Nonstandard creation–annihilation operator algebras and intermediate quantum statistics

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We discuss the problems involved in the mathematical description of systems of a variable number of physical objects of type k, obeying quantum statistics of ranks s_k in the sense that the number n_k of objects of type k in a quantum state can take values only from 0 to s_k. A method is proposed for constructing the algebras A(K) with mutually adjoint generators a_k^+ and a_k, $k \in K$, which can be interpreted as the creation and annihilation operators for an object of type k. A set of specific creation–annihilation operator algebras is constructed, a comparison is made with the algebras already known, and some applications of intermediate $(2 \leqslant s_k < \infty)$ quantum statistics are discussed.

INTRODUCTION

The mathematical formalism of the quantum theory of systems of a variable number of physical objects (particles, quanta, systems, elementary excitations, and so on) involves the concept of creation operators $a^+{}_k$ and annihilation operators a_k for each type of object k. Since the action of these operators is assumed to be defined in the space L of quantum states $|\psi\rangle$ of the system under consideration, the set $a^+{}_k$ and a_k, $k \in K$, can be viewed as the generators of an operator algebra A(K) in the space L, which is naturally referred to as a creation–annihilation operator algebra.

The algebra $A(K)$ contains the operators corresponding to various physical quantities characterizing the system of a variable number of objects, in particular, the operators N_k, which are self-adjoint in L, corresponding to the numbers of objects with integer eigenvalues n_k giving the possible numbers of objects of type k in a single quantum state.

Already at this stage of our discussion we can formulate the problem of studying the different types of quantum statistics for identical (one-level) objects, i.e., objects of fixed type k: Fermi–Dirac statistics,[1] $n_k \in \overline{01}$; intermediate statistics of rank s_k, $n_k \in \overline{0,\ s_k}$ for $1 \leqslant s_k < \infty$; Bose–Einstein statistics, $n_k \in \overline{0, \infty}$.

Owing to the physical meaning of the operators a_k (which decrease the number of objects of type k by one unit), $a^+{}_k$ (which increase the number of objects of type k by one unit), and N_k (which preserve the numbers of objects of all types), the following identities must be valid in the Fock space $L_F \subset L$ spanned by the eigenvectors of the operators N_k:

$$[N_k,\ N_{k'}]_- = 0,\ [N_k,\ a_{k'}]_- = -\delta_{kk'}a_k,\ [N_k,\ \overset{+}{a}_{k'}]_- = \delta_{kk'}\overset{+}{a}_k. \qquad (1)$$

Since $N_k^+ = N_k$, the relations (1) allow us to assume that the operators $a^+{}_k$ and a_k are mutually-adjoint in L_F, i.e., $a_k^+ = a^+{}_k$ and $a^+{}_k = a_k$.

Other features of the algebra $A(K)$, the type of statistics of the objects in question, the explicit form of the operators N_k, and so on, are determined by the system of identity relations for the generators $a^+{}_k$ and a_k.

In present-day quantum theory (see, for example, Refs. 1–5) two types of identity relation are commonly used for the generators of the creation–annihilation operator algebra related to two types of quantum statistics, namely,

$$N_k = \overset{+}{a}_k a_k,\ [a_k,\ \overset{+}{a}_{k'}]_\pm = \delta_{kk'},\ [a_k,\ a_{k'}]_\pm = 0. \qquad (2)$$

Here the " + " sign corresponds to the Fermi algebra $A_F(K)$ and the Fermi–Dirac statistics $(s_k = 1)$, and " − " corresponds to the Bose algebra $A_B(K)$ and the Bose–Einstein statistics $(s_k = \infty)$. In general, it is assumed that all the objects in real physical systems are either fermions or bosons. Here $K = K_1 + K_2$, and the creation and annihilation operators satisfy the relations (2) with the " + " sign for k, $k = k'K_1$ and the " − " for $k, k' \in K_2$. This leads to some product $A(K)$ of the Fermi algebra $A_F(K_1)$ and the Bose algebra $A_B K_2)$. The additional assumption[1–5] that the generators of A_F commute with the generators of A_B makes this product specific and leads to the standard algebra $A_{st}(K)$. However, throughout the development of quantum theory there have been numerous proposals and studies of nonstandard algebras $A_{nst}(K)$ of creation–annihilation operators. Examples of A_{nst} are anomalous algebras,[6–8] algebras of para-Bose operators,[7,9–18] algebras of para-Fermi operators,[10–18] "superoperator" algebras,[19–22] μ algebras.[23–25] μ_S algebras,[20,26,27] Φ algebras,[23,28–31] and M algebras.[28,32,33] The mathematical formulations and possibilities of such nonstandard algebras can be found, for example, in the brief review of Ref. 34.

