Gravitational collapse of a dust sphere in the relativistic theory of gravitation

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It is shown that in the relativistic theory of gravitation with nonzero graviton mass the process of gravitational collapse is very different from the process in the massless theory. Near the Schwarzschild horizon the collapsing dust sphere bounces.

INTRODUCTION

The problem of gravitational collapse, i.e., the process of the self-gravitating contraction of a spherically symmetric body, is one of the most important for any theory of gravitation. It is well known[1,2] that in the general theory of relativity the outcome of collapse is the formation, during a finite proper time, of an object with infinite density and zero radius, which has become known as a black hole. Moreover, if the mass m of the collapsing body exceeds three solar masses, no internal forces are capable of halting the collapse of this body into a point.

A characteristic property of a black hole is the presence of a Schwarzschild sphere of radius $2m$ (also called the gravitational radius), from the interior of which no signals can escape. There is a "gravitational self-closure" of the body, as a result of which the region bounded by the Schwarzschild sphere (Schwarzschild horizon) is in principle inaccessible for investigation by an external observer.

As will be shown in this paper, the nature of gravitational collapse is fundamentally different from the point of view of the relativistic theory of gravitation (RTG); for the collapsing body does not reach its gravitational radius, but "bounces" near it. In other words, in this region the collapse is rapidly halted and, after coming to a stop, develops in the opposite direction. As will be clear from the following exposition, this profound difference between the pictures of collapse in the RTG and in general relativity is due to the presence of a nonzero graviton rest mass of the gravitational field. When the radius of the body is far from the Schwarzschild radius, the influence of the graviton mass on the process of gravitational collapse is negligibly small because of the negligibly small value of this mass: $\mu < 10^{-65}$ g (Refs. 3 and 4). Thus, in the RTG with nonzero graviton mass there cannot exist bodies with radius less than or equal to the gravitational radius, since real bodies differ from the case considered below of the collapse of a dust sphere by the presence of pressure of the matter, which merely weakens the self-gravitation of these bodies. Therefore, no "gravitational self-closure" occurs either. The process of collapse, from onset to its halting near (but outside) the Schwarzschild sphere, occupies a finite time according to the clocks of both a comoving and a distant observer.

We note that these general conclusions were reached earlier by Vlasov and Logunov[5] on the basis of the properties of the vacuum static solution of the RTG with nonzero rest mass.

The remainder of our exposition will be constructed in accordance with the following plan. In Sec. 1 we consider the formulation of the problem and its well-posed character. In Sec. 2 we construct a Riemannian solution in the approximation of zero mass of the graviton, both inside the body and outside it (Tolman solution). In Sec. 3 we find the connection between the radial coordinate of the Minkowski space and the comoving Tolman coordinates for the interior and exterior solutions. In Sec. 4 we seek the general solution for the time coordinate of the Minkowski space with the comoving coordinates in the exterior region and investigate the nature of the singularities of this solution. In Sec. 5 we consider the interior solution in the region of the singularity of the time coordinate of the Minkowski space. In Sec. 6 we show that precisely in this region the influence of the graviton mass on the collapsed process becomes significant and leads to stopping of the process and the bounce.

1. FORMULATION OF THE PROBLEM

We consider the spherically symmetric problem of the collapse of a dust sphere in the RTG, the equations of which have the form[6]

$$R_k^i - \frac{1}{2}\delta_k^i R + \frac{\mu^2}{2}\left[\delta_k^i + g^{ip}\gamma_{pk} - \frac{1}{2}\delta_k^i g^{pq}\gamma_{pq}\right] = 8\pi T_k^i; \quad (1)$$

$$D_i g^{ik}\sqrt{-g} = 0, \quad (2)$$

where the energy–momentum tensor density of ideal-fluid matter has the form

$$T_k^i = (p+\rho)\, u^i u_k - \delta_k^i p.$$

Here, ρ and p are the density and isotropic pressure, u^i is the unit velocity 4-vector, D_i is the covariant derivative with respect to the Minkowski-space metric γ_{ik}, $\sqrt{-g}g^{ik} = \sqrt{-\gamma}(\gamma^{ik} + \phi^{ik})$ is the tensor density of the metric of the Riemannian space–time, and ϕ^{ik}, is the gravitational field. Equations (2) are a consequence of Eqs. (1) and the equations of motion

$$\nabla_i T^{ik} = 0, \quad (3)$$

where ∇_i is the covariant derivative with respect to the Riemannian metric g_{ik}.

We shall seek a solution of the system (1)–(2) in the coordinates τ, R of the Riemannian space–time. We take the interval in diagonal form. Then the general spherically symmetric solution can be represented as

$$ds^2 \equiv g_{ik}\, dx^i\, dx^k = e^{\nu(\tau,\,R)}\, d\tau^2 - e^{\lambda(\tau,\,R)}\, dR^2 - e^{\bar{\mu}(\tau,\,R)}\, d\Omega^2 \quad (4)$$

In this case, the Minkowski-space interval can be written in the form

$$d\sigma^2 \equiv \gamma_{ik}\, dx^i\, dx^k = dt^2 - dr^2 - r^2\, d\Omega^2$$
$$= (\dot{t}^2 - \dot{r}^2)\, d\tau^2 - 2(\dot{t}t' - \dot{r}r')\, d\tau\, dR - (r'^2 - t'^2)\, dR^2 - r^2\, d\Omega^2, \quad (5)$$

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$$d\Omega^2 \equiv d\theta^2 + \sin^2\theta d\varphi^2,$$

where t and r are the time and radial coordinates of the Minkowski space, $t = t(\tau, R)$, $r = r(\tau, R)$. Here and in what follows, the dot will denote partial differentiation with respect to τ, and the prime will denote partial differentiation with respect to R. The nontrivial equations (1)–(3), when g_{ik} and γ_{ik} are given in (4) and (5), can be represented in a comoving frame ($u^R = u^\theta = u^\varphi = 0$) in the form

$$8\pi\rho = e^{-\lambda}\left(\bar\mu'' + \frac{3}{4}\bar\mu'^2 - \frac{\bar\mu'\lambda'}{2}\right) + \frac{e^{-\nu}}{2}\left(\frac{\dot{\bar\mu}^2}{2} + \dot{\bar\mu}\dot\lambda\right) + e^{-\bar\mu}$$
$$+\frac{\mu^2}{2}\left[1 - \frac{e^{-\nu}}{2}(\dot r^2 - \dot t^2) - \frac{e^{-\lambda}}{2}(r'^2 - t'^2) - e^{-\bar\mu}r^2\right]; \qquad (1a)$$

$$-8\pi p = -\frac{1}{2}e^{-\lambda}\left(\frac{\bar\mu'^2}{2} + \bar\mu'\nu'\right)$$
$$+e^{-\nu}\left(\ddot{\bar\mu} - \frac{1}{2}\dot{\bar\mu}\dot\nu + \frac{3}{4}\dot{\bar\mu}^2\right) + e^{-\bar\mu}$$
$$+\frac{\mu^2}{2}\left[1 - \frac{e^{-\nu}}{2}(\dot t^2 - \dot r^2) + \frac{e^{-\lambda}}{2}(r'^2 - t'^2) - e^{-\bar\mu}r^2\right]; \qquad (1b)$$

$$-8\pi p = -\frac{e^{-\lambda}}{4}(2\nu'' + \nu'^2 + 2\bar\mu'' + \bar\mu'^2 + \bar\mu'\nu' - \bar\mu'\lambda' - \lambda'\nu')$$
$$-\frac{e^{-\nu}}{4}(\dot\lambda\dot\nu + \dot{\bar\mu}\dot\nu - \dot{\bar\mu}\dot\lambda - 2\ddot\lambda - \dot\lambda^2 - 2\ddot{\bar\mu} - \dot{\bar\mu}^2) +$$
$$+\frac{\mu^2}{2}\left[1 - \frac{e^{-\nu}}{2}(\dot t^2 - \dot r^2) - \frac{e^{-\lambda}}{2}(r'^2 - t'^2)\right]; \qquad (1c)$$

$$0 = \frac{e^{-\lambda}}{2}(-2\dot{\bar\mu}' - \dot{\bar\mu}\bar\mu' + \dot\lambda\bar\mu' + \nu'\dot{\bar\mu}) + \frac{\mu^2}{2}e^{-\lambda}[\dot t t' - \dot r r']; \qquad (1d)$$

$$\left(e^{\bar\mu + \frac{\lambda}{2} - \frac{\nu}{2}}\dot t\right)^{\cdot} = \left(e^{\bar\mu + \frac{\nu}{2} - \frac{\lambda}{2}}t'\right)'; \qquad (2a)$$

$$\left(e^{\bar\mu + \frac{\lambda}{2} - \frac{\nu}{2}}\dot r\right)^{\cdot} = \left(e^{\bar\mu + \frac{\nu}{2} - \frac{\lambda}{2}}r'\right)' - 2re^{\frac{\lambda+\nu}{2}}; \qquad (2b)$$

$$\nu' = -2\frac{p'}{p+\rho}; \qquad (3a)$$

$$2\dot{\bar\mu} + \dot\lambda = -2\frac{\dot\rho}{p+\rho}; \qquad (3b)$$

$$p = p(\rho). \qquad (6)$$

As already noted, two of these equations are a consequence of the others [excluding, of course, the equation of state (6)]. Therefore, this system contains $9 - 2 = 7$ independent equations. There are also seven variables: ν, λ, $\bar\mu$, t, r, p, ρ. The greatest interest attaches to investigation of the collapse of a dust sphere with the equation $p = 0$. Then on the basis of (3a) we can choose $\nu = 0$. Such a choice corresponds to a synchronous comoving frame in which the collapsing dust particles are at rest, while the time τ is directly measured by clocks attached to them. In this case, our system of equations is somewhat simplified. Taking as independent equations (1a), (1d), (2a), and (3b) and, accordingly, replacing the variables $\bar\mu$ and λ by the new variables $B = (\tau, R)$ and $G = G(\tau, R)$,

$$e^{\bar\mu} = B^2(\tau, R), \quad e^{\lambda} = B'^2(\tau, R)/G^2(\tau, R),$$

we obtain a system of five equations with five unknowns (ρ, B, G, t, r):

$$8\pi\rho = -2\frac{(G^2)'}{(B^2)'} + \frac{1-G^2}{B^2} + \frac{(\dot B^2 B)'}{B^2B'} - 2\frac{\dot B}{G}\frac{\dot G}{G}$$
$$+\frac{\mu^2}{2}\left[1 - \frac{1}{2}(\dot r^2 - \dot t^2) - \frac{G^2}{2B'^2}(r'^2 - t'^2) - \frac{r^2}{B^2}\right]; \qquad (7)$$

$$\frac{B'}{B}\frac{\dot G}{G} = \frac{\mu^2}{4}(\dot t t' - \dot r r'); \qquad (8)$$

$$\left(\frac{B^2B'}{G}\dot t\right)^{\cdot} = \left(\frac{B^2}{B'}Gt'\right)'; \qquad (9)$$

$$\left(\frac{B^2B'}{G}\dot r\right)^{\cdot} = \left(\frac{B^2}{B'}Gr'\right)' - 2\frac{B'}{G}r; \qquad (10)$$

$$2\frac{\dot B}{B} + \frac{\dot B'}{B'} + \frac{\dot\rho}{\rho} - \frac{\dot G}{G} = 0. \qquad (11)$$

