Resonances in three-particle systems

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Studies of the theory of resonances in three-particle systems are reviewed. Particular attention is paid to a method which uses analytic continuation of the Faddeev integral equations to the unphysical sheets of the Riemann energy surface. The features of the method are studied in the example of the two-body potential problem. In this case, Fourier transformation, normalization, and calculation of the matrix elements in the momentum representation are generalized to include Gamow states. The main subject of study is systems of nucleons. For these systems the results of experimental investigations during the last 20 years are also summarized. Problems of allowance for the Coulomb interaction are briefly discussed. Applications of the theory to other hadronic systems, including mesons and antinucleons, are mentioned.

INTRODUCTION

Resonance states of few-particle systems are currently attracting more and more interest. The quasistationary levels of the lightest nuclei are important objects of experimental and theoretical investigation in connection with the problem of thermonuclear fusion, and also the search for long-lived states outside the stability region. The important part played by resonances in the physics of nuclear reactions, including astrophysical problems, is well known.

The development of methods of theoretical investigation of resonances has a long history, which goes back to the work of Gamow[1] on the description of the α decay of nuclei and the work of Breit and Wigner.[2,3] The Breit–Wigner formula is traditionally used to analyze peaks in the cross sections of nuclear reactions. During the last 15 years many important results have been obtained and published in many journals. Unfortunately, the well-known books on scattering theory[4,5] devote too little space to resonance processes and the theory of Gamow states. Naturally, the information in these books is incomplete, since they were published more than 15 years ago. This was one of the reasons for writing the present review.

From the formal point of view, the descriptions of a bound state and of an unstable Gamow state are equivalent, since the radial wave functions of these states correspond to the same radiation condition in the form of an outgoing spherical wave, this being reflected in a zero of the Jost function or a pole of the S matrix. However, for unstable states these poles are situated on unphysical sheets of the Riemann surface of the energy E ($\mathrm{Im}\sqrt{E}<0$), so that the Gamow wave function oscillates with an exponentially increasing amplitude. This leads to well-known difficulties, both in the calculation of the Gamow wave functions and in their normalization (for a long time they were regarded as un-normalizable, or they were normalized in a finite region of space[6]). Problems also arise in the calculation of the matrix elements that determine the cross section for a transition to an unstable state. Reactions of such type have been studied in the literature (see, for example, Refs. 7–13). One of the possible ways to solve the problem is to use the method of complex scaling, which is based on the so-called ABC theorem.[14] A fairly detailed exposition of the method of complex scaling is given in Ref. 15. Sufficient conditions for validity of the ABC theorem are formulated with mathematical rigor in Ref. 16. The application to nuclear resonances is considered, for example, in Ref. 17. The method is based on solution of a quantum-mechanical problem, not on the real axis r, but on a ray (or a more complicated line) in the first quadrant of the complex plane of r. The spectrum of bound states is then augmented by the spectrum of resonance poles, which are situated between the real axis and the corresponding line in the fourth quadrant of the complex plane of the momentum. It is important that the transformed wave functions are square-integrable and, therefore, normalizable, although it is rather difficult to calculate them exactly. In addition, the method can be extended to systems with charged particles, since the Coulomb potential satisfies the conditions of the ABC theorem. A more detailed discussion of the method and its applications goes beyond the scope of the present review. We shall give mainly results obtained in the momentum representation on the basis of analytic continuation of the integral equations of scattering theory.

Fundamental progress in the theory of few-particle systems was associated with the work of Faddeev, who formulated mathematically correct integral equations of the three-body problem and proved that they were Fredholm.[18] Earlier, Skornyakov and Ter-Martirosyan[19] had obtained equations for a three-body system in the approximation of a zero range of the forces between the particles. Although these equations are not completely rigorous from the mathematical point of view (since they require truncation of an integral at large momenta), they already contain important properties of the three-particle dynamics. A generalization to a system with a larger number of particles was made in Ref. 20. Unfortunately, the technical difficulties in the solution of integral equations increase greatly with increasing number of particles. The literature contains only a few examples of solution of the Faddeev–Yakubovskiĭ equations for four bodies. The mathematical problems of the quantum theory of scattering and methods of solution of problems for few-particle systems are discussed in Refs. 21–23. However, a theory of resonance states is essentially absent from these studies. For the investigation of resonances in systems with $\geqslant 4$ particles it is necessary to employ some approximate formulation of microscopic description of the type, for exam-

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ple, of the resonating-group method,[24] or one must use nuclear models. We mention here the dynamical multicluster model.[25]

In the present review the main attention is devoted to calculations that use the Faddeev integral equations for three-particle systems with short-range forces. The practical applications are mainly to two- and three-nucleon systems. We study the trajectories of the poles of the S matrix and their general properties, including the Efimov effect[26,27] and the symmetry theorem for bound and virtual levels.[28] First results on application of the method of integral equations to unstable systems were obtained in Refs. 29–38. Allowance for the Coulomb interaction requires a modification of the method even for bound states. One of the possible ways is to use a screened Coulomb potential (see, for example, Ref. 39, in which *pd* scattering is considered, and references therein). One can then overcome the problem of the Coulomb singularities in the kernel of the integral equation. A great many publications, beginning with the studies of Noble[40] and Veselova,[41] have been devoted to few-particle systems with charged particles. An evidently convenient method of such investigations is that developed by Merkur'ev and his collaborators and based on solution of the Faddeev equations in the configuration space (see, for example, Refs. 42 and 43).

The review is arranged as follows. In Sec. 1 we define resonances and present methods of analysis of resonance phenomena, including theoretical methods for the calculation of resonances (we do not attempt to give a complete list of the methods found in the literature). Section 2 contains a review of the main results of experimental investigations of resonances in three-nucleon systems. In Sec. 3 we consider the theory of unstable states for the two-body problem. Information about the two-particle subsystems, including their resonance poles, is needed to solve the three-particle problem. In addition, in considering the two-body potential problem, we can more readily follow the main details in the method of analytic continuation of an integral equation to an unphysical sheet of the energy. We discuss in detail the generalization to the case of Gamow states of Fourier transformation, the calculation of matrix elements, and normalization. In Sec. 4 we present a three-particle theory of resonance states based on analytic continuation of the Faddeev integral equations to the unphysical sheets with respect to the two- and three-particle cuts in the complex plane of the energy. We study the analytic properties of the solutions. In Sec. 5 we consider realization of analytic continuation of an integral equation by deformation of the contour of integration in order to avoid the singularities of its kernel. In Sec. 6, we give numerical results for virtual and resonance states in systems of two and three nucleons, together with their theoretical interpretation. The review of results enables us to evaluate the investigations and draw the conclusion that, except for the virtual pole in *nd* scattering and the pole corresponding to it in the *pd* system with the quantum numbers of ^{3}H and ^{3}He, the discovery near the physical region of resonances in the $3N$ system is unlikely.

In the conclusions, we discuss the extension of the described method to other three-hadron systems and give references to investigations of resonances in these systems.

If we do not say otherwise we use a system of units in which $\hbar = c = 1$.

1. DEFINITIONS OF RESONANCES AND METHOD OF ANALYSIS OF RESONANCE PHENOMENA

Although general questions of the theory of unstable states, in both single-channel and multichannel problems, have been treated in, for example, Refs. 3–6, we have felt it expedient to give here for completeness some well-known elementary results, since they show why it is preferable to define a resonance as a pole of the S matrix. The various existing definitions and approaches to the analysis of resonances give the same results for the positions and widths of resonances if they are sufficiently narrow, i.e., when the complex poles of the S matrix are close to the physical region of the energy. For broad resonances, the results may differ strongly. From the practical point of view, it would appear at the first glance that this is not important; for with increasing width the very concept of a "resonance" loses a definite meaning, simply because the resonance structure in the cross section as a function of the energy disappears. However, for a deeper understanding of the dynamical nature of resonance processes it is important to have the universal definition of a resonance as a pole of the S matrix, since this concept can also be generalized to the case of relativistic theory. We list briefly the most common methods used to analyze resonances and the methods of their theoretical investigation.

Comparison of excitation function with phase space

Distinct peaks in the excitation function [the reaction cross section $\sigma(E)$] in its dependence on the energy are regarded as resonances in the case of small widths Γ and a weak nonresonance background. In the case of a large width, any appreciable deviation of $\sigma(E)$ from the phase space can be interpreted as a resonance structure. The structures may be diverse—for example, the cross section may contain a minimum rather than a maximum if the interference with the background is destructive.[5] In such cases the partial-wave amplitudes require a more careful analysis, and theoretical arguments must be taken into account, since otherwise conclusions are not unambiguous.

Phase-shift analysis

For simplicity, let us consider the scattering of two spinless particles in the case when the interaction between them is described by a central potential. The total cross section can be expressed in terms of the phase shifts $\delta_l(k)$ for orbital angular momentum l by the well-known relation

$$\sigma(k) = 4\pi/k^2 \sum_l (2l+1) \sin^2 \delta_l(k), \qquad (1)$$
$$E = \hbar^2 k^2/2\mu,$$

where μ is the reduced mass. It is obvious that a resonance (maximum of the cross section) at $k = k_R$ corresponds to the condition

$$\sin^2 \delta_l(k) = \max, \ \delta_l(k_R) = \pm (2n+1)\pi/2, \qquad (2)$$

where n is an integer.

Thus, to find the resonance energy we must carry out a phase-shift analysis and find the value of the energy $E = E_R$ at which the condition (2) is satisfied. The width Γ of the resonance is determined by the rate of change of the phase shift as the resonance region is traversed.

Argand diagram

The Argand-diagram method is a modification of phase-shift analysis. We introduce the S and T matrices by

$$S_l(k) = \exp[2i\delta_l(k)]; \tag{3}$$

$$T_l(k) = (S_l - 1)/2i, \tag{4}$$

from which we obtain

$$\left.\begin{aligned} &\operatorname{Re} T_l(k) = \cos\delta_l(k)\sin\delta_l(k), \\ &\operatorname{Im} T_l(k) = \sin^2\delta_l(k). \end{aligned}\right\} \tag{5}$$

From (5) we obtain the equation of a circle in the complex plane of T_l:

$$(\operatorname{Re} T_l(k))^2 + (\operatorname{Im} T_l - 1/2)^2 = 1/4. \tag{6}$$

For a narrow Breit–Wigner resonance we have near the resonance energy

$$T_l(E) = \frac{-\Gamma/2}{E-(E_R - i\Gamma/2)}. \tag{7}$$

Passage through the resonance region in the interval from $E = E_R - \Gamma/2$ to $E = E_R + \Gamma/2$ corresponds to counterclockwise passage around the upper semicircle in the Argand diagram (Fig. 1). The complete (unitarity) circle corresponds to an increment of the phase from 0 to π (with allowance for the indeterminacy $\pm n\pi$ in the phase). The Argand diagram has the "ideal" form only in the single-channel case. In the multichannel problem, i.e., in the presence of absorption, when the phase shift δ_l is complex, the actual diagram always lies within the unitarity circle and may have a strongly distorted shape. The question of whether or not a resonance exists may then no longer have an unambiguous answer.

Poles of the S matrix

By means of (3) and (4), the condition (2) can be rewritten in the alternative form

$$S_l(E_R) = -1, \quad T_l(E_R) = i \tag{8}$$

or, introducing the K matrix,

$$K_l = \frac{2T_lT_l^*}{T_l + T_l^*} = \tan\delta_l, \tag{9}$$

in the form of the equivalent condition

$$K_l(E_R) = \infty. \tag{10}$$

Thus, the K matrix has poles at the resonance points on the real axis of the energy. From (4) and (7) we obtain for the S matrix the expression

$$S_l(E) = \frac{E - E_R - i\Gamma/2}{E - E_R + i\Gamma/2}, \tag{11}$$

which is valid for a narrow isolated resonance when the contribution of the background can be ignored. It can be seen from (11) that the S matrix has a pole at the point $E = E_R - i\Gamma/2$. This is a well-known result. The definition of resonances as poles of the S matrix is the most convenient for theoretical analysis for the following reasons:

1. One frequently encounters situations in which the "idealized" conditions (2), (8), and (10) are nowhere satisfied on the real axis, but the S matrix does have poles in real problems, irrespective of the widths. Thus, the identification of a resonance with a pole of the S matrix is a true generalization of the concept of resonance behavior of the scattering amplitude.

2. Bound, virtual, and resonance states corresponding to the same radiation condition (outgoing wave at infinity) can be treated on a unified basis. The common origin of poles of the S matrix can be seen from the fact that for the three types of state they lie on the common pole trajectories that describe the way in which the position of a pole in the complex plane of the energy depends on the strength of the interaction (or on some other parameter). For this reason, one can regard virtual and resonance states as "nonoccurring" bound states. One can establish a critical strength of the interaction at which a state ceases to be bound, i.e., it enters the continuum.

The appearance of a resonance maximum in the scattering cross section should not be identified with proximity of the resonance pole to the real axis of the energy ($E \geqslant 0$), since the unitarity of the S matrix means that it also has a zero at the same distance [see (11)]. Rather, one should speak of rapid variation of the S matrix and, accordingly, the phase shift $\delta(E)$ for variation of E near the pole [at which the factor $|\sin^2\delta|$ in (1) attains a maximal value]. If in the experiment the energy resolution ΔE corresponding to the width of the incident wave packet is sufficient for observation of a resonance with width Γ ($\Delta E \ll \Gamma$), then one can show[5,6] that the width $\Gamma = \hbar/\tau$ of the quasistationary state is determined by the time of retention of the particle in the potential well:

$$\tau = \frac{1}{v_0}\frac{d\delta_l}{dp} = \frac{d\delta_l}{dE}.$$

It follows from this that an estimate of the width of a quasistationary state can be obtained by measuring the rate of change of the phase shift in the resonance region of the energy.

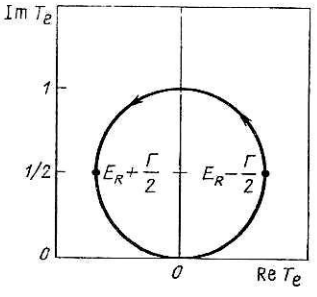

FIG. 1. Argand diagram for the single-channel problem.

Theoretical methods for investigating resonances

We briefly list the theoretical methods that treat resonances in terms of the poles of the S matrix. The effective-range approximation,[44,45] which is convenient for the study of poles situated near a threshold (see also Ref. 46), was applied, after modifications to take into account the Coulomb field, to the theory of the nuclear shift of the levels of the proton–antiproton atom[47] and to the pp system in the spin-singlet state $^1S_0$ in Ref. 48. A first unexpected result was obtained in Ref. 48—the displacement of the well-known virtual level of the np system from the imaginary axis into the fourth quadrant of the complex plane of the momen-

TABLE I. Survey of experimental results on resonances in three-nucleon systems.

