v(r)$ is written in the following form using the two-potential expressions:[106]

$$\hat{\tilde{t}}^{(a)}(z) = \hat{t}_c^{(a)}(z) + [\hat{I} + \hat{v}_c^{(a)}\hat{G}_c^{(a)}(z)]\,\hat{t}^{(a)}(z)\,[\hat{I} + \hat{G}_c^{(a)}(z)\,\hat{v}_c^{(a)}], \quad (75)$$

in which the operators $\hat{t}_c^{(a)}(z)$ and $\hat{t}^{(a)}(z)$ satisfy the equations

$$\hat{t}_c^{(a)}(z) = \hat{v}_c^{(a)} + \hat{v}_c^{(a)} G_0(z)\,\hat{t}_c^{(a)}(z), \quad (76)$$

$$\hat{t}^{(a)}(z) = \hat{v} + \hat{v}\,\hat{G}_c^{(a)}(z)\,\hat{t}^{(a)}(z), \quad (77)$$

where

$$\hat{G}_0(z) = (z/2\mu - \hat{H}_0)^{-1},$$
$$\hat{G}_c^{(a)}(z) = (z/2\mu - \hat{H}_0 - \hat{v}_c^{(a)})^{-1}, \quad (78)$$

$\hat{H}_0$ is the free Hamiltonian, μ is the reduced mass of the colliding particles, and $\hat{v}$ and $v_c^{(a)}$ are local operators in coordinate space with the matrix elements

$$\langle r'|\hat{v}|r\rangle = \delta(r' - r)\,v(r), \quad \langle r'|\hat{v}_c^{(a)}|r\rangle = \delta(r' - r)\,v_c^{(a)}(r).$$

Using the completeness of the in- and out-state vectors $|k,l;a(\pm)\rangle$ for the potential $v_c^{(a)}(r)$ and Eq. (77), we obtain an integral equation for the matrix elements of the operator $\hat{t}^{(a)}(z)$ between these states. Using the expressions

$$\lim_{a\to\infty} \langle r|k,\ l;\ a(\pm)\rangle = \begin{cases} 0 & \text{for} \quad r \leqslant b; \\ \langle r|k,\ l(\pm)\rangle & \text{for} \quad r > b \end{cases} \quad (79)$$

[$\langle r|k,l(\pm)\rangle$ are in the in- and out-solutions of (63) with the potential $v_c(r)$ (74)], in the limit $a \to \infty$ this equation takes the form

$$t_l(k',\ k;\ \nu + \mathrm{i}\eta) = v_l^{(+)}(k',\ k) + \frac{2}{\pi}\int_0^\infty v_l^{(-)}(k,\ p)\,(p^2 - \nu - \mathrm{i}\eta)^{-1} t_l(p,\ k;\ \nu + \mathrm{i}\eta)\,p^2\,dp, \quad (80)$$

where

$$v_l^{(\pm)}(k',\ k) = -\frac{\mu}{2\pi}\lim_{a\to\infty}\langle k',\ l;\ a(-)|\hat{v}|k,\ l;\ a(\pm)\rangle = -\frac{\mu}{2\pi}\int_b^\infty \langle k',\ l(-)|r\rangle\,v(r)\,\langle r|k,\ l(\pm)\rangle\,dr; \quad (81)$$

$$t_l(k'\ k;\ \nu) = -\frac{\mu}{2\pi}\lim_{a\to\infty}\langle k',\ l;\ a(-)|\hat{t}^{(a)}(\nu)|k,\ l;\ a(+)\rangle = -\frac{\mu}{2\pi}\langle k',\ l(-)|\hat{t}(\nu)|k,\ l(+)\rangle. \quad (82)$$

The matrix elements of the potential $v_l^{(\pm)}(k',k)$ (81) in the $\nu = k^2$ ($\nu' = k'^2$) plane for a fixed value of the $k'(k)$ has, in addition to dynamical singularities, a right-hand cut $[0 \leqslant \nu(\nu') < \infty]$. Actually, the solution of (63),m $\chi_l^{(c)}(r)$, with the potential $v_c(r)$ (74) and boundary conditions of the form (65) can be written as

$$\chi_l^{(c)}(r,\ k) = h_{c\,l}^{(\pm)}(bk)\,\langle r \mid k,\ l(\pm)\rangle, \quad (83)$$

where

$$h_{cl}^{(\pm)}(x) = (2l-1)!!\, d_l^{(\pm)}(x)/x^l,$$

and $\langle r|k,l(\pm)\rangle$ are the in- and out-solutions appearing in (79) and (81) of the same equation, and are orthogonal and complete in the region $r>b$. Since the boundary conditions (65) for the solution $\chi_l^{(c)}(r)$ are independent of k, it follows from the Poincaré theorem that for any r it is an entire even function of k. The function $h_{cl}^{(\pm)}(bk)$ have a right-hand cut in the $\nu=k^2$ plane and, owing to (83), so do the functions $\langle r|k,l(\pm)\rangle$; from this, taking into account (81), we arrive at the above statement.

Introducing renormalized matrix elements by means of the relations

$$v_l^{(\pm)}(k',k)=[h_{cl}^{*(-)}(bk)\,h_{cl}^{(+)}(bk)]^{-1}w_l(k',k;b), \quad (84)$$

$$t_l(k',k;\nu)=[h_{cl}^{*(-)}(bk')\,h_{cl}^{(+)}(bk)]^{-1}\,T_l(k',k;\nu,b) \quad (85)$$

and substituting these expressions into (80), we obtain the equation

$$T_l(k',k;\nu+i\eta,b)=w_l(k',k;b)+\frac{1}{\pi}\int_0^\infty w_l(k',\sqrt{\nu'};b)\,T_l(\sqrt{\nu'},k;\nu,b)\,\frac{\xi_l(b,\nu')\,d\nu'}{\nu'-\nu-i\eta}, \quad (86)$$

where

$$\xi_l(b,\nu)=\frac{b^{2l}\nu^{l+1/2}}{[(2l-1)!!]^2\,d_l^{(+)}(bk)\,d_l^{(-)}(bk)}. \quad (87)$$

The matrix elements

$$w_l(k',k;b)=\int_b^\infty \chi_l^{(c)}(r,k')\,\chi_l^{(c)}(r,k)\,v(r)\,dr \quad (88)$$

do not have right-hand cuts in the variables $\nu=k^2$, $\nu'=k'^2$ (i.e., they contain only dynamical singularities), and the amplitude $T_l(k',k;\nu,b)$ on the energy shell ($k'=k=\sqrt{\nu}$) coincides (in the nonrelativistic limit), up to a constant, with the function $T_l(b,s)$ in (13). The solution of (86) can be written as the ratio of two Fredholm series:[132]

$$T_l(k',k;p^2+i\eta,b)=n_l(k',k;p^2,b)\,\Delta_l^{-1}(b,p^2+i\eta), \quad (89)$$

where

$$n_l(k',k;p^2,b)=\sum_{n=0}^{\infty}\frac{(-1)^n}{n!}\times\prod_{i=1}^{n}\frac{2}{\pi}\int_0^\infty dq_i\,\frac{q_i\xi_l(b,q_i^2)}{q_i^2-p^2}\,Q_{n+1}\begin{pmatrix}k', & q_1, & \ldots, & q_n\\ k, & q_1, & \ldots, & q_n\end{pmatrix}; \quad (90)$$

$$\Delta_l(b,p^2+i\eta)=\sum_{n=0}^{\infty}\frac{(-1)^n}{n!}\times\prod_{i=1}^{n}\int_0^\infty dq_i\,\frac{q_i\xi_l(b,q_i^2)}{q_i^2-p^2-i\eta}\,Q_n\begin{pmatrix}q_1, & \ldots, & q_n\\ q_1, & \ldots, & q_n\end{pmatrix},\quad Q_0=1, \quad (91)$$

$$Q_n\begin{pmatrix}q_1, & \ldots, & q_n\\ q_1, & \ldots, & q_n\end{pmatrix}=\det_{(i,j)}\|w_l(q_i,q_j;b)\|\quad\text{for}\quad n\geqslant 1. \quad (92)$$

Introducing the notation $n_l(b,\nu)\equiv n_l(\sqrt{\nu},\sqrt{\nu};\nu,b)$, after purely algebraic transformations we obtain the equation

$$\Delta_l(b,\nu+i\eta)=1-\frac{1}{\pi}\int_0^\infty\frac{\xi_l(b,\nu')\,n_l(b,\nu')}{\nu'-\nu-i\eta}\,d\nu', \quad (93)$$

which is the nonrelativistic analog of (21). Taking into account the analytic properties of the function $n_l(b,\nu)$ following from the representation (90) and its asymptote at infinity [$n_l(b,\nu)\to 0$ for $|\nu|\to\infty$], we can also obtain a second equation of the form (22) relating the functions $\Delta_l(b,\nu)$ and $n_l(b,\nu)$. The proof that the kernel of Eq. (86) is compact, which is sufficient for convergence of the Fredholm series, is carried out as for the Lippmann–Schwinger equation.[133]

It follows from the properties of the functions $d_l^{(\pm)}(x)$ and $T_l(b,s)$ that the amplitude $\overline{A}_l(b,s)$ [see (13)] has an essential singularity at infinity. However, the amplitude $A_l(s)$ for the "analytic" potential $v(r)$ under consideration does not, of course, have such a singularity at infinity. This is a consequence of the cancellation of the singularity of $\overline{A}_l(b,s)$ at the point at infinity analogous to the singularity of the second term on the right-hand side of (8). This cancellation occurs when the P matrix in the limit $|\nu|\to\infty$ tends to the quantity $P_l^{(0)}(b,k)$ corresponding to free motion of the particles.

The investigation in the nonrelativistic potential model can be generalized to the relativistic case by using the quasipotential formalism,[134-136] constructing the quasipotential in such a way that it leads to the required analytic structure of the amplitude $A_l(s)$ (see Appendix B). It can be shown that in this case the function $\widetilde{P}_l(b,s)$ is linearly related to the logarithmic derivative of the wave function describing the relative motion of the particles, $P_l(b,s)$ i.e., to the P matrix

$$\widetilde{P}_l(b,s)=\alpha_l(b,s)\,P_l(b,s)+\beta_l(b,s) \quad (94)$$

with energy-dependent coefficient of proportionality $\alpha_l(b,s)$. In an approximation which corresponds to neglecting effects of nonlocality of the interaction near the point $r=b$, $\alpha_l(b,s)$ in (94) is a purely kinematic function

$$\alpha_l(b,s)\cong\frac{2[(2l-1)!!]^2}{\sqrt{s}\,b^{2l}}. \quad (95)$$

Finally, let us consider some aspects related to the interpretation of the CDD(RA) poles in Eq. (28) using an exactly solvable field-theoretic model like the Dyson model,[110] which takes into account the internal structure of the particles. From the viewpoint of the ideas on which the current theory of strong interactions is based, effects related to rearrangement of the QCD vacuum must play an important role in the intermediate state in the formation of compound systems like the quark–gluon bag.[137]

We shall start from the following physical picture of hadron–hadron scattering. In the initial and final states the hadrons are infinitely far apart and are interpreted as two noninteracting quark–gluon subsystems whose quark fields are localized in cavities B_i and B_j in the spaces of the relative coordinates of the valence quarks. As they approach to a finite distance r the subsystems can interact by exchanging several hadron states (i.e., colorless complexes of quarks and gluons for which the physical vacuum of QCD is transparent). However, this interaction does not lead to any radical rearrangement of the vacuum, and the hadrons preserve their individuality as long as r is larger than some characteristic scale b. At small distance ($r<b$) there is a large probability for rearrangement of the vacuum leading to the formation of a bag type of state, in which the fields of the valence quarks are localized in the same cavity B_k in the space of the relative coordinates of the quarks of the entire system.

Therefore, in this model there are two characteristic configurations of the quark fields and the vacuum (one of them can be termed the hadronic configuration, and the other the bag configuration), which in the interaction process can undergo mutual transformations. It is convenient to formulate this model in the second-quantized formalism. The system is described by a Hamiltonian of the form $\hat{H}=\hat{H}_0+\hat{H}_{\text{int}}$. The first term of this expression corresponds to noninteracting hadron and bag configurations: $\hat{H}=\hat{H}_a+\hat{H}_c$, where

$$\hat{H}_a=\sum_{\lambda}\int \omega_{\lambda}(\mathbf{q})\,a_{\lambda}^{(+)}(\mathbf{q})\,a_{\lambda}^{(-)}(\mathbf{q})\,d\mathbf{q}; \tag{96}$$

$$\hat{H}_c=\sum_{\beta}\varepsilon_{\beta}c_{\beta}^{(+)}c_{\beta}^{(-)}; \tag{97}$$

$a_{\lambda}^{(\pm)}(\mathbf{q})$ are the creation and annihilation operators for pairs of hadrons with relative momentum $\mathbf{q}$ and discrete quantum numbers λ, and $c_{\beta}^{(\pm)}$ are the creation and annihilation operators of an isolated system of the quark–gluon bag type with quantum numbers β; these operators satisfy the commutation relations

$$[a_{\lambda'}^{(-)}(\mathbf{q}'),\ a_{\lambda}^{(+)}(\mathbf{q})]=\delta_{\lambda'\lambda}\delta(\mathbf{q}'-\mathbf{q}); \tag{98}$$

$$[c_{\beta'}^{(-)},\ c_{\beta}^{(+)}]=\delta_{\beta'\beta}; \tag{99}$$

ε_{β} are the energies of the bag-type eigenstates, $\omega(q)=\sqrt{m_i^2+q^2}+\sqrt{m_j^2+q^2}$, and m_i and m_j are the hadron masses. The part of the Hamiltonian responsible for the interaction $\hat{H}_{\text{int}}$ also consists of two terms, one of which describes the mutual transformation of the hadron and bag configurations,

$$\hat{H}_{ac}=\sum_{\lambda\beta}\int d\mathbf{q}\,g_{\lambda\beta}(\mathbf{q})\,a_{\lambda}^{(+)}(\mathbf{q})\,c_{\beta}^{(-)}+\text{h.c}, \tag{100}$$

where h.c. is the Hermitian-conjugate expression, $g_{\lambda\beta}(\mathbf{q})$ are the form factors determining the dynamics of the interaction between the hadrons and the bag states, and the other term corresponds to the interaction mechanism not related to the formation of bag configurations in the intermediate state:

$$\hat{H}_{aa}=\sum_{\lambda\lambda'}\int d\mathbf{q}\int d\mathbf{q}'\,v_{\lambda\lambda'}(\mathbf{q},\ \mathbf{q}')\,a_{\lambda}^{(+)}(\mathbf{q})\,a_{\lambda'}^{(-)}(\mathbf{q}'). \tag{101}$$

For simplicity we assume that the hadrons are spinless particles and have only one scattering channel [the index λ in (96), (98), (100), and (101) in this case takes a single value, and we can drop it]. Let us consider a state $|\Phi_{li}(b)\rangle$ which is an eigenstate of the full Hamiltonian of the system with given angular momentum l:

$$[\hat{H}-\omega_{li}(b)]\,|\,\Phi_{li}(b)\rangle=0, \tag{102}$$

where

$$|\,\Phi_{li}(b)\rangle=\Big[\int d\mathbf{q}\ \langle\mathbf{q}\,|\,\Psi_{li}(b)\rangle\,a^{(+)}(\mathbf{q})+\sum_{\beta}\langle\beta\,|\,\varphi_{li}\rangle\,c_{\beta}^{(+)}\Big]|\,0\rangle, \tag{103}$$

$|0\rangle$ is the vacuum state, and the wave function of the hadron channel in the configuration representation

$$\langle\mathbf{r}\,|\,\Psi_{li}(b)\rangle=(2\pi)^{-3}\int d\mathbf{q}\exp(i\mathbf{qr})\ \langle\mathbf{q}\,|\,\Psi_{li}(b)\rangle \tag{104}$$

is nonzero in the region $r<b$ and on its boundary satisfies the condition

$$\langle\mathbf{r}\,|\,\Psi_{li}(b)\rangle\,|_{r=b}=0. \tag{105}$$

The quantity

$$\chi_{li}^2(b)=\int|\,\langle\mathbf{q}\,|\,\Psi_{li}(b)\rangle\,|^2\,d\mathbf{q} \tag{106}$$

characterizes the probability that a hadron configuration is present, and $\langle\beta\,|\varphi_{li}\rangle^2$ is a bag configuration with quantum numbers β in the state $|\Phi_{li}(b)\rangle$. The operator equation (102) leads to a system of equations for the functions $\langle\mathbf{q}|\Psi_{li}(b)\rangle$, $\langle\beta\,|\varphi_{li}\rangle$, which must be solved with the boundary condition (105).

