An analytic method of studying the stochastic Potts model

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Stochastic n-component lattices are studied. A general classification of annealed and quenched statistical-mechanics systems is given. The Potts model, including its simplex representation, is studied in detail. A number of applications in condensed-matter physics and combinatorial optimization problems are described. A method is developed for studying n-component quenched lattices with arbitrary distributions of random interaction strengths and fields. The results obtained using this method are reviewed. They show that, in contrast to the viewpoint prevailing earlier, the form of the distribution of the quenched parameters of a model significantly affects the thermodynamic properties and structure of the phase diagram of the model.

1. QUENCHED STATISTICAL-MECHANICS SYSTEMS

In a number of statistical systems a state is characterized by more than one type of variable. Depending on how these variables are involved in the calculation of thermodynamic quantities, they can be classified as annealed or quenched variables, and, accordingly, the statistical systems are referred to as annealed or quenched. Let us describe this classification for classical systems within the framework of the canonical ensemble.

Let the state of a system be described by two types of variable, S and J. The Hamiltonian of the system H is a function of these variables, $H = H(S, J)$. If the variables S and J are on an equal footing, the total probability density $P(S, J)$ has the form

$$P\,(S,\ J) = \mathrm{Z}^{-1} \exp\,[-\beta H\,(S,\ J)]. \tag{1}$$

Here Z is the partition function of the system,

$$\mathrm{Z} = \int DSDJ \exp\,[-\beta H\,(S,\ J)], \tag{2}$$

where $\int DSDJ...$ stands for integration over continuous variables and summation over discrete ones. Relations (1) and (2) describe the situation where the objects characterized by the variables S and J are in thermodynamic equilibrium with each other. Accordingly, such systems are referred to as annealed systems. Their free energy has the form

$$\mathcal{F} = -\beta^{-1} \ln \mathrm{Z}. \tag{3}$$

The probability density of one of the two types of variable, say, J, can be obtained by integrating (1) over the variables of the other type:

$$P_J\,(J) = \int DS\mathrm{Z}^{-1} \exp\,[-\beta H\,(S,\ J)] \tag{4}$$

$$= \mathrm{Z}_J^{-1} \exp\,[-\beta H_J\,(J)],$$

where $H_J(J)$ is the effective Hamiltonian,

$$\exp\,[-\beta H_J\,(J)] = \int DS \exp\,[-\beta H\,(S,\ J)], \tag{5}$$

and $Z_J = Z$ is the corresponding partition function

$$\mathrm{Z}_J = \int DJ \exp\,[-\beta H_J\,(J)] \tag{6}$$

or, alternatively, for S,

$$P_S\,(S) = \int DJ\mathrm{Z}^{-1} \exp\,[-\beta H\,(S,\ J)] \tag{7}$$

$$= \mathrm{Z}_S^{-1} \exp\,[-\beta H_S\,(S)],$$

$$\exp\,[-\beta H_S\,(S)] = \int DJ \exp\,[-\beta H\,(S,\ J)], \tag{8}$$

$$\mathrm{Z}_S = \mathrm{Z} = \int DS \exp\,[-\beta H_S\,(S)]. \tag{9}$$

In summary, it can be stated that annealed systems are ordinary statistical-mechanics systems containing objects of several different types.

The relation between the variables S and J can also be different. There exist systems in which the probability density of one type of variable, say, J, is fixed and independent of the state of the objects characterized by variables of the other type—the variables S. We denote this probability density by

$$P_J\,(J). \tag{10}$$

The probability of finding the system in the state S for given values of J, i.e., the conditional probability, is given by the canonical distribution

$$P_S\,(S \mid J)\,DSDJ = \mathrm{Z}_S^{-1}\,(J) \exp\,[-\beta H\,(S \mid J)]\,DSDJ, \tag{11}$$

where $Z_S(J)$, being a function of J, can be termed the partition function of the S-subsystem for given variables J,

$$\mathrm{Z}_S\,(J) = \int DS \exp\,[-\beta H\,(S,\ J)]. \tag{12}$$

In relation to the fact that the distribution density of the variables J (10) is fixed, in other words, that the objects described by these variables are in some given state, the variables J are referred to as "quenched." Therefore, this system is classified as a quenched system.

The total probability density has the form

$$P\,(S,\ J) = P_J\,(J)\,P_S\,(S \mid J)$$

$$= P_J\,(J)\,\mathrm{Z}_S^{-1}\,(J) \exp\,[-\beta H\,(S,\ J)]. \tag{13}$$

Accordingly, the (thermodynamic) expectation value $\mathscr{A}$ of some dynamical quantity $A(S, J)$ is given by the expression

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$$\mathcal{A} = \iint DJDSP_J(J)\, Z_S^{-1}(J) \exp[-\beta H(S, J)]\, A(S, J) = \int DJP_J(J)\, \langle A \rangle_H(J), \qquad (14)$$

where $\langle A \rangle_H(J)$ is the expectation value of this dynamical quantity with respect to the conditional probability distribution $P_S(S|J)$, i.e., its Gibbs average for given values of the variable J,

$$\langle A \rangle_H(J) = \int DSZ_S^{-1}(J) \exp[-\beta H(S, J)]\, A(S, J). \qquad (15)$$

Alternatively, if the expectation value with respect to the probability density of the variables is denoted by $\langle ... \rangle$,

$$\langle \ldots \rangle = \int DJP_J(J)\, (\ldots), \qquad (16)$$

the thermodynamic value $\mathscr{A}$ of the dynamical quantity $A(S, J)$ can be written as

$$\mathcal{A} = \langle \langle A \rangle_H \rangle. \qquad (17)$$

Integrating the total probability density (13) over J, we obtain the probability density of only the variables S:

$$P_S(S) = \int DJP_J(J)\, Z_S^{-1}(J) \exp[-\beta H(S, J)]. \qquad (18)$$

Next, let us consider the quantity

$$F(J) = -\beta^{-1} \ln Z_S(J), \qquad (19)$$

which can be referred to as the free energy of the S-subsystem for given values of the variables J. According to relations (12) and (15), in which the variables J enter as parameters, for any value of J the quantities $\langle A \rangle_H(J)$ and $F(J)$ are related to each other via the usual thermodynamic relations. If we then introduce the quantity

$$\mathcal{F} = \langle F(J) \rangle = \int DJP_J(J)\, F(J), \qquad (20)$$

then, since the thermodynamic relations are linear, in particular, differential, the same relations will connect the thermodynamic values of dynamical quantities A and the quantity $\mathcal{F}$ (20). For this reason it is completely valid to identify $\mathcal{F}$ with the free energy of the quenched system. The quantity $\mathcal{F}$ (20) will perform exactly the same functions. In particular, for a quenched system, starting from the free energy

$$\mathcal{F} = -\beta^{-1} \langle \ln Z_S(J) \rangle = -\beta^{-1} \int DJP_J(J) \ln Z_S(J) \qquad (21)$$

one can use the methods of ordinary thermodynamics to calculate the thermodynamic characteristics.

Therefore, quenched systems differ from annealed systems, i.e., from ordinary statistical-mechanics systems only by the fact that the Gibbs probability density (1) is replaced by the non-Gibbs form (13) with a given distribution density of the variables J (10). A consequence of this (without bringing in any additional physical arguments) is that expression (3) giving the relation to thermodynamics is replaced by expression (20).

An example of a way in which quenched systems can be obtained in practice is the following. A sample is heated to a temperature θ_j at which the relaxation times of the J- and S-objects are comparable, and allowed to reach a state in which they are in thermodynamic equilibrium with each other (although this is not at all necessary). The system is then rapidly cooled to a temperature $\theta_{\rm ch}$, at which the relaxation time of the J-objects will be much longer than the S-object relaxation time, with this relation being valid in some range of temperatures including $\theta_{\rm ch}$ and in certain ranges of the other thermodynamic parameters of the system. If we now work in these ranges on a time scale much smaller than the J-object relaxation time, but much larger than the S-object relaxation time, the state of the J-objects can be assumed fixed, and the state of the S-objects can be assumed to be an equilibrium state described by the densities (10) and (11). Here the probability density $P_J(J)$ will correspond to the equilibrium probability density of the variables J (4) at the temperature θ_i.

Here we shall study the quenched Potts model using the Bogolyubov variational method.[1,2]

2. THE POTTS MODEL

Here we shall give a formulation of the Potts model, which is used to describe a wide range of objects and phenomena in statistical mechanics and condensed-matter physics and which is also applicable to optimization problems.

The Potts model is a natural generalization of the Ising model. In the Ising model there are N discrete objects called lattice points, each of which can be found in one of two states. The Hamiltonian of the model is written as a sum, over pairs of interacting neighbors, of the pair interaction energy of the points. The pair interaction energy takes one value if the interacting points are in the same state, and a different value if they are in different states. In the Potts model each point can be in one of $q \geqslant 2$ states, and the pair interaction energy takes one value if the interacting points are in identical states (no matter what the state is), and a different value if they are in different states (again, this is independent of which states they are in). (The Hamiltonians of the two models can also contain the sum of the energies of the individual lattice points.)

Let us consider a lattice containing N points. Each point of the lattice can be in one of the states $P_1, \ldots, P_q$, $q \geqslant 2$. The state of a point numbered i ($1 \leqslant i \leqslant N$) will be described by the variable S_i: $S_i = P_1, \ldots, P_q$. The Hamiltonian of the Potts model has the form

$$H = \sum_{i<j} H_{ij}(S_i, S_j) + \sum_i H_{0i}(S_i), \qquad (22)$$

where the summation in the first term runs over pairs of different lattice points.

The pair interaction energy $H_{ij}(S_i, S_j)$ is taken to be $E_{ij}^{(1)}$ if the points i and j are in identical states $S_i = S_j$, and $E_{ij}^{(2)}$ if the points i and j are in different states $S_i \neq S_j$:

$$H_{ij}(S_i, S_j) = E_{ij}^{(1)} \delta(S_i, S_j) + E_{ij}^{(2)} [1 - \delta(S_i, S_j)], \qquad (23)$$

where

$$\delta(S_i, S_j) = \begin{cases} 1, & S_i = S_j, \\ 0, & S_i \neq S_j. \end{cases} \qquad (24)$$

Discarding the inessential constant $\Sigma_{i<j} E_{ij}^{(2)}$ in the Hamiltonian H and using the notation $E_{ij} = E_{ij}^{(1)} - E_{ij}^{(2)}$, the Hamiltonian can be written as

$$H=\sum_{i<j} E_{ij}\delta(S_i, S_j)+\sum_i H_{0i}(S_i). \tag{25}$$

Pairs of points (i,j) for which $E_{ij} = 0$ are referred to as pairs of interacting points (neighbors).

The partition function of this model is

$$Z=\sum_{S_1=P_1,\dots,P_q} \cdots \sum_{S_N=P_1,\dots,P_q} \times\exp\Big[\sum_{i<j} E_{ij}\delta(S_i, S_j)+\sum_i H_{0i}(S_i)\Big]. \tag{26}$$

This model was proposed in Ref. 3 and is referred to as the Potts model (more precisely, the standard Potts model).

3. THE SIMPLEX REPRESENTATION OF THE POTTS MODEL

It would be convenient if the states $P_1, \dots, P_q$ could be assigned a vector interpretation, with the delta function $\delta(S_i, S_j)$ represented as the scalar product of vectors describing the states S_i and S_j. For this we need to find a system of vectors for which the scalar product of each pair of (different) vectors is identical.

Let us consider a system of $m+1$ vectors

$$\{\varepsilon_p\} = \{\varepsilon_1, \dots, \varepsilon_{m+1}\} \tag{27}$$

in m-dimensional Euclidean space R^m defined by the relation[4]

$$\varepsilon_p\varepsilon_q = (m+1)^{-1} m\delta_{pq} - m^{-1},$$
$$p, q = 1, \dots, m+1. \tag{28}$$

In the one-dimensional case R^1 the system of vectors $\{\boldsymbol{\varepsilon}_p\}$ has the form

$$\boldsymbol{\varepsilon}_1 = 1, \quad \boldsymbol{\varepsilon}_2 = -1. \tag{29}$$

An example of a system of vectors $\{\boldsymbol{\varepsilon}_p\}$ in R^m for $m>1$ can be constructed by induction in the dimension of the space m, starting from R^1, i.e., from the vectors (29).

Let the system $\{\mathbf{e}_p\} = \{\mathbf{e}_1, \dots, \mathbf{e}_m\}$ in the space R^{m-1} of dimension $m-1$ satisfy the defining relation (28). Namely, in the space R^{m-1} let the vectors $\{\mathbf{e}_p\}$ satisfy the relation

$$\mathbf{e}_p\mathbf{e}_q = m^{-1}(m-1)\delta_{pq} - (m-1)^{-1},$$
$$p, q = 1, \dots, m. \tag{30}$$

Then the desired system $\{\boldsymbol{\varepsilon}_p\} = \{\boldsymbol{\varepsilon}_1, \dots, \boldsymbol{\varepsilon}_{m+1}\}$ in the m-dimensional R^m will be the following:

$$\varepsilon_{pm} = \cos\vartheta_m, \quad p = 1, \dots, m, \tag{31}$$

$$\varepsilon_{px} = e_{px}\sin\vartheta_m, \quad p = 1, \dots, m,$$
$$x = 1, \dots, m-1, \tag{32}$$

$$\varepsilon_{m+1x} = \delta_{mx}, \quad x = 1, \dots, m, \tag{33}$$

where

$$\cos\vartheta_m = -m^{-1} \tag{34}$$

We shall now show that Eq. (28) is satisfied for the vectors (31)–(33). Let us consider the following three cases separately: a) $p, q \neq m+1$; b) $p = m+1$, $q \neq m+1$; c) $p = q = m+1$. If $p, q \neq m+1$, then, according to (31)–(33),

$$\boldsymbol{\varepsilon}_p\boldsymbol{\varepsilon}_q = \sum_{x=1}^{m-1} e_{px}e_{qx}\sin^2\vartheta_m + \cos^2\vartheta_m$$
$$= \mathbf{e}_p\mathbf{e}_q(1-m^{-2}) + m^{-2}, \tag{35}$$

and, owing to (30),

$$\boldsymbol{\varepsilon}_p\boldsymbol{\varepsilon}_q = [(m-1)^{-1} m\delta_{pq} - (m-1)^{-1}]$$
$$\times m^{-2}(m-1)(m+1) + m^{-2}$$
$$= m^{-1}(m+1)\delta_{pq} - m^{-1}. \tag{36}$$

In the case where $p = m+1$ and $q \neq m+1$, we have

$$\boldsymbol{\varepsilon}_p\boldsymbol{\varepsilon}_q = \sum_{x=1}^{m+1} \delta_{mx}\varepsilon_{qx} = \cos\vartheta_m$$

$$= -m^{-1} = m^{-1}(m+1)\delta_{pq} - m^{-1}. \tag{37}$$

If $p = q = m+1$, then

$$\boldsymbol{\varepsilon}_p\boldsymbol{\varepsilon}_q = 1 = m^{-1}(m+1)\delta_{pq} - m^{-1}. \tag{38}$$

Relations (36)–(38) show that (31)–(33) is an example of a system of vectors in R^m satisfying the property (28). The system of vectors $\{\boldsymbol{\varepsilon}_p\}$ is clearly defined in R^m up to rotations.

The vectors of the system $\{\boldsymbol{\varepsilon}_p\}$ are directed toward the vertices of a right $(m+1)$-vertex polyhedron in m-dimensional space (an m-dimensional simplex) from its center.

The vectors $\{\boldsymbol{\varepsilon}_p\}$ possess a number of useful properties. One is that[4]

$$\sum_{p=1}^{m+1} \boldsymbol{\varepsilon}_p = 0. \tag{39}$$

Actually, let us consider the square of the sum $\sum_{p=1}^{m+1}\boldsymbol{\varepsilon}_p$. Owing to the definition (28) of the vectors $\{\boldsymbol{\varepsilon}_p\}$, we have

$$\Big(\sum_{p=1}^{m+1} \boldsymbol{\varepsilon}_p\Big)^2 = \sum_{p,q=1}^{m+1} \boldsymbol{\varepsilon}_p\boldsymbol{\varepsilon}_q = \sum_{p,q=1}^{m+1} [m^{-1}(m+1)\delta_{pq} - m^{-1}]$$
$$= m^{-1}(m+1)(m+1) - m^{-1}(m+1)^2 = 0. \tag{40}$$

Let us now show that the system of vectors (31)–(33) is complete in R^m. First we show that any unit vector $\mathbf{i}^{(m,y)}$ $(i_x^{(m,y)} = \delta_{xy})$, $x, y = 1, \dots, m$, is a linear combination of the vectors $\boldsymbol{\varepsilon}_1, \dots, \boldsymbol{\varepsilon}_{m+1}$. This is true in R^1: $\mathbf{i}^{(1,1)} = \boldsymbol{\varepsilon}_1$. Let it also be true in R^{m-1}:

$$\mathbf{i}^{(m-1,y)} = \sum_{p=1}^{m} a_p^{(y)}\mathbf{e}_p. \tag{41}$$

We note that

$$i_x^{(m,y)} = i_x^{(m-1,y)}, \quad x, y = 1, \dots, m-1; \tag{42}$$

$$i_m^{(m,y)} = 0, \quad y = 1, \dots, m-1; \tag{43}$$

$$i_x^{(m,m)} = \delta_{mx}, \quad x = 1, \dots, m. \tag{44}$$

Using (42) with (32) and (33), we find

$$i_x^{(m,y)} = \sum_{p=1}^{m} (\sin^{-1}\vartheta_m a_p^{(y)})\varepsilon_{px} - \Big(\sum_{q=1}^{m} \operatorname{ctg}\vartheta_m a_q^{(y)}\Big)\varepsilon_{m+1,x},$$
$$x, y = 1, \dots, m-1. \tag{45}$$

According to (43), (31), and (33),

$$i_m^{(m,y)}=\sum_{p=1}^{m}(\sin^{-1}\vartheta_m a_p^{(y)})\,\varepsilon_{pm}-\Big(\sum_{q=1}^{m}\operatorname{ctg}\vartheta_m a_q^{(y)}\Big)\,\varepsilon_{m+1,\,m},\qquad y=1,\ \ldots,\ m-1. \tag{46}$$

Equations (45) and (46) imply that

$$\mathbf{i}^{(m,y)}=\sum_{p=1}^{m+1}b_p^{(y)}\boldsymbol{\varepsilon}_p, \tag{47}$$

where

$$\left.\begin{aligned}&b_p^{(y)}=\sin^{-1}\vartheta_m a_p^{(y)},\quad p=1,\ \ldots,\ m,\\&b_{m+1}^{(y)}=\sum_{q=1}^{m}\operatorname{ctg}\vartheta_m a_q^{(y)},\end{aligned}\right\} \tag{48}$$

i.e., the unit vectors $\mathbf{i}^{(m,1)}, \ldots, \mathbf{i}^{(m,m+1)}$ are linear combinations of the vectors $\boldsymbol{\varepsilon}_1, \ldots, \boldsymbol{\varepsilon}_{m+1}$. However, the same is true for the last unit vector $i^{(m,m)}$ [(33) and (44)]:

$$\mathbf{i}^{(m,m)}=\boldsymbol{\varepsilon}_{m+1}. \tag{49}$$

Owing to the properties (39), any m vectors of the system $\boldsymbol{\varepsilon}_1, \ldots, \boldsymbol{\varepsilon}_{m+1}$ form a complete set of vectors in R^m.

Another useful property of the vectors $\{\boldsymbol{\varepsilon}_p\}$ is that[4]

$$\sum_{p=1}^{m+1}\varepsilon_{px}\varepsilon_{py}=m^{-1}(m+1)\,\delta_{xy},\quad x,\ y=1,\ .\ .\ .\ ,\ m. \tag{50}$$

In order to prove this, we take an arbitrary vector $\mathbf{A}$ and expand it in the first m vectors $\{\boldsymbol{\varepsilon}_p\}$,

$$\mathbf{A}=\sum_{q=1}^{m}a_q\boldsymbol{\varepsilon}_q, \tag{51}$$

and consider the contraction with respect to y of the left-hand side of (44) with A_y:

$$\begin{aligned}\sum_{y=1}^{m}\sum_{p=1}^{m+1}\varepsilon_{px}\varepsilon_{py}A_y&=\sum_{q=1}^{m}a_q\sum_{p=1}^{m+1}\varepsilon_{px}\sum_{y=1}^{m}\varepsilon_{py}\varepsilon_{qy}\\&=\sum_{q=1}^{m}a_q\sum_{p=1}^{m+1}\varepsilon_{px}\,[m^{-1}(m+1)\,\delta_{pq}-m^{-1}]\\&=m^{-1}(m+1)\sum_{q=1}^{m}a_q\varepsilon_{qx}-m^{-1}\sum_{q=1}^{m}a_q\sum_{p=1}^{m+1}\varepsilon_{px}.\end{aligned} \tag{52}$$

Using the property (39), we find

$$\sum_{y=1}^{m}\sum_{p=1}^{m+1}\varepsilon_{px}\varepsilon_{py}A_y=m^{-1}(m+1)\,A_x. \tag{53}$$

Since the vector $\mathbf{A}$ is arbitrary, we obtain the property (50).

The property (50) allows us, in particular, to write down the scalar product of two vectors in an overcomplete basis

$$\mathbf{bc}=(m+1)^{-1}\,m\sum_{p=1}^{m+1}\boldsymbol{\varepsilon}_p\mathbf{b}\boldsymbol{\varepsilon}_p\mathbf{c}, \tag{54}$$

since

$$\mathbf{bc}=\sum_{x,\,y=1}^{m}b_xc_y\delta_{xy}=(m+1)^{-1}\,m\sum_{p=1}^{m+1}\varepsilon_{px}b_x\varepsilon_{py}c_y. \tag{55}$$

In turn, the representation (54) makes it possible, in particular, to reconstruct an arbitrary vector $\mathbf{a}$ from its components $\boldsymbol{\varepsilon}_p\cdot\mathbf{a}$ in the basis $\{\boldsymbol{\varepsilon}_p\}$. Taking $\mathbf{b}$ in (54) to be the unit vector $\mathbf{i}^{(x)}$ in the direction of the x axis, and $\mathbf{c}$ to be equal to $\mathbf{a}$, we have

$$a_x=(m+1)^{-1}\,m\sum_{p+1}^{m+1}\varepsilon_{px}\boldsymbol{\varepsilon}_p\mathbf{a}. \tag{56}$$

After showing that the system of vectors with the properties (28) exists, we can reformulate the Potts model in the language of the simplex vectors.[4,5] With each state $P_1, \ldots, P_q$ we associate one of the vectors $\boldsymbol{\varepsilon}_p$ of the system (27) in the space R^m for $m=q-1$: $P_p\leftrightarrow\boldsymbol{\varepsilon}_p$, $p=1, \ldots, m+1$. Then the state of the ith point is characterized by the vector $\mathbf{s}_i\in R^m$, which can take one of $m+1$ values $\boldsymbol{\varepsilon}_1, \ldots, \boldsymbol{\varepsilon}_{m+1}$:

$$\mathbf{s}_i=\mathbf{s}\,(S_i). \tag{57}$$

Then the delta function (24) can be written as

$$\delta\,(S_i,\ S_j)=(m+1)^{-1}\,[1+m\mathbf{s}\,(S_i)\,\mathbf{s}\,(S_j)]. \tag{58}$$

For each point i the quantity $H_{0i}(S_i)$ can be written as the scalar product of the vector $\mathbf{s}(S_i)$ and some vector h_i:

$$H_{0i}\,(S_i)=\mathbf{h}_i\mathbf{s}\,(S_i). \tag{59}$$

Actually, according to (56) this vector $\mathbf{h}_i$ is

$$h_{ix}=(m+1)^{-1}\,m\sum_{i=1}^{m+1}s_x\,(S_i)\,H_{0i}\,(S_i). \tag{60}$$

The vector $\mathbf{h}_i$ will be referred to as the field at the point i.