Nucleus	Reaction	Energy, MeV	Quantum numbers	Resonance, MeV	Reference
^3n	^{3}H (n, p) $3n$	$E_n = 14.4$	—	$E_B = 1$ (bound)	[65]
		$E_n = 20.8$	—	—	[66]
		$E_n = 15.2$	—	—	[67]
		$E_n = 14.1; 18.0; 21.5$	—	—	[68]
	^{3}He$(\pi^-, \pi^+)3n$	$E_{\pi^-} = 140$	—	$E = 2$; $\Gamma = 12$	[69]
		$E_{\pi^-} = 140$	—	$E = 9 \pm 1$; $\Gamma = 10.5$	[70]
		$E_{\pi^-} = 140$	$T, S, L = 3/2, 1/2, 1$	$E = 12$; $\Gamma = 3$	[71]
		$E_{\pi^-} = 140; 200$ 295	—	"Broad resonance"	[72]
	^{3}H$(\pi^-, \gamma)3n$	$E_{\pi^-} \approx 0$	—	—	[63, 73—76]
	^{4}He (π^-, p) $3n$	$E_{\pi^-} = 140$	—	—	[77]
		$E_{\pi^-} = 140$	—	—	[70]
		$E_{\pi^-} = 60; 100; 200$	—	—	[78]
		$E_{\pi^-} = 285 - 575$	—	—	[79]
		$E_{\pi^-} = 400; 475$	—	—	[80]
	^{7}Li $(\pi^-, ^4$He$)$ $3n+\gamma$	$E_\pi \approx 0$	—	—	[81]
	^{7}Li $(^7$Li, ^{11}C$)$ $3n$	$E_{\mathrm{Li}} = 79.6$	—	—	[82]
	^{14}N $(\pi^-, 3n)$ ^{8}B, ^{3}H	$E_{\pi^-} \approx 0$	—	—	[83]
	^{3}H $(^3$H, ^{3}He$)$ $3n$	$E_{^3\mathrm{H}} = 22.25$	—	$E = 1 - 1.5$	[84]
	^{7}Li $(n, 3n)$ ^{5}Li	$E_n = 19$	—	—	[85]
^{3}H	^{3}He (n, p) ^{3}H	$E_n = 14.4$	—	—	[86]
	^{6}Li $(\pi^-, ^3$H$)$ ^{3}H	$E_\pi = 100$	—	—	[87]
	^{3}He $(^3$H, ^{3}He$)$ ^{3}H	$E_{^3\mathrm{H}} = 22.25$	—	—	[84]
	^{3}H $(^3$H, ^{3}H$)$ ^{3}H	$E_{^3\mathrm{H}} = 22.25$	—	—	[84]
	^{2}H (n, n) ^{2}H	$E_n = 0 - 800$	—	—	[62]
	^{2}H (n, p) $2n$	$E_n = 0 - 152$	—	—	[62]
	^{2}H (p, π^+) ^{3}H	$E_p = 0$—1.5GeV	—	—	[88]
^{3}He	^{3}H (p, n) ^{3}He	$E_p = 30, 50$	$T = 3/2, 1/2$	$E = 16 \pm 1$; $\Gamma = 9 \pm 1$	[89]
		$E_p = 50$		$E = 9.6 \pm 0.7$; $\Gamma = 5 \pm 1$	[90]
	^{3}He (p, p') ^{3}He	$E_p = 30.2$	—	$E = 10.2$; $\Gamma = 0.9$ $E = 12.6$ $E = 8.2$	[91]
		$E_p = 30.9$	—	—	[92]
		$E_p = 30$	—	—	[93]
	^{2}H (p, p) ^{2}H	$E_p = 0 - 2.7$GeV	$T, S, L = 1/2; 3/2; 1$	$E = 14.2 \pm 0.5$; $\Gamma = 10$	[94]
	^{2}H (p, d^*) ^{1}H	$E_p = 9 - 13$	—	$E = 12.4$	[95]
		$E_p = 7 - 17$	—	—	[96]
		$E_p = 7 - 14.5$	—	—	[97]
		$E_p = 3.8 - 5$	—	—	[98]
	^{1}H (d, d^*) ^{1}H	$E_d = 19.2 - 27$	—	—	[99]
	^{2}H (p, γ) ^{3}He	$E_p = 10 - 17.5$	$T, S, L = 1/2; 1/2; 1$	$E = 14.5 \pm 0.5$; $\Gamma = 2$	[100]
		$E_p = 14 - 25.5$	—	—	[101]
	^{1}H (d, γ) ^{3}He	$E_d = 41.1$ $E_d = 45.2$	$T, S, L = 1/2; 1/2; 1$	$E = 19.5 \pm 0.5$; $\Gamma = 2$	[102]
	^{3}He (γ, d) ^{1}H	$E_\gamma = 11 - 65$	—	—	[103]
	^{3}He $(e, e'd)$ ^{1}H	$E_e = 43.07$ $E_e = 87.34$	—	—	[104]

Continuation of TABLE I.

Nucleus	Reaction	Energy, MeV	Quantum numbers	Resonance, MeV	Reference
	^{3}He (e, e') ^{3}He	$E_e=21$ $E_e=56.6$ $E_e=200$	— — —	— — —	[105] [106] [107]
	^{3}He (π^-, π^-) ^{3}He	$E_\pi=140$	T, S, L= 3/2; 1/2; 1	$E=16$; $\Gamma=2$	[71]
	^{4}He (p, d) ^{3}He	$E_p=65$	—	—	[108]
	^{3}He (^{3}He, ^{3}He) ^{3}He	$E_{^3\mathrm{He}}=44$ $E_{^3\mathrm{He}}=53$	—	—	[109]
^{3}He	^{3}He (α, α') ^{3}He	$E_\alpha=115$ $E_\alpha=63.7$	— —	— —	[108] [110, 111]
	^{6}Li (p, pd) ^{4}He	$E_p=9-10$	—	—	[112]
	^{6}Li (p, α) ^{3}He	$E_p=30$ $E_p=45$ $E_p=45$ $E_p=20$ $E_p=35-56$	— — — — —	— — — — —	[113] [58] [114] [115] [116]
	^{6}Li (p, αd) ^{1}H	$E_p=45$	—	—	[58]
^{3}Li	^{3}He (p, n) $3p$	$E_p=13.1$ $E_p=14.1$ $E_p=30-50$	— — $T=3/2$	— — $E=9\pm1$; $\Gamma=10.5$	[117] [118] [89]
	^{2}H (p, π^-) $3p$	$E_p=585$	—	—	[119]
	^{3}He (^{3}He, t) $3p$	$E_{^3\mathrm{He}}=44$; 53 $E_{^3\mathrm{He}}=53$	— —	— —	[120] [121]
	^{6}Li (^{3}He, ^{6}He) $3p$	$E_{^3\mathrm{He}}=53.2$	—	—	[122]

Note. The values of the energy given for the *nnn* and *ppp* systems are measured from the complete-disintegration threshold. For the *nnp* and *npp* systems the energies are measured from the ground states of ^{3}H and ^{3}He, respectively. A dash means that a resonance has not been found.

tum as a result of the additional Coulomb repulsion. In the region of interaction parameters close to the formation of a bound state one can use a calculation of the Regge trajectories[49] for the poles of the S matrix in the complex plane of the angular momentum.

It was proposed in Refs. 50 and 51 that the trajectories of virtual and resonance levels should be calculated by analytic continuation with respect to the coupling constant $\lambda(E)$, which occurs as a factor in the interaction potential: $V=\lambda(E)V(r)$. One starts here from the values of $\lambda(E)$ in the region of the bound states. A Padé approximant is used to make the analytic continuation to the region $\mathrm{Im}\sqrt{E}<0$.

A method of summation of divergent perturbation-theory series for the energy eigenvalues $E=E(\mu)$ corresponding to virtual and resonance states was proposed and developed in Refs. 52 and 53 for a number of screened Coulomb potentials. The power series in the argument μ of the exponential of the potential $V(r)=f(r)\exp(-\mu r)$ was summed by means of the Padé approximant. In the case of resonances, a more accurate result is obtained by considering the power series with respect to the variable $z=(1+w)/(1-w)$, where $w=(1-\mu/\mu_{\mathrm{cr}})^{1/2}$, where μ_{cr} is the value of the parameter μ at which the energy of the level is zero. One then constructs the Padé approximant with respect to the variable z. Below, we shall discuss the results of actual calculations in the framework of this model for the Yukawa potential. It should be emphasized that in this case the solution in the zeroth approximation ($\mu\to0$) is known—it is the Coulomb spectrum. Finally, the complex poles of the S matrix were also studied by means of the N/D method. In this method, the original information is not the potential of the interaction between the particles but the discontinuity of the scattering amplitude across the left-hand dynamical cut. There is an exposition of the fundamentals of the N/D method in Ref. 54, for example. The N/D method was used to investigate systems of "resonance + particle" type,[55] the resonance being treated as a state in the continuum. We wish to emphasize here that study of a system in the continuum (at real energies E) is a quite different problem from the investigation of Gamow states, to which the present review is devoted.

In listing these theoretical methods, we have not attempted to give a complete survey of the methods used to study resonances (we have already mentioned the use of the

ABC theorem in the Introduction). Our main aim is to present the method of analytic continuation of the integral equations and the results obtained by means of it.

It should be noted that the existence of a maximum or any other structure in the cross section need not be associated with a pole of the S matrix. In such cases, one speaks of a pseudoresonance behavior of the amplitude (see, for example, Ref. 56). A resonance-like structure can arise, for example, from the mechanism corresponding to the Feynman triangle diagram when its complex singularity is near the physical region.

Before we turn to the systematic exposition of the method of integral equations, we give a brief review of the experimental data on resonances in few-nucleon systems, these being our prime subject of interest.

2. THE EXPERIMENTAL SITUATION

The principal results of experimental investigations of resonances in three-nucleon systems are given in Table I and in the reviews of Refs. 57–62. Almost all the existing experimental information about resonances in three-nucleon systems up to 1987 can be found in Refs. 60 and 62. An intricate experimental study of the $^3\mathrm{H}(\pi^-, \gamma)3n$ reaction was made at Los Alamos[63] on a gaseous tritium target. The $3n$ mass spectrum for capture of stopped pions was observed with detection of γ rays by a high-resolution spectrometer. The upper limit of the branching ratio for the transition to resonance states with width $\Gamma \leqslant 5$ MeV was found to be 2×10^{-2}.

As follows from Table I, resonances were sought in many nuclear reactions, but only in a few cases were (broad) structures similar to resonances found.[1)] In most of these cases, all deviations from the phase space were interpreted as resonances, though it can be shown theoretically (see, for example, Ref. 64) that such deviations could have another origin, for example, two-particle interaction in the final state (Migdal–Watson effect). Summarizing the results of the experimental search for resonances in the three-nucleon systems over a period of about 20 years, we may conclude that resonances have not been found, and that little hope remains of finding them in the future.

3. RESONANCES IN THE TWO-BODY SYSTEM

Integral equations for resonances and virtual states in a short-range potential

We consider two bodies that interact through a potential $V(r)$ that admits analytic continuation of its Fourier transform to the region of complex momenta. This condition is met, for example, by an "analytic potential" which satisfies the conditions[4]

$$1)\ \int_0^\infty r^2 dr\ |\ V(r)\ | < \infty, \quad \int_0^\infty r dr\ |\ V(r)\ | < \infty, \tag{12}$$

2) $V(r)$ is a regular function in the interval

$$-\pi/2 < \arg r < \pi/2. \tag{12a}$$

We relate the partial-wave t matrix to the off-shell scattering amplitude by

$$t_l(q, q'; z) = -(2\pi/\mu) f_l(q, q'; z). \tag{13}$$

In the physical region $z = E + i0$, where E is the energy of the relative motion, we have the on-shell ($q^2 = q'^2 = p^2 = 2\mu E$) relation

$$f_l(p) \equiv f_l(p, p; E) = p^{-1} e^{i\delta_l} \sin\delta_l. \tag{14}$$

The relations (13) and (14) determine the normalization of the t matrix, which on the physical sheet of the energy ($\mathrm{Im}\sqrt{z} > 0$) satisfies the Lippmann–Schwinger equation

$$t_l(q, q'; z) = V_l(q, q') + \frac{1}{2\pi^2}\int_0^\infty \frac{V_l(q, k)\, t_l(k, q'; z)}{z - k^2/2\mu} k^2 dk, \tag{15}$$

in which the Fourier transform of the potential is

$$V_l(q, q') = 4\pi \int_0^\infty j_l(qr)\, V(r)\, j_l(q'r)\, r^2 dr, \tag{16}$$

and $j_l(x)$ is a spherical Bessel function [$j_0(x) = x^{-1}\sin x$].

The analytic continuation of (15) to the unphysical sheet of $z(\mathrm{Im}\sqrt{z} < 0)$ is determined by the properties of the Cauchy-type integral (see, for example, Ref. 123) on the right-hand side of (15), which has a square-root branch cut along the real axis (the upper edge of the cut is the physical region of the energy). For the second branch of t_l ($\mathrm{Im}\sqrt{z} < 0$) we obtain the equation[35]

$$t_l^{(-)}(q, q'; z) = V_l(q, q') + \frac{1}{2\pi^2}\int_0^\infty \frac{V_l(q, k)\, t_l^{(-)}(k, q'; z)}{z - k^2/2\mu} k^2 dk + (i/\pi)\,\mu p V_l(q, p)\, t_l^{(-)}(p, q'; z), \tag{17}$$

from which it follows that at $q = p$

$$t_l^{(-)}(p, q'; z) = V_l(p, q') + \frac{1}{2\pi^2}\int_0^\infty \frac{V_l(p, k)\, t_l^{(-)}(k, q'; z)}{z - k^2/2\mu} k^2 dk + (i/\pi)\,\mu p V_l(p, p)\, t_l^{(-)}(p, q'; z). \tag{18}$$

Here, $p = \sqrt{2\mu z}$ is the arithmetic value of the root. Figure 2, which shows the deformation of the contour of integration that ensures convergence of the integral, makes the derivation of Eq. (17) transparent. The return of the contour to the real axis leads to an additional [compared with (15)] term in (17). It is associated with the residue of the integrand at the point $k = -p[\mathrm{Im}(-p) < 0]$. Solving (18) for $t_l^{(-)}(p,q';z)$ and substituting the result in (17), we obtain an equation that differs from the Lippmann–Schwinger equation (15) only by the replacement of the potential V_l by the function[35,37]

$$\tilde{V}_l(q, q'; p) = V_l(q, q') + \frac{i\mu p/\pi}{1 + V_l(p, p)\, i\mu p/\pi} V_l(q, p)\, V_l(p, q'), \tag{19}$$

which contains a separable (nonlocal) energy-dependent addition. Although the zero of the denominator in (19) is not a pole of the t matrix,[35] it is more convenient to use the system of equations (17) and (18) near the corresponding values of p. Conversely, near a singularity of $V_l(p, p)$ it is better to use an equation of the form

$$t_l^{(-)}(q, q'; z) = \tilde{V}_l(q, q'; p) + \frac{1}{2\pi^2} \times \int_0^\infty \frac{\tilde{V}_l(q, k; p)\, t_l^{(-)}(k, q'; z)}{z - k^2/2\mu} k^2 dk. \tag{20}$$

