Quasi-exactly solvable models in quantum mechanics

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A general approach to the problem of quasi-exact solvability in quantum mechanics is proposed. A large class of quasi-exactly solvable models is constructed, for which the spectral problem can be solved exactly only for limited parts of the spectrum.

INTRODUCTION

It is well known that exactly solvable models play an important role in quantum theory. First of all, they are interesting in themselves as models of actual physical systems. Second, they can be used as the zeroth-order approximation in constructing perturbation theory. Unfortunately, the number of exactly solvable models presently known is quite small, and therefore the range of applicability of perturbation theory in quantum physics is still rather narrow.

Brute-force attempts to find new exactly solvable models encounter serious difficulties, since the requirement of exact solvability, which is usually understood as the possibility of writing down the entire spectrum of the Hamiltonian in closed form, is too strict. This suggests that the requirements be relaxed and that models be sought for which the spectral problem can be solved exactly, however, not for the entire spectrum, but for only a limited part of it. It is easy to see that such models are no less useful than exactly solvable ones. They can be used successfully to model various physical situations. Moreover, they have certainly proved useful for perturbation theory, which, as is well known, does not require knowledge of the entire spectrum of the unperturbed problem.[1] Finally, they can be of independent value if they reveal the presence of "nonperturbative" effects, which generally are very difficult to study.

Here we shall refer to such models as "quasi-exactly solvable," with the order of the model being the number of states for which the spectral problem can be solved exactly.

We note that one-dimensional quasi-exactly solvable models of first order have been known for a long time.[1,2] They are trivial to construct. For the Schrödinger equation

$$\left[-\frac{\partial^2}{\partial x^2}+V(x)\right]\psi(x)=E\psi(x) \qquad (1a)$$

it is sufficient to know the explicit form of the wave function $\psi(x)$ and energy E of any state (whose order corresponds to the number of nodes of the wave function), and then to use the same equation, rewritten as

$$V(x)=\frac{\partial^2\psi(x)}{\partial x^2}\psi^{-1}(x)+E, \qquad (1b)$$

to reconstruct the potential for which $\psi(x)$ and E are solutions. Of course, when choosing $\psi(x)$ one must take care that the potential obtained using (1b) is physically sensible. However, all such difficulties are easily overcome. This problem has been solved in Refs. 3 and 4 for the ground state and several excited states in models with polynomial potentials.

The example of an infinite series of quasi-exactly solvable models of arbitrary order given in Ref. 5 clearly demonstrated that the potentials of these models are not necessarily exotic "monsters," but can be quite simple and ordinary-looking. For example, the potentials of the models studied in Ref. 5 are even polynomials of order six:

$$V(x)=a^2x^6+2abx^4+[b^2+ap-4aM]x^2,\ a>0. \qquad (2)$$

They contain two real parameters a and b and also a natural number M, which indicates how many energy levels of parity $p(p=\pm1)$ can be calculated exactly in this model. For example, if $M=2$, one can write down explicitly the two energy levels

$$E_\pm=(4-p)b\pm2\sqrt{b^2+4a-2pa}; \qquad (3a)$$

$$\psi_\pm(x)=x^{(1-p)/2}(2ax^2+b\mp2\sqrt{b^2+4a-2pa})\exp\left(-\frac{ax^4}{4}-\frac{bx^2}{2}\right). \qquad (3b)$$

Using the oscillator theorem (direct calculation of the zeros of the wave function), it is easy to show that for $p=+1$ Eqs. (3a) and (3b) describe the zeroth and second energy levels, while for $p=-1$ they describe the first and third levels. Similarly, explicit solutions (involving radicals) can also be written down for $M=3$ and $M=4$. However, beginning with $M=5$, this is not always possibie. The reason is that for a given M the energy levels in the model (2) arise as roots of an algebraic equation of order M, the explicit form of which is easily obtained by seeking solutions of the Schrödinger equation in the form

$$\psi(x)=x^{(1-p)/2}P_{M-1}(x^2)\exp\left(-\frac{ax^4}{4}-\frac{bx^2}{2}\right), \qquad (4)$$

where $P_{m-1}(t)$ is an unknown polynomial of order $M-1$. Equating the terms for identical powers of x^2 and eliminating the coefficients of the polynomial $P_{M-1}(t)$ from the resulting system of equations, we arrive at a single algebraic equation of degree M in E. Therefore, the *differential* Schrödinger equation for the model (2) is exactly solvable when, owing to the choice of a suitable ansatz (4), it can be reduced to the problem of solving *numerical* equations.

The algebraic equation in E can be interpreted as the usual secular equation for the eigenvalues of a finite-dimensional linear problem. The finite dimensionality is due to the felicitous choice of basis in the functional Hilbert space, in which the Hamiltonian of the model (2) takes a block-diagonal form. Here one of the blocks of the Hamiltonian matrix is finite and the other is infinite. Therefore, the Schrödinger problem breaks up into two completely independent linear problems one of which is finite-dimensional and solvable, while the other is infinite-dimensional and nothing about its solutions is known. The manner in which these blocks arise can be seen clearly for the example of the model with a singular potential studied in Ref. 6. We discuss this model in Appendix 1 of this review.

It is easily seen that the model (2) can be viewed as a

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perturbed harmonic oscillator with potential $V_0(x) = b^2x^2$. Here the perturbation parameter is the constant a. Since the perturbation theory in the parameter a is well-defined and easy to construct, we can compare the perturbative results with the exact ones. Such a comparison reveals the presence of a number of nonperturbative effects in the model (2), i.e., effects which do not appear in any finite order of perturbation theory.

One of these effects is already seen in the exact solution (3). This is that the levels E_+ and E_-, viewed as analytic functions of the parameters a and b, are plaited, forming a single two-sheeted Riemann surface. This implies that by analytic continuation of the level E_- along a closed contour it is possible to obtain the level E_+, and vice versa. The plaiting points (where the levels coincide) are square-root singularities and lie on the parabola $b^2/a = 2p - 4$, i.e., outside the region where perturbation theory is applicable. We note that the presence of this degeneracy does not contradict the familiar theorem about the nondegeneracy of the spectra of one-dimensional quantum-mechanical systems,[7] since at the plaiting points the wave functions $\psi_+(x)$ and $\psi_-(x)$ also coincide, and the geometrical multiplicity of the degenerate eigenvalue remains equal to unity. We also note that the normalization integrals for the two wave functions at the plaiting points vanish identically.

These features are easily generalized to the case of arbitrary M: for any given M, the first M levels of identical parity are plaited. These features are typical for most non-exactly solvable problems. They were first discovered by Bender and Wu,[8] and also by Simon[9] in the non-exactly solvable anharmonic-oscillator with potential

$$V(x) = x^2 + gx^4, \tag{5}$$

for which it was found that all levels of identical parity are plaited. This behavior was later discovered also in other models.[10-12]

Another nonperturbative effect which is also present in the exact solution (3) is related to the behavior of the energy levels $E_\pm$ as functions of the parameter b when a is small. Here, if one looks at a poor-resolution graph of the two functions $E_\pm(b)$, one receives the impression that the levels intersect and, in doing so, exchange quantum numbers. For better resolution it becomes clear that there is no such intersection. This effect is well known in nuclear physics, where it is referred to as the quasi-intersection of levels.[13] It cannot be seen in perturbation theory. The existence of exact solutions not only allows us to see it, but also gives a simple explanation for it: the quasi-intersection of levels is a manifestation of their actual intersection (plaiting) at a point lying close to the real axis (in the complex plane of the parameter of the problem).[12]

Finally, another effect worth mentioning is the behavior of high-order terms of the perturbation series. From the exact solution (3) we see that the point $a = 0$ is a regular point for the functions $E_\pm(a)$, so that the perturbation series is convergent for these levels! The radius of convergence, defined in the usual way as the distance to the nearest singularity, is found from (3) to be $b^2/(4 - 2p)$. At first glance this result seems very strange, since naive semiclassical estimates indicate that for models of this type the high-order terms of the perturbation series must grow factorially, so that the series should have zero radius of convergence. This result is confirmed by numerical calculations (using algebraic perturbation theory[8]) for all non-natural values of M. However, if M is a natural number, an amazing cancellation of all the divergent contributions occurs and the series becomes convergent.

Quasi-exactly solvable models possess another rather curious feature. They can be viewed as unique approximations to non-exactly solvable models. As an example, we consider the model (2) and recall that M is the order of the secular equation which determines the spectrum. If $M = \infty$, the problem of finding the spectrum ceases to be algebraic, and the equation becomes non-exactly solvable. Therefore, to obtain a non-exactly solvable model M must be taken to infinity. For the potential to remain finite in this limit, the parameters a and b must be dependent on M. For example, if this dependence is determined by the conditions $b^2 - 4aM = 1$ and $2ab = g$, then for $M \to \infty$ the coefficient of the sextic term in the potential (2) vanishes and we obtain the non-exactly solvable model of the anharmonic oscillator with the potential (5).

The properties of quasi-exactly solvable models that we have listed above show that these models are very interesting objects to study. In this review we attempt to give a unified discussion of the problem of quasi-exact solvability in quantum mechanics and to formulate fairly simple methods for constructing and studying quasi-exactly solvable models, both one-dimensional and multi-dimensional.

Let us give a few remarks about the history of this problem. The first nontrivial model with two exactly calculable energy levels and with a potential expressed in terms of hyperbolic functions was obtained heuristically in Ref. 14. A model with similar properties but with a polynomial potential was found in Ref. 15. The term "quasi-exact solvability" was introduced in Ref. 16, where two-dimensional quasi-exactly solvable models with degenerate spectra were constructed and studied. The first example of an infinite series of one-dimensional models with polynomial potentials and arbitrary, arbitrarily large exactly calculable segments of the spectrum was given in Ref. 5 mentioned above [see Eq. (2)]. Then individual infinite series of quasi-exactly solvable models with potentials expressed as powers of exponential and hyperbolic functions were found in Ref. 17. The list of one-dimensional quasi-exactly solvable models was extended significantly in Ref. 18. The author found new models with trigonometric and hyperbolic potentials, and also a series of models with potentials involving elliptic functions. The existence of finite series of quasi-exactly solvable models was pointed out in that study.

The next stage in the history of quasi-exact solvability is characterized by attempts to understand this phenomenon and to formulate general principles allowing the construction and investigation of all possible quasi-exactly solvable models. These attempts led to the development of two fundamentally different approaches, which we shall refer to as the algebraic and analytic approaches.

The algebraic approach formulated by Turbiner in Ref. 19 is based on the idea of the possible use of finite-dimensional representations of the algebra $SL(2)$. As is well known, finite-dimensional representations of the algebra $SL(2)$ can be realized on the space of polynomials.[20] Here the generators of this algebra take the form of first-order differential operators.[20] For this reason the spectral equation $h\varphi = e\varphi$, where $h = \Sigma_{i,k} a_{ik} S_i S_k + \Sigma_i b_i S_i$ and S_i are the generators of the algebra $SL(2)$, is finite-dimensional and can be solved exactly in the class of polynomials φ. On the other hand, this equation is a second-order differential equation and can be written in the Schrödinger form. As a result, one obtains a one-dimensional quasi-exactly solvable model whose order is equal to the dimension of the representation of the algebra $SL(2)$. The generalization to the multi-dimensional case is obvious. For this the algebra $SL(2)$ must be replaced by algebras of higher rank, whose finite-dimensional representations are realized on the space of polynomials in several variables, and whose generators take the form of first-order multidimensional differential operators.[21,22]

The algebraic approach is attractive primarily because of the simplicity of the idea on which it is based. (We note that the final formulation of this approach[19] preceded the studies of Refs. 23 and 24, which discussed similar ideas.) At the present time this approach has been used for the detailed study of only the one- and two-dimensional cases.[19,21,22] Unfortunately, the algebraic approach is apparently not universal, since it cannot be used to describe the so-called finite series of quasi-exactly solvable models, of which, as was shown in Ref. 25, there exist an infinite number.

The analytic approach was formulated by the present author in Refs. 18 and 25 and is the subject of this review. It is based on the observation that quasi-exactly solvable Schrödinger equations can be viewed as equations with several spectral parameters, some of which are involved in the potential [for example, the parameter M in (2)], while one plays the role of the energy. If the spectra of the "potential" spectral parameters are degenerate relative to the spectrum of the "energy" parameter, the model is quasi-exactly solvable and its order is equal to the degree of degeneracy. Therefore, the construction of quasi-exactly solvable models reduces to the construction of multi-parameter spectral equations and the study of the degeneracies in their spectra. It turns out that the mathematical techniques used in this approach are very similar to those used in the classical multiparticle Coulomb problem, and also in the quantum theory of completely integrable models of magnetic systems based on Lie algebras, so that three seemingly unrelated branches of classical and quantum physics are seen to be equivalent. We stress the fact that Lie algebras arise naturally, but in completely different manners, in both the algebraic and the analytic approaches. The representations of these algebras used in the analytic approach are not finite-dimensional, as in Ref. 19, but infinite-dimensional.

This review is organized as follows. In Secs. 1–3 we formulate our general approach to the problem and give an algorithm which allows exactly and quasi-exactly solvable Schrödinger equations to be constructed, both in one- and in multi-dimensional cases. In Secs. 4–9 we consider a special case in detail. This case is quite rich and includes a large class of both exactly and quasi-exactly solvable models. The concluding sections are devoted to a brief review of other quasi-exactly solvable systems and construction methods.

1. FORMULATION OF THE METHOD

As noted in the Introduction, in our approach a central role is played by spectral equations involving several spectral parameters. The most general form of these equations is

$$(X + \mathbf{eY})\,\varphi = 0, \quad \varphi \in \Omega. \tag{6}$$

Here X is a linear operator from V to V, where V is an infinite-dimensional vector space, and $\mathbf{Y}$ is a linear vector operator from V to $V \otimes V_n$, where V_n is an n-dimensional vector space. The problem is to find all vectors $\mathbf{e} \in V_n$, for which Eq. (6) has a solution $\varphi \in \Omega$, where Ω is some subspace of the space V. The set of vectors $\mathbf{e}$ for which (6) has solutions will be called the spectrum and denoted by $S_n(\Omega)$. Expanding the vectors into components, $\mathbf{e} = \{e_i\}_{n=1}^{n}$ and $\mathbf{Y} = \{Y_i\}_{n=1}^{n}$, can be written in a manifestly multi-parameter form. This can be done in many ways, which is related to the fact that the basis chosen in the space V_n is not unique. Transformations from the group $GL(n)$ take one particular form of Eq. (6) into another.

Let us assume that the subspace Ω has the form $\Omega = \cup_{M=0}^{\infty} \Omega_M$, being the union of finite-dimensional (M-dimensional) surfaces Ω_M in V. If the equations for these surfaces are given explicitly, the solution of the spectral problem (6) reduces to solving various finite systems of numerical equations. In this case Eq. (6) is exactly solvable.

Let us now formulate a theorem establishing a relationship between multi-parameter and one-parameter exactly solvable spectral equations.

Theorem 1. An equation of the type (6) with n spectral parameters and exactly solvable in Ω generates an n-parameter family of equations with a single spectral parameter exactly solvable in $\Omega \otimes \Omega \otimes \ldots \otimes \Omega$ (n times).

Proof. Let $V^n \equiv \otimes_{k=1}^{n} V$. We introduce the vector operator

$$\mathrm{X} = \left\{ \left(\bigotimes_{k=1}^{i-1} I \right) \otimes X \otimes \left(\bigotimes_{k=i+1}^{n} I \right) \right\}_{i=1}^{n}, \tag{7a}$$

acting from V^n to $V^n \otimes V_n$, and the matrix operator

$$\hat{Y} = \left\{ \left(\bigotimes_{k=1}^{i-1} I \right) \otimes \mathbf{Y} \otimes \left(\bigotimes_{k=i+1}^{n} I \right) \right\}_{i=1}^{n}, \tag{7b}$$

acting from $V^n \otimes V_n$ to $V^n \otimes V_n$. Here I is the unit operator. We note that (6) automatically leads to the equations

$$(\mathbf{X} + \hat{Y}\mathbf{e})\,\varphi = 0, \quad \varphi \in \Omega^n, \tag{8}$$

where $\varphi = \otimes_{i=1}^{n} \varphi$, and $\Omega^n = \otimes_{i=1}^{n} \Omega$. Assuming that the operator $\hat{Y}$ is nonsingular, we introduce new vector operators L acting from V^n to $V^n \otimes V_n$, defining them as

$$\mathbf{X} + \hat{Y}\mathbf{L} = 0 \quad \text{or} \quad \mathbf{L} = \hat{Y}^{-1}\mathbf{X}. \tag{9}$$

Applying the operator $\hat{Y}^{-1}$ to both sides of (8) and using (9), we find

$$\mathbf{L}\varphi = \mathbf{e}\varphi, \quad \varphi \in \Omega^n. \tag{10}$$

From (10) we see that the vector $\mathbf{e}$ is an eigenvalue of the vector operator $\mathbf{L}$ on Ω^n. From this it follows that any equation of the form

$$(\gamma\mathbf{L} - e)\,\varphi = 0, \quad \varphi \in \Omega^n, \tag{11}$$

where γ is an arbitrary fixed vector, will be an ordinary equation with a single spectral parameter e whose allowed values are of the form $e = \gamma \cdot \mathbf{e}$ and, therefore, can be calculated exactly. This proves the theorem.

The spectrum $S_1(\Omega^n)$ of Eq. (11) is obviously in one-to-one correspondence with the spectrum $S_n(\Omega)$ of the original equation (6).

We see from Eq. (10) that all the operators L_i (the components of the operator $\mathbf{L}$) have the same set of eigenfunctions. They do not necessarily commute with each other.

Theorem 2. For the operators L_i to commute it is sufficient that the operators Y_i (the components of the vector $\mathbf{Y}$) also commute with each other.

Proof. Let $[Y_l, Y_m] = 0$ for all $l \neq m$. Then we have

$$\sum_{l,m} Y_{il} Y_{km} [L_l, L_m] = 0 \quad \text{for} \quad i = k. \tag{12a}$$

Furthermore, from the commutation relations $[X_i, X_k] = 0$, $[Y_{il}, Y_{km}] = 0$, and $[X_i, X_{km}] = 0$, valid for all $i \neq k$ and following from the definitions (7), it follows that

$$\sum_{l,m} Y_{il} Y_{km} [L_l, L_m] = 0 \quad \text{for} \quad i \neq k. \tag{12b}$$

Combining (12a) and (12b) and taking into account the nondegeneracy of the matrix operator Y_{ik}, we find

$$[L_l, L_m] = 0 \quad \text{for all } l \text{ and } m. \tag{13}$$

This proves the theorem.

It is easy to see that the procedure formulated in Theorem 1 of going from a single "one-dimensional" equation with n spectral parameters to an n-parameter family of "n-dimensional" equations with a single spectral parameter essentially solves the inverse problem of separation of variables. Here the one-dimensional equation (6) is interpreted as the equation arising as a result of separation of variables (in the generalized sense) in the n-dimensional equation (11). The spectral parameters e_i play the role of separation constants, and the operators L_i whose eigenvalues they are play the role of the symmetry operators of Eq. (11), i.e., operators commuting with the operator $\gamma \cdot \mathbf{L}$.

Let us now describe another method of going from multi-parameter exactly solvable spectral equations to one-parameter equations. This method is realized when there is a special degeneracy in the spectrum of the multi-parameter spectral equation. We shall call the subset s_n of the spectrum $S_n(\Omega)$ of Eq. (6) (n/m)-fold degenerate if for all $\mathbf{e} \in s_n$ the following decomposition is valid:

$$\mathbf{e} = \mathbf{e}' \oplus \mathbf{e}'', \ \mathbf{e}' \in V_m, \ \mathbf{e}'' \in V_{n-m}, \ V_n = V_m \oplus V_{n-m}, \tag{14}$$

where $\mathbf{e}'' \in V_{n-m}$ is a vector identical for all $\mathbf{e} \in s_n$. The group interpretation of this degeneracy becomes clear if we note that the vectors $\mathbf{e}'$ and $\mathbf{e}''$ are eigenvalues of the operators $\mathbf{L}'$ and $\mathbf{L}''$ entering into the expansion $\mathbf{L} = \mathbf{L}' \oplus \mathbf{L}''$ corresponding to (14). Degeneracy of the eigenvalue $\mathbf{e}''$ relative to the eigenvalue $\mathbf{e}'$ implies the presence of a (hidden) symmetry group G under which $\mathbf{L}''$ is invariant, while the operator $\mathbf{L}'$ is not.