Substituting (58) and (59) into (26), for the partition function Z we find

$$\begin{aligned}Z=&\sum_{S_1=P_1,\ \ldots,\ P_{m+1}}\cdots\sum_{S_N=P_1,\ \ldots,\ P_{m+1}}\\&\times\exp\Big[\sum_{i<j}(m+1)^{-1}\,mE_{ij}\mathbf{s}\,(S_i)\,\mathbf{s}\,(S_j)\\&+\sum_i\mathbf{h}_i\mathbf{s}\,(S_i)+\sum_{i<j}(m+1)^{-1}\,E_{ij}\Big].\end{aligned} \tag{61}$$

Since the relation

$$\mathrm{P}_p\leftrightarrow\boldsymbol{\varepsilon}_p=\mathbf{s}\,(\mathrm{P}_p) \tag{62}$$

is one-to-one, in (61) we can go from summation over $S_1, \ldots, S_N$ to summation over $\mathbf{s}(S_1)=s_1, \ldots, \mathbf{s}(S_N)=\mathbf{s}_N$. Discarding the inessential constant

$$\sum_{i<j}(m+1)^{-1}\,E_{ij} \tag{63}$$

and introducing the notation

$$J_{ij}=-(m+1)^{-1}\,mE_{ij}, \tag{64}$$

we obtain

$$Z=\sum_{\mathbf{s}_1=\boldsymbol{\varepsilon}_1,\ \ldots,\ \boldsymbol{\varepsilon}_{m+1}}\cdots\sum_{\mathbf{s}_N=\boldsymbol{\varepsilon}_1,\ \ldots,\ \boldsymbol{\varepsilon}_N}\times\exp\Big[-\sum_{i<j}J_{ij}\mathbf{s}_i\mathbf{s}_j+\sum_i\mathbf{h}_i\mathbf{s}_i\Big]. \tag{65}$$

Therefore, the Potts model can be formulated as follows. We have a lattice containing N points. At each point i ($i=1, \ldots, N$) there is a spin $\mathbf{s}_i\in R^m$ which can take one of $m+1$ different values $\{\boldsymbol{\varepsilon}_p\}=\{\boldsymbol{\varepsilon}_1, \ldots, \boldsymbol{\varepsilon}_{m+1}\}$. The state of the entire system is characterized by the N vectors $\{\mathbf{s}_i\}=\mathbf{s}_1, \ldots, \mathbf{s}_N$. The defining relation for the vectors $\{\boldsymbol{\varepsilon}_p\}$ is Eq. (28). Each point i is also characterized by a field $\mathbf{h}_i\in R^m$. The

Hamiltonian of the Potts model in the simplex representation has the form

$$H = -\sum_{i<j} J_{ij}\mathbf{s}_i\mathbf{s}_j + \sum_i \mathbf{h}_i\mathbf{s}_i. \tag{66}$$

Here the summation in the first term runs over pairs of interacting neighbors $\langle i, j\rangle$. The summation in the second term runs over all the lattice points. The quantities $J_{ij} \in R$ are referred to as the interaction strengths.

The partition function is

$$Z = \sum_{s_1} \dots \sum_{s_N} \exp(-\beta H) = \sum_{\{s_i\}} \exp(-\beta H), \tag{67}$$

where the summation runs over the $m+1$ spin vectors $\{\varepsilon_p\}$ at each lattice point.

This representation of the Potts model (66), (67) is referred to as the simplex representation. It has the advantage that the vectors $\{\varepsilon_p\}$ possess the properties described above, owing to the symmetry of the simplex, which greatly simplifies the calculations.

The physics of the system in the representation (25) or the representation (66) is the same.

4. SOME APPLICATIONS OF THE POTTS MODEL

The Potts model serves as the basis for the theoretical description of a number of experimentally observed phenomena. There are several physical systems described by Hamiltonians with the same symmetry properties as the Potts model, which therefore belong to the universality class of this spin model. Here we shall consider several of these systems.

The adsorption of inert gases on adsorbents like graphite can be described by Potts lattice gas models (25). Physically adsorbed films of this type are an experimental realization of phase transitions in two-dimensional systems. The adsorption of krypton on graphite has been studied in Ref. 6. The adsorption centers form a triangular lattice on the basal plane of the graphite crystal (Fig. 1). The adsorbed atoms (adatoms) of krypton interact in pairs: The interaction potential is positive (unfavorable) and very large (350 times larger than the well depth E) for nearest neighbors and negative (favorable) for the others (Fig. 1b); the potential has quite a short range. These properties of the potential lead to the exclusion of nearest neighbors. Therefore, the krypton monolayer is completed when one of the sublattices is filled. Two phases can be distinguished: a synchronized phase, when mainly one of the three sublattices on the lattice of adsorption centers is filled, and the disordered liquid phase, when all three sublattices are equally filled.

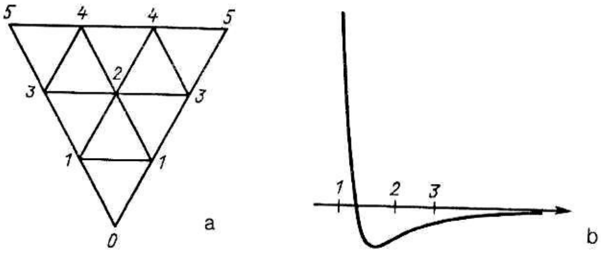

FIG. 1. Modeling of the adsorption process: a—triangular lattice of adsorption centers on a basal atomic plane of graphite consisting of two sublattices; b—the Lennard–Jones interatomic potential.

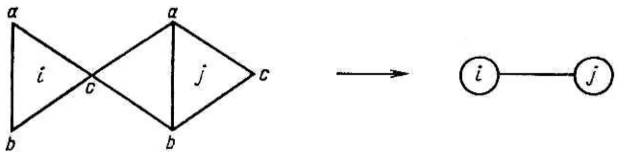

FIG. 2. The description of adsorption based on the Potts model.

The authors of Ref. 6 approximate the problem of the krypton lattice gas by the Potts lattice gas model, using the feature of nearest-neighbor exclusion. The renormalization-group method is applicable to this formulation of the problem.

Let us consider an elementary triangle consisting of three points of the krypton lattice gas (the triangles i and j in Fig. 2). This triangle, viewed as a whole, can be in one of four states: It is possible for it not to contain any krypton adatom, or it can contain one such atom in position a, b, or c. This triangle can be viewed as a single point of the Potts lattice gas. Its state is described by two variables (t_i, s_i). If the triangle is empty, then $t_i = 0$, and if it contains one atom, then $t_i = 1$; in the latter case the additional variable s_i describes the position of the adatom. If the adatom is located at a, then $s_i = 1$, and so on. The points of the Potts lattice gas also form a triangular lattice, which completely covers the lattice of the krypton lattice gas. The Hamiltonian of the Potts lattice gas has the form (25), where H_0 now denotes the chemical potential of the adatoms.

The Potts lattice gas model reflects the fundamental property of nearest-neighbor exclusion in the krypton lattice gas, which leads to the existence of three degenerate states. However, this model is an approximation: Whereas the points (a, b) cannot be occupied simultaneously, the points (j, a) and (i, b) can be, although the probability for this is suppressed. The description of the krypton lattice gas using the Potts lattice gas can be systematically improved by introducing into the Hamiltonian (25) the interaction of next-to-nearest neighbors.

The authors of Ref. 6 studied (25) using the renormalization group, and obtained a phase diagram which gives a good description of the experimental data on adsorption.

The author of Ref. 7 introduced a generalization of the three-component Potts model in which the spin states are coupled to lattice directions. This situation is realized in the adsorption of nitrogen on a graphite substrate having a hexagonal structure. One third of the elementary cells are occupied by nitrogen molecules, which form a triangular sublattice. Here the molecules acquire an orientational ordering, consisting of alternating bands in which the molecules are rotated by angles $\pm\pi/4$ relative to the bands (as in a fish skeleton). The same ordering occurs in quantum liquids at temperatures near zero, in smectic type-E liquid crystals, and in systems of large molecules adsorbed on graphite. The authors of Ref. 7 studied the phase boundaries in this model using the mean-field theory and other methods.

The Potts model is also realized in anisotropic ferromagnets with a cubic structure.[8] In ferromagnetic crystals the magnetization vector tends to be oriented along certain crystal axes, referred to as easy magnetization directions. A finite amount of work is necessary to magnetize the crystal in a different direction.[9] The Hamiltonian of this system has the form[10]

$$H = -\frac{1}{2} r(\theta)\,\mathbf{M}^2 + u\mathbf{M}^4 - v\,(M_x^4 + M_y^4 + M_z^4) - \mathbf{hM}, \qquad (68)$$

where **M** is the magnetization vector. It can be shown that it has the symmetry of the Potts model.[8] In fact, in the new variables

$$\left.\begin{aligned} &\sigma_1 = (M_x + M_y + M_z)/\sqrt{\sigma}, \quad \tilde{\mathbf{h}}_1 = (h_x + h_y + h_z)/\sqrt{\sigma};\\ &\sigma_2 = (M_x - M_y)/\sqrt{2}, \quad \tilde{\mathbf{h}}_2 = (h_x - h_y)/\sqrt{2};\\ &\sigma_3 = (M_x + M_y - 2M_z)/\sqrt{3}, \quad \tilde{\mathbf{h}}_3 = (h_x + h_y - h_z)/\sqrt{3}, \end{aligned}\right\} \qquad (69)$$

related to **M** and **h** by an orthogonal transformation, the Hamiltonian (68) takes the form

$$\begin{aligned} H = &-\tilde{\mathbf{h}}\boldsymbol{\sigma} - \frac{1}{2} r\boldsymbol{\sigma}^2 + u\boldsymbol{\sigma}^4 - 2v\sigma_1^2(\sigma_2^2 + \sigma_3^2)\\ &- 2\sqrt{2/3}\, v\sigma_1^2(\sigma_2^3 - 3\sigma_2^2\sigma_3) + \frac{1}{2} v(\sigma_2^2 + \sigma_3^2)^2 + \frac{4}{3}\sigma_1^4. \qquad (70) \end{aligned}$$

The presence of the term $\sigma_2^3 - 3\sigma_2^2\sigma_3$ leads to a first-order phase transition as h and θ are changed. The dependence of the component of the magnetization parallel to the external field $M_\parallel$ on this field for the system described by the Hamiltonian (68) has the form shown in Fig. 3. When the field is directed along the principal diagonal of the cubic crystal $[h = (H, 0, 0)]$, a change of h and θ is accompanied by a first-order phase transition[8] [the vertical segment on the curve $M_\parallel(H)$]. Of course, the jump is different for different materials.

The magnetization of a rare-earth composite $DyAl_2$ was measured at temperatures below critical in very strong magnetic fields in Ref. 11. At low temperatures the jump of the magnetization in this material is 25% of its total value. As the applied field is rotated from the direction [111] to [110], the jump of the magnetization decreases, and then the phase transition becomes a second-order one. This experiment shows that for the three-dimensional Potts model the predictions of mean-field theory[5] are, at least qualitatively, true.

Structural phase transitions in certain materials such as $SrTiO_3$ (Refs. 12 and 13) and $Pb_3(PO_4)_2$ (Ref. 14) pertain to the universality class of the Potts model with $q = 3$. Composite materials with crystallographic structure A15, such as Nb_3Sn, are superconductors, and undergo a structural phase transition in which their crystallographic structure is transformed from cubic into tetragonal.[15] In the continuum limit this phase transition can be described using the free-energy function constructed from the components of the deformation tensor ε_{xx}, ε_{yy}, and ε_{zz}, which has the symmetry of the Potts model.[16] Therefore, the phase diagram for Nb_3Sn in the variables $(\theta, S_{xx}, S_{yy}, S_{zz})$, where S_{jj} $(i = 1, 2, 3)$ are the components of the stress tensor, coincides with the phase diagram of the three-dimensional Potts model with three states (see Ref. 16). This fact can be used to explain the pressure dependence of the temperature phase transition (related to the temperature transition to the superconducting state; see Ref. 15).[16]

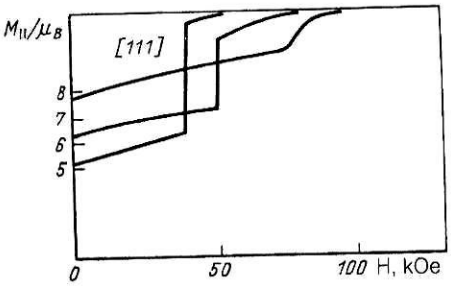

FIG. 3. Magnetization curve of $DyAl_2$ at $\theta = 4.2$ °K (Ref. 11).

The five-component liquid mixture ethylene glycol + lauryl alcohol + water + nitroethane + nitromethane is described by a three-component lattice system.[17,18] The mean-field-theory prediction of a first-order phase transition in this system has been studied intensively, and it has actually been observed.[18]

Recently there has been a great deal of discussion of phase transitions at the boundaries of different phases.[19] These are realized experimentally in multicomponent liquid mixtures and in adsorption systems which, as we have already seen, are related to the Potts model.

A number of polymer problems can be formulated in terms of the Potts model.[20,21]

The statistical-mechanics techniques of lattice systems are applicable to combinatorial optimization problems, in particular, to so-called NP-complete problems.[22] The traveling-salesman problem, the problem of the partitioning of a graph, and the coloring problem have been studied using the techniques of the theory of quenched systems. In most of these studies the basic tool which is used is the q-component Potts model.

The traveling-salesman problem is formulated as a lattice problem in statistical mechanics in Ref. 23. The high- and low-temperature regimes are discussed. The exact solution of the replica method was obtained, and a spin-glass type of phase transition was found. The authors of Ref. 24 obtained a solution of the traveling-salesman problem in the formulation of Ref. 23 using the replica method without symmetry breaking. Good estimates of the distance of the shortest path were obtained. The authors of the key study of Ref. 25 discuss various estimates of the free energy of the traveling-salesman problem, construct the mean-field theory, develop a field-theoretic approach, and discuss the relation to quenched systems.

The application of statistical-mechanics techniques for quenched systems to the problem of the partitioning of a graph is discussed in Refs. 26–29. The method of simulated annealing, which at present is very popular, was used in Refs. 27 and 28. These authors discovered the existence of a low-temperature phase of the spin-glass type, characterized by a set of metastable local extrema and non-self-averaging, which sheds light on the structure of this complicated computational problem.

The authors of Ref. 29 discuss the relation of the problem of the partitioning of a graph into q subgraphs with minimization of the couplings between components, and also the problem of partitioning a random graph, to the theory of a spin glass in the Potts model. Estimates are obtained for the free energy using the mean-field theory for the spin glass and certain other numerical methods.

Another very interesting problem is that of the coloring of a graph with q colors. The author of Ref. 30 obtained an approximate equation for the coloring number of a hypercubic lattice with q colors in d-dimensional space, and Monte Carlo calculations were carried out in Ref. 31.

The relation between the three-coloring problem and the eight-vertex model is studied in detail in Ref. 32. In Refs. 33 and 34 the equivalence of the coloring problem and the problem of the ground state of the antiferromagnet Potts model was used to calculate the chromatic polynomial in the complex plane. In Ref. 35 it was shown that the formulation of the Potts model in terms of the dichromatic polynomial is closely related to the generalized percolation problem.

In Ref. 36 a series in powers of 1/3 was constructed to calculate the entropy of the ground state of the six-vertex model, which reduces to the coloring problem. The exact solution of this problem was later obtained in Ref. 37.

As we have already noted, the thermodynamic properties of weakly condensed media are described by quenched lattice systems. The simplest of these are models of dilute magnetic systems.[38] In particular, the dilute Potts model is described by the Hamiltonian[39]

$$H=-\frac{1}{2}J\sum_{(i,j)}\xi_i\xi_j\delta(S_i,S_j)-h\sum_i\xi_i\delta(S_i,\mathrm{q}),$$

$$S_i=\mathrm{P}_1,\ \ldots,\ \mathrm{P}_q, \qquad (71)$$

where ξ_j are the quenched quantities, which can take the two values 0 and 1. The expectation value $\langle\xi_j\rangle=p$ corresponds to the probability that the point i is occupied by the magnetic spin S_i.

The critical behavior of this system is different for different p. The system can make a transition to the magnetically ordered phase as the temperature is lowered only if $p>p_c$, where p_c is the critical concentration of the problem of percolation through the points (only in this case is there an infinite connected cluster). The critical temperature of the transition from the paramagnetic to the ferromagnetic phase $\theta_c(p)$ must fall off monotonically with decreasing concentration of magnetic atoms p. In Ref. 40 it was shown that for the dilute ferromagnetic Potts model

$$\theta_c(p)\sim p-p_c.$$

The dilute antiferromagnetic model with $q=3$ on a triangular lattice was studied in Ref. 41. States of fractional order were found in which only some of the symmetries are broken.

In Ref. 42 the dilute Potts model in a transverse field was studied by the renormalization-group method of mean-field theory. The critical surfaces in field–temperature–concentration space and some critical exponents were found.

There are more complicated models in which, in contrast to (71), the strength of the interaction of points i and j does not factorize:

$$H=-\frac{1}{2}J\sum_{(i,j)}J_{ij}\delta(S_i,S_j)-\sum_i h_i\delta(S_i,\mathrm{q}),\quad S_i=\mathrm{P}_1,$$

where the strengths J_{ij} and the fields h_i are quenched quantities.

In particular, such models serve as the basis for describing weakly ordered magnetic systems in which atoms possessing a magnetic moment are randomly distributed among the nonmagnetic atoms of a lattice. In such systems a spin-glass phase arises, where the magnetic moments of the impurities are frozen in fixed random directions, i.e., there is a short-range order in the absence of long-range order.[43] The large amount of available experimental data (see, for example, Refs. 44 and 45) indicate that the spin-glass phase is observed at sufficiently low temperatures ranging from a few to several tens of degrees and concentrations of magnetic impurities ranging from a fraction of a percent to several tens of percent. The spins of the randomly distributed magnetic atoms interact with each other by means of an indirect exchange interaction via the conduction electrons. This is referred to as the RKKJ interaction (Refs. 46–48), for which the interaction strength varies with distance as

$$J(r)\sim r^{-3}\cos(2k_Fr).$$

The impurity concentration is small enough to exclude direct exchange (when the spins are located at neighboring points), but large enough that the interaction between the impurity spins is not negligible (in contrast to Kondo systems).

A method of theoretically describing such systems on the basis of the Ising model ($q=2$) was first developed in Ref. 49. Here it was suggested that the spins be viewed as arranged on a regular lattice of smaller dimension, in which each point is occupied by a magnetic atom. Since the RKKJ interaction is long-range and oscillating, it is assumed that on the regular lattice the strengths of the pair exchange interaction J_{ij} are independent random variables with probability densities $P_{ij}(J_{ij})$. The equivalent system should obviously be treated as a quenched system.

The overwhelming majority of studies limit themselves to a Gaussian probability density for the interaction strength. In the case of the Ising model such a lattice is usually referred to as the Sherrington–Kirkpatrick model, since the solution for that model problem was obtained by those investigators in Ref. 50. The distribution

$$P_{ij}(J)=c\delta(J-\bar{J})+(1-c)\,\delta(J+\bar{J})$$

corresponds to the so-called frustration model.[51] The parameter c can describe, for example, the concentration of ferromagnetic couplings.

The spin-glass phase has also been found in the Gaussian Heisenberg[52] and Potts[53,54] models. Quadrupolar spin glasses have been studied in Ref. 55.

Stochastic Potts models, in which the number of spin states is different at different points, have been studied in Refs. 56 and 57. It has been found that the type of phase transition can change as the population of a point changes. The duality properties of quenched models were studied in Ref. 58. The phase diagram for a quenched random two-dimensional vector Potts model was constructed.

5. MULTIDIMENSIONAL GENERALIZATIONS OF THE HYPERBOLIC TRIGONOMETRIC FUNCTIONS

Below in our study of the Potts model an important role will be played by multidimensional generalizations of the hyperbolic trigonometric functions. Let us consider these functions and their properties.[59]

We define a scalar function R of a vector argument from R^m as

$$R(\mathbf{b}) = (m+1)^{-1} m \sum_{p=1}^{m+1} \exp(\mathbf{b}\boldsymbol{\varepsilon}_p). \quad (72)$$

In the one-dimensional case $m = 1$ the system of vectors $\{\boldsymbol{\varepsilon}_p\}$ has the form (29), and the function R is

$$R(\mathbf{b}) = \frac{1}{2}[\exp(b) + \exp(-b)], \quad m = 1, \quad (73)$$

i.e., it coincides with the usual hyperbolic cosine

$$\operatorname{ch} b = \frac{1}{2}[\exp(b) + \exp(-b)]. \quad (74)$$

Therefore, the function $R(\mathbf{b})$ (72) is a natural generalization of the hyperbolic cosine.

Recalling that the derivative of the hyperbolic cosine is the hyperbolic sine

$$\operatorname{sh} b = \frac{\partial}{\partial b} \operatorname{ch} b = \frac{1}{2}[\exp(b) - \exp(-b)], \quad (75)$$

let us consider the derivative of the function $R(b)$ with respect to the xth component of the argument b_x:

$$Q_x(\mathbf{b}) = \frac{b}{\partial b_x} R(\mathbf{b}) = (m+1)^{-1} m \sum_{p=1}^{m+1} \varepsilon_{px} \exp(\mathbf{b}\boldsymbol{\varepsilon}_p),$$

$$x = 1, \ldots, m \quad (76)$$

or, in vector notation,

$$\mathbf{Q}(\mathbf{b}) = \frac{\partial}{\partial \mathbf{b}} R(\mathbf{b}) = (m+1)^{-1} m \sum_{p=1}^{m+1} \boldsymbol{\varepsilon}_p \exp(\mathbf{b}\boldsymbol{\varepsilon}_p). \quad (77)$$

In the one-dimensional case, where the system of vectors $\{\boldsymbol{\varepsilon}_p\}$ reduces to (29), the function $\mathbf{Q}(\mathbf{b})$ takes the form

$$\mathbf{Q}(\mathbf{b}) = \frac{1}{2}[\exp(b) - \exp(-b)], \quad m = 1, \quad (78)$$

and coincides with the hyperbolic sine (75). Therefore, the function $\mathbf{Q}(\mathbf{b})$ is a natural generalization of the hyperbolic sine.

The second derivative of the hyperbolic cosine with respect to its argument gives back the same function:

$$\frac{\partial^2}{\partial b^2} \operatorname{ch} b = \operatorname{ch} b. \quad (79)$$

Let us see how this property is generalized to the multidimensional case. We define the functions $Q_{xy}(b)$ as the second derivatives of $R(\mathbf{b})$:

$$Q_{xy}(\mathbf{b}) = \frac{\partial^2}{\partial b_y \partial b_x} R(\mathbf{b}) = (m+1)^{-1} m \sum_{p=1}^{m+1} \varepsilon_{px}\varepsilon_{py} \exp(\mathbf{b}\boldsymbol{\varepsilon}_p),$$

$$x, y = 1, \ldots, m, \quad (80)$$

and, accordingly,

$$Q_{xy}(\mathbf{b}) = Q_{yx}(\mathbf{b}). \quad (81)$$

Let us take the trace of the matrix $Q_{xy}(\mathbf{b})$. According to the definition (28) of the system of vectors $\{\boldsymbol{\varepsilon}_p\}$, we have

$$\sum_{x=1}^{m} Q_{xx}(\mathbf{b}) = (m+1)^{-1} m \sum_{p=1}^{m+1} \exp(\mathbf{b}\boldsymbol{\varepsilon}_p) \sum_{x=1}^{m} \varepsilon_{px}^2$$

$$= (m+1)^{-1} m \sum_{p=1}^{m+1} \exp(\mathbf{b}\boldsymbol{\varepsilon}_p) \quad (82)$$

or, comparing with (72),

$$\sum_{x=1}^{m} Q_{xx}(\mathbf{b}) = R(\mathbf{b}). \quad (83)$$

In other words, the Laplacian of the function $R(\mathbf{b})$ coincides with $R(\mathbf{b})$ itself:

$$\sum_{x=1}^{m} \frac{\partial^2}{\partial b_x^2} R(\mathbf{b}) = R(\mathbf{b}) \quad (84)$$

or

$$\nabla^2 R(\mathbf{b}) = R(\mathbf{b}). \quad (85)$$

Relations (84) and (85) are the generalization of the property (79).