The relation (19) confirms our expectation that the

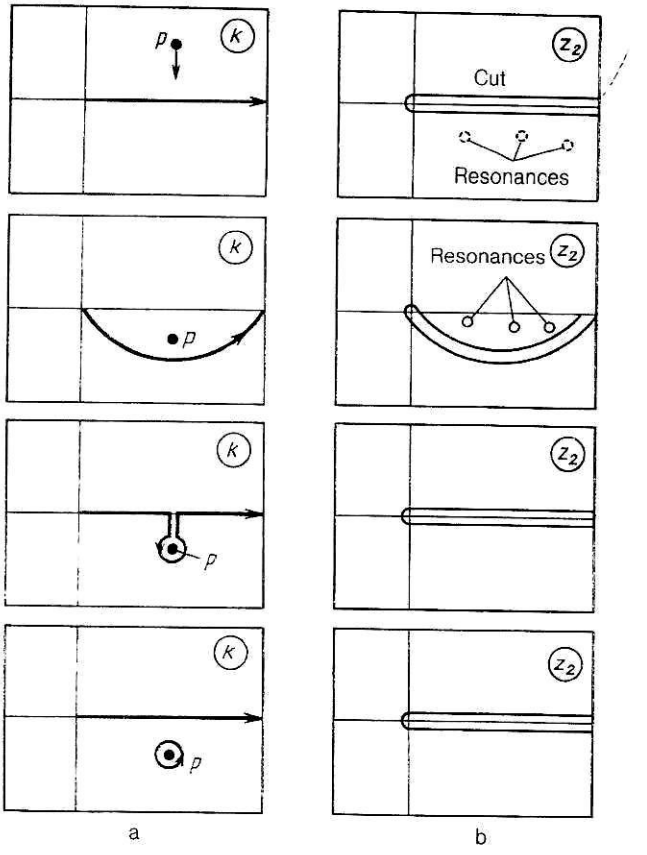

FIG. 2. Deformation of the contour of integration in the Lippmann–Schwinger equation for analytic continuation to an unphysical sheet of the energy in the planes of the complex momentum (a) and complex energy (b).

branches t and $t^{(-)}$ are related through the threshold branch point $z = 0$, where V and $\tilde{V}$ are equal.

Poles of the S matrix

In accordance with the theory of Fredholm integral equations, an inhomogeneous integral equation does not have a solution, i.e., $t = \infty$, if the corresponding homogeneous equation (or system of equations) has a nontrivial solution. Thus, the problem of the poles of the t matrix consists of finding a discrete set of values $(z_1, z_2, \ldots)$ of the complex energy for which there exists a solution of the homogeneous equation

$$F^{(-)}(q, z) = \frac{1}{2\pi^2} \int_0^\infty \frac{\tilde{V}_l(q, k; p) F^{(-)}(k, z)}{z - k^2/2\mu} k^2 \, dk. \qquad (21)$$

This problem can be formulated alternatively as the problem of determination of the eigenvalues $\lambda_n(z)$ of the kernel of the integral equation

$$\lambda_n^{-1}(z) F_n^{(-)}(q, z) = \frac{1}{2\pi^2} \int_0^\infty \frac{\tilde{V}_l(q, k; p) F_n^{(-)}(k, z)}{z - k^2/2\mu} k^2 \, dk, \qquad (22)$$

which exist for any z. The poles of the t matrix are found from the condition

$$\lambda_n(z) = 1, \quad z = z_1, z_2, \ldots \qquad (23)$$

Numerical solution of the integral equations

An approximate solution of the integral equation can be obtained by quadrature (for example, by Gauss's method). Then (21) can be expressed in the form of the algebraic system

$$F(k_j, z) = \sum_{i=1}^{N} M_{ji}(z) F(k_i, z), \; j = 1, 2, \ldots, N, \qquad (24)$$

where

$$M_{ji} = (2\pi^2)^{-1} C_i k_i^2 \tilde{V}_l(k_j, k_i; p)/(z - k_i^2/2\mu).$$

The actual values of C_i are determined by the quadrature formula. The set of eigenvalues z_n is found from the condition on the determinant of the system ($I_{ij} = \delta_{ij}$):

$$\det(I - M) = 0. \qquad (25)$$

In the case of Eq. (22), we seek the eigenvalues of the matrix M:

$$\lambda_n^{-1}(z) F_n(k_j, z) = \sum_{i=1}^{N} M_{ji}(z) F_n(k_i, z), \qquad (26)$$

where the condition (23) determines the positions of the required poles. Note that the relation (19) immediately yields the well-known symmetry of the poles of the S matrix relative to the imaginary axis of the momentum. For the substitution $p \to -p^*$ is tantamount to the complex conjugation $V_l(q, q'; -p^*) = V_l^*(q, q'; p)$ [if we take into account (16) and the relation $j_l(-x) = (-1)^l j_l(x)$]. At the same time, there is a complex conjugation of the determinant of the system, which obviously leaves Eq. (25) in force.

Regularization of the equations

If a resonance is very narrow, we encounter a problem of the accurate calculation of an integral with a singularity near the contour of integration; this is analogous to the problem of the solution of the singular integral equation for the t matrix in the physical region of the energy. In this case one can use some method to derive an equivalent system of nonsingular equations such as the Noyes–Kowalski method (see, for example, Ref. 54). The homogeneous equation corresponding to Eq. (15) can be rewritten, for example, in the form

$$F_l(q, z) = \frac{\mu}{\pi^2} \int_0^\infty dk \left[\frac{k^2 V_l(q, k) F_l(k, z) - p^2 V_l(q, p) F_l(p, z)}{p^2 - k^2} \right] + \frac{\mu}{\pi^2} p^2 V_l(q, p) F_l(p, z) \int_0^\infty \frac{dk}{p^2 - k^2}. \qquad (27)$$

The integrand in the first term on the right-hand side of (27) is now free of poles in the propagator and can be calculated without deformation of the contour, while the second integral can be calculated in analytic form by means of the residue theorem:

$$\int_0^\infty \frac{dk}{p^2 - k^2} = -\frac{i\pi}{2p}, \quad \operatorname{Im} \sqrt{z} > 0. \qquad (28)$$

Since the momentum p occurs explicitly for such a regularization, the equation on the unphysical sheet is obtained by a simple change of sign in front of p, and this now corresponds to $\operatorname{Im}\sqrt{z} < 0$ (Ref. 124):

$$F_l^{(-)}(q, z)$$

$$= \frac{1}{2\pi^2}\int_0^\infty dk\left[\frac{k^2V_l(q, k)F_l^{(-)}(k, z) - p^2V_l(q, p)F_l^{(-)}(p, z)}{z - k^2/2\mu}\right]$$

$$+ (i/2\pi)\,\mu p V_l(q, p)F_l^{(-)}(p, z). \quad (29)$$

It must be emphasized that the possibility of analytic continuation with respect to the energy into the region $\mathrm{Im}\sqrt{z} < 0$ is determined by the analytic properties of not only the kernel of the equation but also its solution, and these properties must be investigated in more detail.

Analytic properties of the t matrix and of the vertex function. Unitarity of the S matrix

Once the pole of the S matrix has been found ($z = z_n$), Eq. (21) for $z = z_n$ or (22) for $\lambda_n(z) = 1$, or the system of homogeneous equations (17) and (18) without the inhomogeneous term $V_l(q, q')$ give an equation (or system of equations) for the vertex function $g_l^{(n)}(q)$ of the decay of the unstable state (and, therefore, for the wave function in the momentum representation). Near the pole, we can write

$$t_l^{(-)}(q, q'; z) = \frac{g_l^{(n)}(q)\,g_l^{(n)}(q')}{z - z_n} + \ldots \quad (30)$$

The expression of the pole term of the t matrix in the form (30) (without complex conjugation of one of the factors) arises because of the use of a biorthogonal system of Gamow wave functions (see, for example, the formulation of the extended completeness condition in Ref. 125).

We shall assume that the analytic properties of the potential $V_l(q, q')$ and of the t matrix are known on the physical sheet. Solving Eq. (17) iteratively and summing the resulting Neumann series, we find[28,126]

$$t_l^{(-)}(q, q'; z) = t_l(q, q'; z)$$

$$+ (i/\pi)\,\mu p t_l(q, p; z)\,t_l^{(-)}(p, q'; z). \quad (31)$$

The relation (31) generalizes the Hilbert identity (see, for example, Refs. 18, 22, and 54) associated with the unitarity of the S matrix. On the energy shell and at physical values of the energy z, we obtain, taking into account (13), the relation $\mathrm{Im}\,f_l(p) = p|f_l(p)|^2$, which is equivalent to the optical theorem. Thus, there is an intimate connection between Eq. (17) and the unitarity of the S matrix. In the derivation of the optical theorem Hermiticity of the Hamiltonian [reality of $V(r)$] is assumed. The use of the complex potential $V(r)$ does not hinder the derivation of (17). However, the optical model destroys the unitarity of the S matrix and is only an approximation to the multichannel problem with a corresponding unitarity condition for the S matrix. For $q = p$, we find from (31) a connection between $t^{(-)}$ and t:

$$t_l^{(-)}(p, q'; z) = t_l(p, q'; z)/[1 - (i/\pi)\,\mu p t_l(p, p; z)]. \quad (32)$$

Equations (31) and (32) completely determine the analytic properties of the t matrix on the unphysical sheet in terms of the analytic properties of the t matrix on the physical sheet. It can be seen from (32) that the resonance pole can be found from the equation

$$1 - (i/\pi)\,\mu p t_l(p, p; z) = S_l(p) = 0, \quad (33)$$

though this is technically more complicated than finding the zeros of the Fredholm determinant for the equations on the unphysical sheet.

The equation corresponding to (17) for the vertex function,

$$g_l^{(n)}(q) = \frac{1}{2\pi^2}\int_0^\infty \frac{V_l(q, k)\,g_l^{(n)}(k)}{z_n - k^2/2\mu}\,k^2\,dk$$

$$+ (i/\pi)\,\mu p_n\,q_l^{(n)}(p_n)\,V_l(q, p_n), \quad (34)$$

can also be solved iteratively. We then obtain

$$q_l^{(n)}(q) = (i/\pi)\,\mu p_n g_l^{(n)}(p_n)\,t_l(q, p_n; z_n). \quad (35)$$

Thus, all the singularities of the vertex function with respect to the variable q are determined by the function $t_l(q, p_n; z_n)$. By means of (30) and (32) we can find[28,126] an expression for the residue:

$$\left.\begin{aligned} g_l^{(n)}(p_n) &= [-(2\mu^2/\pi)\,(d/dp_n)\,(p_n f_l(p_n))]^{-1/2}, \\ \mathrm{Res}\,f_l^{(-)}(p)\,|_{p=p_n} &= [4\mu\,(d/dp_n)\,(p_n f_l(p_n))]^{-1}. \end{aligned}\right\} \quad (36)$$

Symmetry theorem for bound and virtual levels

The relations (35) and (36) completely determine the analytic properties of the vertex function of the unstable state in terms of the analytic properties of the t matrix and of the scattering amplitude on the physical sheet. In addition, (34) and (36) yield a symmetry theorem, proved in Ref. 28 for analytic potentials with asymptotic behavior

$$V(r) \to \mathrm{const}\; r^\beta \exp(-\alpha r),\ \beta \geqslant -2,\ \alpha > 0. \quad (37)$$

Theorem. For potentials of the type (37), the points situated on the intersections of the trajectories $\lambda = -ip = \lambda(\alpha)$ of the bound levels ($\lambda_n = \sqrt{2\mu|E_n|}$) and of the virtual levels ($\lambda_m = -\sqrt{2\mu|E_m|}$) with the lines of the dynamical singularities of the partial-wave amplitudes $\lambda = \pm\nu\alpha/2 (\nu = 1,2,\ldots)$ are reflection-symmetric with respect to the point $p = 0$.

Here, E_k is the energy eigenvalue for the principal quantum number k. The dynamical singularities—the left-hand cuts and false poles (for the exponential potential)—are discussed, for example, in Refs. 4 and 49. If the potential is truncated at a finite distance, all the dynamical singularities recede to infinity, and the symmetry theorem no longer holds. The condition $\beta \geqslant -2$ is needed because only then does the partial-wave scattering amplitude (in the Born approximation) become infinite at the point of the dynamical singularity. Accordingly, the residue of the t matrix vanishes at the pole in accordance with (36), and the correction in (34), which distinguishes the equations on the physical and the unphysical sheets, disappears. From this the symmetry theorem follows, and the vanishing of the residue agrees with the well-known symmetry between the zeros and poles of the S matrix situated at the points p and $-p$. The validity of the symmetry theorem can be verified for the examples of an exponential potential and the Hulthén potential, for which the s-wave solution is known in analytic form (see, for example, Refs. 6 and 127). A literal proof of this theorem is given for the Woods–Saxon potential in Ref. 128. This shows that it can also be extended to potentials that are irregular in the region $\mathrm{Re}\,r > 0$, i.e., do not belong to the class of analytic

(Yukawa) potentials, for which the ABC theorem admits a rotation through an angle $\theta < \pi/2$ by virtue of the condition (12a) (see the Introduction).

Trajectories of poles of the S matrix

Until recently, resonance pole trajectories, i.e., movement of the poles with variation of the interaction strength (or of some other parameter), were known only for potentials that permit an analytic solution. These include the rectangular well (Figs. 3 and 4)[124,129] and separable potentials. The first calculations of the virtual state of the np system in the $^1S_0$ state ("singlet deuteron") were apparently made in Ref. 29. They were based on analytic continuation of the t matrix found by the Bateman separation procedure for series of local NN potentials.

Poles for bound states are always simple (see, for example, Ref. 5), whereas virtual states may have multiple poles. This happens for potentials that have a singularity at $\mathrm{Re}\, r > 0$. The Woods–Saxon potential is an example.[128,130] For potentials truncated at finite distance, when $S(p)$ is represented as a rational function (i.e., for potentials of Bargmann type), the existence of multiple virtual poles was proved in Ref. 131. The trajectories of virtual poles that are "nonoccurring" bound states for the Woods–Saxon potential are shown in Fig. 5. In Refs. 132 and 133, the trajectories of resonance and virtual poles were calculated for the Yukawa potential by means of the integral-equation method and by summation of divergent perturbation-theory series (see Sec. 1). It was shown that the two methods give results that agree to a high accuracy in a wide range of variation of the parameters. The trajectories of the virtual poles for the Hulthén and Yukawa potentials are shown in Figs. 6 and 7.