Analyzing the solution of the Schrödinger equation $[\hat{H}-\omega(k)]|\Phi_l^{(+)}(k)\rangle=0$ corresponding to the scattering state with angular momentum l and energy $\omega(k)\sqrt{s}$, it is easy to verify that the wave function of the hadron channel $\langle r|\Psi_l^{(+)}(k)\rangle$ at the point $r=b$ has a zero at $s=\omega_{li}^2(b)\equiv s_{li}(b)$ and can be represented in the form (71) near this zero. Therefore, $P_l(b,k)=d\ln\langle b\,|\Psi_l^{(+)}(k)\rangle/db$ has a pole at $s=s_{li}(b)$, and, using (73) and (94), we obtain the following for the residue $\gamma_{li}(b)$ in (28):

$$\gamma_{li}(b)=-\alpha_l(b,\ s_{li}(b))\,ds_{li}(b)/db. \tag{107}$$

Therefore, the positions of the CDD(RA) poles of the function $\widetilde{P}_l(b,s)$ (28) are determined by the energies of the states appearing in the system when the boundary condition (105), which corresponds to a confining potential of infinite range in the region $r>b$, is imposed, and the residues $\gamma_{li}(b)$ (107) characterize the strength of the interaction of these states with the scattering channel.

6. STUDY OF NUCLEON–NUCLEON SCATTERING INCLUDING MESON EXCHANGE EFFECTS AND QUARK DEGREES OF FREEDOM AT LOW AND INTERMEDIATE ENERGIES

Let us first of all use our method to study NN scattering for incident-nucleon energies in the lab frame $T\lesssim 1$ GeV (Refs. 94, 96, 98, and 100). In the isotopic formalism for a given total angular momentum J of the system, the scattering in the NN sector has, as is well known (see, for example, Ref. 1), a five independent partial-wave amplitudes: $A_J^{(0)}(S)=\langle\lambda^{(0)}|\hat{A}_J(s)|\lambda^{(0)}\rangle$, the singlet amplitude; $A_J^{(1)}(s)=\langle\lambda^{(1)}|\hat{A}_J(s)|\lambda^{(1)}\rangle$, the uncoupled triplet amplitude; and $A_J^{(11)}(s)=\langle\lambda_1|\hat{A}_J(s)|\lambda_1\rangle$, $A_J^{(22)}=\langle\lambda_2|\hat{A}_J(s)|\lambda_2\rangle$, $A_J^{(12)}(s)=\langle\lambda_1|\hat{A}_J(s)|\lambda_2\rangle$, the coupled triplet amplitudes, where $\lambda^{(0)}=\{l=J,S=0\}$, $\lambda^{(1)}=\{l=J,S=1\}$, $\lambda_1=\{l=J-1,S=1\}$, $\lambda_2=\{l=J+1,S=1\}$, l is the nucleon orbital angular momentum, and S is the total spin of the nucleons. The discontinuities of the partial-wave amplitudes $A_J^{(\alpha)}(s)$ ($\alpha=0,1,11,22,12$) on the dynamical cuts C_L

needed for solving a system of equations of the form (21), (22) have been calculated taking into account the exchange of one, two, and three π mesons in the t and u channels. The contribution of the one-pion exchange mechanism to $\mathrm{Im}\, A_J^{(\alpha)}(s)$ on C_L is determined by the renormalized pion–nucleon coupling constant $g\pi$. The two-pion exchange mechanism was taken into account using a technique based on use of the unitarity condition in the t channel and the available information on the πN and $\pi\pi$ scattering amplitudes. We note that a similar procedure was used to calculate the Paris potential[138,139] at large an intermediate distances ($r \gtrsim 0.8$ fm).

The covariant Feynman amplitude of NN scattering can be written as[1,138]

$$M = \sum_{n=1}^{5} [3p_n^{(+)}(s,\ t,\ u) + 2p_n^{(-)}(s,\ t,\ u)\, \boldsymbol{\tau}^{(1)}\boldsymbol{\tau}^{(2)}]\, P_n, \tag{108}$$

where $\boldsymbol{\tau}^{(j)}$ ($j = 1,2$) are the isospin matrices, P_n are invariants, defined in Ref. 1, constructed from products of bilinear combinations of Dirac spinors and $p_n^{(\pm)}(s,t,u)$ are scalar functions of the Mandelstam invariants, related as $s + t + u = 4m_N^2$ (m_N is the nucleon mass). The functions $p_n^{(\pm)}(s,t,u)$ have the representation

$$p_n^{(\pm)}(s,\ t,\ u) = \delta_{n5} p_\pi^{(\pm)}(t) + \frac{1}{\pi} \int_{4m_\pi^2}^{\infty} \frac{\rho_n^{(\pm)}(s,\ t') \mp (-1)^n \rho_n^{(\pm)}(u,\ t')}{t' - t}\, dt', \tag{109}$$

where $p_\pi^{+}(t) = 0$, $p_\pi^{(-)}(t) = -\frac{1}{2} g_\pi^2/(m_\pi^2 - t)$, and m_π is the π-meson mass. The first term in this expression corresponds to the contribution of the one-pion exchange mechanism, and the second corresponds to more complication interaction mechanisms. The relations between the functions $p_n^{(\pm)}(s,t)$ and the discontinuities of the partial-wave amplitudes $A_J^{(\alpha)}(s)$ on C_L are given in Appendix C.

In the region $s_{2\pi} \leqslant s \leqslant s_\pi$, where $s_\pi = 4m_N^2 - m_\pi^2$ and $s_{2\pi} = 4(m_N^2 - m_\pi^2)$, the quantities $\mathrm{Im}\, A_J^{(\alpha)}(s)$ are determined by the constant $g\pi$. On the segment of the dynamical cut $s_{3\pi} \leqslant s \leqslant s_{2\pi}$, where $s_{3\pi} = 4m_N^2 - 9m_\pi^2$, in addition to the one-pion mechanism, $\mathrm{Im}\, A_J^{(\alpha)}(s)$ receives a contribution from the two-pion exchange mechanism, which can be related to the invariant πN scattering amplitudes $A_{\pi N}^{(\pm)}(s,t,u)$ and $B_{\pi N}^{(\pm)}(s,t,u)$ (Ref. 106) on the mass shell for $t \geqslant 4m_\pi^2$ by using the unitarity condition in the t channel. However, these amplitudes are known in the region $t \leqslant 0$ from the data on πN scattering. Writing the dispersion relations for the invariant functions $A_{\pi N}^{(\pm)}(s,t,u)$ and $B_{\pi N}^{(\pm)}(s,t,u)$, they can be analytically continued to the needed region[8)] of the invariant variables for calculating the two-pion contributions to the discontinuities of the partial-wave NN scattering amplitudes.

It is useful to isolate from the expressions for $\mathrm{Im}\, A_J^{(\alpha)}(s)$ the most important contributions of two-pion states with the lowest angular momenta $J = 0$, 1 in the t channel, which contain resonance parts corresponding to the ρ and σ mesons.[9)] These contributions of two-pion states are expressed in terms of the helicity amplitudes $f_+^J = 0$ and $f_\pm^{J=1}(t)$ of the process $N\bar{N} \to 2\pi$.[1,138] Information on these amplitudes in the region $t \geqslant 4m_\pi^2$ can be obtained using dispersion relations of the Omnès–Muskhelishvili type[103] with the available data on the $\pi\pi$-scattering phase shifts and πN scattering amplitudes in the physical region. The amplitudes $f_+^{J=0}(t)$ and $f_\pm^{J=1}(t)$ have been studied quite thoroughly in the range $4m_\pi^2 \leqslant t < 50m_\pi^2$ (Ref. 140). These data are used to calculate the two-pion contributions to the discontinuities of the partial-wave NN scattering amplitudes on C_L. As regards the contribution of two-pion intermediate states with $J \geqslant 2$ (in the t channel) to the spectral functions $\rho_n^{(\pm)}(s,t)$ [which makes up $\sim$10–20% of the total value of $\rho_n^{(\pm)}(s,t)$], it was calculated using the parameterization of the functions $A_{\pi N}^{(\pm)}(s,t,u)$ and $B_{\pi N}^{(\pm)}(s,t,u)$ suggested in Ref. 141.

The procedure described above automatically includes the contributions from resonance states of two pions in the t and u channels, which in the one-pion exchange model are approximated by ρ- and σ-meson exchanges, but here both the finite width of these mesons (which is especially important in the case of the σ meson) and the nonresonance part of the two-pion exchange contribution are taken into account. A specific analysis shows that the contribution of the two-pion mechanism differs significantly from that of the box diagram (and the corresponding crossing-symmetric diagram) with two pion and two nucleon internal lines, which indicates the importance of including the mechanism of nucleon excitation in the intermediate state contained in the πN scattering amplitudes. The three-pion exchange mechanism was taken into account using the ω-exchange model.

Form factors were not introduced in the meson–nucleon vertices, since the contributions of distant singularities on C_L due to exchange interaction mechanisms are strongly suppressed by the exponential factor in Eq. (19). For this reason the mechanisms of exchange of the heavy mesons ρ, ω, and σ play a considerably more minor role in this approach than in meson theories of nuclear forces.

The effect of inelastic channels was taken into account using the absorptive part of the K matrix (see Sec. 4), which was specified using the following semiphenomenological method: The matrix $\mathrm{Im}\, \hat{K}_J(s)$ (the function in the case where one scattering channel is explicitly included) was approximated by a sum of terms proportional to the absorptive parts of the diagrams corresponding to the exchange of two π mesons in the t channel and containing $NN\pi$ and $N\Delta$ intermediate states in the s channel, where the coefficients of proportionality were chosen on the basis of the data of the energy-dependent phase-shift analysis (EDPSA) of Ref. 127.

Calculations show that to obtain a good description of all the NN phase shifts considered in the energy range $T \lesssim 1$ GeV it is sufficient to include only a single CDD (RA) pole in the expression (28). Setting

$$\widetilde{P}_l(b,\ s) = \widetilde{P}_l^{(0)}(b,\ s) + \frac{\gamma_1}{s - s_1}, \tag{110}$$

where $\gamma_1 \equiv \gamma_{l1}(b)$ and $s_1 = s_{l1}(b)$, we approximate the function $\widetilde{P}_l^{(0)}(b,s)$ by the constant

$$\widetilde{P}_l^{(0)}(b,\ s) \cong a. \tag{111}$$

In the case of the matrix equations (see Sec. 3) used to describe coupled $l = J - 1$ and $l = J + 1$ channels, a similar approximation was used for matrix $\hat{\widetilde{P}}_J(b,s)$. Here, in relations of the type (110) and (111), γ_1 and a must be replaced by the 2×2 matrices $\hat{\gamma}_1$ and $\hat{a}$, respectively. It follows from the results of Sec. 3 that the matrix elements of $\hat{\gamma}_1$ have the form $\langle\lambda|\hat{\gamma}_1|\gamma'\rangle = \xi_\lambda \xi \lambda'$, where $\lambda, \lambda' = 1,2$ and ξ_λ are real constants.

We stress that the approximation (111) is assumed to be valid only in the restricted energy range indicated. It is certainly violated at sufficiently high energies, since, owing to the assumption that there are no singularities at infinity in the amplitude $A_l(s)$, the function $\widetilde{P}_l(b,s)$ must have an infinite number of poles inside a circle of finite radius. However, calculations show that the explicit inclusion of second- and higher-order poles of the function $\widetilde{P}_l(b,s)$ has no effect on the quality of the description of the *NN* scattering phases in the region $T \lesssim 1$ GeV, and the extracted values of the parameters γ_1 and s_1 differ only insignificantly from the results of the analysis based on the approximation (111).

In the calculations we used the following values of the meson–nucleon coupling constants: $g_\pi^2/4\pi = 14.5$, $g_\omega^2/4\pi = 4.77$. We note that this value of the constant g_ω agrees with the predictions of the quark model. Meson theories of nuclear forces require considerably larger values of g_ω (Refs. 2 and 3) ($g_\omega^2/4\pi = 10$–20). This is necessary to ensure the description of the repulsive part of the *NN* interaction at short distances in these theories. In the approach under consideration here the repulsion mechanism has a completely different origin. It is related to the formation in the intermediate state of a compound system like the compound quark–gluon bag.