In order to determine the functional properties of R, Q_x, and Q_{xy}, let us consider a function of one scalar and two vector arguments:

$$T(a, \mathbf{b}, \mathbf{c})$$

$$= (m+1)^{-2} m^2 \sum_{p,q=1}^{m+1} \exp(a\boldsymbol{\varepsilon}_p\boldsymbol{\varepsilon}_q + \mathbf{b}\boldsymbol{\varepsilon}_p + \mathbf{c}\boldsymbol{\varepsilon}_q). \quad (86)$$

We note that when the first argument is zero, this function reduces to the product of two R functions:

$$T(0, \mathbf{b}, \mathbf{c}) = R(\mathbf{b}) R(\mathbf{c}). \quad (87)$$

It is also easy to show that in the one-dimensional case

$$T(a, \mathbf{b}, \mathbf{c}) = \operatorname{ch} a \operatorname{ch} b \operatorname{ch} c + \operatorname{sh} a \operatorname{sh} b \operatorname{sh} c, \quad m = 1. \quad (88)$$

Let us now consider

$$V(a, \mathbf{b}, \mathbf{c}) = \frac{\partial}{\partial a} T(a, \mathbf{b}, \mathbf{c})$$

$$= (m+1)^{-2} m^2 \sum_{p,q=1}^{m+1} \boldsymbol{\varepsilon}_p\boldsymbol{\varepsilon}_q \exp(a\boldsymbol{\varepsilon}_p\boldsymbol{\varepsilon}_q + \mathbf{b}\boldsymbol{\varepsilon}_p + \mathbf{c}\boldsymbol{\varepsilon}_q). \quad (89)$$

We note that V is symmetric in the vector arguments, and also that when the scalar argument is zero, the function V reduces to the scalar product of two $\mathbf{Q}$ functions

$$V(0, \mathbf{b}, \mathbf{c}) = \mathbf{Q}(\mathbf{b}) \mathbf{Q}(\mathbf{c}). \quad (90)$$

According to the definition (28) or the system of vectors $\{\boldsymbol{\varepsilon}_p\}$, we have

$$V(a, \mathbf{b}, \mathbf{c}) = (m+1)^{-1} m \exp a \sum_{p=1}^{m+1} \exp[(\mathbf{b}+\mathbf{c})\boldsymbol{\varepsilon}_p]$$

$$- m^{-1}(m+1)^{-2} m^2 \sum_{p,q=1}^{m+1} \exp(a\boldsymbol{\varepsilon}_p\boldsymbol{\varepsilon}_q + \mathbf{b}\boldsymbol{\varepsilon}_p + \mathbf{c}\boldsymbol{\varepsilon}_q)$$

$$= \exp aR(\mathbf{b}+\mathbf{c}) - m^{-1}T(a, \mathbf{b}, \mathbf{c}). \quad (91)$$

This relation allows us to obtain an expression for the R function of a sum. Setting $a = 0$ and comparing the result with (87) and (90), we find

$$R(\mathbf{b}+\mathbf{c}) = \mathbf{Q}(\mathbf{b})\mathbf{Q}(\mathbf{c}) + m^{-1}R(\mathbf{b})R(\mathbf{c}) \quad (92)$$

or

$$R(\mathbf{b}+\mathbf{c}) = \sum_{x=1}^{m} Q_x(\mathbf{b}) Q_x(\mathbf{c}) + m^{-1}R(\mathbf{b})R(\mathbf{c}). \quad (93)$$

This is the generalization of the addition theorem for the hyperbolic cosine. In fact, for $m = 1$ we have

$$\operatorname{ch}(\mathbf{b}+\mathbf{c}) = \operatorname{sh}\mathbf{b}\operatorname{sh}\mathbf{c} + \operatorname{ch}\mathbf{b}\operatorname{ch}\mathbf{c}, \quad (94)$$

i.e., the usual addition theorem.

To generalize the addition theorem for the sine function we consider the function

$$V_x(a,\ \mathbf{b},\ \mathbf{c}) = \frac{\partial}{\partial b_x} V(a,\ \mathbf{b},\ \mathbf{c})$$

$$= (m+1)^{-2} m^2 \sum_{p,\,q=1}^{m+1} \varepsilon_p\varepsilon_q\varepsilon_{p.c} \exp(a\varepsilon_p\varepsilon_q + \mathbf{b}\varepsilon_p + \mathbf{c}\varepsilon_q). \qquad (95)$$

For $a = 0$ the function V_x takes the form

$$V_x(0,\ \mathbf{b},\ \mathbf{c}) = \sum_{y=1}^{m} Q_{xy}(\mathbf{b})\, Q_y(\mathbf{c}). \qquad (96)$$

Taking into account (28), from (95) we find

$$V_x(a,\ \mathbf{b},\ \mathbf{c}) = (m+1)^{-1} m \exp a \sum_{p=1}^{m+1} \varepsilon_{px} \exp[(\mathbf{b}+\mathbf{c})\,\varepsilon_p]$$

$$- m^{-1}(m+1)^{-2} m^2 \sum_{p,\,q=1}^{m+1} \varepsilon_{px} \exp(a\varepsilon_p\varepsilon_q + \mathbf{b}\varepsilon_p + \mathbf{c}\varepsilon_q). \qquad (97)$$

Setting $a = 0$ here, using (96), we obtain

$$Q_x(\mathbf{b}+\mathbf{c}) = \sum_{y=1}^{m} Q_{xy}(\mathbf{b})\, Q_y(\mathbf{c}) + m^{-1} Q_x(\mathbf{b})\, R(\mathbf{c}). \qquad (98)$$

Using the fact that for $m = 1$ it follows from (83) that $Q_{xx}(\mathbf{b}) = R(\mathbf{b})$, we see that in the one-dimensional case (97) gives

$$\operatorname{sh}(b+c) = \operatorname{ch} b \operatorname{sh} c + \operatorname{sh} b \operatorname{ch} c, \qquad (99)$$

so that relation (98) generalizes the addition theorem for the hyperbolic sine.

Let us see what is the second derivative of the function T with respect to its first argument:

$$W(a,\ \mathbf{b},\ \mathbf{c}) = \frac{\partial^2}{\partial a^2} T(a,\ \mathbf{b},\ \mathbf{c})$$

$$= (m+1)^{-2} m^2 \sum_{p,\,q=1}^{m+1} (\varepsilon_p\varepsilon_q)^2 \exp(a\varepsilon_p\varepsilon_q + \mathbf{b}\varepsilon_p + \mathbf{c}\varepsilon_q). \qquad (100)$$

According to the fundamental property of the simplex vectors ε_p (28), we have

$$(\varepsilon_p\varepsilon_q)^2 = [m^{-1}(m+1)\,\delta_{pq} - m^{-1}]^2$$
$$= m^{-2}(m-1)(m+1)\,\delta_{pq} + m^{-2}, \qquad (101)$$

from which we find

$$W(a,\ \mathbf{b},\ \mathbf{c})$$

$$= \exp a\, m^{-1}(m-1)(m+1)^{-1} m \sum_{p=1}^{m+1} \exp[(\mathbf{b}+\mathbf{c})\,\varepsilon_p]$$

$$+ m^{-2}(m+1)^{-2} m^2 \sum_{p,\,q=1}^{m+1} \exp(a\varepsilon_p\varepsilon_q + \mathbf{b}\varepsilon_p + \mathbf{c}\varepsilon_q)$$

$$= \exp a\, m^{-1}(m-1)\, R(\mathbf{b}+\mathbf{c}) + m^{-2} T(a,\ \mathbf{b},\ \mathbf{c}). \qquad (102)$$

Taking into account the addition theorem (92),

$$W(a,\ \mathbf{b},\ \mathbf{c}) = \exp a\, m^{-1}(m-1)\, \mathbf{Q}(\mathbf{b})\, \mathbf{Q}(\mathbf{c})$$
$$+ \exp a\, m^{-2}(m-1)\, R(\mathbf{b})\, R(\mathbf{c}) + m^{-2} T(a,\ \mathbf{b},\ \mathbf{c}), \qquad (103)$$

so that the function W is expressed in terms of the known functions R, $\mathbf{Q}$, and T. For $a = 0$, according to (87) the function W takes the form

$$W(0,\ \mathbf{b},\ \mathbf{c}) = m^{-1}(m-1)\, \mathbf{Q}(\mathbf{b})\, \mathbf{Q}(\mathbf{c}) + m^{-1} R(\mathbf{b})\, R(\mathbf{c}). \qquad (104)$$

Let us now consider the tensor function

$$Q_{xyz}(\mathbf{b}) = (m+1)^{-1} m \sum_{p=1}^{m+1} \varepsilon_{px}\varepsilon_{py}\varepsilon_{pz} \exp(\mathbf{b}\varepsilon_p), \qquad (105)$$

which has the obvious relation to $Q_{xy}(\mathbf{b})$ (80):

$$Q_{xyz}(\mathbf{b}) = \frac{\partial}{\partial b_z} Q_{xy}(\mathbf{b}), \qquad (106)$$

and is obviously completely symmetric in the indices x, y, and z. Let us also consider the vector and tensor functions

$$U_x(a,\ \mathbf{b},\ \mathbf{c}) = (m+1)^{-2} m^2 \sum_{p,\,q=1}^{m+1} \varepsilon_{px} \exp(a\varepsilon_p\varepsilon_q + \mathbf{b}\varepsilon_p + \mathbf{c}\varepsilon_q), \qquad (107)$$

$$U_{xy}(a,\ \mathbf{b},\ \mathbf{c}) = (m+1)^{-2} m^2$$

$$\times \sum_{p,\,q=1}^{m+1} \varepsilon_{px}\varepsilon_{py} \exp(a\varepsilon_p\varepsilon_q + \mathbf{b}\varepsilon_p + \mathbf{c}\varepsilon_q), \qquad (108)$$

which are derivatives of the function T (86):

$$U_x(a,\ \mathbf{b},\ \mathbf{c}) = \frac{\partial}{\partial b_x} T(a,\ \mathbf{b},\ \mathbf{c}); \qquad (109)$$

$$U_{xy}(a,\ \mathbf{b},\ \mathbf{c}) = \frac{\partial}{\partial b_y} U_x(a,\ \mathbf{b},\ \mathbf{c}), \qquad (110)$$

where U_{xy} is symmetric in the indices x and y. Owing to the normalization of the vectors ε_p, the functions (106) and (108) have the following properties:

$$\sum_{x=1}^{m} Q_{xxy}(\mathbf{b}) = (m+1)^{-1} m \sum_{p=1}^{m+1}$$

$$\times \Big[\Big(\sum_{x=1}^{m} \varepsilon_{px}^2\Big) \varepsilon_{py} \exp(\mathbf{b}\varepsilon_p)\Big] = Q_y(\mathbf{b}); \qquad (111)$$

$$\sum_{x=1}^{m} U_{xx}(a,\ \mathbf{b},\ \mathbf{c})$$

$$= (m+1)^{-2} m^2 \sum_{p,\,q=1}^{m+1}$$

$$\times \Big[\Big(\sum_{x=1}^{m} \varepsilon_{px}^2\Big) \exp(a\varepsilon_p\varepsilon_q + \mathbf{b}\varepsilon_p + \mathbf{c}\varepsilon_q)\Big] = T(a,\ \mathbf{b},\ \mathbf{c}). \qquad (112)$$

Let us also define the function

$$P_x(a,\ \mathbf{b}) = (m+1)^{-1} m \sum_{p=1}^{m+1} \exp(a\varepsilon_{px} + \mathbf{b}\varepsilon_p), \qquad (113)$$

which in the one-dimensional case has the form

$$P_x(a,\ \mathbf{b}) = \operatorname{ch}(a+b), \quad m = 1. \qquad (114)$$

Let us write out its derivatives with respect to b_y and b_z:

$$S_{(x)y}(a,\ \mathbf{b}) = (m+1)^{-1} m \sum_{p=1}^{m+1} \varepsilon_{py} \exp(a\varepsilon_{px} + \mathbf{b}\varepsilon_p); \qquad (115)$$

$$S_{(x)yz}(a,\ \mathbf{b}) = (m+1)^{-1} m \sum_{p=1}^{m+1} \varepsilon_{py}\varepsilon_{pz} \exp(a\varepsilon_{px} + \mathbf{b}\varepsilon_p). \qquad (116)$$

It is easy to see that

$$\sum_{x=1}^{m} S_{(y)xx}(a, \mathbf{b}) = P_{(y)}(a, \mathbf{b}). \quad (117)$$

Finally, we note that

$$S_{(\lambda)x}(a, \mathbf{b}) = \frac{\partial}{\partial a} P_{(x)}(a, \mathbf{b}), \quad S_{(x)xx}(a, \mathbf{b}) = \frac{\partial^2}{\partial a^2} P_{(x)}(a, \mathbf{b}) \quad (118)$$

and

$$P_{(x)}(0, \mathbf{b}) = R(\mathbf{b}), \quad (119)$$

$$S_{(x)x}(0, \mathbf{b}) = Q_x(\mathbf{b}), \quad S_{(x)x}(0, \mathbf{b}) = Q_{xx}(\mathbf{b}). \quad (120)$$

6. THE EFFECTIVE HAMILTONIAN OF THE QUENCHED POTTS MODEL

Let us describe the quenched Potts model. We shall consider the simplex representation of the Potts model (see Sec. 3). Splitting the field $\mathbf{h}_i$ into two terms, we write the Hamiltonian as

$$H = -\sum_{i<j} J_{ij}\mathbf{s}_i\mathbf{s}_j + \sum_i \mathbf{h}_i\mathbf{s}_i + \sum_i \mathbf{v}_i\mathbf{s}_i. \quad (121)$$

According to Sec. 1, the system in question is characterized by variables of different types: s_i, J_{ij} and $\mathbf{h}_i$, $\mathbf{v}_i$. The variables $\mathbf{s}_i$ are ordinary variables, the variables J_{ij} and $\mathbf{h}_i$ are quenched variables, and the variables $\mathbf{v}_i$ are fixed, i.e., the fields $\mathbf{v}_i$ are given parameters. The interaction strengths J_{ij} and fields $\mathbf{h}_i$ are mutually independent random variables. Here we shall not specify their distributions $P_{ij}(J_{ij})$ and $p_{ix}(h_{ix})$. The distributions $P_{ij}(J_{ij})$ can be different for different pairs (i,j), while $p_{ix}(h_{ix})$ can be different for different points i and components x.

The thermodynamic value $\mathscr{A}$ of some quantity A for a quenched system with the Hamiltonian (121) in accordance with (14) is given by the expression

$$\begin{aligned}\mathscr{A} &= \langle A(\{J_{ij}\}, \{h_{ix}\})\rangle \\ &= \int \cdots \int \prod_{i<j} [dJ_{ij}P_{ij}(J_{ij})] \\ &\quad \times \prod_i \prod_{x=1}^{m} [dh_{ix}p_{ix}(h_{ix})] \langle A\rangle_H (\{J_{ij}\}, \{h_{ix}\}),\end{aligned} \quad (122)$$

where $\langle A\rangle_H(\{J_{ij}\}, \{h_{ix}\})$ is the value of this quantity for given values of the interaction strengths J_{ij} and fields $\mathbf{h}_i$.

In particular, for the free energy $\mathscr{F} = N\varphi$ we have [see (21)]

$$\mathscr{F} = \langle F\rangle, \quad \varphi = \langle f\rangle, \quad (123)$$

where

$$Nf = F = -\beta^{-1} \ln Z, \quad (124)$$

β is the inverse temperature,

$$\beta = \theta^{-1}, \quad (125)$$

and Z is the partition function for given values of J_{ij} and $\mathbf{h}_i$,

$$\begin{aligned}Z &= \sum_{\mathbf{s}_1} \cdots \sum_{\mathbf{s}_N} \exp(-\beta H) = \sum_{\{\mathbf{s}_i\}} \exp[-\beta H(\{\mathbf{s}_i\})] \\ &= \operatorname{tr} \exp(-\beta H).\end{aligned} \quad (126)$$

In (126) the summation runs over the $m+1$ spin vectors $\{\boldsymbol{\varepsilon}_p\}$ (28) at each lattice point.

We use the replica method[49] to calculate the free energy (123). Here we give one elementary relation. If the limit for $n \to 0$ of some function a_n is equal to unity,

$$\lim_{n\to 0} a_n = 1, \quad (127)$$

then

$$\lim_{n\to 0} (n^{-1} \ln a_n) = \lim_{n\to 0} [n^{-1}(a_n - 1)]. \quad (128)$$

Setting $a_n = a^n$ in (128), we have

$$\ln a = \lim_{n\to 0} [n^{-1}, (a^n - 1)]. \quad (129)$$

Now, in accordance with (124) we can write

$$-\beta F = \ln Z = \lim_{n\to 0} [n^{-1}(Z^n - 1)], \quad (130)$$

so that the free energy (123) can be written as

$$-\beta\mathscr{F} = \lim_{n\to 0} [n^{-1}(\langle Z^n\rangle - 1)]. \quad (131)$$

Owing to Eq. (128), this expression for $\mathscr{F}$ can be rewritten as

$$-\beta\mathscr{F} = \lim_{n\to 0} [n^{-1} \ln \langle Z^n\rangle]. \quad (132)$$

In contrast to Ref. 49 and many other subsequent studies using the replica method to investigate weakly ordered spin systems (see, for example, Refs. 50, 60, and 61), relation (132) will serve as our starting point.[59,62–64]

We note that it can also be written as

$$\langle \ln Z\rangle = \lim_{n\to 0} \ln \langle Z^n\rangle^{1/n}. \quad (133)$$

For brevity we introduce the notation

$$\mathscr{F}_n = -n^{-1}\beta^{-1} \ln \langle Z^n\rangle, \quad (134)$$

in which the desired quantity, the free energy $\mathscr{F}$, is

$$\mathscr{F} = \lim_{n\to 0} \mathscr{F}_n. \quad (135)$$

The nth power of the partition function Z can be written as

$$\begin{aligned}Z^n &= \sum_{\{\mathbf{s}_i^1\}} \cdots \sum_{\{\mathbf{s}_i^n\}} \exp\Big[-\beta \sum_{\alpha=1}^{n} H(\{\mathbf{s}_i^\alpha\})\Big] \\ &= \sum_{\{\mathbf{s}_1^\alpha\}} \cdots \sum_{\{\mathbf{s}_N^\alpha\}} \exp\Big[-\beta \sum_{\alpha=1}^{n} H(\{\mathbf{s}_i^\alpha\})\Big],\end{aligned} \quad (136)$$

which will be denoted by

$$Z^n = \operatorname{tr}_n \exp\Big[-\beta \sum_{\alpha=1}^{n} H(\{\mathbf{s}_i^\alpha\})\Big]. \quad (137)$$

We introduce the Hamiltonian H_n of a system consisting of nN points whose state is described by the variables $\mathbf{s}_i^\alpha$ ($i = 1, \ldots, N$; $\alpha = 1, \ldots, N$) (Refs. 63, 65, and 69):

$$H_n(\{\mathbf{s}_i^\alpha\}) = \ln \Big\langle \exp\Big[-\beta \sum_{\alpha=1}^{n} H(\{\mathbf{s}_i^\alpha\})\Big]\Big\rangle. \quad (138)$$

Here α is the replica number, and we remind the reader that the averaging $\langle \ldots \rangle$ is carried out in accordance with (122). The Hamiltonian (138) is useful because its trace gives the expectation value of Z^n (137); in fact,

$$\mathrm{tr}_n \exp H_n = \mathrm{tr}_n \left\langle \exp \left[-\beta \sum_{\alpha=1}^{n} H(\{\mathbf{s}_i^\alpha\})\right]\right\rangle$$
$$= \langle \mathrm{tr}_n \exp [\ldots] \rangle = \langle Z^n \rangle. \qquad (139)$$

In accordance with (134) for $\mathcal{F}_n$ we obtain

$$-n\beta\mathcal{F}_n = \ln \mathrm{tr}_n \exp H_n. \qquad (140)$$

Let us now calculate the Hamiltonian H_n. According to (138) we have

$$H_n = \sum_{i<j} \ln \int dJ P_{ij}(J) \exp\left(\beta J \sum_{\alpha=1}^{n} \mathbf{s}_i^\alpha \mathbf{s}_j^\alpha\right)$$
$$+\sum_i \sum_{x=1}^{m} \ln \int dh p_{ix}(h) \exp\left(\beta h \sum_{\alpha=1}^{n} s_{ix}^\alpha\right) + \beta \sum_{\alpha=1}^{n} \sum_i \mathbf{v}_i \mathbf{s}_i^\alpha.$$
$$(141)$$

We use the power expansion

$$g(ay) = \exp(aD)\, g(y\tau)\,|_{\tau=0}, \quad D = \partial/\partial\tau. \qquad (142)$$

Substituting into (142)

$$a = \sum_{\alpha=1}^{n} \mathbf{s}_i^\alpha \mathbf{s}_j^\alpha, \quad y = \beta, \qquad (143)$$

and then

$$a = \sum_{\alpha=1}^{n} s_{ix}^\alpha, \quad y = \beta, \qquad (144)$$

we find

$$H_n = \sum_{i<j} \exp\left(D \sum_{\alpha=1}^{n} \mathbf{s}_i^\alpha \mathbf{s}_j^\alpha\right) \ln \int dJ P_{ij}(J) \exp(\beta J \tau)\,|_{\tau=0}$$
$$+\sum_i \sum_{x=1}^{m} \exp\left(D \sum_{\alpha=1}^{n} s_{ix}^\alpha\right) \ln \int dh p_{ix}(h) \exp(\beta h \tau)\,|_{\tau=0}$$
$$+\beta \sum_{\alpha=1}^{n} \sum_i \mathbf{v}_i \mathbf{s}_i^\alpha. \qquad (145)$$

Introducing the notation

$$\Psi_{ij}(\tau) = \ln \int dJ P_{ij}(J) \exp(\beta J \tau) = \ln \langle \exp(\beta J_{ij}\tau)\rangle, \qquad (146)$$
$$\Phi_{ix}(\tau) = \ln \int dh p_{ix}(h) \exp(\beta h \tau) = \ln \langle \exp(\beta h_{ix}\tau)\rangle, \qquad (147)$$

we obtain

$$H_n = \sum_{i<j} \left(D \sum_{\alpha=1}^{n} \mathbf{s}_i^\alpha \mathbf{s}_j^\alpha\right) \Psi_{ij}\tau\,|_{\tau=0}$$
$$+\sum_i \sum_{x=1}^{m} \exp\left(D \sum_{\alpha=1}^{n} s_{ix}^\alpha\right) \Phi_{ix}(\tau)\,|_{\tau=0} + \beta \sum_{\alpha=1}^{n} \sum_i \mathbf{v}_i \mathbf{s}_i^\alpha. \qquad (148)$$

We have therefore constructed the effective Hamiltonian H_n for the random quenched Potts model (121), (122) (Ref. 59). The free energy (123) is expressed in terms of H_n as

$$\mathcal{F} = \lim_{n\to 0} \mathcal{F}_n, \qquad (149)$$
$$-n\beta\mathcal{F}_n = \ln \mathrm{tr}_n \exp H_n. \qquad (150)$$

This effective Hamiltonian H_n does not contain random quantities. It can be viewed as an average over the interaction strenghts J_{ij} and fields $\mathbf{h}_i$. The form of the Hamiltonian (148) and its relation to the partition function of the original system (149), (150) are independent of the specific form of the distribution functions $P_{ij}(J_{ij})$ and $p_{ix}(h_{ix})$. All the information on the distributions P_{ij} and p_{ix} is contained in only the functions Ψ_{ij} (146) and Φ_{ix} (147).

7. THE FREE ENERGY

Let us calculate the right-hand side of (150), starting from the Bogolyubov variational principle[1,2]:

$$\ln \mathrm{tr} \exp \mathcal{H} - \ln \mathrm{tr} \exp \widetilde{\mathcal{H}} \geqslant \langle \mathcal{H} - \widetilde{\mathcal{H}} \rangle_{\widetilde{\mathcal{H}}}. \qquad (151)$$

For the case under study we write this in the form[59,66]

$$-n\beta\mathcal{F}_n = \ln \mathrm{tr}_n \exp H_n \geqslant \ln \mathrm{tr}_n \exp \widetilde{H}_n$$
$$+\mathrm{tr}_n^{-1} \exp \widetilde{H}_n\, \mathrm{tr}_n [(H_n - \widetilde{H}_n) \exp \widetilde{H}_n] = -n\beta\widetilde{\mathcal{F}}_n, \qquad (152)$$

where $\widetilde{H}_n$ is an arbitrary Hamiltonian serving as the trial Hamiltonian, and the quantity $\widetilde{\mathcal{F}}_n$ is introduced to simplify the notation.