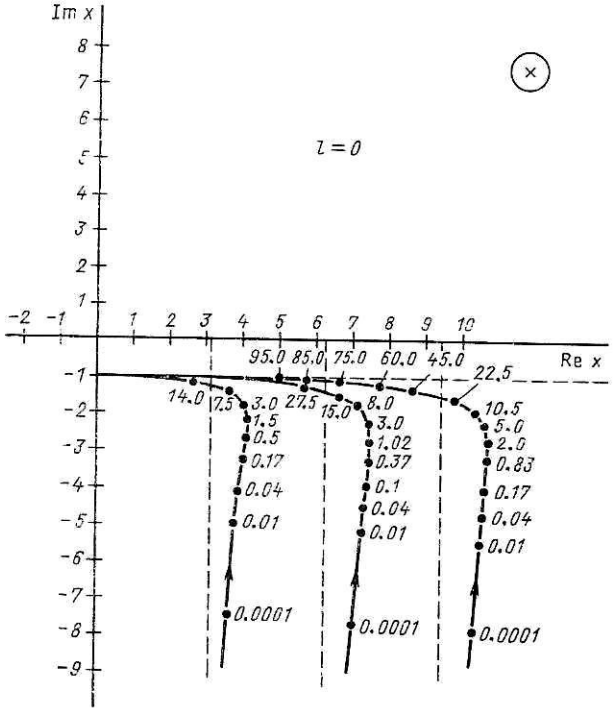

FIG. 3. Pole trajectories in the complex plane of $x = kR = \sqrt{2\mu E/\hbar^2}R$ for a potential in the form of a rectangular well. The numbers next to the points are the interaction strengths $W_0 = \sqrt{2\mu V_0/\hbar^2}R$. The figure is for the s-wave case ($l = 0$). The point of multiplicity of poles is $x = -i$.

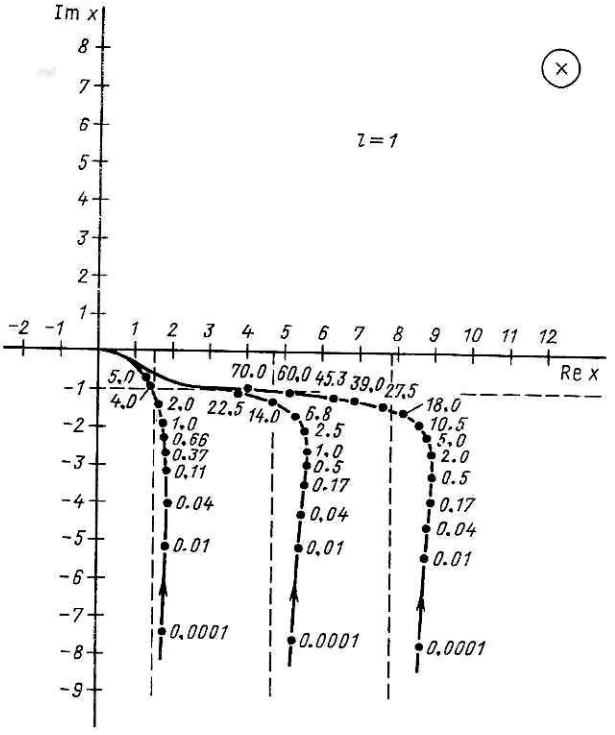

FIG. 4. The same as in Fig. 3 for the p wave ($l = 1$).

One observes a strong difference between the behaviors of these trajectories below the line of the dynamical singularity $\lambda = -ip = -\alpha/2$. For a potential of the type (37) with $\beta = 0$, the trajectories of the virtual poles intersect the lines of the dynamical singularities (false poles) $\lambda = -n\alpha/2$ (n is an integer) at the symmetry points. For the Yukawa potential ($\beta = -1$) the trajectories approach the symmetry points from the left as α is increased. It follows from this that the verv nature of the behavior of the trajectories in the region below the line of the dynamical singularity ($\lambda = -\alpha/2$) nearest the physical region dependsstrongly on the type of asymptotic behavior. All the considered trajectories demonstrate the validity of the symmetry theorem.

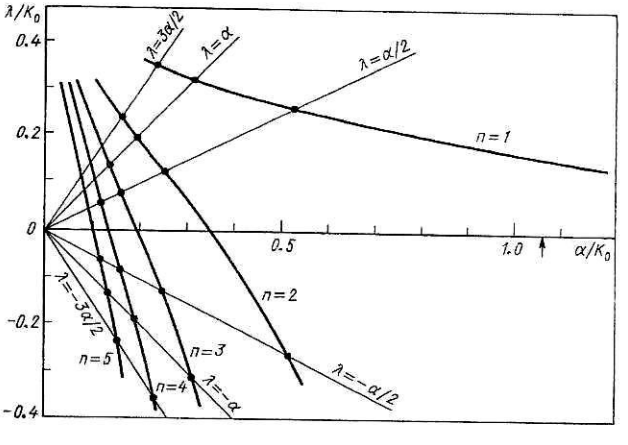

FIG. 5. Trajectories of ns levels with principal quantum numbers $n = 1$, ..., 5 (thick continuous curves) and positions of the dynamical singularities of the partial-wave scattering amplitude (thin continuous lines) (the black circles represent symmetrically situated points) for the Woods–Saxon potential $V(r) = -V_0/[1 + \exp(\alpha(r - R))]$, $k_0 = \sqrt{2\mu V_0/\hbar^2}$.

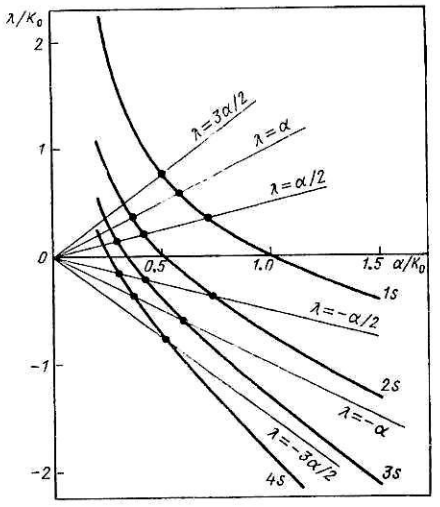

FIG. 6. Trajectories of ns levels for the Hulthén potential $V(r) = -V_0/[\exp(\alpha r) - 1]$. The notation is the same as in Fig. 5.

Trajectories for the exponential potential are given in Ref. 128.

Normalization of the Gamow wave function

The wave function $\varphi_l(q)$ is connected to the vertex function $g_l(q)$ by the well-known relation (with normalization as in Ref. 134)

$$\varphi_l(q) = -g_l(q)/(q^2 - p^2), \quad p^2 = 2\mu z. \tag{38}$$

The relation (36) determines the normalization constant $g_l(p)$ in terms of the partial-wave scattering amplitude on the physical sheet. However, one can directly generalize the rule for normalizing the wave function of a bound state to the case of a "nonoccurring" bound state. The Fourier transform of the wave function of a bound state has the form

$$\psi_l(r) = C\int_0^\infty j_l(qr)\,\varphi_l(q)\,q^2\,dq, \tag{39}$$

where C is a constant.

Using the relations

$$j_l(x) = [h_l^{(1)}(x) + h_l^{(2)}(x)]/2, \qquad h_l^{(2)}(-x) = (-1)^l h_l^{(1)}(x), \tag{40}$$

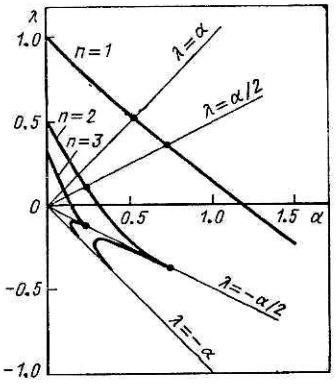

FIG. 7. Trajectories of ns levels for the Yukawa potential $V(r) = -r^{-1}\exp(-\alpha r)$, $\lambda = \sqrt{-2E_{(n)}}$. Atomic units ($\hbar = m_e = e = 1$) have been used, and the remaining notation is as in Fig. 5.

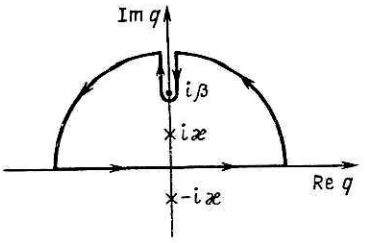

FIG. 8. Contour of integration for calculation of the Fourier transform of a bound-state wave function.

and also (34) and (38), we can rewrite (39) in the form

$$\psi_l(r) = (1/2)C\int_{-\infty}^{\infty} h_l^{(1)}(qr)\,\varphi_l(q)\,q^2\,dq. \tag{41}$$

The poles of $\varphi_l(q)$ at the points $q = \pm p$ determine the asymptotic behavior of the radial wave function if the singularities of $g_l(q)$ are situated further from the point $q = 0$. The generalization of the Fourier transform of the wave function of a bound state to the case of a Gamow state (a "nonoccurring" bound state) reduces to a contour deformation (Figs. 8 and 9) that ensures convgrgence of the integrals (42). The contours for the virtual and resonance levels are different (Figs. 9a and 9b) because of the presence of a centrifugal potential when $l \neq 0$. As a result, we obtain in place of (41) the contour integral (we omit the index l)[37]

$$\psi_p(r) = (C/2)\int_\Gamma h_l^{(1)}(qr)\,\frac{g_l(q)}{q^2 - p^2}\,q^2\,dq. \tag{42}$$

The asymptotic behavior is now determined by the residue at the point $q = -p$, which is situated in the lower half-plane, as a result of which we obtain Gamow growth at infinity:

$$|\psi_p(r)|_{r\to\infty} \simeq \text{const}\; r^{-1}\exp(|\operatorname{Im} p|\, r). \tag{43}$$

Any matrix element containing a Gamow wave function can be transformed similarly. In place of the matrix element $\mathcal{F} = \int_0^\infty \rho(q)dq$ we obtain

$$\mathcal{F} = \mathcal{F} + i\pi[\operatorname{Res}\rho(q)|_{q=-p} - \operatorname{Res}\rho(q)|_{q=p}]. \tag{44}$$

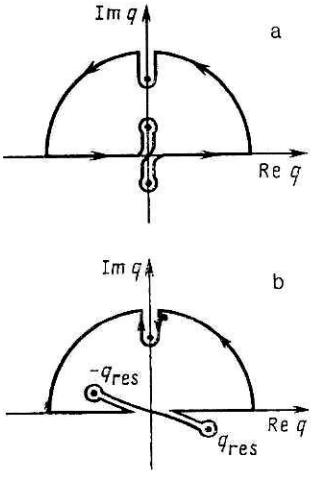

FIG. 9. Contour of integration for calculation of a matrix element containing a Gamow wave function for a virtual state (a) and a resonance state (b).

For an even function $\rho(q)$ we obtain

$$\mathcal{F} = \mathcal{F} + 2i\pi \operatorname{Res} \rho(q)|_{q=p_{res}}. \quad (45)$$

The relation (45) also holds for the normalization of a Gamow wave function when

$$\rho(q) = q^2 g_l^2(q)/(q^2 - p^2)^2, \quad (46)$$

and in this case an integral equation of the type (34) for the vertex function must be used to calculate the residue. A normalization of a Gamow wave function was also given in Ref. 135, in which the technique of orthogonal projection was used to investigate resonances. The proposal (45) is universal, whereas the method of regularization of the normalization integral in the r space depends on the position of the pole. For example, Zel'dovich's[136] regularization procedure, in which a factor $\exp(-\beta r^2)$ is introduced in the integrand and the limit $\beta \to 0$ is taken after calculation of the integral, is valid only under the condition $|\operatorname{Re} p_{res}| > |\operatorname{Im} p_{res}|$.[125]

Multichannel problem. Strong coupling of channels

The rule for analytic continuation of the Lippmann–Schwinger equation can be readily generalized to the system of integral equations of the so-called method of strong channel coupling for the matrix elements of the T operator:

$$T(k_{\alpha'}, k_\alpha; z) = V(k_{\alpha'}, k_\alpha) - 4\pi \times \sum_{\alpha''=1}^{n} \int_0^\infty \frac{V(k_{\alpha'}, k_{\alpha''}) T(k_{\alpha''}, k_\alpha; z)}{k_{\alpha''}^2/2\mu_{\alpha''} - (z - Q_{\alpha''})} k_{\alpha''}^2 dk_{\alpha''}. \quad (47)$$

Here, $Q_\alpha = m_\alpha - m_1$ is the threshold of the $(1, \alpha)$ reaction, m_α is the total mass in channel α, and the channel of the lightest particles has index 1. In the method of strong channel coupling [4,5,137] only binary channels are considered. On the energy shell

$$k_\alpha^2/2\mu_\alpha = z - Q_\alpha, \quad (48)$$

where μ_α is the reduced mass, and z is the energy.

With respect to the variable z, the function $T_{\alpha'\alpha}$ has, in accordance with (47), n cuts, which begin at the thresholds $z = Q_\alpha$ and lie along the real axis of z; accordingly, there are 2^n sheets of the Riemann surface. For brevity, we have in (47) omitted the channel indices appended to the operators V and T (these indices also include the angular-momentum quantum numbers). Because the cuts in (47) are additive, one can pass to the unphysical sheet with respect to any one of them. We denote by $\{\beta\}$ the set of channels with respect to which the system (47) must be analytically continued to the unphysical sheets. By means of matrix algebra we find that the result of such continuation is equivalent to replacement of the potential matrix $V_{\alpha'\alpha}$ by the expression[37]

$$\tilde{V}_{\alpha'\alpha}(k_{\alpha'}, k_\alpha; z) = V_{\alpha'\alpha}(k_{\alpha'}, k_\alpha) + i8\pi^2 \sum_{\beta\beta'} \mu_{\beta'} p_{\beta'} V_{\alpha'\beta'}(k_{\alpha'}, p_{\beta'}) W^{-1}_{\beta'\beta} V_{\beta\alpha}(p_\beta, k_\alpha), \quad (49)$$

where the matrix W^{-1} is the inverse of W:

$$W_{\beta'\beta} = \delta_{\beta'\beta} - i8\pi^2 \mu_\beta p_\beta V(p_{\beta'}, p_\beta), \qquad p_\beta = \sqrt{2\mu_\beta (z - Q_\beta)}. \quad (50)$$

As in the single-channel case, the addition to the potential in (49) is nonlocal (it has the form of a sum of separable terms) and depends on the energy z. The restriction to binary channels is not fundamental, as is evident from examination of the Faddeev equations (see below). In the region of z values in which the matrix W is singular (degenerate), i.e., $\det W = 0$, it is better to use the system of equations that generalizes to the multichannel case the system (17)–(18).