The constant a (111) (the matrix $\hat{a}$ in the case of coupled channels) was chosen so as to guarantee the correct location of the virtual level in the $^1S_0$ channel (the deuteron binding energy in the $^3S_1$–$^3D_1$ channels). The parameters b, s_1, and γ_1 ($\hat{\gamma}_1$) were chosen so as to obtain the best description of the data of the phase-shift analysis of *NN* scattering in the range $T = 0$–1050 MeV.[127] In Table I we give the results of the calculation of the $^1S_0$, $^3S_1$–$^3D_1$, $^3P_1$, $^3P_0$, and $^1P_1$ phase shifts and the mixing parameter ε_1 corresponding to the values of the parameters b, s_1, and γ_2 given in Table II. For comparison, in the same table we also give the data of the EDPSA (Ref. 127) and of the phase-shift analysis at fixed energy (PSAFE).[128] The inclusion of inelasticity has practically no effect on the quality of the description of the phases and leads only to a slight change of the parameters. Meson exchange effects significantly influence the locations of the poles of the function $\widetilde{P}_l(b,s)$ (28) in the *P*-wave amplitudes, but have relatively little effect in the *S* waves. Moreover, in some cases meson exchange interactions qualitatively change the behavior of $\widetilde{P}_l(b,s)$ at low energies. As can be seen from Table II, the lowest CDD(RA) poles, found from a fit to the data of the phase-shift analysis including meson exchange interactions, are located at intermediate energies both in the *S* waves and in the *P* waves. However, if the functions $P_l(b,s)$ are reconstructed for scattering in the $^3P_1$ and $^1P_1$ states, neglecting the dynamical cuts and using the same phase-shift data, they acquire additional poles in the low-energy region.[98]

The data of Table III illustrate the effect of meson exchange interactions on the calculations of the $^1S_0$ and $^3P_1$ phase shifts in the low-energy region. For comparison, in this table we also give the results of calculations [variants (D)] of these phase shifts in which the two-pion exchange interaction mechanism was approximated by ρ- and σ-meson exchanges.[10)] As can be seen from Table III, the inclu-

TABLE I. Phase shifts of *np* scattering* (in deg): (a) results of calculation using parameters of Table II; (b) EDPSA data (Ref. 127); (c) PSAFE data (Ref. 128).

T, MeV		100	200	300	400	500	600	800	1000
$^1S_0$	(a)	25,6	6,1	−7,1	−17,6	−26,4	−33,9	−46,6	−58,1
	(b)	26,4	6,0	−7,8	−17,0	−25,6	−33,8	−48,1	−59,8
	(c)	26,7	7,9	−5,6	−15,5	−22,9	−30,9	−43,6	−49,0
$^3S_1$	(a)	44,0	20,3	6,2	−2,9	−9,1	−13,8	−22,5	−32,8
	(b)	43,7	29,8	7,3	−2,3	−9,7	−15,7	−24,5	−30,5
	(c)	42,7	19,8	4,4	−3,0	−11,5	−20,6	−30,1	—
$^3D_1$	(a)	−13,2	−22,6	−26,7	−27,8	−27,8	−29,4	−37,3	−49,7
	(b)	−11,5	−19,4	−24,6	−28,0	−30,8	−33,1	−36,7	−39,5
	(c)	−12,7	−19,4	−24,3	−25,2	−28,1	−32,1	−26,1	—
ε_1	(a)	1,6	2,9	4,1	5,3	6,3	7,0	7,6	7,6
	(b)	1,8	4,2	5,2	5,7	5,6	5,3	4,4	3,4
	(c)	1,2	4,4	6,1	5,0	7,2	8,2	11,4	—
$^3P_1$	(a)	−14,3	−22,0	−28,5	−34,2	−39,4	−43,9	−56,9	−58,0
	(b)	−13,9	−22,1	−28,7	−34,1	−39,2	−45,1	−53,9	−59,8
	(c)	−14,4	−22,1	−29,2	−34,4	−40,7	−46,0	−52,1	−56,5
$^3P_0$	(a)	9,6	1,9	−7,1	−16,3	−25,6	−34,3	−63,6	−72,7
	(b)	12,3	1,6	−9,8	−19,0	−27,0	−38,1	−59,5	−72,2
	(c)	11,1	−0,5	−10,1	−19,4	−26,6	−33,4	−47,0	−56,2
$^1P_1$	(a)	−13,1	−21,9	−29,7	−35,2	−38,2	−38,2	−31,0	−24,6
	(b)	−12,2	−23,9	−31,0	−35,0	−37,1	−38,1	−38,3	−37,2
	(c)	−11,4	−21,5	−29,2	−32,5	−33,5	−28,7	−40,8	—

*Here and below, the definition of the phase shifts proposed in Ref. 123 is used.

TABLE II. Values of the parameters b, $\sqrt{s_1}$, and γ_1 for the variants of calculating the np phase shifts given in Table I.

Channel	b, GeV^{-1}	$\sqrt{s_1}$, GeV	γ_1,(GeV)$^{2(l+1)}$
$^1S_0$	5,3	2,455	1,133
$^3S_1$	5,6	2,222	0,450
$^3D_1$	5,9	2,222	0,00015
$^3P_1$	5,3	2,487	0,016
$^3P_0$	5,7	2,410	0,004
$^1P_1$	5,7	2,575	0,086

sion of meson exchange effects is very important for obtaining a good description of the $^1S_0$ phase shifts in the low-energy region. In particular, the systematic inclusion of the two-pion exchange mechanism [variant (A)] leads to closer agreement of the results of the calculations with the phase-shift data than variant (D). This is the case because the contribution of the segment of the dynamical cut closest to the physical region, which corresponds to the nonresonance part of the two-pion interaction mechanism, is not taken into account in the one-boson exchange model. As regards the resonance part of the two-pion interaction, it corresponds to segments of the dynamical cut far from the physical region, and therefore, as already emphasized in Sec. 2, it is strongly suppressed.

We see from these results that in this approach it is possible to obtain a good description of the energy dependences of the NN-scattering phase shifts simultaneously at both intermediate energies and low energies. Therefore, the parameters b, s_1, and γ_1 controlling the behavior of the phase shifts at intermediate energies ($T \sim 1$ GeV) are to a considerable degree also responsible for the energy dependence of the scattering amplitudes in the near-threshold region. The influence of meson exchange effects on the results of calculating the scattering lengths a_t, a_s and the effective ranges r_t, r_s in the triplet and singlet states has been studied in Refs. 88, 89, and 91. The analysis shows that these effects are largest ($\sim 25\%$) for the effective range in the singlet state, r_s, while for the scattering lengths a_t, a_s and the effective range r_t in the triplet state they are considerably less important. It is therefore interesting to derive approximate analytic expressions for the low-energy parameters without meson exchange effects.

If we neglect the influence of the dynamical cut in the partial-wave scattering amplitude and also the contribution of the second term in Eq. (19), which arises from relativistic effects, and use the approximation (111), for a_t and r_t we obtain the expressions

$$a_t = b + (m_N \rho_0 h)^{-1}; \tag{112}$$

$$r_t = 2\left\{\frac{1}{3}[b + (m_N \rho_0 h)^{-1}] + m_N^{-1}(1 + m_N \rho_0 b h)^{-2} \times \left[\frac{2}{\pi} + \frac{\xi_1}{\rho_1^4} + \frac{1}{2}\rho_0 h - \frac{1}{3\rho_0 h}\right]\right\}, \tag{113}$$

TABLE III. The $^1S_0$ and $^3P_1$ phase shifts of np scattering in the low-energy region.*

T, MeV		1	5	10	15	20	50	75	100
$^1S_0$	(A)	61,2	63,3	59,8	56,6	53,6	40,4	32,3	25,6
	(B)	51,6	58,8	57,2	54,9	52,7	41,5	34,1	27,8
	(C)	45,2	57,4	57,8	56,4	54,6	44,0	36,4	29,9
	(D)	58,2	65,1	61,3	57,4	54,6	41,3	33,2	26,4
	exp.	61,9	63,2	59,3	55,9	52,9	40,4	32,9	26,4
$^3P_1$	(A)	−0,13	−1,08	−2,36	−3,52	−4,56	−9,17	−11,95	−14,30
	(B)	−0,14	−1,17	−2,58	−3,90	−5,10	−10,71	−14,18	−17,05
	(C)	−0,01	−0,15	−0,42	−0,75	−1,13	−3,87	−6,40	−8,97
	(D)	−0,13	−1,08	−2,36	−3,52	−4,54	−8,96	−11,45	−13,48
	exp.	−0,13	−1,09	−2,36	−3,49	−4,49	−8,85	−11,54	−13,92

*(A) Calculation taking into account inelasticity and meson exchange effects using the method described in the text; the parameters b, $\sqrt{s_1}$, and γ_1 are given in Table II; (B) calculation with the same parameters as in variant (A), but including the dynamical cut due only to the one-pion exchange mechanism; (C) calculation with the same parameters as in variant (A), but neglecting the dynamical cut; in variant (D) the 2π exchange interaction mechanism is approximated by ρ- and σ-meson exchanges. The following values of the coupling constants are used: $g_\rho^2/\pi = 0.53$, $f_\rho = 1.5 g_\rho$, $g_\sigma^2/4\pi = 6$ (the definition of these constants is the same as in Ref. 1), and the σ-meson mass is taken to be 550 MeV. The values of the parameters b, $\sqrt{s_1}$, and γ_1 are, respectively, 5.7 GeV^{-1}, 2.35 GeV, and 1.28 GeV^2 (in the $^1S_0$ channel) and 7.3 GeV^{-1}, 2.37 GeV, and 0.019 GeV^2 (in the $^3P_1$ channel). The rows labeled "exp." give the EDPSA data of Ref. 127.

where

$$h = 1 + \frac{\rho_0 \xi_1}{\rho_1^2 (\rho_0^2 + \rho_1^2)} - \frac{2}{\pi} \operatorname{arctg} \rho_0;$$

$$\xi_1 = \frac{\gamma_1}{4m_N^2} (1 - \rho_1^2)^2, \quad \rho_1 = \sqrt{1 - \frac{4m_N^2}{s_1}}; \qquad (114)$$

$\rho_0 = \sqrt{\varepsilon_d / m_N}$, and ε_d is the deuteron binding energy. The expression for a_s and r_s are obtained by the substitution $\rho_0 \to \rho_0' = -\sqrt{\varepsilon_d' / m_N}$ in Eqs. (112)–(114), where $\varepsilon_d' = 66$ keV (Ref. 142) is a quantity determining the position of the virtual level in the np system—the so-called "singlet deuteron." Equations analogous to (112) and (113) in the context of the nonrelativistic (P-matrix) formalism have been obtained in Ref. 66.

An interesting aspect of this method is that it allows the features of various interaction mechanisms to be studied in the space of relative internucleon separations by variation of the parameter b using data on the energy dependence of the phase shifts. Let us assume that for a given value of the parameter b the interaction in the region $r > b$ is determined by meson exchange mechanisms, which have been appropriately taken into account by specifying the discontinuity of $A_l(s)$ on C_L. Then the behavior of the phase shift must be related to disc $A_l(s)$ on C_L in such a way that the following condition holds at all energies [see (28)]:

$$d\widetilde{P}_l(b, s)/ds < 0, \qquad (115)$$

which is a consequence of the principle of analyticity, which we assume is a reflection of the requirement of causality. It is natural to view the inequality (115) as a generalization of the condition satisfied by the logarithmic derivative of the wave function in nonrelativistic quantum mechanics.[55]

If a fundamentally different interaction mechanism not reducing to meson exchanges acts in the region $r > b$, then for a given value of b the condition (115) can be violated. The range of values of the parameter $b (b < b_0)$ in which (115) is violated must be assumed to be "forbidden." This means that the interaction at distances $r < b_0$ can be described in terms of CDD(RA) poles of the function $P_l(b,s)$ (for $b > b_0$) but cannot be attributed to the "external" interaction, which is determined by the functions $\Delta_l(b,s)$ and $n_l(b,s)$ [(15) and (16)] (for $b < b_0$). Of course, in practice it is possible to reliably calculate the contribution to disc $A_l(s)$ on C_L from only the dynamical singularities closest to the physical region, which are responsible for the interaction in the peripheral region. The energy dependence of the phase shift is also known with some finite accuracy from the data of phase-shift analyses. A favorable factor is that the contributions of segments of the dynamical cut far from the physical region are suppressed by the exponential factor in (19). However, the degree of this suppression depends on the value of b. As b decreases the distant singularities become more important, and their contribution is poorly known. It is therefore necessary to make a special analysis of the most probable reasons for the possible violation of the condition (115) in each specific case.

As an example, in Table IV we give the data on the energy dependence of the function $\widetilde{P}_l(b,s)$ for NN scattering in the $^1P_1$ state obtained by the EDPSA (Ref. 127) for two values of b (7.3 and 5.7 GeV^{-1}) taking into account [variant (a)] and neglecting [variant (b)] the dynamical cut. We see from these data, first of all, that, as noted above, the function $P_l(b,s)$ acquires an additional pole at low energies if the influence of meson exchange effects is neglected and, second, that the condition (115) is violated in the low-energy region when the dynamical cut is taken into account even in the case $b = 7.3$ GeV^{-1}. These regularities are not qualitatively changed when the meson–nucleon coupling constants are varied within reasonable limits and also for different variants of including two-pion exchange effects. Since it is difficult to assume that in the region $r > 7.3$ GeV^{-1} there is some interaction mechanism which has not been included, in our opinion the most likely reason for the violation of the condition (115) in this case is that the EDPSA (Ref. 127) does not adequately represent the energy dependence of the $^1P_1$ phase shift at low energies. The theoretical curves given in Fig. 1 can serve as examples of the behavior of the $^1P_1$ phase shift which do not violate (115). In the same figure we also give the data from various phase-shift analyses. Aside from the behavior in the low-energy region, it is certainly interesting to refine the energy dependence of this phase shift in the range 600–800 MeV, where there is a rather large discrepancy between the various data.

TABLE IV. Energy dependence of the function $\widetilde{P}_l(b,s) \cdot 10^3$ (GeV^2) for np scattering in the $^1P_1$ state: (a) results of the calculation based on the EDPSA data of Ref. 127, including the dynamical cut; (b) neglecting the dynamical cut.*

T, MeV	$b = 7,3$ GeV^{-1}		$b = 5,7$ GeV^{-1}	
	(a)	(b)	(a)	(b)
1	5,22	−10,88	9,85	−5,12
5	5,63	−42,73	10,31	−9,42
10	5,83	32,15	10,95	−21,67
20	6,27	12,52	12,52	176,64
50	7,29	8,66	16,91	25,19
100	8,04	7,93	21,48	22,22
200	7,68	6,83	22,24	20,07
300	6,14	5,27	19,05	17,06
400	3,90	3,31	15,58	14,16
600	−5,39	−4,27	6,94	6,73
800	−26,12	−20,05	1,07	1,46
1050	57,93	53,47	−13,89	−12,53

*The subtraction point s_0 in (22) and (27) is chosen to lie at the start at the left-hand cut.