Theorem 3. When an (n/m)-fold degeneracy is present in the spectrum, Eq. (6) with n spectral parameters and exactly solvable in Ω generates an m-parameter family of equations with a single spectral parameter which are exactly solvable in $\Omega \otimes \Omega \otimes ... \otimes \Omega$ (m times).

Proof. Expanding the operator $\mathbf{Y}$ as in (14), $\mathbf{Y} = \mathbf{Y}' \oplus \mathbf{Y}''$, we rewrite Eq. (6) as

$$(X' + \mathbf{e}'\mathbf{Y}')\,\varphi = 0, \quad \varphi \in \Omega, \tag{15}$$

introducing the operator $X' \equiv X + \mathbf{e}'' \cdot \mathbf{Y}''$. Since Eq. (15) is an equation of the type (6) with m spectral parameters, using Theorem 1 it can be reduced to an m-parameter family of equations

$$(\gamma' \mathbf{L}' - e')\,\varphi = 0, \quad \varphi \in \Omega^m \tag{16}$$

with a single spectral parameter $e' = \gamma' \cdot \mathbf{e}'$. This proves the theorem.

The spectrum $S_1(\Omega^m)$ of Eq. (16) is obviously in one-to-one correspondence with the set s_n, which is smaller than $S_n(\Omega)$. Therefore, the spectrum $S_1(\Omega^m)$ is smaller than the spectrum $S_n(\Omega)$ of the original equation (6). We note that when $m = 1$, we obtain "one-dimensional" exactly solvable equations with a single spectral parameter.

In general, the exactly solvable equations (11) and (16) are not equations of the Schrödinger type. They become equations of this type only in one case, when the sets of exact solutions of (11) or (16) have nonzero intersections with the sets of all solutions of the equations

$$(\gamma \mathbf{L} - e)\,\varphi = 0, \quad \varphi \in W,$$

or

$$(\gamma' \mathbf{L}' - e')\,\varphi' = 0, \quad \varphi' \in W', \tag{11'}$$

in which W and W' are vector spaces with scalar products ensuring the Hermiticity of the operators $\gamma \cdot \mathbf{L}$ and $\gamma' \cdot \mathbf{L}'$ (the Hamiltonians). We shall refer to Schrödinger equations of the type (11) and (16) as exactly solvable (quasi-exactly solvable) if all (some) of the solutions of these equations are contained in the set of exact solutions of Eqs. (11) and (16).

We have formulated the general features of the method of constructing exactly and quasi-exactly solvable models. It can be divided into three steps: 1) the construction of equations with n spectral parameters which are exactly solvable in Ω; 2) the transformation from these equations to equations with a single spectral parameter which are exactly solvable in Ω^n (or in Ω^m, when there is an (n/m)-fold degeneracy); 3) the determination of whether or not the exactly solvable equation which is obtained can be interpreted as an exactly or quasi-exactly solvable Schrödinger equation. This method was first formulated in Ref. 25, and its final form was given in Ref. 26.

2. THE CASE OF DIFFERENTIAL EQUATIONS

Let us consider the case when the space V is a space of functions of a single variable λ, $X = \partial^2/\partial\lambda^2 + U(\lambda)$, $\mathbf{Y} = \mathbf{U}(\lambda)$. In this case Eq. (6), written in terms of components, becomes

$$(\partial^2/\partial\lambda^2 + U(\lambda) + e_1 U_1(\lambda) + \ldots + e_n U_n(\lambda))\,\varphi(\lambda) = 0. \tag{17}$$

We assume that the system of spectral parameters contains an (n/m)-fold degeneracy, with the parameters $e_{m+1}, ..., e_n$ being degenerate. According to Theorem 3, from Eq. (17) we can construct an m-dimensional equation (i.e., a differential equation in m-dimensional space) with a single spectral parameter e. We can take e to be any linear combination of the parameters $e_1, ..., e_m$. Omitting the elementary algebra, we give the explicit form of the resulting equation for the case in which $e = e_\alpha$:

$$\left\{-\sum_{ik=1}^{m} \sqrt{g_\alpha}\frac{\partial}{\partial\lambda_i}\left(\frac{g_\alpha^{ik}}{\sqrt{g_\alpha}}\frac{\partial}{\partial\lambda_k}\right)+V_\alpha\right\}\psi=e\psi. \quad (18)$$

Here $g_\alpha \equiv \det\|g_\alpha^{ik}\|$; $g_\alpha^{ii} \equiv h^{-1}f_\alpha^i$, $i = 1, ..., m$; $g_\alpha^{ik} = 0$, $i \neq k$; $h \equiv \det\| U_i(\lambda_k)\|$, and f_α^i is the cofactor of the element $U_i(\lambda_\alpha)$ in the matrix $\|U_i(\lambda_k)\|$. The function V_α has the form

$$V_\alpha = -\sum_{i=1}^{m} h^{-1}f_\alpha^i\left[U(\lambda_i)+\sum_{k=m+1}^{n} e_k U_k(\lambda_i)\right]-\sum_{i=1}^{m} g_\alpha^{ii}(g_\alpha h^2)^{1/4}$$
$$\times\frac{\partial^2}{\partial\lambda_i^2}(g_\alpha h^2)^{-1/4}-\sum_{i=1}^{m}\frac{1}{h}\left[\frac{\partial}{\partial\lambda_i}(g_\alpha^{ii}h)\right]$$
$$\times(g_\alpha h^2)^{1/4}\frac{\partial}{\partial\lambda_i}(g_\alpha h^2)^{-1/4}, \quad (19)$$

and the solutions of Eq. (18) are related to the solutions of Eq. (17) as

$$\psi=(g_\alpha h^2)^{1/4}\prod_{i=1}^{m}\varphi(\lambda_i), \quad e=e_\alpha. \quad (20)$$

It is easy to see that Eq. (18) has the same form as the Schrödinger equation on an m-dimensional, in general, curved manifold specified by the metric tensor $g_{ik} = (g^{ik})^{-1}$. The off-diagonal components of this tensor are equal to zero, so that the coordinate system parametrizing this manifold is orthogonal. Equation (18) admits separation of variables in the coordinates λ_i (by construction). For this equation to actually be the Schrödinger equation on a curved manifold (in a gravitational field), the operator L_α, which plays the role of the Hamiltonian, must be Hermitian on the space of functions normalized according to

$$\int\frac{1}{\sqrt{g_\alpha}}\psi^2(\lambda)\,d^m\lambda<\infty. \quad (21)$$

From the fact that instead of the sets of spectral parameters $e_1, ..., e_m$ and weight functions $U_1(\lambda), ..., U_m(\lambda)$ we can use any $GL(m)$-transformed sets, its follows that any linear combination of the Hamiltonians L_α can be reduced to the form (18). We also note that, owing to the commutativity of the weight functions, all the Hamiltonians L_α commute with each other (a corollary of Theorem 2).

In the multi-dimensional case ($m > 1$) the choice of coordinate system, has no fundamental meaning and is dictated only by considerations of simplicity. In the one-dimensional case ($m = 1$) it is convenient, after an additional coordinate transformation $\lambda = \lambda(x)$,

$$x=\int^{\lambda(x)}\frac{d\lambda}{\sqrt{g(\lambda)}}, \quad (22)$$

to write Eq. (18) in the usual Schrödinger form:

$$\left\{-\frac{\partial^2}{\partial\lambda^2}+V(\lambda(x))\right\}\psi(\lambda(x))=e\psi(\lambda(x)). \quad (23)$$

In this case $g^{11} = g$, $h = g^{-1}$, $f_1 = 1$ and the expression (19) for the potential is simplified:

$$V=-g\left[U+\sum_{k=2}^{n}e_kU_k\right]-g^{3/4}\frac{\partial^2}{\partial\lambda^2}g^{1/4}. \quad (24)$$

Our discussion in this section is based on that of Ref. 27.

3. CONSTRUCTION OF MULTI-PARAMETER EXACTLY SOLVABLE DIFFERENTIAL SPECTRAL EQUATIONS

In this section we formulate an analytic method allowing the construction of exactly solvable differential spectral equations with several spectral parameters. The central object in this method is a functional equation of the form

$$\Delta(\lambda_1)\Delta(\lambda_2)+\Delta(\lambda_2)\Delta(\lambda_3)+\Delta(\lambda_3)\Delta(\lambda_1)+\rho^2=0,$$
$$\lambda_1+\lambda_2+\lambda_3=0, \quad (25)$$

supplemented by the condition

$$\Delta(\lambda_1)+\Delta(\lambda_2)=0, \quad \lambda_1+\lambda_2=0. \quad (26)$$

We shall refer to Eq. (25) as the scalar triangle equation and to Eq. (26) as the scalar unitarity relation. The meaning of these terms will become clear later. It turns out that the solutions of Eqs. (25) and (26) are the elementary building blocks from which both the multi-parameter spectral equations themselves and their solutions can be constructed.

It is easy to show[28] that any solution of the system (25), (26) can be orthonormalized in such a way that it satisfies a special Riccati equation:

$$\Delta'(\lambda)+\Delta^2(\lambda)=\rho^2. \quad (27)$$

This follows from the easily proved fact that any solution of this system has a simple pole at the origin.[28] Orthonormalizing the function $\Delta(\lambda)$ such that the residue at the pole is unity, we rewrite Eq. (25) as $\Delta(\lambda)\Delta(\lambda+\varepsilon) + \Delta(\varepsilon)[\Delta(\lambda+\varepsilon)-\Delta(\lambda)]=\rho^2$. Taking ε to zero and using the fact that $\Delta(\varepsilon)\approx\varepsilon^{-1}$, we obtain (27).

Let us now formulate the fundamental theorem of this method.[25,26]

Theorem 4. The equation with $n = 2N + 1$ spectral parameters

$$\left\{\frac{\partial^2}{\partial\lambda^2}+\sum_{\alpha=1}^{N}e_\alpha^{(2)}\Delta'(\lambda-a_\alpha)+\sum_{\alpha=1}^{N}e_\alpha^{(1)}\Delta(\lambda-a_\alpha)+e^{(0)}\right\}$$
$$\varphi(\lambda)=0 \quad (28)$$

has solutions in the class of functions of the form

$$\varphi(\lambda)=\prod_{\alpha=1}^{N}\exp\left\{\eta_\alpha\int\Delta(\lambda-a_\alpha)\,d\lambda\right\}\prod_{i=1}^{M}$$
$$\times\exp\left\{\int\Delta(\lambda-\xi_i)\,d\lambda\right\}, \quad (29)$$

where ξ_i, $i = 1, ..., M$, and η_α, $\alpha = 1, ..., N$, are unknown numerical parameters. The system of numerical equations for the spectral parameters $e_\alpha^{(2)}$, $e_\alpha^{(1)}$, $e^{(0)}$ and the parameters ξ_i and η_α have the form

$$\sum_{k=1}^{M}{}'\,\Delta(\xi_i-\xi_k)+\sum_{\alpha=1}^{N}\eta_\alpha\Delta(\xi_i-a_\alpha)=0, \quad i=1,\ \ldots,\ M; \quad (30)$$

$$e_\alpha^{(2)}=\eta_\alpha(\eta_\alpha-1); \quad (31a)$$

$$e_\alpha^{(1)}=-2\left\{\sum_{\beta=1}^{N}\eta_\alpha\eta_\beta\Delta(a_\alpha-a_\beta)+\sum_{i=1}^{M}\eta_\alpha\Delta(a_\alpha-\xi_i)\right\}; \quad (31b)$$

$$e^{(0)}=3\left(\sum_{\alpha=1}^{N}\eta_\alpha+M\right). \qquad (31c)$$

Proof. Acting on the function (29) with the operator $\partial^2/\partial\lambda^2$ and using Eqs. (25)–(27), we verify that the result corresponds to the function (29) multiplied by a sum of four types of terms: 1) constants; 2) terms proportional to $\Delta(\lambda-a_\alpha)$; 3) terms proportional to $\Delta'(\lambda-a_\alpha)$; and 4) terms proportional to $\Delta(\lambda-\xi_i)$. Equating the coefficients of the M terms of the fourth type to zero, we obtain Eq. (30). The coefficients of the remaining terms determine the spectral parameters $e_\alpha^{(2)}$, $e_\alpha^{(1)}$, and $e^{(0)}$. This proves the theorem.

It is easy to see that the spectrum of Eq. (28) is continuous, because $2N+1+M$ conditions are imposed on the $3N+1+M$ unknown quantities. Discreteness of the spectrum is ensured by imposing an additional N conditions on the parameters of the system. This can be done in many ways. For example, it is possible to fix N arbitrarily chosen linear combinations of the spectral parameters $e_\alpha^{(2)}$, $e_\alpha^{(1)}$, and $e^{(0)}$. Then Eq. (28) becomes an equation with $N+1$ spectral parameters. To complete the construction of this equation, we still need to find the explicit form of the function $\Delta(\lambda)$. Solving Eq. (27), we obtain

$$\Delta(\lambda)=\rho\cot\rho\lambda. \qquad (32)$$

Since the number ρ is arbitrary, we can take it to be real, imaginary, or zero. These three choices give us trigonometric solutions $\Delta(\lambda)=|\rho|\cot|\rho|\lambda$, hyperbolic solutions $\Delta(\lambda)=|\rho|\coth|\rho|\lambda$, and rational solutions $\Delta(\lambda)=\lambda^{-1}$ (Refs. 25 and 28).

In the following sections we shall make a detailed study of the quasi-exactly solvable models associated with rational solutions of the scalar triangle equation. To ensure that the spectrum of the multi-parameter spectral equation (28) is discrete, we require that the spectral parameters $e_\alpha^{(2)}$ be fixed by the conditions $e_\alpha^{(2)}=b_\alpha(b_\alpha-1)$, where b_α and also a_α are externally specified parameters. We shall show below that the set of quasi-exactly solvable models arising for this method of fixing the parameters is quite large and includes a variety of both one-dimensional and multi-dimensional models. Other methods of fixing the spectral parameters which also lead to quasi-exactly solvable models of a different type will be discussed later in Sec. 10.

4. RATIONAL QUASI-EXACTLY SOLVABLE MODELS (ONE-DIMENSIONAL CASE)

In the rational case Eq. (28) is (because $\rho=0$) an equation involving N spectral parameters and has the form[25,27]

$$\left\{\frac{\partial^2}{\partial\lambda^2}-\sum_{\alpha=1}^{N}\frac{b_\alpha(b_\alpha-1)}{(\lambda-a_\alpha)^2}+\sum_{\alpha=1}^{N}\frac{e_\alpha}{\lambda-a_\alpha}\right\}\varphi(\lambda)=0. \qquad (33)$$

In accordance with (29), a solution of (33) is sought in the form

$$\varphi(\lambda)=\prod_{\alpha=1}^{N}(\lambda-a_\alpha)^{b_\alpha}\prod_{i=1}^{M}(\lambda-\xi_i). \qquad (34)$$

Substitution of (34) into (33) leads to the following system of spectral equations[1)]:

$$\sum_{k=1}^{M}{}'\frac{1}{\xi_i-\xi_k}+\sum_{\alpha=1}^{N}\frac{b_\alpha}{\xi_i-a_\alpha}=0,\ i=1,\ \ldots,\ M; \qquad (35)$$

$$e_\alpha=2\sum_{i=1}^{M}\frac{b_\alpha}{\xi_i-a_\alpha}-2\sum_{\beta=1}^{N}{}'\frac{b_\alpha b_\beta}{a_\alpha-a_\beta},\ \alpha=1,\ \ldots,\ N. \qquad (36)$$

Following the general prescriptions formulated in Sec. 1, we attempt to determine the presence of degeneracies in the system of spectral parameters. For this, instead of the system of spectral parameters $e_1, \ldots, e_N$, it is convenient to consider another system of spectral parameters $r_0, \ldots, r_{N-1}$, related to the first one as

$$\sum_{\alpha=1}^{N}\left[e_\alpha+2\sum_{\beta=1}^{N}{}'\frac{b_\alpha b_\beta}{a_\alpha-a_\beta}\right]\frac{1}{\lambda-a_\alpha}=\left[\omega\prod_{\alpha=1}^{N}(\lambda-a_\alpha)\right]^{-1}\sum_{n=0}^{N-1}r_n\lambda^n. \qquad (37)$$

Here ω is a numerical factor, which for the moment is arbitrary. Using Eqs (35)–(37), it can easily be shown that

$$r_{N-1}=0; \qquad (38a)$$

$$r_{N-2}=-2\omega M\left[\sum_{\alpha=1}^{N}b_\alpha+\frac{M-1}{2}\right]; \qquad (38b)$$

$$r_{N-3}=-2\omega M\left[\sum_{\alpha=1}^{N}b_\alpha a_\alpha-\sigma_1(a)\left(\sum_{\alpha=1}^{N}b_\alpha+\frac{M-1}{2}\right)\right]-2\omega\left(\sum_{\alpha=1}^{N}b_\alpha+M-1\right)\sigma_1(\xi), \qquad (38c)$$

and so on (σ_n are symmetric nth order polynomials). We see that the group of transformations $GL(N)$ allows the hidden degeneracy in the system of spectral parameters to be exposed. It follows from (38a) that the parameter r_{N-1} is infinitely degenerate: it is equal to zero for all solutions and can simply be dropped. The parameters r_{N-2}, expressed in terms of the number M, has a finite degree of degeneracy equal to the number of different inequivalent solutions of (35) for a given M. Finally, the parameter r_{N-3} is in general not degenerate, owing to the presence in it of a term proportional to $\sigma_1(\xi)$, but it can be taken to be degenerate if the coefficient of this term vanishes:

$$\sum_{\alpha=1}^{N}b_\alpha+M-1=0. \qquad (39)$$

The presence of degeneracies in the system of parameters $r_0, \ldots, r_{N-1}$ allows one-dimensional equations of the Schrödinger type to be constructed from (33). One of the most interesting cases is realized when the parameter r_0 is taken as the energy spectral parameter. Then the potential appearing in the Schrödinger equation has the form

$$V(x)=\omega\prod_{\alpha=1}^{N}(\lambda-a_\alpha)\sum_{\alpha=1}^{N}\left\{\frac{\left(b_\alpha-\frac{1}{4}\right)\left(b_\alpha-\frac{3}{4}\right)}{(\lambda-a_\alpha)^2}+2\left(\sum_{\beta=1}^{N}{}'\frac{b_\alpha b_\beta-\frac{1}{16}}{a_\alpha-a_\beta}\right)\frac{1}{\lambda-a_\alpha}\right\}-\sum_{n=1}^{N-1}r_n\lambda^n, \qquad (40)$$

and the solutions of the Schrödinger equation with this potential take the form

$$\psi(x)=\prod_{\alpha=1}^{N}(\lambda-\alpha_\alpha)^{b_\alpha-\frac{1}{4}}\prod_{i=1}^{M}(\lambda-\xi_i),\ E=r_0. \qquad (41)$$

We see from (41) that the numbers ξ_i determine the zeros of the wave functions. According to (22), the function $\lambda = \lambda(x)$ is given by the equation

$$x = \int_{\lambda_0}^{\lambda} \frac{d\lambda'}{\sqrt{\omega \prod_{\alpha=1}^{N} (\lambda' - a_\alpha)}}. \quad (42)$$

Let us look at some specific examples.

1) $N = 2$. The potential contains an infinitely degenerate spectral parameter r_1, and we find equations with an infinite number of exact solutions.

2) $N = 3$. The potential contains an infinitely degenerate spectral parameter r_2, and also the parameter r_1, which has a finite degree of degeneracy. We obtain equations with a finite number of exact solutions.

3) $N = 4$. The potential contains three spectral parameters: r_3, r_2, and r_1. Let us consider three cases: 1) the condition (39) is satisfied; 2) the condition (39) is not satisfied, but the equation possesses a Z_2 symmetry: $b_\alpha = b_{5-\alpha}$, $a_\alpha = -a_{5-\alpha}$, $\alpha = 1, 2$; 3) the condition (39) is not satisfied and there is no Z_2 symmetry in the equation. It is easy to see that in the first two cases the degree of degeneracy of the potential spectral parameters is finite as a whole, and we again obtain equations with a finite number of exact solutions. In the third case, owing to the explicit dependence of the potential on $\xi_1, ..., \xi_M$, it is possible to obtain an equation with K exact solutions if on any K solutions $\{\xi_i^{(1)}\}, ..., \{\xi_i^{(k)}\}$ of Eq. (35) we impose $K - 1$ constraints: $\sigma_1(\xi^{(1)}) = \sigma_1(\xi^{(2)}) = \cdots = \sigma_1(\xi^{(K)})$. Although each solution is a function of eight parameters a_α and b_α, owing to the presence of the two-parameter group of transformations $a_\alpha \to Aa_\alpha + B$ leaving these equations unchanged, there are only six independent parameters. From the obvious constraint $K - 1 \leqslant 6$ it follows that the maximum number of exact solutions of the spectral equation in this case cannot be greater than seven: $K_{\max} = 7$ (Ref. 18).