From the viewpoint of the statistics of identical objects of fixed type, of greatest interest are the para-Fermi algebras

$$N_k = \frac{1}{2}[\overset{+}{a}_k,\ a_k]_- + \frac{1}{2}p;\quad a_k = \sum_{j=1}^{p} b_k^{(j)}; \qquad (3a)$$

$$[b_k^{(j)},\ \overset{+}{b}_{k'}^{(j)}]_+ = \delta_{kk'};\ [b_k^{(j)},\ b_{k'}^{(j)}]_+ = 0; \qquad (3b)$$

$$[b_k^{(j)},\ \overset{+}{b}_{k'}^{(j')}]_- = [b_k^{(j)},\ b_{k'}^{(j')}]_- = 0,\quad j \neq j', \qquad (3c)$$

and the superoperator algebras (according to the terminology of Ref. 19)

$$N_k = \overset{+}{a}_k a_k,\ [a_k,\ \overset{+}{a}_k] = 1 - \frac{p+1}{p!}\overset{+}{a}{}_k^p a_k^p; \qquad (4a)$$

$$[a_k,\ \overset{+}{a}_{k'}]_- = [a_k,\ a_{k'}]_- = 0,\quad k \neq k'. \qquad (4b)$$

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In both of these algebras p is an integer, $a_k^{p+1} = a^{+\,p+1}_k = 0$ and $n_k \in \overline{0,p}$. Therefore, both algebras decribe objects obeying intermediate quantum statistics of rank p.

The fact that two different creation–annihilation operator algebras corresponding to the same intermediate statistics exist naturally leads to the problem of seeking all possible algebras $A(K)$ suitable for describing physical systems with a variable number of objects of type $k_1, k_2, \ldots \in K$, obeying statistics of rank $s_{k_1}, s_{k_2}, \ldots, \in s(K)$, respectively.

Some aspects of this problem have been studied for the case of identical (one-level) objects in Refs. 20, 27, and 35–37.

In the case of non-identical (or many-level) objects obeying different statistics, the general problem of constructing the corresponding nonstandard creation–annihilation operator algebras has hardly been studied at all. The only exceptions are the M algebras, Φ algebras, and special cases of them related to the so-called fractional statistics[23,28–31,38,39] where the subalgebras of the algebra $A(K)$ are linked by relations of the form

$$\overset{+}{a}_k \overset{+}{a}_{k'} = e^{-i\theta_{kk'}} \overset{+}{a}_{k'} a_k, \; a_k a_{k'} = e^{i\theta_{kk'}} a_{k'} a_k, \quad k \neq k' \qquad (5)$$

with real numbers $\theta_{kk'}$.

It should be noted that the interest in intermediate statistics and the related nonstandard creation–annihilation operator algebras has increased significantly in the last decade in connection with the quantum Hall effect,[38,3] the theory of exotic field quanta,[28,32,33,42] charged excitons,[29,31] and monopoles,[30] and also in connection with the possible (weak) violation of the Pauli principle[43–48]

Here we shall describe mainly the method of constructing algebras $A(K)$ of creation operators $a^+{}_k$ and annihilation operators a_k which are useful in the formal description of quantum systems of a variable number of objects of various types $k \in K$ obeying given quantum statistics of rank $s_k \in s(K)$. The method is based on the use of *a priori* intuitive definitions related to the fundamental concepts of the algebras under construction.

In later sections of this paper we compare the constructed algebras with algebras already known in the literature and discuss some specific applications.

1. INITIAL PROPOSITIONS

For constructing and studying possible mathematical formalisms applicable to the quantum description of systems of a variable number of objects, w shall begin with the following intuitive definitions.

Definition 1. The space of states $|\psi\rangle$ of a system with a variable number of objects of types $k_1, k_2, \ldots \in K$ obeying quantum statistics of rank $s_1, s_2, \ldots \in s(K)$, $s_k \geqslant 1$, respectively, is a complex vector space $L[s(K)]$ with positive metric spanned by an orthogonal basis of $(s_1+1)(s_2+1)\ldots$ nondegenerate vectors $|\varphi_{n_1, n_2 \ldots}\rangle$, such that

$$|\varphi_{n_1, n_2, \ldots}\rangle \in L(s(K)), \quad n_k \in \overline{0, s_k}; \qquad (6a)$$

$$\langle \varphi_{n_1', n_2', \ldots} | \varphi_{n_1, n_2, \ldots}\rangle = \| \varphi_{n_1, n_2, \ldots}\rangle|^2 \delta_{n_1' n_1} \delta_{n_1' n_2} \ldots; \qquad (6b)$$

$$|\psi\rangle = \sum_{n_1=0}^{s_1} \sum_{n_2=0}^{s_2} \ldots \psi_{n_1, n_2, \ldots} |\varphi_{n_1, n_2, \ldots}\rangle \in L(s(K)); \qquad (6c)$$

$$\langle \psi | \psi \rangle = \sum_{n_1=0}^{s_1} \sum_{n_2=0}^{s_2} \ldots |\psi_{n_1, n_2, \ldots}|^2 \| \varphi_{n_1, n_2, \ldots}\rangle|^2 \geqslant 0. \qquad (6d)$$

