Vacuum structure in gauge theories and colorless variables

A. N. Tavkhelidze and V. F. Tokarev

Institute of Nuclear Research, USSR Academy of Sciences, Moscow

Fiz. Elem. Chastits At. Yadra **21**, 1126–1186 (September–October 1990)

Gauge models in space–time with $d = 2, 3, 4$ in which the vacuum has a complicated structure are considered. It is shown that allowance for the vacuum structure leads to a description of the models in colorless variables. The semiclassical method is used to find effective Lagrangians that are expressed in these variables and take into account the vacuum structure explicitly.

INTRODUCTION

It is well known that in some gauge theories the vacuum has a periodic structure.[1,2]

What this means is that the energy functional has not only the trivial minimum $A = 0$ but also an infinite countable number of minima

$$\mathbf{A}_1 = \frac{1}{i} U^{-1} \partial U(x), \quad \ldots, \quad \mathbf{A}_{(n)} = U^{-n} \partial U^n(x), \tag{1}$$

where $U(x)$ is an element of the corresponding gauge group.

It can be shown that this occurs in any gauge theory. Indeed, from the minimum $A = 0$ it is possible, by means of a gauge transformation that does not change the energy, to obtain an infinite set of minima $\mathbf{A}_\alpha \neq 0$. The operator that implements the gauge transformation with parameter $\alpha(x)$ has the form

$$V = \exp \{ i \int \alpha(x) L(x) dx \},$$

$$L = \partial E - \rho(x), \quad \alpha(\infty) = 0.$$

However, in gauge theories there are constraints that must be solved if one is to obtain the physical Hamiltonian and energy. The constraint equation has the form $L = 0$ (Gauss law).

But in this case we can no longer make gauge transformations, since in the physical space the generator is $L \equiv 0$. We are therefore left with a single minimum, for example, $\mathbf{A} = 0$.

The existence of the other minima [see Eq. (1)] can be understood if it is assumed that to the element $U(x)$ there corresponds a parameter $\alpha(x)$ that does not decrease at infinity: $\alpha(\infty) \neq 0$. In this case, the condition $L = 0$ does not forbid the existence of an operator of a gauge transformation T that implements a transformation $U(x)$ with parameter $\alpha(x)$ which does not decrease at infinity. Therefore, in some theories there may exist a well-defined gauge-transformation operator T and, hence, an infinite number of minima (1).

Then the vacuum state is uniformly dispersed near the minima (1). But in such a case charged states cannot occur in the theory.

Indeed, the Hamiltonian of a charged particle is a function of the variable $\mathbf{p} - e\mathbf{A}$. For example, for a nonrelativistic particle $H = (\mathbf{p} - e\mathbf{A})^2/2m$. Then a state with definite momentum p cannot be an eigenstate of the Hamiltonian $H(\mathbf{p} - e\mathbf{A})$, since $e\mathbf{A}$ does not have a definite value [superposition of the quantities (1)]. This assertion can be formulated in a different way by means of the operator T.

Since the Hamiltonian is gauge-invariant,

$$THT^+ = H,$$

and the operator T is unitary, $T^+T = 1$, it commutes with the Hamiltonian:

$$[T, H] = 0.$$

This means that the operators H and T can be diagonalized simultaneously. We denote by $\exp(i\theta)$ the eigenvalue of the unitary operator T. Then for the ground state of the system we have

$$T \mid \mathrm{vac} \rangle = e^{i\theta} \mid \mathrm{vac} \rangle. \tag{2}$$

We shall show that charged excitations cannot exist over such a vacuum.

Let $\Phi(x)$ be a charged field that is transformed under the action of the gauge transformation in accordance with the law

$$\begin{aligned} T^+ \Phi(x) T &= U(x) \Phi(x), \\ T^+ \Phi^+(x) T &= \Phi^+(x) U^+(x). \end{aligned} \tag{3}$$

For the two-point correlation function of this field, we obtain from Eqs. (2) and (3) the chain of identities

$$\begin{aligned} &\langle \mathrm{vac} \mid \Phi^+(y) \Phi(x) \mid \mathrm{vac} \rangle \\ &= \langle \mathrm{vac} \mid T\, {}^+\Phi^+(y)\, TT^+ \Phi(x)\, T \mid \mathrm{vac} \rangle \\ &= \langle \mathrm{vac} \mid \Phi^+(y) \Phi(x) \mid \mathrm{vac} \rangle\, U^+(y)\, U(x). \end{aligned} \tag{4}$$

It follows that this correlation function is equal to $\mathrm{const} \cdot \delta(x - y)$ or is undefined. In either case, a particle described by the field Φ cannot move. This is the case because the momentum of the particle over the vacuum (2) does not have a definite value.

Indeed, the invariance of the vacuum (2) with respect to a shift by the operator T means that the vacuum is uniformly distributed over all the minima $\mathbf{A}_n$ (1). If near the minimum $\mathbf{A} = 0$ the particle is described by a wave function with definite momentum $\mathbf{p}$, $\psi_0(x) \sim \exp(i\mathbf{p} \cdot \mathbf{x})$, then near the minimum $\mathbf{A}_n$ it will be described by the wave function $\psi_n = U^n(x) \psi_0$. This state also has a definite momentum, since the phase associated with the momentum is separated from the phase of $U^n(x)$. But for a superposition of the states $\psi_n(x)$ such separation is no longer possible, and the phase of the momentum is lost on the background of the phase of $U^n(x)$, which fluctuates from minimum to minimum. But this means that the momentum is indefinite.

There are two possible ways to resolve the difficulty associated with the relation (4). The first is confinement. In this case, only a gauge-invariant bound state of Φ and Φ^+ is

 0090-4759/90/050480-25$04.00

observed in the spectrum. The second is color rearrangement. In this case, the charged variable Φ is transformed into a neutral, gauge-invariant variable, which no longer "feels" the phase of U^n and can therefore move.

The dynamics determines which of these two possibilities is realized in reality.

Another aspect of the vacuum structure is the chiral properties of fermions. As a rule, in theories with a complicated vacuum structure the fermions have an anomaly in the chiral current: $\partial_\mu j^5_\mu = 2\partial_\mu K_\mu$, where K_μ is the topological current, and the topological charge $\int K_0 dx$ changes by unity on the passage from minimum to minimum. This means that the chiral charge changes at the same time by ± 2. Therefore, the chiral charge over the θ vacuum in indefinite, and this may be manifested in the appearance of a quark condensate and a mass of physical variables coupled to fermions.

In Sec. 1 we consider the exactly solvable Schwinger model, in which there is confinement of the charged fermions. In this example, one can understand how the operator T of the gauge transformation is constructed, find a Lagrangian of the model expressed in terms of the physical colorless variables, establish operator identities that relate the physical variables to the original variables, and trace the connection between the chiral anomaly and the acquisition of mass by the physical variables.

In Sec. 2 we consider two-dimensional scalar electrodynamics, in which the alternative to confinement—color rearrangement—is realized. The semiclassical method is used to take into account the complicated vacuum structure and find an effective low-energy Lagrangian expressed in terms of physical variables. Approximate operator identities that relate the physical variable to the original variables are established.

If the properties of the Schwinger model make it resemble what is expected in QCD, those of scalar electrodynamics make it similar to the Weinberg–Salam model.

In Sec. 3, we consider a unification of these two models. The resulting model has the properties of both the Schwinger model and scalar electrodynamics. On the one hand, there is color rearrangement and the effective Lagrangian contains the field of a colorless fermion; on the other hand, the presence of a chiral anomaly leads to acquisition by this fermion of a dynamical mass.

In Sec. 4, we consider the three-dimensional Georgi–Glashow model, in which, on the one hand, there is confinement and the effective Lagrangian can be expressed in terms of colorless variables, while, on the other hand, the chiral symmetry is spontaneously broken, and the nonlinear σ model is realized.

In Sec. 5, we consider quantum chromodynamics with one and two light quarks. The Lagrangian of this model can be expressed in terms of colorless collective variables, the effective quark having a nonzero chromomagnetic moment, while the chiral anomaly is manifested by the acquisition by this quark of a dynamical mass.

Except for the Schwinger model, none of the models are exactly solvable. In such a case, a convenient tool for studying the vacuum and excitations over it is the method of functional integration (path integrals).

We recall the basic features of this method for the example of a quantum-mechanical particle in a periodic potential. The Hamiltonian is

$$H = p^2/2 + U(x),$$

where $U(x)$ is a periodic function with period a: $U(X + a) = U(x)$. Suppose that the potential is non-negative, $U \geqslant 0$, and that its minima are at the points $x_n = na$, $U(x_n) = 0$.

In this model, the particle momentum is not a quantum number, since the operator p does not commute with the Hamiltonian. However, the Hamiltonian does commute with the unitary translation operator $T = \exp(ipa)$, and therefore the two operators can be diagonalized simultaneously:

$$\left.\begin{aligned} &T\,|n,\ \theta\rangle = e^{-i\theta}\,|n,\ \theta\rangle, \quad 0 \leqslant \theta < 2\pi, \\ &H\,|n,\ \theta\rangle = \varepsilon_n(\theta)\,|n,\ \theta\rangle, \\ &\sum_n \int \frac{d\theta}{2\pi}\,|n,\ \theta\rangle\langle\theta,\ n| = 1, \end{aligned}\right\} \tag{5}$$

where the angle θ parametrizes the eigenstates of the unitary operator T, and $\varepsilon_n(\theta)$ is the spectrum of the Hamiltonian. The state with the lowest energy is called the θ vacuum. Consider

$$Z = \sum_{q=-\infty}^{\infty} \langle x_q|e^{-\hat{H}\tau}|x_0\rangle\, e^{iq\theta}. \tag{6}$$

Using the property of the translation operator,

$$\langle\, x_0 \mid T^q = \langle x_q \mid = \langle qa \mid,$$

and also the relations (5), we obtain

$$Z = \sum_q \sum_n \frac{d\alpha}{2\pi}\, e^{-iq(\alpha-\theta)} e^{-\varepsilon_n(\alpha)\tau} |\langle x_0\,|n,\ \alpha\rangle|^2$$

$$= \sum_n e^{-\varepsilon_n(\theta)\tau} |\langle x_0\,|n,\ \theta\rangle|^2.$$

Thus, Z is directly related to the spectrum of the model constructed over the θ vacuum. In the limit $\tau\to\infty$, only the vacuum contribution survives in the sum:

$$Z \to e^{-\varepsilon_{\text{vac}}(\theta)\tau}\,|\langle 0\,|\text{vac}\rangle|^2. \tag{7}$$

It is convenient to represent Z as a functional integral. The amplitude of tunneling from the $x_0 = 0$ minimum to the x_q minimum has a representation as an integral over the paths $x(t)$ that join these minima:

$$\langle x_q\,|e^{-H\tau}|x_0\rangle = \int_{x_0}^{x_q} e^{-S(x)} D(x);$$

$$S = \int_{-\tau/2}^{\tau/2} dt\left(\frac{\dot{x}^2}{2} + U(x)\right).$$

By definition,

$$q = \frac{1}{a}\int_{-\tau/2}^{\tau/2} \dot{x}\,dt,$$

from which, taking into account (6), we obtain representations for Z:

$$Z = \sum_q \int_0^{qa} \exp\left[-S(x) + i\theta q(x)\right] D(x). \tag{8}$$

This integral can be estimated by the semiclassical method. For this, it is necessary to find the path that has the least action: $\delta S = 0$.

We denote it by $\bar{x}$, and its action by $\bar{S}$. Taking into account the paths near the extremal, $x = \bar{x} + \delta x$, we obtain their contribution to Z:

$$e^{-\bar{S}} \det{}^{-1/2}\left(\frac{\delta^2 \bar{S}}{\delta \bar{x}\, \delta \bar{x}}\right) e^{-iq\theta}. \tag{9}$$

The solution with $q = 0$ has the form $\bar{x} = 0, \bar{S} = 0$. We denote its contribution to (9) by $Z_{\rm p,th} = \det^{-1/2}[-\partial_t^2 + U''(0)]$.

The solution $\bar{x}_1(t)$, $[-\bar{x}_1(t)]$ joining the neighboring minima of the potential has $q = +1(-1)$ and $\bar{S}_1 \neq 0$. Its contribution to Z is given by the corresponding formula (9).

Note that if $\bar{x}_1(t)$ solves the equations of motion $\delta S = 0$, then the function $\bar{x}_1(t+\varepsilon)$ is also a solution of the equations of motion. This follows from the translational invariance. Therefore, these functions must also be summed in Z. In the expression (9), this is manifested by the fact that the determinant contains a zero mode. Indeed, using the invariance of the equations of motion $\delta S/\delta \bar{x} = 0$ with respect to time translations $\bar{x}_1(t+\varepsilon) \simeq \bar{x}_1(t) + \varepsilon \bar{x}_1'(t)$, we obtain

$$\int \frac{\delta^2 S}{\delta \bar{x}_1(t)\, \delta \bar{x}_1(t')} \bar{x}_1'(t')\, dt' = 0,$$

and this signifies the existence of a zero mode. It must be taken into account by means of the method of collective coordinates.[3,4]

As a result, the single-instanton contributions can be represented in the form

$$Z_{\pm 1} = Z_{\rm p.th} e^{\pm i\theta}\, k \int dt, \tag{10}$$

where $\int dt = \tau$ is the contribution of the zero mode, while the constant k takes into account the contributions of the other modes:

$$k = \det{}'^{-\frac{1}{2}}\left(\frac{\delta^2 S}{\delta \bar{x}_1\, \delta \bar{x}_1}\right) \frac{\sqrt{N_0}}{Z_{\rm p.th}} e^{-\bar{S}_1}, \tag{11}$$

where $N_0 = \int \dot{\bar{x}}_1^2 dt$ is the normalization of the zero mode, and det′ means that the zero mode is not taken into account in the determinant.

To take into account configurations with $q \neq 0$, 1, we use the approximation of a rarefied instanton gas. For this, we consider a composition of N^+ instantons and N^- anti-instantons. Such a configuration carries the topological number $q = N^+ - N^-$. Summing over N^+ and N^-, we take into account all possible topological configurations. We obtain

$$Z = Z_{\rm p.th} \sum_{N^+,\, N^-} \frac{k^{N^+ + N^-}}{N^+!\, N^-!} \exp\left[-i\theta (N^- - N^+)\right] \tau^{N^+ + N^-}$$
$$= Z_{\rm p.th} \exp\left[2k \cos(\theta)\, \tau\right]. \tag{12}$$

Comparing this with (7), we obtain for the vacuum energy the expression

$$\varepsilon_{\rm vac}(\theta) = \varepsilon_{\rm p.th} - 2k \cos\theta.$$

In contrast to perturbation theory, the spectrum of the model is continuous. For despite the potential barriers, the particle can move by below-barrier penetration. Since the tunneling probability is low, the effective mass of the particle will be large. Indeed, at small θ we have the approximate equations

$$\varepsilon(\theta) \approx \varepsilon_{\rm p.th} + k\theta^2,\ p \simeq -\theta/a,$$

from which we obtain $m_{\rm tun} \simeq 1/(2ka^2)$.

1. SCHWINGER MODEL

A. Operator solution

The Lagrangian of the Schwinger model has the form[5]

$$\mathcal{L} = -\frac{1}{4} F_{\mu\nu}^2 + i\bar{\psi}\gamma_\mu (\partial_\mu - ieA_\mu)\psi, \tag{13}$$

where A_μ and ψ are the electromagnetic and fermion fields, $F_{\mu\nu} = \partial_\mu A_\mu - \partial_\nu A_\mu$, $\mu = 0, 1$, and the two-dimensional γ matrices are taken in the form

$$\gamma^0 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix},\quad \gamma^1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix},\quad \gamma^5 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

The equations of motion that follow from the Lagrangian (1) are

$$\partial^\mu F_{\mu\nu} = -eJ_\nu, \tag{14}$$

$$i\gamma^\mu(\partial_\mu - ieA_\mu)\psi = 0. \tag{15}$$

The current J_μ is defined gauge-invariantly:[5]

$$J_\mu(x) = \frac{1}{2} \lim_{\varepsilon \to 0,\, \varepsilon^2 \neq 0} \{J_\mu(x|\varepsilon) + J_\mu(x|-\varepsilon)\}, \tag{16}$$

$$J_\mu(x|\varepsilon) = \bar{\psi}(x)\gamma_\mu \exp\left(-ie\int_x^{x+\varepsilon} A_\nu d\xi^\nu\right)\psi(x+\varepsilon).$$

In this paper, we use the transverse gauge

$$\partial_\mu A^\mu = 0. \tag{17}$$

In this gauge, the fields φ and A_μ satisfy the equal-time commutation relations

$$\{\psi_i(x_1, t), \psi_j^+(y_1, t)\} = +\delta_{ij}\delta(x_1 - y_1),\ i, j = 1,2;$$
$$[\partial_0 A_1(x_1, t), A_1(y_1, t)] = -i\delta(x_1 - y_1);$$
$$[\partial_0 A_0(x_1, t), \partial_0 A_1(y_1, t)] = i\frac{\partial}{\partial x^1}\delta(x^1 - y^1),$$

while the remaining commutators vanish.

The Schwinger model in the transverse gauge has the operator solution[6]

$$A_\mu = \frac{\sqrt{\pi}}{e} \varepsilon_{\mu\nu}\partial^\nu(\Sigma + \eta), \tag{18}$$
$$\psi(x) = K : \exp\{-i\sqrt{\pi}\gamma_5(\Sigma(x) + \eta(x))\} : \psi_0(x),$$

$\varepsilon_{01} = \varepsilon_{10} = 1$, where Σ is a free pseudoscalar field with $m = e/\sqrt{\pi}$, and η is a free massless field quantized with a negative metric,

$$[\eta(y^1, t), \partial_0 \eta(x^1, t)] = -i\delta(x^1 - y^1);$$

ψ_0 is a free massless fermion field (free scalar and fermion fields in two-dimensional space–time are described in detail in the studies of Ref. 9). The negative-frequency part of the commutator function of the field η is

$$D^-(x) = -\frac{i}{4\pi} \ln(-\mu^2 x^2 + i\varepsilon x_0),$$

where μ is an arbitrary parameter with the dimensions of a mass.[9] The coefficient K in (18) is expressed in terms of this parameter by

$$K=(e\gamma/2\mu\sqrt{\pi})^{1/4},$$

where γ is Euler's constant ($\gamma\simeq 1.781$).

Calculation of J_μ in accordance with (16) leads[6,10] to the expression

$$J_\mu=j_\mu+\frac{e}{\pi}A_\mu, \quad (19)$$

where $j_\mu = :\bar{\psi}_0(x)\gamma_\mu\psi_0(x)$, the current of the free fermions, satisfies the equations

$$\partial^\mu j_\mu=0, \quad \varepsilon^{\mu\nu}\partial_\mu j_\nu=0, \quad \partial^2 j_\mu=0. \quad (20)$$

Strictly speaking, the ansatz (18) solves an extended dynamical system in which the Maxwell equation (14) is replaced by

$$\varepsilon^{\nu\alpha}\partial_\alpha(\partial^\mu F_{\mu\nu}+eJ_\nu)=0.$$

This makes it possible to exclude from consideration [see (19) and (20)] the current j_μ of the free fermions.

If we now return to the original system (14), (15), (17), then the ansatz (18) satisfies Eqs. (15) and (17) identically, while Eq. (14) reduces to the operator equation

$$L_\mu = 0, \quad (21)$$

where

$$L_\mu=j_\mu+\frac{1}{\sqrt{\pi}}\varepsilon_{\mu\nu}\partial^\nu\eta. \quad (22)$$

Since the original operators A_μ and ψ can be expressed by (18) in terms of the free operators Σ, η, ψ_0, we have at our disposal the Hilbert space of states

$$\mathscr{H}=\mathscr{H}(\Sigma)\otimes\mathscr{H}(\eta)\otimes\mathscr{H}(\psi_0).$$

Since the fields η and ψ_0 are independent in this entire space, the relation (21) is not satisfied in $\mathscr{H}$, but there must exist a subspace $\mathscr{H}_{\rm ph}\subset\mathscr{H}$ in which the condition (21) is satisfied identically. It is in this subspace that the ansatz (18) solves the equations of motion (14) and (15).

B. Vacuum and observables

We turn to the construction of the physical state space. The operator equation (21) is to be understood as the vanishing in the physical space of all matrix elements of the operator L_μ, i.e., $\langle\Phi|L_\mu|\Phi'\rangle = 0$ for all states $|\Phi\rangle$ and $|\Phi'\rangle$ in the physical space $\mathscr{H}_{\rm ph}$. Thus, in the space $\mathscr{H}$, this equation is a constraint equation that imposes definite restrictions on the state vectors.

These must be imposed[6] in the form

$$L_\mu^-\,|\,\Phi\rangle = 0. \quad (23)$$

Here, L_μ^- is the negative-frequency part of the operator L_μ. For all vectors $|\Phi\rangle$ that satisfy the condition (23), fulfillment of Eq. (21) is guaranteed:

$$\langle\,\Phi\,|\,L_\mu\,|\,\Phi^1\rangle \equiv 0.$$

We define the vacuum in the space $\mathscr{H}$ by the condition

$$\Sigma^-|\,0\,\rangle=\eta^-\,|\,0\,\rangle=\psi_0^-\,|\,0\,\rangle=0.$$

Then $|0\rangle$ is one of the solutions of Eq. (23). The remaining vectors $|\Phi\rangle$ can be obtained from $|0\rangle$ by applying to it operators $\hat{\mathscr{O}}$ that do not carry $|0\rangle$ out of the physical space.