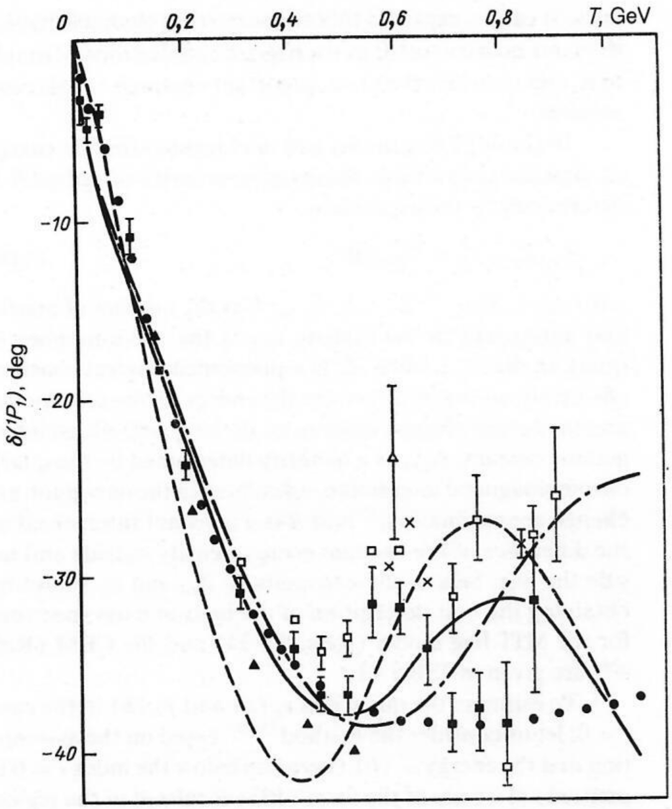

FIG. 1. Energy dependence of the $^1P_1$ phase shift of np scattering: solid line–calculation for $b = 7.3\ \mathrm{GeV}^{-1}$ ($\sqrt{s_1} = 2.289$ GeV, $\gamma_1 = 0.012\ \mathrm{GeV}^2$), taking into account the dynamical cut; dashed line—result of the calculation with the same parameters, but neglecting the dynamical cut; dot-dash line—result of the calculation with the parameters given in Table II and including the dynamical cut. The data of the phase-shift analyses are taken from Ref. 127 (●—EDPSA; ■—PSAFE), Ref. 128 (□—PSAFE), Ref. 153 (▲), Ref. 154 (×), and Ref. 155 (△).

Let us now use (115) to find the "forbidden" range of values of the parameter b in the $^1S_0$ state. In Table V we give the results of calculating the function $P_l(b,s)$ obtained using the EDPSA data (Ref. 127) for the $^1S_0$ phase shift with four values of b, taking into account meson exchange interactions. We see from this table that the values of b equal to 5.0 and 4.5 GeV^{-1} lie in the "allowed" region, while b equal to 4.0 and 3.0 GeV^{-1} already lie in the "forbidden" region. This result can be interpreted in the following way. In order to explain the behavior of the $^1S_0$ phase shift from the EDPSA (Ref. 127), at distances $r \lesssim 0.8$ fm there must be an interaction mechanism which does not reduce to the meson exchange interactions taken into account in the manner described above in the calculation of the discontinuity of the partial-wave amplitude on the dynamical cut.

We see from Table II that the optimal values of b obtained from analysis of both the S-wave and the P-wave NN scattering amplitudes at low and intermediate energies are practically identical and lie in the range 5.3–5.7 GeV^{-1}, which is about 20–30% smaller than the values obtained using Eqs. (1) and (2) if the MIT bag model is used to estimate the radius R of the six-quark bag.[32–34] This is the usual way of estimating the radius of the equivalent hadronic bag, which is the analog of the parameter b in P-matrix analyses (see, for example, Ref. 43). We also note that in the analysis based on the CQB model[73,76] the extracted values of

TABLE V. Energy dependence of the function $\widetilde{P}_l(b,s)$ for np scattering in the $^1S_0$ state for various values of the parameters b (GeV^{-1}).*

T, MeV	b			
	5,0	4,5	4,0	3,0
1	0,5173	0,5207	0,5267	0,3632
5	0,5156	0,5203	0,5268	0,5646
10	0,5155	0,5198	0,5269	0,5662
20	0,5130	0,5184	0,5265	0,5686
50	0,5033	0,5114	0,5223	0,5708
100	0,4856	0,4986	0,5141	0,5722
200	0,4524	0,4765	0,5019	0,5821
300	0,4129	0,4498	0,4854	0,5839
400	0,3579	0,4115	0,4584	0,5701
600	0,2328	0,3344	0,4098	0,5566
800	0,0703	0,2506	0,3627	0,5494
1050	−0,2884	0,1084	0,2891	0,5301

*The subtraction point s_0 in (22) and (27) is chosen to lie at the start at the left-hand cut.

the radius of the equivalent hadronic bag in the $^1S_0$ and $^3S_1$–$^3D_1$ channels are 6.1–6.7 GeV^{-1}.

In relation to this it is interesting to analyze the question of what role is played by the systematic inclusion of effects of the relativistic dynamics in our dispersion approach, which is manifested in the fact that we start from the required analytic structure for the invariant Feynman amplitudes. For this purpose we carried out calculations of the NN phase shifts in the "nonrelativistic" version of the equations, which in the one-channel case are obtained from the equations formulated in Sec. 2 by replacement of the function $\rho_l(s)$ (4) by $k^{2l}\sqrt{s/4m_N^2-1}$ and modification of (19) [in the "nonrelativistic" version the second term in the expression for Im $T_l(b,s)$ must be dropped]. The rest of the calculation was carried out according to the scheme described above (taking into account the relativistic kinematics with the same discontinuities of the partial-wave amplitudes as in the relativistic case). It follows from analysis of the S- and P-wave NN phase shifts on the basis of the nonrelativistic version of the equations that the values of b which are optimal (in the χ^2 sense) lie in the range 6.6–7.0 GeV^{-1}. Therefore, the inclusion of effects of the relativistic dynamics makes it possible to decrease b by about 0.25–0.3 fm.

It follows from the analysis of Sec. 5 that the poles of the function $P_l(b,s)$ (28) can be associated with states of a compound system like the compound quark–gluon bag, which makes it possible to use the quark Hamiltonians of the bag model to estimate the quantities $s_{li}(b)$ and $\gamma_{li}(b)$. In the MIT model[32-34] the quark bag is treated as an isolated system whose size is determined by the balance of the vacuum pressure and the pressure of the quark–gluon plasma located in a spherical cavity of radius R. The positions of the lowest levels of the six-quark bag in the S-wave NN scattering channels in this model are 2.243 GeV ($^1S_0$) and 2.165 GeV ($^3S_1$) (Refs. 40 and 43). For the P-wave channels we can use the calculation of the energies of the lowest states in the rotating-bag model:[65] 2.11 GeV ($^1P_1$), 2.20 GeV ($^3P_1$), and 2.245 GeV ($^3P_0$). The reason why these energies are somewhat smaller than the values of $\sqrt{s_1}$ given in Table II is closely related to the fact that the radii of six-quark bags in the MIT model are larger than the values given by (1) and (2) and the data on the parameter b given in Table II. In fact, if we calculate R from the equilibrium condition for the six-quark bag in the MIT model, then by using (1) and (2) for NN scattering in the $^1S_0$ and $^3S_1$ states we obtain $b\simeq 7.4$ GeV^{-1}.

A drawback of this method of theoretically estimating the parameter b and the locations of the CDD(RA) poles is, in our opinion, that the equilibrium condition in the MIT bag model does not include the coupling of the six-quark bag to the NN scattering channels. We note that in chiral models (CM) (Refs. 45–49) and, in particular, in the Cloudy bag model (CBM) (Refs. 46 and 47) the quark bag is a system in dynamical equilibrium with the pion field surrounding it. In other words, the coupling of the quark bag to the corresponding pion scattering channel is taken into account in the CM. For example, for the nucleon bag this is the πN scattering channel. According to this logic, the quark bag can be treated as a system of quark and gluon fields in dynamical equilibrium with the hadron fields of all the scattering channels (including those closed at a given energy) to which this system can be coupled in accordance with the conservation laws. It can be expected that the scattering channels having the same quark content as the bag are coupled most strongly to it, and these are the most important channels to take into account.

In the MIT bag model and in chiral models the energy of massless quark fields localized in a cavity of radius R is determined by the expression

$$H_Q(R)=\frac{A}{R}+\frac{4\pi}{3}BR^3, \tag{116}$$

where $A=N\omega_Q-Z_0+\alpha_s\Delta_{\rm CM}$; N is the number of quarks and antiquarks in the system, ω_Q is the wave number of quark in the $S_{1/2}$ state, Z_0 is a phenomenological constant effectively taking into account the energy of the zero modes and the center-of-mass motion, α_s in the quark–gluon interaction constant, $\Delta_{\rm CM}$ is a quantity determined by the quark chromomagnetic interaction, calculated in the one-gluon exchange approximation,[35] and B is a constant interpreted as the difference of the vacuum energy density outside and inside the bag. Sets of the constants B, Z_0, and α_s found by obtaining the best description of the hadron mass spectrum for the MIT bag model (Refs. 32–34) and the CBM (Ref. 47) are given in Table VI.

To estimate the quantities $s_{li}(b)$ and $\gamma_{li}(b)$ in the case $l=0$, let us consider the method[99,100] based on the assumption that the energy $\omega_i(b)$ (here and below the index $l=0$ is omitted) of a state of the form (103) localized in the region of relative hadron separations $r<b$ is determined by the Hamiltonian $H_Q(R)$ (116) via the equation

$$\omega_i(b)=H_Q(R), \tag{117}$$

where the radius R of a compound system of the quark–gluon bag type and the parameter b are related by (1), in which the constant c is either estimated theoretically using (2) or treated as a phenomenological parameter. Taking into account (1), (95), (116), and (117), Eq. (107) (for $l=0$) takes the form

$$\frac{A}{R^2}=4\pi BR^2+\frac{1}{4}c\gamma_i(cR). \tag{118}$$

The physical meaning of this equation is that the pressure created by the quarks in the interior of the cavity is balanced by the sum of the vacuum pressure [first term on the right-hand side of (118)] and the pressure due to the coupling to the scattering channel. In the limit $\gamma_i(b)\to 0$ (i.e., in the absence of coupling to the scattering channel), Eq. (118) reduces to the usual equilibrium condition for an isolated bag of the MIT model. The requirement that the residues $\gamma_i(b)$ be non-negative (see Sec. 2), which is a consequence of the principle of causality, implies that the additional pressure due to the coupling to the hadronic scattering channel always acts in the direction so as to compress the quark bag.

In Tables VII and VIII we give the values of $\sqrt{s_1}$ and γ_1 for NN scattering in the $^1S_0$ and $^3S_1$ states found from a fit to the data of the phase-shift analysis[127] by the χ^2 method at fixed values of b, together with the predictions based on Eqs. (117) and (118) with the parameters of the MIT bag model and the CBM given in Table VI. For comparison, in Tables VII and VIII we also give the results of the analogous analysis carried out using the nonrelativistic version of the equations described above. For the Hamiltonian of the MIT bag model the parameters $\sqrt{s_1}$, γ_1, and c were estimated in three ways: In variant (a) the parameter c was taken to be 1.1

TABLE VI. Sets of parameters $B^{1/4}$, α_s, Z_0 for the MIT (Refs. 32–34) and CBM (Ref. 47) models.*

Model	$B^{1/4}$, GeV	α_s	Z_0
MIT	0,146	2,2	1,84
CBM (1)	0,169	1,69	1,80
CBM (2)	0,151	1,41	1,31

*In the second and third rows of the table we give two parameter sets for the CBM model which give practically equally good descriptions of the hadron mass spectrum.

(1.0) for NN scattering in the $^1S_0(^3S_0)$ state, and $\sqrt{s_1}$ and γ_1 were calculated using Eqs. (117) and (118). In variant (b), γ_1 was taken equal to the "experimental" value, i.e., the value found from fitting to the phase-shift data (for a given b), and $\sqrt{s_1}$ and c were calculated using these equations. In variant (c), $\sqrt{s_1}$ was set equal to the experimental value, and predictions were obtained for γ_1 and c. When the Hamiltonian (116) was used with the parameters of the models CBM(1) the CBM(2) (see Table VI) the calculations of $\sqrt{s_1}$ and γ_1 were carried out in the same way as in variant (a). In variants (b) and (c) the predictions for the parameter c practically coincide, and the results of the calculations change only slightly when the parameter b is varied near the minimum of χ^2. The values obtained for the parameter c are slightly smaller than the theoretical estimate from (2) ($c \cong 1.1$), but this discrepancy is small, especially for scattering in the $^1S_0$ state.

We see from the data of tables VII and VIII that, owing to the inclusion of the coupling of the six-quark bag to the NN scattering channel, the predictions based on Eqs. (117) and (118) for the location of the CDD(RA) pole s_1 and the residue γ_1 are on the whole in satisfactory agreement with the experimental values for values of b near the minimum χ^2.

We conclude this section by giving the results of the calculation of the $^1S_0$ and $^3S_1$ phase shifts (see Fig. 2 and Table IX), in which the parameters s_1 and γ_1 were calculated using (117) and (118). The constant a (111), as above, was determined from the condition that the system have the correct position of the virtual level in the $^1S_0$ channel or the experimental deuteron binding energy in the $^3S_1$ channel (the D wave was neglected in this calculation). The remaining undetermined parameter b was chosen in such a way that the scattering length in each channel considered coincided with its experimental value.[1]

7. ANALYSIS OF MESON–MESON AND MESON–BARYON SCATTERING PROCESSES AT INTERMEDIATE ENERGIES

The approach discussed here is quite universal and can be applied not only to NN scattering, but also to other ha-

TABLE VII. Parameters of the CDD(RA) poles* in the $^1S_0$ channel for np scattering at fixed value of b.

b, GeV^{-1}	4,9	5,1	5,3	5,5	5,7	5,9
Fit to the data of the phase-shift analysis of Ref. 127						
$\sqrt{s_1}$, GeV	2,56	2,51	2,45	2,40	2,34	2,30
γ_1, GeV^2	1,32	1,25	1,12	0,98	0,83	0,71
χ^2	2,07	1,78	1,55	1,43	1,42	1,63
$\sqrt{\tilde{s}_1}$, GeV	2,58	2,55	2,52	2,48	2,43	2,38
$\tilde{\gamma}_1$, GeV^2	1,77	1,87	1,95	1,91	1,80	1,63
χ^2	5,49	6,41	5,68	4,96	4,24	3,55
Predictions based on Eqs. (117) and (118)						
$\sqrt{s_1}$ MIT (a)	2,67	2,60	2,53	2,47	2,42	2,38
$\sqrt{s_1}$ MIT (b)	2,51	2,50	2,46	2,42	2,37	2,34
$\sqrt{s_1}$ CBM (1)	2,78	2,72	2,67	2,63	2,61	2,58
$\sqrt{s_1}$ CBM (2)	2,76	2,68	2,62	2,56	2,51	2,47
γ_1 MIT (a)	1,63	1,44	1,27	1,11	0,95	0,81
γ_1 MIT (c)	1,43	1,27	1,10	0,93	0,74	0,56
γ_1 CBM (1)	1,28	1,06	0,86	0,67	0,49	0,32
γ_1 CBM (2)	1,62	1,42	1,24	1,07	0,91	0,76
c (b)	1,00	1,04	1,05	1,06	1,06	1,06
c (c)	1,03	1,04	1,05	1,04	1,03	1,02

*The tilde labels parameters of the CDD(RA) poles in the nonrelativistic version of the formalism.