4. THREE-PARTICLE THEORY OF RESONANCE STATES

Faddeev equations. Two-particle cut

The method just described for analytic continuation of the system of integral equations for coupled binary channels can be extended to the three-body problem without particular difficulty.[37] We shall illustrate it for the simple example of the Faddeev equation for three identical spinless particles whose interaction is described by the separable potential

$$\langle \mathbf{p}' | V | \mathbf{p}'' \rangle = \lambda \chi(\mathbf{p}') \chi^*(\mathbf{p}''). \quad (51)$$

The problem of the scattering of a particle by a bound subsystem reduces to solution of a one-dimensional integral equation for the partial-wave amplitude $X_L(q,q';E+i0)$ (see, for example, Ref. 23):

$$X_L(q, q'; E+i0) = 2Z_L(q, q'; E+i0) + 8\pi \int_0^\infty q''^2 dq'' \times \left[Z_L(q, q''; E+i0) \tau\left(E + i0 - \frac{3}{4}\frac{q''^2}{m}\right) X_L(q'', q'; E+i0) \right]. \quad (52)$$

Here and in what follows L is the total angular momentum of the system, and

$$Z_L(q, q'; E+i0) = (1/2) \int_{-1}^{1} dy P_L(y) Z(\mathbf{q}, \mathbf{q}'; E+i0); \quad (53)$$

$$y = \mathbf{q}\mathbf{q}'/qq';$$

$$Z(\mathbf{q}, \mathbf{q}'; E+i0) = \frac{\chi^*(\mathbf{q}' + \mathbf{q}/2) \chi(-\mathbf{q}'/2 - \mathbf{q})}{E + i0 - [q^2 + (\mathbf{q} + \mathbf{q}')^2 + q'^2]/2m}. \quad (54)$$

Equation (52) is similar to the Lippmann–Schwinger equation, with Z_L playing the part of the potential, and $\tau(z_2)$ that of the propagator. The function $\tau(z_2)$, whose argument is the subsystem energy

$$z_2(q) = E - 3q^2/4m, \quad (55)$$

is connected to the t matrix by the simple relation

$$\langle \mathbf{p}' | t(z_2) | \mathbf{p} \rangle = \chi(\mathbf{p}') \tau(z_2) \chi^*(\mathbf{p}), \quad (56a)$$

$$\tau(z_2) = \left[\lambda^{-1} - \int d\mathbf{p} \frac{\chi(\mathbf{p}) \chi^*(\mathbf{p})}{z_2 - p^2/m} \right]^{-1}. \quad (56b)$$

The symbol of complex conjugation in (56) applies only to the spherical function [see the comments on Eq. (30)]. Suppose that the subsystem has one bound s-wave state. Then $\tau(z_2)$ has a pole at $z_2 = -\varepsilon_b$ (ε_b is the binding energy):

$$\tau(z_2) = \{(z_2 + \varepsilon_b) C(z_2)\}^{-1}, \quad (57)$$

where

$$C(z_2) = -\int d\mathbf{p} \frac{\chi(\mathbf{p}) \chi^*(\mathbf{p})}{(z - p^2/m)(\varepsilon_b + p^2/m)}. \quad (58)$$

Thus, the pole of the function $\tau(E + i0 - 3q''^2/4m)$, which

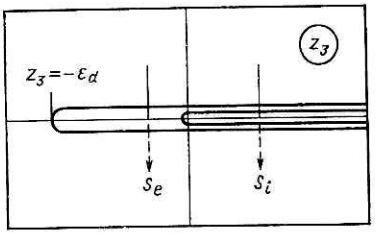

FIG. 10. Two- and three-particle cuts in the plane of the complex energy $z_3 \equiv E$ of the nd system ($\varepsilon_b \equiv \varepsilon_d$).

occurs in the kernel of Eq. (52), induces the two-particle propagator

$$\frac{1}{z_2(q'')+\varepsilon_b}=\frac{1}{E+i0-3q''^2/4m+\varepsilon_b}=\frac{1}{E_{12,3}+i0-3q''^2/4m}, \quad (59)$$

where $E_{12,3}=E+\varepsilon_b$ is the energy of the motion of particle 3 relative to the center of mass of the bound pair $\{12\}$ and, accordingly, a two-particle right-hand cut that begins at $E=-\varepsilon_b$ (Fig. 10) is also induced. It is obvious that the pole term can always be separated from the t matrix (Fig. 11, A). The separability of the potential is used merely to express the t matrix in closed form.

Separating the two-particle propagator in Eq. (52) and introducing the "potential"

$$V_L(q, q'; E) = 2Z_L(q, q'; E)\, C^{-1}(z_2(q')), \quad (60)$$

we obtain an equation of Lippman–Schwinger type (there is a difference in the inhomogeneous term and in the explicit dependence of V_L on E) ($\mathrm{Im}\sqrt{E+\varepsilon_b}>0$):

$$X_L(q, q'; E)=2Z_L(q, q'; E)+4\pi \times \int_0^\infty \frac{V_L(q, q''; E)\, X_L(q'', q'; E)}{E+\varepsilon_b-q''^2/2\mu}\, q''^2\, dq''. \quad (61)$$

Here, $\mu=2m/3$. Continuing (61) analytically into the region s_e (Fig. 10) with respect to the variable $E_{12,3}=E+\varepsilon_b$ as in the case of the Lippman–Schwinger equation, we obtain the following equation on the unphysical sheet of $z_3 \equiv E$:

$$X_L^{(-)}(q, q'; E)=2Z_L(q, q'; E)+4\pi \times \int_0^\infty \frac{V_L(q, q''; E)\, X_L^{(-)}(q'', q'; E)}{E+\varepsilon_b-q''^2/2\mu}\, q''^2\, dq'' + i8\pi^2\mu p V_L(q, p)\, X_L^{(-)}(p, q'; E); \quad p=\sqrt{2\mu(E+\varepsilon_b)}. \quad (62)$$

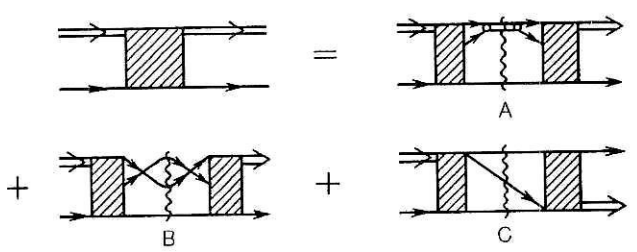

FIG. 11. Diagrams for scattering of a particle by a two-particle bound subsystem that illustrate the appearance of the two-particle cut (A) and the three-particle cuts corresponding to the direct (B) and crossed (C) terms in the unitarity relation.

We emphasize that Z_L and V_L do no have a two-particle right-hand cut. Expressing $X_L^{(-)}(p,q';E)$ by means of (62), we obtain an equation in the same form as (52) but with new inhomogeneous terms

$$\tilde{Z}_L(q, q'; E) = Z_L(q, q'; E) + F(q, p; E)\, Z_L(p, q'; E) \quad (63)$$

and the "potential"

$$\tilde{V}_L(q, q'; E) = V_L(q, q'; E) + F(q, p; E)\, V_L(p, q'; E), \quad (64)$$

where

$$F(q, p; E)=\frac{i8\pi^2\mu p V_L(q, p; E)}{1-i8\pi^2\mu p V_L(p, p; E)}. \quad (65)$$

For the same reasons as for the Lippmann–Schwinger equation, the denominator in (65) is not a pole of the S matrix. However, in the neighborhood of the zero of the denominator of (65) it is expedient to use the system of equations from (62), augmented by Eq. (62) at the point $q=p$. The eigenvalue problem for the kernel of the Faddeev equation corresponding to such a system was also considered in Ref. 3 for the special case of a three-particle system with a Yamaguchi potential. The function $V_L(p, p; E)$ has a logarithmic cut, which connects in pairs the branch points $p_1^{(\pm)}=\pm i(2/3)\sqrt{m\varepsilon_b}$ and $p_2^{(\pm)}=\pm i2\sqrt{m\varepsilon_b}$ for a system of particles of equal mass m [see (68)]. Near the singularities of $V_L(p,p;E)$ it is preferable to use the equation with the potential (64), since the large-valued $V_L(p,p;E)$ occurs in the denominator of (65). Thus, choosing the formulation of the Faddeev integral equations on the unphysical sheet that is most appropriate for the E under consideration, we can use the standard methods of solution known for bound-state problems. At the point of the corresponding pole, $E=E_n$, we obtain a homogeneous equation or a system of equations for the vertex function of the decay $\{123\}\to\{12\}+3$, as in the case of the two-body problem, and, therefore, we obtain the same for the wave function of a resonance (or virtual) state, which can also be normalized with allowance for the modification considered above or by means of the T matrix.

As in the two-body problem, the analytic properties of the solution of the three-body homogeneous equation are determined by the properties of the kernel of the integral equation, i.e., the "potential" (60). By analogy with the two-body potential problem (see Sec. 3), we solve Eq. (62) iteratively and sum the resulting series. We find the relation

$$X_L^{(-)}(q, q'; E) = X_L(q, q'; E) + i8\pi^2\mu p C^{-1}(z(p))\, X_L(q, p; E)\, X_L^{(-)}(p, q'; E). \quad (66)$$

Since $z_2(p)=-\varepsilon_b$ [see (55) and (62)], and from (58) we have $C(-\varepsilon_b)=1$ in our normalization, we arrive at an expression analogous to (31), which is a generalization of the well-known relation for the partial-wave scattering amplitude associated with the condition of unitarity of the S matrix for a binary channel, and we also obtain the same relationship between the scattering amplitudes (but now in the three-body problem) on the physical and unphysical sheets with respect to the channel $3+\{12\}$. The relations (32), (33), (35), and (36) can also be directly generalized to the case of a binary channel in a three-body problem. In

contrast to a two-particle potential, for which the singularities of the Fourier transform depend on the particular model (and recede to infinity when the potential is truncated), the nearest singularities of the exchange "potential" $Z_L(q, q'';E)$ and of the partial-wave amplitude $X_L^{(-)}(q, q'';E)$ are determined by real physical properties of the three-body system that are independent of the specific form of the short-range two-body potential. This potential is always specified in such a way as to ensure, at the least, the correct value of the binding energy ε_b for the $\{12\}$ subsystem. But ε_b also determines the positions of the "dynamical" singularities of the potential $V_L(p, p; E)$ (see above). Therefore, the symmetry theorem, which can be generalized to the case of a three-body system for the binary channel $\{12\}+3$, reflects true dynamical properties that do not depend on the particular model of the two-body interaction.[128] The effective two-particle interaction has the asymptotic behavior (43) with $\beta = -2$.

Faddeev equations. Three-particle cut

The three-particle cut (see Fig. 10) is related to the three-particle propagator, which occurs both in $\tau(z_2)$ and in $Z_L(q, q'';E)$. The right-hand cut of the t matrix, which occurs in $\tau(z_2)$ [specifically in $C(z_2)$ in (58)], induces a three-particle cut, which corresponds to the so-called direct term in the unitarity relation (Fig. 11, B). The crossed term in the unitarity relation on the three-particle cut (Fig. 10) arises from the function $Z_L(q, q'';E)$, the projection onto the L th partial wave of the exchange potential of the interaction of the bound pair $\{12\}$ with the spectator particle 3, as shown in Fig. 12. In contrast to the two-body problem, the potential depends explicitly on the energy. Its singularities are determined (if we forget about the singularities of the vertex functions χ, which do not lead to a right-hand cut with respect to E) by $Q_L(y)$, a Legendre function of the second kind with argument

$$y = (mE - q^2 - q''^2)/qq''. \tag{67}$$

The function $Q_L(y)$ has a logarithmic cut along the real axis of y from -1 to $+1$, and this induces two cuts of $Z_L(q, q''; E)$ in the complex plane of q''. The positions of these cuts are determined from the condition of vanishing of the denominator in (54):

$$q'' = -qy/2 \pm \sqrt{q^2y^2/4 + B^2 + i0}, \tag{68}$$

where $B^2 = mE - q^2$.

The cuts connect the branch points $q_{\min}$ and $q_{\max}$ pairwise:

$$\left.\begin{aligned} q_{\min}^{(1)} &= -\frac{q}{2} + \sqrt{\frac{q^2}{4} + B^2 + i0}; \\ q_{\max}^{(1)} &= \frac{q}{2} + \sqrt{\frac{q^2}{4} + B^2 + i0}; \\ q_{\min}^{(2)} &= -\frac{q}{2} - \sqrt{\frac{q^2}{4} + B^2 + i0}; \\ q_{\max}^{(2)} &= \frac{q}{2} - \sqrt{\frac{q^2}{4} + B^2 + i0} \end{aligned}\right\} \tag{69}$$

allowance being made for the sign of the imaginary addition for real $E(>0)$. The shape and position of the cuts depend both on q and on E, and they are therefore called moving cuts. On transition with respect to E to a neighboring unphysical sheet s_i (see Fig. 10), the logarithmic cut of Z_L dips into the lower half-plane, deforming the contour of integration in (52), as is shown in Fig. 13 for the case when the points q and $\sqrt{mE}$ are on one ray, and $|q|^2 < |mE|$. As a result, the homogeneous equation for the eigenvalues of the kernel of the Faddeev equation acquires an additional term $S_n(q, E)$ (Ref. 37):

$$\left.\begin{aligned} &\lambda_n^{-1}(E)\, X_n(q, E) = R_n(q, E) + S_n(q, E); \\ &R_n(q, E) = 8\pi \int_0^\infty q''^2\, dq'' Z_L(q, q''; E)\, \tau^{(-)} \\ &\qquad \times \left(E - \frac{3}{4}\frac{q''^2}{m}\right) X_n(q'', E); \\ &S_n(q, E) = 8\pi \int_{C(q, E)} \tilde{q}^2\, d\tilde{q}\, \tau^{(-)} \\ &\qquad \times \left(E - \frac{3}{4}\frac{\tilde{q}^2}{m}\right) X_n(\tilde{q}, E)\, \mathrm{disc}\, Z_L(q, \tilde{q}; E). \end{aligned}\right\} \tag{70}$$

In the integrand, we must take $\tau^{(-)}$ instead of τ if we pass simultaneously to the unphysical sheet with respect to the square-root cut of the function $C(z_2)$ in (58). In Eq. (70), $X_n(q, E)$ is an eigenfunction of the kernel, and $C(q, E)$ is the region with nonzero discontinuity disc $Z_L(q, \tilde{q}; E) \neq 0$. The variables $\tilde{q}$ and q separate in disc Z_L, and, as a consequence, $S_n(q, E)$ acquires a factor that includes $\chi(\sqrt{mE - 3/4q^2})$, which depends explicitly on q and determines the far (for short-range forces) potential singularities. "Nonpotential" singularities of $S_n(q, E)$ arise if a subsystem has a virtual or resonance pole, i.e., $\tau^{(-)}(z_2) \sim (z_2 - \nu_b)^{-1}$. At the same time, $S_n(q, E)$ is a Cauchy-type integral whose logarithmic singularities correspond to coincidence of the pole $\tau^{(-)}(z_2)$ with one of the ends of the integration. The corresponding branch points

$$\left.\begin{aligned} q_{p1,2}^{(a)} &= \sqrt{m(E - \nu_p)/3} \pm \sqrt{m\nu_p}; \\ q_{p1,2}^{(b)} &= -\sqrt{m(E - \nu_p)/3} \pm \sqrt{m\nu_p} \end{aligned}\right\} \tag{71}$$

also correspond to the Legendre function $Q(y)$, where

$$y_p = [(4\nu_p - E)/3 - q^2/m]/2\,(q/\sqrt{m})\,\sqrt{(E - \nu_p)/3}, \tag{72}$$

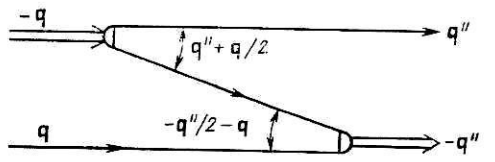

FIG. 12. Graph showing the exchange potential in the Faddeev equations.