As we have already noted, the graviton mass μ is very small, and we therefore first consider the solution of the system (7)–(11) as given by the massless theory, and then analysis of it will show the region in which the graviton mass can distort this massless solution.

2. TOLMAN'S FLAT SOLUTION

Thus, in (7) and (8) we set $\mu^2 = 0$. Then $\dot G = 0$, whence

$$G^2 = 1 + f_1(R). \qquad (12)$$

From (11) we find in this case

$$\rho B^2 B' = f_2(R). \qquad (13)$$

Substituting (12) and (13) in (7), we obtain

$$8\pi\frac{f_2(R)}{B^2B'} = -2\frac{f_1'(R)}{(B^2)'} - \frac{f_1(R)}{B^2} + \frac{(\dot B^2 B)'}{B'B^2}. \qquad (14)$$

The first integral of (14) has the form

$$\dot B^2 = f_1(R) + \frac{f_3(R)}{B}, \qquad (15)$$

where

$$f_3(R) = \int^R f_2(R)\,dR. \qquad (16)$$

The functions $f_1(R)$ in (12) and $f_2(R)$ in (13) are as yet arbitrary. In vacuum, as follows from (13), $f_2(R) = 0$. In this case, we find, on the basis of (16),

$$f_3 = 2m = \text{const}. \qquad (17)$$

The function $B(\tau, R)$ can be determined more accurately than (15)–(17) by using the condition for matching the exterior and interior solutions at the boundary of the body. The function $f_1(R)$ is determined by the initial conditions.

To simplify the calculations, all the further analysis will be constructed by considering the case $f_1 = 0$. Then from (15)–(17) we find

$$B(\tau, R) = \begin{cases} R(1 - \tau/\tau_0)^{2/3}, & R \leqslant R_0, \quad (18) \\ (R^{3/2} - R_0^{3/2}\tau/\tau_0)^{2/3}, & R > R_0. \quad (19) \end{cases}$$

It follows from (13) that the matter density is homogeneous:

$$\rho = \frac{1}{6\pi(\tau - \tau_0)^2}. \qquad (20)$$

Here

$$\frac{9}{2}\frac{m}{R_0^3} = \frac{1}{\tau_0^2}. \qquad (21)$$

We note some characteristic features of the solution (18)–(21). The function $B(\tau, R)$ can take all values from 0 to ∞. For $\tau = \tau_0$ and all $R \leqslant R_0$, the function B vanishes, while the density ρ is infinite, and a black hole is formed.

We now consider how the function $B(\tau, R)$ and its partial derivatives behave on the boundary of the body. As follows from (18) and (19), the function B itself and all its derivatives with respect to τ are continuous at the point $R = R_0$, while B' has a discontinuity on the passage through the boundary of the body:

$$B'(\tau, R) = \begin{cases} \left(1 - \frac{\tau}{\tau_0}\right)^{2/3}, & R - R_0 \to -0; \quad (18a) \\ \left(1 - \frac{\tau}{\tau_0}\right)^{-1/3}, & R - R_0 \to +0. \quad (19a) \end{cases}$$

We can readily understand why there must be a discontinuity on the basis of Eq. (13), since on the surface $R = R_0$ the matter density ρ changes abruptly. We should not be disturbed by this behavior of B', since, as was shown in Ref. 7, the solution (18)–(21) satisfies the invariant matching condition of Darmois and the equivalent condition of Lichnerowicz and, thus, is completely physical. Using the solution (18)–(21) of Einstein's equations, we can find the behavior of the functions $t(\tau, R)$ and $r(\tau, R)$.

3. CALCULATION OF THE CONNECTION BETWEEN THE RADIAL COORDINATE OF THE MINKOWSKI SPACE AND THE COMOVING COORDINATES

In this section, we solve Eq. (10) in the vacuum and inside the body and match the solutions on its boundary.

We first consider the exterior solution. It is more conveniently analyzed, not in the coordinates τ and R, but in new (Schwarzschild) coordinates (T, B):

$$T = \tau - 2\sqrt{2mB} + 2m \ln \frac{\sqrt{B} + \sqrt{2m}}{\sqrt{B} - \sqrt{2m}}. \quad (22)$$

In this case, the Riemannian interval

$$ds^2 = d\tau^2 - B'^2 dR^2 - B^2 d\Omega^2$$

takes the Schwarzschild form

$$ds^2 = \left(1 - \frac{2m}{B}\right) dT^2 - \frac{dB^2}{1 - \frac{2m}{B}} - B^2 \, d\Omega^2. \quad (23)$$

In the new variables, Eq. (10) can be written as

$$\frac{B - 2m}{B}\left[\frac{\partial}{\partial B}\left[B(B - 2m)\frac{\partial r}{\partial B} - 2r\right]\right] = \frac{\partial^2 r}{\partial T^2}, \quad (24)$$

where, obviously, the variables separate, this being the explanation for their introduction.

From the condition that the gravitational field

$$\phi^{ik} = \sqrt{g/\gamma}\,(g^{ik} - \gamma^{ik})$$

should tend to zero at spatial infinity, there follow boundary conditions on the derivatives of r:

$$\frac{\partial r}{\partial T} \xrightarrow[B \to \infty]{} 0, \quad \frac{\partial r}{\partial B} \xrightarrow[B \to \infty]{} 1.$$

We first consider the solutions of Eq. (24) with vanishing separation parameter. In this case, r depends only on B, and the function $r = r(B)$ satisfies the equation

$$B(B - 2m)\frac{d^2 r}{dB^2} + 2(B - m)\frac{dr}{dB} - 2r = 0,$$

whose solutions have, in accordance with the boundary conditions, the form[6,8,9]

$$r = B - m + C_1\left(1 + \frac{B - m}{2m} \ln \frac{B - 2m}{B}\right), \quad (25)$$

where $C_1 = \text{const}$. The general solution of (24) is obtained by adding to (25) an integral over the separation parameter:

$$\int_0^\infty r(\omega, B)\,[s_1(\omega)\, e^{\omega T} + s_2(\omega)\, e^{-\omega T}]\, d\omega, \quad (26)$$

in which the function $r(\omega, B)$ satisfies the equation

$$(B - 2m)^2 \frac{d^2 r(\omega, B)}{dB^2} + 2(B - 2m)\frac{(B - m)}{B}\frac{dr(\omega, B)}{dB} - \left[2\left(1 - \frac{2m}{B}\right) + \omega^2 B^2\right] r(\omega, B) = 0$$

and the boundary condition $r(\omega, B)_{B \to \infty} \to 0$.

To construct the interior solution of Eq. (10), we proceed as follows. By means of the Riemann function,[10] we obtain in accordance with the Riemann–Green formula a solution that is automatically matched as it should be to the exterior solution, and we choose from the set of such solutions [corresponding to the set of exterior solutions (25), (26)] the unique one that satisfies the one further boundary condition $r(\tau, R = 0) = 0$.

Thus, for $B = R(1 - \tau/\tau_0)^{2/3}$ and $G = 1$, Eq. (10) can be written in the form

$$\left(1 - \frac{\tau}{\tau_0}\right)^{2/3}\left[\left(1 - \frac{\tau}{\tau_0}\right)^2 \dot{r}\right]^{\cdot} = \frac{1}{R^2}(R^2 r')' - \frac{2r}{R^2}. \quad (27)$$

It is convenient to go over to the new function u and the new variable ξ:

$$u = R^2 \xi^3 r, \quad \xi = 3\tau_0\left(1 - \frac{\tau}{\tau_0}\right)^{1/3}. \quad (28)$$

Then the hyperbolic equation (27) can be represented in the first canonical form:

$$u_{\xi\xi} - u_{RR} - \frac{2}{\xi} u_\xi + \frac{2}{R} u_R = 0.$$

Introducing the mixed variables

$$\bar{A} = (\xi + R)^2, \quad \bar{B} = (\xi - R)^2,$$

we write it in the second canonical form:[10]

$$u_{\bar{A}\bar{B}} + \frac{u_{\bar{A}} - u_{\bar{B}}}{\bar{A} - \bar{B}} = 0. \quad (29)$$

For this equation, the Riemann functions are contained in Ref. 10:

$$R(\bar{A}, \bar{B}, \lambda, \varkappa) = \frac{\lambda - \varkappa}{\bar{A} - \bar{B}}\left[1 - 2\frac{(\lambda - \bar{A})(\varkappa - \bar{B})}{(A - \bar{B})(\lambda - \varkappa)}\right]. \quad (30)$$

In order to find $r = r(\tau, R)$ inside the body by means of (30) in accordance with the Riemann–Green formula, it is necessary to specify correctly the boundary conditions on the surface. In what follows, we shall use the subscript "in" to identify variables relating to the exterior solution, and the subscript "out" to identify those of the exterior solution.