We take the trial Hamiltonian H_n to be of the form[59]

$$\widetilde{H}_n = \widetilde{H}_n(\{\eta_i\}; \{\zeta_i\}; \{\mathbf{s}_i^\alpha\}) = \sum_i \left[\eta_i \sum_{\alpha=1}^{n} \mathbf{s}_i^\alpha + \frac{1}{2}\zeta_i^2 \left(\sum_{\alpha=1}^{n} \mathbf{s}_i^\alpha\right)^2\right],$$
$$(153)$$

where η_i and ζ_i are variational parameters. According to relations (149), (150), and (152) the free energy $\mathcal{F}$ can be calculated as $\mathcal{F}(\{\bar{\eta}_i\},\{\zeta_i\})$, where

$$\widetilde{\mathcal{F}}(\{\eta_i\}, \{\zeta_i\}) = \lim_{n\to 0} \widetilde{\mathcal{F}}_n(\{\eta_i\}, \{\zeta_i\}), \qquad (154)$$

and the quantities $\eta_i = \eta_i$ and $\zeta_i = \bar{\zeta}_i$ are determined from the equations

$$\frac{\partial}{\partial \eta_i}[-\beta\widetilde{\mathcal{F}}(\{\eta_i\}, \{\zeta_i\})] = 0, \qquad (155)$$
$$\frac{\partial}{\partial \zeta_i}[-\beta\widetilde{\mathcal{F}}(\{\eta_i\}, \{\zeta_i\})] = 0. \qquad (156)$$

Let us calculate the right-hand side of the inequality (152), i.e., $\widetilde{\mathcal{F}}_n$ with the trial Hamiltonian $\widetilde{H}_n$ in the form (153). Here we shall use the relation

$$\exp\left(\frac{1}{2}\mathbf{b}^2\right) = \int dG_{\mathbf{u}} \exp(\mathbf{bu}), \qquad (157)$$

where $\mathbf{b}$ is an arbitrary vector from R^m, and $dG_{\mathbf{u}}$ is the normalized Gaussian measure

$$dG_{\mathbf{u}} = (2\pi)^{-m/2} \exp\left(-\frac{1}{2}\mathbf{u}^2\right), \quad u \in R^m, \qquad (158)$$

$$\int dG_{\mathbf{u}} = 1. \qquad (159)$$

We consider the quantity

$$\exp(\omega\widetilde{H}_n) = \prod_i \exp\left(\omega\eta_i \sum_{\alpha=1}^{n} \mathbf{s}_i^\alpha\right) \exp\left[\frac{1}{2}\left(\sqrt{\omega}\,\zeta_i \sum_{\alpha=1}^{n} \mathbf{s}_i^\alpha\right)^2\right].$$
$$(160)$$

Starting from Eq. (157), we have

$$\exp(\omega\widetilde{H}_n)$$

$$= \prod_i \left[\exp\left(\omega\boldsymbol{\eta}_i \sum_{\alpha=1}^{n} \mathbf{s}_i^\alpha\right) \int dG_{\mathbf{u}} \exp\left(\sqrt{\omega}\,\zeta_i\mathbf{u} \sum_{\alpha=1}^{n} \mathbf{s}_i^\alpha\right)\right]$$

$$= \prod_i \int dG_{\mathbf{u}} \exp\left[(\omega\boldsymbol{\eta}_i + \sqrt{\omega}\,\zeta_i\mathbf{u}) \sum_{\alpha=1}^{n} \mathbf{s}_i^\alpha\right] = \prod_i f_i(\mathbf{s}_i^\alpha), \tag{161}$$

i.e., each factor in the product over i is a function of s_u^α ($\alpha = 1, 2, ..., n$) with a single value of i. In this case for the trace tr_n we obtain

$$\mathcal{A}(\omega) = \mathrm{tr}_n \exp(\omega\widetilde{H}_n) = \sum_{\{\mathbf{s}_1^\alpha\}} \cdots \sum_{\{\mathbf{s}_N^\alpha\}} \prod_i f_i(\mathbf{s}_i^\alpha)$$

$$= \prod_i \sum_{\{\mathbf{s}_i^\alpha\}} f_i(\mathbf{s}_i^\alpha)$$

$$= \prod_i \sum_{\{\mathbf{s}_i^\alpha\}} \int dG_{\mathbf{u}} \exp\left[(\omega\boldsymbol{\eta}_i + \sqrt{\omega}\,\zeta_i\mathbf{u}) \sum_{\alpha=1}^{n} \mathbf{s}_i^\alpha\right]. \tag{162}$$

Similarly, performing the summation over $\mathbf{s}_i^1, ..., \mathbf{s}_i^n$, we find

$$\mathcal{A}(\omega) = \prod_i \int dG_{\mathbf{u}} \sum_{\mathbf{s}_i^1} \cdots \sum_{\mathbf{s}_i^n} \prod_{\alpha=1}^{n} \exp[(\omega\boldsymbol{\eta}_i + \sqrt{\omega}\,\zeta_i\mathbf{u})\,\mathbf{s}_i^\alpha]$$

$$= \prod_i \int dG_{\mathbf{u}} \prod_{\alpha=1}^{n} \sum_{\mathbf{s}_i^\alpha} \exp[(\omega\boldsymbol{\eta}_i + \sqrt{\omega}\,\zeta_i\mathbf{u})\,\mathbf{s}_i^\alpha]. \tag{163}$$

Taking into account the fact that here the summation over $\mathbf{s}_i^\alpha$ gives the same results for any $\alpha = 1, 2, ..., n$, we obtain

$$\mathcal{A}(\omega) = \prod_i \int dG_{\mathbf{u}} \left\{\sum_{p=1}^{m+1} \exp[(\omega\boldsymbol{\eta}_i + \sqrt{\omega}\,\zeta_i\mathbf{u})\,\boldsymbol{\varepsilon}_p]\right\}^n. \tag{164}$$

Using the definition (72) of the multidimensional generalization of the hyperbolic cosine $R(\mathbf{b})$, we write $\mathrm{tr}_n \exp(\omega\widetilde{H}\sigma)$ as

$$\mathcal{A}(\omega) = \prod_i \int dG_{\mathbf{u}} [m^{-1}(m+1) R(\omega\boldsymbol{\eta}_i + \sqrt{\omega}\,\zeta_i\mathbf{u})]^n. \tag{165}$$

The quantity $\mathscr{F}_n$ involves the expression $\mathrm{tr}_n \exp\widetilde{H}_n$, which is equal to $\mathcal{A}(1)$, and can therefore be written as

$$\mathrm{tr}_n \exp\widetilde{H}_n = \left(\frac{m+1}{m}\right)^{Nn} \prod_k \int dG_{\mathbf{u}} R^n(\mathbf{b}_k), \tag{166}$$

where for brevity we have introduced the notation

$$\mathbf{b}_i = \boldsymbol{\eta}_i + \zeta_i\mathbf{u}. \tag{167}$$

The trace $\mathrm{tr}_n(\widetilde{H}_n \exp\widetilde{H}_n)$ which also enters into $\mathscr{F}_n$ can be calculated by writing it as

$$\mathrm{tr}_n(\widetilde{H}_n \exp\widetilde{H}_n) = \frac{\partial}{\partial\omega} \mathrm{tr}_n \exp(\omega\widetilde{H}_n)\,|_{\omega=1} = \mathcal{A}'(1). \tag{168}$$

Then, noting that

$$\left(\prod_i g_i\right)' = \left(\prod_j g_j\right) \sum_i (g_i^{-1} g_i'), \tag{169}$$

and also that according to the definition (77) of the function $\mathbf{Q}(\mathbf{b})$

$$\frac{\partial}{\partial\omega} R(\omega\boldsymbol{\eta}_i + \sqrt{\omega}\,\zeta_i\mathbf{u}) = \left(\boldsymbol{\eta}_i + \frac{1}{2}\omega^{-1/2}\mathbf{u}\right) \mathbf{Q}(\omega\boldsymbol{\eta}_i + \sqrt{\omega}\,\zeta_i\mathbf{u}), \tag{170}$$

we have

$$\mathcal{A}'(\omega) = \mathcal{A}(\omega) \sum_i$$

$$\times \left\{\left[\int dG_{\mathbf{u}} (m^{-1}(m+1) R(\omega\boldsymbol{\eta}_i + \sqrt{\omega}\,\zeta_i\mathbf{u}))^n\right]^{-1}\right.$$

$$\times n \int dG_{\mathbf{u}} [m^{-1}(m+1) R(\omega\boldsymbol{\eta}_i + \sqrt{\omega}\,\zeta_i\mathbf{u})]^{n-1} m^{-1}(m+1)$$

$$\left. \times \left(\omega\boldsymbol{\eta}_i + \frac{1}{2}\omega^{-1/2}\mathbf{u}\right) \mathbf{Q}(\omega\boldsymbol{\eta}_i + \sqrt{\omega}\,\zeta_i\mathbf{u})\right\}, \tag{171}$$

so that

$$\mathbf{tr}_n(\widetilde{H}_n \exp\widetilde{H}_n) = n\,\mathrm{tr}_n \exp\widetilde{H}_n \sum_i \left\{\left[\int dG_{\mathbf{u}} R^n(\boldsymbol{\eta}_i + \zeta_i\mathbf{u})\right]^{-1}\right.$$

$$\left. \times \int dG_{\mathbf{u}} R^{n-1}(\boldsymbol{\eta}_i + \zeta_i\mathbf{u}) \left(\boldsymbol{\eta}_i + \frac{1}{2}\zeta_i\mathbf{u}\right) \mathbf{Q}(\boldsymbol{\eta}_i + \zeta_i\mathbf{u})\right\}. \tag{172}$$

We can get rid of the factor **u** in the integrand by using the relation

$$\int dG_{\mathbf{u}} u_x f(\mathbf{u}) = \int dG_{\mathbf{u}} \frac{\partial}{\partial u_x} f(\mathbf{u}). \tag{173}$$

Noting that

$$\frac{\partial}{\partial u_x} [R^{n-1}(\mathbf{b}_i)\,Q_x(\mathbf{b}_i)] = (n-1) R^{n-2}(\mathbf{b}_i)\,Q_x(\mathbf{b}_i) \frac{\partial}{\partial u_x} R(\mathbf{b}_i)$$

$$+ R^{n-1}(\mathbf{b}_i) \frac{\partial}{\partial u_x} Q_x(\mathbf{b}_i), \tag{174}$$

$$\frac{\partial}{\partial u_x} R(\mathbf{b}_i) = \zeta_i Q_x(\mathbf{b}_i), \tag{175}$$

$$\frac{\partial}{\partial u_x} Q_x(\mathbf{b}_i) = \zeta_i Q_{xx}(\mathbf{b}_i), \tag{176}$$

and using the property (83) of the functions Q_{xx}, we find

$$\sum_{x=1}^{m} \frac{\partial}{\partial u_x} [R^{n-1}(\mathbf{b}_i)\,Q_x(\mathbf{b}_i)]$$

$$= \zeta_i [(n-1) R^{n-2}(\mathbf{b}_i)\,\mathbf{Q}^2(\mathbf{b}_i) + R^n(\mathbf{b}_i)], \tag{177}$$

so that for the trace $\mathrm{tr}_n(\widetilde{H}_n \exp\widetilde{H}_n)$ we have

$$\mathbf{tr}_n(\widetilde{H}_n \exp\widetilde{H}_n) = n\,\mathrm{tr}_n \exp\widetilde{H}_n \sum_i \left\{\left[\int dG_{\mathbf{u}} R^n(\mathbf{b}_i)\right]^{-1}\right.$$

$$\times \int dG_{\mathbf{u}} \left[R^{n-1}(\mathbf{b}_i)\,\boldsymbol{\eta}_i\mathbf{Q}(\mathbf{b}_i) + \frac{1}{2}\zeta_i^2 ((n-1) R^{n-2}(\mathbf{b}_i)\,\mathbf{Q}^2(\mathbf{b}_i)\right.$$

$$\left.\left. + R^n(\mathbf{b}_i))\right]\right\}. \tag{178}$$

Let us now consider the trace $\mathrm{tr}_n(s_{ix_1}^{\alpha_1} ... s_{ix_l}^{\alpha_l} \exp\widetilde{H}_n)$, where among the indices $\alpha_1, ..., \alpha_l$ there is no pair of identical values. Using expression (157), we write

$$\mathbf{tr}_n(s_{ix_1}^{\alpha_1} \ldots s_{ix_l}^{\alpha_l} \exp\widetilde{H}_n) = \sum_{\{\mathbf{s}_1^\alpha\}} \cdots \sum_{\{\mathbf{s}_N^\alpha\}} \prod_{k(\neq i)} \int dG_{\mathbf{u}}$$

$$\times \exp\left(\sum_{\alpha=1}^{n} \mathbf{b}_k\mathbf{s}_k^\alpha\right) \int dG_{\mathbf{u}} s_{ix_1}^{\alpha_1} \ldots s_{ix_l}^{\alpha_l} \exp\left(\sum_{\alpha=1}^{n} \mathbf{b}_i\mathbf{s}_i^\alpha\right). \tag{179}$$

Using the fact that here each factor in the summed expressions depends on only $s_{ix_k}^{\alpha_k}$ with a single i, we interchange the product and summation:

$$\mathrm{tr}_n\,(s_{ix_1}^{\alpha_1}\ldots s_{ix_l}^{\alpha_l}\exp\widetilde{H}_n)$$

$$=\prod_{k(\neq i)}\sum_{\{\mathbf{s}_k^{\alpha}\}}\int dG_{\mathbf{u}}\exp\Big(\sum_{\alpha=1}^{n}\mathbf{b}_k\mathbf{s}_k^{\alpha}\Big)$$

$$\times\sum_{\{\mathbf{s}_i^{\alpha}\}}\int dG_{\mathbf{u}}s_{ix_1}^{\alpha_1}\ldots s_{ix_l}^{\alpha_l}\exp\Big(\sum_{\alpha=1}^{n}\mathbf{b}_i\mathbf{s}_i^{\alpha}\Big)$$

$$=\prod_{k(\neq i)}\int dG_{\mathbf{u}}\sum_{\{\mathbf{s}_k^{1}\}}\ldots\sum_{\{\mathbf{s}_k^{n}\}}\prod_{\alpha=1}^{n}\exp\,(\mathbf{b}_k\mathbf{s}_k^{\alpha})$$

$$\times\int dG_{\mathbf{u}}\sum_{\{\mathbf{s}_i^{1}\}}\ldots\sum_{\{\mathbf{s}_i^{n}\}}\prod_{\gamma(\neq\alpha_1,\ldots,\alpha_l)}$$

$$\times\exp\,(\mathbf{b}_i\mathbf{s}_i^{\gamma})\prod_{r=1}^{l}s_{ix_r}^{\alpha_r}\exp\,(\mathbf{b}_i\mathbf{s}_i^{\alpha_r}).$$

Interchanging the product over α and the corresponding summation in the analogous manner, we obtain

$$\mathrm{tr}_n\,(s_{ix_1}^{\alpha_1}\ldots s_{ix_l}^{\alpha_l}\exp\widetilde{H}_n)=\prod_{k(\neq i)}\int dG_{\mathbf{u}}\prod_{\alpha=1}^{n}\sum_{\mathbf{s}_k^{\alpha}}\exp\,(\mathbf{b}_k\mathbf{s}_k^{\alpha})$$

$$\times\int dG_{\mathbf{u}}\prod_{\gamma(\neq\alpha_1,\ldots,\alpha_l)}\sum_{\mathbf{s}_i^{\gamma}}\exp\,(\mathbf{b}_i\mathbf{s}_i^{\gamma})\prod_{r=1}^{l}\sum_{\mathbf{s}_i^{\alpha_r}}s_{ix_r}^{\alpha_r}\exp\,(\mathbf{b}_i\mathbf{s}_i^{\alpha_r}).\tag{180}$$

Taking into account the definitions (72) and (74) of the functions R and $\mathbf{Q}$, we finally obtain

$$\mathrm{tr}_n\,(s_{ix_1}^{\alpha_1}\ldots s_{ix_l}^{\alpha_l}\exp\widetilde{H}_n)=\prod_{k(\neq i)}\int dG_{\mathbf{u}}\Big[\frac{m+1}{m}R\,(\mathbf{b}_k)\Big]^{n}$$

$$\times\int dG_{\mathbf{u}}\Big[\frac{m+1}{m}R\,(\mathbf{b}_i)\Big]^{n-l}\Big(\frac{m+1}{m}\Big)^{l}Q_{x_1}(\mathbf{b}_i)\ldots Q_{x_l}(\mathbf{b}_i)$$

$$=\Big[\int dG_{\mathbf{u}}R^{n}\,(\mathbf{b}_i)\Big]^{-1}\mathrm{tr}_n$$

$$\times\exp\widetilde{H}_n\int dG_{\mathbf{u}}R^{n-l}\,(\mathbf{b}_i)\,Q_{x_1}(\mathbf{b}_i)\ldots Q_{x_l}(\mathbf{b}_i).\tag{181}$$

In order to calculate the trace $\mathrm{tr}_n\,(H_n\exp\widetilde{H}_n)$ appearing in $\mathscr{F}_n$, we use (148) to write it as

$$\mathrm{tr}_n\,(H_n\exp\widetilde{H}_n)=\sum_{i<j}\mathscr{A}_{ij}\Psi_{ij}(\tau)|_{\tau=0}$$

$$+\sum_{i}\sum_{x=1}^{m}\mathscr{B}_{ix}\Phi_{ix}(\tau)|_{\tau=0}+\beta\sum_{\alpha=1}^{n}\sum_{i}\mathbf{v}_i\mathscr{C}_{i\alpha},\tag{182}$$

where

$$\mathscr{A}_{ij}=\mathrm{tr}_n\Big[\exp\Big(D\sum_{\alpha=1}^{n}\mathbf{s}_i^{\alpha}\mathbf{s}_j^{\alpha}\Big)\exp\widetilde{H}_n\Big];\tag{183}$$

$$\mathscr{B}_{ix}=\mathrm{tr}_n\Big[\exp\Big(D\sum_{\alpha=1}^{n}s_{ix}^{\alpha}\Big)\exp\widetilde{H}_n\Big];\tag{184}$$

$$\mathscr{C}_{i\alpha}=\mathrm{tr}_n\,(\mathbf{s}_i^{\alpha}\exp\widetilde{H}_n).\tag{185}$$

Using (157) to write $\exp\widetilde{H}_n$ as

$$\exp\widetilde{H}_n=\prod_{k}\int dG_{\mathbf{u}}\exp\Big(\sum_{\alpha=1}^{n}\mathbf{b}_k\mathbf{s}_k^{\alpha}\Big),\tag{186}$$

for $\mathscr{A}_{ij}$ we have

$$\mathscr{A}_{ij}=\mathrm{tr}_n\Big[\exp\Big(D\sum_{\alpha=1}^{n}\mathbf{s}_i^{\alpha}\mathbf{s}_j^{\alpha}\Big)\prod_{k}\int dG_{\mathbf{u}}\exp\Big(\sum_{\alpha=1}^{n}\mathbf{b}_k\mathbf{s}_k^{\alpha}\Big)\Big].\tag{187}$$

Isolating the factors with $k=i,j$ in the product over k and combining them with $\exp(D\,\Sigma_{\alpha=1}^{n}\mathbf{s}_i^{\alpha}\mathbf{s}_j^{\alpha})$, we write

$$\mathscr{A}_{ij}=\sum_{\{\mathbf{s}_1^{\alpha}\}}\ldots\sum_{\{\mathbf{s}_N^{\alpha}\}}\prod_{k(\neq i,\,j)}\int dG_{\mathbf{u}}\exp\Big(\sum_{\alpha=1}^{n}\mathbf{b}_k\mathbf{s}_k^{\alpha}\Big)$$

$$\times\int dG_{\mathbf{u}}\,dG_{\mathbf{v}}\exp\Big[\sum_{\alpha=1}^{n}(D\mathbf{s}_i^{\alpha}\mathbf{s}_j^{\alpha}+\mathbf{b}_i\mathbf{s}_i^{\alpha}+\mathbf{c}_j\mathbf{s}_j^{\alpha})\Big]$$

$$=\sum_{\{\mathbf{s}_1^{\alpha}\}}\ldots\sum_{\{\mathbf{s}_N^{\alpha}\}}\prod_{k(\neq i,\,j)}f_k\,(\mathbf{s}_k^{\alpha})\,f_{ij}\,(\mathbf{s}_i^{\alpha},\ \mathbf{s}_j^{\alpha}),\ \ \mathbf{c}_j=\boldsymbol{\eta}_j+\zeta_j\mathbf{v},\tag{188}$$

i.e., $\mathscr{A}_{ij}$ is the sum over $\{\mathbf{s}_1^{\alpha}\}$, ..., $\{\mathbf{s}_N^{\alpha}\}$ of the product of $N-2$ factors, each of which depends only on $\mathbf{s}_k^{\alpha}$ ($\alpha=1, 2, \ldots, n$) with a single value of k, and a single factor depending on $\mathbf{s}_i^{\alpha}$ and $\mathbf{s}_j^{\alpha}$ ($\alpha=1,2,\ldots,n;\ i\neq j$). Interchanging the product and summation, we find

$$\mathscr{A}_{ij}=\prod_{k(\neq i,\,j)}\sum_{\{\mathbf{s}_k^{\alpha}\}}f_k\,(\mathbf{s}_k^{\alpha})\sum_{\{\mathbf{s}_i^{\alpha}\}}\sum_{\{\mathbf{s}_j^{\alpha}\}}f_{ij}\,(\mathbf{s}_i^{\alpha},\ \mathbf{s}_j^{\alpha}).\tag{189}$$

Then in turn, summing $f_k\,(\mathbf{s}_k^{\alpha})$ over $\{\mathbf{s}_k^{\alpha}\}$, we obtain

$$\sum_{\{\mathbf{s}_k^{\alpha}\}}f_k\,(\mathbf{s}_k^{\alpha})=\int dG_{\mathbf{u}}\sum_{\mathbf{s}_k^{1}}\ldots\sum_{\mathbf{s}_k^{n}}\prod_{\alpha=1}^{n}\exp\,(\mathbf{b}_k\mathbf{s}_k^{\alpha})$$

$$=\int dG_{\mathbf{u}}\prod_{\alpha=1}^{n}\sum_{\mathbf{s}_k^{\alpha}}\exp\,(\mathbf{b}_k\mathbf{s}_k^{\alpha})=\Big(\frac{m+1}{m}\Big)^{n}\int dG_{\mathbf{u}}R^{n}\,(\mathbf{b}_k),\tag{190}$$

and summing $f_{ij}\,(\mathbf{s}_i^{\alpha},\mathbf{s}_j^{\alpha})$ over $\{\mathbf{s}_i^{\alpha}\}$, $\{\mathbf{s}_j^{\alpha}\}$ gives

$$\sum_{\{\mathbf{s}_i^{\alpha}\},\,\{\mathbf{s}_j^{\alpha}\}}f_{ij}\,(\mathbf{s}_i^{\alpha},\ \mathbf{s}_j^{\alpha})$$

$$=\int dG_{\mathbf{u}}\,dG_{\mathbf{v}}\sum_{\mathbf{s}_i^{1},\,\mathbf{s}_j^{1}}\ldots\sum_{\mathbf{s}_i^{n},\,\mathbf{s}_j^{n}}\prod_{\alpha=1}^{n}\exp\,(D\mathbf{s}_i^{\alpha}\mathbf{s}_j^{\alpha}+\mathbf{b}_i\mathbf{s}_i^{\alpha}$$

$$+\mathbf{c}_j\mathbf{s}_j^{\alpha})=\int dG_{\mathbf{u}}\,dG_{\mathbf{v}}\prod_{\alpha=1}^{n}\sum_{\mathbf{s}_i^{\alpha},\,\mathbf{s}_j^{\alpha}}\exp\,(D\mathbf{s}_i^{\alpha}\mathbf{s}_j^{\alpha}+\mathbf{b}_i\mathbf{s}_i^{\alpha}+\mathbf{c}_j\mathbf{s}_j^{\alpha})$$

$$=\sum_{\{\mathbf{s}_1^{\alpha}\}}\ldots\sum_{\{\mathbf{s}_N^{\alpha}\}}\prod_{k(\neq i,\,j)}f_k\,(\mathbf{s}_k^{\alpha})\,f_{ij}\,(\mathbf{s}_i^{\alpha},\ \mathbf{s}_j^{\alpha}),\tag{191}$$

where the function T is defined as in (87), and

$$\mathbf{c}_j=\boldsymbol{\eta}_j+\zeta_j\mathbf{v}.\tag{192}$$

Therefore,

$$\mathscr{A}_{ij}=\Big(\frac{m+1}{m}\Big)^{Nn}\prod_{k(\neq i,\,j)}\int dG_{\mathbf{u}}R^{n}\,(\mathbf{b}_k)\int dG_{\mathbf{u}}dG_{\mathbf{v}}T^{n}(D,\mathbf{b}_i,\mathbf{c}_j)$$

$$=\Big(\frac{m+1}{m}\Big)^{Nn}\prod_{k}\int dG_{\mathbf{u}}R^{n}\,(\mathbf{b}_k)\Big[\int\int dG_{\mathbf{u}}\,dG_{\mathbf{v}}R^{n}\,(\mathbf{b}_i)R^{n}\,(\mathbf{c}_j)\Big]^{-1}$$

$$\times\int\int dG_{\mathbf{u}}\,dG_{\mathbf{v}}T^{n}\,(D,\ \mathbf{b}_i,\ \mathbf{c}_j).\tag{193}$$