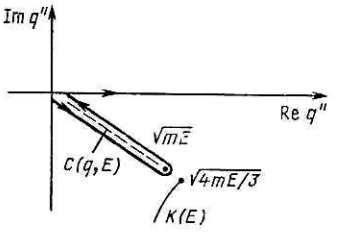

FIG. 13. Structure of the singularities with respect to q'' of the kernel of the Faddeev equation on the sheet s_i; $K(E)$ is the two-particle cut for the subsystem. The heavy line with the arrows is the contour of integration.

with cuts along lines $-1 \leqslant y_p \leqslant 1$,

$$q_{p1,2}(y_p) = -y_p \sqrt{m(E-\nu_p)/3} \pm [m(E-\nu_p)(y_p^2-1)/3 + m\nu_p]^{1/2}. \quad (73)$$

Finally, $S_n(q, E)$ has a square-root cut with respect to $z_2 = E - (3/4)(q^2/m)$, since $\sqrt{z_2}$ occurs in the limits of the integration along the curve $C(q, E)$. Thus, the form of the equations for the unstable states enables us to find the nearest singularities of the solution with respect to the variable q explicitly. Since the contour of integration must necessarily pass through the point $q = 0$, for which the cut $C(q, E)$ contracts into the point $\sqrt{mE}$, the wave function has the singularities of the function $S_n(\sqrt{mE}, E)$, i.e., the singularities with respect to E associated with the singularities with respect to q. Substituting $q_{p1}^{a,b} = \sqrt{mE}$ in (71), we obtain a logarithmic cut with respect to E for the first term in the expansion of $X(q, E)$ in powers of q, the cut running between the points $E = \nu_p$ and $E = 4\nu_p$ (Fig. 14). In addition, there is a pole at $E = 4\nu_p$ of the function $\tau^{(-)}(E/4)$, which occurs in $S_n(\sqrt{mE}; E)$. The following terms of the expansion in a power series in q will contain derivatives at the point $\tilde{q} = \sqrt{mE}$, i.e., the position of the singularities is not changed, but the singularity becomes stronger. Thus, the segment in the E plane between the points ν_p and $4\nu_p$ is a forbidden region for analytic continuation with respect to E. In addition, the solution has a cut that begins at the point $E = \nu_p$ and goes to the right, parallel to the real axis. This cut is analogous to the two-particle unitary cut situated on the physical sheet. The analytic structure of the singularities is shown in Figs. 14 and 15. For fixed E and varying q, the curve $C(q, E)$ wanders quite "capriciously" in the complex plane of q''. Therefore, the Faddeev equation on the sheet s_i cannot be solved by the usual determinant method, since the problem is not closed with respect to the set of quadrature points on the contour of integration. However, knowing the nearest singularities of the solution, we can use the analytic representation (β^{-1} is the range of the two-particle interaction)

$$X_n(q, E) = \sum_{m=1}^{\infty} \chi(p_1, \beta m)\left[a_m + \sum_p b_m^{(p)} Q_L(y_p)\right], \quad (74)$$

$$p_1 = \sqrt{mE - (3/4)q^2},$$

where Σ_p the sum over the poles. The expansion basis is constructed using the vertex function $\chi(p_1, \beta)$ [we note that $\chi(p_1, \beta) \equiv \chi(p_1)$], and is completely analogous to the basis that arises in the two-body problem in Martin's method.[49] In actual problems, the sum over m must be truncated in accordance with the required accuracy. Substitution of (74) in the Faddeev equation gives a system of linear equations for the unknown coefficients a_m and $b_m^{(p)}$, which can be solved in the usual way. Since the position of the three-particle pole with respect to E does not depend on q, the simplest procedure is to expand the left- and right-hand sides of the Faddeev equation in powers of q near $q = 0$. The number of terms of the expansion is determined by the number of unknown parameters a_m and $b_m^{(p)}$ for fixed a maximal number of terms in the sum over m.

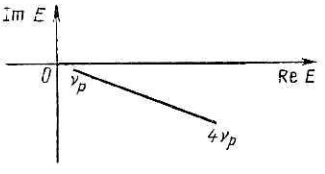

FIG. 14. Structure of the nearest singularities with respect to the energy for the solution of the eigenvalue problem for the kernel of the Faddeev equation in s_i that arise because of the term $S(q, E)$ in (70) for the case when the subsystem has a pole at the energy $z_2 = \nu_p$.

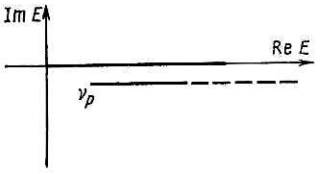

FIG. 15. The same as in Fig. 14 for a term $R(q, E)$ of ordinary type.

Thus, we evidently have a practical and fairly rigorous method for the study of three-particle resonances. It uses analytic continuation of the Faddeev integral equations directly into the region of the energies s_c and s_i that contain the corresponding poles of the S matrix. A more detailed exposition of the method is given in Refs. 37 and 126.

5. NUMERICAL SOLUTION OF THE FADDEEV EQUATIONS ON THE UNPHYSICAL SHEETS BY DEFORMATION OF THE CONTOUR OF INTEGRATION

Determination of the resonance poles

As an alternative to the formulation presented, the resonance poles of the T matrix for three particles can be calculated by the approach proposed by Möller and Glöckle.[31,33,128] This approach proceeds from the Faddeev equations (52) on the physical sheet and uses a deformation of the contour of integration. The deformation of the contour has two purposes in this case. First, the contour can be chosen in such a way that the integrand contains no rapidly varying structure, i.e., is as far as possible from the singularities of the kernel of the equation. Second, the deformation of the contour leads to a deformation of the cuts in the energy plane for the three-particle amplitude, thus "uncovering" part of the unphysical sheet. This means that the deformation of the contour enables us to pass to the unphysical sheet without changing the original form of the equations, in contrast to the approach presented above. In addition, the deformation of the contour of integration gives the values of the T matrix on the unphysical sheet not on the complete Riemann surface of the energy but only in a restricted region, which depends on the contour of integration. However, there always exists in principle a contour that enables one to reach a necessary point with respect to the energy, provided that at this point all the integrands, including the solution, are analytic. To explain the method, we proceed from the equations for the eigenvalues of the kernel of the Faddeev integral equation. We change the notation somewhat from (52). For a separable potential $V(p, p') = -\lambda g(p) g(p')$ we have ($m = 1$)

$$\lambda_{Ln}(z_3) F_{Ln}(q; z_3) = 4\pi \int_0^{\infty} k^2\, dk v_L(q, k; z_3)\, \tau \times \left(z_3 - \frac{3}{4}k^2\right) F_{Ln}(k; z_3), \quad (75)$$

where

$$v_L(q, k; z_3) = \int_{-1}^{1} dy \frac{g(p_1)\, g(p_2)\, P_L(y)}{[z_3 - (yqk + q^2 + k^2)]}; \quad (76)$$

$$p_1^2 = ykq + q^2 + k^2/4; \; p_2^2 = ykq + k^2 + q^2/4 \quad (77)$$

and

$$\tau(z_2) = -\lambda \left[1 + \lambda 4\pi \int_0^{\infty} \frac{k^2\, dk\, g^2(k)}{(z_2 - k^2)} \right]^{-1}. \quad (78)$$

In actual calculations the Yamaguchi potential with the form factor

$$g(p) = (p^2 + \beta^2)^{-1} \quad (79)$$

was used. Then the integrals in (76) and (78) can be calculated analytically. We obtain

$$v_L(q, k; z_3) = \frac{1}{\pi^2 q^3 k^3} \times \left[\frac{Q_L(y_3)}{(y_3 + y_1)(y_3 + y_2)} + \frac{Q_L(-y_1)}{(y_3 + y_1)(y_1 - y_2)} + \frac{Q_L(-y_2)}{(y_3 + y_2)(y_2 - y_1)} \right], \quad (80)$$

where

$$y_1 = \frac{\beta^2 + q^2 + k^2/4}{kq}; \quad y_2 = \frac{\beta^2 + k^2 + q^2/4}{kq}; \quad y_3 = \frac{z_3 - k^2 - q^2}{kq}. \quad (81)$$

For τ we have

$$\tau(z_2) = -\frac{1}{\beta^2 \pi^2} \frac{(\alpha + \beta)^2}{[1 - (\alpha + \beta)^2/(\beta - ip)^2]}, \quad (82)$$

where

$$p = \sqrt{z_2}; \quad \operatorname{Im} p > 0; \quad \varepsilon_d = \alpha^2.$$

As we have already noted, the analytic structure of the eigenvalues $\lambda_{Ln}(z_3)$ that is shown in Fig. 10 results in the existence of two unphysical sheets, which adjoin the physical sheet and which we denote by s_e and s_i. In what follows, we shall restrict ourselves to considering only the analytic continuation into s_i. The continuation into s_e can evidently be done without having to deform the contour. It has already been considered in Sec. 4 (see also Ref. 138). As we have already noted, the use of the method of contour deformation requires a careful analysis of the positions of singularities in the plane of integration. In the given case, we have the following singularities of the kernel of the equation in the complex plane of k ($-1 \leqslant y \leqslant 1$): the kinematic cuts (68)

$$k_{1,2}^{(k)} = -\frac{1}{2} yq \pm \sqrt{q^2 \left(\frac{1}{4} y^2 - 1 \right) + z_3}$$

and the dynamical cuts

$$k_{1,2}^{(A)} = 2\left(-yq \pm \sqrt{q^2 (y^2 - 1) - \beta^2} \right); \quad (83)$$

$$k_{1,2}^{(B)} = -\frac{1}{2} yq \pm \sqrt{\frac{q^2}{4} (y^2 - 1) - \beta^2}. \quad (84)$$

As we have already said, the singularities (68), (83), and (84) arise from the functions Q_L in (80). The function τ in (75) gives the following singularities: kinematic cuts

$$k_{1,2}^{(s)} = \pm \frac{2}{\sqrt{3}} \sqrt{z_3 - s}, \; 0 \leqslant s < \infty, \quad (85)$$

and two poles

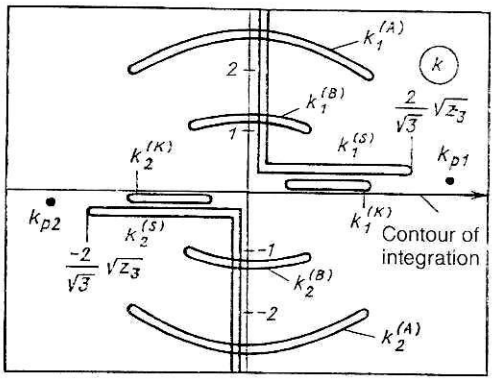

FIG. 16. Structure of the singularities of the kernel of the Faddeev eigenvalue equation in the plane of the complex momentum k (the variable of integration) for $z_3 = E + i\varepsilon (E, \varepsilon > 0)$, $0 < q < \sqrt{mE}$ for undeformed contour of integration (Yamaguchi potential).

$$k_{p1,2} = \pm \frac{2}{\sqrt{3}} \sqrt{z_3 + \alpha^2}, \; \alpha > 0. \quad (86)$$

The positions of the singularities for one value of q and $z_3 = E + i0$ are shown in Fig. 16. In a numerical solution of the integral equation (75), the variable q takes all the values that k takes. Thus, instead of individual cuts an entire region of singularities prevents solution of the integral equations. This situation is shown in Fig. 17, from which it can be seen that direct proximity of the singularities to the contour of integration makes it possible to obtain reliable numerical results by applying the ordinary methods of numerical integration. In contrast to this situation, the deformed contour of integration shown in Fig. 18 gives results with great numerical accuracy.[31,33,124] The path of integration shown in Fig. 18 is suitable for investigating the positions of the resonance poles of the scattering amplitude only in a restricted region of the unphysical sheet. For other regions, it is necessary to use a different deformation of the contour.[33,124] The method works reliably in practice. If the contour of integration is appropriately chosen, it even permits analytic continuation with respect to cuts situated on the unphysical sheet s_i. This was shown by the example of calculation of the pole trajectories for the ^3n system in Ref. 139. In this case, as in a number of others, the analytic continuation requires deformation of the contour not only in (75) but also in the inte-

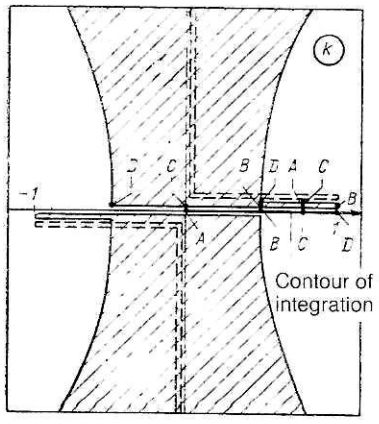

FIG. 17. Singularities of the kernel of Eq. (83) for $z_3 = E + i\varepsilon$ ($E, \varepsilon > 0$) for the undeformed contour of integration (Yamaguchi potential).

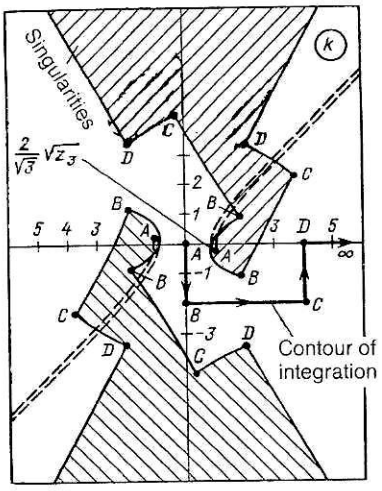

FIG. 18. Positions of singularities for the deformed contour of integration shown in the figure.

gral (78), i.e., for the two-particle T matrix, as we have already discussed in Sec. 4.

Vertex functions and coupling constants

The vertex functions and coupling constants for the decay of Gamow states of two- and three-particle systems can be calculated in the method that uses contour deformation in complete analogy with the calculations for bound states. The only difference from the formalism presented, for example, in Ref. 134 is in the replacement of the contour of integration along the real axis of the momentum by a deformed contour. Fourier transformation and the normalization of the wave function in the momentum representation are generalized similarly.