4) $N \geqslant 5$. In this case the potential contains $N - 3$ independent spectral parameters which are expressed explicitly in terms of the quantities

$$\sigma_1(\xi), \sigma_2(\xi), \ldots, \sigma_{N-3}(\xi).$$

Repeating the foregoing arguments, it can be shown that to obtain equations with K exact solutions it is necessary to impose $(K - 1)(N - 3)$ constraints on the $2N - 2$ essential parameters of the system. From this it follows that the maximum order of a quasi-exactly solvable model in the general case (for arbitrary N) is given by

$$K_{\max} = 3 + \left[\frac{4}{N-3}\right] \quad (43)$$

(see Ref. 25).

Up to now we have discussed only the formal solutions of Schrödinger-type equations with potentials (40). We have not considered the question of whether or not these equations and their solutions are physically meaningful. Let us now attempt to remedy this deficiency and derive conditions for the potential (40) to be the potential of a stable quantum-mechanical system, and for the wave functions (41) to be normalizable and to satisfy zero boundary conditions at the ends of the interval in which the Schrödinger problem is formulated.

The first requirement which must be satisfied by the potential is that it must be real. We see from (40) that for this the parameters a_α and b_α, $\alpha = 1, ..., N$, must be real or must (simultaneously) correspond to complex-conjugate pairs. Real numbers a_α divide the λ axis into a set of intervals, which we shall refer to as the fundamental intervals and which can be finite, semi-finite, or infinite. Let $[\lambda_-, \lambda_+]$ be a fundamental interval. We choose the sign of ω in Eq. (37) in such a way that the expression under the square root in (42) is positive in this interval. Then, under the condition that $\lambda_0 \in [\lambda_-, \lambda_+]$, Eq. (42) describes a continuous and one-to-one mapping of the interval $[\lambda_-, \lambda_+]$ of th λ axis to an interval $[x_-, x_+]$ of the x axis. From this it follows that the potential $V(x)$ and the solution $\psi(x)$ of the formal Schrödinger equation are regular functions inside the interval $[x_-, x_+]$ and can have singularities only at its ends. The nature of these singularities cannot be arbitrary, but must conform to the requirement that the Hamiltonian be Hermitian on the solutions in question in the interval $[x_-, x_+]$. This requirement, which is equivalent to the condition that the surface integral arising in Hermitian conjugation vanish, can be written as

$$\left\{\prod_{\alpha=1}^{N} (\lambda - a_\alpha)\right\}^{1/2} \frac{\partial}{\partial\lambda} \left\{\prod_{\alpha=1}^{N} (\lambda - a_\alpha)^{b_\alpha - (1/4)} \prod_{i=1}^{M} \times (\lambda - \xi_i)\right\}^2 \Bigg|_{\lambda = \lambda_{\lim}} = 0, \quad (44)$$

where $\lambda_{\lim} = \lambda_+$ or λ_- is a limit of the fundamental interval. When $\lambda_{\lim}$ is finite ($\lambda_{\lim} = a_\alpha$), the condition (44) leads to the constraints

$$b_\alpha > 1/2. \quad (45)$$

If the point $\lambda_{\lim}$ is infinite ($\lambda_{\lim} = \pm\infty$), the constraint takes the form

$$\sum_{\alpha=1}^{N} b_\alpha + M < 1/2. \quad (46)$$

Using the explicit expression for the potential (40) together with Eq. (42), it can be shown that if the end point $\lambda_{\lim}$ of the fundamental interval is finite ($\lambda_{\lim} = a_\alpha$), the end point $x_{\lim}$ corresponding to it is also finite, and the potential in its vicinity behaves as

$$V(x) \approx \frac{(2b_\alpha - 1/2)(2b_\alpha - 3/2)}{(x - x_{\lim})^2}, \quad x \to x_{\lim}. \quad (47)$$

However, if the point $\lambda_{\lim}$ is infinite ($\lambda_{\lim} = \pm\infty$), the point $x_{\lim}$ can be either finite or infinite, depending on which case is being considered: $N > 2$ or $N = 2$. In the first case the potential in the vicinity of $x_{\lim}$ behaves as

$$V(x) \approx \left\{-\frac{1}{4} + \left(\frac{2}{N-2}\right)^2 \left[\sum_{\alpha=1}^{N} b_\alpha + M - \frac{1}{2}\right]\right\} \times \frac{1}{(x - x_{\lim})^2}, \quad x \to x_{\lim}, \quad (48)$$

while in the second it becomes a constant. We see that in the cases described by Eqs. (47) and (48) the potentials do not always grow near the limits. However, since the coefficient of the singular term $(x - x_{\lim})^{-2}$ is always larger than $-\frac{1}{4}$, there is no falloff at the center and the system remains stable.

Therefore, our final statement is the following. For a stable quantum-mechanical system to be associated with the fundamental interval, it is necessary and sufficient that the condition (45) or (46) be satisfied at its end points. Intervals in which these conditions are satisfied will be termed quantum-mechanically stable. There may be several such intervals. The interval mapped to a segment of the x axis, where the Schrödinger boundary-value problem is formulated, will be referred to as the physical interval. By choosing the physical interval in different ways and ensuring that it is stable, we obtain different exactly and quasi-exactly solvable models. We note that the numbers ξ_i lying in the physical interval determine physical zeros, i.e., nodes of the wave functions.

The entire set of potentials of exactly or quasi-exactly solvable models corresponding to a given N can be obtained by allowing the function $b(\lambda)$, which contains all the information about the system, to be degenerate.

In the nondegenerate case this function is given by the expression

$$b(\lambda)=\sum_{\alpha=1}^{N}\frac{b_\alpha}{\lambda-a_\alpha}, \tag{49}$$

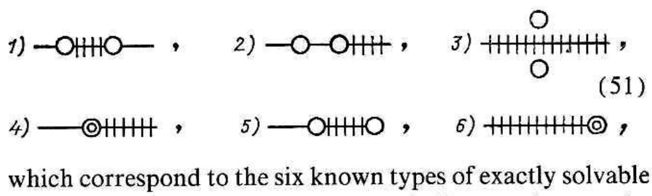
and has N simple poles in a finite region. In the degenerate case the simple poles can merge or go off to infinity, so that new functions $b(\lambda)$ arise. All the results of this section, given for the nondegenerate case, can also be generalized to a degenerate case. In particular, the equation for the parameters ξ_i preserves its original form:

$$\sum_{k=1}^{M}{}'\frac{1}{\xi_i-\xi_k}+b(\xi_i)=0,\quad i=1,\ \ldots,\ M \tag{50}$$

[all the information about the degeneracy is concentrated in the function $b(\lambda)$]. The other equations describing the degenerate case are too awkward, and we shall not give them here, but refer the interested reader to Ref. 30.

We shall use a graphical method to classify the exactly and quasi-exactly solvable models obtained by this method. The real λ axis is denoted by a horizontal line whose end points lie at infinity. The points a_α will be depicted by circles, and points formed by the merging of several points a_α will be depicted by several concentric circles. This rule holds for both finite points and infinite points. The physical interval will be denoted by a series of vertical lines.

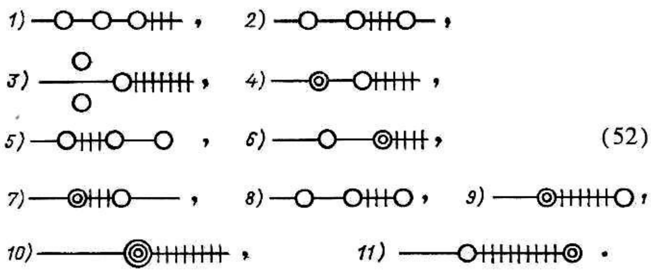
It is easily seen that in the case $N=2$ there are six different types of diagram:



(51)

which correspond to the six known types of exactly solvable models. In order to save space we shall not write out the potentials of these models, which are listed in Refs. 30 and 31. We only note that diagram 1 describes the trigonometric Pöschl–Teller potential, diagrams 2 and 3 describe hyperbolic Pöschl–Teller potentials, 4 is the Morse potential, 5 is the harmonic oscillator with centrifugal barrier, and 6 is the simple harmonic oscillator.

Let us now consider the case $N=3$ in more detail. For this case there are 11 different types of diagram:



(52)

The diagrams not written down here do not satisfy the stability criterion.[30] The diagrams in (52) describe 11 different types of quasi-exactly solvable models. Let us write down the potentials of these models together with the wave functions and energies of the exactly calculable states. We shall also give the form of the functions $b(\lambda)$, the physical intervals, and the stability conditions.

1. The elliptic potential of the first type:

$$\begin{aligned}V(x)=&4\left(\delta-\tfrac14\right)\left(\delta-\tfrac34\right)\mathrm{cs}^2 x\\&-4\left(\gamma-\tfrac14\right)\left(\gamma-\tfrac34\right)m\,\mathrm{cn}^2 x\\&+4\left(\beta-\tfrac14\right)\left(\beta-\tfrac34\right)m\,\mathrm{cd}^2 x\\&+4(1-m)\left(\beta+\gamma+\delta+M-\tfrac14\right)\\&\times\left(\beta+\gamma+\delta+M-\tfrac34\right)\mathrm{sc}^2 x,\end{aligned}$$

$$\psi(x)\propto[\mathrm{sc}^2 x]^{\delta-\frac14}[\mathrm{nc}^2 x]^{\gamma-\frac14}[\mathrm{dc}^2 x]^{\beta-\frac14}\prod_{i=1}^{M}[\mathrm{sc}^2 x-\xi_i].$$

$$m=\frac{\alpha-1}{\alpha},\quad \alpha>1,\quad x\in[0,\ K(m)],$$

$$\begin{aligned}E=&-\frac{\alpha+1}{4\alpha}-\frac{4}{\alpha}(\beta+\gamma+\delta+M-1)\{(\alpha+1)(M+\delta)\\&+(\alpha-1)(\gamma-\beta)+2\sigma_1(\xi)\},\end{aligned}$$

$$b(\lambda)=\frac{\beta}{\lambda+\alpha}+\frac{\gamma}{\lambda+1}+\frac{\delta}{\lambda},\quad \lambda\in[0,\ \infty],$$

$$\delta>\frac12,\quad \beta+\gamma+\delta+M<\frac12.$$

2. The elliptic potential of the second type:

$$\begin{aligned}V(x)=&4\left(\delta-\tfrac14\right)\left(\delta-\tfrac34\right)\mathrm{dc}^2 x\\&+4\left(\gamma-\tfrac14\right)\left(\gamma-\tfrac34\right)\mathrm{ds}^2 x\\&-4\left(\beta-\tfrac14\right)\left(\beta-\tfrac34\right)\mathrm{dn}^2 x\\&+4m(m-1)\left(\beta+\gamma+\delta+M-\tfrac14\right)\\&\times\left(\beta+\gamma+\delta+M-\tfrac34\right)\mathrm{sd}^2 x,\end{aligned}$$

$$\psi(x)\propto(\mathrm{sd}^2 x)^{\gamma-\frac14}(\mathrm{nd}^2 x)^{\beta-\frac14}(\mathrm{cd}^2 x)^{\delta-\frac14}\prod_{i=1}^{M}[\mathrm{sd}^2 x-\xi_i],$$

$$m=(1+\alpha)^{-1},\quad \alpha>0,\quad x\in[0,\ K(m)],$$

$$\begin{aligned}E=&\frac14\,\frac{\alpha-1}{\alpha+1}+\frac{4}{\alpha+1}(\beta+\gamma+\delta+M-1)\\&\times\{(\alpha-1)(M+\gamma)+(\alpha+1)(\delta-\beta)+2\sigma_1(\xi)\},\end{aligned}$$

$$b(\lambda)=\frac{\beta}{\lambda+\alpha}+\frac{\gamma}{\lambda}+\frac{\delta}{\lambda-1},\quad \lambda\in[0,\ 1],$$

$$\gamma>\frac12,\quad \delta>\frac12.$$

3. The elliptic potential of the third type:.

$$V(x)=\frac{4\left(\gamma-\frac{1}{4}\right)\left(\gamma-\frac{3}{4}\right)(\alpha^2-1)}{\mathrm{sc}^2 x\,\mathrm{dn}^2 x-\alpha}$$
$$+\frac{4\left(\gamma^*-\frac{1}{4}\right)\left(\gamma^*-\frac{3}{4}\right)(\alpha^{*2}-1)}{\mathrm{sc}^2 x\,\mathrm{dn}^2 x-\alpha^*}+\frac{4\left(\beta-\frac{1}{4}\right)\left(\beta-\frac{3}{4}\right)}{\mathrm{sc}^2 x\,\mathrm{dn}^2 x}$$
$$+4\left(\beta+\gamma+\gamma^*+M-\frac{1}{4}\right)\left(\beta+\gamma+\gamma^*+M-\frac{3}{4}\right)\mathrm{sc}^2 x\,\mathrm{dn}^2 x,$$
$$\psi(x)\propto(\mathrm{sc}^2 x\,\mathrm{dn}^2 x-\alpha)^{\gamma-\frac{1}{4}}(\mathrm{sc}^2 x\,\mathrm{dn}^2 x-\alpha^*)^{\gamma^*-\frac{1}{4}}\times(\mathrm{sc}^2 x\,\mathrm{dn}^2 x)^{\beta-\frac{1}{4}}\prod_{i=1}^{M}(\mathrm{sc}^2 x\,\mathrm{dn}^2 x-\xi_i),$$
$$m=\frac{1}{4}(\alpha+\alpha^*+2),\ |\alpha|=1,\ x\in[0,\ K(m)],$$
$$E=\frac{1}{4}(\alpha+\alpha^*)-4(\beta+\gamma+\gamma^*+M-1)\times\{-(\alpha+\alpha^*)(\beta+M)+(\alpha-\alpha^*)(\gamma-\gamma^*)+2\sigma_1(\xi)\},$$
$$b(\lambda)=\frac{\beta}{\lambda}+\frac{\gamma}{\lambda-\alpha}+\frac{\gamma^*}{\lambda-\alpha^*},\ \lambda\in[0,\ \infty],$$
$$\beta>\frac{1}{2},\ \beta+\gamma+\gamma^*+M<\frac{1}{2}.$$

(In all these equations, m is the modulus of the elliptic functions and $K(m)$ is the complete elliptic integral; see Ref. 32.)

4. The trigonometric potential of the first type:

$$V(x)=-4\gamma^2\cos^4 x+[4\gamma^2-8\gamma(\beta-1)]\cos^2 x$$
$$+4\left(\delta-\frac{1}{4}\right)\left(\delta-\frac{3}{4}\right)\mathrm{ctg}^2 x+4\left(\beta+\delta+M-\frac{1}{4}\right)\times\left(\beta+\delta+M-\frac{3}{4}\right)\mathrm{tg}^2 x,$$
$$\psi(x)\propto(\mathrm{tg}^2 x)^{\delta-\frac{1}{4}}(\cos^2 x)^{-\beta+\frac{1}{4}}e^{-\gamma\cos^2 x}\prod_{i=1}^{M}(\mathrm{tg}^2 x-\xi_i),$$
$$x\in\left[0,\ \frac{\pi}{2}\right],$$
$$E=-8\left(\delta-\frac{1}{4}\right)\left(\delta-\frac{3}{4}\right)-8\gamma(\beta-1)-8\delta(\gamma+\beta)+1-8M(M-1+\beta+\gamma+2\delta)-8(M-1+\beta+\delta)\sigma_1(\xi),$$
$$b(\lambda)=\frac{\beta}{\lambda+1}+\frac{\gamma}{(\lambda+1)^2}+\frac{\delta}{\lambda},\ \lambda\in[0,\ \infty],$$
$$\delta>\frac{1}{2},\ \beta+\delta+M<\frac{1}{2}.$$

5. The trigonometric potential of the second type:

$$V(x)=4\left(\gamma-\frac{1}{4}\right)\left(\gamma-\frac{3}{4}\right)\frac{1}{\sin^2 x}+4\left(\delta-\frac{1}{4}\right)\left(\delta-\frac{3}{4}\right)\frac{1}{\cos^2 x}+4\beta[2(\gamma+\delta+M)+\beta]\sin^2 x-4\beta^2\sin^4 x,$$
$$\psi(x)\propto(\sin^2 x)^{\gamma-\frac{1}{4}}(\cos^2 x)^{\delta-\frac{1}{4}}e^{-\beta\sin^2 x}\prod_{i=1}^{M}(\sin^2 x-\xi_i),$$
$$x\in[0,\ \pi/2],$$
$$E=4\left(\gamma-\frac{1}{4}\right)\left(\gamma-\frac{3}{4}\right)+4\left(\delta-\frac{1}{4}\right)\left(\delta-\frac{3}{4}\right)+8\gamma(\beta+\delta)-\frac{1}{2}+4M(M-1)+8M(\gamma+\delta+\beta)-8\beta\sigma_1(\xi),$$
$$b(\lambda)=-\beta+\frac{\gamma}{\lambda}+\frac{\delta}{\lambda-1},\ \lambda\in[0,\ 1],$$
$$\gamma>\frac{1}{2},\ \delta>\frac{1}{2}.$$

6. The hyperbolic potential of the first type:

$$V(x)=4\left(\beta-\frac{1}{4}\right)\left(\beta-\frac{3}{4}\right)\mathrm{th}^2 x+4[\delta^2+2\delta(\gamma-1)]\mathrm{sh}^2 x+4\delta^2\mathrm{sh}^4 x+4\left(\beta+\gamma+M-\frac{1}{4}\right)\left(\beta+\gamma+M-\frac{3}{4}\right)\frac{1}{\mathrm{sh}^2 x},$$
$$\psi(x)\propto(\mathrm{cth}^2 x)^{\beta-\frac{1}{4}}(\mathrm{sh}^2 x)^{-\gamma+\frac{1}{2}}e^{-\delta\,\mathrm{sh}^2 x}\prod_{i=1}^{M}\left(\frac{1}{\mathrm{sh}^2 x}-\xi_i\right),$$
$$x\in[0,\ \infty],$$
$$E=4\left(\beta-\frac{1}{4}\right)\left(\beta-\frac{3}{4}\right)-4\left(\gamma-\frac{1}{2}\right)^2-8\delta(\gamma-1)-8\beta\delta-8(\delta+\gamma)M-4M(M-1)-8(M-1+\beta+\gamma)\sigma_1(\xi),$$
$$b(\lambda)=\frac{\beta}{\lambda+1}+\frac{\gamma}{\lambda}+\frac{\delta}{\lambda^2},\ \lambda\in[0,\ \infty],$$
$$\delta>0,\ \beta+\gamma+M<\frac{1}{2}.$$

7. The hyperbolic potential of the second type:

$$V(x)=4\left(\beta-\frac{1}{4}\right)\left(\beta-\frac{3}{4}\right)\mathrm{cth}^2 x-4[\delta^2+2\delta(\gamma-1)]\mathrm{ch}^2 x+4\delta^2\mathrm{ch}^4 x+4\left(\beta+\gamma+M-\frac{1}{4}\right)\left(\beta+\gamma+M-\frac{3}{4}\right)\mathrm{th}^2 x,$$
$$\psi(x)\propto(\mathrm{th}^2 x)^{\beta-\frac{1}{4}}(\mathrm{ch}^2 x)^{-\gamma+\frac{1}{2}}e^{\delta\,\mathrm{ch}^2 x}\prod_{i=1}^{M}(\mathrm{th}^2 x-\xi_i),$$
$$x\in[0,\ \infty],$$
$$E=8\left(\beta-\frac{1}{4}\right)\left(\beta-\frac{3}{4}\right)-8\delta(\gamma-1)+8\beta(\gamma-\delta)-1+8M(M-1)+8M(2\beta+\gamma-\delta)-8(M-1+\beta+\gamma)\sigma_1(\xi),$$
$$b(\lambda)=\frac{\beta}{\lambda}+\frac{\gamma}{\lambda-1}+\frac{\delta}{(\lambda-1)^2},\ \lambda\in[0,\ 1],$$
$$\beta>\frac{1}{2},\ \delta<0.$$