For this, they must commute with L_μ^+:

$$[\hat{\mathscr{O}},\ L_\mu^\pm]=0. \quad (24)$$

One can show that these operators can be constructed from the operators Σ, $\sigma(x)$, Q_F, $\tilde{Q}_5$. Here

$$Q_F = -\int j_0 dx^1;\quad \tilde{Q}_5 = \int j_1\,dx^1 \quad (25)$$

are the operators of the fermion number and of the chirality of the free massless fermions, while $\sigma(x)$ are defined as[7]

$$\sigma_\pm(x)=\left(\frac{2\pi}{\mu}\right)^{1/2}\frac{1\pm\gamma_5}{2}:\exp\{-i\sqrt{\pi}\gamma_5\eta(x) -i\sqrt{\pi}\tilde{\eta}(x)\}:\psi_0(x)\exp\left(i\frac{\pi}{2}\tilde{Q}_5\right). \quad (26)$$

By direct calculation, we can verify[7] the following properties of these operators:

$$\left.\begin{aligned}&[\sigma_\pm(x),\ \sigma_\pm(y)]=[\sigma_\pm(x),\ \sigma_\mp(y)]=0;\\&[\sigma_\pm(x),\ \sigma_\pm^+(y)]=[\sigma_\pm(x),\ \sigma_\mp^+(y)]=0;\\&\sigma_\pm(x)\,\sigma_\pm^+(x)=1.\end{aligned}\right\} \quad (27)$$

The commutation relations of the unitary operators $\sigma_\pm$ with operator charges have the form

$$[\sigma_\pm,\ Q_F] = -\sigma_\pm,\ [\sigma_\pm,\ Q_5] = \mp\,\sigma_\pm, \quad (28)$$

from which it follows that the operators $\sigma_\pm$ carry a fermion number and chirality.

In addition, $\sigma_\pm$ in the physical space $\mathscr{H}_{\rm ph}$ commute with the generators of translations and therefore are constant unitary operators.

Indeed, defining P_μ in accordance with $P_\mu \int T_{\mu 0}dx^1$, $T_{\mu\nu}(x) = T_{\mu\nu}(\Sigma) + T_{\mu\nu}(\psi_0) - T_{\mu\nu}(\eta)$, we obtain

$$[P_\mu\sigma_\pm(x)]=-\pi\,(L_\mu\mp\varepsilon_{\mu\nu}L^\nu(x))\,\sigma_\pm(x).$$

From the condition $L_\mu = 0$ we find that in $\mathscr{H}_{\rm ph}$, $\sigma_\pm$ do not depend on x.

Note that the fermion field $\psi(x)$ (18) does not commute with L_μ and therefore does not belong to the physical space. It is therefore meaningless to consider the fermion propagator in the physical space.

All that has remained from the fermions is the operators $\sigma_\pm$, which in accordance with (28) carry fermion and chiral numbers. But these operators cannot "propagate," since in the physical space they do not depend on x. It is clear that this phenomenon is intimately related to the confinement that holds in the model. Indeed, the only field that can propagate is the neutral (colorless) field $\Sigma(x)$.

Since the field Σ commutes with $\sigma_\pm$, Q_F, $\tilde{Q}_5$, the physical space is the product of the Fock space of the field Σ and the space of the operators $\sigma_\pm$, Q_F, $\tilde{Q}_5$ with the commutation relations (27) and (28):

$$\mathscr{H}_{\rm ph}=\mathscr{H}(\Sigma)\otimes\mathscr{H}(\sigma_\pm,\ Q_F,\ \tilde{Q}_5).$$

Since the Hamiltonian commutes with Q_F and Q_5, there exists a common set of eigenvectors of these operators and the Hamiltonian.

One of them is $|0\rangle$:

$$Q_F\,|\,0\,\rangle=\tilde{Q}_5\,|\,0\,\rangle=0.$$

It follows from the commutation relations (28) that the operators change the fermion and chiral numbers by ± 1. Thus, the basis in the space $\mathscr{H}(\sigma, Q_F, \widetilde{Q}_5)$ has the form

$$| n_+, n_- \rangle = (\sigma_+)^{n_+}(\sigma_-)^{n_-} | 0 \rangle.$$

Here, $n_\pm$ are arbitrary integers. The vectors $|n^+, n^-\rangle$ have a definite fermion number and chirality:

$$\left.\begin{aligned} Q_F\, |n^+, n^-\rangle &= (n^+ + n^-)\, |n^+, n^-\rangle, \\ \widetilde{Q}_5\, |n^+, n^-\rangle &= (n^+ - n^-)\, |n^+, n^-\rangle. \end{aligned}\right\} \tag{29}$$

Since the state $|0,0\rangle$ has zero energy, and the operators $\sigma_\pm$ commute with the Hamiltonian, all the vectors $|n^+, n^-\rangle$ have zero energy. Thus, the ground state of the system is degenerate with respect to the fermion number and chirality.

As the ground state, any linear superposition of the states $|n^+, n^-\rangle$ may be chosen.

In particular, it can be chosen in such a way that the Hamiltonian and the operators $\sigma_\pm$ are diagonalized simultaneously. This is possible because the Hamiltonian commutes with $\sigma_\pm$.

Since the operators $\sigma_\pm$ are unitary, the eigenvalue equation is

$$\sigma_\pm \,|\, \theta_+, \theta_- \rangle = \exp(-i\theta_\pm)\, |\, \theta_+, \theta_- \rangle, \tag{30}$$

$0 \leqslant \theta_\pm < 2\pi$.

This equation is solved by the vector

$$|\, \theta_+, \theta_- \rangle = \sum_{n^+, n^-} \exp(in^+\theta^+ + in^-\theta^-)\, |\, n^+, n^- \rangle,$$

the so-called θ vacuum.[2,6,7]

It follows from the relations (2.17) [*sic*] that the different θ vacua are related by the unitary operators

$$U_F = \exp(i\alpha Q_F), \quad \widetilde{U}_5(\widetilde{\alpha}) = \exp(i\widetilde{\alpha}\widetilde{Q}_5) \tag{31}$$

in accordance with

$$|\theta_+ + \alpha + \widetilde{\alpha},\ \theta_- + \alpha - \widetilde{\alpha}\rangle = U_F(\alpha)\, \widetilde{U}_5(\widetilde{\alpha})\, |\theta_+, \theta_-\rangle. \tag{32}$$

Therefore, all the θ vacua are equivalent, i.e., by means of the unitary operators they can be obtained from a single vacuum, for example, from $|\theta_+, \theta_-\rangle|_{\theta_\pm = 0}$.

It follows from (32) that after the choice of a definite θ vacuum the symmetry (31) associated with the conserved charges Q_F and $\widetilde{Q}_5$ is broken, since the vacuum is not invariant with respect to these transformations.

The breaking of the symmetry is manifested in the fact that the vacuum expectation value of $\sigma_\pm$, which is not invariant with respect to the phase transformations (31),

$$U_F \sigma_\pm U_F^+ = \exp(i\alpha)\, \sigma_\pm; \quad \widetilde{U}_5 \sigma_\pm \widetilde{U}_5^+ = \exp(\pm i\widetilde{\alpha})\, \sigma_\pm, \tag{33}$$

is nonzero:

$$\langle \theta_+, \theta_- \,|\, \sigma_\pm \,|\, \theta_+, \theta_- \rangle = \exp(-i\theta_\pm). \tag{34}$$

Thus, the symmetry is spontaneously broken, but the Goldstone particles corresponding to this symmetry breaking are absent in the physical spectrum. This happens because the conditions of Goldstone's theorem are not satisfied in the given case.[8]

Indeed, if the theorem is to hold, conserved currents must correspond to the transformations (31). These currents are j_μ and $j_\mu^5 \varepsilon_{\mu\nu} j^\nu$ [see (25)], the currents of the free fermions, but they do not commute with L_μ and therefore do not belong to the physical space. It is here that the conditions of Goldstone's theorem fail, since they require the existence of matrix elements of conserved currents.

There is here another interesting circumstance that we should like to emphasize.

The existence of the unitary operators (31) is related to the symmetry of the Lagrangian (13) with respect to phase transformations of the form

$$\psi(x) \to \exp(i\alpha)\, \psi(x), \tag{35}$$

$$\psi(x) \to \exp(i\widetilde{\alpha}\gamma_5)\, \psi(x). \tag{36}$$

Indeed, in the space $\mathscr{H}$ it is precisely the operators (31) that realize these transformations:

$$U_F \psi(x) U_F^+ = e^{i\alpha}\psi(x); \quad U_5 \psi(x) U_5^+ = e^{i\widetilde{\alpha}\gamma_5}\psi(x). \tag{37}$$

In accordance with Noether's theorem, to the group of phase transformations (35) there corresponds a locally conserved current $J_\mu = \bar{\psi}\gamma_\mu\psi$, and to the group of chiral transformations there corresponds the current $J_\mu^5 = \bar{\psi}\gamma_\mu\gamma_5\psi$.

Note that these currents are related:

$$J_\mu^5 = \varepsilon_{\mu\nu} J^\nu. \tag{38}$$

To the current J_μ there corresponds the conserved charge $Q = -\int J_0 dx^1$, which (by analogy with QCD) we call the color. Thus, the transformations (35) can be satisfied by the two charges Q_F (37) and Q.

However, in the physical space there remains only the charge Q_F. Indeed, it follows from (18) and (19) that

$$J_\mu = L_\mu + \frac{1}{\sqrt{\pi}}\, \varepsilon_{\mu\nu}\partial^\nu \Sigma.$$

Then in the physical space, where $L_\mu = 0$, the color operator is a null operator:

$$Q = -\int J_0\, dx^1 = \frac{1}{\sqrt{\pi}} \int \partial_1 \Sigma\, dx^1 = 0.$$

The integral vanishes, since the field Σ is massive and therefore decreases exponentially at large distances.

The essence of this result is that $\mathscr{H}_{\rm ph}$ is a space of colorless states, and this is a necessary property for confinement.

With regard to the chiral current J_μ^5, it is conserved only at the classical level. At the quantum level there is an anomaly:

$$\partial_\mu J_\mu^5 = \frac{e}{2\pi}\, \varepsilon_{\mu\nu} F^{\mu\nu}. \tag{39}$$

This identity is readily verified by means of (19), (20), and (38).

Thus, the Noether charge $Q_5 = -\int J_0^5 dx^1$ is not conserved. Nevertheless, there is the conserved charge $\widetilde{Q}_5$ (25), which satisfies the γ_5 transformations (33) and (37). As we have already said, this symmetry is spontaneously broken over the θ vacuum.

From all the above it follows that the Schwinger model, formulated as a gauge model in the language of gauge (A_μ) and fermion (ψ) fields, is ultimately equivalent to the theory of the free massive pseudoscalar field Σ.

The properties of the Σ boson are intimately related to the chiral properties of the model. From (39) and the operator solution (18) there follows an identity which establishes a direct proportionality between the divergence of the axial current and the field Σ:

$$\partial_\mu J_\mu^5 = -2f_\Sigma m^2 \Sigma(x), \quad f_\Sigma = 1/2\sqrt{\pi}. \tag{40}$$

In its form, this relation is reminiscent of the identity of partial conservation of the axial current[11] that holds in the theory of strong interactions at the level of a hypothesis.

In addition, from the operator solution (18) it is possible to obtain a representation for the composite fermion operators in terms of the operator[12]

$$J(x) = \bar{\psi}(x)\psi(x) = -A{:}\cos(\Sigma/f_\Sigma + \pi - \theta){:},$$
$$J_5(x) = -i\bar{\psi}(x)\gamma_5\psi(x) = A : \sin(\Sigma/f_\Sigma + \pi - \theta): \tag{41}$$

Here, $A = m\gamma/2\pi$, and $\theta = \theta_+ - \theta_-$ is related to the eigenvalue of the operator

$$T = \sigma_-^+ \sigma_+ \tag{42}$$

by

$$T \mid \theta_+, \theta_-\rangle = e^{-i\theta} \mid \theta_+, \theta_-\rangle. \tag{43}$$

It follows from (41) that the vacuum expectation values of the operators J and J_5 are nonzero:

$$\langle J\rangle = A\cos\theta; \ \langle J_5\rangle = A\sin\theta.$$

The sum of the squares of these quantities does not depend on the parameter θ:

$$(\langle J\rangle)^2 + (\langle J_5\rangle)^2 = A^2.$$

This relation reflects the existence of the circle of equally valid vacua whose properties we considered in detail earlier.

Since nothing depends on θ, we can set $\theta = \pi$. In this case, there is the quark condensate

$$\langle\bar{\psi}\psi\rangle = -A, \text{but} \langle\bar{\psi}\gamma_5\psi\rangle = 0.$$

We now consider the expression (41) for $\theta = \pi$ in the limit of large distances, where $\Sigma(x)$ is exponentially small. Expanding (41) in a series in Σ, we have

$$\bar{\psi}(x)\psi(x) \simeq \langle\bar{\psi}\psi\rangle = \Psi(0) f_\Sigma,$$
$$-i\bar{\psi}\gamma_5\psi(x) \simeq \Psi(0)\,\Sigma(x). \tag{44}$$

Here $\Psi(0) = -A/f_\Sigma$.

It can be seen from (44) that the Σ boson can be regarded as a fermion–antifermion state with wave function equal at the origin to $\Psi(0)$.

It follows from comparison of (39) and (40) that the Σ boson can also be regarded as pseudoscalar "gluonium":

$$\frac{e}{4\pi}\varepsilon_{\mu\nu}F^{\mu\nu} = -f_\Sigma m^2 \Sigma(x). \tag{45}$$

C. Gauge transformations

In this part, we trace the connection between the constraint equation $L_\mu = 0$ (22) and the requirement of gauge invariance of the theory. This is also the key to the significance of the operators $\sigma_\pm$ and T.

We consider in $\mathscr{H}$ the gauge transformation $U[\alpha(x)]$:

$$U\psi(x)U^+ = \exp(i\,\alpha(x))\,\psi(x),$$
$$UA_\mu U^+ = A_\mu(x) + \frac{1}{e}\partial_\mu\alpha(x), \tag{46}$$

which leaves the Lorentz condition (17) invariant. It is readily seen that for this we require fulfillment of the condition

$$\partial^2\alpha(x) = 0. \tag{47}$$

All the gauge transformations with gauge function $\alpha(x)$ that satisfies the condition (47) can be classified in accordance with the behavior of the function $\alpha(x)$ at infinity. The functions $\alpha(x)$ can be divided into classes that are each characterized by two numbers:[7]

$$n_\pm[\alpha] = -\frac{1}{2\pi}\int_{-\infty}^{\infty} dx^1\,[\partial_0\alpha \mp \partial_1\alpha]. \tag{48}$$

The meaning of these numbers can be clarified by noting that $\alpha(x)$ can always be represented in the form

$$\alpha(x) = \alpha_-(x_-) + \alpha_+(x_+),$$
$$x_\pm = x_0 \pm x^1.$$

Then $n_\pm[\alpha]$ are determined by the behavior of these functions at infinity:

$$n_\pm[\alpha] = -\frac{1}{\pi}[\alpha_\mp(\infty) - \alpha_\mp(-\infty)].$$

We shall prove that $n_\pm[\alpha]$ must be integers.

From the gauge invariance of the current J_μ (19) and (46) there follows a transformation law for the current j_μ:

$$Uj_\mu U^+ = j_\mu - \frac{1}{\pi}\partial_\mu\alpha,$$

whence

$$UQ_\pm U^+ = Q_\pm - n_\pm[\alpha], \tag{49}$$

where $Q_\pm = (Q_F \pm \tilde{Q}_5)/2$.

It follows from the integer nature of the spectrum of the operators $Q_\pm$ and from the relation (49) that $n_\pm[\alpha]$ are integers.

Thus, at the quantum level there exist only operators $U[\alpha]$ for which $n_\pm[\alpha]$ are integers.

It can also be seen from (49) that the operator $U[\alpha]$ carries charges $Q_\pm$ equal to $n_\pm[\alpha]$, i.e., for $n_\pm \neq 0$ it is constructed from fermion fields.

For $\alpha(x)$ which decrease at infinity ($n_\pm[\alpha] = 0$), the operator $U[\alpha]$ has the form[7]

$$U[\alpha_\pm] = \exp\left\{-i\int dx^1\alpha_\pm(x)\,[L^0(x) \pm L_1(x)]\right\}, \tag{50}$$

where L_μ is given by (22).

The condition (24) on the physical operators $\mathscr{O}$ can be expressed in the form

$$U\mathscr{O}U^+ = \mathscr{O}. \tag{51}$$

This means that the operator $\mathscr{O}$ can be translated from $\mathscr{H}$ to the physical space $\mathscr{H}_{ph}$, if in $\mathscr{H}$ it is invariant with respect to the topologically trivial ($n_\pm[\alpha] = 0$) gauge transformations.

Note that invariance with respect to gauge transformations with nondecreasing $\alpha(x)$ ($n[\alpha] \neq 0$) is not required.

Calculating the commutator

$$\left[\int dx^1\alpha_\pm(x)(L^0+L_1),\ \eta^+(y)\right]=\frac{1}{2\pi}\int\frac{dx^+\alpha_+(x_+)}{x_+-y_+-i\varepsilon},$$

we see that the operator $U[\alpha]$ (50) is defined only if $\alpha_+\to 0$ as $x^+\to\infty$. Therefore, for $n_\pm[\alpha]\neq 0$ it is not defined.

We shall clarify the meaning of the unitary (at one point) operators $\sigma(y)$ (26). It follows from the commutation relations (28) that

$$\left.\begin{aligned}\sigma_\pm(y)Q_F\sigma_\pm^+(y)&=Q_F-1,\\ \sigma_\pm(y)\tilde{Q}_5\sigma_\pm(y)&=\tilde{Q}_5\mp 1.\end{aligned}\right\}\qquad(52)$$

Comparison of this expression with (49) shows that $\sigma_\pm$ are the operators of a gauge transformation. The topological numbers of the operator $\sigma_+(y)$ are $n_+=1$, $n_-=0$, and those of the operator $\sigma_-(y)$ are $n_+=0$, $n_-=1$.

Having at our disposal the operator representation (18) and (26), we can calculate[7] the transformation law

$$\sigma_\pm(y)A_\mu(x)\sigma_\pm^+(y)=A_\mu(x)+\frac{1}{e}\partial_\mu F_\pm(x/y);$$
$$\sigma_\pm(y)\psi(x)\sigma_\pm^+(y)=\exp(iF_\pm(x/y))\psi(x);$$
$$F_+(x/y)=-\pi\theta(x_\mp+y_\mp)$$

and show that this is indeed a gauge transformation with the necessary topological numbers.

Note that the unitary operator T (42) changes only the chirality:

$$T\tilde{Q}_5T^+=\tilde{Q}_5-2.\qquad(53)$$

In conclusion, we should like to note the following:

1. Topologically trivial gauge transformations (50) are not extended to the physical space $\mathscr{H}_{\rm ph}$, where $L_\mu=0$:

$$U[\alpha]\equiv 1.$$

2. The topologically nontrivial gauge transformations $\sigma_\pm$ and T realize a symmetry in the *physical* space.

3. There are the pairs of charges (Q_F, Q) and $(\overline{Q}_5, Q_5)$ (25).

4. The physical space is the space of colorless $(Q\equiv 0)$ variables Σ, $\sigma_\pm$, Q_F, $\tilde{Q}_5$.

5. The anomaly (39) (nonconservation of Q_5) in the theory (13) is taken into account manifestly in the colorless variables as acquisition of mass by the Σ boson [see (40)].

6. The chiral number Q_5 is not defined over the θ vacuum [cf. (43) and (53)]. This leads to the quark condensate (44).

2. TWO-DIMENSIONAL SCALAR ELECTRODYNAMICS

We now consider the scalar version of the Schwinger model. It realizes an alternative to confinement—charge screening (color rearrangement); in addition, in it there is no degeneracy with respect to the parameter θ.

It is traditionally assumed that spontaneous symmetry breaking plays a decisive role in the mechanism of mass acquisition by vector bosons. However, the idea of unbroken gauge symmetry also has its adherents.[13,14]

There are several different approaches to this problem; we develop one based on allowance for the effects of the complicated vacuum structure. For the example of scalar electrodynamics in two-dimensional space–time, we show that allowance for the effects of tunneling between minima that differ by a gauge transformation makes it possible to go over to an effective Lagrangian that contains only neutral fields. Thus, it will be shown that consideration of the θ vacuum and excitations over it leads to a description of the model in terms of colorless variables.

We find the mass spectrum of the particles and show that if the tunneling is small, then it is practically the same as in the Higgs phase.

A. Description of the model

The Lagrangian of scalar electrodynamics in two-dimensional Euclidean space has the form

$$\begin{aligned}\mathscr{L}&=\frac{1}{4}F_{\mu\nu}^2+|D_\mu\varphi|^2+\frac{\lambda}{2}\left(\varphi^+\varphi-\frac{c^2}{2}\right)^2;\\ F_{\mu\nu}&=\partial_\mu A_\nu-\partial_\nu A_\mu,\quad D_\mu=\partial_\mu-ieA_\mu;\\ x&=(x_0,\ x^1),\quad \varphi=(\varphi_1+i\varphi_2)/\sqrt{2}.\end{aligned}\qquad(54)$$

If we make the standard assumption that the system is concentrated near the minimum,

$$\varphi=c/\sqrt{2},\quad A_1=0,\qquad(55)$$

then the field A_μ acquires mass $m_v=ec$, and the scalar field acquires mass $m_s=\sqrt{\lambda c}$.