As we have already noted in the Introduction, the vertex functions for the decay of unstable systems can be used for a separable representation of the t matrix that is particularly helpful at energies near the corresponding pole of the t matrix when a restriction is made to a single separable term. A separable expansion of the t matrix in Hilbert–Schmidt form is often used in the literature. For the scattering of two particles,

$$t(\mathbf{q}, \mathbf{q}'; z_2) = \sum_{n=1}^{\infty} \frac{\lambda_n(z_2)}{1-\lambda_n(z_2)} F_n(\mathbf{q}, z_2) F_n(\mathbf{q}', z_2), \qquad (87)$$

where the eigenfunction F_n is a normalized solution of Eq. (22). For a three-particle system, for example, for a process with breakup (Fig. 19), we have

$$T(\mathbf{f}, \mathbf{k}; \mathbf{k}_0; z_3) = \sum_{\alpha} [1-\lambda_\alpha(z_3)]^{-1} \tilde{F}_\alpha(\mathbf{k}_0; z_3) F_\alpha(\mathbf{f}, \mathbf{k}; z_3). \qquad (88)$$

The index α represents the set of conserved quantum numbers, $F_\alpha(\mathbf{f}, \mathbf{k}; z_3)$ is an eigenfunction of the kernel of the Faddeev equation and is proportional to the vertex function of decay of the state $|\alpha\rangle$ into three free particles, and $F_\alpha(\mathbf{k}_0; z_3)$ corresponds to the vertex function of decay into the bound subsystem $\{12\}$ and particle 3. Bearing in mind the definition of the vertex function (30) and the fact that near the pole at $z = z_n$

$$\lambda_n(z) = 1 + (z - z_n)(d\lambda_n/dz)_{z=zn} + \ldots, \qquad (89)$$

we obtain the following relation for the two-particle problem:

$$g^{(n)}(\mathbf{q}) = C_n F_n(\mathbf{q}, z_n), \qquad (90)$$

where

$$C_n = (-d\lambda_n/dz)^{1/2}_{z=z_n}.$$

An analogous relationship between the vertex functions and eigenfunctions is also obtained for both stable and unstable three-particle states. The Hilbert–Schmidt procedure was first used in scattering problems by Weinberg.[140] The further name of the method—the quasiparticle method—arose because Weinberg called all poles of the S matrix, including bound, antibound (virtual), and resonance states, quasiparticles. Details of the application of the Hilbert–Schmidt expansion can be found in Refs. 141–147. To conclude this section, we note that some studies with calculations of vertex functions and coupling constants for the three-nucleon system were not included in the review of Ref. 134 (many of them were published later). Among these are the studies of Refs. 29 and 148–157, which include calculations with "realistic" NN potentials. We recall that the coupling constants have the status of independent physical quantities that determine the asymptotic behavior of the coordinate wave function. They can be extracted from analysis of experiments. For example, in Ref. 158 the vertex function and constant for the virtual decay $^3\mathrm{He} \to p + d$ were obtained from analysis of the reaction $^3\mathrm{He} + p \to p + p + d$ at ^{3}He momenta 2.5 and 5 GeV/c.

6. NUMERICAL RESULTS FOR NUCLEON SYSTEMS

Virtual two-nucleon state

It is well known that the np system can be either in a bound $^3S_1$ state (deuteron) or in a virtual $^1S_0$ state ("singlet deuteron"). Because of the particle identity, the corresponding nn and pp systems can be only in the $^1S_0$ state. Table II gives values for the energy E_s of the singlet deuteron for different NN potentials. Note that the rather large value of $|E_s|$ for the Reid potential arises because the scattering

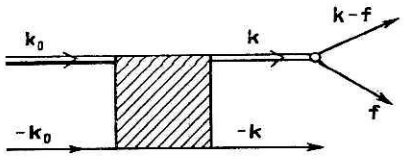

FIG. 19. Diagram for a process with disintegration.

TABLE II. Pole values of the energy of a singlet pair of nucleons for different potentials.

Potential	$-E_s$, keV	Reference
Darewich–Green	78	[29]
Malfliet–Tjon	71	[29]
Bressel–Kerman–Ruben (modified)	81	[29]
Reid	122	[36]
Yukawa	69	[133]

TABLE III. Review of theoretical results on the study of resonances in the ^3n system.

Method	Quantum numbers	Resonance, MeV	Reference
5) Solution of Faddeev equation with separable potential, $\lambda < 1$, ^3P interaction + terms containing $\mathbf{L}\cdot\mathbf{S}$ and $(\mathbf{L}\cdot\mathbf{S})^2$	L, S, J = 1; 3/2; 1/2	A bound state exists	[161—163]
Variational calculus, local potential, systematics for separation energy of last neutron, $\lambda \approx 2$	L, S, J = 1; 3/2; 1/2	—	[164]
Solution of Euler–Lagrange equation, $\lambda \approx 5$	L, S, J = 1; 3/2; 1/2	—	[165]
Solution of Schrödinger equation with resonance boundary conditions, method of K harmonics,[167] potentials: rectangular well, exponential, Yukawa, $\lambda = 2$	L, S, J = 1; 3/2; 1/2	—	[166]
Calculation of cross section for the $^3\mathrm{H}(n,p)3n$ reaction		Cross section for formation of ^3n below experimental sensitivity	[168, 169]
Investigation of $^3\mathrm{He}(\pi^-,\pi^+)3n$ reaction, Faddeev equation. Deviation from phase space interpreted as nn interaction in the final state	—	—	[64]
Special model for calculation of resonances, variational calculus, potentials: Afnan–Tang,[171] Eikemaier–Hakenbroich[172]	T, S, L = 3/2; 1/2; 1	$E = 2.1$; $\Gamma = 1.6$ $E = 3.5$; $\Gamma = 2.7$	[170]
K matrix for three particles, calculation of eigenvalues of K matrix	T, S, L = 3/2; 1/2; 1	$E = 6.1$	[173, 174]
Faddeev equation, calculation of eigenvalues of kernel of Faddeev equation for real energies	$L = 0$; 1	—	[30]
1. Schrödinger equation, rectangular-well potential, first eigenvalue[173] 2. Skornyakov–Ter-Martirosyan integral equation,[19] zero-range potential[173]	T, S, L = 3/2; 1/2; 1	—	[175]
Faddeev equation, Yamaguchi potential (separable), pole trajectory for S matrix	T, S, L = 3/2; 1/2; 1	—	[31] [33]
Faddeev equation, Reid potential, $^2P_3$–$^3F_2$ interaction, $\lambda = 4$	T, J^π = 3/2; 3/2$^-$	—	[176]
Faddeev equation and variational calculus; comparison of the results showed that the differences is due to the difference of the potentials, and not of the approaches	L, S, J = 1; 1/2; 1/2	—	[177]
Faddeev equation, coordinate space, Reid potential with soft core[176]	$1/2 \leqslant J \leqslant 7/2$	—	[178]
Calculations with a potential that reproduces the binding energies of the nuclei ^{3}H, ^{3}He, ^{4}H, ^{4}He, ^{4}Li (Ref. 177)	L, S, J = 1; 1/2; 3/2	—	[179]

Continuation of TABLE III.

Method	Quantum numbers	Resonance, MeV	Reference
Lower bounds for the binding energy of the ^3n system for various potentials		No contradiction to existence of ^3n resonance	[180]
Faddeev equation, pole trajectories of S matrix	T, S, L^π = 3/2; 1/2; 1⁻	$E=(-88.0\pm$ $\pm i27.9)$ keV; $C_0^2=-0.05\pm i0.1$	[139]
Calculation of the spectrum E_ν from the $\mu^- + {}^3\mathrm{H}\to\nu+3n$ reaction	—	—	[181]

length chosen in Ref. 36 is much smaller than in the remaining calculations of Table II ($a_s \simeq -17$ F instead of $a_s \simeq -23$ F). This indicates that the parameters in Ref. 36 relate to the *nn*, and not the *np*, system, since in the literature (see, for example, Ref. 159) such a value for a_s is given for the *nn* system. The possibility of taking into account differences between a_s for the *nn* and *np* interactions in the Faddeev equations for ^{3}H is discussed in Ref. 160. It was noted in Sec. 1 that allowance for the Coulomb *pp* interaction shifts the virtual level from the imaginary axis to the right into the complex plane of the momentum p, as was first shown in Ref. 48. However, it must be said that it does not become a "true" resonance (quasistationary state), since $|\mathrm{Im}\, p/\mathrm{Re}\, p| > 1$:

$$\left.\begin{aligned} P_{^3\mathrm{He}} &= (0.0647 - i\,0.0870)\ \mathrm{F}^{-1}, \\ E_{^3\mathrm{He}} &= (-140 - i467)\ \mathrm{keV}. \end{aligned}\right\} \qquad (91)$$

These numbers were obtained from analysis of experimental data on *pp* scattering in the framework of the effective-range approximation, modified to take into account the Coulomb interaction. The position of the pole (91) is fairly close to the result for the Reid potential in Table II.

Unstable three-nucleon states

The main numerical results obtained in theoretical studies of resonances in the three-nucleon systems $3n$ and ^{3}H (without the Coulomb interaction) are given in Tables III and IV, respectively, and in the review papers of Refs. 57, 59, 60, 62, and 188. As regards unstable states in the ^{3}He and ^{3}Li systems, practically no work has yet been done on the basis of the Faddeev integral equations. In the Introduction we noted that allowance for the Coulomb interaction in even the bound-state problem requires modification of the theory either by the use of a screened Coulomb potential[39] (see also Ref. 162 and the references in it) or by means of equations in the configuration space.[42,43]

A modification is also required for the dispersion theory of nuclear reactions with charged particles, including the definition of vertex functions with allowance for the Coulomb interaction. A review devoted to Coulomb effects in nuclear reactions is given in Ref. 190 (see also Ref. 191). In Ref. 192, the *pd* system was treated in the framework of the N/D method, also with allowance for the Coulomb interaction, and it was shown that, as in the case of the *pp* interaction, the tritium virtual pole is displaced into the complex plane of the momentum. However, it should be noted that at the present level of accuracy achieved in the calculations of resonances, and in connection with the experimental situation, it is hardly necessary to treat the ^{3}He and ^{3}Li systems in the framework of complicated modified equations. One can use charge independence with subsequent allowance for a shift of the levels by the Coulomb energy. We therefore turn to discussion of the $3N$ system without the Coulomb interaction. Tables III and IV show that most theoretical studies conclude that there are no resonances in the ^3n and ^{3}He systems near the physical region. An exception is the pole for ^{3}He that we mentioned above, which corresponds to the ^{3}H virtual pole. The bound state of the ^3n system obtained by Mitra[161] was not confirmed in further studies. It is asserted in Ref. 188 that Mitra's calculations contain an error. Nevertheless, some authors (see, for example, Ref. 193) regard the existence of a localized (i.e., situated near the threshold at which all three neutrons have zero energy) trineutron as possible. The only general theoretical result is the virtual pole for the *nd* system with the quantum numbers of bound tritium near ($\sim$0.5 MeV) the threshold of the $n+d$ channel. The position of this pole depends on the form of the *NN* potential, but the actual existence of virtual tritium can be regarded as reliably established. It should be noted that observation of this state in an experiment is not easy, as the following arguments show:

1. If the virtual pole is to be observed in the physical region of energy near the $n+d$ threshold, the corresponding pole term in the Hilbert–Schmidt expansion of the *nd* amplitude must be dominant at low relative energies. But it was shown in Ref. 194 that the background contribution of the other terms in the Hilbert–Schmidt expansion is of the same order as the contribution of the pole term, having, moreover, the opposite sign, so that there is a rather accurate compensation. This fact is due to the small value of the residue at the virtual pole. Moreover, the compensation leads to a strong sensitivity of the *nd* doublet scattering length to the form of the *NN* potential.

2. In a number of studies (see, for example, Refs. 195–197) it is asserted that the effective-range expansion for the doublet S-wave phase shift of *nd* scattering, $^{(2)}\delta_0$, has a pole situated in the unphysical region near the threshold:

$$k\cot^{(2)}\delta_0(k) = -A + Bk^2 - C/(1+Dk^2). \qquad (92)$$

The expansion parameters ($A = 0.3105$ F^{-1}, $B = 0.85$ F, $C = 3.138$ F^{-1}, $D = 478.5$ F^2) obtained in Ref. 196 by fitting the data at low energies correspond to a virtual pole at energy -0.515 MeV if the energy is measured from the

TABLE IV. Review of theoretical results on the study of resonances in the ^{3}H system.

Method	Quantum numbers	Resonance, MeV	Reference
See Table III	$S=1/2;\ L=1$	—	[166]
Dispersion relations	$T,\ S,\ L=$ 1/2; 3/2; 1 1/2; 1/2; 1	$E=15-i5$ $E=18-i8$	[182]
See Table III, potentials, Afnan–Tang	$T,\ S,\ L=$ 3/2; 1/2; 1	$E=10.6;\ \Gamma=1.6$	[170]
Eikemaier–Hakenbroich	1/2; 3/2: 1 3/2; 1/2; 1 1/2; 3/2; 1	$E=9.3$ $E=12.0;\ \Gamma=2.7$ $E=9.3$	
Generalization of Jost function to the case of the three-body problem, analytic continuation of the Jost function to the complex plane of the energy, determination of zeros of the Jost function	$T,\ S,\ L=$ 1/2; 3/2; 1 1/2; 1/2; 0	$E=-(0.335+$ $+i0.28)$ * $E=-0.49$ * (virtual state)	[183]
Investigation of the ^{3}He(π^-,γ) ^{3}H and ^{3}He$(\pi^-,\pi^0)^3$H reactions	—	—	[184]
Faddeev equation, pole trajectories of the S matrix	$T,\ S,\ L=$ 1/2; 3/2; 1 1/2; 1/2; 0	$E=-(0.235+$ $+i0.93)$ $E=-0.355$ (virtual state) $C_0^2=0.1058$	[124, 185]
Determination of position of pole and residue for doublet nd scattering: Faddeev equation, N/D	$T,\ S,\ L=$ 1/2; 1/2; 0	$E=-0.482$ *; $C_0^2=0.0504$	[34]
Calculation of poles of T matrix in the complex plane of the energy by the method of Refs. 36 and 186	$T,\ S,\ L=$ 1/2; 1/2; 0	$E=-0.381$ *; $C_0^2=0.0589$	[36]
Formulation of Faddeev equation on unphysical sheet, calculation of poles of the T matrix, Malfliet–Tjon local potential	$T,\ S,\ L=$ 1/2; 1/2; 0	$E=-0.502$ *	[37]
Faddeev equation with three-nucleon forces	$T,\ S,\ L=$ 1/2; 1/2; 0	Three-particle forces raise somewhat the bound level and the virtual level	[187]

Note (on Tables III and IV). The quantity λ is the factor by which one must multiply the potential in order to obtain a three-particle system with zero binding energy, so that $\lambda > 1$ corresponds to an unbound state; the asterisk indicates that the energy of the level is measured from the two-particle threshold; C_0^2 is the residue at the resonance pole associated with the scattering amplitude (14): $\lim_{k\to k_{res}}[(k-k_{res})f_0(k)] = 3/2iC_0^2$.

elastic threshold. Since the scattering amplitude at low energies is $f(k) = [k\cot^{(2)}\delta_0(k) - ik]^{-1}$, the pole in the expression (92) corresponds to a zero of the amplitude $f(k)$. On the other hand, at the point of the virtual pole $f(k)$ becomes infinite. Thus, near the threshold $f(k)$ must vary strongly and, therefore, must depend on the details of the calculation.