Definition 2. The creation–annihilation operator algebra for the objects of this system is an algebra $A(K)$ having a representation in $L[s(K)]$ with mutually-adjoint generators a_k and $a^+{}_k$, which realize motion in the basis (6a) with change of the corresponding index by one unit, namely:

$$a_k |\varphi_{\ldots, n_{k-1}, 0, n_{k+1}, \ldots}\rangle = 0; \qquad (7a)$$

$$a_k |\varphi_{\ldots, n_k, \ldots}\rangle = |\varphi_{\ldots, n_{k-1}, \ldots}\rangle, \; n_k \in \overline{1, s_k}; \qquad (7b)$$

$$\overset{+}{a}_k |\varphi_{\ldots, n_k, \ldots}\rangle = |\varphi_{\ldots, n_{k+1}, \ldots}\rangle, \quad n_k \in \overline{0, s_k - 1}, \qquad (7c)$$

$$\overset{+}{a}_k |\varphi_{\ldots, n_{k-1}, s_k, n_{k+1}, \ldots}\rangle = 0, \quad \text{if} \quad s_k < \infty. \qquad (7d)$$

These definitions allow us to prove a series of statements, which below are divided into theorems and corollaries. The proofs of all these statements are omitted, since they are quite elementary. Detailed proofs of some statements are given in Refs.[35–37].

Theorem 1. In $L[s(K)]$ there exists an orthonormal basis

$$|n_1, n_2, \ldots\rangle = \alpha(n_1, n_2, \ldots) |\varphi_{n_1, n_2, \ldots}\rangle, \quad n_k \in \overline{0, s_k}; \qquad (8a)$$

$$\langle n_1', n_2', \ldots | n_1, n_2, \ldots \rangle = \delta_{n_1' n_1} \delta_{n_2' n_2} \ldots, \qquad (8b)$$

where $\alpha(n_1, n_2, \ldots)$ are certain complex numbers such that

$$a_k |\ldots, n_{k+1}, 0, n_{k+1}, \ldots\rangle = 0; \qquad (9a)$$

$$a_k |\ldots, n_k, \ldots\rangle = \sqrt{\lambda_k(\ldots, n_k, \ldots)} \times e^{i\theta_k(\ldots, n_k, \ldots)} |\ldots, n_k - 1, \ldots\rangle, \quad n_k \in \overline{1, s_k}; \qquad (9b)$$

$$\overset{+}{a}_k |\ldots, n_k, \ldots\rangle = \sqrt{\lambda_k(\ldots, n_k + 1, \ldots)} \times e^{-i\theta_k(\ldots, n_k+1, \ldots)} |\ldots, n_k + 1, \ldots\rangle, \quad n_k \in \overline{0, s_k - 1}; \qquad (9c)$$

$$\overset{+}{a}_k |\ldots, n_{k-1}, s_k, n_{k+1}, \ldots\rangle = 0, \quad \text{if} \quad s_k < \infty, \qquad (9d)$$

where $\lambda_k(n_1, n_2, \ldots)$ are non-negative numbers and $\theta_k(n_1, n_2, \ldots)$ are real numbers, henceforth referred to as the parameters of the algebra (moduli and phases).

Corollary 1.1. The creation–annihilation operator algebra $A(K)$ is uniquely determined by specifying the ranks of the statistics $s_k \in s(K)$ and the parameters of the algebra—the moduli $\lambda_k(n_1, n_2, \ldots) > 0$ and phases $\theta_k(n_1, n_2, \ldots) = \theta_k^*$, where $k \in K$, $n_k \in \overline{1, s_k}$, and $n_{k'} \in \overline{0, s_{k'}}$, for $k' \neq k$.

In fact, if the ranks of the statistics and the parameters of the algebra are specified, then a unique representation of $A(K)$ in $L[s(K)]$ is specified, since the action of all a_k and $a^+{}_k$ and, consequently, of any operator from $A(K)$ on any vector from $L[s(K)]$ is uniquely specified by the relations (9).

Corollary 1.2. A given, fixed quantum statistics [where all the ranks $s(K)$ are specified] can correspond to an infinite set of creation–annihilation operator algebras $A(K)$, differing from each other in the values of the parameters.