On the boundary of the body, the function $r(\tau, R)$ itself must obviously be continuous:

$$r_{\text{in}}(\tau, R_0) = r_{\text{out}}(\tau, R_0).$$

From this it also follows in particular that all the partial derivatives of r with respect to τ (or, equivalently, with respect to ξ) will be continuous.

However, as follows from Eq. (10), the derivative of r with respect to R must have a discontinuity, since the function B' [see (18a), (19a)] is not continuous at the point $R = R_0$. Therefore

$$\frac{r'_{\text{in}}(\tau, R_0)}{B'_{\text{in}}(\tau, R_0)} = \frac{r'_{\text{out}}(\tau, R_0)}{B'_{\text{out}}(\tau, R_0)}$$

or

$$r'_{\text{in}}(\tau, R_0) = r'_{\text{out}}(\tau, R_0)\,(1 - \tau/\tau_0).$$

We mention once more that such a discontinuity of r' is needed to ensure fulfillment on the surface of the body of the RTG Eq. (2):

$$D_i \tilde{g}^{ik} = 0.$$

The corresponding boundary conditions for the function u will be

$$u(\xi, R_0) \equiv u_0(\xi) = R_0^2 \xi^3 r_{\text{out}}(\tau, R_0); \tag{31}$$

$$u'(\xi, R_0) \equiv w_0(\xi) = \frac{R_0^2 \xi^6}{(3\tau_0)^3} r'_{\text{out}}(\tau, R_0) + 2R_0 \xi^3 r_{\text{out}}(\tau, R_0).$$

In accordance with the Riemann–Green formula, we obtain the final expression for r_{in} in a form that automatically satisfies (31):

$$r_{\text{in}}(\xi, R) = \frac{1}{2R_0 R \xi^2}\left[\frac{u_0(\xi + R_0 - R)}{\xi + R_0 - R} + \frac{u_0(\xi - R_0 + R)}{\xi - R_0 + R}\right]$$
$$- \frac{1}{2R_0 R \xi^2} \int\limits_{\xi - R_0 + R}^{\xi + R_0 - R} \frac{w_0(x)\,dx}{x}$$
$$\times \left\{1 - \frac{[(\xi + R)^2 - (R_0 + x)^2]\,[(\xi - R)^2 - (R_0 - x)^2]}{8\xi R R_0 x}\right\}$$
$$+ \frac{1}{4R_0 R^2 \xi^3} \int\limits_{\xi - R_0 + R}^{\xi + R_0 - R} u_0(x)\,dx\, \frac{\xi^2 + R^2 + x^2 - R_0^2}{x^2}. \tag{32}$$

Substitution in this formula of the exterior solution (25) (in terms of the corresponding values of u_0 and w_0) shows, after a very lengthy calculation, that the second boundary condition $z_{\text{in}}(\xi, 0) = 0$ is satisfied by the unique exterior solution with $C_1 = 0$:

$$r_{\text{out}} = B_{\text{out}} - m. \tag{33}$$

In this case (32) gives

$$r_{\text{in}} = R\left(1 - \frac{\tau}{\tau_0}\right)^{2/3} + \frac{R^2}{(3\tau_0)^2} - \frac{3R_0^2 R}{(3\tau_0)^2}$$
$$\equiv B_{\text{in}} + \frac{m}{2R_0^3} R^3 - \frac{3m}{2R_0} R. \tag{34}$$

We note that this solution was first given in Ref. 13 without derivation. This enables us to assume (see Sec. 6) that the given solution will not significantly change its form when a graviton mass is "switched on."

4. SINGULARITY OF THE TIME COORDINATE OF THE MINKOWSKI SPACE ON THE SCHWARZSCHILD SPHERE (IN VACUUM)

In this section, we shall find the general solution of Eq. (9),

$$\left(\frac{B^2 B'}{G} \dot{t}_{\text{out}}\right)^{\cdot} = \left(\frac{B^2}{B'} G t'_{\text{out}}\right)',$$

in vacuum under the assumption that the graviton has no mass ($\mu^2 = 0$). Recalling that for the exterior problem $B = (R^{3/2} - R_0^{3/2}\tau/\tau_0)^{2/3}$, $G = 1$ and going over, as in the previous section, to the Schwarzschild variables T, B (22), we transform Eq. (9) to an equation with separated variables:

$$\frac{B - 2m}{B^2} \frac{\partial}{\partial B}\left[B(B - 2m)\frac{\partial t_{\text{out}}}{\partial B}\right] = \frac{\partial^2 t_{\text{out}}}{\partial T^2}. \tag{35}$$

The condition that the gravitational field of our system tend to zero at spatial infinity leads to the following boundary conditions for the derivatives of t_{out}:

$$\frac{\partial t_{\text{out}}}{\partial T} \xrightarrow[B \to \infty]{} 1, \quad \frac{\partial t_{\text{out}}}{\partial B} \xrightarrow[B \to \infty]{} 0.$$

We consider first the solutions of (35) that correspond to a vanishing separation parameter. We denote them by t^{0}_{out}. Taking into account the boundary conditions, we obtain

$$t^{(0)}_{\text{out}} = T + C_2 \ln \frac{4m^2}{B(B - 2m)},$$

where C_2 is an arbitrary constant. We shall show that C_2 must be equal to zero. Bearing in mind that $r_{\text{out}} = B - m$, we reduce the Riemannian metric in vacuum for such t_{out} and r_{out} to the form (omitting here for simplicity the index "out" of t and r)

$$ds^2 = \frac{r - m}{r + m} dt^2 - 4C_2 \frac{m^2}{(r + m)^2} dt\,dr -$$
$$- \frac{r + m}{r - m}\left(1 - 4C_2^2 \frac{m^4}{(r + m)^4}\right) dr^2 - (r + m)^2 d\Omega^2. \tag{36}$$

The Minkowski-space interval has the usual form

$$d\sigma^2 = dt^2 - dr^2 - r^2 d\Omega^2.$$

It can be seen that the solution (36) is static and nondiagonal. Going over to Cartesian coordinates, we find that in the given case

$$g_{0\alpha} = -2\frac{C_2 m^2}{(r + m)^2} \frac{r^\alpha}{r}, \quad \alpha = x, y, z$$

and, accordingly,

$$\sqrt{-g}\, g^{0\alpha} = -2C_2 m^2 \frac{r^\alpha}{r^3}.$$

We now consider the RTG equation (2). In Cartesian coordinates it takes for $k = 0$ the form

$$\frac{\partial}{\partial t}\sqrt{-g}\, g^{00} + \frac{\partial}{\partial r^\alpha}\sqrt{-g}\, g^{0\alpha} = 0. \tag{37}$$

The term $\sqrt{-g}g^{00}$ does not depend on the time t, since the metric (37) is static, and the calculation of the three-dimensional divergence gives

$$\frac{\partial}{\partial r^\alpha}\sqrt{-g}\, g^{0\alpha} = 8\pi m^2 C_2 \delta(r).$$

Thus, Eq. (37) is satisfied everywhere except at the point $r = 0$. However, for the considered exterior solution r is strictly positive. Thus, Eq. (37) appears to be satisfied. However, this is not so, since the constant C_2 acquires the meaning of a charge (by analogy with electrodynamics), the presence of which could also be noted in the exterior solution in vacuum.

Following electrodynamics, we integrate Eq. (37) over the volume of a sphere that has a radius greater than the radius of our body. Using Gauss's theorem, we reduce the integrated equation (37) to the form

$$8\pi C_2 m^2 = 0.$$

Thus, we conclude that $C_2 = 0$.

As will be shown in the following section, the solution

$$t_{\rm out}=T=\tau-2\sqrt{2mB}+2m\ln\frac{\sqrt{B}+\sqrt{2m}}{\sqrt{B}-\sqrt{2m}}$$

of Eq. (35) cannot be fitted to a function $t_{\rm in}$ that is regular at the origin ($R=0$), and therefore it does not have physical meaning. In this case, we must necessarily consider the general solution of (35),

$$t_{\rm out}=T+\int_0^\infty t(\omega,b)\,t(\omega,T)\,d\omega,$$

where the integral itself and its partial derivatives with respect to T and with respect to B must tend to zero in the limit $B\to\infty$. Here, $t(\omega,T)=\exp(\pm\omega T)$, and the function $t(\omega,B)$ satisfies, as follows from (35), the equation

$$(B-2m)^2\frac{d^2t(\omega,B)}{dB^2}+2(B-2m)\frac{(B-m)}{B}\frac{dt(\omega,B)}{dB}-\omega^2B^2t(\omega,B)=0 \qquad (38)$$

and the boundary conditions

$$t(\omega,B)\xrightarrow[B\to\infty]{}0,\quad \frac{\partial t}{\partial B}(\omega,B)\xrightarrow[B\to\infty]{}0.$$

It follows from the general theory of differential equations[11] that the point $B=2m$ is a regular singular point of this equation. We briefly recall some fundamental theoretical results. For an equation that can be represented in the form

$$(x-x_1)^2\frac{d^2y}{dx^2}+(x-x_1)\,p_1(x)\frac{dy}{dx}+p_2(x)\,y=0, \qquad (39)$$

where p_1 and p_2 are analytic at x_1, the indicial equation reduces to a quadratic equation:

$$\varepsilon^2+\varepsilon\,[p_1(x_1)-1]+p_2(x_1)=0.$$

It has two roots. For the root with the larger real part, Eq. (39) admits a solution in the form of a series

$$y=y_1(x)=(x-x_1)^{\varepsilon_1}\sum_{k=0}^\infty a_k(x-x_1)^k.$$

The coefficients of the series, a_k, can be found if the series is substituted in the original equation (39) and corresponding recursion relations are obtained. The second root leads to a solution that is linearly independent of y_1 (if $\varepsilon_2-\varepsilon_1\neq z$, where z is an integer):

$$y_2(x)=(x-x_1)^{\varepsilon_2}\sum_{k=0}^\infty b_k(x-x_1)^k.$$

But if $\varepsilon_1-\varepsilon_2=z$, then the solution linearly independent of y_1 can be written in the form

$$y_2(x)=A\,y_1(x)\ln(x-x_1)+(x-x_1)^{\varepsilon_2}\sum_{k=0}^\infty b_k(x-x_1)^k.$$