Similarly, for $\mathscr{B}_{ix}$ we have

$$\mathscr{B}_{ix}=\sum_{\{s_1^{\alpha}\}}\cdots\sum_{\{s_N^{\alpha}\}}\prod_{k(\neq i)}\int dG_{\mathbf{u}}\exp\Big(\sum_{\alpha=1}^{n}\mathbf{b}_k\mathbf{s}_k^{\alpha}\Big)$$

$$\times\int dG_{\mathbf{u}}\exp\Big[\sum_{\alpha=1}^{n}(Ds_{ix}^{\alpha}+\mathbf{b}_i\mathbf{s}_i^{\alpha})\Big]$$

$$=\prod_{k(\neq i)}\int dG_{\mathbf{u}}\prod_{\alpha=1}^{n}\sum_{s_k^{\alpha}}\exp\Big(\sum_{\alpha}\mathbf{b}_k\mathbf{s}_k^{\alpha}\Big)$$

$$\times\int dG_{\mathbf{u}}\prod_{\alpha=1}^{n}\sum_{s_i^{\alpha}}\exp(Ds_{ix}^{\alpha}+\mathbf{b}_i\mathbf{s}_i^{\alpha}),\tag{194}$$

or, according to the definitions (72), (113) of the functions R and $P_{(x)}$,

$$\mathscr{B}_{ix}=m^{-Nn}(m+1)^{Nn}\prod_{k}\int dG_{\mathbf{u}}R^{n}(\mathbf{b}_k)$$

$$\times\Big[\int dG_{\mathbf{u}}R^{n}(\mathbf{b}_i)\Big]^{-1}\int dG_{\mathbf{u}}P_{(x)}^{n}(D,\,\mathbf{b}_i).\tag{195}$$

In accordance with (181), for $\mathscr{C}_{i\alpha}$ we have

$$\mathscr{C}_{i\alpha x}=\Big(\frac{m+1}{m}\Big)^{Nn}\prod_{k}\int dG_{\mathbf{u}}R^{n}(\mathbf{b}_k)$$

$$\times\Big[\int dG_{\mathbf{u}}R^{n}(\mathbf{b}_i)\Big]^{-1}\int dG_{\mathbf{u}}R^{n-1}(\mathbf{b}_i)\,Q_x(\mathbf{b}_i).\tag{196}$$

Therefore, using (182), (193), (195), and (196) and taking into account (166), for $\mathrm{tr}_n(H_n\exp\widetilde{H}_n)$ we ultimately obtain

$$\mathrm{tr}_n(H_n\exp\widetilde{H}_n)=\mathrm{tr}_n\exp\widetilde{H}_n\sum_{i<j}$$

$$\times\Big\{\Big[\int\int dG_{\mathbf{u}}\,dG_{\mathbf{v}}R^{n}(\mathbf{b}_i)\,R^{n}(\mathbf{c}_j)\Big]^{-1}$$

$$\times\int\int dG_{\mathbf{u}}\,dG_{\mathbf{v}}T^{n}(D,\,\mathbf{b}_i,\,\mathbf{c}_j)\,\Psi_{ij}(\tau)|_{\tau=0}\Big\}$$

$$+\,\mathrm{tr}_n\exp\widetilde{H}_n\sum_{i}\sum_{x=1}^{m}\Big\{\Big[\int dG_{\mathbf{u}}R^{n}(\mathbf{b}_i)\Big]^{-1}$$

$$\times\int dG_{\mathbf{u}}P_{(x)}^{n}(D,\,\mathbf{b}_i)\,\Phi_{ix}(\tau)|_{\tau=0}\Big\}$$

$$+\,\beta\,\mathrm{tr}_n\exp\widetilde{H}_n\sum_{\alpha=1}^{n}\sum_{i}\Big\{\mathbf{v}_i\Big[\int dG_{\mathbf{u}}R^{n}(\mathbf{b}_i)\Big]^{-1}$$

$$\times\int dG_{\mathbf{u}}R^{n-1}(\mathbf{b}_i)\,\mathbf{Q}(\mathbf{b}_i)\Big\}.\tag{197}$$

Finally, combining (166), (178), and (197), for $\mathscr{F}_n$ (152) we obtain

$$-n\beta\widetilde{\mathscr{F}}_n=\sum_{i<j}\Big\{\Big[\int\int dG_{\mathbf{u}}\,dG_{\mathbf{v}}R^{n}(\mathbf{b}_i)\,R^{n}(\mathbf{c}_j)\Big]^{-1}$$

$$\times\int\int dG_{\mathbf{u}}\,dG_{\mathbf{v}}T^{n}(D,\,\mathbf{b}_i,\,\mathbf{c}_j)\,\Psi_{ij}(\tau)|_{\tau=0}\Big\}$$

$$+\sum_{i}\sum_{x=1}^{m}\Big\{\Big[\int dG_{\mathbf{u}}R^{n}(\mathbf{b}_i)\Big]^{-1}\int dG_{\mathbf{u}}P_{(x)}^{n}(D,\,\mathbf{b}_i)\,\Phi_{ix}(\tau)|_{\tau=0}\Big\}$$

$$+\,n\beta\sum_{i}\Big\{\mathbf{v}_i\Big[\int dG_{\mathbf{u}}R^{n}(\mathbf{b}_i)\Big]^{-1}\int dG_{\mathbf{u}}R^{n-1}(\mathbf{b}_i)\,\mathbf{Q}(\mathbf{b}_i)\Big\}$$

$$-\,n\sum_{i}\Big\{\Big[\int dG_{\mathbf{u}}R^{n}(\mathbf{b}_i)\Big]^{-1}\int dG_{\mathbf{u}}\Big[\boldsymbol{\eta}_iR^{n-1}(\mathbf{b}_i)\,\mathbf{Q}(\mathbf{b}_i)$$

$$+\frac{1}{2}\zeta_i^2((n-1)\,R^{n-2}(\mathbf{b}_i)\,\mathbf{Q}^2(\mathbf{b}_i)+R^{n}(\mathbf{b}_i))\Big]\Big\}$$

$$+\sum_{i}\ln\int dG_{\mathbf{u}}R^{n}(\mathbf{b}_i)+Nn\ln\frac{m+1}{m}.\tag{198}$$

Now in accordance with the variational principle we must calculate the limit $\lim\limits_{n\to0}\widetilde{\mathscr{F}}_n=\widetilde{\mathscr{F}}$ (Ref. 59). Here we shall use relations (128) and (129), and also the fact that, according to the definitions (146) and (147), the functions $\Psi_{ij}(\tau)$ and $\Phi_{ix}(\tau)$ satisfy the equations

$$\Psi_{ij}(0)=0,\tag{199}$$

$$\Phi_{ix}(0)=0.\tag{200}$$

Therefore, owing to (199), (200), and (129),

$$\lim_{n\to0}n^{-1}T^{n}\Psi_{ij}(\tau)|_{\tau=0}$$

$$=\lim_{n\to0}n^{-1}(T^{n}-1)\,\Psi_{ij}(\tau)|_{\tau=0}=\ln T\Psi_{ij}(\tau)|_{\tau=0},\tag{201}$$

$$\lim_{n\to0}n^{-1}P_{(x)}^{n}\,\Phi_{ix}(\tau)|_{\tau=0}$$

$$=\lim_{n\to0}n^{-1}(P_{(x)}^{n}-1)\,\Phi_{ix}(\tau)|_{\tau=0}=\ln P_{(x)}^{n}\Phi_{ix}(\tau)|_{\tau=0}.\tag{202}$$

We further take into account the fact that the measure $dG_{\mathbf{u}}$ is normalized as in (159) and, therefore,

$$\lim_{n\to0}\int dG_{\mathbf{u}}R^{n}=1,\tag{203}$$

so that according to (128) we can write

$$\lim_{n\to0}n^{-1}\ln\int dG_{\mathbf{u}}R^{n}=\lim_{n\to0}n^{-1}\Big(\int dG_{\mathbf{u}}R^{n}-1\Big)$$

$$=\lim_{n\to0}\int dG_{\mathbf{u}}n^{-1}(R^{n}-1),\tag{204}$$

or, again using relation (129),

$$\lim_{n\to0}\;n^{-1}\ln\int dG_{\mathbf{u}}R^{n}=\int dG_{\mathbf{u}}\ln R.\tag{205}$$

On the basis of (201), (202), and (205) we can now write down the limit of $\widetilde{\mathscr{F}}_n$:

$$-\beta\widetilde{\mathscr{F}}=\lim_{n\to0}(-\beta\mathscr{F}_n)$$

$$=\sum_{i<j}\int\int dG_{\mathbf{u}}\,dG_{\mathbf{v}}\ln T(D,\,\mathbf{b}_i,\,\mathbf{c}_j)\,\Psi_{ij}(\tau)|_{\tau=0}$$

$$+\sum_{i}\sum_{x=1}^{m}\int dG_{\mathbf{u}}\ln P_{(x)}(D,\,\mathbf{b}_i)\,\Phi_{ix}(\tau)|_{\tau=0}$$

$$+\sum_{i}\Big[(\beta\mathbf{v}_i-\boldsymbol{\eta}_i)\int dG_{\mathbf{u}}R^{-1}(\mathbf{b}_i)\,\mathbf{Q}(\mathbf{b}_i)\Big]$$

$$+\sum_{i}\Big[\frac{1}{2}\zeta_i^2\int dG_{\mathbf{u}}R^{-2}(\mathbf{b}_i)\,\mathbf{Q}^2(\mathbf{b}_i)\Big]-\sum_{i}\Big(\frac{1}{2}\zeta_i^2\Big)$$

$$+\sum_{i}\int dG_{\mathbf{u}}\ln R(\mathbf{b}_i)+N\ln\frac{m+1}{m}.\tag{206}$$

We note that in the expression for the free energy $\widetilde{\mathscr{F}}_n$ in the process of taking the limit $n\to0$ the factor $(n-1)$ in front of $R^{n-2}(\mathbf{b}_i)\mathbf{Q}^2(\mathbf{b}_i)$ in the term proportional to ζ_i^2 changes sign. That is, the term

$$-\frac{1}{2}\zeta_i^2\int dG_{\mathbf{u}}\,[(n-1)\,R^{n-2}(\mathbf{b}_i)\,\mathbf{Q}^2(\mathbf{b}_i)+R^n(\mathbf{b}_i)] \quad (207)$$

becomes

$$+\frac{1}{2}\zeta_i^2\int dG_{\mathbf{u}}\,[R^{-2}(\mathbf{b}_i)\,\mathbf{Q}^2(\mathbf{b}_i)-1]. \quad (208)$$

As a result, the sign in front of all the terms containing ζ_i^2 reverses. This also causes the maximum with respect to ζ_i in $-\beta\widetilde{\mathscr{F}}_n$ to become a minimum with respect to ζ_i in $-\beta\widetilde{\mathscr{F}}$. Therefore, if $(\boldsymbol{\eta}_i, \zeta_i) = (\bar{\boldsymbol{\eta}}_i, \bar{\zeta}_i)$ is the point at which $-\beta\widetilde{\mathscr{F}}_n$ reaches its maximum, then for $-\beta\widetilde{\mathscr{F}}$ the point $(\boldsymbol{\eta}_i, \zeta_i) = (\bar{\boldsymbol{\eta}}_i, \bar{\zeta}_i)$ will be a saddle point (a maximum with respect to $\boldsymbol{\eta}_i$ and a minimum with respect to ζ_i).

In order to write (206) in a more convenient form, we use the relation

$$\mathscr{A}(D)\,\mathscr{B}(\tau)|_{\tau=0}=\mathscr{B}(D)\,\mathscr{A}(\tau)|_{\tau=0}. \quad (209)$$

Therefore, in accordance with (206) and (209) for the free energy we have the expression

$$\begin{aligned}-\beta\widetilde{\mathscr{F}}=&\sum_{i<j}\iint dG_{\mathbf{u}}\,dG_{\mathbf{v}}\Psi_{ij}(D)\ln T(\tau,\ \mathbf{b}_i,\ \mathbf{c}_j)|_{\tau=0}\\&+\sum_i\sum_{x=1}^m\int dG_{\mathbf{u}}\Phi_{ix}(D)\ln P_{(x)}(\tau,\ \mathbf{b}_i)|_{\tau=0}\\&+\sum_i(\beta\mathbf{v}_i-\boldsymbol{\eta}_i)\int dG_{\mathbf{u}}R^{-1}(\mathbf{b}_i)\,\mathbf{Q}(\mathbf{b}_i)\\&+\sum_i\Big[\frac{1}{2}\zeta_i^2\int dG_{\mathbf{u}}R^{-2}(\mathbf{b}_i)\,\mathbf{Q}^2(\mathbf{b}_i)\Big]-\sum_i\Big(\frac{1}{2}\zeta_i^2\Big)\\&+\sum_i\int dG_{\mathbf{u}}\ln R(\mathbf{b}_i)+N\ln\frac{m+1}{m}.\end{aligned} \quad (210)$$

Let us see how the quenched Ising model is described in our scheme. We obtain the Ising model by setting the dimension of the spin space m equal to unity, $m = 1$.

Using the fact that for $m = 1$ the function $R(\mathbf{b})$ is the hyperbolic cosine $\cosh b$ and $\mathbf{Q}(\mathbf{b})$ is the hyperbolic sine $\sinh b$, and also taking into account Eqs. (88) and (114), for the free energy of the quenched Ising model we obtain

$$\begin{aligned}-\beta\widetilde{\mathscr{F}}=&\sum_{i<j}\iint dG_u\,dG_v\Psi_{ij}(D)\ln(\operatorname{ch}\tau\operatorname{ch}b_i\operatorname{ch}c_j\\&\qquad+\operatorname{sh}\tau\operatorname{sh}b_i\operatorname{sh}c_j)|_{\tau=0}\\&+\sum_i\int dG_u\Phi_i(D)\ln\operatorname{ch}(\tau+b_i)|_{\tau=0}\\&+\sum_i\Big[(\beta v_i-\eta_i)\int dG_u\operatorname{th}b_i\Big]+\sum_i\Big[\frac{1}{2}\zeta_i^2\int dG_u\operatorname{th}^2b_i\Big]\\&-\sum_i\Big(\frac{1}{2}\zeta_i^2\Big)+\sum_i\int dG_u\ln\operatorname{ch}b_i+N\ln 2,\ \ m=1,\end{aligned} \quad (211)$$

where

$$\begin{aligned}&dG_w=(2\pi)^{-1/2}\exp\Big(-\frac{1}{2}w^2\Big)\,dw;\\&b_i=\eta_i+\zeta_iu;\quad c_j=\eta_j+\zeta_iv;\\&\Psi_{ij}(\xi)=\ln\int dJP_{ij}(J)\exp(\beta J\xi);\\&\Phi_i(\xi)=\ln\int dhp_i(h)\exp(\beta J\xi);\end{aligned} \quad (212)$$

η_i and ζ_i are scalar variational parameters, and P_{ij} and p_i are the distribution densities of the interaction strengths of the Ising spins at points i and j and of the field at the point i, respectively.

Let us now obtain the explicit equations for the variational parameters $\bar{\boldsymbol{\eta}}_i$ and $\bar{\zeta}_i$, (155) and (156).

Differentiation of (206) with respect to η_{ix} gives Nm equations:

$$\begin{aligned}&\sum_{j(\neq i)}\iint dG_{\mathbf{u}}\,dG_{\mathbf{v}}T^{-1}(D,\ \mathbf{b}_i,\ \mathbf{c}_j)\,U_x(D,\ \mathbf{b}_i,\ \mathbf{c}_j)\,\Psi_{ij}(\tau)|_{\tau=0}\\&+\int dG_{\mathbf{u}}\sum_{y=1}^m[P_{(y)}^{-1}(D,\ \mathbf{b}_i)\,S_{(y)x}(D,\ \mathbf{b}_i)\,\Phi_{iy}(\tau)|_{\tau=0}]\\&+\int dG_{\mathbf{u}}\Big\{\sum_{y=1}^m[(\beta\mathbf{v}_{iy}-\eta_{iy})(-R^{-2}Q_xQ_y+R^{-1}Q_{xy})]\\&+\zeta_i^2\Big[-R^{-3}\mathbf{Q}^2Q_x+\sum_{y=1}^m(R^{-2}Q_yQ_{xy})\Big]\Big\}(\mathbf{b}_i)=0,\end{aligned} \quad (213)$$

where the functions U_x and $S_{(y)x}$ are defined by (107) and (115). Differentiating (206) with respect to ζ_i and eliminating the factor $\mathbf{u}$ from the integrand using Eq. (173), we find the other N equations:

$$\begin{aligned}&\zeta_i\sum_{j\neq(i)}\iint dG_{\mathbf{u}}\,dG_{\mathbf{v}}\sum_{x=1}^m[-T^{-2}(D,\ \mathbf{b}_i,\ \mathbf{c}_j)\,U_x^2(D,\ \mathbf{b}_i,\ \mathbf{c}_j)\\&+T^{-1}(D,\ \mathbf{b}_i,\ \mathbf{c}_j)\,U_{xx}(D,\ \mathbf{b}_i,\ \mathbf{c}_j)]\,\Psi_{ij}(\tau)|_{\tau=0}\\&+\zeta_i\int dG_{\mathbf{u}}\sum_{x,\,y=1}^m[-P_{(y)}^{-2}(D,\ \mathbf{b}_i)\,S_{(y)x}^2(D,\ \mathbf{b}_i)\\&+P_{(y)}^{-1}(D,\ \mathbf{b}_i)\,S_{(y)xx}(D,\ \mathbf{b}_i)]\,\Phi_{iy}\\&\times(\tau)|_{\tau=0}+\zeta_i\int dG_{\mathbf{u}}\Big\{\sum_{x,\,y=1}^m[(\beta\nu_{iy}-\eta_{iy})(2R^{-3}Q_x^2Q_y\\&-R^{-2}Q_{xx}Q_y-2R^{-2}Q_xQ_{xy}+R^{-1}Q_{xxy})]\\&+\zeta_i^2\Big[3R^{-4}(\mathbf{Q}^2)^2-\sum_{x=1}^m(R^{-3}Q_{xx}\mathbf{Q}^2)\\&+\sum_{x,\,y=1}^m(-4R^{-3}Q_xQ_yQ_{xy}+R^{-2}Q_{xy}^2+R^{-2}Q_yQ_{xxy})\Big]\\&+\sum_{x=1}^m(R^{-1}Q_{xx})\Big\}(\mathbf{b}_i)-\zeta_i=0,\end{aligned} \quad (214)$$

where Q_{xyz}, $S_{(x)yz}$, and U_{xy} are defined as in (105), (116), and (109). The properties (83), (111), (117), and (112) of the functions Q_{xy}, Q_{xyz}, $\mathrm{S}_{(x)yz}$, and U_{xy} allow Eq. (214) to be rewritten as

$$\begin{aligned}&-\zeta_i\sum_{j(\neq i)}\iint dG_{\mathbf{u}}dG_{\mathbf{v}}T^{-2}(D,\mathbf{b}_i,\ \mathbf{c}_j)\,\mathbf{U}^2(D,\ \mathbf{b}_i,\ \mathbf{c}_j)\,\Psi_{ij}(\tau)|_{\tau=0}\\&-\zeta_i\int dG_{\mathbf{u}}\sum_{x=1}^m P_{(x)}^{-2}(D,\ \mathbf{b}_i)\,S_{(x)}^2(D,\ \mathbf{b}_i)\,\Phi_{ix}(\tau)|_{\tau=0}\end{aligned}$$

$$+\zeta_i \int dG_{\mathbf{u}} \Big\{ 2 \sum_{y=1}^{m} \Big[(\beta\nu_{iy} - \eta_{iy}) \Big(R^{-3}\mathbf{Q}^2 Q_y - \sum_{x=1}^{m} R^{-2} Q_x Q_{xy} \Big) \Big]$$

$$+\zeta_i^2 \Big[3R^{-4}(\mathbf{Q}^2)^2 - 4 \sum_{x,\, y=1}^{m} (R^{-3} Q_x Q_y Q_{xy} + R^{-2} Q_{xy}^2) \Big] \Big\} (\mathbf{b}_i) = 0. \tag{215}$$

Let us see what the equations for the variational parameters look like for the Ising model. Using the fact that in the one-dimensional case (see Sec. 5)

$$U_x(a,\ \mathbf{b},\ \mathbf{c}) = \operatorname{ch} a \operatorname{sh} b \operatorname{ch} c + \operatorname{sh} a \operatorname{ch} b \operatorname{sh} c, \quad m = 1; \tag{216}$$

$$S_{(y)x}(a,\ \mathbf{b}) = \operatorname{sh}(a + b), \quad m = 1; \tag{217}$$

$$Q_{xy}(\mathbf{b}) = \operatorname{ch} b, \quad m = 1, \tag{218}$$

we have

$$\sum_{j(\neq i)} \iint dG_{\mathbf{u}}\, dG_{\mathbf{v}} (1 + \operatorname{th} D \operatorname{th} b_i \operatorname{th} c_j)^{-1}$$

$$\times (\operatorname{th} b_i + \operatorname{th} D \operatorname{th} c_j)\, \Psi_{ij}(\tau)|_{\tau=0}$$

$$+ \int dG_{\mathbf{u}} \operatorname{th}(D + b_i)\, \Phi_i(\tau)|_{\tau=0} + \int dG_{\mathbf{u}}$$

$$\times \{(\beta\nu_i - \eta_i)(-\operatorname{th}^2 b_i + 1) + \zeta_i(-\operatorname{th}^3 b_i + \operatorname{th} b_i)\} = 0,\ m = 1; \tag{219}$$

$$-\zeta_i \sum_{j(\neq i)} \iint dG_{\mathbf{u}}\, dG_{\mathbf{v}} (1 + \operatorname{th} D \operatorname{th} b_i \operatorname{th} c_j)^{-2} (\operatorname{th} b_i$$

$$+ \operatorname{th} D \operatorname{th} c_j)^2\, \Psi_{ij}(\tau)|_{\tau=0} - \zeta_i \int dG_{\mathbf{u}} \operatorname{th}^2(D + b_i)\, \Phi_i(\tau)|_{\tau=0}$$

$$+ \zeta_i \int dG_{\mathbf{u}} \{2(\beta\nu_i - \eta_i)(\operatorname{th}^3 b_i - \operatorname{th} b_i)$$

$$+ \zeta_i^2 (3 \operatorname{th}^4 b_i - 4 \operatorname{th}^2 b_i + 4)\} = 0, \quad m = 1. \tag{220}$$

8. THE ORDER PARAMETERS

Let us make a few remarks before considering the order parameters.

For the trace of the product $s_{ix_1}^{\alpha_1} \ldots s_{ix_l}^{\alpha_l} \exp \widetilde{H}_n$, in which all the indices $\alpha_1, \ldots, \alpha_l$ are different we have expression (181), where the trace $\operatorname{tr}_n \exp \widetilde{H}_n$ can be written in the form (166). In this case

$$\lim_{n \to 0} \langle s_{ix_1}^{\alpha_1} \ldots s_{ix_l}^{\alpha_l} \rangle_{\widetilde{H}_n} = \int dG_{\mathbf{u}} R^{-l}(\mathbf{b}_i)\, Q_{x_1}(\mathbf{b}_i) \ldots Q_{x_l}(\mathbf{b}_i). \tag{221}$$

On the other hand, within the framework of the variational principle

$$\langle s_{ix_1}^{\alpha_1} \ldots s_{ix_l}^{\alpha_l} \rangle_{\widetilde{H}_n(\overline{\eta}_i,\, \overline{\zeta}_i)} = \langle s_{ix_1}^{\alpha_1} \ldots s_{ix_l}^{\alpha_l} \rangle_{H_n}, \tag{222}$$

so that

$$\lim_{n \to 0} \langle s_{ix_1}^{\alpha_1} \ldots s_{ix_l}^{\alpha_l} \rangle_{H_n} = \int dG_{\mathbf{u}} R^{-l}(\overline{\mathbf{b}}_i)\, Q_{x_1}(\overline{\mathbf{b}}_i) \ldots Q_{x_l}(\overline{\mathbf{b}}_i), \tag{223}$$

where

$$\overline{\mathbf{b}}_i = \overline{\boldsymbol{\eta}}_i + \overline{\zeta}_i \mathbf{u}, \tag{224}$$

and the parameters $\overline{\boldsymbol{\eta}}_i$ and $\overline{\zeta}_i$ are determined from Eqs. (213) and (215).