However, besides the minimum (55) there exists a discrete set of gauge-rotated minima:

$$\begin{aligned}\varphi(x)&=\exp(i\alpha(x^1))\,c/\sqrt{2},\\ A_1(x)&=\frac{1}{e}\partial_1\alpha(x^1),\end{aligned}\qquad(56)$$

which have[3,15] topological number

$$n[A_1]=\frac{e}{2\pi}\int A_1\,dx_1.\qquad(57)$$

Requiring that the field $\varphi(x)$ be single-valued at spatial infinity, we obtain a restriction on the function $\alpha(x_1)$:

$$\alpha(\infty)-\alpha(-\infty)=2\pi m,$$

where m is an integer.

From this expression and from (56) and (57), we find that the number $n[A_1]$ is an integer, which is the number of the minimum.

We denote by T the operator of the gauge transformation that realizes a "translation" from the minimum (55) with number $n[A]=0$ to the minimum (56) with number $n[A]=1$.

Repeating the arguments made earlier for the periodic potential, we arrive at the construction of the θ vacuum.

To write down a functional integral [the analog of (8)], we must describe the Euclidean configurations A_μ that connect different minima.

It can be seen from (57) that the number $n[A]$ can be regarded as a charge constructed from the zeroth component of the current:

$$K_\mu=\frac{e}{2\pi}\varepsilon_{\mu\nu}A_\nu.\qquad(58)$$

The change in the number $n[A]$ on the transition from $n=0$ to $n=q$ can be calculated in terms of the divergence of the current:

$$q=\int\partial_\mu K_\mu\,d^2x=\frac{e}{4\pi}\int\varepsilon_{\mu\nu}F^{\mu\nu}\,d^2x=\frac{e}{2\pi}\oint A_\mu\,dx_\mu.\qquad(59)$$

This is an analog of the expression (11). Note that the topological number q is expressed in terms of A_μ (59) in a gauge-invariant manner. In what follows, we shall use the Lorentz gauge:

$$\partial_\mu A_\mu = 0. \tag{60}$$

The analog of (8) will be

$$Z = \int dA d\varphi^+ d\varphi \exp\{-S(A, \varphi) + i\theta q(A)\}, \tag{61}$$

where the Euclidean action S is related to the Lagrangian (54) in the standard manner ($S = \int \mathscr{L} d^2x$), and $q(A)$ is given by (59). In the integral (61), we assume the presence of the gauge-fixing term (60), and the integration is over fields that possess all possible values of q.

The maximal contribution to Z is made by configurations with minimal action S, i.e., ones that satisfy the equations of motion $\delta S = 0$.

The solution with $q = 0$ has the form $A_\mu = 0$, $\varphi = c/\sqrt{2}$, $S = 0$.

Taking into account the small fluctuations near this solution, we obtain the well-known result

$$Z_{\rm p.th} = \det{}^{-1/2}(-\partial^2 + m_v^2)\det{}^{-1/2}(-\partial^2 + m_s^2). \tag{62}$$

The solutions with $q = \pm 1$ are known[16] as Nielsen–Olesen strings.

The basic configuration with $q = 1$ (instanton) has the form

$$A_\mu = \frac{2\pi}{e}\varepsilon_{\mu\nu}\partial^\nu\Phi(x),$$
$$\varphi(x) = \frac{c}{\sqrt{2}}\exp(i\alpha(x))(1 - \Psi(x)). \tag{63}$$

Here, $\tan\alpha = x_1/x_0$, while the functions Φ and Ψ satisfy the conditions $\Psi(0) = 1$ and $\partial_\nu\Phi(0) = 0$.

The explicit forms of the functions Φ and Ψ are unknown, but the behaviors at $|x| \gg 1/m_s$, $1/m_v$ follow from the equations of motion

$$\left.\begin{aligned}\Phi(x) &\simeq D(x) - \beta_v\Delta_v(x),\\ \Psi(x) &\simeq \beta_s\Delta_s(x).\end{aligned}\right\} \tag{64}$$

Here, $\beta_{s,v}$ are certain constants,

$$D(x) = -\ln(x^2\mu^2)/4\pi,\ \Delta_{v,s} = K_0(m_{s,v}|x|)/2\pi, \tag{65}$$

K_0 is the Macdonald function of zeroth order, and μ is a constant with the dimensions of mass. Note that μ and $\beta_{v,s}$ can in principle be calculated.

The functions D and $\Delta_{v,s}$ are the propagators of the massless and massive particles in Euclidean space and satisfy the equations

$$-\partial^2 D = \delta(x),\ (-\partial^2 + m_{v,s})\Delta_{v,s} = \delta(x). \tag{66}$$

The solution with $q = -1$ (anti-instanton) is obtained from (63) by the substitution $A \to -A$, $\varphi \to \varphi^+$.

As approximate solutions, we can take a superposition of N widely spaced instantons with $q_i = \pm 1$:

$$\left.\begin{aligned}A_\mu(x) &= \sum_{i=1}^{N}\frac{2\pi}{e}\varepsilon_{\mu\nu}\partial_\nu\Phi(x - x_i)q_i,\\ \varphi(x) &= \frac{c}{\sqrt{2}}\exp\left(i\sum_{i=1}^{N}\alpha(x - x_i)q_i\right)\prod_{i=1}^{N}(1 - \Psi(x - x_i)).\end{aligned}\right\} \tag{67}$$

The topological number of this configuration is $q = N^{\pm} - N^-$, where $N^+(N^-)$ is the number of instantons with $q_i = +1(-1)$.

We denote by $\overline{S}$ the action of the single-instanton configuration (63). In order of magnitude, $\overline{S} = \mathscr{O}(c^2)$. We shall assume that c is a number sufficiently large that the semiclassical treatment is justified.

By analogy with (10), we represent the single-instanton contribution to (61) in the form

$$Z_1 = Z_{\rm p.th}\, e^{i\theta} k \int d^2x_1. \tag{68}$$

Here, $k = \exp(-\overline{S})$const, and const is the ratio of the corresponding determinants [see (11)], from which the zero mode associated with the translational invariance has been eliminated. The zero mode is separated explicitly in (68) in the form of an integral over the position x_1 of the center of the instanton.

To take into account the contribution of the configurations (67) to the functional integral (61), it is necessary to estimate the action $S(A,\varphi)$ of these configurations. For this, it is necessary to substitute the fields (67) into (54) and calculate the integral.

We represent the result in the form

$$S_N = \overline{S}N + U_N(x_i - x_j), \tag{69}$$

where the first term is the sum of the actions of the individual instantons, while the second describes the deviation from this law. If the instantons are widely spaced ($|x_i - x_j| \to \infty$), the interaction potential U_N tends to zero.

We shall give a method for calculating U_N in which it is not necessary to calculate explicitly any integrals. This means that the form of U_N at large distances is completely determined by the structure of the theory. We shall then demonstrate this method for the example of QCD.

To calculate the action, it is convenient to use a gauge in which the field φ in (67) is purely real. Since the action is gauge-invariant, this does not affect the result. However, it must be borne in mind that such a gauge is singular because of the indeterminacy of the phase $\alpha(x - x_i)$ at the point $x = x_i$. Therefore, strictly speaking, it is not a gauge at all. But for the calculation of S_N we can show that the singularities do not contribute and are therefore unimportant.

Somewhat later, we shall give an example in which the use of such a "gauge" transformation is impossible (see Sec. 3).

For the moment, noting that for $x \neq x_i$ we have the identity

$$\partial_\mu\alpha(x - x_i) = 2\pi\varepsilon_{\mu\nu}\partial_\nu D(x - x_i),$$

we reduce the configurations (67) to the form

$$\left.\begin{aligned}A_\mu &= \sum_{i=1}^{N}\frac{2\pi}{e}\varepsilon_{\mu\nu}\partial_\nu(\Phi(x - x_i) - D(x - x_i))q_i,\\ \varphi &= \frac{c}{\sqrt{2}}\prod_{i=1}^{N}(1 - \Psi(x - x_i)).\end{aligned}\right\} \tag{70}$$

In this gauge, the field A_μ is exponentially small at large distances.

For simplicity, we consider two instantons ($N = 2$) with charges q_1 and q_2. We must separate from the integral the term that depends on the positions x_1 and x_2 of the instantons.

We divide the plane of the x integration into three parts (Fig. 1). Region I is bounded by the circle S_1, which is concentric with the first instanton. The radius of the circle is in the interval $1/m_{v,s} \ll R \ll |x_1 - x_2|$. Region II is bounded by the circle S_2, which is concentric with the second instanton. Region III is the remaining part of the plane.

We denote by $\Delta S_{\rm I}, \Delta S_{\rm II}, \Delta S_{\rm III}$ the parts of the interaction energy accumulated in regions I, II, III, respectively. We calculate $\Delta S_{\rm I}$. In region I, the influence of the second instanton reduces to small corrections to the fields A and φ produced by the first instanton (70):

$$\left.\begin{aligned} \delta A_\mu^{(2)} &= q_2 \frac{2\pi}{e} \varepsilon_{\mu\nu} \partial_\nu (-\beta_v \Delta_v (x - x_2)), \\ \delta\varphi^{(2)} &= \frac{c}{\sqrt{2}} (1 - \Psi(x - x_1))(-\beta_s \Delta_s (x - x_2)). \end{aligned}\right\} \quad (71)$$

In the first order, the influence of the corrections to the action S is determined by the variation δS on the background of the first instanton. But in region I the equations of motion are satisfied, so that there we have $\delta S = 0$. There remain, however, surface terms, which have the form

$$\Delta S_{\rm I} = \delta S = \int_{S_1} d\sigma^k \left[\frac{\partial \mathscr{L}^{(1)}}{\partial \varphi_{;k}} \delta\varphi^{(2)} + \frac{\partial \mathscr{L}^{(1)}}{\partial A_{\mu;k}} \delta A_\mu^{(2)}\right]. \quad (72)$$

Here, $d\sigma^k$ is an element of the boundary S_1 directed along the outer normal to S_1.

The "canonical momenta" of the fields are readily found from the Lagrangian:

$$\left.\begin{aligned} \frac{\partial \mathscr{L}^{(1)}}{\partial \varphi_{;k}} &= 2\partial_k \varphi^{(1)} = -\sqrt{2}\, c\beta_s \partial_k \Delta_s (x - x_1), \\ \frac{\partial \mathscr{L}^{(1)}}{\partial A_{\mu;k}} &= F_{k\mu}^{(1)} = -\varepsilon_{k\mu} \frac{2\pi}{e} \partial^2 \Phi = q_1 \frac{2\pi}{e} \beta_v m_v^2 \Delta_v (x - x_1). \end{aligned}\right\} \quad (73)$$

We have here taken the asymptotic behaviors of the fields, since we need to know these quantities only on the circle S_1, where we are certainly in asymptotia ($R \gg 1/m_v$).

For the same reason, the function $\Psi(x - x_1)$ in the expression (71) for $\delta\varphi^{(2)}$ must be assumed to be equal to zero.

From the requirement of symmetry, we obtain for region II an expression analogous to (72):

$$\Delta S_{\rm II} = \delta S = \oint_{S_2} d\sigma^i \left[\frac{\partial \mathscr{L}^{(2)}}{\partial \varphi_{;i}} \delta\varphi^{(1)} + \frac{\partial \mathscr{L}^{(2)}}{\partial A_{\mu;i}} \delta A_\mu^{(1)}\right].$$

Here, $\delta\varphi^{(1)}$, $\delta A^{(1)}$ are the small (in region II) fields of the first instanton given by expressions analogous to (71), while the canonical momenta of the second instanton are given by expressions analogous to (73).

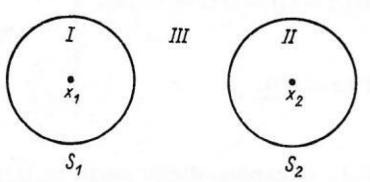

FIG. 1. Regions of integration: S_1 and S_2 are circles that bound the regions of integration I, II, and III; x_1 and x_2 are the positions of the centers of the instantons.

In region III, both instantons are in asymptotia, and therefore either can be regarded as a perturbation. We shall regard the field of the first instanton, $\delta\varphi^{(1)}$, $\delta A^{(1)}$, as a perturbation.

Since the equations of motion for an individual instanton (the second) are also satisfied in region III, it follows, as before, that only surface terms remain. The only difference is that now the boundary of the region consists of two parts and the normals to it have the directions opposite to those in regions I and II. Therefore

$$\Delta S_{\rm III} = -\oint_{S_1} \left[\frac{\partial \mathscr{L}^{(2)}}{\partial \varphi_{;i}} \delta\varphi^{(1)} + \frac{\partial \mathscr{L}^{(2)}}{\partial A_{\mu;i}} \delta A_\mu^{(1)}\right] d\sigma^i - \oint_{S} d\sigma^i \left[\frac{\partial \mathscr{L}^{(2)}}{\partial \varphi_{;i}} \delta\varphi^{(1)} + \frac{\partial \mathscr{L}^{(2)}}{\partial A_{\mu;i}} \delta A_\mu^{(1)}\right].$$

As a result, we obtain

$$\Delta S = \oint_{S_1} d\sigma_i \left[\frac{\partial \mathscr{L}^{(1)}}{\partial \varphi_{;i}} \delta\varphi^{(2)} + \frac{\partial \mathscr{L}^{(1)}}{\partial A_{\mu;i}} \delta A_\mu^{(2)} - \frac{\partial \mathscr{L}^{(2)}}{\partial \varphi_{;i}} \delta\varphi^{(1)} - \frac{\partial \mathscr{L}^{(2)}}{\partial A_{\mu;i}} \delta A_\mu^{(1)}\right]. \quad (74)$$

Thus, the interaction energy is expressed in terms of the asymptotic behavior of the fields at large distances. Substituting here the asymptotic expressions (71) and (73), we obtain

$$\Delta S_1 = \left(\frac{2\pi}{e}\right)^2 \beta_v^2 m_v^2 q_1 q_2 (I_2^v - I_1^v) - c^2 \beta_s^2 (I_2^s - I_1^s),$$

where we have introduced the notation

$$I_1^{v,s} = \oint_{S_1} d\sigma_i\, \partial_i \Delta^{v,s} (x - x_1)\, \Delta^{v,s} (x - x_2);$$

$$I_2^{v,s} = \oint_{S_1} d\sigma_i\, \partial_i \Delta^{v,s} (x - x_2)\, \Delta^{v,s} (x - x_1).$$

To find $I_2 - I_1$, we consider the following integral over region I:

$$\int_{\rm I} \{\partial_\mu \Delta^{v,s} (x - x_1)\, \partial_\mu \Delta^{v,s} (x_1 - x_2) + m_{v,s}^2 \Delta^{v,s} (x - x_1)\, \Delta^{v,s} (x - x_2)\}\, d^2x.$$

We shall integrate by parts in two ways, omitting differently the derivatives in the first term of this integral. Then, taking into account the definition of the propagator, $(-\partial^2 + m_{v,s}^2)\Delta^{v,s} = \delta(x)$, we can readily see that in one case the singularity $\delta(x - x_1)$ is in the region of integration, but in the other $\delta(x - x_2)$ is not. Comparison of these methods of integration leads to the identity

$$I_2^{v,s} - I_1^{v,s} = \Delta^{v,s} (x_1 - x_2).$$

Thus, the interaction energy can be expressed in terms of the propagators $\Delta_{v,s}$:

$$U_{q_i q_j} = q_i q_j (2\pi c \beta_v)^2 \Delta_v (x_i - x_j) - (c\beta_s)^2 \Delta_s (x_i - x_j). \quad (75)$$

The total interaction energy will be

$$U_N = \sum_{i<j}^{N} U_{q_i q_j} (x_i - x_j). \quad (76)$$

We are now ready to write down the contribution of an arbitrary N-instanton configuration (67) to the path integral Z (61).

By analogy with the quantum-mechanical model (12), taking into account (61), (68), and (69), we obtain

$$Z = Z_{\text{p.th}} \sum_{N^+,\,N^-} \frac{k^{N^+ + N^-} \exp[i\theta(N^+ - N^-)]}{N^+!\,N^-!} \times \int e^{-U_N} \prod_{i=1}^{N^+} d^2x_i^+ \prod_{i=1}^{N^-} d^2x_i^-, \tag{77}$$

where U_N is given by the expressions (75) and (76).

Formally, Z can be regarded as the grand partition function of a gas of classical particles of two species (+, −) in two-dimensional Euclidean space. The particles interact through the Yukawa forces (75) and have chemical activity equal to k. The chemical potential μ is related to the chemical activity by

$$k = \exp(\mu). \tag{78}$$

The system (77) can be studied by the standard methods of classical statistical physics. However, there is a different method.

We shall show that the system (77) is equivalent to a certain quantum field theory in two-dimensional Euclidean space.

B. Effective Lagrangian

Here, we find the effective low-energy Lagrangian of the model (54), which explicitly takes into account the effects of the complicated vacuum structure.

These effects have the consequence that the Lagrangian can be expressed in terms of colorless variables. Namely, it contains the scalar field $\rho(x)$ and the pseudoscalar field $\Sigma(x)$:

$$\mathcal{L}_{\text{eff}} = \frac{1}{2}(\partial_\mu\Sigma)^2 + \frac{m_v^2}{2}\Sigma^2 + \frac{1}{2}(\partial_\mu\rho)^2 + \frac{1}{2}m_s^2\rho^2 + \mathcal{L}^+ + \mathcal{L}^-, \tag{79}$$

where the interaction Lagrangian has the form

$$\mathcal{L}^\pm = -k\exp(\pm i\Sigma/F_\Sigma + \rho/F_\rho \pm i\theta), \tag{80}$$

$$F_\rho = (c\beta_s)^{-1},\ F_\Sigma = (2\pi c\beta_v)^{-1}. \tag{81}$$

We shall prove that a system with such a Lagrangian is equivalent to the system (57). For this, we consider the functional integral

$$Z_{\text{eff}} = \int d\Sigma\, d\rho\, \exp\left(-\int d^2x \mathcal{L}_{\text{eff}}\right). \tag{82}$$

The effective action in (82) can be represented as the sum of the free action $S_0(\Sigma,\rho)$ and interaction terms:

$$S_{\text{eff}} = S_0 + \int \mathcal{L}^+ d^2x + \int \mathcal{L}^- d^2x.$$

We expand the exponential in (82) in a double series in powers of the interaction:

$$\exp(-S_{\text{eff}}) = e^{-S_0} \sum_{N^+N^-} \frac{1}{N^+!\,N^-!} \prod_{i=1}^{N^+} \left\{-\int \mathcal{L}^+(x_i^+)\,d^2x_i^+\right\} \times \prod_{i=1}^{N^-} \left\{-\int \mathcal{L}^-(x_i^-)\,d^2x_i^-\right\}.$$

Using the explicit form (80) of the interaction Lagrangian $\mathcal{L}^\pm$, we can reduce this expansion to the form

$$\begin{aligned} e^{-S_{\text{eff}}} &= e^{-S_0} \sum_{N^+,\,N^-} \int \frac{k^{N^+ + N^-}}{N^+!\,N^-!} \times e^{i\theta(N^+ - N^-)} \prod_{i=1}^{N^+} \exp\{\rho(x_i^+)/F_\rho + i\Sigma(x_i^+)/F_\Sigma\}\, d^2x_i^+ \\ &\quad \times \prod_{i=1}^{N^-} \exp\{\rho(x_i^-)/F_\rho - i\Sigma(x_i^-)/F_\Sigma\}\, d^2x_i^- \\ &= e^{-S_0} \sum_{N^+N^-} \frac{k^{N^+ + N^-}}{N^+!\,N^-!} e^{i\theta(N^+ - N^-)} \times \exp\left\{\int \rho(x) J_\rho(x)\, d^2x + \int \Sigma(x) J_\Sigma(x)\, d^2x\right\} \times \prod_{i=1}^{N^+} d^2x_i^+ \prod_{i=1}^{N^-} d^2x_i^-. \end{aligned} \tag{83}$$

We have here introduced the notation

$$\left.\begin{aligned} J_\rho(x) &= \left(\sum_{i=1}^{N^+} \delta(x - x_i^+) + \sum_{i=1}^{N^-} \delta(x - x_i^-)\right)\Big/ F_\rho; \\ J_\Sigma(x) &= \left(\sum_{i=1}^{N^+} \delta(x - x_i^+) - \sum_{i=1}^{N^-} \delta(x - x_i^-)\right)\frac{i}{F_\Sigma}. \end{aligned}\right\} \tag{84}$$

It follows from (83) that the integral over the fields Σ and ρ has become Gaussian and can be readily calculated in accordance with the formula

$$\begin{aligned} &\int d\Sigma\, d\rho e^{-S_0} \exp\left\{\int d^2x\, (\rho(x) J_\rho(x) + \Sigma(x) J_\Sigma(x))\right\} \\ &= Z_{\text{p.th}} \exp\left\{\frac{1}{2}\int d^2x\, d^2x' \left[J_\rho(x)\Delta^s(x - x') J_\rho(x') + J_\Sigma(x)\Delta^v(x - x') J_\Sigma(x')\right]\right\}, \end{aligned} \tag{85}$$

where $Z_{\text{p,th}}$ is given by (62), and $\Delta^{v,s}$ by (66). It follows from the explicit form (84) of the currents that the expression (85) reproduces the interaction potential

$$Z_{\text{p.th}} e^{-U_N}, \tag{86}$$

where U_N is defined by the expressions (75) and (76). Collecting together the expressions (82), (83), (85), and (86), we find that Z_{eff} (82) is equal to the partition function Z (77):

$$Z_{\text{eff}} = Z.$$

Thus, in the approximation of a rarefied instanton gas (the region of large distances, or the low-energy limit), the model (61) is equivalent to the statistical system (77), which, in its turn, is equivalent to the theory (79) and (82).