We apply these general results to Eq. (38). The indicial equation [$p_1(B)=2(B-m)/B,\ p_2=-\omega^2B^2$] has the form

$$\varepsilon_{1,2}=\pm 2m\omega.$$

If $\omega\neq\omega_n=n/4m$ (where n is a natural number), then the general solution has the form

$$t_{\rm out}=T+\int_0^\infty d\omega\,(B-2m)^{2m\omega}\sum_{k=0}^\infty a_k^+(B-2m)^k$$

$$\times(s^+(\omega)\,e^{-\omega T}+g^+(\omega)\,e^{\omega T})+\int_0^\infty\frac{d\omega}{(B-2m)^{2m\omega}}$$

$$\times\sum_{k=0}^\infty a_k^-(B-2m)^k\,(s^-(\omega)\,e^{-\omega T}+g^-(\omega)\,e^{\omega T}), \qquad (40)$$

where $s^\pm(\omega)$ and $g^\pm(\omega)$ are (as yet) arbitrary functions of ω. The numerical coefficients $a_k^\pm$, which also depend on ω, correspond to the choice of the sign in front of the exponent in $(B-2m)^{\pm 2m\omega}$.

Substitution of (40) in (38) gives recursion relations for the coefficients $a_k^\pm$:

$$2mk(k\pm 4m\omega)\,a_k^\pm-[(k\pm 2m\omega)(k\pm 2m\omega-1)-3(2m\omega)^2]\,a_{k-1}^\pm-6m\omega^2a_{k-2}^\pm-\omega^2a_{k-3}^\pm=0.$$

We write down the first few coefficients:

$$a_0^\pm=1;$$

$$a_1^\pm=\mp\omega\frac{4m\omega\pm 1}{4m\omega\mp 1}=\pm\omega\left(1\pm\frac{1}{2m\omega}+O\left(\frac{1}{m^2\omega^2}\right)\right);$$

$$a_2^\pm=\pm\omega\frac{(-1\pm 6m\omega+(2m\omega)^2\pm 2(2m\omega)^3)}{4m(1\mp 4m\omega)(1\mp 2m\omega)}=\frac{\omega^2}{2}\left(1\pm\frac{1}{m\omega}+O\left(\frac{1}{m^2\omega^2}\right)\right);$$

$$\cdots$$

$$a_n^\pm=\frac{(\pm\omega)^n}{n!}\left(1+O\left(\frac{1}{m\omega}\right)\right).$$

$$\cdots$$

Such behavior of the coefficients indicates that at large ω

$$\sum_{k=0}^\infty a_k^\pm(B-2m)^k\underset{\omega\gg 1/m}{\simeq}e^{\pm\omega(B-2m)}. \qquad (41)$$

We now find the asymptotic behavior of the solutions of Eq. (38) at large values of B. Following the standard method,[12] we readily obtain

$$t(\omega,B)\underset{B\to\infty}{\simeq}e^{\pm\omega B}.$$

This can also be seen if in (38) we ignore the mass m in comparison with the large value of B:

$$\frac{d^2t(\omega,B)}{dB^2}+\frac{2}{B}\frac{dt(\omega,B)}{dB}-\omega^2t(\omega,B)=0.$$

This equation has the linearly independent solutions

$$t_{1,2}=\frac{e^{\pm\omega B}}{B}=e^{\pm\omega B-\ln B}. \qquad (42)$$

Thus, the asymptotic behaviors of the solution $t(\omega,B)$ as $\omega\to\infty$ and $B\to\infty$ are the same ($e^{\pm\omega B}$). This means that the limits $\omega\to\infty$ and $B\to\infty$ are equivalent in the sense that they "lead" to the same asymptotic behaviors. Therefore, in the limit $B\to\infty$ we obtain, replacing $\lim_{B\to\infty}$ by $\lim_{\omega\to\infty}$,

$$(B-2m)^{\pm 2m\omega}\sum_{k=0}^\infty a_k^\pm(B-2m)^k$$

$$\xrightarrow[B\to\infty]{}e^{\pm\omega(B+2m\ln(B-2m))}\simeq e^{\pm\omega B}.$$

It follows from the boundary conditions that the decreasing exponential must be chosen. Therefore, the final expression for $t_{\rm out}$ that satisfies the boundary conditions at infinity has the form

$$t_{\text{out}} = T + \int_0^{\infty} \frac{d\omega}{(B-2m)^{2m\omega}} \sum_{k=0}^{\infty}{}' a_k^- (B-2m)^k$$

$$\times\, s^-(\omega)\, e^{-\omega T} + g^-(\omega)\, e^{\omega T}).$$

Substituting here $T = T(\tau, B)$ (22), we rewrite this expression in the form

$$t_{\text{out}} = \tau - 2\sqrt{2mB} + 2m \ln \frac{(\sqrt{B} + \sqrt{2m})^2}{B-2m}$$

$$+ \int_0^{\infty} d\omega \sum_{k=0}^{\infty}{}' a_k^- (B-2m)^k s^-(\omega)\, e^{-\omega(\tau - 2\sqrt{2mB})}$$

$$+ \int_0^{\infty} \frac{d\omega}{(B-2m)^{4m\omega}} \sum_{k=0}^{\infty} a_k^- (B-2m)^k g^-(\omega)\, e^{\omega(\tau - 2\sqrt{2mB})}. \quad (43)$$

It follows from (43) that a singularity of t_{out} on the Schwarzschild sphere $(B = 2m)$ is unavoidable. If $g^-(\omega) = 0$, then the singularity is logarithmic, but if $g^-(\omega) \neq 0$, then it is a pole singularity. Near the point $B = 2m$, we obtain

$$t_{\text{out}} \simeq -2m \ln (B-2m) + \int_0^{\infty} \frac{d\omega g^-(\omega)\, e^{\omega(\tau - 4m)}}{(B-2m)^{4m\omega}}. \quad (44)$$

For convenience in the following investigation, we represent (44) in the form

$$t_{\text{out}} \underset{B \simeq 2m}{\simeq} -2m \ln (B-2m) - \frac{F(B,\tau)}{(B-2m)^{\alpha}} \simeq \frac{F(B,\tau)}{(B-2m)^{\alpha}}, \quad (45)$$

where $F(B, \tau)$ is a bounded function, and the order of the pole α must be determined from the matching condition, since it is specified by the function $g^-(\omega)$. Using the fact that $\alpha > 0$, we can in general omit the logarithmic term in (45).

The general form of the singularity (45) on the Schwarzschild sphere enables us to establish the type of the singularity of t_{in}, and it is this in which we are interested in the first place, since, as follows from Eq. (7), it is a consequence of the "smooth" behavior of r_{in} that the only region in which a graviton mass can influence the development of gravitational collapse must be the region of singularity of the partial derivatives of t_{in} (the terms $\mu^2 \dot{r}_{\text{in}}^2$, $\mu^2 r_{\text{in}}'^2$, $\mu^2 r_{\text{in}}^2$ are always small), when the terms $\mu^2 \dot{t}_{\text{in}}^2$ and $\mu^2 t_{\text{in}}'^2$ become of the same order as the remaining terms of (7).

5. SINGULARITY OF THE INTERIOR SOLUTION FOR THE TIME COORDINATE OF THE MINKOWSKI SPACE

We recall that since the complete class of solutions t_{in} has not been found (the recursion relations are rather complicated), it has not been possible to find, as in the case of the radial coordinate $r_{\text{in}}(\tau, R)$, a unique solution for the function t that is regular at the center of the sphere and at spatial infinity and also suitably fitted ($t'_{\text{in}}/B'_{\text{in}} = t'_{\text{out}}/B'_{\text{out}}$) on its boundary. Nevertheless, a general analysis of the singularities of this solution make it possible to draw a number of fundamental conclusions about the development of gravitational collapse without even having the exact solution for t, since, as we have noted, it is the region of singularity of t_{in} that changes the picture of "massless" collapse.

Thus, we consider the equation that is satisfied by t_{in}:

$$\left(1 - \frac{\tau}{\tau_0}\right)^{-2/3} \left[\left(1 - \frac{\tau}{\tau_0}\right)^2 \dot{t}_{\text{in}}\right]^{\cdot} = \frac{1}{R^3} (R^2 t'_{\text{in}})'. \quad (46)$$

Introducing the variable $\xi = 3\tau_0 (1 - \tau/\tau_0)^{1/3}$ (28) and matching ξ and R,

$$\bar{\bar{A}} = \xi - R, \quad \bar{\bar{B}} = \xi + R,$$

we obtain for the function $V = R\xi^3 t_{\text{in}}$ an equation identical to (29):

$$V_{\bar{\bar{A}}\bar{\bar{B}}} + \frac{V_{\bar{\bar{A}}} - V_{\bar{\bar{B}}}}{\bar{\bar{A}} - \bar{\bar{B}}} = 0.$$

Using the boundary conditions

$$t_{\text{in}}(R_0, \xi) \equiv t_0(\xi) = t_{\text{out}}(R_0, \xi), \quad (47)$$

$$t'_{\text{in}}(R_0, \xi) \equiv t_1(\xi) = t'_{\text{out}}(R_0, \xi)\, \xi^3/(3\tau_0)^3, \quad (48)$$

we find by means of the Riemann–Green formula

$$t_{\text{in}}(R, \xi) = \frac{R_0}{2R\xi^2} [(\xi + R_0 - R)^2 t_0(\xi + R_0 - R) + (\xi - R_0 + R)^2 t_0(\xi - R_0 + R)] -$$

$$- \frac{1}{4\xi^3 R} \int_{\xi - R_0 + R}^{\xi + R_0 - R} x\, dx\, t_0(x) [\xi^2 + R^2 + x^2 - R_0^2]$$

$$- \frac{R_0}{4\xi^3 R} \int_{\xi - R_0 + R}^{\xi + R_0 - R} x t_1(x)\, dx\, [\xi^2 + x^2 - (R - R_0)^2]. \quad (49)$$

The function $t_{\text{in}}(R, \xi)$ determined by (49) automatically satisfies the boundary conditions (47) and (48). This formula was first found, in a somewhat different form, in Ref. 13.