8. The hyperbolic potential of the third type:

$$V(x)=4\left(\delta-\frac{1}{4}\right)\left(\delta-\frac{3}{4}\right)\frac{1}{\mathrm{sh}^2 x}-4\left(\gamma-\frac{1}{4}\right)\left(\gamma-\frac{3}{4}\right)\frac{1}{\mathrm{ch}^2 x}+4\beta^2\mathrm{sh}^4 x+4[\beta^2-2\beta(\gamma+\delta)-2\beta M]\mathrm{sh}^2 x,$$
$$\psi(x)\propto(\mathrm{sh}^2 x)^{\delta-\frac{1}{4}}(\mathrm{ch}^2 x)^{\gamma-\frac{1}{4}}e^{-\beta\,\mathrm{sh}^2 x}\prod_{i=1}^{M}(\mathrm{sh}^2 x-\xi_i),$$
$$x\in[0,\ \infty],$$
$$E=-4\left(\gamma-\frac{1}{4}\right)\left(\gamma-\frac{3}{4}\right)-4\left(\delta-\frac{1}{4}\right)\left(\delta-\frac{3}{4}\right)+8\delta(\beta-\gamma)+1+8M(\beta-\gamma-\delta)-4M(M-1)+8\beta\sigma_1(\xi),$$
$$b(\lambda)=-\beta+\frac{\gamma}{\lambda+1}+\frac{\delta}{\lambda},\ \lambda\in[0,\ \infty],$$
$$\beta>0,\ \delta>\frac{1}{2}.$$

9. The exponential potential:

$$V(x)=\delta^2e^{-2x}+2\delta(\gamma-1)e^{-x}-2\beta(\gamma+M)e^{x}+\beta^2e^{2x},$$
$$\psi(x)\propto\exp\left\{\left(\gamma-\frac{1}{2}\right)x-\beta e^{x}-\delta e^{-x}\right\}\prod_{i=1}^{N}(e^{x}-\xi_i),$$
$$x\in[-\infty,\ \infty],$$
$$E=-\left(\gamma-\frac{1}{2}\right)^2+2\beta\delta-2\gamma M-M(M-1)+2\beta\sigma_1(\xi),$$
$$b(\lambda)=-\beta+\frac{\gamma}{\lambda}+\frac{\delta}{\lambda^2},\ \lambda\in[0,\ \infty],\ \delta>0,\ M-\beta<0.$$

10. The rational potential of the first type:

$$V(x)=4\delta^2x^6+8\gamma\delta x^4+4(\gamma^2+2\delta\beta-3\delta)x^2+4\left(\beta+M-\frac{1}{4}\right)\left(\beta+M-\frac{3}{4}\right)\frac{1}{x^2},$$

$$\psi(x)\propto(x^2)^{-\beta+\frac{3}{4}}e^{-\gamma x^2-\frac{1}{2}x^4}\prod_{i=1}^{M}\left(\frac{1}{x^2}-\xi_i\right),$$

$$x\in([0,\ \infty],$$

$$E=-8(\beta+M-1)[\gamma+\sigma_1(\xi)],$$

$$b(\lambda)=\frac{\beta}{\lambda}+\frac{\gamma}{\lambda^2}+\frac{\delta}{\lambda^3},\ \lambda\in[0,\ \infty],\ \delta>0,\ \beta+M<\frac{1}{2}.$$

11. The rational potential of the second type:

$$V(x)=4\gamma^2x^6+8\beta\gamma x^4+4(\beta^2-\gamma-2\gamma\delta-2\gamma M)x^2+4\left(\delta-\frac{1}{4}\right)\left(\delta-\frac{3}{4}\right)\frac{1}{x^2},$$

$$\psi(x)\propto(x^2)^{\delta-\frac{1}{4}}e^{-\beta x^2-\frac{\gamma x^4}{2}}\prod_{i=1}^{M}(x^2-\xi_i),$$

$$x\in[0,\ \infty],$$

$$E=8\beta(\delta+M)+8\gamma\sigma_1(\xi),$$

$$b(\lambda)=-\beta-\gamma\lambda+\frac{\delta}{\lambda},\ \lambda\in[0,\ \infty],\ \delta>\frac{1}{2},\ \gamma>0.$$

These models were written down in Ref. 18. We note that some of them, namely, models 9, 11, and also special cases of models 5 and 7 (without the singular term in the potential) were found earlier.[17]

Thus, we therefore have obtained 11 types of quasi-exactly solvable models and have derived their stability conditions. The following rule can be formulated for these models: if the stability condition of any model does not involve M explicitly, we have an infinite series of quasi-exactly solvable models of any, arbitrarily high order (the potentials 2, 5, 7, 8, and 11). Otherwise, there is only a finite series of stable quasi-exactly solvable models with some maximum order (the potentials 1, 3, 4, 6, 9, and 10). There is a particular analog of this rule also for exactly solvable models: if the stability condition does not involve M explicitly, the model has an infinite discrete spectrum (the potentials 1, 5, and 6). Otherwise, the exactly solvable model admits only a finite number of bound states, and this situation corresponds to potential wells of finite depth (the potentials 2, 3, and 4).

Earlier we made the remark that in the case $N=4$ it is also possible to have infinite series of quasi-exactly solvable models if the parameters b_α satisfy the condition (39). The potentials of these models in the nondegenerate case have the form

$$V(x)=\omega\sum_{\alpha=1}^{4}\frac{\left(b_\alpha-\frac{1}{4}\right)\left(b_\alpha-\frac{3}{4}\right)\prod_{\beta=1}^{4}{}'(a_\alpha-a_\beta)}{\lambda-a_\alpha},\qquad(53)$$

where $\lambda=\lambda(x)$ is the function determined from (42). However, all attempts to explicitly construct these potentials have convinced us that we do not obtain any new quasi-exactly solvable models which are different from the 11 models listed above. This result is a special case of a more general theorem proved in Ref. 30. According to this theorem, when the condition (39) is satisfied, Eq. (33) with $N=n$ reduces to Eq. (33) with $N=n-1$. This can be proved by a linear-fractional substitution of the variable λ in Eq. (33).

If in the case $N=4$ the parameters b_α do not satisfy the relation (39), the potential (53) acquires additional terms of the form

$$\left(\sum_{\alpha=1}^{4}b_\alpha+M-1\right)\left\{-\sum_{\alpha=1}^{4}a_\alpha\left(\sum_{\alpha=1}^{4}b_\alpha+M\right)+2\sum_{\alpha=1}^{4}b_\alpha a_\alpha+2\sigma_1(\xi)\right\}\lambda+\left(\sum_{\alpha=1}^{4}b_\alpha+M-1\right)\left(\sum_{\alpha=1}^{4}b_\alpha+M\right)\lambda^2.\qquad(54)$$

It is extended potentials of this type which are described by finite series of quasi-exactly solvable models up to seventh order.

If in the case $N=4$ the parameters b_α do not satisfy the relation (39), but there is a Z_2 symmetry in the problem, certain new series of quasi-exactly solvable models of arbitrary order arise which are not present in the list (52). An example of such a model is the model with the linear sextic potential for $p=+1$ discussed in the Introduction.

5. RATIONAL ONE-DIMENSIONAL QUASI-EXACTLY SOLVABLE MODELS AND THE COULOMB PROBLEM

In the preceding section we constructed a series of exactly and quasi-exactly solvable models and wrote down their solutions. However, from these solutions we cannot see to what states of the quantum systems they correspond. Moreover, we do not even know the orders of our quasi-exactly solvable models, not to speak of the numbering of the states described by them.

According to the oscillation theorem, the number of a state in the one-dimensional case is determined by the number of real zeros of the wave function inside the physical interval in which the boundary-value problem is formulated. On the λ axis one of the fundamental intervals corresponds to this interval, and the numbers ξ_i play the role of the wave-function nodes. Therefore, the classification problem can be reduced to the simple calculation of the real numbers ξ_i lying in the physical interval.

In the nondegenerate case the numbers ξ_i satisfy a system of algebraic equations

$$\sum_{i=1}^{M}\frac{1}{\xi_i-\xi_k}+\sum_{\alpha=1}^{N}\frac{b_\alpha}{\xi_i-a_\alpha}=0,\ i=1,\ \ldots,\ M,\qquad(55)$$

in which a_α and b_α are, in general, complex numbers:

$$a_\alpha=a_\alpha^{(1)}+ia_\alpha^{(2)},\ b_\alpha=b_\alpha^{(1)}+ib_\alpha^{(2)}.\qquad(56)$$

Therefore, ξ_i should also be taken to be complex:

$$\xi_i=\xi_i^{(1)}+i\xi_i^{(2)}.\qquad(57)$$

Substitution of (56) and (57) into (55) leads to a system of real equations, which can be written as

$$\sum_{k=1}^{M}{}'\frac{\boldsymbol{\xi}_i-\boldsymbol{\xi}_k}{|\boldsymbol{\xi}_i-\boldsymbol{\xi}_k|^2}+\sum_{\alpha=1}^{N}b_\alpha^{(1)}\frac{\boldsymbol{\xi}_i-\mathbf{a}_\alpha}{|\boldsymbol{\xi}_i-\mathbf{a}_\alpha|^2}+\sum_{\alpha=1}^{N}b_\alpha^{(2)}\hat{\varepsilon}\frac{\boldsymbol{\xi}_i-\mathbf{a}_\alpha}{|\boldsymbol{\xi}_i-\mathbf{a}_\alpha|^2}=0,\qquad(58)$$

where $\boldsymbol{\xi}_i=(\xi_i^{(1)},\ \xi_i^{(2)})$ and $\mathbf{a}_\alpha=(a_\alpha^{(1)},\ a_\alpha^{(2)})$ are real two-dimensional vectors, and $\hat{\varepsilon}$ is the matrix rotating the vectors by 90° counterclockwise.

Equation (58) can be interpreted as the condition for an extremum of the function

$$V(\xi_1, \ldots, \xi_M) = -\sum_{i<k} q_i q_k \ln|\xi_i - \xi_k|$$
$$-\sum_{i,\alpha} q_i b_\alpha^{(1)} \ln|\xi_i - \mathbf{a}_\alpha|$$
$$-\sum_{i,\alpha} q_i b_\alpha^{(2)} f(\xi_i - \mathbf{a}_\alpha),$$
$$q_i \equiv 1,\ i = 1, \ldots, M, \qquad (59)$$

in which $f(\mathbf{x}) \equiv \tan^{-1}(x^{(2)}/x^{(1)})$ is the angular coordinate of the vector $\mathbf{x}$ and $q_i = 1$ are unit constants. It is easy to see that (59) is none other than the potential of a two-dimensional (logarithmic) Coulomb system consisting of M moving particles with coordinates ξ_i and charges q_i and N stationary particles with coordinates $\mathbf{u}_\alpha$ and two types of charge: ordinary electric charges $b_\alpha^{(1)}$, and magnetic charges $b_\alpha^{(2)}$ creating a vortex electrostatic field. It can be verified that $b_\alpha^{(1)}$ and $b_\alpha^{(2)}$ do actually correspond to electric and magnetic charges by writing down the potential produced by a single particle located at the origin,

$$\Phi = b^{(1)} \ln|\mathbf{x}| + b^{(2)} f(\mathbf{x}), \qquad (60)$$

and noting that this potential can be obtained from the equations of $(2+1)$-dimensional magnetoelectrodynamics,

$$\partial_\mu F_{\mu\nu} = j_\nu,\ \partial_\mu \tilde{F}_\mu = g,$$
$$F_{\mu\nu} \equiv \partial_\mu A_\nu - \partial_\nu A_\mu,\ \tilde{F}_\mu \equiv \frac{1}{2}\varepsilon_{\mu\nu\lambda} F_{\nu\lambda} \qquad (61)$$

in the static limit. In fact, taking $g \sim b^{(2)}\delta(r)$ and $j_0 \sim b^{(1)}\delta(r)$, $j_{1,2} = 0$, and finding the static solution (61) in the class of functions of the form $A_0 = \Phi$, $A_{1,2} = 0$, we obtain (60).

We therefore see that the problem of finding a solution to the system of algebraic equations (55) is equivalent to the problem of finding the equilibrium positions of a system of Coulomb particles moving in the field of stationary dyons. In general, this problem is quite complicated. However, in the special case in which the parameters a_α and b_α either are real or are complex-conjugate pairs (we recall that this is the condition for the quantum-mechanical potential to be real) it simplifies considerably. In fact, the presence of a Z_2 symmetry in the system leads to the existence of a straight line (coinciding in the present case with the real λ axis) on which all the Coulomb forces (from the stationary dyons) are longitudinal. The problem of the equilibrium of the particles moving on this line therefore becomes one-dimensional, which allows us to seek solutions of Eq. (55) in the class of real numbers.[2)]

Let us now consider the structure of the (real) λ axis in more detail. The real numbers u_α, i.e., the coordinates of the stationary particles lying on this axis, divide it into some number of intervals, which above were called fundamental intervals. We shall say that a fundamental interval is classically stable if the moving Coulomb particles contained in it cannot reach its limits. A limit of a fundamental interval can be finite (it can coincide with one of the real points u_α) or infinite. The condition for the classical stability of an interval near its finite limit can obviously be written in the form

$$b_\alpha > 0 \qquad (62)$$

as the condition for the charges of the moving and stationary particles to be the same. In the case of the stability condition for an interval near an infinite limit, it can be expressed as the condition that the total charge of the system be negative:

$$\sum_{\alpha=1}^{N} b_\alpha + M < 0. \qquad (63)$$

We note that the conditions for classical and quantum stability of the fundamental intervals nearly coincide! The only difference is that in the quantum case [see Eqs. (45) and (46)] the right-hand sides of the inequalities contain $\frac{1}{2}$ rather than 0.

It is easy to verify that the maximum number of stable fundamental intervals is $N-1$. To each distribution of particles in stable intervals there corresponds a position of stable equilibrium. Therefore, the number of different (inequivalent) solutions of Eq. (55) is equal to the number of ways to distribute M particles in $N-1$ intervals. Of these $N-1$ intervals, only one is physical. Since the number of moving particles lying in the physical interval determines the number of nodes of the wave functions, i.e., the number of states, it is possible for us to solve the classification problem completely. Here it is worth noting that the familiar phenomenon of repulsion of wave-function nodes, i.e., the fact that any decrease of the distance between nodes requires a large change in the potential, has a simple explanation in the language of the electrostatic analog: this phenomenon is nothing but ordinary Coulomb repulsion. Let us consider some specific cases.

1) $N = 2$. There is only a single stable fundamental interval, which must be identified with the physical interval. It can be either finite or infinite. If the interval is finite, it can contain any number of moving particles, and a position of stable equilibrium will exist for each number of particles. We therefore obtain problems in which all the levels of the system are numbered. If the interval is infinite, then all the levels up to a certain one are numbered. We note that for $N = 2$ the energy levels are expressed directly in terms of the number of particles M.

2) $N = 3$. In this case two intervals can be stable, and only one of them is physical. M particles can be distributed between these two intervals in $M + 1$ ways. Here the physical interval contains 0, 1, ..., M particles. Therefore, to each M there corresponds a quasi-exactly solvable model of order $M + 1$ in which the first $M + 1$ energy levels, beginning with the ground state, can be calculated exactly. Here the energy levels are expressed in terms of the centers of mass of the corresponding particle configurations $\sigma_1(\xi)$.

Cases with larger N can be treated in a similar manner. To summarize, we can conclude that the one-dimensional quantum systems studied in the preceding section are exactly equivalent to two-dimensional classical systems of Coulomb particles possessing electric and magnetic charges. By solving the purely classical problem of the equilibrium of such a system of particles, it is possible to obtain detailed information both about the wave functions and about the energies of exactly and quasi-exactly solvable models.

The discussion can easily be generalized to the degenerate case. As we have already noted, when degeneracy is present, certain points merge or move out to infinity. In the language of the Coulomb problem, the merging of points a_α corresponds to the formation of all possible dipoles, multipoles, and so on. Movement of points a_α to infinity corresponds to the appearance of an external uniform or nonuniform electrostatic field in the system.

The electrostatic analog makes it easy to construct the trajectories in complex parameter space along which energy levels transform into other levels. Let us consider the example of the quasi-exactly solvable model with $N = 3$, when there are six parameters a_α and b_α, $\alpha = 1, 2, 3$. We take the two finite intervals $[a_1 a_2]$ and $[a_2 a_3]$ to be the nonphysical and physical intervals, respectively. Their stability is ensured by the requirement $b_{1,2,3} > 0$. Let the initial position of the ξ-particles correspond to the Kth level, i.e., let K ξ-particles be located in the right-hand interval, and $M - K$ in the left-hand one. (We assume that $M > K$.) Let us consider the following trajectory: 1) the moving particle a_2 leaves its original location and moves to the position of stable equilibrium: $a_2 \to a_2'$; 2) its charge decreases to zero: $b_2 \to b_2' \equiv 0'$, with the equilibrium position corrected accordingly: $a_2' \to a_2''$; 3) the now-neutral a_2 particle moves to the left through L ξ-particles: $a_2'' \to a_2'''$; 4) its charge is restored: $0 \equiv b_2' \to b_2$; 5) its original location is restored: $a_2''' \to a_2$. The final configuration of the system of ξ-particles obviously corresponds to the $(K + L)$th energy level. The same result can be obtained by changing other parameters of the system, for example, the parameters a_1, a_3, b_2, and b_3 for constant a_2 (Ref. 33).

We can also use the electrostatic analog to study the spectral singularities in quasi-exactly solvable models. This is related to the fact that the singularities of Bender and Wu, which arise in the intersection (plaiting) of the energy levels as functions of the parameters of the system, can be interpreted as points where the system of particles becomes classically unstable. In order to illustrate this, we imagine that after the first two steps of the procedure described above the charge b_2 is made small and negative. This leads to the appearance of a weak attractive center, which does not spoil the existing particle equilibrium. If we now begin to move the particle a_2 to the left, at some instant the stable state of the ξ-particle closest to it becomes unstable, and it "falls into" particle a_2. It can be shown that: 1) two equilibrium positions—one stable. and one unstable—merge at the point where the instability arises; 2) the coordinates of all the ξ-particles, and also of the energy levels have a square-root singularity at this point related to the plaiting of the given level with the following one. The electrostatic analog helps in the calculation of the positions of all such singularities. It is easy to see that for small b_2 they lie at the points $\xi_1^{(0)} \pm c_i\sqrt{-b_2}$, where $\xi_i^{(0)}$ are the positions of stable equilibrium of all the ξ_i-particles in the absence of the particle a_2, and c_i are constants easily calculated using perturbation theory. We note that for positive b_2 the singularities are located at complex conjugate points.[33]

Another type of singularity arises in quasi-exactly solvable models when two or more nodes of a wave function merge. The L merging nodes ξ_i, $i = 1, \ldots, L$, as functions of the parameters of the system have root singularities of order L at the points of merging. Here the energy levels themselves as symmetric functions of the node coordinates remain regular at these points. By analyzing Eq. (55), it can easily be shown that such nodal singularities can arise only when the charge of one of the stationary particles is $b_\alpha = -(L-1)$. A stationary particle of this charge can accumulate L moving particles with unit charge, as a result of which a compound particle arises, having charge $+1$ and binding energy equal to zero. For such a particle to exist without decaying, the total force exerted on it by all the other particles must vanish. This condition leads to all the possible positions a_α (Ref. 33).