From the Lagrangian (79), we can find the energy of the θ vacuum and the spectrum of excitations over it. To this end, we investigate the minimum of the potential part of the Lagrangian (79):

$$U_{\text{eff}} = \frac{1}{2}m_v^2 \Sigma^2 + \frac{1}{2}m_s^2\rho^2 - 2k\cos\left(\frac{\Sigma}{F_\Sigma} + \theta\right) e^{\rho/F_\rho}. \tag{87}$$

It can be seen from this expression that the field ρ always has a nonzero condensate ρ_0, while the field Σ has a nonzero condensate only when $\theta \neq 0$. Since the approximation of a rarefied instanton gas is valid when $k \ll m_{v,s}^2$, in this case the condensates are weak:

$$\rho_0 \simeq \frac{2k}{m_s^2 F_\rho} \cos\theta; \quad \Sigma_0 \simeq -\frac{2k}{m_v^2 F_\Sigma} \sin\theta. \tag{88}$$

The value of the potential U_{eff} at the minimum gives the vacuum energy density:

$$\varepsilon_{\text{vac}} \simeq -2k\cos\theta. \tag{89}$$

The second derivatives of the effective potential at the minimum give the masses of the fields Σ and ρ:

$$m_\Sigma^2 \simeq m_v^2 + \frac{2k}{F_\Sigma^2}\cos\theta, \quad m_\rho^2 \simeq m_s^2 - \frac{2k}{F_\rho^2}\cos\theta. \tag{90}$$

The expressions (89) and (90) determine the spectrum of the system (54) over the θ vacuum. In what follows, for simplicity we restrict ourselves to the case $\theta = 0$, when the vacuum energy (89) is minimal.

It now remains to show that by means of the Lagrangian (79) we can calculate the Green's functions for the gauge-invariant operators constructed from the original fields A_μ, φ (54) and show that the positions of the poles in them are determined by the masses (90) of the fields Σ and ρ.

It is clear that the Green's functions are related to the correlation functions of the statistical system (77). In what follows, we shall show that the s-particle Green's function is directly related to the s-particle statistical distribution function.

The distribution functions are determined by the statistical means of the operators of the instanton densities, which have the form

$$\hat{n}^+(x) = \sum_{i=1}^{N^+} \delta(x - x_i^+), \quad \hat{n}^-(x) = \sum_{i=1}^{N^-} \delta(x - x_i^-), \tag{91}$$
$$\hat{n}(x) = \hat{n}^+(x) + \hat{n}^-(x).$$

The statistical means of the operators are determined by introducing these operators under the sign of the statistical sum in (77).

To achieve a unified description of these means, we introduce into the statistical system (77) an additional inhomogeneous chemical potential $\mu^\pm(x)$ in accordance with the formula

$$Z(\mu^\pm) = Z_{\text{p.th}} \sum_{N^+N^-} \times \int \frac{k^{N^+ + N^-} e^{-U_N}}{N^+!\, N^-!} \prod_{i=1}^{N^+} e^{\mu^+(x_i^+)}\, d^2x_i^+ \prod_{i=1}^{N^-} e^{\mu^-(x_i^-)}\, d^2x_i^-. \tag{92}$$

We have here set $\theta = 0$, although this is not fundamental.

It follows from the explicit form (92) of the functional $Z(\mu^+, \mu^-)$ that it is the generating functional for the statistical means of the density operators $\hat{n}^\pm(y)$;

$$\langle\langle \hat{n}^+(y_1)\, \hat{n}^+(y_2) \ldots \hat{n}^-(y_m)\rangle\rangle = \frac{1}{Z} \frac{\delta^m Z[\mu^+, \mu^-]}{\delta\mu^+(y_1)\,\delta\mu^+(y_2) \ldots \delta\mu^-(y_m)}\bigg|_{\mu^\pm = 0}. \tag{93}$$

The correlations of the operators (93) can be studied by the methods of statistical physics, for example, by the method of BBGKY hierarchies. However, a simpler method is based on the fact that the functional $Z(\mu^+, \mu^-)$ (92) can be represented in the form of the functional integral (82), $Z_{\text{eff}}(\mu^+, \mu^-)$. The only difference from (82) is that now the interaction Lagrangian will depend on $\mu^\pm$:

$$\mathscr{L}^\pm(\mu^\pm) = -k\exp\{\rho(x)/F_\rho \pm i\Sigma(x)/F_\Sigma + \mu^\pm(x)\}. \tag{94}$$

This is readily seen by comparing (77) with (92); the latter is obtained from the former by the substitution

$$k \to k\exp(\mu^\pm(x)).$$

Making such a substitution in (80), we arrive at (94).

Since $Z(\mu^+, \mu^-)$ is represented in the form of the functional integral (82), $Z_{\text{eff}}(\mu^+, \mu^-)$, it follows on the basis of (93) that everything reduces to the field theory (79), (94) with sources $u^\pm(x)$.

We shall demonstrate this for the example of the single- and two-particle correlation functions (93).

The single-particle correlation function, which determines the mean density of instantons, is obtained from the expressions (79), (82), (93), and (94) in the form

$$\langle\langle \hat{n}^\pm(x)\rangle\rangle = k\langle \exp(\rho(x)/F_\rho \pm i\Sigma(x)/F_\Sigma)\rangle, \tag{95}$$

where the mean $\langle \ldots \rangle$ is understood as the average over the fields Σ and ρ under the sign of the functional integral (82).

Developing a perturbation theory for the effective Lagrangian near the minimum (88) for $\theta = 0$, we obtain from (95) the instanton density:

$$\left.\begin{aligned} n^\pm &= \langle\langle \hat{n}^\pm\rangle\rangle = k\exp(\rho_0/F_\rho) \approx k; \\ n &= \langle\langle \hat{n}\rangle\rangle = 2n^+ = 2k. \end{aligned}\right\} \tag{96}$$

We now find the two-particle correlation functions. From (93) and (94), we obtain

$$\langle\langle \hat{n}^\pm(x)\, \hat{n}^\pm(y)\rangle\rangle = \delta(x-y)\langle\langle \hat{n}^\pm\rangle\rangle + k^2\langle \exp\{\rho(x)/F_\rho \pm i\Sigma(x)/F_\Sigma\} \times \exp\{\rho(y)/F_\rho \pm i\Sigma(y)/F_\Sigma\}\rangle.$$

In the leading order in the interaction, this reduces to averaging over the free fields with masses m_Σ and m_ρ (90) and leads to the result

$$\langle\langle \hat{n}^\pm(x)\, \hat{n}^\pm(y)\rangle\rangle = \delta(x-y)\frac{n}{2} + \left(\frac{n}{2}\right)^2 \exp\{\Delta_\rho(x-y)/F_\rho^2 - \Delta_\Sigma(x-y)/F_\Sigma^2\}, \tag{97}$$

where n is determined by (96), and $\Delta_{\Sigma,\rho}$, the propagators of the particles Σ and ρ, satisfy (66) with masses $m_{\Sigma,\rho}$, respectively.

Similarly, we obtain

$$\langle\langle \hat{n}^+(x)\, \hat{n}^-(y)\rangle\rangle = \left(\frac{n}{2}\right)^2 \exp\left\{\frac{\Delta_\rho(x-y)}{F_\rho^2} + \frac{\Delta_\Sigma(x-y)}{F_\Sigma^2}\right\}. \tag{98}$$

It is readily verified that in the region of large distances, where the functions $\Delta_{\Sigma,\rho}$ are small, these correlation functions satisfy the principle of correlation weakening.

C. Green's functions

Here we calculate the tow-point Green's functions for the gauge-invariant operators

$$q(x)=\frac{e}{4\pi}\varepsilon_{\mu\nu}F_{\mu\nu}(x)_{\Sigma}\quad \varphi^{+}(x)\varphi(x)$$

and show that the particles Σ and ρ are reflected in them as poles.

For the operator of the density of the topological number in the rarefied-gas approximation (67), we have

$$q(x)=-\int\partial^2\Phi(x-z)(\hat{n}^{+}(z)-\hat{n}^{-}(z))\,d^2z, \quad (99)$$

where $\hat{n}^{\pm}$ are determined by the expression (91).

From the expressions (96) we readily obtain $\langle q(x)\rangle=0$. We note that a zero condensate q is obtained only for $\theta=0$.

For the two-point Green's function, we obtain from (99)

$$\langle q(x)q(y)\rangle=\int\partial^2\Phi(x-u)\,\partial^2\Phi(y-v)\,d^2u\,d^2v$$
$$\times\langle\langle(\hat{n}^{+}(u)-\hat{n}^{-}(u))(\hat{n}^{+}(v)-\hat{n}^{-}(v))\rangle\rangle$$
$$\simeq\int\partial^2\Phi(x-u)\,\partial^2\Phi(y-v)\,d^2u\,d^2v$$
$$\times\left\{n\delta(u-v)-\left(\frac{n}{F_{\Sigma}}\right)^2\Delta_{\Sigma}(u-v)\right\}. \quad (100)$$

In deriving this result, we have used the expressions for the means (97) and (98), having first expanded them in powers of $\Delta_{\Sigma,\rho}$. Such expansions enable us to separate the pole contribution in the correlation function (100), and they are valid at large $|x-y|$, where $\Delta_{\Sigma,\rho}(x-y)$ are small.

It is convenient to express (100), as an equality for the corresponding Fourier components with momentum p:

$$\langle q(p)q(-p)\rangle=(\partial^2\Phi(p))^2 n\,\frac{m_v^2+p^2}{p^2+m_{\Sigma}^2}. \quad (101)$$

Here we have used the expression (90) for the mass m_{Σ}, which with allowance for (96), can be written in the form

$$m_{\Sigma}^2=m_v^2+n/F_{\Sigma}^2.$$

Taking into account the behavior (64) of $\Phi(x)$ at large distances, we obtain $\partial^2\Phi(x)\simeq-\beta_v m_v^2\Delta_v(x)$, or, in the p space,

$$\partial^2\Phi(p)\simeq-\beta_v\frac{m_v^2}{p^2+m_v^2}.$$

As a result, we have

$$\langle q(p)q(-p)\rangle\simeq(\beta_v m_v^2 F_{\Sigma})^2\frac{m_{\Sigma}^2-m_v^2}{(p^2+m_{\Sigma}^2)(p^2+m_v^2)}. \quad (102)$$

It can be seen from this that there are two poles at nearby points. The residues of these poles are equal in modulus but opposite in sign, and therefore one of the poles is not physical.

We shall show that m_{Σ}^2 is the physical pole. In calculating the Green's function (100), we restricted ourselves essentially to just the classical contributions (99) and ignored the fluctuations around them.

In the first approximation, this contribution can be taken into account by developing perturbation theory near the minimum of (55);

$$\langle q(p)q(-p)\rangle_{\text{p.th}}=\left(\frac{e}{2\pi}\right)^2\frac{p^2}{p^2+m_v^2}=\frac{(\beta_v m_v^2 F_{\Sigma})^2p^2}{m_v^2(p^2+m_v^2)}.$$

Adding this contribution to (102), we obtain the final result:

$$\langle q(p)q(-p)\rangle=(\beta_v m_v^2 F_{\Sigma})^2\frac{p^2+m_{\Sigma}^2-m_v^2}{(p^2+m_{\Sigma}^2)\,m_n^2}. \quad (103)$$

The residue at the pole is $-\beta_v^2 m_v^4 F_{\Sigma}^2$. The negative sign of the residue does not contradict the condition of positivity of the spectral function for the correlation function (103), since it must be borne in mind on the translation of (103) to Minkowski space that the pseudoscalar operator q is related to the corresponding operator q_M in Minkowski space by $q_M=iq$. The imaginary unit in this expression explains the "incorrect" sign of the residue in Euclidean space.

Near the pole, the expression (103) can be reproduced by means of the approximate operator equation

$$q_M(x)=\frac{e}{4\pi}\varepsilon_{\mu\nu}F^{\mu\nu}(x)\simeq\beta_v m_v^2 F_{\Sigma}\Sigma(x). \quad (104)$$

It is interesting to compare this equation with the corresponding identity (45) in the Schwinger model.

We turn to a discussion of the operator $\varphi^{+}\varphi$. In the framework of the approximation (67), (70) this operator has the form

$$\varphi^2(x)=\varphi^{*}\varphi(x)=\frac{c^2}{2}\prod_i(1-\Psi(x-x_i))^2.$$

Assuming the gas to be rarefied, we can approximate it by

$$\varphi^2(x)\simeq\frac{c^2}{2}\left\{1-\sum_{i=1}^{N}(2\Psi(x-x_i)-\Psi^2(x-x_i))\right\}$$
$$=\frac{c^2}{2}\left\{1-\int(2\Psi(x-x_1)-\Psi^2(x-x_1))\,\hat{n}(x_1)\,d^2x_1\right\},$$
$$(105)$$

where the operator $\hat{n}$ is defined by (91). Averaging this relation by means of (96), we obtain the value of the scalar condensate:

$$\langle\varphi^2(x)\rangle\simeq\frac{c^2}{2}\left\{1-n\int(2\Psi(x)-\Psi^2(x))\,d^2x\right\}. \quad (106)$$

By means of the representation (105) and the averaging formulas (97) and (98), we can also readily calculate the two-point function:

$$\langle\varphi^2(x)\varphi^2(y)\rangle=\langle\varphi^2\rangle^2$$
$$+\int\{2\Psi(x-u)-\Psi^2(x-u)\}\left\{n\delta(u-v)\right.$$
$$\left.+\frac{n^2}{F_{\rho}^2}\Delta_{\rho}(u-v)\right\}\{2\Psi(y-v)-\Psi^2(y-v)\}\,d^2u\,d^2v.$$
$$(107)$$

Here, as in the case of the correlation function (100) of the operator q, we have retained only the pole contribution.

It is convenient to go over to the momentum space. We denote by $\Psi(p)$ and $\Psi^2(p)$ the Fourier components of the functions $\Psi(x)$ and $\Psi^2(x)$, respectively. It follows from (64) that

$$\Psi(p)\simeq\beta_s\frac{1}{p^2+m_s^2}.$$

With allowance for this, we reduce (107) to the form

$$\langle\varphi^2(p)\varphi^2(-p)\rangle=(2\pi)^2\langle\varphi^2\rangle\,\delta(p)$$
$$+\frac{c^2}{p^2+m_{\rho}^2}\left\{\frac{m_s^2-m_{\rho}^2}{p^2+m_s^2}-c^2n\beta_s\Psi^2(p)\right.$$
$$\left.+c^2n(p^2+m_s^2)\Psi^2(p)\Psi^2(p)\right\}. \quad (108)$$

If we add to this expression the perturbative contribution

$$\langle\varphi^2(p)\,\varphi^2(-p)\rangle_{\text{p.th}}=\frac{c^2}{p^2+m_s^2},$$

then the spurious pole at the point $p^2=-m_s^2$ disappears, while the pole at the point $p^2=-m_\rho^2$ remains. The corresponding residue at this point is $c^2\left(1-(n/2)c^2\beta_s\Psi^2(p)\,|_{p^2=-m_\rho^2}\right)^2$, and therefore near the pole the following approximate operator equation is valid:

$$\varphi^2(x)\approx\langle\varphi^2\rangle+c\left(1-\frac{nc^2}{2}\beta_s\Psi^2(-m_\rho^2)\right)\rho(x). \qquad (109)$$

Thus, a study of the functional integral (61) has led to a description of the model in terms of the colorless variables (79) and has enabled us to determine the spectrum (89), (90). We have then demonstrated that the resulting spectrum is manifested in the form of the poles (103) and (108) in the Green's functions for the gauge-invariant operators. Near the poles, the approximate operator equations (104) and (109), which relate the original fields A_μ, $\varphi(x)$ to the colorless Σ and ρ, hold.

3. INCLUSION OF MASSLESS FERMIONS

Here we consider the two-dimensional electrodynamics of fermion and scalar fields. As in scalar electrodynamics, in this model there is color rearrangement, and the spectrum contains colorless scalar, pseudoscalar, and fermion particles.

The presence of massless fermions has the consequence that, as in the Schwinger model, the vacuum is degenerate with respect to the chiral number. As a result, the θ vacuum does not have a definite chiral number, and this means that the chiral symmetry is broken. As in the Schwinger model, this leads to acquisition of dynamical mass by the colorless variables (in the present case, the fermions). We note that acquisition of dynamical mass by two-dimensional fermions was considered earlier in Ref. 17.

We add to the scalar electrodynamics (54) fermion fields that interact with the gauge field in the standard manner:

$$\mathcal{L}_\psi=i\bar\psi\gamma_\mu(\partial^\mu-ieA_\mu)\psi. \qquad (110)$$

Such an addition significantly changes the properties of the original model (54). Since the fermion fields are massless, it is to be expected that in the region of large distances there will appear degrees of freedom constructed from these fermion fields.

The investigation of the model is simplified by the fact that the fermion sector (110) is, in a certain sense, exactly solvable. Indeed, the Green's function of the fermions in an external field,

$$A_\mu=\frac{2\pi}{e}\varepsilon_{\mu\nu}\partial^\nu\Phi, \qquad (111)$$

has[5] the form

$$G(x,y\mid A)=G_0(x-y)\exp\{2\pi\gamma_5(\Phi(x)-\Phi(y))\}. \qquad (112)$$

Here, G_0 is the Green's function of the free massless fermions:

$$G_0(x)=-\gamma_\mu x_\mu/(2\pi^2x^2). \qquad (113)$$

The expression (112) makes it possible to calculate exactly the fermion determinants in an external field:[5]

$$\frac{\det(\gamma_\mu(\partial_\mu-ieA_\mu))}{\det(\gamma_\mu\partial_\mu)}=\exp\left\{-2\pi\int(\partial_\mu\Phi)^2\,d^2x\right\}. \qquad (114)$$

In addition, the same formula can be used to calculate[5] the axial anomaly, which has the same form as (39) in the Schwinger model:

$$\partial_\mu J_\mu^5=\frac{e}{2\pi}\varepsilon_{\mu\nu}F_{\mu\nu}=2q(x). \qquad (115)$$

Here, $J_\mu^5=\bar\psi\gamma_\mu\gamma_5\psi$, and $q(x)$ is the density of the topological number (59).

It follows from (115) that the gauge-invariant current is not conserved, but a current that is conserved is

$$j_5^\mu=J_5^\mu-\frac{e}{\pi}\varepsilon^{\mu\nu}A_\nu.$$

The charge constructed from the zeroth component of this current has the form

$$\tilde Q_5=-\int j_0^5\,dx^1=Q_5-2n\,[A_1], \qquad (116)$$

where $n[A_1]$ is given by (57). Under the action of the topologically nontrivial gauge transformation T the number $n[A_1]$ changes by unity, while Q_5 remains constant, and therefore $\tilde Q_5$ changes by two:

$$T^+\tilde Q_5T=\tilde Q_5-2. \qquad (117)$$

On the other hand, $\tilde Q_5$ is observed in time, and therefore (117) means that in the system there is degeneracy with respect to the number $\tilde Q_5$. Indeed, from the state $|0\rangle$ with zero value of $\tilde Q_5$ we can construct, by means of a gauge transformation that does not change the energy, states

$$|\,n\rangle=(T^+)^n\,|\,0\rangle, \qquad (118)$$

which carry the nonzero chiral number $\tilde Q_5=2n$. As the ground state, one can take any state $|n\rangle$ or a superposition of them. Thus, for a system with degeneracy we have for the θ vacuum the representation

$$|\,\theta\rangle=\sum_n\exp(i\,n\theta)\,|\,n\rangle. \qquad (119)$$

It follows from this that the energy of the θ vacuum does not depend on the parameter θ.

The state $|\theta\rangle$, in contrast to the states (118), does not have a definite value of the chiral number $\tilde Q_5$. Therefore, a fermion propagating over the θ vacuum also does not have a definite chirality. In what follows, we shall see that this is explained by the fact that the colorless fermion acquires a dynamical mass.

We now turn directly to the construction of the approximation of a rarefied instanton gas in this model. Here, we shall follow Ref. 18, in which an effective low-energy Lagrangian describing effective fermion fields was obtained.

We shall then establish approximate operator identities relating these fields to gauge-invariant operators constructed from the original fields ψ, φ, A_μ. This will show that the effective fermion fields are colorless.

To calculate Z (61) in this model, we note that it differs from the corresponding quantity (77) in scalar electrodynamics only by the fermion determinant, which is known exactly [Eq. (114)].