TABLE VIII. Parameters of the CDD(RA) poles in the $^3S_1$ channel for np scattering at fixed values of b.

b, GeV^{-1}	4,9	5,1	5,3	5,5	5,7
Fit to the data of the phase-shift analysis of Ref. 127					
$\sqrt{s_1}$, GeV	2,30	2,27	2,24	2,21	2,19
γ_1, GeV^2	0,52	0,51	0,48	0,45	0,41
χ^2	6,8	4,6	2,9	1,6	1,1
$\sqrt{\tilde{s}_1}$, GeV	2,33	2,31	2,28	2,26	2,23
$\tilde{\gamma}_1$, GeV^2	0,67	0,72	0,73	0,72	0,69
χ^2	42,9	35,1	27,8	21,9	16,7
Predictions based on Eqs. (117) and (118)					
$\sqrt{s_1}$ MIT (a)	2,40	2,34	2,29	2,25	2,22
$\sqrt{s_1}$ MIT (b)	2,20	2,20	2,20	2,19	2,19
$\sqrt{s_1}$ CBM (1)	2,57	2,54	2,51	2,50	2,50
$\sqrt{s_1}$ CBM (2)	2,52	2,46	2,42	2,38	2,35
γ_1 MIT (a)	1,22	1,04	0,87	0,72	0,57
γ_1 MIT (c)	0,98	0,85	0,71	0,56	0,39
γ_1 CBM (1)	0,78	0,57	0,36	0,16	—
γ_1 CBM (2)	1,22	1,02	0,84	0,67	0,52
c (b)	0,83	0,87	0,90	0,93	0,96
c (c)	0,93	0,95	0,86	0,96	0,95

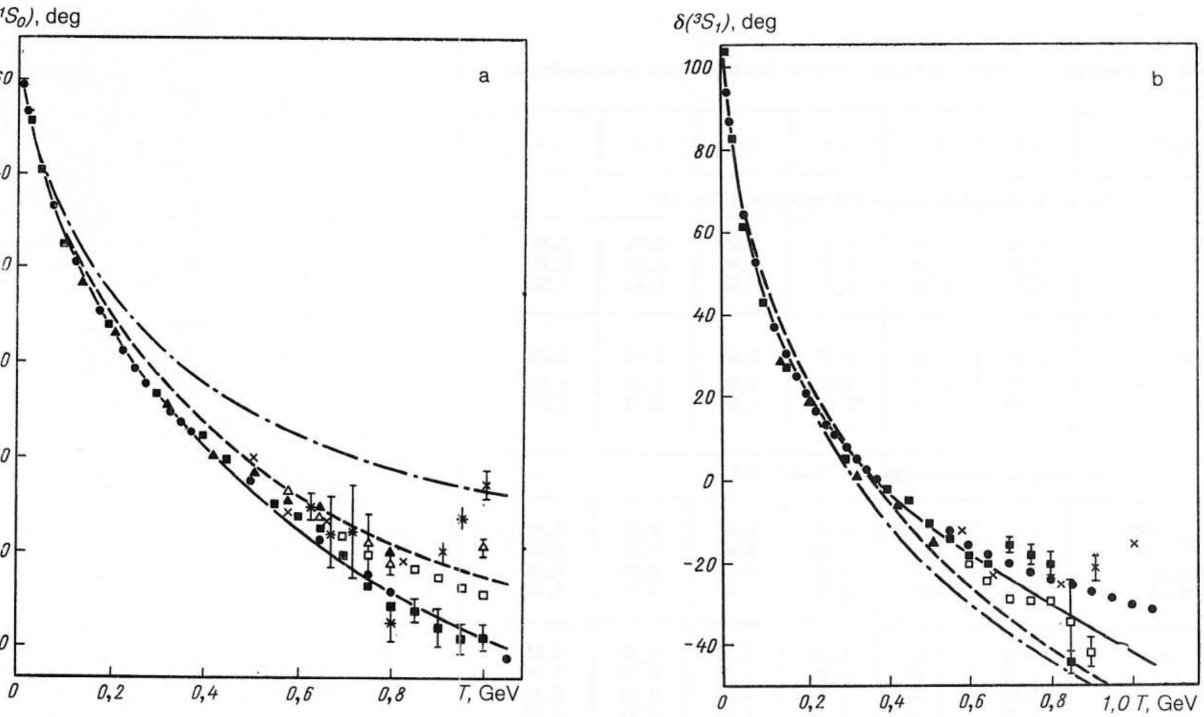

FIG. 2. Results of calculating the $^1S_0$ (a) and $^3S_2$ (b) phase shifts of np scattering with the parameters given in Table IX: the method of calculating the parameters is described in the text and is based on the use of the quark Hamiltonians of the bag models. a—the solid line corresponds to the CBM(1) model, the dashed line corresponds to the CBM(2) model, and the dot–dash line corresponds to the MIT model; b—the solid line (the variant with $b = 5.7$ GeV^{-1}) and the dot–dash line (the variant with $b = 5.1$ GeV^{-1}) correspond to the MIT model, and the dashed line corresponds to the CBM(1) model. The phase-shift data are taken from Ref. 127 (●—EDPSA; ■—PSAFE), Ref. 128 (□—PSAFE), Ref. 153 (▲), Ref. 154 (×), Ref. 156 (△), and Ref. 157 (*).

TABLE IX. Values of the parameters b, $\sqrt{s_1}$, γ_1, and c for the variants of calculating the $^1S_0$ and $^3S_1$ phase shifts of np scattering given in Fig. 2.

Phase shift	Model	b, GeV^{-1}	$\sqrt{s_1}$, GeV	γ_1, Gev^2	c
$^1S_0$	MIT	5,1	2,629	1,490	1,1
	CBM (1)	4,9	2,811	1,367	1,1
	CBM (2)	4,4	2,760	1,599	1,1
$^3S_1$	MIT	5,7	2,220	0,568	1,0
	MIT	5,1	2,485	1,354	1,1
	CBM (1)	4,1	2,828	1,839	1,0

dron–hadron scattering processes at low and intermediate energies. By studying these processes we can find new, interesting qualitative and quantitative regularities. It follows from the results of Sec. 6 that a characteristic feature of NN scattering is the fact that at intermediate energies $T \lesssim 1$ GeV, where phase-shift data exist and inelastic channels play a relatively small role, the contribution of only a single CDD(RA) pole is important in all the partial waves studied. However, it is interesting to analyze processes which are characterized by the presence of several CDD(RA) poles in some scattering channel in the energy range where there is information on the phase shifts. An important example of such a process is $\pi\pi$ scattering. In recent years the interest in this problem has been closely connected to the problem of interpreting and describing the properties of scalar mesons (see, for example, Refs. 143 and 144).

During the last 10–15 years more than ten phase-shift analyses of $\pi\pi$ and $K\bar{K}$ scattering have been carried out in the region $\sqrt{s} \lesssim 1.8$ GeV. The available experimental data do not yet permit an unambiguous phase-shift analysis at fixed energy (PSAFE). Analyses of this type, as a rule, lead to several solutions. For example, the PSAFE of Ref. 145 leads to four solutions for the δ_0^0 phase shift (δ_J^I, where I is the isospin and J is the angular momentum). The first of these solutions in the mass range 1.0–1.4 GeV almost completely coincides with the result of the energy-dependent phase-shift analysis (EDPSA) of Ref. 146. The authors of Ref. 147 presented arguments in favor of this solution. However, for masses above 1.4 GeV the EDPSA data[146] are close to another PSAFE solution.[145] The more recent EDPSA of Ref. 148 is based on experiments with large statistics on pion and kaon production in πN interactions. The data of the analyses of Refs. 146 and 148 for the δ_0^0 phase do not differ strongly from each other, and the main difference is in the detailed behavior of this phase shift in the vicinity of the $S^*(980)$ resonance. On the other hand, one of the recent analyses[149] gives values for δ_0^0 in the $\pi\pi$ mass ranges 0.4–0.7 and 1.1–1.3 GeV which differ markedly from the data of Ref. 146 and 148.

Calculations of the $S(I=0,2)$ and $P(I=1)$ phase shifts for $\pi\pi$ scattering have been carried out using the equations of Sec. 2, taking into account inelasticity effects due mainly to the $K\bar{K}$ and $\eta\bar{\eta}$ channels. Information on the inelasticity parameter in these equations was taken from the data of phase-shift analyses of $\pi\pi$ and $K\bar{K}$ scattering.[146,148] The discontinuities of the partial-wave amplitudes on the dynamical cut were calculated with the ρ-meson exchange mechanism in the t channel. However, owing to the above-mentioned effect of suppression of distant dynamical singularities, the contribution of ρ-meson exchange [first term in (19)] only very slightly affects the results of the calculation of the partial-wave amplitudes. The second term in (19) arising from relativistic effects significantly influences the results of the calculation of the phase shifts.

In Fig. 3 we show the form of the function $\tilde{P}_0(b,s)$ for $b = 5$ GeV^{-1} in the channel $I = J = 0$, obtained from the EDPSA of Ref. 146. We see that $\tilde{P}_l(b,s)$ has the behavior characteristic of an R function: It falls off with increasing argument for all values of s and has discontinuities corresponding to poles. In Fig. 4 the solid line [variant (A)] shows the fit to the EDPSA data[146] for the δ_0^0 phase shift of $\pi\pi$ scattering, taking into account the contribution of four CDD(RA) poles for $b = 5$ GeV^{-1} and the parameters $\sqrt{s_i} \equiv \sqrt{s_{0i}(b)}$ and $\gamma_i \equiv \gamma_{0i}(b)$ $(i = 1,2,3,4)$ given in Table X. In the table we also give the values of the masses of the primitives of the $Q^2\bar{Q}^2$ system in the $J^P = 0^+$ channel calculated using the MIT bag model.[43] We note that, in contrast to the Jaffe calculation,[39] the mixing due to the dependence of the strength of the chromomagnetic interaction on the quark flavor was taken into account in Ref. 43. However, for the levels of the $Q^2\bar{Q}^2$ system given in Table X, the results of Refs. 39 and 43 are very similar. It can be expected that near the third CDD(RA) pole there is also a primitive of the $Q\bar{Q}$ system corresponding to the $\varepsilon(1300–1400)$ resonance.

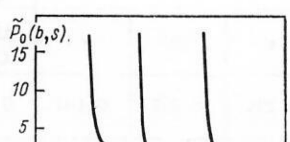

FIG. 3. Energy dependence of the function $\tilde{P}_0(b,s)$ in (27) for $\pi\pi$ scattering in the channel with $J = I = 0$ for $b = 5$ GeV^{-1}, constructed from the data of the phase-shift analysis of Ref. 146.

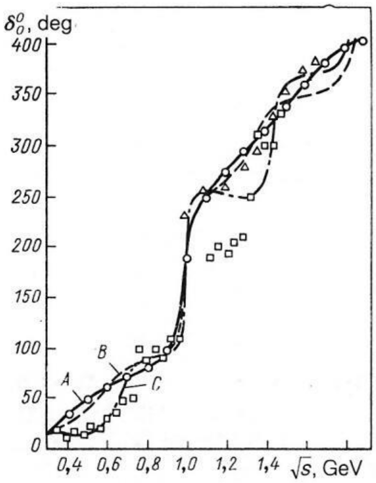

FIG. 4. Energy dependence of the δ_0^0 phase shift of $\pi\pi$ scattering for $b = 5$ GeV^{-1} for variants A, B, and C of the calculation with the parameters $\sqrt{s_i}$ and γ_i ($i = 1,2,3,4$) given in Table X. The phase-shift data were taken from Ref. 146 (○), Ref. 148 (△), and Ref. 149 (□).

In order to demonstrate the degree to which the calculated δ_0^0 phase shift is sensitive to the choice of the various parameters (especially the residues γ_i), in Fig. 4 we give two alternative variants of the calculation, B and C. As can be seen from Table X, the positions of the CDD(RA) poles found from the fit to the phase-shift data are in fairly good agreement with the calculations with the mass spectrum of the $Q^2\overline{Q}^2$ system in the MIT quark bag model.[39,43]

Let us try to estimate γ_i by taking into account (94) and (95) and using the P-matrix method proposed in Ref. 54 to estimate the residues. As a result, we obtain the expression[95]

$$\gamma_i = 3\sqrt{s_i}\,\langle\Lambda\rangle/b, \qquad (119)$$

where $\langle\Lambda\rangle$ is the fractional admixture of the quark configuration corresponding to the hadron scattering channel in the cluster expansion of the wave function of the bag state. Assuming that the residue at the first CDD(RA) pole comes from the coupling to the $C^0(\underline{9})$ primitive, and that the second CDD(RA) pole is due to the $C^0(\underline{36})$, by using (119) we obtain the following values for these residues: $\gamma_1 = 0.16$ GeV^2 and $\gamma_2 = 0.06$ GeV^2 (Ref. 95).

It should be noted that the residues γ_i are more sensitive to the details of the energy dependence of the phase shift than the positions of the CDD(RA) poles are. We see from Fig. 4 that the behavior of phase shifts like those in variants B or C does not contradict the phase-shift data. Moreover, the values of the residues in these variants differ considerably from their values in variant A. Therefore, to obtain more accurate data on the residues it is necessary to improve the experimental information on the δ_0^0 phase shift in $\pi\pi$ scattering.