To conclude this section, we note that whereas the problem of finding the wave-function nodes for one-dimensional quantum systems is related to solving the problem of the equilibrium of charged particles in an external field, the problem of finding the nodal lines or the wave-function surfaces for systems of dimension $D \geqslant 2$ turns out to be related to the problem of the equilibrium of classical charged strings or membranes in an external field. Here the Schrödinger equation is not necessarily an equation admitting separation of variables.[34] In fact, let us consider the problem of constructing a D-dimensional Schrödinger equation $[-\Delta + V(x)]\psi(\mathbf{x}) = E\psi(\mathbf{x})$ which is exactly solvable for any one state. It is easily seen that for the choice

$$V(\mathbf{x}) = E + \Delta\psi(\mathbf{x})/\psi(\mathbf{x}), \tag{64}$$

where $\psi(\mathbf{x})$ is a smooth function, the Schrödinger equation has the function $\psi(\mathbf{x})$ itself as a formal solution. The requirement that the potential $V(\mathbf{x})$ be smooth imposes a number of constraints on the admissible form of the nodal surfaces of $\psi(\mathbf{x})$. To derive these we assume that the M nodal surfaces of $\psi(\mathbf{x})$ are described by the equations $\mathbf{x}_i = \mathbf{x}_i(t)$, $\mathbf{x}_i \in R_D$, $t \in R_{D-1}$, $i = 1, \ldots, M$. Then the wave function $\psi(\mathbf{x})$ (up to a sign) can be written as

$$\psi(\mathbf{x}) = \exp\left\{-\sum_{i=1}^{M}\int\frac{\sigma[\mathbf{x}_i(t)]\,d^{D-1}t}{|\mathbf{x}_i - \mathbf{x}_i(t)|^{D-1}}\right\}\exp F(\mathbf{x}), \tag{65}$$

where $\sigma[\mathbf{x}(t)]d^{D-1}t$ is an element of the nodal surface and $F(\mathbf{x})$ is a smooth function. The substitution of (65) into (64) and the requirement that $V(\mathbf{x})$ be smooth lead to the following system of integral equations in $\mathbf{x}_i(t)$:

$$\mathbf{n}[\mathbf{x}_i(t)]\left\{P\int\frac{[\mathbf{x}_i(t)-\mathbf{x}_i(t')]\,\sigma[\mathbf{x}_i(t')]\,d^{D-1}t'}{|\mathbf{x}_i(t)-\mathbf{x}_i(t')|^{D+1}} + \sum_{k=1}^{M}{}'\int\frac{[\mathbf{x}_i(t)-\mathbf{x}_k(t')]\,\sigma[\mathbf{x}_k(t')]\,d^{D-1}t'}{|\mathbf{x}_i(t)-\mathbf{x}_k(t')|^{D+1}} + \mathbf{b}(\mathbf{x}_i(t))\right\} = 0, \tag{66}$$

where $\mathbf{b}(\mathbf{x}) \equiv \nabla F(\mathbf{x})$. It is easy to see that for $D = 1$ the first term in (66) vanishes, while the remainder of the equation degenerates into Eq. (50), describing the equilibrium of Coulomb particles with coordinates x_i in an external force field. For $D > 1$ the resulting system can be interpreted as the equilibrium condition for M absolutely inelastic massless charged strings ($D = 2$) or membranes ($D \geqslant 3$) interacting according to the laws of $(D + 1)$-dimensional electrostatics in a D-dimensional subspace. The charge is distributed uniformly along the strings (surfaces of the membranes) with unit density. Equation (66) expresses the condition that the normal component of a force acting on each element of a string (membrane) due to the other strings (membranes) and also the external potential $F(\mathbf{x})$ vanish. This electrostatic analog allows us to understand the features of the nodal surfaces and the singularities associated with their relative location, and also to follow the variation in the nodal-surface shape as the potential is varied.[34]

6. CLASSICAL FORMULATION OF QUANTUM-MECHANICAL PROBLEMS

In the preceding section we showed that quasi-exactly solvable models can be formulated in the language of electrostatics. Since quasi-exactly solvable models are a limiting case of non-exactly solvable models (see the Introduction), a classical formulation is also possible for the former.[33,35,36]

As an example, let us consider the eleventh model in the list (52). For convenience, we replace the quantities ξ_i by their inverses $\nu_i = \xi_i^{-1}$. It is easy to show that the spectrum of this model is given by

$$E = 8\delta\left(\beta + \sum_{i=1}^{M} \nu_i\right), \tag{67}$$

where the numbers ν_i satisfy the system of equations

$$\sum_{k=1}^{M}{}' \frac{1}{\nu_i - \nu_k} + \frac{\beta}{\nu_i^2} + \frac{\gamma}{\nu_i^3} - \frac{M+\delta-1}{\nu_i} = 0, \quad i = 1, \ldots, M. \tag{68}$$

This system can be viewed as the equilibrium condition for M Coulomb particles with unit charge moving in an external potential corresponding to two wells separated by a barrier which is singular at the origin. We know that to each distribution of the particles between these two wells (for example, K particles in the right-hand well and $M-K$ particles in the left-hand one) there corresponds a certain position of stable equilibrium describing the kth energy level.

In the limit $M \to \infty$, Eq. (68) becomes infinitely complicated and a non-exactly solvable model arises. The finiteness of the potential of this model is ensured by the dependence of β and γ on M. Determining it from the conditions $4(\beta^2 - 2\gamma M) = g$ and $8\beta\gamma = 1$, we find that $\beta \sim M^{1/3}$ and $\gamma \sim M^{-1/3}/2$. Therefore, the potential of the limiting model has the form

$$V(x) = \left(2\delta - \frac{1}{2}\right)\left(2\delta - \frac{3}{2}\right)\frac{1}{x^2} + gx^2 + \frac{1}{2}x^4. \tag{69}$$

As before, the spectral problem for the non-exactly solvable potential (69) can be formulated in classical language. Making the substitutions $\beta = bM^{1/3}$, $\gamma = (b^2/2)M^{-1/3}$, and $\nu_i = b\tau_i M^{-2/3}$ in (67) and (68), we find

$$E = 4\delta b M^{1/3}\left[1 + \sum_{i=1}^{M} \frac{\tau_i}{M}\right]; \tag{70}$$

$$\sum_{k=1}^{M} \frac{1}{\tau_i - \tau_k}\frac{1}{M} + \frac{1}{\tau_i^2} + \frac{1}{2}\frac{1}{\tau_i^3} - \left(1 + \frac{\delta-1}{M}\right)\frac{1}{\tau_i} = 0, \quad i = 1, \ldots, M. \tag{71}$$

Introducing the particle distribution density $\rho(\tau)$, in the limit $M \to \infty$ we obtain (in leading order)

$$E = 4\delta b M^{1/3} e, \quad e = 1 + \int_{-\infty}^{\infty} \rho(\tau)\,\tau\,d\tau, \tag{72}$$

where $\rho(\tau)$ satisfies the integral equation

$$\int_{-\infty}^{\infty} \frac{\rho(\tau')}{\tau - \tau'}d\tau' + \frac{1}{\tau^2} + \frac{1}{2}\frac{1}{\tau^3} - \frac{1}{\tau} = 0, \quad \int_{-\infty}^{\infty} \rho(\tau)\,d\tau = 1. \tag{73}$$

It is easy to see that this equation is the equation for the equilibrium of a charged "liquid" (with total charge 1) distributed between two separate wells. The charge $Q = K/M$ of the liquid in the right-hand well determines the number of the energy level, and the center of mass of the liquid is the value of the energy. It is obvious that for any excitations with finite number K in the model (69) the charge of the liquid in the right-hand well (in leading order) is equal to zero. Then the solution of Eq. (73) has the form

$$\rho(\tau) = \frac{2}{\pi}(-\tau)^{-3}(1+4\tau)^{1/2},$$
$$-\infty < \tau < -\frac{1}{4}; \quad \rho(\tau) = 0, \quad \tau > -\frac{1}{4}. \tag{74}$$

Equation (74) can be used to find the distribution of unphysical zeros of the wave function: $x_n \sim in^{1/3}$, which agrees with the semiclassical result. Substitution of (74) into (72) gives $e = 0$, which is consistent with the finiteness of the energy levels in the model (69). By including the corrections to the solution (74) obtained by iterating Eq. (71) in small (for $M \to \infty$) deviations of this equation from its limiting variant (73), it can be shown that $e \sim M^{-1/3}$, which leads to a finite expression for the energy levels coinciding with the semiclassical result for $K \gg 1$. For $K \gtrsim 1$ the corrections do not form a decreasing series (owing to the fact that the model (69) is not exactly solvable). It can be shown that of the quasi-exactly solvable models (52), only models with rational and trigonometric potentials can be reduced to stable, non-exactly solvable models.[36]

The procedure described above for reducing quasi-exactly solvable models to non-exactly solvable models is possible for models with arbitrary N. As an example, let us consider the rational model corresponding to the case $N = 4$ and obtained as a result of the degeneracy of the potentials described by Eqs. (53) and (54):

$$V(x) = \left(2\delta - \frac{1}{2}\right)\left(2\delta - \frac{3}{4}\right)\frac{1}{x^2} - \left[2\beta\left(M + \delta + \frac{1}{2}\right) - \alpha^2 + 2\gamma\sum_{i=1}^{M}\frac{1}{\nu_i}\right]x^2 + [\gamma(M+\delta+1) - \alpha\beta]\,x^4 + \frac{1}{4}[\beta^2 + 2\alpha\gamma]\,x^6 + \frac{1}{4}\beta\gamma x^8 + \frac{1}{16}\gamma^2 x^{10}. \tag{75}$$

This model has the solution

$$E = 4\delta\left(\alpha + \sum_{i=1}^{M}\nu_i\right),$$

where the ν_i are numbers satisfying the system of equations

$$\sum_{k=1}^{M}{}'(\nu_i - \nu_k)^{-1} + \gamma\nu_i^{-4} + \beta\nu_i^{-3} + \alpha\nu_i^{-2} - (M+\delta-1)\,\nu_i^{-1} = 0. \tag{76}$$

This model differs from the one discussed above in that its potential depends on the form of the solution. Nevertheless, the M dependence of the parameters α, β, and γ can be chosen in such a way that the dependence of the potential on the form of the solution becomes vanishingly small in the limit $M \to \infty$. This dependence is found from the equations $\beta^2 + 2\alpha\gamma = 4$, $\gamma M - \alpha\beta = B$, and $2\beta M - \alpha^2 + 2\gamma\Sigma_{i=1}^{M}\nu_i^{-1} = A$. The non-exactly solvable model arising in the limit $M \to \infty$ has the form

$$V(x) = Ax^2 + Bx^4 + x^6 + \left(2\delta - \frac{1}{2}\right)\left(2\delta - \frac{3}{2}\right)\frac{1}{x^2}. \tag{77}$$

Going to the particle distribution density $\rho(\tau)$, for it we can obtain a system of equations which, as in the preceding case, can be solved exactly. In the case $K \gg 1$ the inclusion of corrections to our solution leads to results for both the energy levels and the distribution of the wave-function nodes which coincide with the semiclassical results. Similarly, it can be shown that quantum-mechanical models with even polynomial potentials of degree $2n$ can be obtained as limiting cases of quasi-exactly solvable models with $N = n + 1$. This implies that the spectral problem for any non-exactly solvable model in one-dimensional quantum mechanics can be formulated in purely classical language, in terms of the problem of finding the equilibrium of an infinite number of charged Coulomb particles in an external classical potential.[36]

7. RATIONAL ONE-DIMENSIONAL QUASI-EXACTLY SOLVABLE MODELS AND MAGNETIC SYSTEMS BASED ON THE ALGEBRA $SL(2)$

Let us return again to the system of algebraic equations (35), (36) describing the spectra of exactly and quasi-exactly solvable models. It turns out that these equations coincide exactly with the Bethe-ansatz equation for completely integrable, nonlocal spin systems on a finite one-dimensional lattice. The Hamiltonian of such a system can be chosen to be any linear combination

$$H_\alpha = \sum_{\beta=1}^{N}{}' \frac{S_\alpha^+ S_\beta^- + S_\alpha^- S_\beta^+ - 2S_\alpha^0 S_\beta^0}{a_\alpha - a_\beta} \tag{78}$$

of the operators acting in the direct product $W = W_1 \otimes W_2 \otimes \ldots \otimes W_N$ of representation spaces of the algebra $SL(2)$. Here S_α^-, S_α^+, and S_α^0 are the generators of this algebra acting in the representation space W_α of the site α. It is easily verified that all the operators (78) commute with each other, $[H_\alpha, H_\beta] = 0$, so that they have a common spectrum. As we shall see, the parameters a_α enter explicitly into the Hamiltonian and play the role of coupling constants characterizing the strength of the interaction between spins located at different sites. The parameters b_α do not appear explicitly in the Hamiltonian; they are included in the definition of the generators, characterizing the representations in which they act. They are related to the "spins" of infinite-dimensional irreducible representations of the algebra $SL(2)$, which can be realized as

$$S_\alpha^+ = t_\alpha, \quad S_\alpha^0 = t_\alpha \frac{\partial}{\partial t_\alpha} + b_\alpha, \quad S_\alpha^- = t_\alpha \frac{\partial^2}{\partial t_\alpha^2} + 2b_\alpha \frac{\partial}{\partial t_\alpha} \tag{79}$$

on the space of all analytic functions regular near the origin. In this space there exists a vector of lowest weight $|0\rangle_\alpha \equiv 1$ such that $S_\alpha^- |0\rangle_\alpha = 0$. The eigenvalue of the operator for the z projection of the spin S_α^0 on $|0\rangle$ is $-b_\alpha$, so that b_α is the spin of the irreducible representation of the algebra $SL(2)$ with opposite sign. This is confirmed by the fact that the eigenvalue of the Casimir operator S_α^2 on $|0\rangle_\alpha$ is $(-b_\alpha)(-b_\alpha + 1)$. The representation (79) is infinite-dimensional, owing to the absence of a vector of highest weight, i.e., an analytic function, regular near the origin, on which the operator S_α^+ would give zero.

The spectra of the operators (78) in the case when the generators of the algebra $SL(2)$ act in a finite-dimensional representation have been calculated by Goden in Ref. 37 by the Bethe-ansatz method. This method can also be trivially generalized to the infinite-dimensional case in which we are interested. Following Goden, we introduce the following operator-valued functions of the parameter λ:

$$S^\pm(\lambda) = \sum_{\alpha=1}^{N} \frac{S_\alpha^\pm}{\lambda - a_\alpha}, \quad S^0(\lambda) = \sum_{\alpha=1}^{N} \frac{S_\alpha^0}{\lambda - a_\alpha}, \tag{80}$$

which are easily seen to satisfy the commutation relations

$$\left.\begin{aligned} &[S^\pm(\lambda),\ S^0(\mu)] = \pm \frac{1}{\lambda - \mu}[S^\pm(\lambda) - S^\pm(\mu)], \\ &[S^+(\lambda),\ S^-(\mu)[= \frac{2}{\lambda - \mu}[S^0(\lambda) - S^0(\mu)]. \end{aligned}\right\} \tag{81}$$

It follows from these relations that the operators

$$S^2(\lambda) \equiv \frac{1}{2}\{S^+(\lambda) S^-(\lambda) + S^-(\lambda) S^+(\lambda) - 2S^0(\lambda) S^0(\lambda)\} \tag{82}$$

commute

$$[S^2(\lambda),\ S^2(\mu)] = 0 \tag{83}$$

for all λ and μ. Therefore, $S^2(\lambda)$ can be viewed as an arbitrary function of the integrals of the motion. In particular, the residues at the simple poles of the function $S^2(\lambda)$ give the operators H_α. If we seek eigenfunctions of the operators $S^2(\lambda)$ in the Bethe form

$$|M\rangle = S^-(\xi_1) \times \ldots \times S^-(\xi_M)\ |0\rangle, \quad |0\rangle \equiv \prod_{\alpha=1}^{N} |0\rangle_\alpha, \tag{84}$$

where $\xi_1, \ldots, \xi_M$ are unknown numerical parameters (the quasimomenta of the magnons), then, using the commutation relations (81), we can obtain solvability conditions for the spectral equation in the class of functions of the form (84) which exactly coincide with Eqs. (35) and (36).

The models of magnetic systems considered here are not local spin systems. In the Hamiltonians of these systems each spin interacts with all the other spins, i.e., there is a long-range force and the situation is apparently a typical semiclassical one. This is also confirmed by the fact that the complete integrability of these models is related to the solutions of not the usual quantum Yang–Baxter equation (the triangle equation), but the so-called classical triangle equation[38] arising in the limit $\hbar \to 0$. The quantum S matrix $S^{\alpha\beta}(\lambda)$ is related to the classical matrix $X^{\alpha\beta}(\lambda)$ as $S^{\alpha\beta}(\lambda) \approx 1 + \hbar X^{\alpha\beta}(\lambda)$ (Ref. 38).

The classical triangle equation for the matrix $X^{\alpha\beta}(\lambda)$ has the form

$$\begin{aligned} &[X^{\alpha\beta}(\lambda_1),\ X^{\beta\gamma}(\lambda_2)] + [X^{\beta\gamma}(\lambda_2), \\ &X^{\gamma\alpha}(\lambda_3)] + [X^{\gamma\alpha}(\lambda_3),\ X^{\alpha\beta}(\lambda_1)] = 0, \\ &\lambda_1 + \lambda_2 + \lambda_3 = 0. \end{aligned} \tag{85}$$

If we seek a solution to it in the form $X^{\alpha\beta}(\lambda) = \mathbf{S}_\alpha \mathbf{S}_\beta \Delta(\lambda)$, then for the function $\Delta(\lambda)$ we find Eq. (25), whose solutions were used to construct exactly and quasi-exactly solvable models. We therefore see that in the theory of exactly and quasi-exactly solvable models Eq. (25) plays the same role as the triangle equation in the theory of completely integrable systems.

We have therefore succeeded in relating some of the exactly and quasi-exactly solvable models considered in Sec. 4 and characterized by the numbers N to magnetic systems based on algebras of the form $SL(2) \otimes \ldots \otimes SL(2)$ (N times). We see that by solving the spectral problem for these systems we can obtain an exhaustive amount of information on the spectra of the associated exactly and quasi-exactly solvable systems. So far we have considered only the nondegenerate case. It can be shown that an analogous correspondence holds also when degeneracy is present, but in this case the magnetic systems are based on other (contracted) Lie algebras. As before, the Hamiltonians of degenerate magnetic systems can be obtained from the operator generating function $S^2(\lambda)$ defined by (82). The operators $S^{\pm}(\lambda)$ and $S^0(\lambda)$ entering into (82) satisfy the same commutation relations as in (81). However, in the degenerate case the form of these operators is different from (80). We recall that a degeneracy always reduces either to the merging of poles, $a_{\alpha_1}, \dots, a_{\alpha_K} \to a$, or to the departure of poles to infinity, $a_{\beta_1}, \dots, a_{\beta_L} \to \infty$. To explicitly construct the operators $S^I(\lambda)$, $I = +, -, 0$, arising when such degeneracies are present, in (80) we need to make the substitution

$$\sum_{i=1}^{K} \frac{S^I_{\alpha_i}}{\lambda - a_{\alpha_i}} \to \frac{U^I_1}{\lambda - a} + \dots + \frac{U^I_K}{(\lambda - a)^K}, \tag{86}$$

or

$$\sum_{i=1}^{L} \frac{S^I_{\beta_i}}{\lambda - U_{\beta_i}} \to V^I_1 + \dots + \lambda^{L-1} V^I_L, \tag{87}$$

where U^I_n and V^I_n are new operators. The commutation relations for these operators are found by substituting the expansions (86) and (87) into the commutation relations (81).[39] (The classical analog of this method was formulated in Ref. 40, where it was used to study degeneracies in classical Hamiltonian systems.)

8. RATIONAL QUASI-EXACTLY SOLVABLE MODELS (MULTIDIMENSIONAL CASE)

In Sec. 4 we constructed a class of one-dimensional exactly and quasi-exactly solvable models related to linear multi-parameter equations of the form (33). According to the results of Secs. 1 and 2, these equations can also be used to construct multi-dimensional exactly and quasi-exactly solvable equations of the Schrödinger type on, in general, curved manifolds. There are an infinite number of inequivalent ways of transforming from (33) to a D-dimensional Schrödinger equation. We shall first consider one of the simplest of such transformations,[36] based on identifying the spectral parameter r_{D-1} as the energy and the parameters $r_0, \dots, r_{D-2}$ as separation constants. Using the explicit equations given in Sec. 2, it is easy to show that the resulting equation has the form[31,41]

$$\left\{ -\sqrt{g} \sum_{i=1}^{D} \frac{\partial}{\partial \lambda_i} \left[\frac{g^{ii}}{\sqrt{g}} \frac{\partial}{\partial \lambda_i} \right] + V(\lambda) \right\} \psi(\lambda) = E \psi(\lambda), \tag{88}$$

where

$$g^{ii} = \omega \prod_{\alpha=1}^{N} (\lambda_i - a_\alpha) \prod_{k=1}^{D}{}' (\lambda_i - \lambda_k)^{-1}, \quad g \equiv \prod_{i=1}^{D} g^{ii}, \tag{89}$$

and

$$V(\lambda) = \omega \sum_{i=1}^{D} \frac{\prod_{\alpha=1}^{N} (\lambda_i - a_\alpha)}{\prod_{k=1}^{D}{}' (\lambda_i - \lambda_k)} \left\{ \sum_{\alpha=1}^{N} \frac{\left(b_\alpha - \frac{1}{4}\right)\left(b_\alpha - \frac{3}{4}\right)}{(\lambda_i - a_\alpha)^2} + 2 \sum_{\alpha=1}^{N} \left(\sum_{\beta=1}^{N} \frac{b_\alpha b_\beta - \frac{1}{16}}{a_\alpha - a_\beta} \right) \frac{1}{\lambda_i - a_\alpha} \right\} - \sum_{n=D}^{N-1} r_n \left(\sum_{i=1}^{D} \frac{\lambda_i^{D-1+n}}{\prod_{k=1}^{D}{}' (\lambda_i - \lambda_k)} \right). \tag{90}$$

The solutions of Eq. (88) are

$$\psi(\lambda) = \prod_{i=1}^{D} \prod_{\alpha=1}^{N} (\lambda_i - a_\alpha)^{b_\alpha - \frac{1}{4}} \prod_{i=1}^{D} \prod_{j=1}^{M} (\lambda_i - \xi_j); \tag{91}$$

$$E = r_{D-1}. \tag{92}$$

The numbers ξ_i, $i = 1, \dots, M$, satisfy the system of algebraic equations (35), and the dependence of the spectral parameter r_{D-1} on these numbers is given by (38).