As Φ in this expression, it is necessary to take the many-instanton configuration (67). Note that in the given case it is not possible to use the singular gauge (70), since the expression (114) is very sensitive to singularities. At large distances, $(\partial_\mu \overline{\Phi})$ has the asymptotic behavior of (120): $(N^+ - N^-)\mathscr{O}(1/r)$. Therefore, the determinant (114) is nonzero only for $N^+ = N^-$, i.e., when the number of instantons is equal to the number of anti-instantons. To calculate the interaction potential of the instantons that arises from the fermion determinant,

$$U^F = 2\pi \int (\partial_\mu \Phi)^2 \, d^2\bar{x}, \tag{120}$$

we use the asymptotic behavior

$$\Phi(x) = \sum_{i=1}^{N} q_i D(x - x_i). \tag{121}$$

Such an approximation corresponds to the approximation of point instantons. In this case, we obtain from (120) and (121) the expression[18]

$$U_F^N = -2\pi \int \Phi \, \partial^2 \Phi \, \partial^2 x = 4\pi \sum_{i<j} D(x_i - x_j) q_i q_j$$
$$= -\sum_{i<j}^{N} q_i q_j \ln [(x_i - x_j)^2 \mu^2]. \tag{122}$$

The expression (122) gives only the leading asymptotic behavior of the potential at large distances. The following terms of the asymptotic behavior of the potential have order $\mathscr{O}(\exp(-m_{s,v} r))$. In principle, they can be calculated by means of (120), but for this it is insufficient to know the asymptotic behavior of the instanton solution (64); it is necessary to have an exact solution, which is not available.

The approximation of point instantons corresponds to neglect of corrections of order $\mathscr{O}(\exp(-m_{v,s} r))$. This means that to the given accuracy we must ignore the interaction potential (76) calculated earlier in Sec. 2.

Thus, our approximation corresponds to neglect of the contributions of the heavy particles with masses $\mathscr{O}(m_{v,s})$. This is justified, since at large distances only light particles are manifested.

In the given approximation, Z has the form

$$Z = \sum_{N^+, N^-} \int \delta_{N^+, N^-} \frac{k^{N^+ + N^-} e^{i\theta (N^+ - N^-)} e^{-U_N^F}}{N^+! \, N^-!} \det(\gamma_\mu \partial_\mu)$$
$$\times \prod_{i=1}^{N^+} d^2 x_i^{\pm} \prod_{i=1}^{N^-} d^2 x_i^{-}, \tag{123}$$

where U_N^F is given by (122).

It can be seen from (123) that Z does not depend on θ, in complete agreement with the expression (119).

It can be verified[18] that the partition function (123) is equivalent to a fermion model, namely,

$$Z = \int d\sigma^+ d\sigma \exp\left\{ \int \sigma^+(x) \hat{\partial} \sigma(x) \, d^2 x \right\}$$
$$\times \sum_{N^+, N^-} \frac{1}{N^+, N^-} \prod_{i=1}^{N^+} \left\{ \frac{2\pi}{\mu} k e^{i\theta} \sigma^+(x_i^+) \right.$$
$$\left. \times \frac{1-\gamma_5}{2} \sigma(x_i^+) \, d^2 x_i^+ \right\}$$
$$\times \prod_{i=1}^{N^-} \left\{ \frac{2\pi}{\mu} k e^{-i\theta} \sigma^+(x_i^-) \frac{1+\gamma_5}{2} \sigma(x_i^-) \right\} d^2 x_i^-. \tag{124}$$

Proof of the equivalence of (123) and (124) is readily given for the terms of the series with $N^+ = N^- = 1$. In the general case, it is simplest to use the bosonization representation[19] for the two-dimensional fermions $\sigma(x)$.

It follows from (124) that the parameter θ can be eliminated by redefining the fields $\sigma(x)$ by means of a γ_5 rotation. Since nothing depends on θ, we choose $\theta = \pi$. Summing the series (124) over $N^\pm$, we obtain the effective Lagrangian of the fermion model:

$$\mathscr{L}_{\text{eff}} = \sigma^+ \hat{\partial} \sigma + \mathscr{L}^+ + \mathscr{L}^-; \tag{125}$$

$$\mathscr{L}^\pm = -\frac{2\pi k}{\mu} \sigma^+ \frac{1 \mp \gamma_5}{2} \sigma. \tag{126}$$

It can be seen from (125) and (126) that the model is equivalent to the theory of a free fermion field with mass

$$m = -2\pi k/\mu. \tag{127}$$

Thus, the appearance of the γ_5 anomaly in the model leads to acquisition of dynamical mass by the colorless quark. As a result of this, the conserved charge $\widetilde{Q}_5$ (116) is not defined for such a fermion. Note, however, that if we had constructed the theory over the vacuum $|n\rangle$ (118), and not over the θ vacuum, the number would be defined for the complete spectrum.

It can be verified that introduction of the inhomogeneous chemical potential $\mu^\pm(x)$ (92) leads to modification of the interaction Lagrangian (126):

$$\mathscr{L}^\pm = -m\sigma^+(x) \frac{1 \mp \gamma_5}{2} \sigma(x) \exp(\mu^\pm(x)). \tag{128}$$

We now establish approximate operator identities between the field $\sigma(x)$ and the fields ψ, φ, A_μ. We consider first the density of the topological charge (59).

In the approximation of point instantons (121) we have

$$q(x) = \frac{e}{4\pi} \varepsilon_{\mu\nu} F_{\mu\nu} = \sum_{i=1}^{N} q_i \delta(x - x_i) = \hat{n}^+(x) - \hat{n}^-(x), \tag{129}$$

where the operators $\hat{n}^\pm$ are given by the expressions (91). By means of (93) and (128), we obtain

$$\langle q(x) \rangle = \langle \mathscr{L}^+ - \mathscr{L}^- \rangle = m \langle \sigma^+ \gamma_5 \sigma \rangle = 0. \tag{130}$$

Similarly, for the two-point correlation function we obtain

$$\langle q(x) q(y) \rangle = \delta(x - y) \langle m\sigma^+ \sigma \rangle$$
$$+ \langle m\sigma^+(x) \gamma_5 \sigma(m) \, m\sigma^+(y) \gamma_5 \sigma(y) \rangle. \tag{131}$$

It follows from (130) and (131) that in the region of large distances

$$q(x) \simeq m\sigma^+(x) \gamma_5 \sigma(x). \tag{132}$$

In Minkowski space, this relation has the form

$$q_M(x) = im\bar{\sigma}(x) \gamma_5 \sigma(x).$$

In terms of the new degrees of freedom, the identity (115) for the anomaly of the axial current takes the simple form

$$\partial_\mu J_\mu^5 \simeq 2im\bar{\sigma} \gamma_5 \sigma, \tag{133}$$

the significance of which is the acquisition of dynamical mass by the effective fermion [analog of (40)].

We now consider the operators

$$J^+(z^+)=\overline{\psi}(z^+)\,\frac{1-\gamma_5}{2}\,\psi(z^+),$$
$$J^-(z^-)=\overline{\psi}(z^-)\,\frac{1+\gamma_5}{2}\,\psi(z^-),\quad z^\pm=(z_0^\pm,\ z_1^\pm),$$

which are composed of left and right fermions, respectively.

The two-point correlation function of these operators in Euclidean space has the form

$$\langle J^+(z^+)\,J^-(z^-)\rangle$$
$$=\left\langle \psi^+(z^+)\,\frac{1-\gamma_5}{2}\,\psi(z^+)\,\psi^+(z^-)\,\frac{1+\gamma_5}{2}\,\psi(z^-)\right\rangle.$$

Using the explicit expression for the Green's function of the fermions in the external field (112), we obtain

$$\langle J^+(z^+)\,J^-(z^-)\rangle=\exp\{-4\pi\,(\Phi(z^+)-\Phi(z^-))\}\,\frac{1}{4\pi^2\,(z^+-z^-)^2}\,.$$

In order to immerse this correlation function in the instanton gas, we must take for Φ in this formula the expression (121). We obtain

$$\langle J^+(z^+)\,J^-(z^-)\rangle$$
$$=\frac{\mu^2}{4\pi^2}\exp\Big\{4\pi D\,(z^+-z^-)-4\pi\sum_i q_i D\,(z^+-x_i)$$
$$+4\pi\sum_i q_i D\,(z^--x_i)\Big\}.\tag{134}$$

Comparing this with the expression (122) for the interaction potential, we see that the introduction into the partition function (123) of the correlation function (134) is equivalent to addition to the system (123) of two test particles. As can be seen from (134), the particle at the point z^+ has positive charge, and the one at z^- has a negative charge.

Using the representation of the partition function (123) in terms of the fermion fields (124), we obtain the required representation in terms of the fields σ:

$$\langle J^+(z^+)\,J^-(z^-)\rangle=\left\langle \sigma^+(z^+)\,\frac{1-\gamma_5}{2}\,\sigma(z^+)\,\sigma^+(z^-)\,\frac{1+\gamma_5}{2}\,\sigma(z^-)\right\rangle.$$

Thus, the gauge-invariant operators $J^\pm$ are related to the operator σ:

$$\psi^+\,\frac{1\pm\gamma_5}{2}\,\psi\simeq\sigma^+\,\frac{1\pm\gamma_5}{2}\,\sigma.\tag{135}$$

Note that the field σ occurs bilinearly in the relations (132) and (135).

We shall now find an approximate equation in which σ occurs linearly.

For this, we consider the gauge-invariant operator that interpolates the "white quark" field:[13,14]

$$\left.\begin{aligned}&F(X_-)=\varphi^*(X_-)\,\Pi^+\psi(X_-)=\varphi^+(X_-)\,\psi_R(X_-),\\&\overline{F}(X_+)=\varphi(X_+)\,\overline{\psi}\Pi^-=\varphi(X_+)\,\overline{\psi}_R(X_+),\\&\Pi^\pm=(1\pm\gamma_5)/2.\end{aligned}\right\}\tag{136}$$

With allowance for the expression (112), the two-point correlation function in Euclidean space has the form

$$\langle F(X_-)\,\overline{F}(X_+)\rangle=\varphi^+(X_-)\,\varphi(X_+)$$
$$\times\exp\{2\pi\,(\Phi(X_-)-\Phi(X_+))\}\,\Pi^+G_0'(X_--X_+)\,\Pi^-.\tag{137}$$

In the approximation of the point instantons (67) and (121), we obtain

$$\langle F(X_-)\,\overline{F}(X_+)\rangle=\frac{c^2}{2}\sum_{i=1}^{N}\exp\{-iq_i\alpha\,(X_--x_i)\}$$
$$\times\sum_{i=1}^{N}\exp\{iq_i\alpha\,(X_+-x_i)\}$$
$$\times\exp\Big\{2\pi\sum_{i=1}^{N}q_iD\,(X_--x_i)-2\pi$$
$$\times\sum_{i=1}^{N}D\,(X_+-x_i)\Big\}\,\Pi^+G_0\,(X_--X_+)\,\Pi^-.\tag{138}$$

Introducing the notation $z(x)=x_0+ix_1$ and $z^*=(x_0-ix_1)$, and substituting the explicit expressions (65) and (113) for D and G_0 into (138), we obtain the following expression for the correlation function in the approximation of a rarefied instanton gas:

$$\langle F(X_-)\,\overline{F}(X_+)\rangle=\frac{c^2}{2}\prod_{i=1}^{N^+}\frac{z\,(X_+-x_i^+)}{z\,(X_--x_i^+)}$$
$$\times\prod_{i=1}^{N^-}\frac{z\,(X_--x_i^-)}{z\,(X_+-x_i^-)}\left(\frac{-1}{2\pi z\,(X_--X_+)}\right).\tag{139}$$

After fairly lengthy but simple calculations, it can be shown that in the language of the fermion fields σ (124) the expression (139) is equivalent ot the introduction into the system (124) of fermion fields at the points $X_\pm$, namely,

$$\langle F(X_-)\,\overline{F}(X_+)\rangle=\frac{c^2}{2}\,\langle\Pi^+\sigma\,(X_-)\,\sigma^+(X_+)\,\Pi^-\rangle.\tag{140}$$

The averaging on the right-hand side of (140) is done by means of the Lagrangian (125). We shall demonstrate the equivalence of (139) and (140) in the second order of perturbation theory with respect to the fermion mass m (127). For this, in the expressions (124) and (139) it is necessary to set $N^+=N^-=1$. In this case, it follows from the expression (124) that the right-hand side of (140) has the form

$$\frac{c^2}{2}\left(\frac{2\pi k}{\mu}\right)^2\int\exp\left\{\int\sigma^+\hat{\partial}\sigma\,d^2x\right\}d\sigma^+\,d\sigma$$
$$\times\{\Pi^+\sigma\,(X_-)\,\sigma^+(x^+)\,\Pi^-\sigma\,(x^+)$$
$$\times\sigma^+(x^-)\,\Pi^+\sigma\,(x^-)\,\sigma^+(X_+)\,\Pi^-\}\,d^2x^+\,d^2x^-.$$

The averaging over the massless fermions can be readily done by means of the Green's function (113), as a result of which the upper expression takes the form

$$\int\frac{c^2}{2}\left(\frac{2\pi k}{\mu}\right)^2 d^2x^+\,d^2x^-$$
$$\times\Big\{-\frac{1}{2\pi z\,(X_--X_+)\,|\,2\pi z\,(x^+-x^-)\,|^2}$$
$$-\frac{1}{2\pi z\,(X_-+x^+)\,2\pi z^*(x^+-x^-)\,2\pi z\,(x^--X_+)}\Big\}.$$

After simple algebraic manipulations, this expression reduces to

$$\int k^2 \exp(-U_2^F(x^+ - x^-))\, d^2x^+\, d^2x^- \times$$
$$\times \left[-\frac{c^2 z (X_+ - x^+) z (X_- - x^-)}{2z (X_- - x^+) z (X_+ - x^-) 2\pi z (X_- - X_+)} \right], \qquad (141)$$

where $U_2^F(x^+ - x^-)$ is given by (122). The expression in the square brackets in (141) is identical to the right-hand side of (139) for $N^+ = N^- = 1$, as we needed to prove.

One can similarly prove the equivalence of (139) and (140) for $N^\pm > 1$.

Thus, we have shown that the colorless gauge-invariant operator F (136) can be expressed by means of (140) in terms of the massive fermion field $\sigma(x)$:

$$\varphi^*(x)\, \Pi^+ \psi(x) \approx \Pi^+ \sigma(x) \frac{c}{\sqrt{2}}. \qquad (142)$$

Thus, we have shown how the gauge-invariant operators (132), (133), (135), and (142) can be expressed in terms of the field of the effective fermion σ.

We now consider the effects of the heavy particles. It was shown in the third part that in the absence of massless fermions the spectrum of scalar electrodynamics consists of the heavy scalar and pseudoscalar particles ρ and Σ (79). The inclusion of the massless fermions leads to the appearance of the light fermion σ (125).

The field σ is the main degree of freedom in the region of large distances. In order to take into account the heavy degrees of freedom Σ and ρ, it is necessary to add to the potential U_N^F (122) the potential U_N (76), which we calculated earlier.

Then, combining the expressions (77) and (123), we can conclude [see (83) and (124)] that the system is described by the fields σ, ρ, Σ, and the interaction Lagrangian (80), (125) having the form

$$\mathcal{L}^\pm = -m\sigma^+ \frac{1 \pm \gamma_5}{2} \sigma \exp\{\pm i\Sigma/F_\Sigma + \rho/F_\rho\}, \qquad (143)$$

where m is given by (127), and the constants $F_{\Sigma,\rho}$ by the expressions (81).

In the region of large distances, the heavy degrees of freedom and Σ and ρ are exponentially small. If we expand the exponential in (143) in a series in Σ and ρ, then the first term of the expansion reproduces the result (126), while the following terms describe the interaction of the light and heavy particles:

$$\mathcal{L}^+ + \mathcal{L}^- \simeq -m\sigma^+\sigma - \frac{m}{F_\rho}\rho\sigma^+\sigma + i\frac{m}{F_\Sigma}\Sigma\sigma^+\gamma_5\sigma.$$

Here, we should note that the potential U_N (76) at large distances has order $\mathcal{O}(\exp(-m_{v,s} r))$. On the other hand, in the potential U_N^F (122) we have ignored such terms. It is to be expected that allowance for them will lead to a renormalization of the constant $\beta_{v,s}$ in the expression (76) for U_N. This means that the constants $F_{\Sigma,\rho}$ in the expression (143) must be regarded as certain renormalized effective coupling constants.

4. THREE-DIMENSIONAL GEORGI–GLASHOW MODEL

This model is based on the gauge group $SU(2)$ and contains a triplet of gauge fields A_μ^a and a triplet of scalar fields φ^a. In three-dimensional Euclidean space–time, it is described by the Lagrangian

$$\mathcal{L} = \frac{1}{4g^2}\left(G_{\mu\nu}^a\right)^2 + \frac{1}{2}(D_\mu\varphi_a)^2 + \frac{\lambda}{4}(\varphi_a^2 - c^2)^2. \qquad (144)$$

If it is assumed that spontaneous symmetry breaking has concentrated the field φ_a near

$$\varphi_a = c\delta_a^3,$$

then the scalar field acquires mass $m_H = \sqrt{2\lambda c}$, the fields $A_\mu^{(1,2)}$ acquire mass $m_v = gc$, and the field $A_\mu^{(3)}$ remains massless.

However, it was shown in Ref. 20 that by virtue of tunneling effects the symmetry is restored ($\langle\varphi_a\rangle = 0$), the field $A_\mu^{(3)}$ acquires a small mass, and there is confinement in the model.

This means that the effective Lagrangian of the model can be expressed in colorless variables.

In the region of large distances there remains[20] the lightest degree of freedom, which is associated with $A_\mu^{(3)}$. If $m_H \ll m_v$, then there will also be manifested the light degree of freedom associated with the scalar field φ_a.[21]

If the model (144) is augmented with fermion fields, it acquires nontrivial chiral properties, which are manifested in the spontaneous breaking of the chiral invariance,[22] the role of the Goldstone particle being played by the degree of freedom associated with $A_\mu^{(3)}$.

A. Effective low-energy Lagrangian

The tunneling is described by the solutions of the classical equations of motion that follow from the Lagrangian (144). In this model, they are [20] identical to monopoles:[23]

$$\left.\begin{array}{l} \varphi^a = n^a c(1 + h(r)), \;\; A_\mu^a = \varepsilon_{a\mu\nu}\dfrac{n^\nu}{r}(1 + a(r)), \\ h^\nu = r^\nu/r, \;\; h(0) = a(0) = -1, \;\; a(\infty) = h(\infty) = 0. \end{array}\right\} \qquad (145)$$

Solutions of monopole type are characterized by topological number[3]

$$\left.\begin{array}{c} q = \int dx\, q(x), \;\; q(x) = \partial_\mu H_\mu; \\ H_\mu = \dfrac{g}{8\pi}\varepsilon_{\mu\nu\alpha}\hat{\phi}^a\left(G_{\nu\alpha}^a - \dfrac{1}{g}\varepsilon^{abc}D_\nu\hat{\phi}^b D_\alpha\hat{\phi}^c\right), \\ \hat{\phi}^a = \varphi^a/c. \end{array}\right\} \qquad (146)$$

The monopole and antimonopole have topological number q equal to $+1$ and -1, respectively. The classical action of the monopole is given[3] by

$$S_0 = \frac{m_v}{g^2}\varepsilon\left(\frac{\lambda}{g^2}\right),$$

where ε is a slowly varying function which satisfies[24] the condition $\varepsilon(0) = 4\pi$.

A superposition of widely spaced N^+ monopoles and N^- antimonopoles gives the action

$$S = (N^+ + N^-) S_0 + U_N \frac{4\pi}{g^2}, \qquad (147)$$

where the interaction potential U_N has the Coulomb form[20]

$$U_N = \sum_{i<j} U_{ij}(|x_i - x_j|), \qquad (148)$$

$$U_{ij} = q_i q_j / r. \qquad (149)$$

Here $q_i = +1$ and -1 are the topological numbers of the monopoles and antimonopoles, respectively.

The long-range Coulomb interaction appears in (149) because the component $A_\mu^{(3)}$ is massless.

It was shown in Ref. 21 that for small λ (low mass m_H)

the potential (149) acquires the Yukawa correction

$$U_{ij}=\frac{1}{r}(q_i q_j - e^{-m_H r}). \qquad (150)$$

The approximation of a rarefied instanton gas for the functional integral

$$Z = \int \exp\{-S(A, \varphi)\}\, dA d\varphi$$

was constructed in Ref. 20, where it was shown that in the approximation of an interacting instanton gas the integral has the form

$$Z = \sum_{N^+, N^-} \int \frac{k_0^{N^+ + N^-}}{N^+! N^-!} \exp\left(-\frac{4\pi}{g^2} U_N\right) \prod_{i=1}^{N^+} d^2 x_i^+ \prod_{i=1}^{N^-} d^3 x_i^-, \qquad (151)$$

where U_N is given by (148),

$$k_0 = m_v^3 \left(\frac{m_v}{g^2}\right)^{3/2} \alpha\left(\frac{\lambda}{g^2}\right) \exp(-S_0),$$

and α is some function that can be calculated.

Thus, Z is the grand partition function of a Coulomb gas with Yukawa attraction [see (150)].