Now we use Eqs. (117) and (118) to calculate s_1, γ_1 and also the parameter c in Eq. (1). In Table XI we give the results of calculating the parameters pertaining to the lowest level of the $Q^2\overline{Q}^2$ bag with the quantum numbers $J = I = 0$ for $b = 5$ GeV^{-1}. In variant (a) the parameter c is estimated using (2), and $\sqrt{s_1}$ and γ_1 are calculated using (117) and (118) and the quark Hamiltonian (116) with the parameters of the MIT bag model (see Table VI); in variant (b) the value of γ_1 is taken from the fit to the data of the phase-shift analysis [for variant (A), see Table X], and $\sqrt{s_1}$ and c are predicted using these equations; finally, in variant (c) the value of $\sqrt{s_1}$ is taken from the fit [for variant (A), see Table X], and Eqs. (117) and (118) are used to calculate γ_1 and c. We see from Tables X and XI that the predictions for γ_1 based on Eqs. (117) and (118) and the Hamiltonian of the MIT bag model are in satisfactory agreement with the values found from the fit to the phase-shift data. The predictions for the parameter c are also in satisfactory agreement with the estimate from (2).

A similar analysis was also carried out for the δ_0^2 and δ_1^1 phase shift of $\pi\pi$ scattering. In each of these cases it is possible to restrict ourselves to the inclusion of only two

TABLE X. Parameters of the CDD(RA) poles for $\pi\pi$ scattering in the channel with $I = J = 0$ (see Fig. 4). The predictions for the masses of the $Q^2\overline{Q}^2$ states (primitives) are taken from Ref. 43.

	$\sqrt{s_i}$, GeV			γ_i, GeV^2			Predictions for the masses of the $Q^2\overline{Q}^2$ ($J^P = 0^+$, $I = 0$) primitives	
	(A)	(B)	(C)	(A)	(B)	(C)	Mass, GeV	Primitive notation
1	0,651	0,638	0,685	0,293	0,207	0,150	0,642	$C^0(\underline{9})$
2	1,009	1,005	0,998	0,128	0,091	0,040	1,122 1,115	$C^0(\underline{36})$ $C^s(\overline{\underline{9}})$
3	1,380	1,358	1,431	1,049	0,701	0,298	1,43 1,51	$C^0(\underline{9}^*)$ $C^s(\underline{\overline{36}})$
4	1,913	1,838	1,844	4,177	0,800	1,047	1,82 1,78	$C^0(\underline{36}^*)$ $C^{ss}(\underline{\overline{36}})$

Note. The primitive notation of Ref. 37 is used; the quantity in the parentheses gives the $SU(3)_{fl}$ multiplet to which the state will belong in the absence of the mixing induced by the difference of the chromomagnetic forces for different quark flavors; the stars denote color-spin excitations of a given multiplet; the letter s denotes the presence of a hidden $s\overline{s}$ pair.

TABLE XI. Predictions for the parameters $\sqrt{s_1}$, γ_1, and c in the $J = I = 0$ $\pi\pi$ scattering channel based on Eqs. (117) and (188). The variants (a), (b), and (c) of the calculation are described in the text.

Variant	$\sqrt{s_1}$, GeV	γ_1, GeV^2	c
(a)	0,678	0,265	1,4
(b)	0,687	0,293	1,437
(c)	0,651	0,140	1,260

CDD(RA) poles. The corresponding fits to the phase-shift data from Refs. 146 and 150 with the parameters given in Tables XII and XIII are shown in Figs. 5 and 6. In Table XII we also give the predictions for the masses of the two lowest primitives of the $Q^2\overline{Q}^2$ system with the quantum numbers $J^P = 0^+$, $I = 2$ in the MIT bag model.[43] Therefore, also in the channel with $I = 2$ the positions of the lowest CDD(RA) poles are in good agreement with the predictions of the quark bag model.

It is interesting to compare the behavior of the $\pi\pi$ phase shifts in the S-wave channels with isospins 0 and 2. The qualitatively different behavior of these phase shifts is due to the different arrangements of bag levels: In the channel with $I = 2$ the lowest levels are shifted towards higher energies relative to the levels in the channel with $I = 0$ by about 0.5 GeV.

The equations formulated in Sec. 2 were generalized to the case of the scattering of particles with different masses m_1, m_2 and used to study the πK, πN, and KN scattering processes at intermediate energies.[99] The renormalized amplitude, as in the scattering of particles with equal masses, is determined by Eqs. (27) and (28) with $s_R = (m_1 + m_2)^2$. The special feature of the scattering of particles with different masses is that the partial-wave amplitudes have dynamical cuts in the complex region in the s plane. This to some degree complicates the solution of the system of equations of the form (21), (22). However, it should be noted that the effect of these cuts on the results of the calculation of the S-wave phase shifts of πK, πN, and KN scattering considered below is small. The calculated values of the S_{2I} phase shifts of πK scattering in states with isospin I equal to 1/2 and 3/2 are given in Tables XIV and XV together with the data of the phase-shift analysis of Ref. 151. Two CDD(RA) poles are included in each channel. The corresponding sets of parameters for variants (A) and (B) of the calculation are given in Table XVI. In the calculations we included the dynamical cuts of the partial-wave amplitudes related to the diagram for $K^*(892)$-meson exchange in the u channel. However, this mechanism has little effect on the results of calculating the S-wave phase shifts of πK scattering.

The masses of the two lowest $Q^2\overline{Q}^2$ primitives in the exotic channel with strangeness equal to unity and isospin $I = 1/2$ in the MIT bag model are 0.88 GeV [$C_K(\underline{9})$] and 1.32 GeV [$C_K(\underline{36})$]; those with isospin $I = 3/2$ are 1.32 GeV [$E_{\pi K}(\underline{36})$] and 1.95 GeV [$E_{\pi K}(\underline{36})$].[43]

The data given in Table XVI can be compared with the results of the P-matrix analysis. The values of the parameter $\sqrt{s_1}$ extracted by processing the experimental data using the P-matrix formalism, which determines the locations of the lowest P-matrix poles, are 0.94 GeV for scattering in the channel with $I = 1/2$ and 1.19 GeV for the channel with $I = 3/2$.[43] Therefore, the locations of the lowest poles of the P matrix are in fairly good agreement with the results in Table XVI [especially for variants (A)]. However, for the residues at these poles the difference between the data of the P-matrix analysis and the results of Table XVI is more significant. If Eqs. (94) and (95) are used to relate the P-matrix residues and the function $\widetilde{P}_l(b,s)$ (28), then by using the data on the residues from the P-matrix analysis of Ref. 43 we obtain the following values of the parameter γ_1 in the πN scattering channels considered: $\gamma_1 = 0.165$ GeV^2 ($I = 1/2$) and $\gamma_1 = 0.37$ GeV^2 ($I = 3/2$).

Let us find the predictions for $\sqrt{s_1}$ and γ_1 in the S-wave πK scattering channels using equations of the form (117), (118) and the quark Hamiltonians of the bag model. However, here the expression (116) for $H_Q(R)$ in Eq. (117) must be modified to include the finiteness of the s-quark mass m_s. In the case $m_s \neq 0$ the quantity A in (116) is a function of R. For it we can use the approximate expression[43]

$$A \cong A_0 + \lambda n_s R, \tag{120}$$

TABLE XII. Parameters of the CDD(RA) poles for $\pi\pi$ scattering in the channel with $J = 0, I = 2$ (see Fig. 5). The predictions for the masses of the $Q^2\overline{Q}^2$ states (primitives) are taken from Ref. 43.

i	$\sqrt{s_i}$, GeV	γ_i, GeV^2	Predictions for the masses of the $Q^2\overline{Q}^2$ ($J^P = 0^+, I = 0$) primitives	
			Mass, GeV	Primitive notation
1	1,14	2,03	1,122	$E_{\pi\pi}$ ($\underline{36}$)
2	2,08	10,16	1,8	$E_{\pi\pi}$ ($\underline{36}$ *)

TABLE XIII. Parameters of the CDD(RA) poles for $\pi\pi$ scattering in the channel with $J=I=1$ (see Fig. 6).

$\sqrt{s_i}$, ГэВ		γ_i, ГэВ⁴	
$i=1$	$i=2$	$i=1$	$i=2$
0,805	1,72	0,0038	0,187

TABLE XIV. Results of the calculation of the S_1 phase shift of πK scattering in the state with isospin $I=1/2$. The parameter sets corresponding to variants (A) and (B) of the calculation are given in Table XVI. The row labeled "exp." gives the data of the phase-shift analysis of Ref. 151.

$\sqrt{s}$, GeV	0,730	0,875	0,935	1,1	1,3
S_1 (A)	19,6	37,2	42,7	59,2	76,9
S_1 (B)	22,3	37,1	42,8	59,0	77,4
S_1 exp.	21,0	38,1	40,9	60,2	79,3

TABLE XV. Results of the calculation of the S_3 phase shift of πK scattering in the state with isospin $I=3/2$. The parameter sets corresponding to variants (A) and (B) of the calculation are given in Table XVI. The row labeled "exp." gives the data of the phase-shift analysis of Ref. 151.

$\sqrt{s}$, GeV	0,723	0,98	1,18	1,34	1,72
S_3 (A)	−13,6	−17,7	−22,5	−25,3	−23,6
S_3 (B)	−12,7	−17,9	−22,4	−25,1	−22,4
S_3 exp.	−10,0	−18,0	−22,4	−26,4	−22,8

TABLE XVI. Parameters set [b (GeV^{-1}), $\sqrt{s_i}$ (GeV), γ_i (GeV^2), $i=1,2$] corresponding to variants (A) and (B) of the calculation of the S_{2I} phase shifts ($I=1/2,3/2$) of πK scattering given in Tables XIV and XV.

πK	b	$\sqrt{s_1}$	$\sqrt{s_2}$	γ_1	γ_2
S_1 (A)	6,0	0,945	1,583	0,209	1,879
S_1 (B)	5,0	1,012	1,782	0,304	3,283
S_3 (A)	6,0	1,242	2,323	0,803	9,07
S_3 (B)	5,0	1,421	2,278	1,371	6,3

TABLE XVII. Predictions for $\sqrt{s_1}$ (GeV) and γ_1 (GeV^2) in the S-wave πK-scattering channels based on equations of the form (117) and (118).

Model	b	c	$I=1/2$		$I=3/2$	
			$\sqrt{s_1}$	γ_1	$\sqrt{s_1}$	γ_1
MIT	5,0	1,4	0,927	0,312	1,533	0,797
CBM (1)	5,0	1,4	1,267	0,363	1,732	0,736
CBM (2)	6,0	1,4	1,362	0,324	1,685	0,540
MIT	5,0	1,5	0,958	0,388	1,606	0,907
MIT	6,0	1,5	0,892	0,144	1,433	0,504
IT	5,0	1,6	0,992	0,455	1,684	1,009
IT	6,0	1,6	0,910	0,212	1,487	0,597

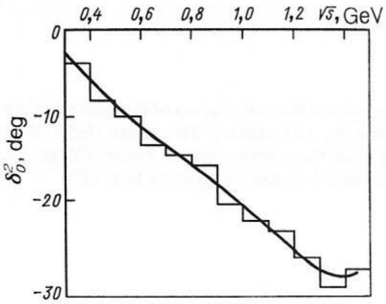

FIG. 5. Energy dependence of the δ_0^2 phase shift of $\pi\pi$ scattering for $b = 5.3$ GeV^{-1} with the parameters $\sqrt{s_i}$ and γ_i ($i = 1,2$) given in Table XII. The phase-shift data, shown as a histogram in the figure, are taken from Ref. 150.

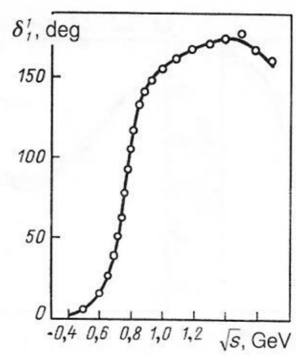

FIG. 6. Energy dependence of the δ_1^1 phase shift of $\pi\pi$ scattering for $b = 5$ GeV^{-1} with the parameters $\sqrt{s_i}$ and γ_i ($i = 1,2$) given in Table XIII. The phase-shift data (○) are taken from Ref. 146.

where A_0 is calculated for $m_s = 0$, n_s is the number of s quarks in the system, and λ is a coefficient equal to about 0.19 GeV. In Eq. (118), A should be replaced by A_0. The results of calculating $\sqrt{s_1}$ and γ_1 for various values of the parameters b and c are given in Table XVII. Comparing the data of Tables XVI and XVII, we see that the theoretical predictions based on Eqs. (117) and (118) and on the bag-model Hamiltonians for the S-wave πK scattering channels are on the whole is satisfactory agreement with the values found from the fit to the experimental scattering phases.

In studying πN and KN scattering we took into account the dynamical singularities of the partial-wave amplitudes closest to the physical region, which arise from the mechanism of one-baryon exchange in the u channel. However, calculations show that this mechanism has little effect on the S-wave phase shifts. In Fig. 7 we show the results of the calculations of the S_{11} ($I = 1/2, J = 1/2$) and S_{31} ($I = 3/2$, $J = 1/2$) phase shifts of πN scattering, taking into account two CDD(RA) poles for the parameter sets given in Table XVIII. These parameters are determined from the fit to the phase-shift data of Ref. 129. The solid lines in Fig. 7 were obtained using the fitting procedure in the region $\sqrt{s} \leqslant 1.8$ GeV, and the dashed line in Fig. 4b was obtained using this procedure in the region $\sqrt{s} \leqslant 1.6$ GeV.

It should be noted that in this energy range there is a significant influence on the behavior of the S-wave πN scattering amplitudes from the coupling to inelastic channels (ηN, $\pi\pi N$) taken into account using the absorptive part of the K matrix (see Sec. 4), which is chosen by taking into account the data of the phase-shift analysis of Ref. 129. In particular, the irregularity in the behavior of the S_{11} phase shift for $\sqrt{s} \cong 1.5$ GeV (which is closely related to the resonance in this partial wave at this energy[152]) is due to the interaction with the ηN channel. The peak in the energy dependence of the S_{31} phase shift in the region $\sqrt{s} \cong 1.57$ GeV is also due to the coupling to inelastic channels.

The masses of the two lowest states of the $Q^4\bar{Q}$ bag with $I = 1/2$ in the MIT model are 1.502 GeV [C_N $\underline{1}8$)] and 1.75 GeV [C_N ($\underline{4}\underline{5}$)]; those with $I = 3/2$ are 1.713 GeV [C_Δ ($\underline{4}\underline{5}$)] and 1.95 GeV [$C_\Delta$ ($\underline{4}\underline{5}$*)].[43] Let us also give the predictions for the parameters of the lowest CDD(RA) poles in the πN scattering channels considered, based on Eqs. (117) and (118) and on the quark Hamiltonian of the MIT bag model: $\sqrt{s_1} = 1.577$ GeV, $\gamma_1 = 0.489$ GeV^2 (in the S_{11} channel for $b = 6.0$ GeV^{-1}), and $\sqrt{s_1} = 1.781$ GeV, $\gamma_1 = 0.475$ GeV^2 (in the S_{31} channel for $b = 6.4$ GeV^{-1}); the parameter c in these calculations was taken to be 1.25 in accordance with (2). Therefore, the predictions based on Eqs. (117) and (118) are in satisfactory agreement with the data of Table XVIII.