Let us consider the case in which all the parameters a_α and b_α are real and find the condition for the metric g_{ik} to be positive-definite, i.e., the condition for Eq. (88) to be elliptic. For this we note that the N points a_α divide the λ axis into $N + 1$ intervals, which above were called fundamental intervals. We define the sign of a fundamental interval as the sign of the expression $(\lambda - a_1) \times \dots \times (\lambda - a_N)$. Obviously, the signs of the fundamental intervals alternate. We now recall that at our disposal we have D independent variables $\lambda_1, \dots, \lambda_D$. We distribute them among the D intervals in such a way that the signs of the intervals occupied by these variables alternate. Then, obviously the sign of g^{ii} will be independent of i. The sign of ω can be chosen so as to ensure that all the diagonal elements of the metric tensor g_{ik} are positive.

Let us now formulate the Hermiticity condition for the Hamiltonian of the model (88) on the solutions (91). By the same arguments as in Sec. 4, it can be shown that for the Hamiltonian to be Hermitian it is necessary that all the intervals occupied by the variables λ_i be quantum-mechanically stable. The definition of quantum-mechanical stability is the same as in the one-dimensional case. From Eq. (91) it follows that the stability of all the intervals occupied by the variables λ_i guarantees that the wave function vanishes on the boundaries of the region in D-dimensional space in which the spectral problem is formulated. By analogy with the one-dimensional case, we shall refer to the fundamental intervals occupied by the variables λ_i as the physical intervals. It should also be noted that the number of stable intervals cannot exceed $N - 1$.

The discussion is easily generalized to the degenerate case arising when the points a_α merge or go to infinity. The graphical method formulated in Sec. 4 for one-dimensional problems can be used to classify the models obtained when degeneracies are present. The only difference from the one-dimensional case is that the number of physical intervals shown on the diagrams is now D rather than 1.

Let us consider some specific cases.

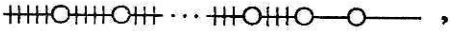
1) $D = N$. We have a single diagram of the form

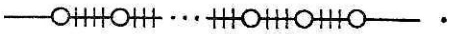
HHOHHOHH···HHOHHO—— •   (93)

The role of the energy parameter is played by the parameter

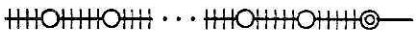
r_{N-1}, which, according to (38a), is zero. Therefore, the spectrum of the model is infinitely degenerate. The number of physical intervals, N, exceeds the number of stable intervals, $N-1$. Therefore, the Hamiltonian of the model is not Hermitian, and the wave functions are not normalizable. The space in which the equation is formulated is flat, as is easily verified by identifying the variables λ_i with generalized N-dimensional ellipsoidal coordinates.[41]

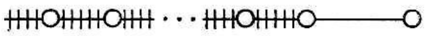
2) $D=N-1$. In this case for each N there are two nondegenerate diagrams:

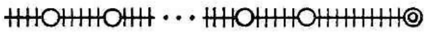
 , (94a)

 . (94b)

The role of the energy parameter is played by the parameter r_{N-2}, expressed in terms of the number M. We obtain exactly solvable models. In each of the diagrams all the physical intervals can be made stable. Therefore, the Hamiltonians of these models are Hermitian, and the wave functions are normalizable. In the nondegenerate case the space in which the equation is formulated is not flat. However, it may become flat when a degeneracy is present. We have the following degenerate diagrams:

 , (95a)

 , (95b)

 . (95c)

As before, the diagram (95a) describes a curved space. However, the diagrams (95b) and (95c) describe flat space, as is easily shown by identifying the variables λ_i with $(N-1)$-dimensional ellipsoidal and paraboloidal coordinates, respectively. In flat space we obtain a model of an $(N-1)$-dimensional spherically asymmetric anharmonic oscillator with centrifugal barriers.

3) $D=N-2$. In this case, for every N there are $N-1$ nondegenerate diagrams. For example, for $N=4$ we have three diagrams of the form

 , (96a)

 , (96b)

 . (96c)

The role of the energy parameter is played by the parameter r_{N-3}, which depends explicitly on ξ_i, $i=1, \ldots, M$. The parameter r_{N-2} entering into the potential depends only on M. We therefore obtain a quasi-exactly solvable model whose order is $K=(M+N-2)![(N-2)!M!]^{-1}$, i.e., the number of ways of distributing the M numbers ξ_i among $N-1$ intervals. All the physical intervals can be made stable, so that the Hamiltonians of these models are Hermitian, and the wave functions are normalizable. The coordinate space, which is not flat in the nondegenerate case, can become flat when degeneracy is present. For example, the diagrams

 (97)

describe multi-dimensional quasi-exactly solvable models in flat space, as can be verified by identifying the variables λ_i with $(N-3)$-dimensional ellipsoidal coordinates. The potentials of these models have the form

$$V(x)=\alpha^2r^6+2\alpha r^2\sum_{i=1}^{D}\beta_i x_i^2+\sum_{i=1}^{D}\beta_i^2x_i^2+\sum_{i=1}^{D}\frac{\gamma_i(\gamma_i-1)}{x_i^2}-\alpha\left\{\sum_{i=1}^{D}(2\gamma_i+1)+2+4M\right\}r^2,\quad r^2=\sum_{i=1}^{D}x_i^2. \tag{98}$$

They contain $2D+1$ independent parameters α, β_i, γ_i, $i=1, \ldots, D$, which can be expressed explicitly in terms of the original parameters of the problem u_α and b_α, $\alpha=1, \ldots, N$. The model is defined in the region specified by the inequalities $x_i\geqslant 0$, $i=1, \ldots, D$. The stability condition for the model can be written as the inequalities $\gamma_i>\frac{1}{2}$, $i=1, \ldots, D$, and $\alpha>0$. Let us consider the case with $M=1$ as an example. It is easy to show that in this case the Schrödinger equation for the potential (98) has the solutions

$$\psi(x)=\prod_{i=1}^{D}x_i^{\gamma_i}\left(1-\sum_{i=1}^{D}\frac{4\alpha x_i^2}{\xi-4\beta_i}\right)\exp\left\{-\frac{\alpha r^4}{4}-\frac{1}{2}\sum_{i=1}^{D}\beta_i x_i^2\right\}; \tag{99}$$

$$E=\xi+\sum_{i=1}^{D}\beta_i(2\gamma_i+1), \tag{100}$$

where the number ξ satisfies the equation

$$\xi=8\alpha\sum_{i=1}^{D}\frac{2\gamma_i+1}{\xi-4\beta_i}, \tag{101}$$

which is equivalent to an algebraic equation of order $M+1$. It is easy to see that if the stability conditions are satisfied, the wave functions vanish at the boundaries of the region in which the model is defined.

We note that if $\gamma_i=0$, the potential becomes a polynomial, which, owing to the $Z_2\times Z_2\times\ldots\times Z_2$ symmetry of the model allows it to be formulated on the entire space. A model of this type is interesting because in the limit $D\to\infty$ it is transformed into a nonlocal, nonrelativistic quasi-exactly solvable "field theory" with gapless excitations.

Multi-dimensional quasi-exactly solvable models also exist for $D<N-2$. For example, in the case $D=N-3$ there exist infinite series of quasi-exactly solvable models if the condition (39) is satisfied. However, as in the one-dimensional case, here it is not possible to construct any new models. They all reduce to the models with $D=N'-2$ studied earlier, where $N'=N-1$. If the condition (39) is not satisfied, we can obtain finite series of quasi-exactly solvable models of bounded maximum order

$$K_{\max}=3+\left[\frac{2D+2}{N-D-2}\right]. \tag{102}$$

This is also true for the general case $D<N-3$.

All the models obtained by the method described above are defined on, in general, curved manifolds. For a manifold to become flat for given D and N, there must be a special degeneracy taking $N-D$ points a_α to infinity.

The D-dimensional manifolds under discussion are embedded in N-dimensional flat space and can be described by a system of $N-D$ equations:

$$\sum_{\alpha=1}^{N}\sigma_\alpha x_\alpha^2 a_\alpha^{N-D-j}=-\delta_{1j},\quad j=1,\ \ldots,\ N-D, \tag{103}$$

where σ_α are the signs of the expressions $\Pi_{i=1}^{D}(\lambda_i - a_\alpha) \times \Pi_{\beta=1}^{N}{}'(a_\beta - a_\alpha)^{-1}$.

Let us now consider a more general method for constructing multi-dimensional exactly and quasi-exactly solvable models. Including the spectral parameters $r_D, ..., r_{N-1}$ in the potential, we take the energy spectral parameter to be an arbitrary linear combination of the parameters $r_0, ..., r_{D-1}$, and we identify the remaining $D-1$ linear combinations of these parameters as separation constants. As a result, we again obtain a Schrödinger equation of the form (88), but in which

$$g^{ii}=\prod_{\alpha=1}^{N}(\lambda_i-a_\alpha)\prod_{k=1}^{D}{}'(\lambda_i-\lambda_k)^{-1}\frac{\partial}{\partial\lambda_i}\sum_{n=1}^{D}c_n\sigma_n(\lambda-\alpha); \tag{104}$$

$$V=-\sum_{i=1}^{D}g^{ii}\times\left\{\sum_{n=D}^{N-1}r_n\lambda^{n-D}\prod_{\alpha=D+1}^{N}(\lambda_i-a_\alpha)^{-1}+(h^2g)^{1/4}\frac{\partial^2}{\partial\lambda_i^2}(h^2g)^{-1/4}\right\}-\sum_{i=1}^{D}\left[\frac{1}{h}\frac{\partial}{\partial\lambda_i}(hg^{ii})\right](h^2g)^{1/4}\frac{\partial}{\partial\lambda_i}(h^2g)^{-1/4}, \tag{105}$$

where

$$h\sim\prod_{i<k}^{D}(\lambda_i-\lambda_k)\prod_{i=1}^{D}\prod_{\alpha=1}^{N}(\lambda_i-a_\alpha)^{2b_\alpha-1} \tag{106}$$

Here $\sigma_n(\lambda - a)$ are symmetric polynomials of order n in D variables $\lambda_i - a$, $i = 1, ..., D$, where $a = \min_\alpha\{a_\alpha\}$, and c_n are arbitrary constants. For the Schrödinger equation to be elliptic it is necessary that: 1) the signs of the physical intervals alternate; 2) all the constants c_n be non-negative; and 3) the interval $[-\infty, a]$ not be a physical interval. The Hermiticity of the Hamiltonian on the solutions and also the normalizability of the wave functions are ensured by the requirement that all the physical intervals be quantum-mechanically stable. It is easy to show that for $D = N-1$ we obtain exactly solvable models. These models differ from the ones discussed above in that now the energy spectral parameter depends not only on M, but also ξ_i. We therefore obtain exactly solvable equations with plaited energy levels. Only the levels pertaining to different values of M are plaited (owing to the multi-dimensionality of the problem). For $c_n = \delta_{n1}$, all the levels become disentangled and we return to the case of models described by Eqs. (89) and (90). Infinite series of quasi-exactly solvable models arise for $D = N-2$, and for $c_n = \delta_{n1}$ they reduce to the models (89), (90). As before, the case $D = N-3$ gives nothing new if the condition (39) is satisfied. However, if this condition is not satisfied, then for $D \leqslant N-3$ finite series of quasi-exactly solvable models arise, with the maximum order being

$$K_{\max}=3+\left[\frac{2D+2}{N-D-2}\right]. \tag{107}$$

By means of variable substitutions $t_\alpha = \Pi_{i=1}^{D}(\lambda_i - a_\alpha)\Pi_{\beta=1}^{D}{}'(\alpha_\beta - a_\alpha)^{-1}$ and similarity transformations, for $D = N-1$ and $D = N-2$ the Hamiltonians of the exactly and quasi-exactly solvable models described by Eqs. (88), (104)–(106) can be written as combinations of the generators S_α^+, S_α^-, and S_α^0 of the algebra $SL(2)$ given by (79). For example, the Hamiltonians of the exactly solvable models reduce to linear combinations of the operators

$$h_\alpha=\sum_{\beta=1}^{D}{}'\frac{S_\alpha S_\beta}{a_\alpha-a_\beta}-\frac{S_\alpha^+S_\alpha^-+2b_{D+1}S_\alpha^0}{a_\alpha-a_{D+1}}-S_\alpha^-,\ \alpha=1,\ \dots,\ D, \tag{108}$$

and the Hamiltonians of the quasi-exactly solvable models ($D = N-2$) reduce to linear combinations of the operators

$$h_\alpha=\sum_{\beta=1}^{D}{}'\frac{S_\alpha S_\beta}{a_\alpha-a_\beta}-\frac{S_\alpha^+S_\alpha^-+2b_{D+1}S_\alpha^0}{a_\alpha-a_{D+1}}-\frac{S_\alpha^+S_\alpha^-+2b_{D+2}S_\alpha^0}{a_\alpha-a_{D+2}}-S_\alpha^- -\frac{S_\alpha^+\{(\sum_{\alpha=1}^{D}S_\alpha^0)^2-(\sum_{\alpha=1}^{D}b_\alpha)^2-M^2+(2b_{D+1}+2b_{D+2}-1)(\sum_{\alpha=1}^{D}S_\alpha^0-\sum_{\alpha=1}^{D}b_\alpha-M)\}}{(a_\alpha-a_{D+1})(a_\alpha-a_{D+2})}. \tag{109}$$

We see that the expressions (108) are bilinear in the generators, while the expressions (109) are *trilinear*. In addition, we stress the fact that the operators $\mathbf{S}_\alpha$ are the generators of *infinite-dimensional* representations of the algebra $SL(2)$. These features lead to an important distinction between representations of the type (109) and the representations used in the algebraic approach[19,21] for the Hamiltonians of quasi-exactly solvable systems.

The Hamiltonians of all the infinite series of quasi-exactly solvable models considered in this study can be represented as infinite-dimensional partitioned matrices $H = \|H_{\alpha\beta}\|$ of a special form acting on block vectors $\varphi = \{\varphi_\alpha\}$. (The indices label blocks of the matrices H of dimension $K_\alpha \times K_\alpha$ and also the block-components of vectors φ of length K_α.) The matrices H are constructed as follows:

$$H=H^+(H^d-e)+H^0. \tag{110}$$

Here H^d is a block-diagonal matrix: $H^d_{\alpha\beta} = \delta_{\alpha\beta}H^d_{\alpha\beta}$; H^+ is a matrix for which all the blocks located above the first block-hyperdiagonal are equal to zero: $H^+_{\alpha\beta} = 0$ if $\alpha > \beta + 1$, and H^0 is a matrix for which all the blocks located above the principal block-diagonal are equal to zero: $H^0_{\alpha\beta} = 0$ if $\alpha > \beta$. We see that the matrix H^+ acts as a raising operator, transforming n-component block vectors φ into $(n+1)$-component ones, while the action of the matrices H^d and H^0 on the vector φ does not increase the number of their block-components. For this reason, the operator H can on the whole be treated as a raising operator for which the spectral problem is, obviously, infinite-dimensional. However, if the spectrum of the Mth diagonal block H^d_{MM} of the matrix H is K_M-fold degenerate, and the value of the parameter e coincides with the eigenvalue of H^d_{MM}, then the operator H is no longer the raising operator for M-component block vectors φ, and the corresponding spectral problem for φ becomes finite-dimensional. As a result, we obtain quasi-exactly solvable models of order $K_1 + K_2 + \cdots + K_M$ (Ref. 26).

It is easy to see that all the operators h_α described by Eq. (109) and also linear combinations of them can be represented in the form (110). Here the role of the space of vectors φ is played by the space of polynomials in D variables, the blocks are formed by homogeneous polynomials of identical degree, and the role of the operators H is played by the operators $(\Sigma_{\alpha=1}^{D} S_\alpha^0)^2 + (2b_{D+1}) + 2b_{D+2} - 1)\Sigma_{\alpha=1}^{D} S_\alpha^0$. It is easy to see that the most general form of the operator H on the space of polynomials which is consistent with the requirement that H be a second-order differential operator is

$$H = (c_n t_n + f)\,[a\,(t_i\partial_i)^2 + b\,(t_i\partial_i) - e] + P_{ik}^{2}(t)\,\partial_i\partial_k + Q_i^{(1)}(t)\,\partial_i \quad (111a)$$

or

$$H = (c_n t_n + d_{nmk} t_n t_m \partial_k + f + g_{mk} t_m \partial_i + h_k\partial_k) \times (a t_i\partial_i - e) + P_{ik}^{(2)}(t)\,\partial_i\partial_k + Q_i^{(1)}(t)\,\partial_i \quad (111b)$$

(there is a summation over repeated indices). Here $a, b, f, c_n, d_{nmk}, g_{mk}$, and h_k are arbitrary parameters, and $P_{ik}^{(2)}(t)$ and $Q_{ik}^{(1)}(t)$ are arbitrary polynomials of second and first order, respectively. The allowed values of the parameter e for which the problem becomes finite-dimensional are, respectively, $e = aM^2 + bM$ for (111a) and $e = aM$ for (111b). Here M is the degree of the polynomials in whose class the solution is sought.

The quasi-exactly solvable spectral equations for the operators (111) are equations with two spectral parameters. One is the "energy" parameter, and the other is the "potential" parameter. It is easy to see that the scheme described above can also be used to construct equations with many spectral parameters. Let $w_{n_1,\ldots,n_R}$, $n_r \geqslant 0$, $r = 1, \ldots, R$, be a sequence of finite-dimensional linear spaces. With each set $m_1, \ldots, m_R$ we associate a space $W_{m_1,\ldots,m_R}$ which is the linear envelope of the spaces $w_{n_1,\ldots,n_R}$ with $n_r \leqslant m_r$, $r = 1, \ldots, R$. Obviously, for all finite $m_1, \ldots, m_R$ the spaces $W_{m_1,\ldots,m_R}$ are finite-dimensional. We denote the infinite-dimensional space $W_{\infty,\ldots,\infty}$ by W. We consider the following operators in W:

1) H_r^{+N}, $r = 1, \ldots, R$, acting from $W_{n_1,\ldots,n_r,\ldots,n_R}$ to $W_{n_1,\ldots,n_r+N,\ldots,n_R}$;

2) H_r^d, $r = 1, \ldots, R$, diagonal in each of the spaces $w_{n_1,\ldots,n_r,\ldots,\sigma_R}$ and possessing the property $H_r^d\varphi = f_r(n_r)\varphi$ if $\varphi \in w_{n_1,\ldots,n_r,\ldots,n_R}$ (the eigenvalues of H_r^d are determined only by the number n_r;)

3) H^0, which transform the spaces $W_{n_1,\ldots,n_R}$ into themselves.