One can show[20–22] that it can be represented as a functional integral over auxiliary fields,

$$Z = \int d\Sigma d\rho \exp\left\{\int \mathcal{L}_{\rm eff}(\Sigma, \rho)\, d^3x\right\}, \qquad (152)$$

where the effective Lagrangian has the form

$$\mathcal{L}_{\rm eff} = \frac{1}{2}(\partial\Sigma)^2 + \frac{1}{2}(\partial\rho)^2 + \frac{m_H^2 \rho^2}{2} + \mathcal{L}^+ + \mathcal{L}^-, \qquad (153)$$

$$\mathcal{L}^\pm = -k_0 \exp\left\{\pm i \frac{\Sigma}{f_\Sigma} + \frac{\rho}{f_\Sigma}\right\}, \qquad (154)$$

$$f_\Sigma = g/4\pi.$$

It follows from (153) and (154) that the potential part

$$U_{\rm eff} = \frac{1}{2} m_H^2 \rho^2 - 2k_0 \cos\frac{\Sigma}{f_\Sigma} \exp\left(\frac{\rho}{f_\Sigma}\right)$$

of the Lagrangian has a minimum at

$$\Sigma = 0, \quad \rho \simeq 2k_0/m_H^2 f_\Sigma.$$

Expanding the potential in a series in Σ and ρ near this minimum, we obtain the particle masses:

$$\left.\begin{array}{l} m_\Sigma^2 \approx 2k_0/f_\Sigma^2, \\ m_\rho^2 \simeq m_H^2 - 2k_0/f_\Sigma^2. \end{array}\right\} \qquad (155)$$

The field Σ acquires a mass[20] because of the Debye screening that occurs in a Coulomb plasma.

If we introduce fermions, the Lagrangian (144) must be augmented by a fermion Lagrangian, which we choose in the form (see Ref. 25)

$$\mathcal{L}_F = -\psi^+\left\{\gamma_\mu\left(\partial_\mu + ig\frac{\tau^a}{2}A_\mu^a\right) + iG\varphi^a\tau^a\beta - m_q\right\}\psi. \qquad (156)$$

Here, ψ is a four-component spinor in the x space and a doublet in the color space. The Dirac matrices have the form

$$\gamma_\mu = \begin{pmatrix} 0 & \tau_\mu \\ \tau_\mu & 0 \end{pmatrix}, \quad \beta = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \gamma_5 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad \mu = 1, 2, 3.$$

The constant G^2 of the Yukawa interaction is assumed to be large compared with λ and g^2.

In the chiral-invariant limit ($m_q \to 0$), the Lagrangian (156) in Minkowski space has the symmetry

$$\psi \to \exp(i\alpha\gamma_5)\psi, \quad \bar\psi \to \bar\psi \exp(i\alpha\gamma_5).$$

The presence of the small mass m_q lifts the degeneracy corresponding to this symmetry:

$$\partial_\mu j_\mu^5 = -2im_q \bar\psi\gamma_5\psi. \qquad (157)$$

The influence of the fermions on Z (151) in the approximation of a rarefied gas reduces to the substitution

$$k_0 \to k = k_0 \det(-\hat{D}_c - i\beta G\varphi_c^a\tau^a + m_q)/\det(-\hat\partial + m_q). \qquad (158)$$

Here, the subscript c means that in the calculation of the determinant it is necessary to take the single-instanton fields (145) as the fields φ^a and A_μ^a.

The fermions have a significant influence on the tunneling effects, since in the field of an instanton (respectively, anti-instanton) they have a zero mode of positive (respectively, negative) chirality:[26]

$$(\hat{D}_c + i\beta\varphi_c^a\tau^a)\psi_0 = 0.$$

For an instanton, ψ_0 has vanishing lower components, while the upper components are

$$\psi^{\alpha i} = \text{const}\exp\left\{-cGr - \int_0^r \left(cGh(r') + \frac{1+a(r')}{r'}\right) dr'\right\} \varepsilon^{\alpha i} \equiv \varepsilon^{\alpha i}(f(r)/\Lambda^2)^{1/2}, \qquad (159)$$

where the function $f(r)$ is normalized to unity:

$$\int f d^3r = 1, \qquad (160)$$

so that $\|\psi_0\|^2 = 1/\Lambda$, where Λ is a certain constant with the dimensions of mass, and it can be calculated by direct determination of the determinant (158).

For small m_q, it follows from (158) that

$$k = k_0 m_q \|\psi_0\|^2 \det{}'(D_c - i\beta G\varphi_c^a\tau^a)/\det(-\hat\partial), \qquad (161)$$

where det′ denotes the product of all eigenvalues of the operator except the zero mode.

We introduce the operators

$$j(x) = \psi(x)\psi(x), \; j_5(x) = i\bar\psi\gamma_5\psi(x).$$

To find the Green's functions of these operators, we use the method of functional derivatives. For this, we introduce in the Lagrangian (156) of the model the additional term

$$\{\psi^+(x)\, m(x)\, \psi(x) + i\psi^+(x)\, \gamma_5 m_5(x)\, \psi(x)\}. \qquad (162)$$

Then

$$\langle j(x_1), \ldots, j(x_n)\, j_5(y_1), \ldots, j_5(y_m)\rangle = (-1)^{n+m} Z^{-1}\delta^{n+m} Z/\delta m(x_1), \ldots, \delta m(x_n)\, \delta m_5(y_1), \ldots, \delta m_5(y_m). \qquad (163)$$

For small m and m_5, the influence of the term (162) on the functional Z (as when allowance is made for the term $m_q\psi^+\psi$) reduces to calculation of the matrix element $\langle\psi_0|m(x) + i\gamma_5 m_5(x)|\psi_0\rangle$.

It can be seen from (159) that for large G the function

$\psi_0(x)$ decreases rapidly at large distances. Therefore, the function $f(x)$ (160) can be approximated by a $\delta(x)$ function.

As a result, the entire effect reduces to the following substitution in the expression (151):

$$\left.\begin{aligned} k_0^{N^+} &\to k^{N^+} \prod_{i=1}^{N^+} \left\{1 + \frac{m(x_i^+) + im_5(x_i^+)}{m_q}\right\}, \\ k_0^{N^-} &\to k^{N^-} \prod_{i=1}^{N^-} \left\{1 + \frac{m(x_i^-) - im_5(x_i^-)}{m_q}\right\}. \end{aligned}\right\} \quad (164)$$

Further, we introduce in the partition function (151) additional inhomogeneous chemical potentials $\mu^+(x)$ and $u^-(x)$ for the particles of species $+$ and $-$, respectively.

It follows from the expressions (151) and (164) that the functional $Z(m, m_5, \mu^\pm)$ can be represented in the form of the functional integral (152), where the effective Lagrangian $\mathscr{L}_{\text{eff}}$ is given by (153). Only the interaction Lagrangian is changed:

$$\mathscr{L}^\pm = -k\left(1 + \frac{m(x) \pm im_5(x)}{m_q}\right) \times \exp\{\pm i\Sigma(x)/f_\Sigma + \rho(x)/f_\Sigma + \mu^\pm(x)\}. \quad (165)$$

It follows from (152), (163), and (165) that in the region of large distances the following operator identities hold:

$$\left.\begin{aligned} j(x) &\simeq -\frac{2k}{m_q}\cos\frac{\Sigma(x)}{f_\Sigma}\exp(\rho(x)/f_\Sigma), \\ j_5(x) &\simeq \frac{2k}{m_q}\sin\frac{\Sigma(x)}{f_\Sigma}\exp(\rho(x)/f_\Sigma). \end{aligned}\right\} \quad (166)$$

It follows from the first expression in (166) that there has been a spontaneous breaking of the γ_5 invariance in the system, manifested in the appearance of a quark condensate:

$$\langle j\rangle = \langle\bar{\psi}\psi\rangle = -\frac{2k}{m_q} = -\frac{m_\Sigma^2 f_\Sigma^2}{m_q}. \quad (167)$$

We note that the expression (167) is a well-known result in the method of current algebra.[11] It is a direct consequence of the spontaneous breaking of the γ_5 symmetry and of the Goldstone nature of the Σ boson. Indeed, in the chiral limit ($m_q \to 0$), the mass of the Σ boson, $m_\Sigma^2 = 2k\bar{f}_\Sigma^2$ (155), tends to zero, since the parameter k is proportional to the quark mass. This means that the Σ boson is a Goldstone particle.

Note that it is precisely the chiral-noninvariant mass term in the Lagrangian (156) that lifts the degeneracy corresponding to the γ_5 symmetry, and thus fixes the sign of the condensate: $\langle\bar{\psi}\psi\rangle < 0$ from $m_q > 0$.

If $m_q < 0$, then k changes sign, and the minimum of the effective potential will no longer be realized at $\Sigma = 0$, but at $\Sigma = \pi f_\Sigma$. It can be seen from (166) that in this case $|\bar{\psi}\psi|$ changes sign.

Since the fields σ and Σ are massive, they are exponentially small at large distances. In this limit, the expressions (166) and (157) take the form

$$\left.\begin{aligned} \bar{\psi}(x)\psi(x) &\simeq \frac{\langle\bar{\psi}\psi\rangle}{f_\Sigma}(f_\Sigma + \rho(x)); \\ -i\bar{\psi}(x)\gamma_5\psi(x) &\simeq \frac{\langle\bar{\psi}\psi\rangle}{f_\Sigma}\Sigma(x); \\ \partial_\mu j_\mu^5(x) &\simeq -2f_\Sigma m_\Sigma^2\Sigma(x). \end{aligned}\right\} \quad (168)$$

It can be seen from these expressions that the ρ particle can be regarded as a scalar meson (quark–antiquark bound state), while the Σ particle is a pseudoscalar meson.

The last of the relations (168) is the PCAC identity [analog of (40)].

Thus, the complicated structure of the vacuum in this model leads to spontaneous breaking of the chiral invariance, and the operator relations (166)–(168), which in the current-algebra method are hypothetical, are realized dynamically.

Note that besides the relations (166) there are other operator identities which relate the gauge-invariant operators constructed from the original fields A, φ, ψ to the fields Σ and ρ, which completely determine the system in the region of large distances.

We consider the operators $G^2(x) = (G_{\mu\nu}^a)^2$, $\varphi^2(x) = [\varphi_a(x)]^2$, and $q(x)$ [see (146)]. In the approximation of a rarefied instanton gas, they can be represented in the form

$$\left.\begin{aligned} q(x) &\simeq \int q_c(x-y)(\hat{n}^+(y) - \hat{n}^-(y))\,dy; \\ G^2(x) &\simeq \int G_c^2(x-y)(\hat{n}^+(y) + \hat{n}^-(y))\,dy; \\ \varphi^2(x) &\simeq c^2\left\{1 + 2\int h(x-y)(\hat{n}^+(y) + \hat{n}^-(y))\,dy\right\}, \end{aligned}\right\} \quad (169)$$

where the operators of the density of the instantons and anti-instantons have the form

$$\hat{n}^\pm(y) = \sum_{i=1}^{N^\pm}\delta(y - x_i^\pm). \quad (170)$$

The statistical means of these operators can be found by varying the functional (151) with respect to the inhomogeneous chemical potentials $\mu^\pm(x)$.

Using the representation of the functional Z in the form (152) and (153), we obtain

$$\langle n^\pm(x)\rangle = \frac{1}{Z}\frac{\delta Z}{\delta\mu^\pm(x)} = -\langle\mathscr{L}^\pm(x)\rangle \simeq k, \quad (171)$$

from which we can readily find the magnitudes of the condensates (169):

$$\left.\begin{aligned} &\langle q\rangle = 0, \ \langle G^2\rangle \simeq 2k\int G_c^2(x)\,dx; \\ &\langle\varphi^2\rangle \simeq c^2\left(1 + 4k\int h(x)\,dx\right). \end{aligned}\right\} \quad (172)$$

Similarly, we find the correlation functions

$$\begin{aligned} &\langle\hat{n}^\pm(x)\,\hat{n}^\pm(x')\rangle = -\delta(x-x')\langle\mathscr{L}^\pm\rangle + \langle\mathscr{L}^\pm(x)\,\mathscr{L}^\pm(x')\rangle, \\ &\langle\hat{n}^+(x)\,\hat{n}^-(x')\rangle \\ &= \langle\mathscr{L}^+(x)\,\mathscr{L}^-(x')\rangle. \end{aligned} \quad (173)$$

In the region of large distances, the expression (165) for $\mathscr{L}^\pm$ can, in the absence of sources ($m = m_5 = \mu^\pm = 0$), be expanded in a series with respect to Σ and ρ. In this approximation, (173) has the form

$$\begin{aligned} \langle\hat{n}^\pm(x)\,\hat{n}^\pm(x')\rangle &\simeq k\delta(x-x') \\ &+ k^2\{1 - \langle\Sigma(x)\,\Sigma(x')\rangle/f_\Sigma^2 + \langle\rho(x)\,\rho(x')\rangle/f_\Sigma^2\}; \\ \langle n^+(x)\,n^-(x')\rangle &\approx k^2\{1 + \langle\Sigma(x)\,\Sigma(x')\rangle/f_\Sigma^2 \\ &+ \langle\rho(x)\,\rho(x')\rangle/f_\Sigma^2\}. \end{aligned} \quad (174)$$

The propagators of the Σ and ρ particles in these expressions can be calculated by perturbation theory. In a first approximation, they can be assumed to be the propagators of free particles with masses m_Σ and m_ρ, respectively.

We can now calculate the Green's functions of the operators (169). For example, from the first expression in (169) and (174) we can obtain the two-point correlation function of the density of the topological charge in the momentum space:

$$\langle q(p)\, q(-p)\rangle = f_\Sigma^2 m_\Sigma^2 \tilde{q}_c^2 (p^2) \frac{p^2}{p^2+m_\Sigma^2}.$$

It can be seen from this that the Σ meson is manifested as a pole in this correlation function. The residue of the pole is

$$\langle 0 \mid q(0) \mid \Sigma\rangle = f_\Sigma m_\Sigma^2 \tilde{q}_c(-m_\Sigma^2). \tag{175}$$

Since the mass m_Σ is small, on the basis of the obvious identity $\tilde{q}_c(0) = 1$ the last factor in (175) can be set equal to unity.

It follows from (175) that near the pole the following approximate operator equation holds:

$$q(x) \simeq f_\Sigma m_\Sigma^2 \Sigma(x). \tag{176}$$

This means that the Σ particle can be regarded as a Higgs gluon (146) state. Note the similarity between (175) and the corresponding exact expression (45) in the Schwinger model.

Thus, on the basis of allowance for the tunneling effects this model can be formulated in the region of large distances in terms of the colorless variables (152). The effects of the complicated vacuum structure also lead to spontaneous breaking of the chiral invariance, as a result of which there is a dynamical realization of the well-known nonlinear σ model, its parameters being calculated exactly. It is also possible to establish the approximate operator relations (166) and (176) between the colorless variables and the original variables.

5. COMPLICATED VACUUM STRUCTURE AND EFFECTIVE LAGRANGIAN OF QCD

In the models considered above, the gauge fields fall off exponentially at large distances. This means that the gauge charge is screened completely symmetrically.

However, there may still be a possibility of asymmetric screening if a particle can retain a dipole or magnetic moment. Then the fields will fall off as a power, and therefore, despite the charge screening, there remains in the symmetry a global symmetry group associated with rotations of the dipole or magnetic moments.

We shall show that such a possibility is realized in QCD. It is well known that the vacuum in QCD has a complicated structure,[1,2] manifested in the fact that besides the trivial minimum $\mathbf{A} = 0$ of the energy there are other minima:

$$\mathbf{A}_{(n)} = \frac{1}{i} U^n(\mathbf{x})\, dU^{-n}(\mathbf{x}),$$

where n is an integer, and $U(\mathbf{x})$ is a certain color matrix.[1,2]

The existence of instanton configurations joining different minima means that the system cannot remain near one of the minima ($A = 0$) but is uniformly distributed over all the minima at once. In such a case, a charged quark cannot move, since its wave function, which is a superposition of the states $U^n(\mathbf{x})q(\mathbf{x})$, does not have a definite phase, and hence, momentum. But if the quark is decolored, it ceases to feel the phase of the matrix $U^n(x)$, and such a state can have a definite momentum.

Let us consider the process of tunneling from the minimum $n = 0$ to the minimum $n = 1$ in more detail. The instanton that joins these minima has nonzero classical fields **A**, **E**, and **H** below the potential barrier.

Since the gluons are charged, this means that below the barrier there exists a classical density of the gluon color charge: $\rho_G^a = \varepsilon^{abc} A^b E^c$. This changes the color charge density of a tunneling fermion.

If on penetration below the barrier the charge density of a fermion was $\rho^a(x)$, on emergence from below the barrier it will be $V^{ab}(x)\rho^b(x)$, where the color matrix V_{ab} is related to the matrix U by

$$V_{ab} = \frac{1}{2} \mathrm{Tr}\, (\tau^a U \tau^b U^+).$$

Thus, the density vector $\rho^a(x)$ undergoes a rotation in the color space. If we now average ρ^a over all minima at once, the answer will be "zero." This is the case because the contributions from the various instantons responsible for rotations of the vector ρ^a will be added randomly and compensate each other.

Thus, in the system there must remain only collective excitations corresponding to uncharged fermions.

To formalize the above, we consider the approximation of an instanton gas.

A. Approximation of a rarefied instanton gas

For simplicity, we consider an $SU(2)$ gauge theory with one quark species. In Euclidean space, it is described by the action

$$S = \int dx \left\{ \frac{1}{2} \mathrm{Tr}\, G_{\mu\nu}^2 - q^+ (\hat{D} - m_q)\, q - i\theta q_T \right\}. \tag{177}$$

Here

$$\hat{D} = \gamma_\mu (\partial_\mu - igA_\mu), \quad A_\mu = \frac{\tau^a}{2} A_\mu^a,$$

$$\gamma_\mu = \begin{pmatrix} 0 & \alpha_\mu^- \\ \alpha_\mu^+ & 0 \end{pmatrix}, \quad \alpha_\mu^\pm = (\pm i\boldsymbol{\tau}, 1),$$

and $q_T(x) = g^2 \mathrm{Tr} G_{\mu\nu} G^*_{\mu\nu}/16\pi^2$ is the density of the topological number.

The vacuum–vacuum transition amplitude can be represented in the form of the functional integral

$$Z = \int \exp(-S_A) \det(\hat{D}(A) - m_q)\, dA. \tag{178}$$

In this expression, S_A is the purely gluonic part of the action (177).

The integral (178) can be estimated by the semiclassical method. As basic configurations possessing nonzero topological number, one can take a composition of instantons and anti-instantons.[27]

This composition takes its simplest form in the singular gauge:[28]

$$A_\mu(N^+, N^-) = -\sum_{+}^{N^+} \frac{m_{\mu\nu}^{+} (x-x_+)^\nu}{((x-x_+)^2+\rho_+^2)(x-x_+)^2} - \sum_{-}^{N^-} \frac{m_{\mu\nu}^{-} (x-x_-)^\nu}{((x-x_-)^2+p_-^2)(x-x_-)^2}. \tag{179}$$

Here, $x_\pm$, $\rho_\pm$, $m^{\pm}_{\mu\nu}$ are the position, size, and four-dimensional magnetic moments of the instantons:

$$\left.\begin{aligned} m^{\pm}_{\mu\nu} &= -\frac{2}{g} S(\omega_\pm)\,\alpha^{\mp}_{\mu\nu} S^{+}(\omega_\pm)\,\rho^2_\pm; \\ \alpha^{\pm}_{\mu\nu} &= \frac{1}{4i}(\alpha^{\pm}_{\mu}\alpha^{\mp}_{\nu} - \alpha^{\pm}_{\nu}\alpha^{\mp}_{\mu}); \\ S(\omega_\pm) &= \exp\left(i\,\frac{\tau^a}{2}\,\omega^a_\pm\right). \end{aligned}\right\} \qquad (180)$$

When the instantons are sufficiently far apart, the configuration (179) has the action

$$S_A = \frac{8\pi^2}{g^2}(N^+ + N^-) - i\theta(N^+ - N^-) + U^G_N, \qquad (181)$$

where the dipole–dipole interaction potential can be obtained by means of (74) and has the form[28,29]

$$U^G_N \simeq \sum_{+-}^{N^+N^-} 8\pi^4\,\mathrm{Tr}\,(m^{+}_{\mu\nu} m^{-}_{\mu\alpha})\,\partial_\nu\partial_\alpha D(x^+ - x^-). \qquad (182)$$

The function D is identical to the propagator of a massless scalar particle in Euclidean space:

$$D(x) = 1/4\pi^2 x^2. \qquad (183)$$

In the literature there are disagreements about the sign of the potential U^G_N. Here, we use (182), the result of Ref. 29. One might think that the sign of the potential is unimportant, since it can be changed by changing, for example, the sign of the magnetic moment $m^{-}_{\mu\nu}$. However, in this case the reaction of the system to an external field $H_{\mu\nu}$ will be proportional to $H^{*}_{\mu\nu} = \varepsilon_{\mu\nu\alpha\beta} H^{\alpha\beta}/2$, and not to $H_{\mu\nu}$, a result that is needed for interpretation of the system as a paramagnetic medium in four dimensions.

The potential (182) is formally identical to the interaction energy of the magnetic moments ($m^{\pm}_{\mu\nu}$) of particles placed in four-dimensional space. However, whereas in a paramagnetic medium an individual magnetic moment attempts to align the nearest neighbors in the direction of its own orientation (Fig. 2), the distant neighbors in the opposite direction, in the case considered here the situation is reversed. The sign of the potential U^G_N is such[29] that the magnetic moment attempts to align the distant neighbors in the direction of its own orientation. Because of the long range of the interaction ($U^G_N \sim 1/R^4$), this may lead to instability if the interaction is sufficiently strong. In what follows, we shall find a condition under which this can occur.