In Table XIX we give the results of the calculation of the S_{11} ($I = 1, J = 1/2$) phase shift of KN scattering, taking into account two CDD(RA) poles. The values of the parameters b, $\sqrt{s_i}$, and γ_i found from the fit to the phase-shift data of Ref. 130 are given in Table XX. The mass of the lowest $Q^4\bar{Q}$ primitive with strangeness and isospin equal to unity is 1.905 GeV [$E_{(NK)'}$ ($\underline{4}\underline{5}$)] in the MIT bag model.[43] The predictions for the parameters of the lowest CDD(RA) pole based on modified equations of the form (117) and (118) (taking into account the finiteness of the s-quark mass) and

TABLE XVIII. Parameters set [b (GeV^{-1}), $\sqrt{s_i}$ (GeV), γ_i (GeV^2)] corresponding to the variants of the calculation of the S_{11} and S_{31} phase shifts of πN scattering given in Fig. 7.

πN	b	$\sqrt{s_1}$	$\sqrt{s_2}$	γ_1	γ_2
S_{11}	6,0	1,507	1,829	0,427	1,291
S_{31} (A)	6,35	1,637	2,982	0,920	0,331
S_{31} (B)	6,40	1,604	2,858	0,511	12,50

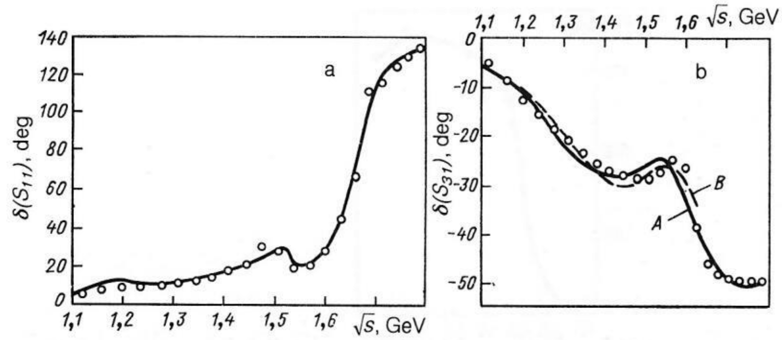

FIG. 7. Results of the calculation of the energy dependence of the S_{11} (a) and S_{31} (b) phase shifts of πN scattering with the parameters of Table XVIII. The phase-shift data (○) are taken from Ref. 129.

on the Hamiltonian of the MIT bag model lead to the following values: $\sqrt{s_1} = 1.965$ GeV, $\gamma_1 = 0.408$ GeV^2 (for $b = 7.0$ GeV^{-1} and $c = 1.35$).

The quantities given in Tables XVIII and XX can also be compared with the results of the P-matrix analyses of πN and KN scattering[43,56] and with the data of Refs. 78 and 79, in which the S-wave amplitudes of these processes were analyzed using the CQB model.[66,67] The locations of the CDD(RA) poles turn out to be in good agreement with the data of these studies on the locations of the P-matrix poles (for similar values of the parameter b). However, the data on the residues differ considerably. For example, calculation of γ_1 in the S_{11} channel of πN scattering using Eqs. (94) and (95) and information about the residue at the P-matrix pole[43] leads to the value 0.19 GeV^2.

CONCLUSION

The equations for the on-shell partial-wave amplitudes formulated in Secs. 2–4 on the basis of the relativistic formalism take into account the mechanisms of hadron exchange in the t and u channels in the peripheral interaction region and the mechanism of production, in the intermediate state, of a compound system like the compound quark–gluon bag in the s channel at distances on the order of the confinement radius. A representation for the S matrix which is compatible with unitarity, analyticity, relativistic invariance, and the requirement of confinement is constructed in such a way that the parameters determining the general solution of the dispersion relations for the partial-wave amplitudes admit interpretation in terms of the characteristics of a quark compound system of the bag type. For fixed values of the parameter b characterizing the dimensions of the region at short distances at which effects of the quark structure of hadrons are important, this solution is determined by specifying the discontinuities of the partial-wave amplitudes on the dynamical cuts corresponding to the contribution of hadron exchange interactions, and by the poles of the R function $\widetilde{P}_l(b,s)$ (28) [or the matrix $\hat{\widetilde{P}}_j(\mathbf{b},s)$ in the multichannel case]—the relativistic analog of the Jaffe–Low P matrix.[54]

The poles of $\widetilde{P}_l(b,s)$ $[\hat{\widetilde{P}}_J(b,x)]$, which we refer to as CDD(RA) poles, are located in the s plane on the real semiaxis, the origin of which is determined by the mass of the most deeply bound state (or by the threshold of the scattering channel in the absence of bound states), and the residues [the diagonal elements of the residue matrix $\hat{P}_J(\mathbf{b},s)$] at these poles are positive constants (causality). The fact that the general solution of the dispersion relations is, in general, determined by an infinite number of parameters [CDD(RA) poles] is essentially a reflection of the fact that the initial fundamental principles do not prevent the system from having an infinite number of "internal" degrees of freedom. However, in a restricted energy range (at low and intermediate energies ~ 1 GeV) we can with good accuracy restrict ourselves to the inclusion of only a small number of CDD(RA) poles. These poles carry information about the interaction at short distances ($r \lesssim b$) and can be associated with the eigenstates of the quark compound system, which allows the behavior of the scattering phase shifts to be related to calculations of the mass spectrum of multiquark systems and the bag-model Hamiltonians to be used for this purpose, with the parameters in them fixed so as to obtain the best description of the hadron states.

We have used the approach developed here to analyze NN, $\pi\pi$, πK, πN, and KN scattering processes at low and intermediate energies. It follows from these examples, first of all, that the locations of the bag levels predicted by the quark Hamiltonians successfully explain the characteristic qualitative regularities in the energy dependence of the phase shifts at intermediate energies (in our opinion, this is most clearly seen by comparing the behavior of the δ_0^0 and δ_0^2 $\pi\pi$-scattering phase shifts). Second, the predictions not only of the locations of the CDD(RA) poles, but also of the resi-

TABLE XIX. Results of the calculation of the S_{11} phase shift ($J = 1/2$, $I = 1$) of KN scattering with the parameter sets given in Table XX. The row labeled "exp." gives the data of the phase-shift analysis of Ref. 130.

$\sqrt{s}$, GeV		1,527	1,673	1,933	2,119	2,247
S_{11}	(A)	−23,75	−39,48	−56,00	−58,46	−41,98
	(B)	−22,63	−40,62	−53,33	−63,69	−35,19
	exp.	−23,23	−38,15	−55,74	−58,21	−41,44

TABLE XX. Parameter sets [b (GeV^{-1}), $\sqrt{s_i}$ (GeV), γ_i (GeV^2), $i = 1,2$] for the variants of the calculation of the S_{11} ($I = 1, J = 1/2$) phase shifts of KN scattering given in Table XIX.

KN	b	$\sqrt{s_1}$	$\sqrt{s_2}$	γ_1	γ_2
S_{11} (A)	7,0	1,852	2,286	0,572	12,713
S_{11} (B)	7,0	1,834	2,528	0,527	3,701

dues at these poles, based on Eqs. (117) and (118), which include the coupling of the bag states to the scattering channel, are, for the lowest states, in satisfactory agreement with the values found from the fit to the data of phase-shift analyses.

In the calculation of the meson exchange interactions in the t and u channels this approach allows the effective use of the methods of relativistic quantum field theory and the inclusion of model-independent information about the renormalized coupling constants of mesons and baryons, and also about the processes determining the discontinuities of the partial-wave scattering amplitudes on the dynamical cuts. The mechanism of formation of a compound system of the quark–gluon bag type in the intermediate state leads to strong suppression of the contribution of dynamical singularities of the scattering amplitude far from the physical region, which removes the main difficulty of the traditional approaches related to the need to calculate meson exchange interactions at short distances. Because of this, in our approach exchanges of heavy mesons (ρ,ω) in NN interactions play a considerably smaller role than in meson theories of nuclear forces. In particular, the short-range repulsion in NN scattering is mainly due to the interaction of the NN channels with bag states, and not to ω-meson exchange, as in meson theories.

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APPENDIX A

The functions $d_l^{(\pm)}(x)$ (12) can be written as

$$d_l^{(\pm)}(x)=\left[1+\sum_{n=1}^{l}(\mp ix)^n a_n^{(l)}\right]\exp(\pm ix), \tag{A1}$$

where

$$a_n^{(l)}=\frac{(2l-n)!}{2^{l-n}n!\,(l-n)!\,(2l-1)!!} \tag{A2}$$

are positive coefficients. In the complex x plane the function $d_l^{(\pm)}(x)$ has the following properties: It is an entire function of x, it has l zeros in the lower half-plane (i.e., all its zeros are located on the second sheet in the z plane, where $z = x^2$), it falls off (grows) exponentially in the upper (lower) half-plane for $|x| \to \infty$ in any direction not parallel to the real axis, and it oscillates for $|x| \to \infty$ along directions parallel to the real axis. The absence of zeros of the function $d_l^{(+)}(x)$ on the real axis in the z plane (on the first sheet) follows directly from the representation (A1) if the positivity of the coefficients $d_n^{(l)}$ (A2) is taken into account. Using the properties of the Riccati–Hankel functions, it can be shown that $d_l^{(+)}(x)$ does not have zeros on the first sheet also in the complex region.

Using the fact that in the z plane $d_l^{(+)}(x)$ has a cut on the real axis ($0 \leqslant z < \infty$), and that it does not have poles and zeros on the first sheet, and also taking into account the normalization condition $d_l^{(+)}(0) = 1$, we can write down the Omnés–Muskhelishvili representation for it (see, for example, Ref. 103):

$$d_l^{(+)}(x)=\exp\left\{\frac{x^2}{\pi}\int_0^\infty\frac{\alpha_l(z')\,dz'}{z'(z'-x^2-i\eta)}\right\}, \tag{A3}$$

where the phase $\alpha_l(z)$ has the form

$$\alpha_l(x^2)=x-\operatorname{arctg}\left\{\frac{x\sum\limits_{m=0}^{N_1}(-1)^m x^{2m}a_{2m+1}^{(l)}}{1+\sum\limits_{m=1}^{N_2}(-1)^m x^{2m}a_{2m}^{(l)}}\right\}; \tag{A4}$$

$$N_1=\begin{cases} l/2-1, & \text{if } l \text{ is even;}\\ (l+1)/2, & \text{if } l \text{ is odd,}\end{cases}\qquad N_2=N_1+(-1)^l.$$

The product $d_l^{(+)}(bk)d_l^{(-)}(bk)$ involved in the definition of the functions $\sigma_l(b,s)$ and $\xi_l(b,\nu)$ [see (18) and (87)] is a polynomial of degree l in the variable $\nu(\nu = k^2)$, which is always positive on the real axis for $\nu \geqslant 0$.

APPENDIX B

The investigation using the nonrelativistic potential model (see Sec. 5) can be generalized to the relativistic case using the quasipotential formalism.[134–136] We shall use a version of this formalism in which the equation for the partial-wave amplitude describing the scattering of spinless particles with equal masses m has the form

$$A_l(k',\,k;\,\nu+i\eta)=W_l(k',\,k)+\frac{2}{\pi}\int_0^\infty W_l(k',\,q)\,(q^2-\nu-i\eta)^{-1}A_l(q,\,k;\,\nu+i\eta)\,\varepsilon(q)\,q^2\,dq, \tag{B1}$$

where $\nu = s/4 - m^2$, $\varepsilon(q) = (m^2 + q^2)^{-1/2}$. The wave functions of the in- and out-states in the momentum representation are given by

$$\langle q\mid\Psi_{lk}^{(\pm)}\rangle=\frac{\pi}{2}\,\frac{1}{k^2}\,\varepsilon^{-1}(k)\,\delta(q-k)+\frac{A_l(q,\,k;\,k^2\pm i\eta)}{q^2-k^2\mp i\eta}. \tag{B2}$$

The kernel of the integral transformation in the coordinate representation

$$\langle r\mid\Psi_{lk}^{(\pm)}\rangle=\frac{2}{\pi}\int_0^\infty\langle r\mid q,\,l\rangle\langle q\mid\Psi_{lk}^{(\pm)}\rangle\,\varepsilon(q)\,q^2\,dq \tag{B3}$$

is chosen in the form[136]

$$\langle r\mid q,\,l\rangle=q^{-1}j_l(rq)\,\varepsilon^{-1/2}(q), \tag{B4}$$

where $j_l(x) = \sqrt{(\pi x/2)}J_{l+1/2}(x)$ are the Riccati–Bessel functions.

From (B1)–(B4) it follows that the functions $\langle r|\Psi_{lk}^{(\pm)}\rangle$ are solutions of the equation

$$\left(\frac{d^2}{dr^2}-\frac{l(l+1)}{r^2}+k^2\right)\langle r\mid\Psi_{lk}^{(\pm)}\rangle=-\int_0^\infty V_l(r,r')\langle r'\mid\Psi_{lk}^{(\pm)}\rangle\,dr' \tag{B5}$$

with the boundary conditions at infinity

$$\langle r\mid\Psi_{lk}^{(\pm)}\rangle\underset{r\to\infty}{\cong}\frac{1}{k\varepsilon^{1/2}(k)}\left\{\sin\left(kr-\frac{\pi l}{2}\right)\right.$$
$$\left.+\rho_l(k^2)A_l(k^2\pm i\eta)\exp\left[\pm i\left(kr-\frac{\pi l}{2}\right)\right]\right\}, \tag{B6}$$

where

$$\rho_l(k^2)=k^{2l+1}\varepsilon(k);\ A_l(k^2)=k^{-2l}A_l(k,k;k^2), \tag{B7}$$

and $V_l(r,r')$ in (B5) and $W_l(k,k')$ in (B1) are related by a transformation of the form (B3). The solution of (B1) can be written as

$$A_l(k',k;\nu+i\eta)=N_l(k',k;\nu)D_l^{-1}(\nu+i\eta), \tag{B8}$$

where $N(k',k;\nu)$ and $D_l(\nu)$ are determined by Fredholm series of the form (90) and (91), in which $\xi_l(b,q^2)$ must be replaced by $\rho_l(q^2)$ (B7) and we set

$$Q_n\begin{pmatrix}q_1\cdots q_n\\ q_1\cdots q_n\end{pmatrix}=\det_{(i,j)}\|W_l(q_i,q_j)\|. \tag{B9}$$

The functions (B6) can be written as

$$\langle r\mid\Psi_{lk}^{(\pm)}\rangle=\frac{1}{D_l(k^2\pm i\eta)}\frac{k^l\langle r\mid\Phi_{lk}\rangle}{\varepsilon^{1/2}(k)(2l+1)!!}, \tag{B10}$$

where

$$\langle r\mid\Phi_{lk}\rangle=\frac{(2l+1)!!}{2ik^{2l+1}}\left[D_l(k^2-i\eta)\langle r\mid h_{lk}^{(+)}\rangle-D_l(k^2+i\eta)\langle r\mid h_{lk}^{(-)}\rangle\right], \tag{B11}$$

and $\langle r|h_{lk}^{\pm}\rangle$ are the solutions of Eq. (B5) with boundary conditions of the form (69).