3. The cross section for nd scattering at the threshold is given by the expression $\sigma_0 = (4/3)\pi\times[2^{(4)}a^2 + {}^{(2)}a^2]$. For the adopted experimental values of the quartet and doublet scattering lengths, $^{(4)}a = 6.35$ F and $^{(2)}a = 0.65$ F (Ref. 198), the first term in σ_0 is approximately 190 times greater than the second. Thus, it is difficult to detect a virtual pole when this cross section is measured, even if it strongly influences the value of $^{(2)}a$. To avoid this difficulty, it is necessary to make measurements with polarized particles. Thus, analysis of the theoretical calculations indicates that one can hardly expect observation of resonances in the $3N$ system, since the calculations of the trajectories by variation of the

dynamical parameters show that the poles of the S matrix (apart from the virtual pole for ^{3}H and the corresponding one for ^{3}He) are situated far from the physical region.

Correlation between the energy parameters in *nd* scattering

For more than 20 years there has been intense discussion in the literature on the connection between the *nd* doublet scattering length $^{(2)}a$ and the tritium binding energy. The dependence $E_T = f(^{(2)}a)$ is known as the Phillips[199] line, on which the results of calculations by means of the Faddeev equations using different *NN* potentials lie. The Phillips curve is close to a straight line near the experimental values of E_T and $^{(2)}a$.

In Ref. 200, the correlation between E_T and $^{(2)}a$ is interpreted in terms of the effective-range expansion. A similar question arises in connection with the relationship between $^{(2)}a$ and the energy E_v of the virtual level. The existence of such a connection already follows from Eq. (92). Finally, in the recent study of Ref. 201 a correlation between E_v and $^{(2)}a$ was obtained in the so-called minimal approximation for the three-nucleon system by solving the Skornyakov–Ter-Martirosyan equations truncated with respect to the momentum at the upper limit (the cutoff parameter q_c was regarded as a parameter). The resulting relation reproduces the correlation exhibited in Ref. 34. This result explains the weak dependence of the three-particle low-energy parameters on the particular model of the *NN* interaction, provided that the binding energies and *NN* scattering lengths are correctly reproduced in the doublet and triplet S states. The effective-range expansion for doublet *nd* scattering was also considered in Ref. 202.

Summarizing our discussion of the properties of *nd* scattering at low energy, we can conclude that these properties are determined mainly by the singularities which appear in the kernel of the Faddeev equation on account of the effective single-nucleon exchange potential, as discussed in Sec. 4 (see also Ref. 34). This circumstance evidently explains the successful application of the two-body approximation to the three-particle system.[203–206] Since the effective *nd* interaction that reflects the dynamics of the three-particle problem can be expressed in the form[205]

$$V(r) = -V_0 \frac{\exp\left(-\frac{r_0}{|a|}\beta r\right)}{\beta^2 (Cr_0^2 + r^2)} \qquad (93)$$

where a and r_0 are the scattering length and effective range in the two-body subsystem (with the greatest scattering length), it can be concluded that the symmetry theorem considered in Sec. 3 must also hold in the three-particle problem, the positions of the dynamical singularities being determined by the binding energy of the subsystem (see Sec. 4).

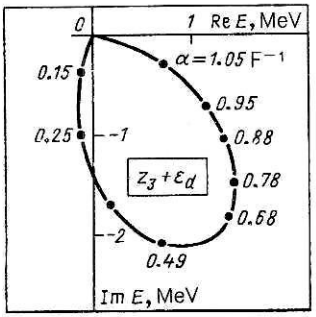

FIG. 20. Trajectory of first pole of the $3n$ system on the unphysical sheet s_e.

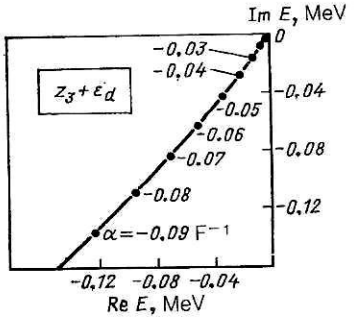

FIG. 21. Continuation of the pole trajectory shown in Fig. 20 to the unphysical sheet s_{ie}.

Trajectories of poles for three-nucleon systems

We give some examples of such trajectories for different parameter values of the *NN* interaction. The Faddeev equations with an s-wave single-term separable potential for the $3n$ system and for the quartet state of the *nnp* system are, apart from a spin factor ($-\frac{1}{2}$), identical to the one-dimensional integral equation for three spinless bosons. For the doublet *nnp* state there is a system of two equations.

The 3*n* system

The pole trajectory of the first pole (ground state) of the $3n$ system with the quantum numbers $(T, S, L^\pi) = (3/2, 1/2, 1^-)$ is shown in Figs. 20–22 (Refs. 32 and 33), in which $\alpha = \sqrt{m\varepsilon_d^*/\hbar^2}$. It can be seen from Fig. 20 that for $\alpha \geqslant 1.1$ F^{-1} the pole of the $3n$ system becomes bound. In the interval $0 < \alpha < 1.1$ F^{-1} the pole is on the unphysical sheet s_e. For $\alpha < 0$, the pole does not pass to the unphysical sheet s_i, as one would have expected, but passes directly from the threshold to the sheet that we denote by s_{ie} (Fig. 21). The sheet s_{ie} is associated with the elastic cut, which for $\alpha < 0$ is situated on the sheet s_i. For the real singlet interaction $\alpha = -0.04$ F^{-1} we obtain the pole position and scattering-amplitude residue given in Table III (Ref. 139):

$$E = (-88.0 \mp i27.9) \text{ keV},$$
$$C_0^2 = -0.05 \pm i0.1.$$

The *nnp* system

In this system there are not only singlet but also triplet parameters of the *NN* interaction. For the state $(T, S, L^\pi) = (1/2, 3/2, 1^-)$, the Faddeev equation can be ob-

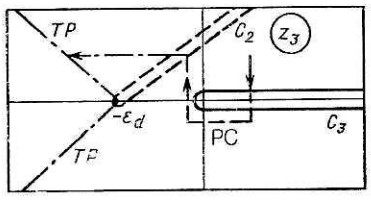

FIG. 22. Analytic continuation for the trajectory shown in Fig. 20 (*TP* is the trajectory, and PC is the path of analytic continuation).

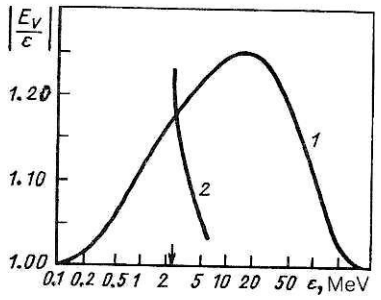

FIG. 23. Trajectory of the virtual pole E_v as a function of the binding energy ε in the $^1S_1$ state for the Yamaguchi potential (1) and the Malfliet–Tjon potential (2).

tained from the equation for the $3n$ system by simple replacement of the singlet parameters by the triplet parameters. Thus, from the trajectory in Fig. 20 it immediately follows that there is a pole at $E = (0.235 + i0.93)$ MeV (see Table IV, Ref. 124). It is difficult to detect this pole experimentally, since it is fairly far from the elastic threshold. In addition, since we are now dealing with a P state ($L = 1$), the corresponding partial amplitude for nd scattering vanishes at the threshold, and the presence of the pole cannot be clearly manifested.

The Faddeev integral equation for the state $(T, S, L^{\pi}) = (1/2, 1/2, 0^+)$, i.e., with the quantum numbers of ^{3}H, contains both a triplet and a singlet interaction. This means that complete investigations of the trajectories require variation of two parameters. So far as we know, there are no such calculations in the literature. Trajectories of the virtual level of ^{3}H as a function of the triplet parameter of the binding energy ε of the "deuteron," calculated for the Yamaguchi NN potential[208] and the local Malfliet–Tjon potential[209] (set of parameters I, III), are given in Refs. 186 and 207, respectively. The equations from Ref. 37 with the Bateman separation procedure for the Malfliet–Tjon potential (four separable terms) were used. The results of the calculations are compared in Fig. 23. It can be seen that in the region of the physical value of the triplet parameter, $\varepsilon = \varepsilon_d$, the two potentials give similar results. However, the behavior of the trajectories is different. In the case of the Malfliet–Tjon potential the trajectory behaves normally (i.e., the pole moves to the $n + d$ threshold with increasing attraction in the two-particle subsystem, approaching the region of the three-particle bound state), while in the case of the Yamaguchi potential the behavior is anomalous. The anomalous behavior of the trajectory in the region of small ε is due to the Efimov nature of the level. It must therefore also occur for the Malfliet–Tjon potential, but at smaller values of ε. The true value $\varepsilon = \varepsilon_d$ leads to the values of E_v given in Table IV (Refs. 37, 185, and 186) for the Malfliet–Tjon and Yamaguchi potentials.

The Efimov effect

Investigation of the pole trajectories reveals a pathological behavior as $\alpha \sim \varepsilon_{12}^{1/2} \to 0$, which is associated with the Efimov effect. In 1970, Efimov[26,27] showed that in the limit $a \to \infty$ (i.e., $\alpha \to 0$), where a is the scattering length in the two-body subsystem, the number of three-particle bound states also tends to infinity. The qualitative explanation is that when one particle moves in the field of a two-particle subsystem with binding energy near zero it is acted on by an effective long-range (of the type $1/r^2$) potential [see (93)] irrespective of the range of the interaction potential of the two particles. A very simple explanation of the effect is given by Amado *et al.*,[210–213] who showed that in the Efimov limit ($\varepsilon_{12} \to 0$, $z_3 \to 0$) the kernel of the Faddeev integral equation becomes non-Fredholm, i.e., $\mathrm{Sp}(K^n) \to \infty$ ($n = 1,2...$). Since

$$\mathrm{Sp}\,(K^n) = \sum_{i=1}^{\infty} \lambda_i^n, \quad n = 2, 3 \ldots, \tag{94}$$

where λ_i is the eigenvalue of the kernel, in the limiting case we find that the accumulation point of the eigenvalues λ_i for $i \to \infty$ is not at the value zero, as it is for integral kernels of Fredholm type, but at a finite value. For the S-wave state, the accumulation point is greater than 1, from which there follows an infinite number of bound states, $N \sim (1/\pi)\ln(|a|/r_0)$. The first illustration of the Efimov effect in numerical calculations was given in Ref. 214, in which it was shown that in the limit $\alpha \to 0$ a large number of bound levels do indeed arise. With decreasing α ($\alpha \gtrsim 0$), these states come out of the continuum and become bound states, but for $\alpha \lesssim 0$ they go back into the continuum. The question arises of the way in which the Efimov poles behave in the continuum. The trajectories of the first three poles for a system of three identical spinless bosons for $L = 0$ are shown in Fig. 24.[215] It can be seen that as the strength of the two-particle interaction is increased from zero to infinity the poles of the S matrix situated in the complex plane of the energy approach from infinity on the unphysical sheet s_i. At $\alpha = -0.15\ \mathrm{F}^{-1}$ $\alpha = 0.011\ \mathrm{F}^{-1}$, and $\alpha = -0.000\,55\ \mathrm{F}^{-1}$ the first, second, and third poles, respectively, become bound states. With further increase of α, the first two poles remain bound states. At $\alpha = 0.0103\ \mathrm{F}^{-1}$, the third pole passes to the unphysical sheet s_e, while at $\alpha = 4.25\ \mathrm{F}^{-1}$ it returns to the physical sheet and becomes a bound state. The investigation of the behavior of the trajectory of the third pole on the sheet s_e is a complicated problem, since it does not remain all the time on the sheet s_e, but passes to other sheets of the Riemann surface that are associated with different cuts. For the other Efimov poles the same qualitative behavior as for the third pole is expected.

There is an unexpected property of the trajectory that is a characteristic aspect of the Efimov effect. In a certain region of strength of the two-particle interaction there arises a situation in which, when the strength is increased, the given

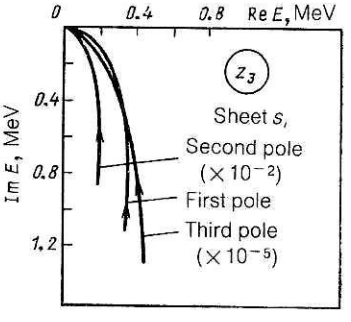

FIG. 24. Trajectories of the first three poles on the unphysical sheet s_i that characterize the behavior of the Efimov poles in the region $\alpha \lesssim 0$.

three-particle bound state does not become more strongly bound, as one would expect, but enters the continuum. This is due to the fact that in the plane of the energy the origin of the two-particle cut is shifted to the left more rapidly than the pole of the bound state. The anomalous behavior of the pole trajectory associated with the Efimov effect has already been discussed above for virtual states (see also Refs. 215 and 216). Searches for real physical systems in which Efimov levels are manifested have been made in many other studies that used various theoretical models (Refs. 34, 36, and 217–222). As Efimov noted[223] already in 1975, one of the possible candidates is the system of three α particles, since there are a quasistationary state of ^{8}Be with low energy (-95 keV) and a three-particle ^{12}C level with binding energy 7.65 MeV quite close to the threshold of 3α disintegration. However, for the final experimental proof of the existence of the Efimov effect it is necessary to find a system in which, not one, but several levels with the Efimov effect are manifested.

CONCLUSIONS

The approach that we have described here to the search for and the investigation of resonances in three-particle systems in terms of poles of the T matrix can be extended to different three-particle (or three-cluster) physical systems. For example, the contour-deformation method for transition to an unphysical sheet of the energy was used in Refs. 224 and 225 to look for dibaryon resonances in the framework of the three-particle $NN\pi$ dynamics. In fact, a modified and simpler way of deforming the contour of integration was used in Ref. 225. The method of integral equations for resonance states can also be applied to other hadronic systems, for example, $NN\bar{N}$, $\pi\pi N$, 3π, $K\bar{K}\pi$, $K\pi\pi$, etc. The Faddeev equations for the $NN\bar{N}$ system were applied in Refs. 226–229. For completeness, we also mention Refs. 230–236 on this subject, although in most of these studies other approaches to the investigation of resonances were used.

With regard to $3N$ systems, on which we have mainly concentrated in this review (see also Ref. 237), we may conclude that they do not contain resonances near the physical region (except for the ^{3}H virtual level). In particular, a negative result was also obtained in the recent study of Ref. 238 devoted to experimental search for the tri- and tetraneutron in reactions on a ^{7}Li target induced by ^{11}B and ^{9}Be ions (see also the theoretical study of Ref. 239).

Thus, at the present time there are several fairly accurate and practical methods of theoretical investigation of Gamow states, among which an important place is occupied by the method of analytic continuation of the integral equations of scattering theory, the exposition of which has taken up a large proportion of the present review. The further development of this method requires more accurate allowance for the Coulomb interaction, which plays a particularly important part near the threshold with respect to the channel of charged particles.

[1)]In the tables of this paper we have used the standard notation for the quantum numbers of a three-particle system: T is the isospin, S is the spin, L is the orbital angular momentum, J is the total angular momentum, and π is the parity.

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