The relation (49) makes it possible to investigate the region of singularity of t_{in}, since it is clear that the singularity of t_{out} on the Schwarzschild sphere leads to corresponding singularities of $t_0(\xi)$ and $t_1(\xi)$. Then, writing out the integrals in (49) by means of the mean-value theorem, we find the "propagation" of the singularity of t_{in} into the interior of the collapsar.

Using (18), (21), and (45), we can readily obtain (in the region $B \simeq 2m$)

$$t_0(x) \underset{x \sim 2R_0}{=} \frac{F(x)}{(x - 2R_0)^{\alpha}}; \quad (50)$$

$$t_1(x) = \frac{\alpha F(x)\, x^2}{(2m/R_0)^{\alpha} (3\tau_0)^2} \frac{1}{(x - 2R_0)^{\alpha+1}}, \quad (51)$$

where $F(\xi) = F(2m, \tau)$ (45). The singularities of $t_0(x)$ and $t_1(x)$ at the point $x = 2R_0$ (horizon) lead in accordance with Eq. (49) to a divergence of t_{in} at the upper and lower limits of integration, and also, because of the analogous singularities, of the first two terms. However, it is obvious that the $\xi - R_0 + R$ divergence in t_{in} occurs earlier in the time τ than the $\xi + R_0 - R$ divergence. Thus, the singular behavior is determined by the factor $\xi - R_0 + R$, a wave of the t_{in} singularity that travels from the center of the collapsing body to its surface. Having reached the surface, it is, as it were, reflected and returns to the center of the sphere. This reflected wave is none other than the $\xi + R_0 - R$ divergence of the function t_{in} at the upper limit of integration. We emphasize once more that for any dust particle the singularity corresponding to the $\xi - R_0 + R$ wave occurs earlier in the proper time τ. From (49)–(51), we find

$$t_{\rm in}^{\rm sing} = \frac{F^*(R,\xi)}{(\xi - 3R_0 + R)^\alpha} = \frac{F^{**}(R,\xi)}{\{\tau_0[1-(R_0-R/3)^3/\tau^3]-\tau\}^\alpha}, \quad (52)$$

where the functions F^* and F^{**} are smooth bounded functions of their variables.

We shall analyze (52). It follows from this expression that in the range of variation of the variable ξ, $2R_0 < \xi < 3R_0$, corresponding to the interval of proper time

$$\tau_0\left[1-\left(\frac{9}{2}\frac{m}{R_0}\right)^{3/2}\right] < \tau < \tau_0\left[1-\left(\frac{2}{3}\right)^3\left(\frac{9}{2}\frac{m}{R_0}\right)^{3/2}\right], \quad (53)$$

the "wave of the singularity" of $t_{\rm in}$ passes through the body—from the center of the sphere ($R=0$) at the time $\tau_1 = \tau_0[1-(9m/2R_0)^{3/2}]$, reaching its surface ($R=R_0$) at the time $\tau_2 = \tau_0[1-(2/3)^3(9m/2R_0)^{3/2}]$ with mean velocity

$$\vartheta = \frac{R_0}{\tau_2-\tau_1} = \frac{38}{243m/R_0}. \quad (54)$$

Substituting in this formula the parameters of the sun (which is a typical star), $m/R_0 = 10^{-6}$, we find that the propagation velocity of the $t_{\rm in}$ singularity wave is huge: 10^6c! The fact that this velocity exceeds the speed of light does not contradict anything, since this velocity is not the propagation velocity of any real physical object.

To end this section, we consider one further question: How does the motion of the dust particles of the collapsing sphere appear to an exterior observer?

We express the interval of the Riemannian space–time in the coordinates of the Minkowski space:

$$\begin{aligned} ds^2 &= d\tau^2 - B'^2\,dR^2 - B^2\,d\Omega^2 \\ &= J^{-2}[(r'^2 - \dot r^2B'^2)\,dt^2 - 2(r't' - \dot r\dot tB'^2)\,dt\,dr \\ &\quad + (t'^2 - \dot t^2B'^2)\,dr^2 - B^2\,d\Omega^2], \end{aligned} \quad (55)$$

where $J \equiv \dot t r' - t'\dot r$. The Hamilton–Jacobi function S of a particle moving in the field (55) depends, by virtue of the symmetry of the problem, on two variables: $S = S(t, r)$. In this case, the equations of motion

$$\frac{dx^i}{ds} = -g^{ik}\partial_k S \quad (56)$$

take the form

$$\frac{dr}{ds} = -(\dot r^2 - r'^2B'^2)\,\partial_r S - (\dot t\dot r - t'r'B'^2)\,\partial_t S; \quad (57)$$

$$\frac{dt}{ds} = -(\dot t^2 - t'^2B'^2)\,\partial_t S - (\dot t\dot r - t'r'B'^2)\,\partial_r S. \quad (58)$$

Since the function S is a scalar, and in comoving coordinates is equal to the proper time with the opposite sign,

$$S = -\tau, \quad (59)$$

and, in addition, $\partial\tau/\partial t = r'/J$ and $\partial\tau/\partial r = -t'/J$, we obtain

$$\frac{dt}{ds} = \dot t \equiv \frac{\partial t(\tau, R)}{\partial\tau}, \quad \frac{dr}{ds} = \dot r \equiv \frac{\partial r(\tau, R)}{\partial\tau}.$$

For the exterior observer, the velocity of a particle of the collapsing sphere,

$$\frac{dr}{dt} = \frac{\dot r_{\rm in}}{\dot t_{\rm in}} \quad (60)$$

will tend to zero when $t_{\rm in}$ enters its singularity region ($t_{\rm in} \to \infty$). At the same time, as follows from (34), $\dot r_{\rm in}$ is equal to $\dot B_{\rm in}$ and is finite (nonzero). As we see, the existence of the graviton mass significantly changes the behavior of $\dot r_{\rm in}$. In the region close to the singularity of $t_{\rm in}$, the function $\dot r_{\rm in}$ vanishes. As a consequence, so does the velocity dr/dt. At the same time, although $t_{\rm in}$ is large, it is finite. Thus, the graviton mass eliminates the degeneracy of the massless theory expressed in the fact that from the point of view of an exterior observer the surface of the collapsing sphere approaches the Schwarzschild surface for an infinitely long time, and, as we have shown, all the particles of the collapsing sphere approach their limiting radii for an infinitely long time, while from the point of view of a comoving observer there are no physical reasons for such halting of the collapse. Massive gravity provides such a reason—the effective mass–energy density of gravitons with $\mu \neq 0$ is negative and proportional to $\mu^2(\dot t_{\rm in}^2 + G^2t'^2_{\rm in}/B'^2)$. Therefore, when $t_{\rm in} \to \infty$, the total density of the matter (dust) and gravitons will, because of the rapid growth of the negative term, tend to zero, and as a consequence of this the collapse process will initially slow down, and then, after stopping (when the total density is equal to zero), will begin to develop in the opposite direction. A gravitational bounce occurs. It is in this that the new mechanism of halting of gravitational collapse through the presence of a graviton rest mass consists.

6. GRAVITATIONAL BOUNCE

We shall study the behavior of the solution in the region in which the Minkowski time is close to the singularity. As we have noted, it is in precisely this region that the graviton mass plays a part in the collapse process and can radically change it.

We write down the equations that determine the development of the gravitational collapse, omitting in a first approximation, for the reasons given above, the terms $\mu^2r_{\rm in}^2$, $\mu^2\dot r_{\rm in}^2$, $\mu^2r'^2_{\rm in}$, and μ^2. From (7)–(11) we obtain

$$8\pi\rho - \frac{\mu^2}{4}\left(\dot t_{\rm in}^2 + t'^2_{\rm in}\frac{G^2}{B'^2}\right) = -2\frac{(G^2)'}{(B^2)'} + \frac{1-G^2}{B^2} + \frac{(\dot B^2B)'}{B^2B'} - 2\frac{\dot B}{B}\frac{\dot G}{G}; \quad (61)$$

$$\left(\frac{B^2B'}{G}\dot t_{\rm in}\right)^{\cdot} = \left(\frac{B^2G}{B'}t'_{\rm in}\right)'; \quad (62)$$

$$\frac{\dot B'}{B}\frac{\dot G}{G} = \frac{\mu_2}{4}\dot t_{\rm in}t'_{\rm in}; \quad (63)$$

$$2\frac{\dot B}{B} + \frac{\dot B'}{B'} + \frac{\dot\rho}{\rho} - \frac{\dot G}{G} = 0. \quad (64)$$

From (64), integrating over τ, we find

$$\rho = \frac{\varkappa(R)\,G}{B'B^2}. \quad (65)$$

This relation is satisfied rigorously at any time. Therefore, at the start of the collapse, when the influence of the graviton mass is negligible, we obtain from (65), using (18) and (20),

$$\varkappa(R) = R^2/6\pi\tau_0^2. \quad (66)$$

We now consider the left-hand side of Eq. (61), which is the effective matter density that determines the development of the gravitational collapse. We use the expressions (18), (20), and (52) of the massless theory (omitting in what follows for simplicity the index "in"):

$$8\pi\rho = 12\,\frac{(3\tau_0)^4}{(\xi)^6},$$

$$\frac{\mu^2}{4}\left(\dot{t}^2 + t'^2\frac{G^2}{B'^2}\right) = \frac{\mu^2}{2}\,\frac{F^{*2}\alpha^2\,(3\tau_0)^4}{(\xi+R-3R_0)^{2(1+\alpha)}\xi^4}.$$

It can be seen from this that as $\xi \to 3R_0 - R$ the matter density ρ tends to $3(3\tau_0)^4/2\pi(3R_0-R)^6$, while the μ^2 terms tend to infinity. On the basis of this, we estimate the interval of variation of ξ (and, accordingly, of the proper time $\tau = \tau_0[1-(\xi/3\tau_0)^3]$), in which, without reaching the singularity $\xi^{\rm sing} = 3R_0 - R$, the effective density vanishes (the point $\xi = \xi_0$), and the bounce commences. For it is in this interval that our solution will differ from the massless case. In what follows we shall investigate precisely this region $\xi = 3R_0 - R$.