After substitution of (179) and (181) in (178) and calculation of the quantum fluctuations around the classical configuration (179), the amplitude Z takes the form

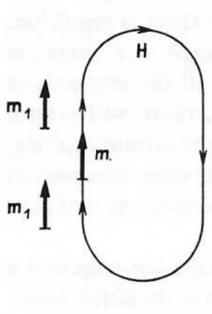

FIG. 2. A paramagnet: **m** is the magnetic moment **H** is the magnetic field, $\mathbf{m}_1$ is a nearest neighbor, and $\mathbf{m}_2$ is a distant neighbor.

$$Z = Z_{\mathrm{p.th}} \sum_{N^+N^-} \int \exp\{-U^G_N + i\theta(N^+ - N^-)\} \det B \, \frac{1}{N^+!\,N^-!} \times \prod_{+}^{N^+} d\Gamma^{+}_0 \prod_{-}^{N^-} d\Gamma^{-}_0. \qquad (184)$$

Here, $d\Gamma^{\pm}_0 = d^4x_\pm\, d\omega^\pm\, dn_0(\rho^\pm)$, $d\omega$ is the invariant measure on the group $SU(2)$, and dn_0 is the size distribution function of the instantons. For the group $SU(N)$ with N_f quarks it has, in the single-loop approximation and under the assumption $m_q\rho \ll 1$, the form[30]

$$dn_0(\rho) \simeq \frac{4{,}6\exp(-1{,}68N)}{\pi^2 (N-1)!\,(N-2)!}\left(\frac{8\pi^2}{g^2}\right)^{2N}\exp\left(-\frac{8\pi^2}{g^2(\rho)}\right) \times \prod_{i=1}^{N_f}(1{,}34\rho)\,\frac{d\rho}{\rho^5}.$$

The appearance of the matrix B in the expression (184) is associated with the calculation of the fermion determinant (178) in the many-instanton field (179). These calculations were made in Ref. 31. The result is

$$B_{ij} = -m_q\delta_{ij} - i a_{ji}. \qquad (185)$$

The indices i and j take values from 1 to $N^+ + N^-$, and the matrix a_{ij} has nonzero elements between instanton and anti-instanton states:

$$\overset{*}{a}_{+-} = a_{-+} = 4\pi^2\, i\rho_+\rho_- \times \frac{\partial}{\partial x^{-}_{\alpha}} D(x_+ - x_-)\,\mathrm{Tr}\,(\alpha^{+}_{\alpha} S^{+}(\omega_+)\, S(\omega_-)). \qquad (186)$$

The region of applicability of this formula is

$$\rho \ll |x_+ - x_-| \ll 1/m_q.$$

We shall show that the partition function (184) can be represented as a functional integral with respect to the collective variables X_μ and σ, where X_μ is a triplet of vector fields, and σ a doublet of fermion fields. There is the following representation for the potential U^G_N.

$$\begin{aligned} e^{-U^G_N} = \det{}^{-1/2}(\partial^2) \int dX_\mu \exp\Big\{-\frac{1}{2}\int (\mathrm{Tr}\,X^2_{\mu\nu} &+ 2\,\mathrm{Tr}\,(\partial_\mu X_\mu)^2\,dx\Big\} \prod_{+}^{N^+} \exp\{2\pi^2\,\mathrm{Tr}\,(m^{+}_{\mu\nu} X_{\mu\nu}(x_+))\} \\ &\times \prod_{-}^{N^-} \exp\{2\pi^2\,\mathrm{Tr}\,(m^{-}_{\mu\nu} X_{\mu\nu}(x_-))\}, \\ X_{\mu\nu} &= \partial_\mu X_\nu - \partial_\nu X_\mu. \end{aligned} \qquad (187)$$

Here $X_\mu = X^i_\mu \tau^i/2$ is a triplet vector field. This formula can be readily verified by noting that the products ($\Pi_+ \Pi_-$) in it can be rewritten in the form

$$\exp\left\{2\int \mathrm{Tr}\,(X_\mu J_\mu)\,dx\right\},\quad J_\mu = -\sum_{+} 2\pi^2 m^{+}_{\mu\nu}\frac{\partial}{\partial x^{+}_{\nu}}\delta(x - x^+) - \sum_{-} 2\pi^2 m^{-}_{\mu\nu}\frac{\partial}{\partial x^{-}_{\nu}}\delta(x - x^-).$$

After this, the integral on the right-hand side of (187) becomes manifestly Gaussian and can be calculated. Direct calculation shows the validity of (187).

A second source of interaction in the partition function (184)—the determinant of the matrix B—can be represented in the form of a functional integral over a certain fermion field $\sigma(x)$.

Indeed, let us introduce the $N^+ + N^-$ Grassmann variables

$$\Omega = (u_1, \ldots, u_{N+}, v_1, \ldots, v_{N-}).$$

Then we have the identity

$$\det B = \int d\Omega^+ d\Omega \exp\Big\{-\sum_{+} m_q u_+^+ u_+ - m_q \sum_{-} v_-^+ v_- - \mathrm{i} \sum_{+-} (u_+^+ v_- a_{-+} + v_-^+ u_+ a_{+-})\Big\}. \tag{188}$$

We introduce a massless fermion field $\sigma_a(x)$, which is a doublet $(a = 1,2)$. It is convenient to separate from it the right (σ_+) and left (σ_-) components:

$$\sigma^a = \begin{pmatrix} \sigma_+^{a\alpha} \\ \sigma_-^{a\alpha} \end{pmatrix}.$$

The Lorentz index α takes values from 1 to 2.

We have the identity

$$\exp\Big\{-\mathrm{i}\sum_{+-} (u_+^+ v_- a_{-+} + v_-^+ u_+ a_{+-})\Big\}$$

$$= \det{}^-(\hat\partial) \int d\sigma^+ d\sigma \exp\Big\{\int dx \sigma^+ \hat\partial\sigma + \sum_{+} 2\pi\rho_+ [u_+^+ (S^T(\omega_+)\tau_2)_{\beta a}\sigma_{+a\beta}(x_+) - u_+ \sigma_{+\beta a}^+ (x_+)(\tau_2 S^*(\omega_+))_{a\beta}] + \sum_{-} 2\pi\rho_- [v_-^+ (S^T(\omega_-)\tau_2)_{\beta a}\sigma_{-a\beta}(x_-) - v_- \sigma_{-\beta a}^+ (x_-)(\tau_2 S^*(\omega_-))_{a\beta}]\Big\}. \tag{189}$$

This can be readily verified by direct calculation of the Gaussian integral over the fermion fields on its right-hand side.

Substituting (189) in (188) and integrating over the variables Ω and Ω^+, we obtain

$$\det B = \det{}^{-1}(\hat\partial) \int d\sigma^+ d\sigma \exp\Big(\int dx \sigma^+ \hat\partial\sigma\Big) \times \prod_{+}^{N^+} \{-m_q + (2\pi\rho_+)^2 \times \mathrm{Tr}(\sigma_+^+(x_+)\tau_2 S^*(\omega_+))\,\mathrm{Tr}(S^T(\omega_+)\tau_2\sigma_+ \times (x_+))\} \prod_{-}^{N^-} \{-m_q + (2\pi\rho_-)^2\,\mathrm{Tr}(\sigma_-^+(x_-) \times \tau_2 S^*(\omega_-))\,\mathrm{Tr}(S^T(\omega_-)\tau_2\sigma_-(x_-))\}. \tag{190}$$

In the evaluation of Tr in this expression, the spinor $\sigma_{\pm a\beta}$ is to be understood as a 2×2 matrix.

Substituting (187) and (190) in (184) and summing over $N^\pm$, we obtain

$$Z = \int dX d\sigma^+ d\sigma \exp\Big(\int \mathscr{L}_{\rm eff}\, dx\Big). \tag{191}$$

Here, the effective Lagrangian is given by the expressions

$$\mathscr{L}_{\rm eff} = -\frac{1}{2}\mathrm{Tr}\, X_{\mu\nu}^2 + \sigma^+\hat\partial\sigma + \mathscr{L}^+ + \mathscr{L}^-,$$

$$\mathscr{L}^\pm = e^{\pm i\theta}\int d\omega\, dn_0 \exp\{2\pi^2\,\mathrm{Tr}(m_{\mu\nu}^\pm X_{\mu\nu})\}[-m_q + (2\pi\rho)^2\,\mathrm{Tr}(\sigma_\pm^+\tau_2 S^*(\omega))\,\mathrm{Tr}(S^T(\omega))\tau_2\sigma_\pm]. \tag{192}$$

Using this formula, we can verify the invariance of the $SU(2)$ Lagrangian with respect to the global $SU(2)$ group:

$$X'_{\mu\nu} = U X_{\mu\nu} U^+, \quad \sigma' = U\sigma, \quad U^+U = 1.$$

Another property of the Lagrangian is the absence of invariance with respect to γ_5 transformations:

$$\sigma' = \exp(\mathrm{i}\alpha\gamma_5)\,\sigma, \quad \bar\sigma' = \bar\sigma\exp(\mathrm{i}\alpha\gamma_5). \tag{193}$$

It is assumed that this transformation is made with the Lagrangian (192), translated to Minkowski space in accordance with the rules

$$(X_{4i}, X_{ij}) \to (\mathrm{i}X_{0i}, -X_{ij}),\ \sigma \to \sigma,\ \sigma^+ \to \bar\sigma.$$

The entire dependence of the Lagrangian (192) on the parameter θ is contained in the first term, which is proportional to the quark mass m_q. The θ dependence of the remaining terms of the Lagrangian is fictitious. It can be eliminated by a redefinition of the fields σ in accordance with (193). This result can be understood by recalling that in the chiral limit $(m_q \to 0)$ there can be no dependence on θ.

In what follows, we set $\theta = \pi$. To elucidate the physical content of the theory (192), we consider the region of large distances, where in the absence of condensates of the field $X_{\mu\nu}(x)$ it decreases at least as $1/x^2$.

We expand in a series with respect to $X_{\mu\nu}$ the exponential in (192), retaining only the first few terms in the expansion, and we integrate over ω in the standard manner:

$$\mathscr{L}_{\rm eff} \simeq -\frac{1}{2}\mathrm{Tr}\, X_{\mu\nu}^2(1-\chi) + \mathrm{i}\bar\sigma\hat\partial\sigma - m\bar\sigma\sigma + \mu\bar\sigma\sigma_{\mu\nu}X_{\mu\nu}\sigma + 2m_q n_0. \tag{194}$$

We have here introduced the notation

$$\left.\begin{aligned} &n_0 = \int dn_0,\quad m = \frac{1}{2}(2\pi\bar\rho)^2 n_0,\quad \mu = -\frac{1}{3}(2\pi\bar\rho)^4 n_0/g, \\ &\chi = \frac{2}{3g^2} m_q n_0 (2\pi\bar\rho)^4,\quad \sigma_{\mu\nu} = (\gamma_\mu\gamma_\nu - \gamma_\nu\gamma_\mu)/4\mathrm{i}, \end{aligned}\right\} \tag{195}$$

where $\bar\rho$ is the mean size of the instantons, defined below. It can be seen from the explicit form of the function $dn_0/d\rho$ that the integral over $dn_0(\rho)$ diverges at the upper limit (large ρ). However, there are grounds for believing[32,33] that at short distances between the instantons there is repulsion, which leads to the appearance of a factor $\exp(-\rho^2\,\mathrm{const})$ in the expression for $dn_0(\rho)$. In this case, all the integrals in (195) will be well defined. In addition, there will exist a mean size $\bar\rho$ of the instantons, defined by the formula $\int\rho^m dn_0 \approx \bar\rho^m n_0$. We shall assume that there exists some mechanism for suppressing large instanton sizes and treat n_0 and $\bar\rho$ as semiphenomenological parameters.

It follows from (194) that the σ quark has acquired a dynamical mass m, which does not vanish in the chiral limit.

In the language of the statistical physics of an instanton gas, this result can be formulated as follows: In the system

(184) there was an attraction between the particles described by the function det B, which had a long range; there then occurred a phenomenon analogous to the screening of electric charge in a Coulomb plasma, and this interaction became one with a short range of the order of $1/m$, where m is the dynamical mass.

Another feature of the Lagrangian (194) is the presence of the chromomagnetic interaction with coupling constant μ. We showed earlier that the Lagrangian is invariant with respect to global $SU(2)$ transformations in the space of the global color, in which σ is a doublet, and X_μ is a triplet. In this space, the σ quarks possess chromomagnetic moments equal to μ (195), and these moments interact through the field X_μ. Thus, the Lagrangian describes Abelian $U^3(1)$ magnetodynamics.

It was noted earlier that in the statistical system (184) an instability may arise. It can be seen from (194) that it arises for $\chi > 1$, when the renormalized field X_μ has negative metric, and a nonvanishing vacuum expectation value of this field develops in the system.

The parameter χ is proportional to m_q and vanishes in the chiral limit. However, once the σ quark has acquired the mass m the condensate $\langle\bar{\sigma}\sigma\rangle$ is no longer zero, and this leads to a renormalization of the parameter m_q and χ. Indeed, we write

$$\bar{\sigma}_{+\beta a}\sigma_{\pm b\alpha} = \frac{1}{8}\,\delta^{ab}\delta_{\alpha\beta}\langle\bar{\sigma}\sigma\rangle + :\bar{\sigma}_{\pm\beta a}\sigma_{\pm ba}:, \tag{196}$$

$$\langle\bar{\sigma}\sigma\rangle = -\frac{8m}{(2\pi)^4}\int\frac{d^4p}{p^2+m^2}. \tag{197}$$

Substituting (196) in the expression (192) for the effective Lagrangian, we obtain the renormalization prescription

$$m_q \to \bar{m}_q = m_q - \frac{1}{4}(2\pi\bar{\rho})^2\langle\bar{\sigma}\sigma\rangle. \tag{198}$$

This effect is related to the fact that the normal product in (196) is defined relative to the massive field $\sigma(x)$, i.e., it takes into account the circumstance that the part of the interaction Lagrangian $\mathscr{L}^+ + \mathscr{L}^-$ that gives mass to the field $m\bar{\sigma}\sigma$, relates to the free part of the Lagrangian (192).

As a result, $\chi(m_q)$ (195) is replaced by $\bar{\chi} = \chi(\bar{m}_q)$.

We now calculate the quark condensate $\langle\bar{q}q\rangle$ in accordance with the formula

$$\langle\bar{q}q\rangle = -\left(Z\int d^4x\right)^{-1} dZ/dm_q.$$

Using for Z the representation (191), (194), we obtain

$$\langle\bar{q}q\rangle = -2n_0 - \frac{n_0}{3g^2}(2\pi\bar{\rho})^4\langle\mathrm{Tr}\,X^2_{\mu\nu}\rangle \simeq -2n_0. \tag{199}$$

We have here assumed the absence of condensates of the field $X_{\mu\nu}$ (it is assumed that $\bar{\chi} < 1$).

Thus, the existence of the dynamical mass of the field σ (the condensate $\langle\bar{\sigma}\sigma\rangle$) has had no influence at all on $\langle\bar{q}q\rangle$. The absence of correlation between these quantities would be impossible if there were a symmetry associated with γ_5 rotations. But it is anomalously broken,

$$\partial_\mu(\bar{q}\gamma_\mu\gamma_5 q) = \frac{g^2}{8\pi^2}\,\mathrm{Tr}\,G_{\mu\nu}G^*_{\mu\nu} + 2im_q\bar{q}\gamma_5 q, \tag{200}$$

for the field $q(x)$ and manifestly (the nonvanishing mass m in the chiral limit) for the field σ.

From the expressions (195) and (199) we obtain the following representation for the dynamical mass:

$$m = -\frac{1}{4}(2\pi\bar{\rho})^2\langle\bar{q}q\rangle. \tag{201}$$

We note the similarity between this expression and (198); however, $\bar{m}_q$, in contrast to the dynamical mass m, is only a parameter of the Lagrangian (194) and does not have the significance of a pole of any propagator.

Note that with allowance for the renormalization of X_μ by the factor $\sqrt{1-\bar{\chi}}$ the magnetic moment of the σ quark, $\bar{\mu} = \mu/\sqrt{1-\bar{\chi}}$, can be expressed in the form

$$\bar{\mu} = -\frac{\bar{g}}{2m_q}\,\bar{\chi}.$$

We have here used the expressions of (195) and introduced the notation $\bar{g} = g/\sqrt{1-\bar{\chi}}$. Thus, the magnetic moment of the σ quark is equal to the renormalized magnetic moment of the q quark.

B. Approximate operator equations

Here we establish a connection between the gauge-invariant operators constructed from the fields q, A_μ and the fields $\sigma(x)$, $X_{\mu\nu}(x)$.

Consider the operator $G^2_{\mu\nu}(x)$. In the field of one instanton it has the value

$$\frac{g^2}{16\pi^2}\,\mathrm{Tr}\,G^2(x) = f(x,\rho) = \frac{6}{\pi^2(x^2+\rho^2)^4}. \tag{202}$$

In the many-instanton approximation (179),

$$\frac{g^2}{16\pi^2}\,\mathrm{Tr}\,G^2(x) \simeq \sum_{+}^{N^+} f(x-x_+,\rho_+) + \sum_{-}^{N^-} f(x-x_-,\rho_-).$$

By means of the density operator

$$d\hat{n}^\pm/d\rho = \sum_{+-}^{N^{+-}}\delta(x-x_\pm)\,\delta(\rho-\rho_\pm)$$

this expression can be rewritten as

$$\frac{g^2}{16\pi^2}\,\mathrm{Tr}\,G^2(x) \simeq \int f(x-z,\rho)\left\{\frac{d\hat{n}^+(z,\rho)}{d\rho} + \frac{d\hat{n}^-(z,\rho)}{d\rho}\right\}dz\,d\rho. \tag{203}$$

Since the partition function (184) is equivalent to the effective theory (191), all the statistical operators can be expressed in terms of σ and X. Using the method of external sources employed above to investigate the two-and three-dimensional models, we can obtain an approximate operator equation valid in the region of large distances:[29]

$$\frac{d\hat{n}^\pm}{d\rho} \approx l^\pm(z,\rho), \tag{204}$$

where $l^\pm$ is the Lagrangian density $\mathscr{L}^\pm$ (192) with respect to the variables ρ and z:

$$\mathscr{L}(X(z),\sigma(z)) = \int d\rho\, l^\pm(X(z),\sigma(z),\rho). \tag{205}$$

As in the derivation of (194), we expand $\mathscr{L}^\pm$ (192) in a series with respect to X. Then from (192) and (205) we obtain

$$l^\pm \simeq \frac{dn_0}{d\rho}\Big\{m_q - (2\pi\rho)^2\sigma^+\frac{1\pm\gamma_5}{2}\sigma - \frac{(2\pi\rho)^4}{3g}\sigma^+X_{\mu\nu}\sigma_{\mu\nu}\frac{1\pm\gamma_5}{2}\sigma + \frac{1}{3}\frac{(2\pi\rho)^4}{4g^2}m_q\,\mathrm{Tr}\,(X_{\mu\nu}\pm X^*_{\mu\nu})^2 + \dots\Big\}. \tag{206}$$

From (203), (204), and (206) we obtain the approximate operator equation

$$\frac{g^2}{16\pi^2} G^2(x) \simeq \int dz\, f(x-z,\ \bar{\rho}) \times \left\{ -m\sigma^+(z)\,\sigma(z) + \frac{1}{2}\chi \operatorname{Tr} X^2_{\mu\nu}(z) \ \ldots \right\} + 2m_q n_0. \tag{207}$$

We have retained only the terms bilinear in the fields and have omitted all the higher terms. In obtaining the last term in (207), we also used the realtion $\int f(x,\rho)dx = 1$, which follows from the definition (202).

Similarly, we obtain

$$\frac{g^2}{16\pi^2} \operatorname{Tr} GG^*(x) \simeq \int f(x-z,\ \rho)\{l^+(z,\ \rho) - l^-(z,\ \rho)\}\, d\rho\, dz \simeq \int f(x-z,\ \rho)\left\{ -m\sigma^+\gamma_5\sigma(z) - \frac{\chi}{2}\operatorname{Tr} X_{\mu\nu}X^*_{\mu\nu}(z) + \ldots \right\} dz. \tag{208}$$

It follows from (207) and (208) that in the gluon channel there is a large admixture of quark–antiquark pairs of σ quarks. This has the consequence that the gluon condensate is intimately related to the condensate (197) of the σ quark. Indeed, it follows from (207) that

$$\left\langle \frac{g^2}{16\pi^2} G^2 \right\rangle \simeq -m\langle\bar{\sigma}\sigma\rangle + 2m_q n_0 = -m\langle\bar{\sigma}\sigma\rangle - m_q\langle\bar{q}q\rangle = -\bar{m}_q\langle\bar{q}q\rangle. \tag{209}$$

As can be seen from what has been written above, there exist several forms of expression of the same effect of the dynamical mass on the gluon condensate.

The appearance of the dynamical mass also clarifies the significance of the axial anomaly (200).

In the chiral limit there remains on the right-hand side of Eq. (200) only the interpolating operator of the pseudoscalar gluonium, which can be approximated by a quark operator by means of (208).

Then (200) takes the form

$$\partial_\mu(\bar{q}\gamma_\mu\gamma_5 q) \simeq -2im \int f(x-z,\ \rho)\,\sigma^+\gamma_5\sigma(z)\, dz.$$

The appearance of the imaginary unit in this expression is explained by the rules of passage to Euclidean space. Thus, in the language of collective variables the γ_5 anomaly is interpreted as acquisition by the collective quark of dynamical mass.

Another feature of the relations (207) and(208) is the presence of a nonlocal connection between the original and the collective variables. The reason for this is that the collective variables σ, X describe the motion of the centers of these instantons, and therefore these fields are local, whereas an instanton as a whole is a nonlocal object, so that the connection between the original and the collective variables must be nonlocal. Therefore, in the original variables the σ quark will appear as an extended object of size $\bar{\rho}$ and possess a nontrivial distribution of the color charge.