We now construct the new operator

$$H = H^0 + \sum_{r=1}^{R} H_r^{+1}(H_r^d - e_r^1) + \sum_{r=1}^{R} H_r^{+2}(H_r^d - e_r^2)(H_r^d - e_r^1) + \ldots,$$

acting from W to W. It is easy to see that the spectral problem for this operator is, in general, infinite-dimensional. However, if $e_r^1 = f_r(N_r)$, $e_r^2 = f_r(N_r - 1)$, and so on, the operator H acts inside the space $W_{N_1,\ldots,N_R}$, and the spectral problem for it becomes finite-dimensional. We obtain a quasi-exactly solvable spectral equation. The numbers $e_r^1, e_r^2, \ldots$ play the role of the "potential" spectral parameters. The operators H_r^{+N}, H_r^d, and H^0 can be constructed from the generators of the Lie algebras. As an example, let us consider the case of a semi-simple Lie algebra G of rank R with the generators e_r^0, $e_r^\pm$, $r = 1, \ldots, R$, satisfying the commutation relations

$$[e_i^0, e_j^0] = 0,\ [e_i^0, e_j^\pm] = \pm A_{ij} e_j^\pm,\ [e_i^+, e_j^-] = \delta_{ij} e_j^0$$

(here we have written the commutation relations only for the generators related to the simple roots of the algebra G; A_{ij} is the Cartan matrix). Let $w_{0,\ldots,0}$ be the lowest-weight vector of the infinite-dimensional representation of the algebra G:

$$e_i^- w_{0,\ldots,0} = 0,\ e_i^0 w_{0,\ldots,0} = E_i w_{0,\ldots,0},\ i = 1, \ldots, R.$$

We define $w_{n_1,\ldots,n_R}$ as a linear space spanned by vectors obtained from the vector

$$(e_1^+)^{n_1}(e_2^+)^{n_2}\ldots(e_R^+)^{n_R}\,w_{0,\ldots,0}$$

as a result of all possible permutations of the $n_1 + n_2 + \cdots + n_R$ operators e_r^+. It is easily seen that

$$e_i^0 w_{n_1,\ldots,n_R} = \Big(E_i + \sum_{j=1}^{R} A_{ij} n_j\Big)\, w_{n_1,\ldots,n_R}.$$

Multiplying both sides of this equation by the matrix B_{rj}, the inverse of the Cartan matrix, we find

$$\Big(\sum_{j=1}^{R} B_{rj} e_j^0\Big) w_{n_1,\ldots,n_R} = \Big(\sum_{j=1}^{R} B_{rj}E_j + n_r\Big)\, w_{n_1,\ldots,n_R},$$

from which it follows that the operators H_r^d can be defined as

$$H_r^d \equiv \sum_{j=1}^{R} B_{rj} e_j^0.$$

The operators H^0 and H_r^{+N} can be represented as polynomials in the generators $e_r^\pm$ and e_r^0.

Let us take the case of the algebra $SL(3)$ as an example. The generators $e_{1,2}^\pm$ and $e_{1,2}^0$ can be realized as first-order differential operators:

$$e_1^0 = \alpha - 2x\frac{\partial}{\partial x} - y\frac{\partial}{\partial y} + z\frac{\partial}{\partial z};$$

$$e_2^0 = \beta + x\frac{\partial}{\partial x} - y\frac{\partial}{\partial y} - 2z\frac{\partial}{\partial z};$$

$$e_1^+ = \alpha x - x^2\frac{\partial}{\partial x} - xy\frac{\partial}{\partial y} + (xz - y)\frac{\partial}{\partial z};$$

$$e_2^+ = \beta z + y\frac{\partial}{\partial x} - z^2\frac{\partial}{\partial z};$$

$$e_1^- = \frac{\partial}{\partial x};$$

$$e_2^- = x\frac{\partial}{\partial y} + \frac{\partial}{\partial z}.$$

The parameters α and β characterize the infinite-dimensional representation of the algebra $SL(3)$. The constant plays the role of the vector of lowest weight. From the general equations it follows that

$$H_1^d = \Big(x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y}\Big),\ H_2^d = \Big(y\frac{\partial}{\partial y} + z\frac{\partial}{\partial z}\Big).$$

Requiring that the operator H be a differential operator of order no greater than two, we find that

$$H=H_1^{+1}(H_1^d-e_1)+H_2^{+1}(H_2^d-e_2)+H^0,$$

where

$$H_1^{+1}=A^1e_1^+ +\sum_i B_i^1e_i^0+\sum_i C_i^1e_i^-;$$

$$H_2^{+1}=A^2e_2^+ +\sum_i B_i^2e_i^0+\sum_i C_i^2e_i^-,$$

and H^0 is a sum of terms of the form

$$e_i^+e_i^-,\ e_i^0e_k^0,\ e_i^0e_k^-,\ e_i^-e_k^-,\ e_i^0,\ e_i^-.$$

We note that the spectral equations for the operators (111) can sometimes, by obvious transformations, be reduced to quasi-exactly solvable equations of the Schrödinger or Pauli type on, in general, curved manifolds. The question of the Hermiticity of the Hamiltonians involved in the resulting equations obviously requires special consideration (see the Conclusions).

9. QUASI-EXACT SOLVABILITY AND COMPLETE INTEGRABILITY

In Sec. 7 we discussed the relation between the exactly and quasi-exactly solvable models generated by the multiparameter spectral equation (33) and completely integrable quantum models of magnetic systems based on the algebra $SL(2)$. It was pointed out that Eqs. (35) and (36) determining the spectra of quasi-exactly solvable models coincide with the Bethe-ansatz equations for spin models with the Hamiltonians (78). However, it was not explained why this is so. In this section we attempt to give this explanation, using the statement of Sec. 1 that to each exactly solvable spectral equation with N spectral parameters and commuting weight functions there corresponds a system of N commuting operators with exactly calculable spectra. It is easy to show that the operators L_α have the form of second-order differential operators in the variables λ_i. Changing to new variables $t_\alpha=\Pi_{i=1}^N(\lambda_i-a_\alpha)\Pi_{\beta=1}^{N}{}'(a_\beta-a_\alpha)^{-1}$, we obtain[25,42,43]

$$L_\alpha=\sum_{\beta=1}^{N}{}'\frac{S_\alpha^-S_\beta^+ +S_\alpha^+S_\beta^- -2S_\alpha^0S_\beta^0}{a_\alpha-a_\beta}-S_\alpha^-, \tag{112}$$

where the operators $S_\alpha^\pm$ and S_α^0 are given in terms of the variables t_α by (79) and therefore are the generators of infinite-dimensional representations of the algebra $SL(2)$. The commuting operators L_α can thus be viewed as integrals of the motion of a completely integrable model of a magnetic system based on the algebra $SL(2)$. We see that the operators L_α differ from the Hamiltonians H_α studied in Sec. 7 only by terms of the form S_α^-, which act as lowering operators. Therefore, the spectra of the operators L_α and H_α coincide (the equations for these spectra also coincide), as was pointed out in Sec. 7. Here it is helpful to use the fact that the operators L_α can be obtained from a generating function—an operator of the form

$$\tilde{S}^2(\lambda)=S^2(\lambda)-S^-(\lambda).$$

Introducing the new operator functions

$$\tilde{S}^-(\lambda)=S^-(\lambda),\ \tilde{S}^0(\lambda)=S^0(\lambda),\ \tilde{S}^+(\lambda)=S^+(\lambda)-1,$$

it is easy to verify that they satisfy the same commutation relations as $S^-(\lambda)$, $S^0(\lambda)$, and $S^+(\lambda)$ and allow the generating function $\tilde{S}^2(\lambda)$ to be written as

$$\tilde{S}^2(\lambda)=\frac{1}{2}[\tilde{S}^+(\lambda)\tilde{S}^-(\lambda)+\tilde{S}^-(\lambda)\tilde{S}^+(\lambda)-2\tilde{S}^0(\lambda)\tilde{S}^0(\lambda)],$$

in complete analogy with (82).[43]

In Sec. 1 we have already mentioned that the phenomenon of quasi-exact solvability is a consequence of the presence of a degeneracy in the system of spectral parameters. For example, the degeneracy of the spectral parameter r_{N-2} [see Eq. (38b)] relative to the other spectral parameters leads to the existence of quasi-exactly solvable models of order equal to the degree of degeneracy. The parameter r_{N-2} is the eigenvalue of the operator $R_{N-2}=\Sigma_\alpha a_\alpha H_\alpha$; therefore, to find the symmetry which is related to this degeneracy we should consider operators which commute with R_{N-2}, but not with each of the H_α separately. To find such operators we note that R_{N-2} commutes with all the operators $H_\alpha=H_\alpha(a_1,...,a_N)$. On the other hand, owing to (78), this operator has the form $R_{N-2}=(\Sigma_\alpha S_\alpha)^2$, so it is independent of a_1, ..., a_N. Therefore, it will also commute with all operators of the form $H_\alpha(x_1,...,x_N)$, where x_1, ..., x_N are arbitrary parameters. Here if $N\geqslant 3$, the sets of operators $H_\alpha(a_1,...,a_N)$ and $H_\alpha(x_1,...,x_N)$ will, in general, not commute with each other. Therefore, the operators $H_\alpha(x_1,...,x_N)$ can be viewed as generating functions for the generators of the algebra associated with the hidden symmetry responsible for the degeneracy. Attempts to make this algebra closed convince us that it is infinite-dimensional.[26]

The relation between quasi-exactly solvable models and completely integrable systems can also be used to *construct* quasi-exactly solvable equations. In fact, if R_0, ..., R_{N-1} is any set of commuting operators with an exactly calculable spectra, an operator of the form

$$H=R_0+U_1(R_1-e_1)+\ldots+U_{N-1}(R_{N-1}-e_{N-1}), \tag{113}$$

where U_1, ..., U_{N-1} are arbitrary operators, will be an operator of a quasi-exactly solvable equation if the spectra of the operators R_1, ..., R_{N-1} are degenerate relative to the spectrum of the operator R_0. For R_α we can take, for example, linear combinations of the operators

$$H_\alpha=\sum_{\beta=1}^{N}{}'\frac{K_{ik}S_\alpha^iS_\beta^k}{a_\alpha-a_\beta}, \tag{114}$$

which can be interpreted as the Hamiltonians of magnetic systems based on a Lie algebra with generators S_α^i (K_{ik} is the Killing–Cartan tensor).[20] Since, in general, the generators of infinite-dimensional representations of Lie algebras can be realized as first-order differential operators, the operator H can take the form of a second-order multi-dimensional differential operator, with arbitrary functions chosen for the operators U_1, ..., U_{N-1}. Requiring that for certain α the operators S_α^i be the generators of finite-dimensional matrix representations of the Lie algebra and taking matrix functions as the U_α, we can obtain H in the form of a second-order matrix differential operator. Finally, taking the S_α^i to be the generators of graded Lie algebras, we can construct supersymmetric generalizations of quasi-exactly solvable equations.[39]

We note that it is not at all necessary to use operators of the type (114) to construct the commuting operators R_α. For example, the R_α can be taken to be second-order differential operators of various variables λ_α which commute with each other in an obvious manner. If the spectral problems for each of these operators are exactly or quasi-exactly solvable, we obtain finite or infinite series of quasi-exactly solvable equations of finite or infinite order. It is easy to show that such equations can always be reduced to equations of the Schrödinger form on certain curved manifolds if the functions $U_1(\lambda), \ldots, U_{N-1}(\lambda)$ are assumed to have a factorized form: $U_\alpha(\lambda) = u_{\alpha 1}(\lambda_1)\times\ldots\times u_{\alpha N-1}(\lambda_{N-1})$, $\alpha = 1, \ldots, N-1$ (Ref. 27).

10. SOME OTHER RATIONAL EXACTLY AND QUASI-EXACTLY SOLVABLE MODELS

Up to now we have used only one very special method of discretizing the spectrum of the $2N$-parameter spectral equation (28), which in the rational case has the form

$$\left\{\frac{\partial^2}{\partial\lambda^2} - \sum_{\alpha=1}^{N} \frac{e_\alpha^{(2)}}{(\lambda-a_\alpha)^2} + \sum_{\alpha=1}^{N} \frac{e_\alpha^{(1)}}{\lambda-a_\alpha}\right\}\varphi(\lambda) = 0. \tag{115}$$

In this method we fixed the parameters $e_\alpha^{(2)}$ and declared the remaining parameters $e_\alpha^{(1)}$ to be spectral parameters. This method allowed us to obtain the set of exactly and quasi-exactly solvable models studied in detail in Secs. 4–9. However, as was noted in Sec. 3, this is not the only possible approach, since any N linearly independent combinations of the parameters $e_\alpha^{(2)}$ and $e_\alpha^{(1)}$ can be fixed. This greatly enlarges the set of exactly and quasi-exactly solvable models associated with Eq. (115). We note that, since the change from one system of parameters $e_\alpha^{(2)}, e_\alpha^{(1)}$ to another $\tilde{e}_\alpha^{(2)}, \tilde{e}_\alpha^{(1)}$ is effected by a transformation of the group $GL(2N)$, each way of fixing N parameters $\tilde{e}_\alpha^{(2)}$ and defining the other N parameters $\tilde{e}_\alpha^{(1)}$ to be spectral parameters can be uniquely characterized by elements of this group g.

It is easy to see that, owing to the identity $e_1^{(1)} + \ldots + e_N^{(1)} = 0$, all the solutions of Eq. (115) for any choice of g are $[N/(N-1)]$-fold degenerate. According to Theorem 3 of Sec. 1, this leads us to a set of $(N-1)$-dimensional exactly solvable equations of the Schrödinger type, which can easily be constructed using the explicit equations of Sec. 2. In particular, for $N=2$ we obtain a family of one-dimensional exactly solvable models, including both the models listed in Sec. 4 and models of a more general type related to the hypergeometric equation. The Coulomb, Kratzer, and Eckart potentials and many other potentials found in Ref. 44 are special cases of these models. A special choice of the element g can lead to an even greater degeneracy in the system of parameters. For example, we know that fixing all the $e_\alpha^{(2)}$ ensures a $[N/(N-2)]$-fold degeneracy for the $(M+N-2)!\{M!(N-2)!\}^{-1}$ solutions at each M. This leads to the large class of $(N-2)$-dimensional quasi-exactly solvable models of any order described in Secs. 4 ($N=3$) and 8 ($N>3$). A different choice of g corresponding to fixed $e_\alpha^{(2)}$, $\alpha = 1, \ldots, N-1$, and vanishing $\Sigma_{\alpha=1}^N (a_\alpha e_\alpha^{(1)} + e_\alpha^{(2)})$ ensures $[N/(N-2)]$-fold degeneracy for all the solutions of Eq. (115). This leads to $(N-2)$-dimensional exactly solvable models. In particular, for $N=3$ we obtain one-dimensional exactly solvable models, which in Ref. 19 were erroneously attributed to a class of quasi-exactly solvable models with a finite number of unplaited equations. Actually, all the levels in these models are disentangled and can be obtained explicitly. However, it is not easy to show this using the algebraic approach of Ref. 19. Furthermore, it is easy to show that the vanishing of the expression $\Sigma_{\alpha=1}^N (a_\alpha e_\alpha^{(1)} + e_\alpha^{(2)})$ is equivalent to imposing a condition analogous to (39) on the parameters of the problem. This reduces all the exactly solvable models arising for $N=3$ to the above-mentioned exactly solvable models corresponding to the case $N=2$ and completely described in Ref. 44.

Let us now consider finite series of quasi-exactly solvable models with finite maximum order. We recall that earlier we obtained quasi-exactly solvable models of order K for $N>3$ by choosing some combination of the parameters $r_0, \ldots, r_{D-1}$ to be the energy spectral parameter, and requiring that the $N-D-2$ potential parameters $r_D, \ldots, r_{N-3}$ coincide for the K solutions, thereby imposing $(K-1)(N-D-2)$ conditions on the $2N-2$ parameters of the system. Now we make use of the possibility of transforming from one system of parameters to another by mixing them via transformations from the group $GL(N)$. Obviously, it is meaningless to mix the parameters $r_D, \ldots, r_{N-3}$, since this does not lead to a change in the potential. However, we can mix each of the potential parameters with all the energy parameters. As a result, the number of free parameters of the system increases from $2N-2$ to $2N-2+D(N-2-D)$. We then easily find that

$$K_{\max} = D+3+\left[\frac{2D+2}{N-D-2}\right]. \tag{116}$$

We shall discuss this equation later.

11. TRIGONOMETRIC AND ELLIPTIC QUASI-EXACTLY SOLVABLE MODELS

Up to now we have been exclusively concerned with models related to rational solutions of the scalar triangle equation (25). However, as was shown in Sec. 3, this equation also has trigonometric and hyperbolic solutions which, as in the rational case, can be used to construct multi-parameter spectral equations of the type (115). The spectra of these equations become discrete when any N linear combinations of the parameters $e_\alpha^{(2)}$, $e_\alpha^{(1)}$, and $e^{(0)}$ are fixed. If (as in the rational case) we fix the parameters $e_\alpha^{(2)}$, the system of spectral equations takes the form

$$\sum_{k=1}^{M}{}' \rho\cot\rho(\xi_i-\xi_k) + \sum_{\alpha=1}^{N} b_\alpha\rho\cot\rho(\xi_i-\alpha_\alpha) = 0, \quad i=1,\ldots,M; \tag{117}$$

$$e_\alpha^{(1)} = \sum_{i=1}^{M} b_\alpha\rho\cot\rho(\xi_i-a_\alpha), \quad \alpha=1,\ldots,N; \tag{118a}$$

$$e^{(0)} = 3\left(\sum_{\alpha=1}^{N}\eta_\alpha + M\right). \tag{118b}$$

From the general theorems of Secs. 1 and 2 it follows that each exactly solvable multi-parameter spectral equation can be associated with various families of exactly and quasi-exactly solvable equations of the Schrödinger type. The higher the degree of degeneracy in the system of spectral parameters, the richer are these families. Therefore, since we have at our disposal a trigonometric multi-parameter spectral equation with a sufficiently degenerate spectrum [see Eq. (118b), and also the identity $e_1^{(1)} + \ldots + e_N^{(1)} = 0$], it is natural to ask the question of what new exactly and quasi-exactly solvable models arise from this equation. The answer is none! This is true because the trigonometric spectral equation with

$N=n$ can, by means of variable substitution, easily be reduced to a rational equation of the same type with $N=n+1$. The reverse transition from a rational equation to a trigonometric one is possible only when the parameters b_α of the rational equation satisfy any of the N conditions

$$\sum_{\alpha=1}^{N} b_\alpha + M - 1 = -b_\alpha, \quad \alpha = 1, \ldots, N. \tag{119}$$

The fact that the trigonometric models are equivalent to rational ones does not mean that they are not worth studying in detail. In fact, from the trigonometric form (117), (118) of the spectral equations for these models it follows that they possess a series of special properties which are not characteristic of rational models in general. For example:

1. For trigonometric models there are simultaneously two types of analog Coulomb problem. On the one hand, the problem of the spectrum of a trigonometric model of N order is equivalent to the problem of the equilibrium of M particles with unit charge in the field produced by $N+1$ stationary particles with charges $b_1, \ldots, b_N$ and $b_{N+1} = -\frac{1}{2}(\Sigma_{\alpha=1}^{N} b_\alpha + M - 1)$. On the other hand, this same problem is equivalent to the problem of the equilibrium of infinite one-dimensional periodic lattices with unit charges at the vertices, located in the field of N stationary lattices of this type with particles of charges $b_1, b_2, \ldots, b_N$ located at the vertices.

2. Trigonometric models also admit two different formulations in the language of spin systems. A trigonometric model of order N is equivalent, on the one hand, to a model of a magnetic system based on the algebra $SU(2) \otimes \ldots \otimes SU(2)$ ($N+1$ times), characterized by the "spins" $-b_1, \ldots, -b_N$ and $-b_{N+1} = \frac{1}{2}(\Sigma_{\alpha=1}^{N} b_\alpha + M - 1)$. On the other hand, this model is equivalent to a model of a magnetic system based on the algebra $SU(2) \otimes \ldots \otimes SU(2)$ (N times), characterized by the "spins" $-b_1, \ldots, -b_N$. Here the first $(N+1)$-vertex magnetic system is isotropic and possesses a global $SU(2)$ symmetry conserving both the total spin and its z projection. The second magnetic system is anisotropic, and the remaining global symmetry group $U(1)$ conserves only the z projection of the total spin. We note that the anisotropic magnetic system, like the isotropic system, is completely integrable, and the Bethe-ansatz equations for it coincide with the system (117), (118). All of what we have said for one-dimensional problems remains valid also in the multi-dimensional case.[25,43]

In conclusion, we note that the elliptic models associated with elliptic solutions of the generalized scalar triangle equation can be studied in a similar manner.[28] It can be shown that these models can be systematically reduced to the trigonometric and rational case, but again there are a number of features peculiar to the elliptic case. For example, these models are equivalent to Coulomb systems describing the interaction of two-dimensional lattices with charges at the vertices. In addition, the Hamiltonians of these models can be rewritten in terms of the Hamiltonians of completely anisotropic magnetic systems with no continuous global group.