Thus, from the condition

$$\rho \sim \mu^2\left(\dot{t}^2 + \frac{G^2}{B'^2}t'^2\right)$$

we obtain

$$\xi_0(R) = \xi^{\rm sing}(R) + \varepsilon(R),$$

where

$$\varepsilon(R) \sim [\mu F^*(R)\,\alpha\,(3R_0 - R)]^{\frac{1}{\alpha+1}} \sim (\mu R_0)^{\frac{1}{\alpha+1}}. \qquad (67)$$

It follows from this that the region is fairly small:

$$\delta\xi \sim \delta\tau \sim \mu^{\frac{1}{\alpha+1}}.$$

This enables us to draw certain conclusions about the behavior of the function G. It will differ from unity only near the point of the bounce, at distances of order $\varepsilon(R)$. We shall estimate this change. Since at the point of the bounce the derivatives $\dot{t}$ and t' have not yet reached an infinite value (in fact, either of them could even decrease appreciably), we obtain from (63) an upper bound on the function $\dot{G}/G$, choosing the maximal values of t' and $\dot{t}$:

$$\frac{\dot{G}}{G} < \mu^2\,\frac{B}{B'}\,\frac{F^{*2}\alpha^2}{\varepsilon^{2(\alpha+1)}}\,\frac{(3\tau_0)^4}{\xi_0^4} = \frac{R\,(3\tau_0)^4}{(3R_0-R)^6} < \frac{(3\tau_0)^4}{(2R_0)^5}. \qquad (68)$$

It follows from this that although the number on the right-hand side of this inequality is fairly large over a distance (with respect to ξ) of order $\varepsilon(R)$, the function G varies little:

$$\Delta G \sim \left(\frac{\varepsilon_0}{3\tau_0}\right)^2\frac{\dot{G}}{G}\,\varepsilon(R) < R\,\frac{(3\tau_0)^2}{\xi_0^4}\,[\mu F^*\xi]^{\frac{1}{\alpha+1}} \ll 1,$$

since the majorant itself is proportional to the small parameter $(\mu R_0)1/(\alpha+1)$. Therefore

$$G = 1 + O[(\mu R_0)^{\frac{1}{\alpha+1}}],$$
$$G' = O[(\mu R_0)^{\frac{1}{\alpha+1}}], \qquad (69)$$

although the derivative $\dot{G}$ near the point of the bounce may still, in accordance with (68), have an appreciable value.

Our estimates enable us to simplify the system (61)–(65) and represent it in the form

$$8\pi\rho - \frac{\mu^2}{4}(\dot{t}^2 + t'^2/B'^2) = \frac{\dot{B}}{B}\left(\frac{\dot{B}}{B} + 2\,\frac{\dot{B}'}{B'} - \dot{G}\right); \qquad (70)$$

$$(2\dot{B}BB' + B^2\dot{B}')\,\dot{t} + B^2B'\,\ddot{t}$$
$$= t'\left(2B - \frac{B''B^2}{B'^2} + \frac{\mu^2}{4}B^3\dot{t}^2\right) + t''\,\frac{B^2}{B'}; \qquad (71)$$

$$\dot{G} = \frac{\mu^2}{4}\,\frac{B}{B'}\,\dot{t}t'; \qquad (72)$$

$$\rho = \frac{R^2}{6\pi B^2B'}. \qquad (73)$$

Unfortunately, even this simplified system of equations cannot be solved exactly because of its nonlinearity. Nevertheless, the nature of the solution and its asymptotic behavior near the bounce can be established.

We shall analyze the right-hand side of Eq. (70). As follows from (72), the sign of $\dot{G}$ is determined by the function $B\dot{t}t'/B'$. Since B can be physically measured directly—it is the circumference of the circle of radius R at the time τ divided by 2π, it follows that $B > 0$. Its partial derivative B' also cannot change sign ($B' > 0$), since in the case $B' = 0$ the Jacobian $g = -B^4\sin^2\theta B'^2/G^2$ vanishes. Thus, sign $\dot{G}$ = sign $\dot{t}t' = -$ sign $t't_\xi$, since $\dot{t} = t_\xi\dot{\xi} = -(3\tau_0/\xi)^2 t_\xi$. Here $t_\xi \equiv \partial t/\partial\xi$. In the singular region $t \sim F^*/(\xi + R - 3R_0)^\alpha$. Therefore $t' \simeq t_\xi$ and $\dot{G} < 0$. For the massless solution $\dot{B}/B = \dot{B}'/B' = (\xi/3\tau_0)^2$. Therefore, all three terms in the round brackets and the entire right-hand side of (70) are non-negative. On the other hand, the left-hand side (the effective density) decreases with increasing t, but since the right-hand side is non-negative, the left-hand side cannot become negative, i.e., it cannot "pass" through zero. Therefore, if at this point $\ddot{B}, \ddot{B}' > 0$, and $\dddot{t} < 0$, then, having reached zero, the effective density must then begin to increase. We consider the instant of time at which the left-hand side of (70) is equal to zero. In this case $\dot{B} = 0$.

This relation shows that the collapse is halted—the circumference of the circle of fixed radius R has ceased to decrease with increasing proper time τ. This occurred for the given radius R at the time $\xi = 3R_0 - R$. The wave of the halting of the collapse passes through the body, in the direction from the center of the sphere to its surface, reaching the surface precisely at the time at which it approaches the Schwarzschild sphere. Ahead of the front of this wave (its velocity was considered in Sec. 5) the matter is still compressed, but behind it expansion already takes place.

We consider the nature of the solution of the system (70)–(73) near the point of the bounce $\dot{B} = 0$. It is obvious that in this case the functions B and B' will have the form

$$B = R\,\frac{\xi_0^2}{(3\tau_0)^2} + (\xi - \xi_0)^2\,b(R) + O[(\xi-\xi_0)^3], \qquad (74)$$

$$B' = \frac{\xi_0^2}{(3\tau_0)^2} + (\xi - \xi_0)^2\,b'(R) + O[(\xi-\xi_0)^3], \qquad (75)$$

where $\xi_0 = 3R_0 - R + \varepsilon(R)$, $\varepsilon(R) \ll R_0$. Note that ξ_0 is a parameter and that its dependence on R is substituted in the expression for B' after B has been differentiated. The fact that the first terms in the expressions (74) and (75) are taken in the form of the values of B and B' for the massless solution at the point ξ_0 is again explained by the fact that the region of "influence" of the graviton mass on the massless solution is very small, $\sim(\mu R_0)1/(\alpha+1)$, and (as a consequence of this) the functions B and B' cannot change at all significantly. This is reflected in Figs. 1 and 2. In accordance with them, the choice of the first terms in (74) and (75) corresponds to the choice of the values B_2 and B_2' instead of the exact values B_1 and B_1'.

It follows from the boundary condition $B(\tau, 0) = 0$, $B'(\tau, 0) < \infty$ that

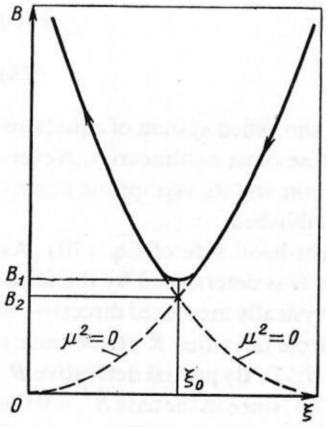

FIG. 1. Dependence of the Schwarzschild radius on the parameter ξ, which is related to the proper time τ by means of Eq. (28). The broken curve corresponds to the solution of the massless theory.

$$b(R) \underset{R=0}{\sim} R^{\nu}, \quad \nu \geqslant 1. \tag{76}$$

We shall show below that this is indeed correct. From (75) we find that at the points $\xi=\xi_0$

$$\dot{B}'=\dot{B}=0. \tag{77}$$

From the physical point of view it is obvious that at the same time we must have $\dot{\rho}=0$ and, as follows from (64), $\dot{G}=0$ as well. We shall show that at $\xi=\xi_0$ the relation $\dot{G}=0$ holds, and, therefore, also $\dot{\rho}=0$. For this, we consider Eq. (71) in the neighborhood of the points of the bounce $\xi=\xi_0$. As was noted earlier, at $\xi \approx \xi_0$ the derivatives $\dot{t}$ and t' are finite. Therefore, near ξ_0 we must be able to represent the derivative t_ξ in the form of its Taylor expansion:

$$t_\xi=-\frac{\alpha F^*}{\varepsilon^{\alpha+1}}+c(R)(\xi-\xi_0)+a(R)(\xi-\xi_0)^2+O[(\xi-\xi_0)^3]. \tag{78}$$

The choice of the first term in (78) can be justified by analogy with the choice of the first terms of the expansions in Eqs. (74) and (75). It follows from (78) that (ξ_0 is not differentiated)

$$t'_\xi=-\alpha\left(\frac{F^*}{\varepsilon^{\alpha+1}}\right)'+c'(R)(\xi-\xi_0)+a'(R)(\xi-\xi_0)^2+O[(\xi-\xi_0)^3].$$

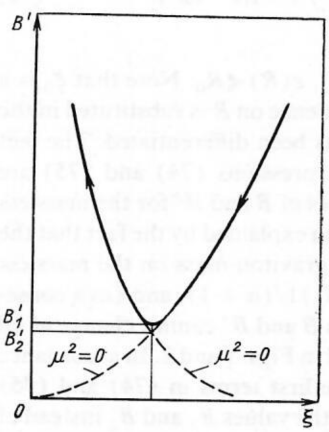

FIG. 2. Dependence of the function $B'(\xi, R)\equiv\partial B(\xi, R)/\partial R$ on the parameter ξ for a fixed value of R. The broken curve corresponds to the solution of the massless theory.