Let us now consider the quantities $\langle \langle s_{ix} \rangle_H \rangle$ and $\langle \langle s_{ix} \rangle_H \langle s_{iy} \rangle_H \rangle$. Since

$$\langle s_{ix} \rangle_H = -\frac{\partial F}{\partial \nu_{ix}}, \tag{225}$$

where F is the free energy for given values of the interaction strength j_{ij} and the fields $\mathbf{h}_i$ (124), according to (129) and using expression (137) for Z^n, we can write

$$\langle s_{ix} \rangle_H = \lim_{n \to 0} \Big\{ n^{-1} \operatorname{tr}_n \sum_{\alpha=1}^{n} \Big\{ s_{ix}^{\alpha} \exp\Big[-\beta \sum_{\delta=1}^{n} H(\{\mathbf{s}_i^{\delta}\}) \Big] \Big\} \Big\}, \tag{226}$$

from which

$$\langle s_{ix} \rangle_H = \lim_{n \to 0} \operatorname{tr}_n \Big\{ s_{ix}^{\alpha} \exp\Big[-\beta \sum_{\delta=1} H(\{\mathbf{s}_i^{\delta}\}) \Big] \Big\}. \tag{227}$$

From (226) we find

$$\langle s_{ix} \rangle_H \langle s_{iy} \rangle_H = \lim_{\substack{n \to 0 \\ l \to 0}} \Big\{ n^{-1} l^{-1} \sum_{\alpha=1}^{n} \sum_{\gamma=n+1}^{n+l} \operatorname{tr}_{n+l} \Big\{ s_{ix}^{\alpha} s_{iy}^{\gamma}$$

$$\times \exp\Big[-\beta \sum_{\delta=1}^{n+l} H(\{\mathbf{s}_i^{\delta}\}) \Big] \Big\} \Big\}$$

$$= \lim_{k \to 0} \Big\{ \operatorname{tr}_k \Big\{ s_{ix}^{\alpha} s_{iy}^{\gamma} \exp\Big[-\beta \sum_{\delta=1}^{k} H(\{\mathbf{s}_i^{\delta}\}) \Big] \Big\} \Big\}, \quad \alpha \neq \gamma. \tag{228}$$

In accordance with (227) and (228), using the definition (138) we find

$$\langle \langle s_{ix} \rangle_H \rangle = \lim_{n \to 0} \operatorname{tr}_n (s_{ix}^{\alpha} \exp H_n) = \lim_{n \to 0} [\langle \mathrm{Z}_n \rangle^{-1} \operatorname{tr}_n (s_{ix}^{\alpha} H_n)]$$

$$= \lim_{n \to 0} [\operatorname{tr}_n^{-1} \exp H_n \operatorname{tr}_n (s_{ix}^{\alpha} \exp H_n)] = \lim_{n \to 0} \langle s_{ix}^{\alpha} \rangle_{H_n}; \tag{229}$$

$$\langle \langle s_{ix} \rangle_H \langle s_{iy} \rangle_H \rangle = \lim_{n \to 0} \operatorname{tr}_n (s_{ix}^{\alpha} s_{iy}^{\gamma} \exp H_n)$$

$$= \lim_{n \to 0} [\langle \mathrm{Z}^n \rangle^{-1} \operatorname{tr}_n (s_{ix}^{\alpha} s_{iy}^{\gamma} \exp H_n)]$$

$$= \lim_{n \to 0} [\operatorname{tr}_n^{-1} \exp H_n \operatorname{tr}_n (s_{ix}^{\alpha} s_{iy}^{\gamma} \exp H_n)]$$

$$= \lim_{n \to 0} \langle s_{ix}^{\alpha} s_{iy}^{\gamma} \rangle_{H_n}, \quad \alpha \neq \gamma. \tag{230}$$

Therefore, owing to (221),

$$\langle \langle s_{ix} \rangle_H \rangle = \int dG_{\mathbf{u}} R^{-1}(\overline{\mathbf{b}}_i)\, Q_x(\overline{\mathbf{b}}_i), \tag{231}$$

$$\langle \langle s_{ix} \rangle_H \langle s_{iy} \rangle_H \rangle = \int dG_{\mathbf{u}} R^{-2}(\overline{\mathbf{b}}_i)\, Q_x(\overline{\mathbf{b}}_i)\, Q_y(\overline{\mathbf{b}}_i). \tag{232}$$

Now let us discuss the order parameters. According to (122) the magnetic moment of the ith point is

$$\mathbf{m}_i = \langle \langle \mathbf{s}_i \rangle_H \rangle, \tag{233}$$

and from (231) we find for it

$$\mathbf{m}_i = \int dG_{\mathbf{u}} R^{-1}(\overline{\boldsymbol{\eta}}_i + \overline{\zeta}_i \mathbf{u})\, \mathbf{Q}(\overline{\boldsymbol{\eta}}_i + \overline{\zeta}_i \mathbf{u}). \tag{234}$$

To describe the state of a spin-glass in the Potts model we need to find the quantities

$$\mathrm{q}_i^{(p)} = \langle \langle \mathbf{s}_i \boldsymbol{\varepsilon}_p \rangle_H^2 \rangle, \tag{235}$$

which are the expectation values of the squared statistical averages of the spin projections at the ith point on the direction of the vectors $\boldsymbol{\varepsilon}_p$. Here, in contrast to the Ising model, instead of one parameter we have $m+1$ parameters. This obviously is because in the Potts model we are dealing with a multicomponent spin, and the number of possible orientations of it is $m+1$.

Since

$$\langle\langle \mathbf{s}_i\boldsymbol{\varepsilon}_p\rangle_H^2\rangle = \sum_{x,\,y=1}^{m} \langle\langle s_{ix}\rangle_H \langle s_{iy}\rangle_H\rangle\, \varepsilon_{px}\varepsilon_{py}, \tag{236}$$

then, owing to (232), we obtain

$$\mathrm{q}_i^{(p)} = \int dG_{\mathbf{u}} R^{-2}(\bar{\boldsymbol{\eta}}_i + \bar{\zeta}_i\mathbf{u})\, [\boldsymbol{\varepsilon}_p \mathbf{Q}(\bar{\boldsymbol{\eta}}_i + \bar{\zeta}_i\mathbf{u})]^2. \tag{237}$$

Therefore, the values of $\mathbf{m}_i$ and $q_i^{(p)}$ at the ith point are expressed only in terms of the values of the variational parameters $\bar{\boldsymbol{\eta}}_i$ and $\bar{\zeta}_i$ at the same point.

Together with $q_i^{(p)}$ let us consider the quantity

$$\mathrm{q}_i = (m+1)^{-1} m \sum_{p=1}^{m+1} \mathrm{q}_i^{(p)}. \tag{238}$$

We note that for an arbitrary vector $\mathbf{a}$, according to (54) we have

$$\sum_{p=1}^{m+1} (\mathbf{a}\boldsymbol{\varepsilon}_p)^2 = m^{-1}(m+1)\,\mathbf{a}^2. \tag{239}$$

In this case for

$$\mathrm{q}_j = \langle (m+1)^{-1} m \sum_{p=1}^{m+1},\, (\langle \mathbf{s}_i\rangle_H\, \boldsymbol{\varepsilon}_p)^2\rangle \tag{240}$$

we obtain

$$\mathrm{q}_i = \langle\langle \mathbf{s}_i\rangle_H^2\rangle. \tag{241}$$

This order parameter has the same form as the usual spin-glass parameter for the Ising model.

Using (239), it follows from the representation (237) for q_i that

$$\mathrm{q}_i = \int dG_{\mathbf{u}} R^{-2}(\bar{\boldsymbol{\eta}}_i + \bar{\zeta}_i\mathbf{u})\, \mathbf{Q}^2(\bar{\boldsymbol{\eta}}_i + \bar{\zeta}_i\mathbf{u}). \tag{242}$$

Now we need the property (39) of the system of vectors $\{\boldsymbol{\varepsilon}_p\}$:

$$\sum_{p=1}^{m+1} \boldsymbol{\varepsilon}_p = 0. \tag{243}$$

Let us consider the possible states of the ith spin as a function of the values of the parameters $\bar{\boldsymbol{\eta}}_i$ and $\bar{\zeta}_i$ determined from Eqs. (213) and (215).

Case 1: $\bar{\boldsymbol{\eta}}_i = \mathbf{0}$, $\bar{\zeta}_i = \mathbf{0}$. According to the definition (77) of the function $\mathbf{Q}$,

$$\mathbf{Q}(0) = \mathbf{0}. \tag{244}$$

Therefore, if $\bar{\boldsymbol{\eta}}_i = \mathbf{0}$ and $\bar{\zeta}_i = \mathbf{0}$, then from (234), (237), and (244) it follows that

$$\mathbf{m}_i = \mathbf{0}, \quad \mathrm{q}_i^{(p)} = 0, \tag{245}$$

and we say that the ith spin is in the N-state.

Case 2: $\bar{\boldsymbol{\eta}}_i = \mathbf{0}$, $\bar{\zeta}_i \neq 0$. From (72), (77), and (234) it follows that

$$\begin{aligned}\mathbf{m}_i &:= \sum_{p=1}^{m+1} \boldsymbol{\varepsilon}_p \int dG_{\mathbf{u}} \Big[\sum_{q=1}^{m+1} \exp(\bar{\zeta}_i\boldsymbol{\varepsilon}_p\mathbf{u})\Big]^{-1} \exp(\zeta_i\boldsymbol{\varepsilon}_p\mathbf{u})\\ &= \sum_{p=1}^{m+1} [\boldsymbol{\varepsilon}_p V(\bar{\zeta}_i)],\end{aligned} \tag{246}$$

where $V(\zeta_i)$ is independent of p, so that the property (243) of the system of vectors $\{\boldsymbol{\varepsilon}_p\}$ gives

$$\mathbf{m}_i = 0. \tag{247}$$

Moreover, the parameters $q_i^{(p)}$ have the form

$$\mathrm{q}_i^{(p)} = \int dG_{\mathbf{u}} R^{-2}(\bar{\zeta}_i\mathbf{u})\, [\boldsymbol{\varepsilon}_p \mathbf{Q}(\bar{\zeta}_i\mathbf{u})]^2 \tag{248}$$

and, therefore, take the same value for all p and are nonzero:

$$\mathrm{q}_i^{(p)} = \int dG_{\mathbf{u}} R^{-2}(\bar{\zeta}_i\mathbf{u})\, [\boldsymbol{\varepsilon}_p \mathbf{Q}(\bar{\zeta}_i\mathbf{u})]^2. \tag{249}$$

In this case, i.e., if $\mathbf{m}_i = 0$ and $q_i^{(p)} \neq 0$, we shall say that the ith spin is in the SG-state.

Case 3: $\boldsymbol{\eta}_i \neq \mathbf{0}$. In this case, again owing to (234) and (237), we have

$$\mathbf{m}_i \neq 0, \quad \mathrm{q}_i^{(p)} \neq 0, \tag{250}$$

and we shall say that the ith spin is in the M-state. For definiteness let $\boldsymbol{\eta}_i \,\|\, \boldsymbol{\varepsilon}_r$, i.e.,

$$\bar{\boldsymbol{\eta}}_i = \bar{\eta}_i\boldsymbol{\varepsilon}_r. \tag{251}$$

For the magnetic moment of the ith point, taking into account the definition (39) of the system of vectors $\{\boldsymbol{\varepsilon}_p\}$, we have

$$\begin{aligned}\mathbf{m}_i = \sum_{p=1}^{m+1} \Big\{\boldsymbol{\varepsilon}_p \exp(\bar{\eta}_i\boldsymbol{\varepsilon}_r\boldsymbol{\varepsilon}_p) \int dG_{\mathbf{u}}\\ \times \Big[\sum_{q=1}^{m+1} \exp(\bar{\eta}_i\boldsymbol{\varepsilon}_r\boldsymbol{\varepsilon}_q) \exp(\bar{\zeta}_i\mathbf{u}\boldsymbol{\varepsilon}_p)\Big]^{-1}\\ \times \exp(\bar{\zeta}_i\mathbf{u}\boldsymbol{\varepsilon}_p)\Big\} = \sum_{p=1}^{m+1} \Big\{\boldsymbol{\varepsilon}_p \exp\Big(\bar{\eta}_i\,\frac{m+1}{m}\,\delta_{rp}\Big) \int dG_{\mathbf{u}}\\ \times \Big[\sum_{q=1}^{m+1} \exp\Big(\bar{\eta}_i\,\frac{m+1}{m}\,\delta_{qr}\Big) \exp(\bar{\zeta}_i\mathbf{u}\boldsymbol{\varepsilon}_q)\Big]^{-1} \exp(\bar{\zeta}_i\mathbf{u}\boldsymbol{\varepsilon}_p)\Big\},\end{aligned} \tag{252}$$

and, therefore,

$$\mathbf{m}_i = \sum_{\substack{p=1\\(p\neq r)}}^{m+1} [\boldsymbol{\varepsilon}_p \mathrm{U}(\bar{\eta}_i, \bar{\zeta}_i)] + \boldsymbol{\varepsilon}_r \mathrm{W}(\bar{\eta}_i, \bar{\zeta}_i), \tag{253}$$

where the function U is independent of p. Then, owing to the property (39),

$$\mathbf{m}_i = [\mathrm{W}(\bar{\eta}_i, \bar{\zeta}_i) - \mathrm{U}(\bar{\eta}_i, \bar{\zeta}_i)]\, \boldsymbol{\varepsilon}_r \tag{254}$$

and, therefore,

$$\mathbf{m}_i \,||\, \bar{\boldsymbol{\eta}}_i. \tag{255}$$

We also note that the quantities $q_i^{(p)}$ for $p \neq r$ are equal to each other:

$$\begin{aligned}\mathrm{q}_i^{(p)} &= \int dG_{\mathbf{u}} R^{-2}(\bar{\eta}_i\boldsymbol{\varepsilon}_r + \bar{\zeta}_i\mathbf{u})\, [\boldsymbol{\varepsilon}_p \mathbf{Q}(\bar{\eta}_i\boldsymbol{\varepsilon}_i + \bar{\zeta}_i\mathbf{u})]^2\\ &= \begin{cases} \mathrm{q}_{ir}, & p = r,\\ \mathrm{q}_{i0}, & p \neq r.\end{cases}\end{aligned} \tag{256}$$

Therefore, the number of different parameters $q_i^{(p)}$ is not at all equal to $m+1$. In the N- and SG-states all the parameters are equal to each other. In the M-state they are all identical, except for one: the expectation value of the squared statistical average of the spin projection on the magnetization direction.

9. THE SPATIALLY UNIFORM CASE

Let us now consider the special case where the regular field v_i is uniform and directed along one of the vectors $\boldsymbol{\varepsilon}_p$, the distribution densities of the interaction strengths J_{ij} are identical for any pair $\langle i,j\rangle$ of interacting neighbors, while for the other pair the interaction strengths are equal to zero, and the distribution densities of the random fields $\mathbf{h}_i$ are identical for all points and components.[59,67,68] That is, we take

$$\mathbf{v}_i = \mathbf{v} \parallel \boldsymbol{\varepsilon}_r. \tag{257a}$$

For definiteness we set $r = m + 1$:

$$\mathbf{v}_j = \mathrm{v}\boldsymbol{\varepsilon}_{m+1}. \tag{257b}$$

Let the distribution density of the interaction strength have the form

$$P_{ij}(J_{ij}) = \begin{cases} P(J_{ij}), & \text{for pairs of interacting neighbors} \\ (i,\ j)\in\{\langle i,\ j\rangle\}, & \\ \delta(J_{ij}-\bar{J}_{ij}),\ \bar{J}_{ij}=0, & \text{for the other pairs} \\ (i,\ j)\bar{\in}\{\langle i,\ j\rangle\}, & \end{cases} \tag{258}$$

with the number of interacting neighbors of each spin equal to z. We also assume that

$$p_{ix}(h_{ix}) = p(h_{ix}). \tag{259}$$

From the assumption (258) it follows that

$$\Psi_{ij}(\tau) = \begin{cases} \Psi(\tau) = \ln \int dJP(J)\exp(\beta J\tau), & (i,j)\in\{\langle i,\ j\rangle\}; \\ \beta J_{ij}\tau = 0, & (i,\ j)\bar{\in}\{\langle i,\ j\rangle\}, \end{cases} \tag{260}$$

and from the assumption (259) it follows that

$$\Phi_{ix}(\tau) = \Phi(\tau) = \ln \int dhp(h)\exp(\beta h\tau). \tag{261}$$

With the conditions (257)–(259), taking into account symmetry considerations, the solution of Eqs. (213) and (215) can be sought among variational parameters of the form $\boldsymbol{\eta}_i = \eta\boldsymbol{\varepsilon}_{m+1}$, $\zeta_i = \zeta_i$. In this case instead of $Nm + N$ parameters η_{ix}, ζ_i ($i = 1, \ldots, N$; $x = 1, \ldots, m$) we have only two parameters, η and ζ. In accordance with (210), (260), and (261) the free energy (3) is written as

$$\begin{aligned} -N^{-1}\beta\tilde{\mathcal{F}}(\eta,\ \zeta) &= \frac{1}{2} z \iint dG_{\mathbf{u}}\, dG_{\mathbf{v}}\Psi(D)\ln T(\tau,\ \mathbf{B},\ \mathbf{C})|_{\tau=0} \\ &+ \int dG_{\mathbf{u}}\Phi(D)\sum_{x=1}^{m}\ln P_{(x)}(\tau,\ \mathbf{B})|_{\tau=0} + \int dG_{\mathbf{u}} \\ &\times\left\{(\beta\mathrm{v}-\eta)R^{-1}Q_m + \frac{1}{2}\zeta^2 R^{-2}Q^2 + \ln R\right\} \\ &\times(\mathbf{B}) - \frac{1}{2}\zeta^2 + \ln\frac{m+1}{m}, \end{aligned} \tag{262}$$

where we have introduced the notation

$$\mathbf{B} = \bar{\eta}\boldsymbol{\varepsilon}_{m+1} + \bar{\zeta}\mathbf{u};\quad \mathbf{C} = \bar{\eta}\boldsymbol{\varepsilon}_{m+1} + \bar{\zeta}\mathbf{v}. \tag{263}$$

Instead of the N equations (215) we have the single equation

$$\begin{aligned} \zeta\Big\{ &-z\iint dG_{\mathbf{u}}\, dG_{\mathbf{v}}\Psi(D)\mathbf{T}^{-2}(\tau,\ \mathbf{B},\ \mathbf{C})U^2(\tau,\ \mathbf{B},\ \mathbf{C})|_{\tau=0} \\ &-\int dG_{\mathbf{u}}\Phi(D)\sum_{x=1}^{m}P_{(x)}^{-2}(\tau,\ \mathbf{B})S_{(x)}(\tau,\ \mathbf{B})|_{\tau=0} \\ &+\int dG_{\mathbf{u}}\Big\{2(\beta\mathrm{v}-\eta)\Big(R^{-3}\mathbf{Q}^2Q_m - \sum_{x=1}^{m}R^{-2}Q_xQ_{mx}\Big) \\ &+\zeta^2\Big[3R^{-4}(\mathbf{Q}^2)^2 - 4\sum_{x,\ y=1}^{m}Q_xQ_yQ_{xy}\Big]\Big\}(\mathbf{B}) = 0, \end{aligned} \tag{264}$$

while the Nm equations (213) give m equations

$$\begin{aligned} z&\iint dG_{\mathbf{u}}\, dG_{\mathbf{v}}\Psi(D)T^{-1}(\tau,\ \mathbf{B},\ \mathbf{C})U_x(\tau,\ \mathbf{B},\ \mathbf{C})|_{\tau=0} \\ &+\int dG_{\mathbf{u}}\Phi(D)\sum_{y=1}^{m}P_{(y)}^{-1}(\tau,\ \mathbf{B})S_{y(x)}(\tau,\ \mathbf{B})|_{\tau=0} \\ &+\int dG_{\mathbf{u}}\Big[(\beta\mathrm{v}-\eta)(-R^2Q_mQ_x + R^{-1}Q_{mx}) \\ &+\zeta^2\Big(-R^{-3}\mathbf{Q}^2Q_x + R^{-2}\sum_{y=1}^{m}Q_yQ_{xy}\Big)\Big](\mathbf{B}) = 0. \end{aligned} \tag{265}$$

However, the left-hand sides of these equations are identically equal to zero for all components x except $x = m$. This can be verified by exactly the same method as was used to show that $\boldsymbol{\eta}_i \parallel \boldsymbol{\varepsilon}_r$ [see (252)–(255)]. In fact, rewriting (265) in vector form, just as in (252)–(255), we find that each integration on the left-hand side gives a vector parallel to $\boldsymbol{\varepsilon}_{m+1}$. Therefore, instead of the m equations (265) we have the single equation

$$\begin{aligned} z&\iint dG_{\mathbf{u}}\, dG_{\mathbf{v}}\Psi(D)T^{-1}(\tau,\ \mathbf{B},\ \mathbf{C})U_m(\tau,\ \mathbf{B},\ \mathbf{C})|_{\tau=0} \\ &+\int dG_{\mathbf{u}}\Phi(D)\sum_{x=1}^{m}P_{(x)}^{-1}(\tau,\ \mathbf{B})S_{(x)m}(\tau,\ \mathbf{B})|_{\tau=0} \\ &+\int dG_{\mathbf{u}}\Big[(\beta\mathrm{v}-\eta)(-R^2Q_m^2 + R^{-1}Q_{mm}) \\ &+\zeta^2\Big(-R^{-3}\mathbf{Q}^2\mathbf{Q}_m + R^{-2}\sum_{x=1}^{m}Q_xQ_{mx}\Big)\Big](\mathbf{B}) = 0. \end{aligned} \tag{266}$$

Equations (264) and (266) form a system of two equations for the two unknowns $\bar{\eta}$ and $\bar{\zeta}$.

In accordance with the representation (231) for the magnetization

$$\mathbf{m} = \Big\langle\Big\langle\frac{1}{N}\sum_i s_i\Big\rangle_H\Big\rangle \tag{267}$$

we obtain

$$\mathbf{m} = \int dG_{\mathbf{u}}R^{-1}(\bar{\eta}\boldsymbol{\varepsilon}_{m+1} + \bar{\zeta}\mathbf{u})\,\mathbf{Q}(\bar{\eta}\boldsymbol{\varepsilon}_{m+1} + \bar{\zeta}\mathbf{u}), \tag{268}$$

or

$$\mathbf{m} = \int dG_{\mathbf{u}}R^{-1}(\mathbf{B})\,\mathbf{Q}(\mathbf{B}). \tag{269}$$

The spin-glass order parameters $q^{(p)}$ (235) and q (241) are independent of the point number i:

$$q^{(p)} = \langle\langle s_i\boldsymbol{\varepsilon}_p\rangle_H^2\rangle, \tag{270}$$

$$\mathbf{q} = \langle\langle \mathbf{s}_i \rangle_H^2 \rangle \tag{271}$$

and have the form

$$\mathbf{q}^{(p)} = \int dG_{\mathbf{u}} R^{-2} (\bar{\eta}\boldsymbol{\varepsilon}_{m+1} + \bar{\zeta}\mathbf{u}) [\boldsymbol{\varepsilon}_p \mathbf{Q} (\bar{\eta}\boldsymbol{\varepsilon}_{m+1} + \bar{\zeta}\mathbf{u})]^2, \tag{272}$$

$$\mathbf{q} = \int dG_{\mathbf{u}} R^{-2} (\bar{\eta}\boldsymbol{\varepsilon}_{m+1} + \bar{\zeta}\mathbf{u}) \mathbf{Q}^2 (\bar{\eta}\boldsymbol{\varepsilon}_{m+1} + \zeta\mathbf{u}), \tag{273}$$

or

$$\mathbf{q}^{(p)} = \int dG_{\mathbf{u}} R^{-2} (\mathbf{B})[\boldsymbol{\varepsilon}_p \mathbf{Q} (\mathbf{B})]^2, \tag{274}$$

$$\mathbf{q} = \int dG_{\mathbf{u}} R^{-2} (\mathbf{B}) \mathbf{Q}^2 (\mathbf{B}). \tag{275}$$

According to (256), all the $q^{(p)}$ with $p \neq m+1$ are equal to each other:

$$\mathbf{q}^{(p)} = \tilde{\mathbf{q}}, \quad p = 1, \ldots, m, \qquad \mathbf{q}^{(m+1)} = q_0. \tag{276}$$

For $\bar{\eta} = 0$, i.e., in the absence of magnetic ordering, all the $q^{(p)}$ $(p = 1, \ldots, m, m+1)$ are identical:

$$\mathbf{q}^{(p)} = \tilde{\mathbf{q}}, \quad p = 1, \ldots, m+1, \quad \mathrm{m} = 0. \tag{277}$$

With the conditions (257)–(259) for studying antiferromagnetic ordering, the spin-glass phase, or the paramagnetic phase (for $\bar{J} < 0$), the solution of the variational problem can be sought among variational parameters of the form[69,67,68]

$$\boldsymbol{\eta}_{2k-1} = \eta\boldsymbol{\varepsilon}_{m+1}, \quad \boldsymbol{\eta}_{2k} = \eta\boldsymbol{\varepsilon}_m.$$

Therefore, as in the ferromagnetic case we have only the two scalar parameters η and ζ.