12. THE NUMBER OF QUASI-EXACTLY SOLVABLE MODELS

In Sec. 10 we derived Eq. (116), from which it follows that there exists an infinite set of D-dimensional quasi-exactly solvable models of order $D+3$. There is a simple explanation for this fact. Let us consider the most general form of the Riccati equation:

$$y'(\lambda) + a(\lambda) y^2(\lambda) + b(\lambda) y(\lambda) + c(\lambda) + e_1 d_1(\lambda) + \ldots + e_D d_D(\lambda) = 0, \tag{120}$$

in which $a(\lambda)$, $b(\lambda)$, $c(\lambda)$, and $d_\alpha(\lambda)$, $\alpha = 1, \ldots, D$, are certain functions and e_α, $\alpha = 1, \ldots, D$, are numerical parameters. We arbitrarily choose the $D+3$ sets

$$y^{(i)}(\lambda),\ e_1^{(i)},\ e_2^{(i)},\ \ldots,\ e_D^{(i)},\quad i = 1, \ldots, D+3 \tag{121}$$

and require that each of these satisfy Eq. (120). Substituting (121) into (120), we note that the resulting equations can be interpreted as a system of $D+3$ linear equations for $D+3$ unknown functions $a(\lambda)$, $b(\lambda)$, $c(\lambda)$, and $d_\alpha(\lambda)$, $\alpha = 1, \ldots, D$. By solving this system and finding these functions, we obtain the Riccati equation with D spectral parameters having $D+3$ exact (obvious) solutions. After the substitution $y = (1/a)(\varphi'/\varphi + \frac{1}{2}(a'/a - b))$, this equation becomes a linear D-parameter spectral equation with $D+3$ exact solutions. According to the results of Sec. 1, this equation is equivalent to a D-dimensional quasi-exactly solvable model of order $D+3$. We therefore have obtained even more than we wanted. In fact, from the arbitrariness of the functions belonging to the specified set (121), it follows that there are not just infinitely many, but *functionally many* quasi-exactly solvable models of order $D+3$. From this it follows that, in particular, in the one-dimensional case there exist functionally many quasi-exactly solvable models of order 4 (Refs. 27 and 34).

Curiously, in the one-dimensional case for constructing quasi-exactly solvable models of order 3 it is sufficient to specify only one arbitrary function. Let us assume that the Schrödinger equation $-\psi'' + V\psi = E\psi$ has three explicit solutions (ψ_i, E_i), $i = 1, 2, 3$. Transforming to the logarithmic derivative of the wave functions $y_i = \psi_i'/\psi_i$, we rewrite the equation for the three solutions in the form

$$y_i' + y_i^2 + E_i = V, \quad i = 1, 2, 3. \tag{122}$$

We subtract the first equation from the other two:

$$(y_i - y_1)' + (y_i - y_1)(y_i + y_1) + E_i - E_1 = 0, \quad i = 2, 3 \tag{123}$$

and introduce the functions

$$z_i = y_i - y_1, \quad g_i = y_i + y_1, \quad i = 2, 3. \tag{124}$$

Substituting (124) into (123), we find the relations

$$g_i = -\frac{z_i' + E_i - E_1}{z_i}, \quad i = 2, 3, \tag{125}$$

from which, using (124), we obtain

$$y_i = \frac{1}{2}\left[z_i - \frac{z_i' + E_i - E_1}{z_i}\right], \quad i = 1, 2; \tag{126a}$$

$$y_1 = -\frac{1}{2}\left[z_2 + \frac{z_2' + E_2 - E_1}{z_2}\right] = -\frac{1}{2}\left[z_3 + \frac{z_3' + E_3 - E_1}{z_3}\right]. \tag{126b}$$

Introducing the function

$$t = z_3/z_2, \tag{127}$$

we rewrite (126b) as

$$(t-1)\, z_2^2+\frac{t'}{t}\, z_2+\frac{(E_3-E_1)}{t}-(E_2-E_1)=0, \tag{128}$$

from which we find

$$z_2=\frac{-\frac{t'}{t}\pm\sqrt{\left(\frac{t'}{t}\right)^2-4\,(t-1)\left(\frac{E_3-E_1}{t}-(E_2-E_1)\right)}}{2\,(t-1)}. \tag{129}$$

We see that the specification of a single function of t formally solves the problem. In fact, knowing t, from (129) we can find z_2, and then from (127) we can reconstruct z_3. Then, after finding the functions y_i from Eqs. (124) and (125), we can construct the solutions (ψ_i, E_i) of the Schrödinger equation with the potential reconstructed using (122). The constraints on the function of t allowing the construction of stable quantum-mechanical models with three exactly calculable states with *a priori* specified numbering were derived in Ref. 45.

We conclude this section by describing a simple method allowing every quasi-exactly solvable equation of a given order to be put into correspondence with another quasi-exactly solvable equation of the same order. The method is based on the use of the form-invariance of the Riccati equation under a linear-fractional substitution of the unknown function. Let us consider a Riccati equation

$$y' + ay^2 + 2by + c + Ed = 0,$$

having a certain number of exact solutions. Let y_0, E_0 be one of these solutions. Transforming to a new function $\tilde{y}$ via the equation $\tilde{y} = -(E-E_0)/(y-y_0)$, we obtain a new Riccati equation of the form

$$\tilde{y}'+\tilde{a}\tilde{y}^2+2\tilde{b}\tilde{y}+\tilde{c}+\tilde{E}\tilde{d}=0,$$

in which $\tilde{a}=d$, $\tilde{b}=b+ay_0$, $\tilde{c}=-E_0a$, and $\tilde{d}=a$. If $y=y_i$, $E=E_i$, $i=0,...,K$, are exact solutions of the original equation, then $\tilde{y}=0$, $\tilde{E}=E_0$, $\tilde{y}=-(E_i-E_0)/(y_i-y_0)$, $\tilde{E}_i=E_i$, $i=1,...,K$, will be exact solutions of the new equation. The linear quasi-exactly solvable equations associated with the old and new Riccati equations obviously do not reduce to each other as the result of the homogeneous substitution of the function and the independent variable.[26] The question of whether or not the solutions of these equations are normalizable obviously requires special consideration.

CONCLUSIONS

This concludes the exposition of our approach to the problem of quasi-exact solvability in quantum mechanics. Here we summarize a number of important features.

The approach is based on the interpretation of quasi-exactly solvable Schrödinger equations as exactly solvable equations with several spectral parameters, one of which is identified with the energy, while the others are included in the potential. To construct nontrivial quasi-exactly solvable models of order higher than the first it is necessary that there be a degeneracy in the system of parameters, i.e., that a single set of "potential" spectral parameters correspond to several values of the "energy" parameter. The allowed values of these parameters can be interpreted as the eigenvalues of commuting operators with exactly calculable spectra. The degeneracy responsible for the quasi-exact solvability is present because of a hidden symmetry in the problem under which the "potential" operators are invariant, while the "energy" operator is not. The full set of "potential" and "energy" operators can be thought of as the integrals of the motion of some completely integrable system. In this sense, almost all the quasi-exactly solvable models considered here are equivalent to completely integrable models of magnetic systems based on the algebra $SL(2)$. The Bethe-ansatz equations for these systems exactly coincide with the equations determining the spectra of the quasi-exactly solvable systems. These equations also coincide with the equilibrium equations for a system of Coulomb particles in an external electrostatic field, so that the problem of the spectrum in quasi-exactly solvable models can be posed in purely classical language. If the order of a quasi-exactly solvable model tends to infinity, a non-exactly solvable model arises. Therefore, the equivalence between the problems in nonrelativistic quantum mechanics, the theory of completely integrable quantum spin systems, and multi-particle Coulomb problems of classical physics, discussed here for the example of quasi-exactly solvable models, is also preserved in the non-exactly solvable case.

The construction of one-dimensional and multi-dimensional quasi-exactly solvable models is equally simple in our approach. We therefore think that a generalization of our approach may be applicable to the case of systems with an infinite number of degrees of freedom and, ideally, to the case of field theory.

In this review we have only briefly touched upon other methods of constructing exactly and quasi-exactly solvable models. The methods which by now have reached a high level of development are: a) the Turbiner–Shifman method, described in the Introduction (see also Refs. 21 and 22), which is based on the use of differential realizations of finite-dimensional representations of Lie algebras; b) the method, described at the end of Sec. 8, using differential realizations of infinite-dimensional representations of Lie algebras; c) the method, discussed at the end of Sec. 9 (see also Refs. 39 and 43), in which the construction tools are sets of commuting operators—the integrals of the motion of completely integrable models of magnetic systems based on Lie algebras. Each of these methods naturally splits into two stages. In the first (constructive) stage a definite algorithm is stated which allows the construction of second-order N-dimensional spectral differential equations

$$\left\{\sum_{i,\,k=1}^{N} P_{ik}(x)\frac{\partial^2}{\partial x_i\,\partial x_k}+\sum_{i=1}^{N} Q_i(x)\frac{\partial}{\partial x_i}+R(x)\right\}\varphi(x)=E\varphi(x), \tag{130}$$

having a finite or infinite number of exact solutions in a certain class of functions. In the second (nonconstructive) stage attempts are made to reduce Eq. (130) to exactly or quasi-exactly solvable equations of the Schrödinger type

$$\left\{\sqrt{P(x)}\sum_{i,\,h=1}^{N}\frac{\partial}{\partial x_i}\left(\frac{P_{ik}(x)}{\sqrt{P(x)}}\frac{\partial}{\partial x_i}\right)+V(x)\right\}\psi(x)=E\psi(x) \tag{131}$$

on N-dimensional manifolds with the metric $\|g_{ik}\| = \|P_{ik}\|^{-1}$ $[P(x)\equiv\det\|P_{ik}(x)\|]$. Unfortunately, not every equation of the type (130) can be reduced to an equation of the type (131). For the substitution of $\varphi(x) = \{P(x)\}^{-1/4}\{U(x)\}^{-1/2}\psi(x)$ into (130) to give Eq.

(131), the function $U(x)$ must satisfy the equations

$$\sum_{k=1}^{N} P_{ik}(x) \frac{\partial U(x)}{U(x)\partial x_k} + \sum_{k=1}^{N} \frac{\partial}{\partial x_k} P_{ik}(x) = Q_i(x),$$
$$i = 1, \ldots, N. \quad (132)$$

Obviously, the main difficulty is to solve this over-determined system, the compatibility requirement for which imposes quite stringent constraints on the allowed form of the functions $P_{ik}(x)$, $Q_i(x)$, and $U(x)$ (see Refs. 21 and 22).

However, it is remarkable that all these difficulties can easily be avoided by dropping the requirement that the dimensions of the spaces in which Eqs. (130) and (131) are formulated coincide. It is possible to develop a simple procedure which allows each N-dimensional equation of the type (130) to be put in correspondence with an $(N+1)$-dimensional equation of the type (131). In fact, we can write Eq. (130) in the $(N+1)$-dimensional form, while preserving all the solutions of the N-dimensional equation:

$$\sum_{i,k=0}^{N} P_{ik}(x, x_0) \frac{\partial^2}{\partial x_i \partial x_k} + \sum_{i=0}^{N} Q_i(x, x_0) \frac{\partial}{\partial x_i} + R(x, x_0)\Big\} \varphi(x) = E\varphi(x). \quad (133)$$

Here $P_{ik}(x, x_0) \equiv P_{ik}(x)$ and $Q_i(x, x_0) \equiv Q_i(x)$ for all i, $k = 1, \ldots, N$, $R(x, x_0) \equiv R(x)$, and $P_{i0}(x, x_0)$ and $Q_0(x, x_0)$ are arbitrary functions of $x = (x_1, \ldots, x_N)$ and of the newly introduced, extra variable x_0. Since Eq. (133) has the same form as (130), but is formulated in $(N+1)$-dimensional space, it can be reduced to the $(N+1)$-dimensional Schrödinger equation if a function $U(x, x_0)$ is found for which the $(N+1)$-dimensional analogs of (132) are satisfied:

$$\sum_{k=0}^{N} P_{ik}(x, x_0) \frac{\partial U(x, x_0)}{U(x, x_0)\partial x_k} + \sum_{k=0}^{N} \frac{\partial}{\partial x_k} P_{ik}(x, x_0) = Q_i(x, x_0),$$
$$i = 0, 1, \ldots, N \quad (134)$$

In contrast to (132), the system of equations (134) can always be solved, since the components $P_{i0}(x, x_0)$ and $Q_0(x, x_0)$ are arbitrary. The solutions depend on two arbitrary functions, for which it is convenient to choose the function $U(x, x_0)$, which *a priori* ensures the normalizability of the wave functions, and the function $P_{00}(x, x_0)$. In this case the other unknown functions $P_{i0}(x, x_0)$, $i = 1, \ldots, N$, and $Q_0(x, x_0)$ are found explicitly.

Thus, we see that every equation of the type (130) can be reduced to the Schrödinger form. We then can use any of the second-order exactly and quasi-exactly solvable differential equations obtained by the three methods listed above to generate exactly and quasi-exactly solvable quantum-mechanical models.

In conclusion, we would like to express our special gratitude to V. G. Kadyshevskiĭ for his interest in this study. We also take this opportunity to thank T. I. Maglaperidze and A. V. Turbiner for their interest and fruitful collaboration, and also P. B. Vigman, N. M. Gel'fand, V. I. Man'ko, A. A. Nersesyan, V. M. Savel'ev, L. A. Slepchenko, V. Ya. Faĭnberg, E. S. Fradkin, G. A. Kharadze, and M. A. Shifman for useful discussions and valuable remarks.

APPENDIX 1. EXPLICITLY TRIDIAGONALIZABLE HAMILTONIANS AND QUASI-EXACT SOLVABILITY

In this appendix we study models with Hamiltonians of the form

$$\hat{H} = -\frac{\partial^2}{\partial x^2} + \frac{\alpha}{x^2} + \beta x^2 + \gamma x^6, \quad (A1)$$

which possess a unique property: the squares of these Hamiltonians admit explicit tridiagonalization for all values of the parameters α, β, and γ. For simplicity, we shall demonstrate this for the special case of the model (A1) with the Hamiltonian

$$\hat{H} = -\frac{\partial^2}{\partial r^2} - \frac{1}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}(r^4 - g)^2 - 4\nu r^2, \quad (A2)$$

which can be interpreted as the operator of the radial Schrödinger equation for a two-dimensional spherically symmetric anharmonic oscillator with centrifugal barrier. Let us consider the trial function, $\psi_0(r) = r^g e^{-r^4/4}$, and also the operator $\hat{S} = \hat{H}^2$ and sequence generated by it:

$$\psi_n(r) = \hat{S}^n \psi_0(r). \quad (A3)$$

We orthogonalize the terms of this sequence by the standard Gram–Schmidt procedure. The resulting orthonormalized functions $\varphi_n(r)$ have the form

$$\varphi_n(r) = Q_n\left(\frac{r^4}{2}\right) r^g e^{-r^4/4}, \quad (A4)$$

where $Q_n(t)$ are certain n-th order polynomials. Since these polynomials (by construction) are orthogonal with weight

$$\omega(t) = t^{\frac{g-1}{2}} e^{-t}, \quad (A5)$$

they are Laguerre polynomials. The explicit form of the functions $\varphi_n(r)$ is thereby determined. According to the well-known Lanczos theorem (Ref. 40; see also Ref. 41), the operator $\hat{S}$ in the basis (A4) has a tridiagonal form. The nonzero matrix elements of the operator $\hat{S}$ are easily calculated using the familiar properties of the Laguerre polynomials. The result has the form

$$S_{nn} = 32\left[(2n+1-\nu)^2\left(n + \frac{g+1}{2}\right) + (2n-\nu)^2 n\right]; \quad (A5a)$$

$$S_{n,n+1} = -32(2n+1-\nu)(2n+2-\nu)\sqrt{(n+1)\left(n + \frac{g-1}{2}\right)}. \quad (A5b)$$

Obviously, if ν is a natural number,

$$\nu = \left[\frac{n-1}{2}\right], \quad (A6)$$

the matrix S_{nm} is block-diagonal. One of the blocks is finite, so that we arrive at a quasi-exactly solvable problem of finite order. We go from the operator $\hat{S}$ to the original operator $\hat{H}$ by discarding the extraneous solutions which do not satisfy the condition that $\hat{H}$ be Hermitian. The details of the calculations and also the explicit form of the solutions obtained are given in Ref. 6.

APPENDIX 2. QUASI-EXACTLY SOLVABLE MODELS OF FINITE ORDER

For the explicit construction of quasi-exactly solvable models of order K with the potentials described by Eqs. (53) and (54) when the condition (39) is not satisfied, it is necessary that for the K solutions $\{\xi_i^{(k)}\}_{i=1}^M$ $(k=1,...,K)$ the values of the symmetric polynomials $s_n(\xi^{(k)}) \equiv \Sigma_{i=1}^M (\xi_i^{(k)})^n$ of order $n=1,...,N-3$ entering into the potential be made independent of k, and the entire k dependence be concentrated in the polynomial s_{N-2} defining the energy of the system. To do this we multiply Eq. (35) by $\xi_i^N \Pi_{\alpha=1}^N (\xi_i - a_\alpha)$, sum over i, and use the expression

$$\sum_{i,\,k=1}^{M}{}' \xi_i^{n+1}(\xi_i-\xi_k)^{-1} = -\frac{n+1}{2} s_n + \frac{1}{2}\sum_{l=0}^{n} s_{n-l}s_l.$$

As a result, we obtain a system of equations expressing s_n with $n>N-2$ in terms of s_n with $n \leqslant N-2$. A different system of conditions on the polynomials s_n can be obtained by noting that s_n with $n>M$ are expressed in terms of s_n with $n \leqslant M$. Assuming that $M>N-2$ and combining these two systems, we arrive at $N-2$ algebraic equations of the form

$$s_{N-2}^{K_i} + f_{1i} s_{N-2}^{K_i-1} + \ldots + f_{K_i i} = 0, \quad i=1, \ldots, N-2,$$

where $K_i = [(M+i)/(N-2)]$. The coefficients in these equations depend explicitly on the $3N-5$ quantities $a_1, ..., a_{N-2}, b_1, ..., b_N$ and $s_1, ..., s_{N-3}$. For each equation to have at least K different solutions for a fixed set of these quantities, it is necessary that the inequality $[(M+i)/(N-2)] \geqslant K$ be satisfied for all $i=1, ..., N-2$. This leads to a restriction on M: $M \geqslant K(N-2)-1$. If $M = K(N-2)-1$, the degree of all the equations is the same and equal to K. For all the $N-2$ equations to be compatible, it is necessary to require that the coefficients of identical powers of s_{N-2} coincide. This is possible when $K(N-3)$ conditions are imposed on the $3N-5$ quantities. From this we find the limit $K \leqslant 3 + [4/(N-3)]$ on the order of the quasi-exactly solvable model.

As an example, let us construct a second-order ($K=2$) quasi-exactly solvable model characterized by a degenerate function (49) of the form $B(\lambda) = \alpha\lambda^{-1} - \beta - \gamma\lambda - \lambda^2$. In this case $N=4$, so that $M=3$ and, therefore, we obtain a system of two quadratic equations in s_2. The coefficients in these equations depend on α, β, γ, and s_1. We have at our disposal a sufficient number of parameters; therefore, for definiteness we can set $s_1 = 0$, reducing this system to the form

$$s_2^2 - 2(\gamma^2-\beta)s_2 + 3\gamma(\alpha+1) = 0,$$
$$s_2^2 + \frac{6}{5\gamma}\left[2\beta\gamma - \gamma^3 - \frac{3\alpha}{2} - 1\right]s_2 + \frac{18}{5\gamma}(\gamma^2-\beta)(\alpha+1) = 0.$$

Equating the coefficients of identical powers of s_2, we find that $\alpha = (8/27)\gamma^3 - 2/3$ and $\beta = -(2/3)\gamma^2$. The potential of the corresponding model has the form

$$V(x) = 4x^{10} + 8\gamma x^8 - \frac{4}{3}\gamma^2 x^6 - \left(\frac{208}{27}\gamma^3 + \frac{80}{3}\right)x^4 - \left(\frac{16}{27}\gamma^4 + \frac{68}{3}\gamma\right)x^2 + \left(\frac{16}{27}\gamma^3 - \frac{11}{6}\right)\left(\frac{11}{27}\gamma^3 - \frac{17}{6}\right)\frac{1}{x^2},$$

and the solutions are given by

$$E_\pm = -\frac{32}{81}\gamma^5 + \frac{80}{9}\gamma^2 \pm 8\sqrt{\gamma^4 - 2\gamma}.$$

Using the electrostatic analog, it is easy to show that the levels found in this way describe the first and second excited states. It is also easy to verify that this model does not reduce to the models obtained using the algebraic approach of Ref. 19.

[1)]Similar equations with b_α equal to $\frac{1}{2}$ and $\frac{1}{4}$ arise in solving the elimination problem in the theory of Lamé functions.[29]

[2)]The magneto-electrostatic analog that we consider is a generalization of the familiar Stieltjes electrostatic analog.[48]

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