Hence, integrating over ξ, we find

$$t'=d(R)-\alpha\left(\frac{F^*}{\varepsilon^{\alpha+1}}\right)'(\xi-\xi_0)+\frac{c'(R)}{2}(\xi-\xi_0)^2+O[(\xi-\xi_0)^3]. \tag{79}$$

This expression satisfies equality of the mixed derivatives with the expression (78): $t_{\xi R}=t_{R\xi}$.

We substitute the expansions (74), (75), (78), and (79) in the RTG equation (71). Equating to zero the coefficient of the zeroth power of $(\xi-\xi_0)$ and the coefficient of the first power of $(\xi-\xi_0)$, we obtain accordingly

$$d'+\left[\frac{2}{R}+q(R)\right]d=c; \tag{80}$$

$$b'+\frac{2}{R}b-\frac{\xi_0^2}{(3\tau_0)^2}a=\frac{1}{2}\left(\frac{\xi_0}{3\tau_0}\right)^2\left\{\left(\frac{F^*}{\varepsilon^{\alpha+1}}\right)'\left[\frac{2}{R}+q(R)\right]+\left(\frac{F^*}{\varepsilon^{\alpha+1}}\right)''\right\}\Big/\frac{F^*}{\varepsilon^{\alpha+1}}, \tag{81}$$

where

$$q(R)=\frac{\mu^2}{4}\frac{\alpha^2F^{*2}}{\varepsilon^{2(\alpha+1)}}R.$$

A similar procedure for Eq. (70) [or, more precisely, (61)] enables us, using the expansions (74), (75), (78), and (79) and

$$G=1-\frac{\alpha F^*}{\varepsilon^{\alpha+1}}\frac{\mu^2}{4}Rd(\xi-\xi_0)+O((\xi-\xi_0)^2), \tag{82}$$

to obtain the equations

$$\frac{\alpha^2F^{*2}}{\varepsilon^{2(\alpha+1)}}+d^2=\frac{48}{\mu^2\xi_0^2}; \tag{83}$$

$$c=d'+\frac{2}{R}d+\frac{2d}{\xi_0^2}(3R+b(3\tau_0)^2). \tag{84}$$

Subtracting (84) from (80), we obtain

$$d\left[\frac{3}{\xi_0^2}R+b\frac{(3\tau_0)^2}{\xi_0^2}-\frac{\mu^2}{8}\frac{F^{*2}\alpha^2}{\varepsilon^{2(\alpha+1)}}R\right]=0.$$

This equation can have the root $d=0$. Suppose, however, that

$$d\neq 0. \tag{85}$$

Then

$$\frac{3}{\xi_0^2}+\frac{b}{R}\left(\frac{3\tau_0}{\xi_0}\right)^2=\frac{\mu^2}{8}\frac{F^{*2}\alpha^2}{\varepsilon^{2(\alpha+1)}}. \tag{86}$$

We now consider the field equation (1c), which can be represented in the form

$$\frac{1-G^2}{B^2}+\frac{2\ddot{B}B+\dot{B}^2}{B^2}+\frac{\mu^2}{2}\left[1-\frac{(\dot{t}^2-\dot{r}^2)}{2}+\frac{G^2}{2B'^2}(r'^2-t'^2)-\frac{r^2}{B^2}\right]=0. \tag{87}$$

Ignoring in a first approximation the small terms μ^2, μ^2r^2, $\mu^2\dot{r}^2$, and $\mu^2r'^2$, we find that at the point $\xi=\xi_0$ this equation gives

$$\frac{4b}{R}\left(\frac{3\tau_0}{\xi_0}\right)^2=\frac{\mu^2}{4}\frac{F^{*2}\alpha^2}{\varepsilon^{2(\alpha+1)}}+\frac{\mu^2d^2}{4}.$$

Eliminating from this and from Eq. (86) the term proportional to b, we obtain

$$\frac{\alpha^2F^{*2}}{\varepsilon^{2(\alpha+1)}}-d^2=\frac{48}{\mu^2\xi_0^2}. \tag{88}$$

Comparing this equation with the relation (83), we arrive at the conclusion that

$$d = 0,$$

which contradicts the original assumption (85). Therefore

$$d(R) = c(R) = 0. \quad (89)$$

Thus, the RTG equation (71) has led us to an important conclusion: near the point at which the collapse is halted the derivative t' must tend to zero. At the points of the bounce, we have

$$t' = 0.$$

On the basis of (72), we immediately conclude that at the same time

$$\dot{G} = 0,$$

and it follows from (64) that

$$\dot{\rho} = 0,$$

the result that we needed to prove.

We now show that at the point $\xi = \xi_0$ the second derivatives with respect to the proper time $\ddot{B}$ and $\ddot{B}'$ are positive, while the third derivative $\dddot{t}$ is negative, and therefore this point of halting really is a point of bouncing. Bearing in mind that at the point of halting $G = 1$, $\dot{B} = t' = 0$, we find from (87)

$$\frac{\ddot{B}}{B} = \frac{\mu^2 \dot{t}^2}{8}.$$

Thus, the second derivative $\ddot{B}$ is positive. It follows from (83) and (89) that at the point $\xi = \xi_0$

$$\frac{\alpha^2 F^{*2}}{\varepsilon^{2(\alpha+1)}} = \frac{48}{\mu^2 \xi_0^2}. \quad (90)$$

Hence, and also from (74), we find the function $b(R)$:

$$b(R) = \frac{6}{(3\tau_0)^2} R > 0. \quad (91)$$

Thus, the boundary condition (76) is satisfied. It follows from (75) and (91) that at $\xi = \xi_0$

$$b'(R) = \frac{6}{(3\tau_0)^2} > 0,$$

whence $\ddot{B}'(\xi = \xi_0) > 0$.

It is shown in the Appendix that

$$a > 0. \quad (92)$$

By means of (78), (90), and (92) we conclude that at $\xi = \xi_0$ the derivative $\dot{t}$ has a maximum:

$$\dot{t}_{\xi=\xi_0} = \left(\frac{3\tau_0}{\xi_0}\right)^2 \left(\frac{\sqrt{48}}{\mu \xi_0} - a(\xi - \xi_0)^2\right). \quad (93)$$

After the bounce, the collapse process will develop in the opposite direction.

We now show that at the point of bounce $(\xi = \xi_0)$ $\dot{r} = 0$. We use an expansion of the function $r(\xi, R)$ in a series:

$$r = P(R) + Q(R)(\xi - \xi_0) + L(R)(\xi - \xi_0)^2 + M(R)(\xi - \xi_0)^3 + O((\xi - \xi_0)^4). \quad (94)$$

Substituting this expression and the expansions of the functions B and G in the RTG equation (40), and equating to zero the coefficients of the zeroth and first powers of the factor $(\xi - \xi_0)$, we obtain

$$P'' + \frac{2}{R} P' - 2\frac{P}{R^2} = 2L; \quad (95)$$

$$Q'' + \frac{2}{R} Q' - Q\left(1 + \frac{2}{R^2}\right) = \frac{M \xi_0^2}{6}. \quad (96)$$

As we have noted, the "region of influence" of the graviton mass on the collapse process is very small, $\sim (\mu R_0) 1/(\alpha + 1)$, and therefore, for the first term of the expansion of $P(R)$, we can take its expression from the massless solution:

$$P(R) = R\left(\frac{\xi_0}{3\tau_0}\right)^2 + \frac{R}{(3\tau_0)^2}(R^2 - 3R_0^2). \quad (97)$$

Then from (95) and (97) we obtain

$$L = 5R/(3\tau_0)^2. \quad (98)$$

To find Q and, accordingly, M, we must consider one of the modified Hilbert–Einstein equations, for example, (7) or (8), in the second order of perturbation theory.

Then, following the standard scheme, we must, retaining the second-order terms μ^2, $\mu^2 r^2/B^2$, $\mu^2 \dot{r}^2$, and $\mu^2 r'^2/B^2$ (the small parameter is $\mu^2 R_0^2$), substitute in the chosen equation the values for the functions ρ, B, G, and t obtained earlier in the first approximation. Application of this procedure to the exact equation (7),

$$\frac{\dot{B}'}{B}\,\frac{\dot{G}}{G} = \frac{\mu^2}{4}(\dot{t}t' - \dot{r}r'),$$

immediately gives

$$P'(R)\, Q(R) = 0.$$

Hence, and from (96), we find

$$Q(R) = M(R) = 0. \quad (99)$$

Therefore

$$\dot{r}\,|_{\xi=\xi_0} = Q\dot{\xi} = 0, \quad (100)$$

as we needed to prove.

Thus, it follows from the expression (60) that the velocity of the particles of the collapsing sphere vanishes for an exterior observer too at the points of the bounce: $dr/dt = 0$.

Allowance for the P and L terms in Eqs. (7) and (8) leads to an unimportant modification of the expressions (74) and (75) (in the second order) that does not disturb the boundary condition (76). There is no point in giving this contribution in explicit form, since the first terms of the expansions (74) and (75) already have an error corresponding to the magnitude of this contribution, $\sim \mu^2 R_0^2$ (see Figs. 1 and 2).