We construct the quasipotential $W_l(k',k)$ in (B1) in such a way that it leads to the required analytic structure of the scattering amplitude $A_l(k^2)$ (B7) on the energy shell (see Sec. 2). We write $W_l(k',k)$ in the form

$$W_l(k',k)=\frac{m}{k'k}\int_0^\infty j_l(k'r)j_l(kr)v(r)\,dr, \tag{B12}$$

where the auxiliary local potential $v(r)$ is chosen in the form of a superposition of Yukawa potentials. The quasipotential $V_l(r,r')$ in (B5) is related to $v(r)$ as

$$V_l(r,r')=\int_0^\infty\tilde\delta_l(r,r'')\tilde\delta_l(r',r'')v(r)\,dr, \tag{B13}$$

where

$$\tilde\delta_l(r,r')=\frac{2\sqrt{m}}{\pi}\int_0^\infty\frac{j_l(rq)j_l(r'q)}{(q^2+m^2)^{1/4}}\,dq. \tag{B14}$$

The functions $\tilde\delta_l(r,r')$ determine the effects of the relativistic "smearing" of the interaction in configuration space. In the nonrelativistic limit $\lim_{m\to\infty}\tilde\delta_l(r,r')=\delta(r-r')$. Analysis of the Fredholm series for $N_l(k',k;\nu)$ and $D_l(\nu)$ (B8) leads to the conclusion that $A_l(\nu)$ (B7) has the required analytic structure on the physical sheet.

It can be shown that for the "analytic" quasipotential (B13) the logarithmic derivative $P_l(b,\nu)$ of the wave function $\langle r|\Phi_{lk}\rangle$ (B11) at $r=b$ is a meromorphic function in the $\nu=k^2$ plane with a cut on the real axis for $-\infty<\nu\leqslant-m^2$. This cut is a consequence of the relativistic effect of "smearing" of the interaction (B13).

We define the functions $\tilde A_l(b,\nu)$ and $\bar A_l(b,\nu)$ by relations of the form (5)–(9), in which

$$D_l^{(\pm)}(b,s)\to D_l^{(\pm)}(b,\nu)=\frac{(bk)^l\langle b\mid h_{lk}^{(\pm)}\rangle}{(2l-1)!!}. \tag{B15}$$

Here it can be shown that

$$\tilde A_l^{-1}(b,\nu+i\eta)=\alpha_l(b,\nu)\left[P_l(b,\nu)-f_l^{(+)}(b,\nu)\right], \tag{B16}$$

where

$$\alpha_l(b,\nu)=\frac{2i\tilde\rho_l(b,\nu)}{f_l^{(+)}(b,\nu)-f_l^{(-)}(b,\nu)}; \tag{B17}$$

$$\tilde\rho_l(b,\nu)=\rho_l(\nu)\left[D_l^{(+)}(b,\nu)D_l^{(-)}(b,\nu)\right]^{-1}; \tag{B18}$$

$f_l^{(\pm)}(b,\nu)$ are the logarithmic derivatives of the functions $\langle r|h_{lk}^{(\pm)}\rangle$ at $r=b$. The discontinuity of $\tilde A_l^{-1}(b,\nu)$ on the right-hand cut C_R $(0\leqslant\nu<\infty)$ is given by the expression

$$\operatorname{disc}\tilde A_l^{-1}(b,\nu)=-\tilde\rho_l(b,\nu)\,\theta(\nu). \tag{B19}$$

Therefore, the function $\tilde P_l(b,\nu)$, determined by the equation

$$\tilde P_l(b,\nu)=\tilde A_l^{-1}(b,\nu+i\eta)+J_l(b,\nu+i\eta), \tag{B20}$$

where

$$J_l(b,\nu+i\eta)=\frac{\nu-\nu_0}{\pi}\int_0^\infty\frac{\tilde\rho_l(b,\nu')\,d\nu'}{(\nu'-\nu-i\eta)(\nu'-\nu_0)}, \tag{B21}$$

does not have a right-hand cut. From (B16) and (B20) it follows that in the physical region $\tilde P_l(b,\nu)$ and $P_l(b,\nu)$ are related as

$$\tilde P_l(b,\nu)=\alpha_l(b,\nu)\left[P_l(b,\nu)-\operatorname{Re}f_l^{(+)}(b,\nu)\right]-\operatorname{Re}J_l(b,\nu). \tag{B22}$$

For the "analytic" quasipotential (B13), $\tilde P_l(b,\nu)$, like $P_l(b,\nu)$, is a meromorphic function in the ν plane with a cut for $-\infty<\nu\leqslant-m^2$.

The quantity $\alpha_l(b,\nu)$ in (B22) can be written as

$$\alpha_l(b,\nu)=\frac{[(2l-1)!!]^2\varepsilon(k)}{b^{2l}[1+Q_l(b,\nu)]}, \tag{B23}$$

where

$$Q_l(b,\nu)=\frac{1}{2ik}\int_0^\infty dr\int_0^\infty dr'V_l(r,r')\langle r\mid h_{lk}^{(-)}\rangle\langle r'\mid h_{lk}^{(+)}\rangle$$
$$\times[\theta(b-r)\theta(r'-b)-\theta(b-r')\theta(r-b)]. \tag{B24}$$

Therefore, the function $Q_l(b,\nu)$ is nonzero, owing to the above-mentioned effects of relativistic "smearing" of the interaction in the vicinity of the point $r=b$. If we neglect these effects, $\alpha_l(b,\nu)$ is a purely kinematic function and is defined by (95). Finally, we note that the cut of $\tilde P_l(b,\nu)$ in the region $-\infty<\nu\leqslant-m^2$ can be estimated by redefinition of the functions $D_l^{(\pm)}(b,\nu)$ (B15).

APPENDIX C

The discontinuities of the partial-wave NN scattering amplitudes $A_J^{(\alpha)}(s)$ $(\alpha=0,1,11,22,12)$ on the dynamical cuts are related to the spectral functions $\rho_n^{(\pm)}(s,t)$ in (109) as

$$\mathrm{Im}\, A_J^{(\alpha)}(s)=\frac{\theta(s_{2\pi}-s)}{s-s_R}\sum_{\alpha'}\beta_J(\alpha,\alpha')F_J^{(\alpha')}(s), \qquad \text{(C1)}$$

where $s_R = 4m_N^2$, $s_2 = 4(m_N^2 - m_\pi^2)$, the matrix $\beta_J(\alpha,\alpha')$ has the nonzero matrix elements

$$\begin{aligned}
&\beta_J(0,\ 0)=\beta_J(1,\ 1)=1,\\
&\beta_J(11,22)=\beta_J(22,11)=\frac{J+1}{2J+1},\\
&\beta_J(11,12)=\beta_J(12,11)=-\beta_J(22,12)=-\beta_J(12,22)=\frac{\sqrt{J(J+1)}}{2J+1},\\
&\beta_J(12,12)=\frac{4}{2J+1},
\end{aligned} \qquad \text{(C2)}$$

and the functions $F_J^{(\alpha)}(s)$ have the form

$$F_J^{(\alpha)}(s)=\sum_{n=1}^{4}\frac{1}{16\pi}\int_{t_1}^{t_2(s)}D_n^{(I)}(s,\ t)\,R_{Jn}^{(\alpha)}(s,\ t)\,dt, \qquad \text{(C3)}$$

in which

$$\begin{aligned}
&t_1=4m_\pi^2;\quad t_2(s)=s_R-s;\\
&D_n^{(I)}(s,\ t)=3\Delta_n^{(+)}(s,\ t)+2\lambda^{(I)}\Delta_n^{(-)}(s,\ t);\\
&\Delta_n^{(\pm)}(s,\ t)=\rho_n^{(\pm)}(s,\ t)\mp(-1)^n\rho_n^{(\pm)}(s,\ t);
\end{aligned} \qquad \text{(C4)}$$

$\lambda^{(I)}$ is the isospin factor ($\lambda^{(0)} = 1, \lambda^{(1)} = -3$), $R_{Jn}^{(\alpha)}(s,t)$ are kinematic functions given by the equations

$$\left.\begin{aligned}
&R_{Jn}^{(0)}(s,\ t)=f_n^{(1)}(s,\ t)\,h_J^{(+)}(s,\ t);\\
&R_{Jn}^{(1)}(s,\ t)=f_n^{(3)}(s,\ t)\,h_J^{(-)}(s,\ t)+f_n^{(4)}(s,\ t)\,h_J^{(+)}(s,\ t);\\
&R_{Jn}^{(11)}(s,\ t)=f_n^{(2)}(s,\ t)\,h_J^{(+)}(s,\ t);\\
&R_{Jn}^{(22)}(s,\ t)=f_n^{(3)}(s,\ t)\,h_J^{(+)}(s,\ t)+f_n^{(4)}(s,\ t)\,h_J^{(-)}(s,\ t);\\
&R_{Jn}^{(12)}(s,\ t)=f_n^{(5)}(s,\ t)\,h_J^{(0)}(s,\ t),
\end{aligned}\right\} \qquad \text{(C5)}$$

where $h_J^{(+)}(s,t) = P_J[z(s,t)]$,

$$\left.\begin{aligned}
&h_J^{(-)}(s,\ t)=\frac{1}{2J+1}\{JP_{J+1}[z(s,\ t)]+(J+1)P_{J-1}[z(s,\ t)]\};\\
&h_J^{(0)}(s,\ t)=\frac{1}{4}\frac{\sqrt{J(J+1)}}{2J+1}\{P_{J+1}[z(s,\ t)]-P_{J-1}[z(s,\ t)]\};
\end{aligned}\right\} \qquad \text{(C6)}$$

$P_J(z)$ are the Legendre polynomials, $z(s,t) = 1 + t/(s - s_R)$, and

$$\left.\begin{aligned}
&f_1^{(m)}(s,\ t)=\delta_{1m}(s_R-t)+\delta_{2m}\left(s_R+\frac{s+s_R}{s-s_R}t\right)\\
&-\frac{1}{2}\delta_{3m}(s-s_R)+\frac{1}{2}\delta_{4m}(s+s_R)-\delta_{5m}\sqrt{ss_R};\\
&f_2^{(m)}(s,\ t)=\sqrt{s_R}\left\{\delta_{1m}(2s+t-s_R)+\delta_{2m}\left(2s-s_R+\frac{3s-s_R}{s-s_R}t\right)\right.\\
&\left.+\frac{1}{2}\delta_{3m}(s-s_R)+\frac{1}{4}\delta_{4m}(3s-s_R)\right\}-\delta_{5m}s\sqrt{s};\\
&f_3^{(m)}(s,t)=\frac{1}{4}(2s-s_R)^2\left\{(\delta_{1m}+\delta_{2m})\left(1+\frac{t}{s-s_R}\right)+\frac{1}{2}(\delta_{3m}+\delta_4)\right\}\\
&-\frac{1}{4}(\delta_{1m}-\delta_{2m})\frac{ss_R}{s-s_R}t-\frac{1}{8}(\delta_{3m}-\delta_{4m})ss_R-\frac{1}{4}\delta_{5m}\sqrt{ss_R};\\
&f_4^{(m)}(s,\ t)=\delta_{1m}(2s-s_R)+2\delta_{2m}\left(s-s_R+\frac{ts_R}{s-s_R}\right)\\
&+\delta_{3m}(s-s_R)+\delta_{4m}s-\delta_{5m}\sqrt{ss_R};\ \delta_{mn}=1\ (0)\ \text{for}\ \ m=n\ (m\neq n).
\end{aligned}\right\} \qquad \text{(C7)}$$

[1)] The quantity b is usually referred to as the radius of the equivalent hadron bag.

[2)] The presence of this ambiguity was first illustrated in the solution of the Low equation.[108]

[3)] In the usual N/D equations the spectral function, which is the analog of the function $\sigma_l(b,s)$ in Eq. (22), grows for $s\to\infty$ as $s^{l+1/2}$; because of this, the equations in the case $l>0$ do not, strictly speaking, have solutions if the discontinuity of the partial-wave amplitude on the left-hand cut does not satisfy certain special requirements depending on the interaction dynamics. This difficulty is usually overcome by introducing a cutoff in the spectral function at large s.

[4)] For simplicity, the particle masses m_λ in all the two-particle channels are assumed to be equal.

[5)] Methods of including also three-particle channels in the dispersion equations have been discussed in the literature (see, for example, Refs. 118 and 119). However, these methods usually reduce (owing to their complexity) to some modification of the two-particle approaches.

[6)] The methods used to parametrize the contribution of inelastic channels in earlier phase-shift analyses of NN scattering either do not satisfy the generality criterion, or they spoil the unitarity condition if no special constraints are imposed on the range of variation of certain parameters as functions of the other parameters.[123]

[7)] Here, of course, all the equations of Sec. 2 are considered in the nonrelativistic limit.

[8)] It should be noted that this region is unphysical for the process $N\bar{N}\to 2\pi$. Therefore, the relation between the discontinuities of the NN amplitudes on C_L and the πN scattering amplitudes is found by using either the so-called extended unitarity relation in the t channel or the usual unitarity condition (i.e., in the region $t\geqslant 4m_N^2$), and then analytically continuing to the region $4m_\pi^2\leqslant t<4m_N^2$; the two procedures are equivalent.

[9)] In theoretical studies a σ meson of width $\sim$300 MeV and mass 500–600 MeV is sometimes introduced to approximately take into account the two-pion exchange mechanism in the state with $I=J=0$.

[10)] A detailed description of calculations of the NN phase shifts in this approach, taking into account meson exchange interactions in the one-boson exchange model, can be found in Refs. 94 and 98.

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