The free energy takes the form

$$\begin{aligned} -N^{-1}\beta\mathcal{F}(\eta, \zeta) = {} & \frac{1}{2} z \int\int dG_{\mathbf{u}}\, dG_{\mathbf{v}} \Psi(D) \ln T(\tau, \mathbf{B}, \mathbf{E})|_{\tau=0} \\ & + \frac{1}{2} \sum_{x=1}^{m} dG_{\mathbf{u}} \Phi(D) \ln P_{(x)}(\tau, \mathbf{B})|_{\tau=0} \\ & + \frac{1}{2} \sum_{x=1}^{m} dG_{\mathbf{u}} \Phi(D) \ln P_{(x)}(\tau, E)|_{\tau=0} \\ & - \frac{1}{2} \eta \int dG_{\mathbf{u}} R^{-1} (\mathbf{B}) \boldsymbol{\varepsilon}_{m+1} \mathbf{Q} (\mathbf{B}) \\ & - \frac{1}{2} \sum_{x=1}^{m} \int dG_{\mathbf{u}} R^{-1} (\mathbf{E}) \boldsymbol{\varepsilon}_m \mathbf{Q} (\mathbf{E}) \\ & + \frac{1}{4} \zeta^2 \int dG_{\mathbf{u}} R^{-2} (\mathbf{B}) \mathbf{Q}^2 (\mathbf{B}) \\ & + \frac{1}{4} \zeta^2 \int dG_{\mathbf{v}} R^{-2} (\mathbf{E}) \mathbf{Q}^2 (\mathbf{E}) - \frac{1}{2} \zeta^2 \\ & + \frac{1}{2} \int dG_{\mathbf{u}} \ln R(\mathbf{B}) + \frac{1}{2} \int dG_{\mathbf{v}} \ln R(\mathbf{E}) + \ln \frac{m+1}{m}, \end{aligned} \tag{278}$$

$$\mathbf{B} = \eta\boldsymbol{\varepsilon}_{m+1} + \zeta\mathbf{u}, \quad \mathbf{E} = \eta\boldsymbol{\varepsilon}_m + \zeta\mathbf{v}. \tag{279}$$

If we make use of the symmetry of the simplex vectors, we can write

$$\begin{aligned} -N^{-1}\beta\mathcal{F}(\eta, \zeta) = {} & \frac{1}{2} z \int\int dG_{\mathbf{u}}\, dG_{\mathbf{v}} \Psi(D) \ln T(\tau, \mathbf{B}, \mathbf{E})|_{\tau=0} \\ & + \sum_{x=1}^{m} \int dG_{\mathbf{u}} \Phi(D) \ln P_{(x)}(\tau, \mathbf{B})|_{\tau=0} \\ & - \eta \int dG_{\mathbf{u}} R^{-1} (\mathbf{B}) \varepsilon_{m+1} \mathbf{Q} (\mathbf{B}) \\ & + \frac{1}{2} \zeta^2 \int dG_{\mathbf{v}} R^{-2} (\mathbf{E}) \mathbf{Q}^2 (\mathbf{E}) - \frac{1}{2} \zeta^2 \\ & + \int dG_{\mathbf{u}} \ln R(\mathbf{B}) + \ln \frac{m+1}{m}. \end{aligned} \tag{280}$$

Owing to the representation (231), the magnetization vector of the first sublattice

$$\mathbf{m}^{(1)} = \left\langle \left\langle \frac{1}{N/2} \sum_k \mathbf{s}_{2k-1} \right\rangle_H \right\rangle = \mathbf{m}_{2k-1} = \langle\langle \mathbf{s}_{2k-1} \rangle_H \rangle \tag{281}$$

has the form

$$\mathbf{m}^{(1)} = \int dG_{\mathbf{u}} R^{-1} (\bar{\eta}\boldsymbol{\varepsilon}_{m+1} + \bar{\zeta}\mathbf{u}) \mathbf{Q} (\bar{\eta}\boldsymbol{\varepsilon}_{m+1} + \bar{\zeta}\mathbf{u}), \tag{282}$$

or

$$\mathbf{m}^{(1)} = \int dG_{\mathbf{u}} R^{-1} (\mathbf{B}) \mathbf{Q} (\mathbf{B}). \tag{283}$$

According to (255), it points along the simplex vector $\boldsymbol{\varepsilon}_{m+1}$:

$$\mathbf{m}^{(1)} \parallel \boldsymbol{\varepsilon}_{m+1}. \tag{284}$$

For the same reasons the magnetization vector of the second sublattice

$$\mathbf{m}^{(2)} = \left\langle \left\langle \frac{1}{N/2} \sum_k \mathbf{s}_{2k} \right\rangle_H \right\rangle = \mathbf{m}_{2k} = \langle\langle \mathbf{s}_{2k} \rangle_H \rangle \tag{285}$$

has the form

$$\mathbf{m}^{(2)} = \int dG_{\mathbf{u}} R^{-1} (\bar{\eta}\boldsymbol{\varepsilon}_m + \bar{\zeta}\mathbf{u}) \mathbf{Q} (\bar{\eta}\boldsymbol{\varepsilon}_m + \bar{\zeta}\mathbf{u}), \tag{286}$$

or

$$\mathbf{m}^{(2)} = \int dG_{\mathbf{u}} R^{-1} (\mathbf{E}) \mathbf{Q} (\mathbf{E}), \tag{287}$$

it points along the simplex vector $\boldsymbol{\varepsilon}_m$,

$$\mathbf{m}^{(2)} \parallel \boldsymbol{\varepsilon}_m, \tag{288}$$

and its modulus is equal to that of the magnetization vector of the first sublattice:

$$\mathrm{m}^{(1)} = \mathrm{m}^{(2)} \equiv \mathrm{m}^{(a)}. \tag{289}$$

On the first sublattice the spin-glass order parameters $q^{(p)}$ (235) and (238), in accordance with (237) and (242), have the form

$$\mathbf{q}^{(p,1)} = \int dG_{\mathbf{u}} R^{-2} (\bar{\eta}\boldsymbol{\varepsilon}_{m+1} + \bar{\zeta}\mathbf{u}) [\boldsymbol{\varepsilon}_p \mathbf{Q} (\bar{\eta}\boldsymbol{\varepsilon}_{m+1} + \bar{\zeta}\mathbf{u})]^2, \tag{290}$$

$$\mathbf{q}^{(1)} = \int dG_{\mathbf{u}} R^{-2} (\bar{\eta}\boldsymbol{\varepsilon}_{m+1} + \bar{\zeta}\mathbf{u}) \mathbf{Q}^2 (\bar{\eta}\boldsymbol{\varepsilon}_{m+1} + \bar{\zeta}\mathbf{u}), \tag{291}$$

or

$$\mathbf{q}^{(p,1)} = \int dG_{\mathbf{u}} R^{-2} (\mathbf{B}) [\boldsymbol{\varepsilon}_p \mathbf{Q} (\mathbf{B})]^2, \tag{292}$$

$$\mathbf{q}^{(1)} = \int dG_{\mathbf{u}} R^{-2} (\mathbf{B}) \mathbf{Q}^2 (\mathbf{B}). \tag{293}$$

According to (256), all the $q^{(p,1)}$ with $p \neq m+1$ are equal to each other:

$$\begin{aligned} \mathrm{q}^{(p,\,1)} &= \tilde{\mathrm{q}}^{(1)}, \quad p = 1, \ldots, m, \\ \mathrm{q}^{(m+1,\,1)} &= \mathrm{q}_0^{(1)}. \end{aligned} \tag{294}$$

On the second sublattice for the spin-glass parameters we find

$$\mathrm{q}^{(p,\,2)} = \int dG_{\mathbf{u}} R^{-2} (\bar{\eta}\boldsymbol{\varepsilon}_m + \bar{\zeta}\mathbf{u}) \left[\boldsymbol{\varepsilon}_p \mathbf{Q} (\bar{\eta}\boldsymbol{\varepsilon}_m + \bar{\zeta}\mathbf{u})\right]^2, \tag{295}$$

$$\mathrm{q}^{(2)} = \int dG_{\mathbf{u}} R^{-2} (\bar{\eta}\boldsymbol{\varepsilon}_m + \bar{\zeta}\mathbf{u})\, \mathbf{Q}^2 (\bar{\eta}\boldsymbol{\varepsilon}_m + \bar{\zeta}\mathbf{u}), \tag{296}$$

or

$$\mathrm{q}^{(p,\,2)} = \int dG_{\mathbf{u}}\, R^{-2}\, (\mathbf{E})\, [\boldsymbol{\varepsilon}_p \mathbf{Q}\, (\mathbf{E})]^2, \tag{297}$$

$$q^{(2)} = \int dG_{\mathbf{u}}\, R^{-2}\, (\mathbf{E})\, \mathbf{Q}^2\, (\mathbf{E}). \tag{298}$$

Owing to (256), all the $q^{(p,2)}$ with $p \neq m$ are equal to each other:

$$\begin{aligned} \mathrm{q}^{(p,\,2)} &= \tilde{\mathrm{q}}^{(2)}, \quad p = 1, \ldots, m-1,\ m+1, \\ \mathrm{q}^{(m,\,2)} &= \mathrm{q}_0^{(2)}, \end{aligned} \tag{299}$$

where

$$\left.\begin{aligned} \tilde{\mathrm{q}}^{(1)} = \tilde{\mathrm{q}}^{(2)} &\equiv \tilde{\mathrm{q}}, \\ \mathrm{q}_0^{(1)} = q_0^{(2)} &\equiv q_0. \end{aligned}\right\} \tag{300}$$

It follows from (294) and (299) that the parameters (238) on the first and second sublattices $q^{(1)}$ (291) and $q^{(2)}$ (298) coincide:

$$\mathrm{q}^{(1)} = \mathrm{q}^{(2)} \equiv \mathrm{q}. \tag{301}$$

For $\bar{\eta} = 0$, i.e., in the absence of magnetic ordering, all the $q^{(p)}$ ($p = 1, \ldots, m, m + 1$) are identical:

$$\mathrm{q}^{(p,\,1)} = \mathrm{q}^{(p,\,2)} = \tilde{\mathrm{q}}, \quad p = 1, \ldots, m+1, \quad \mathbf{m} = 0. \tag{302}$$

Therefore, according to (276), (277), (299), (300), and (302) in the paramagnetic phase and the spin-glass phase all the spin-glass parameters $q_i^{(p)}$ are equal to each other. In magnetically ordered phases for each point i there are two different parameters $q_i^{(p)}$, namely, the expectation value of the squared statistical average of the spin projection on the magnetization direction of the point and on any other direction of the other simplex vectors.

10. THE GAUSSIAN DISTRIBUTION

When studying specific distributions P_{ij} and p_{ix}, it is convenient to use the expansion in semi-invariants $\varkappa_w^{(k)}$,

$$\ln \chi_w(t) = \sum_{k=1}^{\infty} (k!)^{-1} (\mathrm{i}t)^k \varkappa_w^{(k)}, \tag{303}$$

where $\chi_w(t)$ is the characteristic function of the distribution $w(z)$ of the random variable z:

$$\chi_w(t) = \int dz w(z) \exp(\mathrm{i}tz).$$

In accordance with the definitions (140), (147) of the functions Ψ_{ij} and Φ_{ix},

$$\left.\begin{aligned} \Psi_{ij}(\tau) &= \ln \chi_{P_{ij}}(-\mathrm{i}\beta\tau), \\ \Phi_{ix}(\tau) &= \ln \chi_{P_{ix}}(-\mathrm{i}\beta\tau), \end{aligned}\right\} \tag{304}$$

so that

$$\left.\begin{aligned} \Psi_{ij}(\tau) &= \sum_{k=1}^{\infty} (k!)^{-1} (\beta\tau)^k \varkappa_{P_{ij}}^{(k)}, \\ \Phi_{ix}(\tau) &= \sum_{k=1}^{\infty} (k!)^{-1} (\beta\tau)^k \varkappa_{P_{ix}}^{(k)}. \end{aligned}\right\} \tag{305}$$

For the Gaussian distribution

$$w(z) = (2\pi\sigma)^{-1/2} \exp\left[-\frac{1}{2}\sigma^{-1}(z - \bar{z})^2\right] \tag{306}$$

with mean value $\bar{z}$ and dispersion σ we have

$$\chi_w(t) = \exp\left(\mathrm{i}zt - \frac{1}{2}\sigma t^2\right), \tag{307}$$

so that

$$\varkappa_w^{(k)} = \begin{cases} \bar{z}, & k = 1, \\ \sigma, & k = 2, \\ 0, & k = 3, 4 \ldots \end{cases} \tag{308}$$

When the distributions P_{ij} and p_{ix} are Gaussians with average values $\bar{J}_{ij}$ and $\bar{h}_{ix}$ and dispersions σ_{ij} and ρ_{ix}, respectively,

$$\left.\begin{aligned} P_{ij}(J) &= (2\pi\sigma_{ij})^{-1/2} \exp\left[-\frac{1}{2}\sigma_{ij}^{-1}(J - \bar{J}_{ij})^2\right], \\ p_{ix}(h) &= (2\pi\rho_{ix})^{-1/2} \exp\left[-\frac{1}{2}\rho_{ix}^{-1}(h - \bar{h}_{ix})^2\right], \end{aligned}\right\} \tag{309}$$

the functions Ψ_{ij} and Φ_{ix} have the form

$$\left.\begin{aligned} \Psi_{ij}(\xi) &= \beta\bar{J}_{ij}\xi + \frac{1}{2}\beta^2\sigma_{ij}\xi^2, \\ \Phi_{ix}(\xi) &= \beta\bar{h}_{ix}\xi + \frac{1}{2}\beta^2\rho_{ix}\xi^2. \end{aligned}\right\} \tag{310}$$

Let us consider the translationally invariant Gaussian case where

$$\bar{J}_{ij} = \begin{cases} \bar{J}, & (i,\ j) \in \{\langle i,\ j\rangle\}, \\ 0, & (i,\ j) \bar{\in} \{\langle i,\ j\rangle\}, \end{cases} \quad \sigma_{ij} = \begin{cases} \sigma, & (i,\ j) \in \{\langle i,\ j\rangle\}, \\ 0, & (i,\ j) \bar{\in} \{\langle i,\ j\rangle\}, \end{cases} \tag{311}$$

$$\bar{h}_{ix} = 0, \quad \rho_{ix} = \rho. \tag{312}$$

We note that setting $\bar{h}_{ix} = 0$ does not lead to loss of generality, since the average value of the random field $\mathbf{h}_i$ can be taken into account in the regular field $\mathbf{v}_i$. The equation $\rho_{ix} = \rho$ implies that the fluctuations of the random field are isotropic.

In expressions (262) and (263) for the free energy we can eliminate the double integration (over $\mathbf{u}$ and $\mathbf{v}$) and the scalar product of the component functions and write it in terms of the functions

$$\mathscr{E}_0(\eta, \zeta) = \int dG_{\mathbf{u}}\, R^{-1}(\mathbf{B}) \exp(\mathbf{B}\boldsymbol{\varepsilon}_{m+1}), \tag{313}$$

$$\mathscr{D}_2(\eta, \zeta) = \int dG_{\mathbf{u}}\, R^{-2}(\mathbf{B}) \exp(2\mathbf{B}\boldsymbol{\varepsilon}_1), \tag{314}$$

$$\mathscr{D}_0(\eta, \zeta) = \int dG_{\mathbf{u}}\, R^{-2}(\mathbf{B}) \exp(2\mathbf{B}\boldsymbol{\varepsilon}_{m+1}), \tag{315}$$

$$\mathscr{B}_0(\eta, \zeta) = \int dG_{\mathbf{u}} \ln R(\mathbf{B}), \tag{316}$$

while for $m = 1$ it can be written only in terms of the functions $\mathscr{E}_0$, $\mathscr{D}_0$, and $\mathscr{B}_0$ (Refs. 70, 67, and 68). Namely, for $m > 1$ in the case of the ferromagnetic, paramagnetic, and spin-glass phases the free energy has the form

$$
\begin{aligned}
&-N^{-1}\beta\mathcal{F}(\eta,\ \zeta)\\
&\quad=-\frac{1}{4}z\sigma\beta^2\left(\frac{m}{m+1}\right)^2\{m^{-1}(m-1)^{-1}(m^2+m-1)\mathcal{D}_0^2\\
&\qquad+(m-1)^{-1}(m^2-m+1)\mathcal{D}_2^2\\
&\qquad+m^{-3}(m-1)^{-1}(10m^2+6m-1)\mathcal{E}_0^2\\
&\quad-2(m-1)^{-1}\mathcal{D}_0\mathcal{D}_2-4m^{-2}(m-1)^{-1}(m^2+5m-2)\mathcal{D}_0\mathcal{E}_0\\
&+4m^{-1}(m-1)^{-1}(3m-1)\mathcal{D}_2\mathcal{E}_0-2m^{-2}(m-1)^{-1}(m+1)^2\mathcal{D}_2\\
&\quad+2m^{-3}(m-1)^{-1}(m+1)^2\mathcal{D}_0\\
&\quad-2m^{-4}(m-1)^{-1}(3m^3+10m^2+m-4)\mathcal{E}_0\\
&\quad-m^{-5}(m-1)^{-1}(m+1)^2(3m-7)\}\\
&\quad+\frac{1}{2}z\left(\beta\bar{J}+\frac{1}{2}\beta^2\sigma\frac{m-1}{m}\right)\\
&\quad\times\frac{m}{m+1}(\mathcal{E}_0^2-2m^{-1}\mathcal{E}_0+m^{-2})\\
&\quad+\frac{1}{2}(\zeta^2-\beta^2\rho)\left[\frac{m}{m+1}(m\mathcal{D}_2+\mathcal{D}_0)-\frac{m+1}{m}\right]\\
&\quad-(\beta\nu-\eta)(\mathcal{E}_0-m^{-1})+\mathcal{B}_0+\frac{1}{4}z\beta^2\sigma m^{-1}+\ln\frac{m+1}{m}.
\end{aligned}
\tag{317}
$$

The magnetization **m** is equal to

$$
\mathrm{m}=\mathcal{E}_0(\bar{\eta},\ \bar{\zeta})-m^{-1}. \tag{318}
$$

In the case of the antiferromagnetic, paramagnetic, and spin-glass phases for the free energy we have

$$
\begin{aligned}
&-N^{-1}\beta\mathcal{F}(\eta,\ \zeta)\\
&\quad=-\frac{1}{4}z\sigma\beta^2\left(\frac{m}{m+1}\right)^2\{m^{-2}(m-1)^{-1}(3m-4)\mathcal{D}_0^2\\
&\qquad+(m-1)^{-1}(m^2-m-1)\mathcal{D}_2^2\\
&\qquad-2m^{-4}(m-1)^{-1}(7m^3-3m^2-11m-1)\mathcal{E}_0^2\\
&\qquad-4m^{-2}\mathcal{D}_0\mathcal{E}_0-6m^{-3}(m+1)\mathcal{D}_2\mathcal{E}_0\\
&\qquad-2m^{-4}(m-1)^{-1}(3m^3+5m^2-m-3)\mathcal{D}_0\\
&\qquad-2m^{-2}(m-1)^{-1}(m^2+1)\mathcal{D}_2\\
&-2m^{-4}(m-1)^{-1}(3m^3+5m^2+m+3)\mathcal{E}_0+m^{-6}(m-1)^{-1}(m^4\\
&+4m^3-12m^2-9m-4)\}-\frac{1}{2}z\left(\beta\bar{J}+\frac{1}{2}\beta^2\sigma\frac{m-1}{m}\right)\frac{m}{m+1}\\
&\times\{m^{-2}(m+1)\mathcal{E}_0^2+2m^{-3}(m-1)\mathcal{E}_0+m^{-4}(m+1)\}\\
&\quad+\frac{1}{2}(\zeta^2-\beta^2\rho)\left[\frac{m}{m+1}(m\mathcal{D}_2+\mathcal{D}_0)-\frac{m+1}{m}\right]\\
&\quad+(\beta\nu-\eta)(\mathcal{E}_0-m^{-1})+\mathcal{B}_0\\
&\quad+\frac{1}{4}z\beta^2\sigma m^{-1}+\ln\frac{m+1}{m}.
\end{aligned}
\tag{319}
$$

The magnetization of the sublattices $\mathbf{m}^{(a)}$ has the form

$$
\mathrm{m}^{(a)}=\mathcal{E}_0(\bar{\eta},\ \bar{\zeta})-m^{-1}. \tag{320}
$$

Independently of the type of ordering, the spin-glass order parameters $\tilde{q}$, q_0, and q are expressed in terms of the solution $(\bar{\eta}, \bar{\zeta})$ of the problem of finding the saddle point of the free-energy function (317), (319) as follows:

$$
\left.
\begin{aligned}
\tilde{q}&=\mathcal{D}_2(\bar{\eta},\ \bar{\zeta})+2m^{-1}\mathcal{E}_0(\bar{\eta},\ \bar{\zeta})-m^{-3}(m+2),\\
q_0&=\mathcal{D}_0(\bar{\eta},\ \bar{\zeta})-2m^{-1}\mathcal{E}_0(\bar{\eta},\ \bar{\zeta})+m^{-2}.\\
q&=\frac{m}{m+1}[m\mathcal{D}_2(\bar{\eta},\ \bar{\zeta})+\mathcal{D}_0(\bar{\eta},\ \bar{\zeta})]-m^{-1}.
\end{aligned}
\right\}
\tag{321}
$$

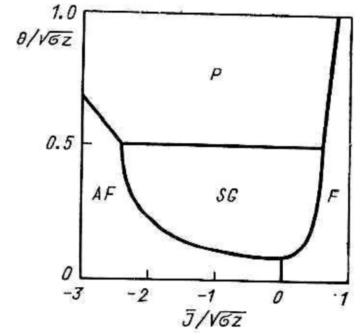

FIG. 4. Phase diagram of the three-component Gaussian stochastic lattice system.

The functions $\mathcal{E}_0$, $\mathcal{D}_0$, $\mathcal{D}_2$ and $\mathcal{B}_0$ are universal in the sense that they are independent of the temperature $1/\beta$, the number of interacting neighbors z, and the parameters of the distributions of the interaction strengths $\bar{J}$ and σ and the random fields ρ. This is why the representations (317)–(321) are convenient.

For given parameters of the distributions of $\bar{J}$, σ, and ρ, the number of interacting neighbors z, and the temperature $\theta = 1/\beta$, the problem of finding the saddle point of the free energy (317) or (319) can be solved. Its solution gives the values of the order parameters of the magnetization (318) or (320), and also the spin-glass order parameters (321). As was shown in Sec. 8, if $\bar{\eta} = 0$ and $\bar{\zeta} = 0$, the lattice is in the paramagnetic phase, if $\bar{\eta} = 0$ and $\bar{\zeta} \neq 0$, it is in the spin-glass phase, and if $\bar{\eta} \neq 0$ it is in the magnetically ordered phase.

The problem of finding the saddle point of the free energy (317), (319) was solved numerically in Refs. 67 and 68 for the three-component model. The result was used to construct the phase diagram shown in Fig. 4. Here the letters P, F, AF, and SG denote the paramagnetic, ferromagnetic, antiferromagnetic, and spin-glass phases.

When the interaction strengths $\bar{J}$ and their dispersion σ are fixed at the average value, at sufficiently high temperatures θ the system is in the paramagnetic state. If $\bar{J}$ is sufficiently large, as the temperature is decreased the system enters the ferromagnetic phase, while if $\bar{J}$ is fairly small it enters the antiferromagnetic phase, and it will remain in these phases down to zero temperature. If the modulus of $\bar{J}$ is sufficiently small, then for $\theta = \frac{1}{2}\sqrt{z\sigma}$ the system goes to the spin-glass phase, but at lower temperatures down to zero a magnetically ordered phase will again be observed: if $\bar{J} > 0$ this phase is ferromagnetic, and if $\bar{J} < 0$ it is antiferromagnetic. This is the fundamental difference between stochastic multicomponent ($q \geqslant 3$) lattice systems and the stochastic Ising model.

This feature is related to the degeneracy of the ground state and the possible existence of long-range correlations in antiferromagnetic and frustrated systems.

The asymmetry of the phase diagram relative to the temperature axis should also be noted.