The final question which we analyze is this: Does our solution describe a physical gravitational field, i.e., a field in which the light cone of the effective Riemannian space–time lies within the light cone of the Minkowski space? Since we have at our disposal an approximate solution only in the singular region, we prove the required property—that the gravitational field is physical, near the point of the bounce. From the condition

$$ds^2 = d\tau^2 - B'^2\, dR^2 - B^2 d\Omega^2 > 0 \quad (101)$$

and the relations (74) and (75), we find a restriction on the coordinate velocity of a particle moving in the metric (101):

$$\left|\frac{dr^\beta}{d\tau}\right|^2 < \left(\frac{3\tau_0}{\xi_0}\right)^4, \quad (102)$$

where $(r^\beta)^2 = x^2 + y^2 + z^2$; $x = R\cos\varphi \sin\theta$; $y = R \sin$

$\varphi \sin\theta$; $z = R\cos\theta$. It follows from the inequality (102) that

$$\dot{R}^2 < \left(\frac{3\tau_0}{\xi_0}\right)^4 \quad \text{and} \quad R^2\dot{\theta}^2 < \left(\frac{3\tau_0}{\xi_0}\right)^4. \qquad (103)$$

We now consider the Minkowski-space interval

$$d\sigma^2 = \dot{t}_{\text{in}}^2\, d\tau^2 - dr_{\text{in}}^2 - r_{\text{in}}^2\, d\Omega^2.$$

In accordance with (94) and (97)–(99),

$$r_{\text{in}} = R\left(\frac{\xi_0}{3\tau_0}\right)^2 + \frac{R}{(3\tau_0)^2}(R^2 - 3R_0^2) + O[(\xi-\xi_0)^2].$$

Hence we find

$$\dot{r}_{\text{in}} = \dot{R}\left[\left(\frac{\xi_0}{3\tau_0}\right)^2 + \frac{3m}{R_0}\left(\frac{R^2}{R_0^2} - 1\right)\right] < \dot{R}\left(\frac{\xi_0}{3\tau_0}\right)^2.$$

By means of this inequality and (103), we conclude that $\dot{r}_{\text{in}} < 1$. Since

$$r_{\text{in}} = R\left[\left(\frac{\xi_0}{3\tau_0}\right)^2 + \frac{m}{2R_0}\left(\frac{R^2}{R_0^2} - 3\right)\right] < R\left(\frac{\xi_0}{3\tau_0}\right)^2,$$

we find from (103) that $r_{\text{in}}\dot{\theta} < 1$.

Thus, the Minkowski-space interval can be represented in the form

$$d\sigma^2 = d\tau^2(\dot{t}_{\text{in}}^2 - \dot{r}_{\text{in}}^2 - r_{\text{in}}^2\dot{\theta}^2) > d\tau^2(\dot{t}_{\text{in}}^2 - 2) > 0.$$

This inequality is certainly satisfied, since in accordance with (93)

$$\dot{t}_{\text{in}}^2 \sim \frac{1}{\mu^2 m^2} \gg 1.$$

We note in conclusion that the approach of the surface of the collapsar to the Schwarzschild sphere, i.e., the duration of the collapse from its onset to the bounce, occupies a fairly long time according to the clock of an exterior observer. The time can be readily estimated by using the power-law dependence of t on ε:

$$t = \frac{F}{\varepsilon^\alpha} = \frac{\dot{t}}{\alpha\dot{\xi}}\varepsilon, \quad \alpha > 0.$$

By means of (67) and (93), we find that the collapse time is

$$t \sim (\mu R_0)^{-\frac{\alpha}{\alpha+1}}.$$

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APPENDIX

We prove that the function $a(R)$ in the expansion (78),

$$t_\xi = -\alpha\frac{F^*(R)}{\varepsilon^{\alpha+1}} + a(R)(\xi-\xi_0)^2 + O((\xi-\xi_0)^3),$$

is positive. It follows from (81) that it is equal to

$$a = \left(\frac{3\tau_0}{\xi_0}\right)^2\left(b' + \frac{2}{R}b\right) - \left[\left(\frac{F^*}{\varepsilon^{\alpha+1}}\right)'\left(\frac{1}{R} + \frac{6R}{\xi_0^2}\right) + \frac{1}{2}\left(\frac{F^*}{\varepsilon^{\alpha+1}}\right)''\right] \Big/ \left(\frac{F^*}{\varepsilon^{\alpha+1}}\right). \qquad (A1)$$

From (91) we find that

$$\left(\frac{3\tau_0}{\xi_0}\right)^2\left(b' + \frac{2}{R}b\right) = \frac{18}{\xi_0^2} > 0.$$

We estimate the last term in the expression (A1). Since $\alpha F^*(R)/\varepsilon^{\alpha+1}$ is none other than the value of the function t_{in} for the massless case [see (52)], for the estimates in which we are interested we write down the solution of the massless equation for t_{in}, separating the variables. We replace t_{in} by the new function

$$U(\xi, R) = \xi^{3/2} R t_{\text{in}}(\xi, R)$$

and substituting it in Eq. (46) for t_{in}, we obtain

$$U_{\xi\xi} + \frac{1}{\xi}U_\xi - \frac{9}{4\xi^2}U = U_{RR}.$$

Separating the variables, we find that the solution regular at the point $R = 0$ ($t = U/R\xi^{3/2}$) has the form

$$U(\xi, R) = \operatorname{sh}\omega R\,[pI_{3/2}(\omega\xi) + qK_{3/2}(\omega\xi)],$$

where p and q are certain constants, I is a Bessel function of imaginary argument, and K is the Macdonald function. Hence

$$t_{\text{in}} = \int_0^\infty \frac{\operatorname{sh}\omega R}{R\xi} \times\left[p(\omega)\left(\frac{\operatorname{sh}\omega\xi}{\omega\xi} - \operatorname{ch}\omega\xi\right) + q(\omega)\left(\frac{\operatorname{ch}\omega\xi}{\omega\xi} - \operatorname{sh}\omega\xi\right)\right]d\omega. \qquad (A2)$$

For the convergence of the integral (A2) at the initial times of the collapse ($\xi \sim 3\tau_0$), we must establish a bound on the rate of growth of the functions $p(\omega)$ and $q(\omega)$:

$$p(\omega),\ q(\omega) < \text{const}\, e^{-3\tau_0\omega}. \qquad (A3)$$

Hence and from the strong inequality

$$\frac{\tau_0}{R_0} = \sqrt{\frac{2}{9}\frac{R_0}{m}} \gg 1$$

it follows that in the region in which we are interested, near the singularity, $\xi \approx \xi_0$, $t \approx \alpha F^*\dot{\xi}/\varepsilon^{\alpha+1}$, the integration over ω in the expression (A2) is effectively cut off at the upper limit $(1/3)\tau_0$. As we have already noted, at $\xi \approx \xi_0$

$$\left(\frac{F^*}{\varepsilon^{\alpha+1}}\right)' \Big/ \left(\frac{F^*}{\varepsilon^{\alpha+1}}\right) \approx \frac{\dot{t}_{\text{in}}'}{\dot{t}_{\text{in}}}.$$

Therefore, using new notation for the ξ-dependent integrand, and also introducing the cutoff at the upper limit,

$$\dot{t}_{\text{in}} \equiv \int_0^\infty \frac{\operatorname{sh}\omega R}{\omega R}\psi(\omega, \xi)\,d\omega \approx \int_0^{(1/3)\tau_0} \frac{\operatorname{sh}\omega R}{\omega R}\psi(\omega, \xi)\,d\omega,$$

we obtain, by differentiating with respect to R in the integrand,

$$\frac{\dot{t}_{\text{in}}'}{\dot{t}_{\text{in}}} = \frac{\displaystyle\int_0^{(1/3)\tau_0} \frac{\left(\operatorname{ch}\omega R - \dfrac{\operatorname{sh}\omega R}{\omega R}\right)}{\omega R}\psi(\omega, \xi)\,d\omega}{\displaystyle\int_0^{(1/3)\tau_0} \frac{\operatorname{sh}\omega R}{\omega R}\psi(\omega, \xi)\,d\omega}.$$

Expanding the R-dependent functions in series,

$$\frac{\dot{t}_{\text{in}}'}{\dot{t}_{\text{in}}} = \frac{R}{3}\,\frac{\displaystyle\int_0^{(1/3)\tau_0} \omega^2\left(1 + \frac{\omega^2R^2}{10} + \frac{\omega^4R^4}{320} + \dots\right)\psi(\omega, \xi)\,d\omega}{\displaystyle\int_0^{(1/3)\tau_0}\left(1 + \frac{\omega^2R^2}{6} + \frac{\omega^4R^4}{120} + \dots\right)\psi(\omega, \xi)\,d\omega},$$

we establish the upper bound

$$\dot{t}_{\text{in}}'/\dot{t}_{\text{in}} < \frac{R}{27\tau_0^2}.$$

Similarly, we find that

$$\frac{\dot{t}''_{\text{in}}}{\dot{t}_{\text{in}}} \simeq \frac{1}{3} \frac{\int_0^{(1/3)\tau_0} \omega^2 \left(1+\frac{3}{10}\omega^2 R^2+\frac{1}{64}\omega^4 R^4+\ldots\right) \psi(\omega, \xi)\, d\omega}{\int_0^{(1/3)\tau_0} \left(1+\frac{\omega^2 R^2}{6}+\frac{\omega^4 R^4}{120}+\ldots\right)^2 \psi(\omega, \xi)\, d\omega}$$

and

$$\frac{\dot{t}''_{\text{in}}}{\dot{t}_{\text{in}}} < \frac{1}{27\tau_0^2}.$$

Finally, we obtain

$$\left[\left(\frac{F^*}{\varepsilon^{\alpha+1}}\right)'\left(\frac{1}{R}+\frac{6R}{\xi_0^2}\right)+\frac{1}{2}\left(\frac{F^*}{\varepsilon^{\alpha+1}}\right)''\right]\Big/\frac{F^*}{\varepsilon^{\alpha+1}} < \frac{1}{18\tau_0^2}.$$

Therefore

$$a > \frac{18}{\xi_0^2} - \frac{1}{18\tau_0^2} = \frac{18}{\xi_0^2}\left(1-\frac{1}{8}\frac{m}{R_0}\right) > 0,$$

as we needed to prove.

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