It follows from (194) that with respect to the fields the σ quark possesses a color magnetic moment $\bar{\mu}$, and its color charge is equal to zero, i.e., the Lagrangian (194) describes pure magnetodynamics.

It is worth studying the color properties of the σ quark with respect to the initial color concept.

The conserved current of the color charge, expressed in terms of $q(x)$ and $A(x)$, is related to $\sigma_{\mu\nu}$ by the Maxwell equation

$$C^a_\mu(x) = \partial_\nu G^a_{\nu\mu}(x). \tag{210}$$

In the field of a single instanton,

$$\partial_\nu G^a_{\nu\mu} = m^a_{\nu\mu}\pi^2\,\partial_\nu F(x,\ \rho), \tag{211}$$

$$F(x,\ \rho) = \frac{4}{\pi^2}\left(\frac{1}{\rho^4}\ln\frac{x^2+\rho^2}{x^2} - \frac{1}{\rho^2(\rho^2+x^2)} - \frac{1}{2(x^2+\rho^2)^2} \right). \tag{212}$$

In the many-instanton sector (179), we obtain

$$C^a_\mu(x) \simeq \int \partial_\nu F(x-z)\,\pi^2 \frac{d\hat{m}^a_{\nu\mu}(z)}{d\rho}\, dz\, d\rho, \tag{213}$$

where the operator of the magnetization density

$$\frac{d}{d\rho}\hat{m}^a_{\nu\mu}(z,\ \rho) = \sum_{+}^{N^+} m^+_{\nu\mu,\,a}\,\delta(z-x_+)\,\delta(\rho-\rho_+) + \sum_{-}^{N^-} m^-_{\nu\mu,\,a}\delta(z-x_-)\,\delta(\rho-\rho_-),$$

and the matrices $m^\pm_{\mu\nu,a}$ are given by the expression (180). Using the method of external sources, as in the case of (94) and (166), we can obtain[29] the approximate operator equation

$$\pi^2 \frac{d}{d\rho}\hat{m}^a_{\nu\mu} \simeq \frac{\partial}{\partial X^a_{\nu\mu}}(l^+ + l^-).$$

Using for $l^\pm$ the expression (206), and taking into account the renormalization of χ, we obtain from (213)

$$C^a_\mu(x) \simeq \int \partial_\nu F(x-z,\ \bar{\rho}) \left\{ \frac{\bar{\chi}}{2} X^a_{\nu\mu} + \mu\sigma^+\sigma_{\nu\mu}\frac{\tau^a}{2}\sigma \right\} dz.$$

Integrating this equation by parts and using the equations of motion that follow from the Lagrangian (194),

$$\partial_\mu X^a_{\mu\nu}(1-\bar{\chi}) \simeq \partial_\mu\left(2\mu\sigma^+\sigma_{\mu\nu}\frac{\tau^a}{2}\sigma\right),$$

we finally obtain

$$C^a_\mu(x) \simeq \int F(x-z,\ \bar{\rho})\frac{\bar{\mu}}{\sqrt{1-\bar{\chi}}}\,\partial_\nu\left(\sigma^+(z)\,\sigma_{\nu\mu}\frac{\tau^a}{2}\sigma(z)\right) dz.$$

We calculate the color form factor of the σ quark in the Euclidean domain with respect to the momentum transfer $q = p_1 - p_2$. Here, p_1 and p_2 are the momenta of the initial and final states of the quark, respectively. From the upper equation we have

$$\langle p_1|C^a_\mu(0)|p_2\rangle = -i\frac{\bar{\mu}}{\sqrt{1-\bar{\chi}}}\tilde{F}(q^2)\, q_\nu\sigma^+_{p_1}\sigma_{\mu\nu}\frac{\tau^a}{2}\sigma_{p_2}.$$

Here $\tilde{F}(q^2)$ is the Fourier transform of the function $F(x)$ (212). It is readily verified that $\tilde{F}(0) = 1$.

In the Breit system, the chromoelectric (F_e) and chromomagnetic (F_m) form factors have the form

$$F_e = -q^2\tilde{F}(q^2)\,\bar{\mu}/4m\sqrt{1-\bar{\chi}}, \quad F_m = \bar{\mu}\tilde{F}(q^2)/2\sqrt{1-\bar{\chi}}. \tag{214}$$

It follows from these expressions that the color charge of the σ quark is equal to zero, while the formal chromomagnetic moment is $F_m(0) = \bar{\mu}/2\sqrt{1-\bar{\chi}}$.

We note also that the formal chromomagnetic moment differs from the dynamical moment $\bar{\mu}$, which is a coupling constant in the effective Lagrangian (194).

Hitherto, we have considered a theory with one quark. However, the results can be generalized to a theory with several quarks. For example, in a theory with two light quarks u and d the effective Lagrangian will also contain two quarks σ_1 and σ_2, and

$$\mathcal{L}^{\pm} = e^{\pm i\theta} \int dn_0\, d\omega \exp\{2\pi^2 \operatorname{Tr}(m^{\pm}_{\mu\nu} X_{\mu\nu})\} [-m_u + (2\pi\rho)^2 \times \operatorname{Tr}(\sigma^+_{1\pm}\tau_2 S^*(\omega)) \operatorname{Tr}(S^T(\omega)\tau_2\sigma_{1\pm})] \times [-m_d + (2\pi\rho)^2 \operatorname{Tr}(\sigma^+_{2\pm}\tau_2 S^*(\omega)) \times \operatorname{Tr}(S^T(\omega)\tau_2\sigma_{2\pm})]. \quad (215)$$

This Lagrangian has the same form as the 't Hooft Lagrangian.[30] A difference is that it depends on other variables.

When there was a single quark, the γ_5-noninvariant term was a two-quark term. This meant that the attraction induced by the fermion determinant $\det B$ led directly to the appearance of a dynamical mass, which, in its turn, ensured screening of this attraction (Debye effect).

It can be seen from (215) that now the γ_5-noninvariant term is, in general, a four-quark term. However, its investigation is difficult because of the appearance of ultraviolet infinities [see (197)] in the effective theory when an attempt is made to use it outside the framework of the tree approximation. This means that the theory must be regularized in the region of short distances.

To estimate the parameters of the Lagrangian, we assume that, by virtue of certain effects, quark condensates $\langle\sigma_1^+\sigma_1\rangle$ and $\langle\sigma_2^+\sigma_2\rangle$ arise in the theory.

We can then use the approximation

$$\sigma_1^+\sigma_1\sigma_2^+\sigma_2 \simeq \sigma_1^+\sigma_1\langle\sigma_2^+\sigma_2\rangle + \sigma_2^+\sigma_2\langle\sigma_1^+\sigma_1\rangle - \langle\sigma_1^+\sigma_1\rangle\langle\sigma_2^+\sigma_2\rangle.$$

Substituting this expression in (215) and integrating, we obtain the relations

$$\left.\begin{aligned} &\bar{m}_u = m_u - \frac{1}{4}(2\pi\bar{\rho})^2\langle\sigma_1^+\sigma_1\rangle;\\ &m_{\sigma_1} = -\frac{1}{4}(2\pi\bar{\rho})^2\langle\bar{u}u\rangle;\quad \mu_{\sigma_1} = -\frac{g\bar{\chi}}{2\bar{m}_u};\\ &\bar{\chi} = -\frac{1}{3g^2}(2\pi\bar{\rho})^4\bar{m}_u\langle\bar{u}u\rangle;\quad \langle\bar{u}u\rangle = -2\bar{m}_d n_0;\\ &\frac{\alpha}{8\pi}\langle G^2\rangle = -\bar{m}_u\langle\bar{u}u\rangle. \end{aligned}\right\} \quad (216)$$

The remaining relations are obtained by the simultaneous substitutions $u\to d$ and $\sigma_1\to\sigma_2$ in these expression.

For the group $SU_c(3)$ there are similar relations; namely, in the expressions for the masses $\bar{m}_u$, $\bar{m}_d$, $\bar{m}_\sigma$ it is sufficient to replace $(2\pi\rho)^2$ by $8\pi^2\rho^2/3$, and in the expression for $\bar{\chi}$ to replace the factor 1/3 by 1/8.

In Ref. 33 it was conjectured and argued that the instanton physics develops at the scale $\bar{\rho} = 1/600$ MeV. The significance of the appearance of this scale is that in this region the current quarks are transformed into constituent quarks with masses of the order of 300 MeV.

Taking this point of view, we equate the mass of the σ quark to 300 MeV. Using also the phenomenological values of the condensates $\langle\bar{u}u\rangle = -1.7\times10^{-2}$ GeV^3, $\langle(\alpha/\pi)G^2\rangle \simeq 1.2\times10^{-2}$ GeV^4, we obtain

$$\bar{m}_u = \bar{m}_d = 90\ \text{MeV},\quad \bar{\rho} = 1/600\ \text{MeV},$$
$$\langle\bar{\sigma}\sigma\rangle = 0{,}3\langle\bar{q}q\rangle,\quad \bar{\chi} \simeq 0.5.$$

Thus, the effects of the complicated vacuum structure lead to the description of QCD in terms of the collective degrees of freedom X and σ. The collective quark has acquired a dynamical mass proportional to the quark condensate.

In terms of the original variables, the σ quark appears as an extended object with size of order $\bar{\rho}$ [see (207), (208), and (214)].

One can also show[35] that allowance for repulsion between the instantons, taken in the form of an absolutely hard core,[32] leads to the formation of a core with size of order $\bar{\rho}$ in the effective quarks, and this also leads to a diffraction picture of quark–quark scattering.

In the gluon channel, there is a large admixture of quark–antiquark pairs of σ particles [see (207 and (208)]. This explains the possibility of treating the η' meson as pseudoscalar gluonium[36] in the sense that the gluon operator $GG^*(x)$ is proportional to the quark operator $im\bar{\sigma}\gamma_5\sigma$. A similar situation obtains in the Schwinger model, in which the operators $\varepsilon_{\mu\nu}F_{\mu\nu}$ and $i\bar{\psi}\gamma_5\psi$ are proportional to the same field Σ [(44) and (45); see also (132)].

The interaction of the collective degrees of freedom is described by the magnetodynamic Lagrangian (194). Despite the fact that the σ quark is colorless, it has a chromomagnetic moment. It arises because there exists around the quark a virtual cloud of a classical gluon field of instanton type. Tunneling below the barrier, the quark magnetizes with its magnetic field this cloud, the magnetic moment $\chi(e/2m_q)$ of which is to be ascribed to the effective quark.

With regard to the electromagnetic properties of the σ quark, one can show[37] that its charge is equal to the charge of the "bare" q quark. Therefore, its electromagnetic moment is equal to $e/2m$, where m is the dynamical mass (195). Note that the magnetic moment $e/2m_q$ of the bare quark, as it must be in the constituent quark model.

Thus, between the region of asymptotically free quarks with the Coulomb interaction and the hadron domain with the Yukawa short-range interaction there is a region of constituent quarks with magnetic interaction $1/R^3$.

CONCLUSIONS

We have considered several gauge models that possess a complicated vacuum structure. They are all united by one property—the physical excitations in them are colorless. However, this property is realized in different ways.

In the Schwinger model, the electromagnetic field can be divided into transverse and longitudinal parts. After this, the phase of the longitudinal part can be used to construct the gauge-invariant fermion $\sigma(x)$ by compensation of the corresponding phase resulting from a gauge transformation. It is clear that the operator $\sigma(x)$ is defined up to a global phase, and therefore a global $U(1)$ group remains in the model. However, physically this symmetry is not observable, since a consequence of the long-range interaction in two dimensions is confinement, only fermion–antifermion pairs can be observed, and the operator $\sigma(x)$ is an x-independent unitary operator.

In the scalar variant of the Schwinger model with a scalar potential of Higgs type, there is charge screening. Because of the specific potential in the system, a condensate $\langle \varphi^+ \varphi \rangle$ of pairs of charged particles develops, as a result of which the gauge field acquires mass. This means that the field is exponentially small at large distances, i.e., the charge is screened symmetrically. As a result, there remain in the system only colorless variables—the scalar and pseudoscalar fields (ρ,Σ).

When a massless fermion is added, a further degree of freedom appears—the colorless fermion (σ), which has acquired dynamical mass by the γ_5 anomaly. In this case, in contrast to the Schwinger model, there does remain in the system a global symmetry, associated with conservation of the fermion number, since the uncharged fermions (σ) can now move.

In the three-dimensional Georgi–Glashow model with fermions there is confinement, and at large distances it is possible to observe the lightest particle, which is associated with the originally massless component of the gauge field and which is colorless, since it can be regarded as a fermion–antifermion bound state.

In the model, the γ_5 symmetry is spontaneously broken, the nonlinear σ model is realized, the PCAC identities are satisfied, and the particle under consideration is a Goldstone boson.

In QCD with one light quark, the instantons lead to a colorless fermion (σ). There is still a global group associated with nonsymmetric screening of color,since there remains a chromomagnetic moment and the corresponding chromomagnetic fields. The fermion has dynamical mass on account of the γ_5 anomaly.

In QCD with two light quarks (u,d) the corresponding γ_5-noninvariant 't Hooft Lagrangian will be a four-quark Lagrangian $(\bar{\sigma}_u\sigma_u\bar{\sigma}_d\sigma_d)$, i.e., it does not directly lead to dynamical mass of the fermion.

One can also assume the occurrence of condensates $\langle\bar{\sigma}_u\sigma_u\rangle$ and $\langle\bar{\sigma}_d\sigma_d\rangle$ through certain effects, and then a dynamical mass appears.

In the literature there are investigations of the 't Hooft Lagrangian by the methods of the Nambu–Jona-Lasinio model. But this is incorrect for the following reason. If a condensate $\langle\bar{\sigma}_u\sigma_u\rangle$ is to appear in the framework of the method, there must be an interaction $(\bar{\sigma}_u\sigma_u\bar{\sigma}_u\sigma_u)$ of attractive type (the sign is important); for the appearance of a condensate $\langle\bar{\sigma}_d\sigma_d\rangle$ it is necessary to have the same interaction $(\bar{\sigma}_d\sigma_d\bar{\sigma}_d\sigma_d)$, but we have only an interaction between the σ_u and σ_d quarks $(\bar{\sigma}_u\sigma_u\bar{\sigma}_d\sigma_d)$.

Moreover, if the 't Hooft Lagrangian is diagonalized, then in the η' (pseudoscalar) channel there will be a repulsion, i.e., there is no hint of formation of an $\eta'(x)$ field in the framework of this method. Note that this repulsion is an inescapable consequence of the γ_5 anomaly, since it is this that must increase the η' mass relative to the pion mass [$U(1)$ problem[30]]. This can be traced if the 't Hooft Lagrangian is augmented by a phenomenological interaction of attractive type $G(\bar{\sigma}_u\sigma_u\bar{\sigma}_u\sigma_u + \bar{\sigma}_d\sigma_d\bar{\sigma}_d\sigma_d)$ and this Lagrangian is investigated by the method under consideration (see, for example, Ref. 38).

It should also be noted that the problem of large instantons also has a fairly complete solution.

In addition, the renormalization of the pre-exponential in the expression for $dn_0(\rho)$ requires a two-loop calculation of the fluctuations around the instanton. Therefore, the instanton calculations are as yet at the semiphenomenological level.

[1]R. Jackiw and C. Rebbi, Phys. Rev. Lett. **37**, 172 (1976).
[2]C. G. Callan, R. F. Dashen, and D. J. Gross, Phys. Lett. **63B**, 334 (1976).
[3]R. Rajarman, *Solitons and Instantons: An Introduction to Solitons and Instantons in Quantum Field Theory* (North-Holland, Amsterdam, 1982), [Russ. transl., Mir, Moscow, 1985].
[4]A. A. Slavnov and L. D. Faddeev, *Gauge Fields: Introduction to Quantum Theory* (Benjamin, Reading, Mass., 1980) [Russ. original, Nauka, Moscow, 1978].
[5]J. Schwinger, Phys. Rev. **128**, 2425 (1962).
[6]J. H. Lowenstein and J. A. Swieka, Ann. Phys. (N.Y.) **68**, 172 (1971).
[7]N. V. Krasnikov, V. A. Matveev, V. A. Rubakov *et al.*, Teor. Mat. Fiz. **45**, 313 (1980).
[8]A. N. Tavkhelidze and V. F. Tokarev, Fiz. Elem. Chastits At. Yadra **16**, 973 (1985) [Sov. J. Part. Nucl. **16**, 431 (1985)].
[9]A. S. Wightman, Lectures at the French Summer School of Theoretical Physics, Cargèse, July 1964 [Russ. transl., Nauka, Moscow, 1968].
[10]G. Velo, Nuovo Cimento **52A**, 1028 (1967).
[11]S. L. Adler and R. F. Dashen, *Current Algebras* (Benjamin, New York, 1968) [Russ. transl., Mir, Moscow, 1970]; V. de Alfaro, S. Fubini, G. Furlan, and C. Rossetti, *Currents in Hadron Physics* (North-Holland, Amsterdam, 1973) [Russ. transl., Mir, Moscow, 1976].
[12]S. Coleman, R. Jackiw, and L. Susskind, Ann. Phys. (N.Y.) **93**, 267 (1975).
[13]J. Frohlich, G. Morchio, and F. Strocchi, Phys. Lett. **97B**, 249 (1980).
[14]B. A. Matveev, A. N. Tavkhelidze, and M. E. Shaposhnikov, Teor. Mat. Fiz. **59**, 323 (1984); V. V. Vlasov, V. A. Matveev, A. N. Tavkhelidze *et al.*, Fiz. Elem. Chastits At. Yadra **18**, 5 (1987) [Sov. J. Part. Nucl. **18**, 1 (1987)].
[15]K. D. Rothe and J. A. Swieka, Phys. Rev. D **15**, 541 (1977).
[16]H. B. Nielsen and P. Olesen, Nucl. Phys. **B61**, 45 (1973).
[17]B. A. Arbuzov, A. H. Tavkhelidze, and R. N. Faustov, Dokl. Akad. Nauk SSSR **139**, 345 (1961) [Sov. Phys. Dokl. **6**, 598 (1962)].
[18]C. G. Callan, R. Dashen, and D. J. Gross, Phys. Rev. D **16**, 2526 (1977).
[19]A. K. Pogrebkov and V. N. Sushko, Teor. Mat. Fiz. **24**, 425 (1975); S. Mandelstam, Phys. Rev. D **11**, 3026 (1975); S. Coleman, Phys. Rev. D **11**, 2088 (1975).
[20]A. Polyakov, Nucl. Phys. **B120**, 429 (1977).
[21]K. Dietz and Th. Filk, Nucl. Phys. **B164**, 535 (1980).
[22]V. F. Tokarev, Teor. Mat. Fiz **54**, 111 (1983).
[23]A. M. Polyakov, Pis'ma Zh. Eksp. Teor. Fiz. **20**, 430 (1974) [JETP Lett. **20**, 194 (1974)]; G. 't Hooft, Nucl. Phys. **B79**, 276 (1974).
[24]M. K. Prasad and C. M. Sommerfield, Phys. Rev. Lett. **35**, 760 (1975); E. B. Bogomol'nyĭ, Yad. Fiz. **24**, 861 (1976) [Sov. J. Nucl. Phys. **24**, 449 (1976)].
[25]S. N. Vergeles, Nucl. Phys. **B152**, 330 (1979).
[26]R. Jackiw and C. Rebbi, Phys. Rev. D **13**, 3398 (1977).
[27]A. A. Belavin, A. M. Polyakov, A. S. Schwartz, and Yu. S. Tyupkin, Phys. Lett. **59B**, 85 (1975).
[28]C. G. Callan, R. F. Dashen, and D. J. Gross, Phys. Rev. D **19**, 1826 (1979).
[29]V. F. Tokarev, Teor. Mat. Fiz. **58**, 354 (1984); in *Proc. of the International Seminar, Tbilisi, 1984*, Vol. 2 [in Russian] Institute of Nuclear Research, Moscow, 1985), p. 14; Preprint R-0406 [in Russian], Institute of Nuclear Research, Moscow (1985); Teor. Mat. Fiz. **73**, 223 (1987).
[30]G. 't Hooft, Phys. Rev. D **14**, 3432 (1976).
[31]C. Lee and W. A. Bardeen, Nucl. Phys. **B153**, 210 (1979).
[32]E. M. Ilgenfritz and M. Mueller-Preüssker, Nucl. Phys. **B184**, 443 (1981).
[33]D. I. Dyakonov and V. Yu. Petrov, Nucl. Phys. **B245**, 259 (1984).
[34]E. V. Shuryak, Phys. Ref. **115**, 151 (1984).
[35]I. V. Musatov and V. F. Tokarev, in *Proc. of the International Seminar, Protvino, 1987: Problems of High Energy Physics and Field Theory* [in Russian] (Nauka, Moscow, 1988), p. 343.
[36]A. I. Vaĭnshteĭn, V. I. Zakharov, V. A. Novikov, and M. A. Shifman, Pis'ma Zh. Eksp. Teor. Fiz. **29**, 649 (1979) [JETP Lett. **29**, 594 (1979)].
[37]I. V. Leonov and V. F. Tokarev, Teor. Mat. Fiz. **74**, 192 (1988).
[38]D. Ebert and M. E. Volkov, Z. Phys. C **16**, 205 (1983).

Translated by Julian B. Barbour