11. THE NATURE OF THE PARAMAGNET–SPIN-GLASS PHASE TRANSITION

In the paramagnetic phase (the N-phase) and the spin-glass phase (the SG-phase) the variational parameter $\bar{\eta}$ is equal to zero: $\bar{\eta} = 0$. This causes the expression for the free-

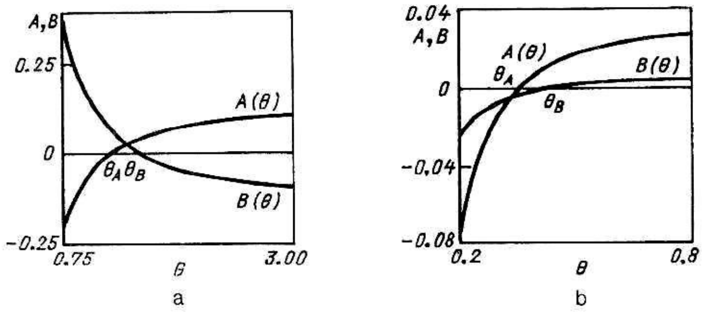

FIG. 5. Dependence of the coefficients A and B in the free-energy expansion (115) on the temperature θ: (a) $m<5$; (b) $m>5$.

energy function to simplify considerably. For the free energy $N^{-1}\mathcal{F}(\eta,\zeta)|_{\eta=0} \equiv f_s(\zeta)$ one obtains the expansion[71]

$$-\beta f_s(\zeta)=\frac{1}{4}m^{-1}z\sigma\beta^2+\ln(m+1)+\zeta^4\frac{1}{4}m^{-1}(1-m^{-2}z\sigma\beta^2)$$
$$+\zeta^6\frac{1}{2}m^{-2}(m-5)\left(\frac{1}{3}-\frac{1}{2}m^{-2}z\sigma\beta^2\right)+\mathcal{O}(\zeta^8). \tag{322}$$

The substitution $x=\zeta^2$ transforms the expansion (322) to the form

$$\left.\begin{aligned}
&-\beta f_s(x)=C+Ax^2+Bx^3+\dots,\\
&A=\frac{1}{4}m^{-1}(1-m^{-2}z\sigma\beta^2),\\
&B=\frac{1}{2}m^{-2}(m-5)\left(\frac{1}{3}-\frac{1}{2}m^{-2}z\sigma\beta^2\right).
\end{aligned}\right\} \tag{323}$$

We note that, in contrast to the regular model, both coefficients A and B depend on the temperature. They vanish at the temperatures $\theta_A=m^{-1}\sqrt{z\sigma}$ and $\theta_B=\sqrt{3/2}\,\theta_A>\theta_A$, respectively. The form of the functions $A(\theta)$ and $B(\theta)$ is shown in Fig. 5.

Let us first consider the case $m<5$. For $0<\theta<\theta_A$ the form of the function $-f_s(\zeta)$ is shown in Fig. 6a. The minimum of the function $-f_s(\zeta)$ is reached for $\bar{\zeta}\neq 0$, so that the system is in the spin-glass phase. For $\theta>\theta_A$ both the coefficients $A(\theta)$ and $B(\theta)$ are positive, the function $-f_s(\zeta)$ grows monotonically, and $\bar{\zeta}=0$, i.e., the system is in the paramagnetic phase (Fig. 6b). For $\theta=\theta_A$ we have the relation

$$\left.\frac{d^2}{d\zeta^2}f_s(\zeta)\right|_{\zeta=0}=0,$$

i.e., for $\theta=\theta_A$ there is a second-order phase transition.

The expansion (323) makes it possible to find the critical exponent associated with the spin-glass order parameter. Namely, for $m<5$ and $\theta\lesssim\theta_A$ the value of $\bar{\zeta}$ is given by the relation

$$\bar{\zeta}^2=-\frac{2A}{3B}=-\frac{2}{3}\,\frac{1}{4m}\,\frac{\theta^2-\theta_A^2}{\theta^2 B}\sim\frac{\theta_A-\theta}{\theta_A},$$

and for the spin-glass order parameter we have

$$q=m\mathcal{D}_1-m^{-1}=m^{-1}\bar{\zeta}^2+\mathcal{O}(\bar{\zeta}^4)\sim\frac{\theta_A-\theta}{\theta_A}, \tag{324}$$

i.e., the critical exponent associated with the spin-glass order parameter is 1 for all $m<5$. In the special case of the Ising model $m=1$, this corresponds to the result of Ref. 50.

For $m>5$ the sign of the coefficient $B(\theta)$ changes. In this case for $\theta<\theta_A$ we have the inequalities $B<0$ and $A<0$, so that the function $-f_s(\zeta)$ first decreases, and then, owing to higher-order terms neglected in (323), must increase [the function $f_s(\zeta)$ is bounded below], and therefore $\bar{\zeta}\neq 0$ (Fig. 7a). For $\theta_A<\theta<\theta_B$ the form of the function $-f_s(\zeta)$ is shown in Fig. 7b. Therefore, for $m>5$ a first-order phase transition occurs in the system at a temperature θ_{SG} in the range $\theta_{SG}\in[\theta_A,\theta_B]$ (Ref. 71).

12. A DISTRIBUTION OF THE INTERACTION STRENGTHS HAVING TWO MAXIMA

In real spin and ferroelectric glasses the magnetic or dipole moments are stochastically distributed on a regular lattice (the so-called matrix). These moments interact with each other via a strongly oscillating interaction which falls off slowly with distance (the RKKJ interaction) (Refs. 46–48). The effective probability density of the interaction strength in such systems was modeled numerically in Ref. 72. It was shown that the probability density has several (two or more, depending on the parameters of the RKKJ interaction and the lattice parameter) maxima.

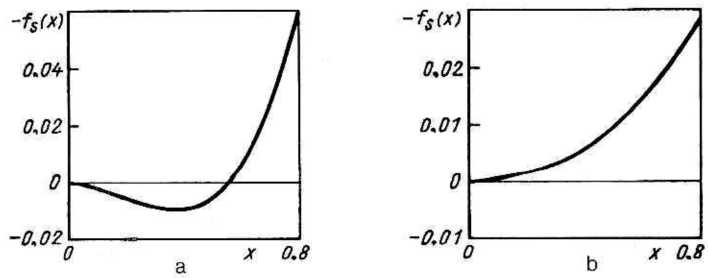

FIG. 6. The function $-f_s(x)$ for $m<5$: (a) $\theta<\theta_A$; (b) $\theta_A<\theta$.

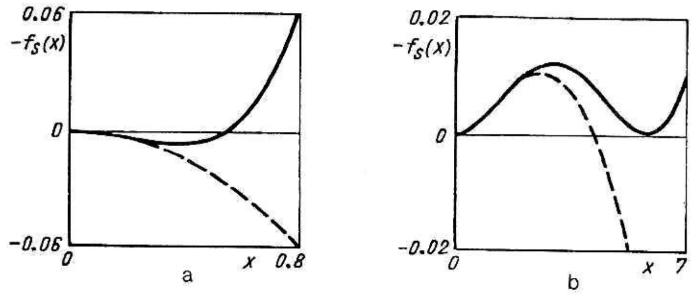

FIG. 7. The function $-f_s(x)$ for $m > 5$: (a) $\theta < \theta_A$; (b) $\theta_A < \theta_B$ [the dashed line corresponds to the series (115), and the solid line corresponds to the inclusion of higher-order terms].

Therefore, in order to approximate the situation occurring in real spin and ferroelectric glasses, let us consider the Potts model with the probability density of the interaction strength $P(J_{ij})$ in the form of the sum of two Gaussians with, in general, different weights and arbitrary location relative to the points $J_{ij} = 0$:

$$P(J) = (2\pi\sigma)^{-1/2}(\varkappa + \varkappa^{-1})^{-1} \times \{\varkappa^{-1}\exp\{-[J - J_0(1 + a)]^2/2\sigma\} + \varkappa\exp\{-[J + J_0(1 - a)]^2/2\delta\}\}. \tag{325}$$

This probability density has a sufficient number of parameters to accurately approximate the distribution of the interaction strengths in real materials. We shall refer to the distribution (325) as the frustration–Gaussian distribution.

Calculations carried out for the frustration–Gaussian distribution using expressions (210), (234), (237), and (242) for the free energy and the order parameters have shown that the vicinity of the triple point and its location undergo certain changes when the parameters of the distribution (325) are varied. For example, in Fig. 8 we show the paramagnet–spin-glass–ferromagnet triple-point trajectories as the parameter a varies for several values of $\varkappa$. Here ΔJ denotes the dispersion of the frustration–Gaussian distribution

$$\Delta J = \sigma + \frac{4J_0^2}{(\varkappa^{-1} + \varkappa)^2}. \tag{326}$$

The dependences of the slope angles of the curves for the paramagnet–spin-glass, paramagnet–ferromagnet, and spin-glass–ferromagnet phase transitions at the triple point on the parameter

$$C = 2\sqrt{z}(\varkappa + \varkappa^{-1})^{-1}J_0/\sqrt{\sigma} \tag{327}$$

for $a = 0$ are shown in Fig. 9. Here the slope angles of the paramagnet–spin-glass, paramagnet–ferromagnet, and spin-glass–ferromagnet curves are denoted by $\pi - \beta$, $\pi/4 - \alpha$, and $\pi/2 + \gamma$, respectively.

The phase diagram for the frustration–Gaussian model is constructed in the coordinates

$$X = \sqrt{z}\bar{J}/\sqrt{\Delta J}, \quad Y = \theta/\sqrt{z\Delta J}, \tag{328}$$

where $\bar{J}$ is the average value of J and ΔJ is the dispersion, or, equivalently,

$$\left.\begin{aligned} X &= \frac{1}{2}\sqrt{z}\frac{C}{(1 + C^2)^{1/2}}[\varkappa^{-1}(1 + a) - \varkappa(1 - a)], \\ Y &= [z\sigma\beta^2(1 + C^2)]^{-1/2}. \end{aligned}\right\} \tag{329}$$

These coordinates are the natural generalizations of the coordinates for the phase diagrams of the Gaussian distribution.

For the curve corresponding to the paramagnet–spin-glass phase transition we obtain

$$Y = 1, \quad X \in (-1, 1),$$

the curve for the paramagnet–ferromagnet phase transition is

$$X = Y, \quad X > 1,$$

and that for the paramagnet–antiferromagnet transition is

$$X = -Y, \quad X < 1.$$

For the curve corresponding to the phase transition from the spin glass to the magnetically ordered phases (the

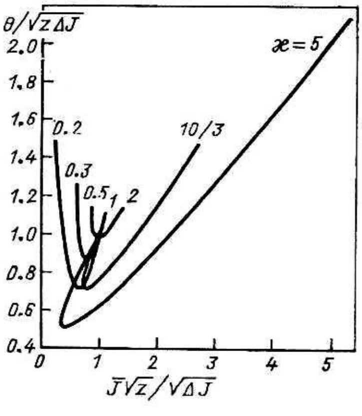

FIG. 8. Triple-point trajectories for various values of $\varkappa = 0.2 - 5$.

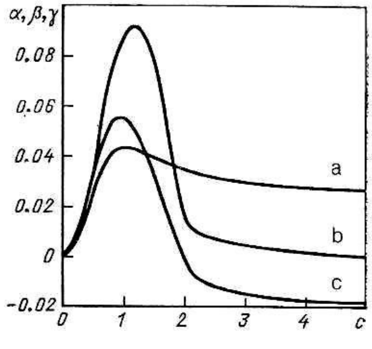

FIG. 9. Slope angles of the curves for the following phase transitions: a—slope angle of the paramagnet–spin-glass curve $\pi - \beta$; b—slope angle of the paramagnet–ferromagnet curve $\pi/4 - \alpha$; c—slope angle of the spin-glass–ferromagnet curve $\pi/2 + \gamma$.

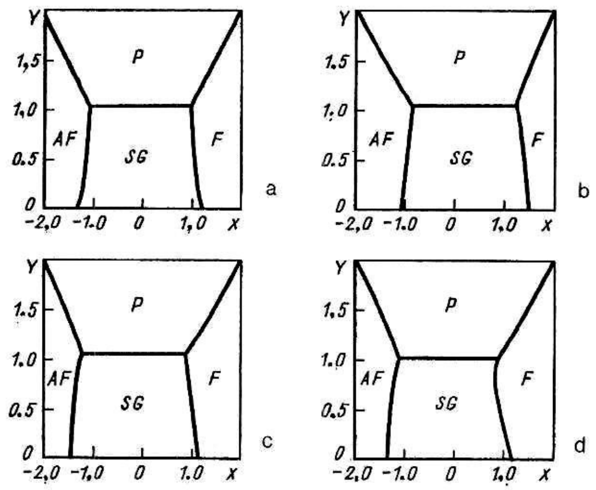

FIG. 10. Phase diagrams of the frustration–Gaussian Ising model: (a) $\varkappa = 1$; $C = 1$; (b) $\varkappa = 3$; $C = 1.5$; (c) $\varkappa = 1/3$; $C = 1.5$; (d) $\varkappa = 1/3$; $C = 0.5$.

ferromagnetic and antiferromagnetic phases) in the variables G and S

$$\left.\begin{aligned} G &= J_0\beta/\zeta, \\ S &= \sigma\beta/\zeta \end{aligned}\right\} \tag{330}$$

we obtain the system of equations

$$G_a = \lambda/z + 1/\sqrt{2}\ln\varkappa + 1/\sqrt{2}\sum_{k=1}^{\infty}(-1)^{k+1}/k\,(\varkappa^{-2k} - \varkappa^{2k}) \times \mathrm{e}^{-4k^2G^2}\psi(4k^2G^2), \tag{331}$$

$$S = 1/z - (2 - \sqrt{2})\ln(\varkappa + \varkappa^{-1}) + \sum_{k=1}^{\infty}(-1)^{k+1}/k\,(\varkappa^{-2k} + \varkappa^{2k})\,\mathrm{e}^{-4k^2G^2} \times [2\psi(2k^2G^2) - \sqrt{2}\,\psi(4k^2G^2)], \tag{332}$$

where

$$\psi(x) = \frac{1}{2\sqrt{\pi}}\int_0^{\infty} du\,\mathrm{e}^{-u}(u+x)^{1/2}, \tag{333}$$

and $\lambda = 1$ for the ferromagnet–spin-glass transition and $\lambda = -1$ for the antiferromagnet–spin-glass transition. This system of equations has been studied numerically. The resulting phase diagrams for the frustration–Gaussian Ising model are shown in Fig. 10.

13. THE FRUSTRATION MODEL

Let us consider the frustration model with the distribution (Refs. 51, 69, 73, and 74)

$$P(J) = c\delta(J - J_0) + (1 - c)\,\delta(J + J_0), \tag{334}$$

where J_0 is a non-negative parameter—the coupling strength, and c is the concentration of positive couplings and takes values in the range $0 \leqslant c \leqslant 1$.

Using the notation

$$\mathscr{A}^{(\lambda)} = 1 - \lambda z\left\{\sum_{k=1}^{\infty}(-1)^k \ln\left(\frac{1 + (1-c)^{-1}c\mathrm{e}^{-4kJ_0\beta}}{1 + (1-c)\,c^{-1}\mathrm{e}^{-4kJ_0\beta}}\right) - \frac{1}{2}\ln[(1-c)^{-1}c]\right\},$$

$$\mathscr{B} = 1 - 2z\sum_{k=1}^{\infty}(-1)^k k\ln[(1 + (1-c)^{-1}c\mathrm{e}^{-4kJ_0\beta}) \times (1 + (1-c)\,c^{-1}\mathrm{e}^{-4kJ_0\beta})] - \frac{1}{2}z\ln(c - c^2)$$

the results of Sec. 12 lead to the following conditions for the phase transition from the paramagnetic phase to the ferromagnetic ($\lambda = 1$) or antiferromagnetic ($\lambda = -1$) phase[75]:

$$\mathscr{A}^{(\lambda)} = 0, \quad \mathscr{B} > 0. \tag{335}$$

The paramagnet–spin-glass coexistence curve is determined by the conditions

$$\mathscr{B} = 0, \quad \mathscr{A}^{(\lambda)} > 0. \tag{336}$$

The phase transition from the spin-glass phase to the ferromagnetic ($\lambda = 1$) or antiferromagnetic ($\lambda = -1$) phase is described by the system

$$\lambda\int dG_u[1 - \mathrm{th}^2(\zeta u)] + z\ln[(1-c)\,c^{-1}]^{1/2}\int dG_u[1 - \mathrm{th}^2(\sqrt{2}\zeta u)] + \frac{1}{4}z\sum_{k=1}^{\infty}(-1)^{k+1}\Big/k\left[\left(\frac{c}{1-c}\right)^k - \left(\frac{1-c}{c}\right)^k\right] \times \zeta^{-2}\int dG_u\,dG_v\ln\left[1 + \mathrm{e}^{-4k\beta J}\frac{\mathrm{ch}(\sqrt{2}\zeta u)}{\mathrm{ch}(\sqrt{2}\zeta v)}\right](u^2 - v^2) = 0; \tag{337}$$

$$\zeta^2\int dG_u\frac{1 - 3\,\mathrm{th}^2\zeta u}{\mathrm{ch}^2\zeta u} - \frac{1}{2}z\ln(c - c^2) \times \int dG_u[\mathrm{th}^2\zeta u - \mathrm{th}^2(\sqrt{2}\zeta u)] + \frac{1}{4}z\sum_{k=1}^{\infty}(-1)^{k+1}\Big/k\left[\left(\frac{c}{1-c}\right)^k + \left(\frac{1-c}{c}\right)^k\right]\zeta^{-2} \times \int dG_u\,dG_v\ln\left[1 + \mathrm{e}^{-4k\beta J}\frac{\mathrm{ch}(\sqrt{2}\zeta u)}{\mathrm{ch}(\sqrt{2}\zeta v)}\right](u^2 + v^2 - 2) = 0. \tag{338}$$

The triple point is determined by the equations

$$\mathscr{A}^{(\lambda)} = 0, \quad \mathscr{B} = 0, \tag{339}$$

where $\lambda = 1$ for the coexistence point of the paramagnetic, ferromagnetic, and spin-glass phases, and $\lambda = -1$ for the coexistence point of the paramagnetic, antiferromagnetic, and spin-glass phases.

The systems (335), (336) and (337), (338) completely determine the phase diagram.

These systems have been solved numerically. It has been found that Eqs. (337) and (338) can be used directly for practical numerical calculations only for values of the parameter ζ which are not too small or too large. Therefore, the limits $\zeta \to 0$ and $\zeta \to \infty$ must be considered separately.

For $\zeta \to 0$ the system (337), (338) is analytically transformed to the form

$$\left.\begin{aligned} \mathscr{B} &= 2\mathscr{A}^{(\lambda)}, \\ \mathscr{B} &< 0, \quad |\mathscr{A}^{(\lambda)}| < 1. \end{aligned}\right\} \tag{340}$$

We note that (340) gives a satisfactory approximation also in the case of finite (not small) values of $\mathscr{A}^{(\lambda)}$.

For $\zeta \to \infty$ the system (337), (338) becomes

$$\lambda/z + 1/\sqrt{8}\ln\frac{1-c}{c} + 1/\sqrt{2}\sum_{k=1}^{\infty}(-1)^{k+1}/k\left[\left(\frac{c}{1-c}\right)^{k} - \left(\frac{1-c}{c}\right)^{k}\right]e^{-4k^2G^2}\psi(4k^2G^2) = 0; \tag{341}$$

$$1/z + (1 - 1/\sqrt{2})\ln(c - c^2) + \sum_{k=1}^{\infty}(-1)^{k+1}/k\left[\left(\frac{c}{1-c}\right)^{k} - \left(\frac{1-c}{c}\right)^{k}\right] \times e^{-4k^2G^2}[2\psi(2k^2G^2) - \sqrt{2}\,\psi(4k^2G^2)] = 0, \tag{342}$$

where the function $\psi(x)$ has the form (333). Here

$$G = \beta \lim_{\zeta\to\infty}(\tilde{J}/\zeta),$$

and $\tilde{J}$ is the solution of the system (337), (338) for J_0 with a given value of ζ.

The phase diagram was constructed in the variables

$$X = c, \quad Y = \frac{\theta}{J_0 z}.$$

It is given in Fig. 11 for the number of interacting neighbors $z = 6$ (a) and $z = 12$ (b) (Ref. 75).

The dependence of the coordinates of the characteristic points of the diagram on the value of z has been obtained. In Fig. 12 we show the z dependence of the triple-point coordinates. The coordinates X of concentration phase transitions from a spin glass to a magnetically ordered phase at zero temperature, and the coordinates Y of the point of the paramagnet–spin-glass transition for equal concentrations of positive and negative couplings ($c = \frac{1}{2}$) are shown in Fig. 13.

It has been shown that for any z when the concentration of positive couplings is close to unity, the end point of the paramagnet–ferromagnet transition curve is the point $(X, Y) = (1, 1)$, while for the concentration of positive couplings near zero it is at the point $(X, Y) = (0, 1)$.

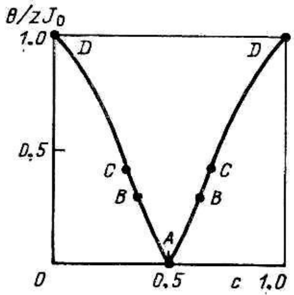

FIG. 12. Dependence of the triple-point location on the number of interacting neighbors z; for $c > \frac{1}{2}$, the point of osculation of the paramagnetic, ferromagnetic, and spin-glass phases; for $c < \frac{1}{2}$, the point of osculation of the paramagnetic, antiferromagnetic, and spin-glass phases. Points: (A) $z = 0$; (B) $z = 6$; (C) $z = 12$; (D) $z \to \infty$.

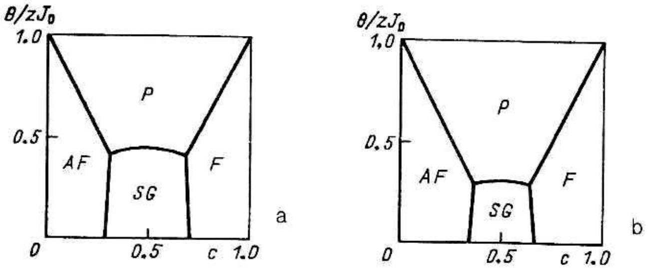

FIG. 11. Phase diagrams of the stochastic frustration Ising model: (a) $z = 6$; (b) $z = 12$.

CONCLUSIONS

It is clear from this review of applications of the Potts model that this model is interesting because it serves as the basis for the theoretical description of a wide range of diverse objects and phenomena in condensed-matter physics. Examples are complicated magnetic systems (for example, anisotropic ferromagnets with cubic structure), ferroelectrics, spin glasses, multicomponent alloys and liquid mixtures, the adsorption of inert gases on adsorbents like graphite, phase transitions on interfaces, structural phase transitions, certain processes in polymer systems, and percolation phenomena. The language of the Potts model has been used to formulate the classical coloring problem, which, in turn, has applications in physics, and for a number of optimization problems. Particular examples are the traveling-salesman problem and the problem of the partitioning of a graph into n parts with the minimal number of couplings between the parts. The latter problem is directly related to designs for the architecture of electronic computation, to structural programming problems, and problems in control theory.

The Potts model is of theoretical interest as a simple but rich model for studying first-order phase transitions and multicritical phenomena. Here it fully satisfies the basic requirement that the model be simple enough that a reliable solution can be obtained, but not so simple that the content is trivial.

Here we have proposed a method for studying stochastic Potts models, i.e., models in which the interaction strengths and fields are random variables. The techniques developed were applied to systems in which these random variables can have arbitrary probability densities, with the possibility of the latter varying from pair to pair and from point to point. Possible ordering structures have been studied.

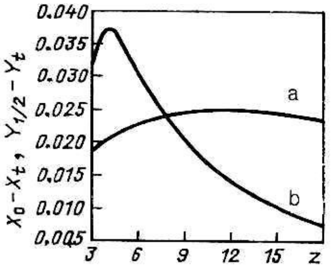

FIG. 13. Dependence of the critical-point locations on z: a—the coordinate X_0 of a point of the boundary between the ferromagnetic and spin-glass phases for $Y = 0$ relative to the coordinate X_t of the tripole point; b—the coordinates $Y_{1/2}$ of a point of the boundary between the paramagnetic and spin-glass phases for $X = 1/2$ relative to the coordinate Y_t of the triple point.

We have used our method to study specific models with various dimensions of the space of states and distributions of the random variables. The corresponding phase diagrams were constructed. In particular, we studied the three-component Gaussian model. We calculated its free energy and obtained its phase diagram. The nature of the transition from the disordered to the weakly ordered phase in multicomponent Gaussian lattices was determined. If the number of possible states of the lattice points is less than six, a second-order phase transition occurs; if this number of states is larger than six, a first-order phase transition occurs. The corresponding critical exponent was calculated. The frustration Ising model was solved. We studied frustration–Gaussian models, i.e., models in which the probability density of the interaction strengths is the sum of two Gaussian distributions, in general, with different weights and arbitrary locations on the intensity axis. This type of probability density gives a fairly good description of the distribution of the interaction strength in real magnetic and ferroelectric systems. The phase diagram was obtained, and the dependence of the characteristics of the phase diagram on the distribution parameters was determined.

The results show that, in contrast to what was thought earlier, the form of the distribution of the lattice parameters significantly affects the thermodynamic properties of the lattice and the structure of the phase diagram.

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