To conclude this section, we note that relations (6)–(9) provide a basis for the physical interpretation of the mathematical concepts which we have used and the standard terminology: the vector $|n_1, n_2, \ldots\rangle$ (or the proportional vector $|\varphi_{n_1, n_2}\rangle$) is a state with fixed numbers of objects n_1, $n_2, \ldots$; a_k is the annihilation operator of an object of type k; $a^+{}_k$ is the creation operator of an object of type k; $|0, 0, \ldots\rangle$ is the vacuum; $|s_1, s_2 \ldots\rangle$ is the saturated state, reached only for all $s_k < \infty$.

2. THE ALGEBRAS $A(1)$, THEIR PROPERTIES, AND METHODS OF DETERMINING THEM

Let us consider a system of a variable number of identical (one-level) objects obeying quantum statistics of rank s (for simplicity, in this section we drop the index denoting the object).

According to the propositions of the preceding section, the states of this system belong to the space L(s) spanned by an orthonormal basis of $s+1$ vectors, and the creation–annihilation operator algebra, henceforth called $A(1)$, has one pair of generators: a and a^+.

The main features and methods of determining the algebra $A(1)$ are reflected in the following set of statements.

Theorem 2. In $L(s)$ there exists a unique basis

$$|n\rangle \in L(s), \quad n \in \overline{0,\ s}; \quad \langle n' \mid n\rangle = \delta_{n'n}, \tag{10}$$

in which the action of the generators a and a^+ of the algebra $A(1)$ is determined by the system of equations

$$a \mid 0\rangle = 0; \quad a \mid n\rangle = \sqrt{\lambda(n)} \mid n-1\rangle, \quad n \in \overline{1,\ s}; \tag{11a}$$

$$\overset{+}{a} \mid n\rangle = \sqrt{\lambda(n+1)} \mid n+1\rangle, \quad n \in \overline{0,\ s-1};$$

$$\overset{+}{a} \mid s\rangle = 0, \quad \text{if} \quad s < \infty. \tag{11b}$$

Corollary 2.1. For the unique determination of the algebra $A(1)$ it is necessary and sufficient to specify the rank of the statistics s and the set of modulus parameters $\lambda(n) > 0$, $n \in \overline{1,s}$.

Corollary 2.2. To a quantum statistics of any rank there corresponds an infinite set of algebras $A(1)$, differing from each other by the values of the modulus parameters $\lambda(n)$.

Theorem 3. The generators a and a^+ of the algebra $A(1)$, corresponding to statistics of rank s, satisfy the identity relations

$$a^{s+1} = 0, \quad \text{if} \quad s < \infty; \; a^k \overset{+}{a}{}^{k+l} = \sum_{m=0}^{s-l} \mu_m^{k,\,k+l} \overset{+}{a}{}^{m+l} a^m, \tag{12}$$

where $1 \leqslant k \leqslant k+l \leqslant s$ and the relations conjugate to (12) for $l \neq 0$, where the coefficients μ are real and are uniquely determined by the parameters of the algebra $\lambda(n)$ by means of the recursion relations

$$\mu_0^{k,\,k+l} = \Gamma_{k+l}/\Gamma_l, \quad \Gamma_0 = 1, \quad \Gamma_n = \prod_{k=1}^{n} \lambda(n); \tag{13a}$$

$$\mu_n^{k,\,k+l} = \Gamma_{n+k+l}/\Gamma_n\Gamma_{n+l} - \sum_{m=0}^{n-1} \mu_m^{k,\,k+l}/\Gamma_{n-m}, \quad 1 \leqslant n < s-k-l+1; \tag{13b}$$

$$\mu_n^{k,\,k+l} = -\sum_{m=0}^{n-1} \mu_m^{k,\,k+l}/\Gamma_{n-m}, \quad s-k-l+1 \leqslant n \leqslant s-l. \tag{13c}$$

Corollary 3.1. Any operator of the algebra $A(1)$ can be written in normal-ordered form, i.e., as an expression in which all the creation operators stand to the left of all the annihilation operators.

Theorem 4. The first s coefficients of the identity relation (13b) with $k=1$ and $l=0$ existing in any $A(1)$,

$$a\overset{+}{a} = \sum_{m=0}^{s} \mu_m^{1,\,1} \overset{+}{a}{}^m a^m, \tag{14a}$$

are related to the parameters of the algebra $\lambda(n)$ by the recursion relations

$$\lambda(n) = \sum_{m=0}^{n-1} \mu_m^{1,\,1} \frac{\Gamma_{n-1}}{\Gamma_{n-m-1}}, \quad n \in \overline{1,\ s}. \tag{14b}$$

Corollary 4.1. The algebra $A(1)$ is uniquely determined by the rank of the statistics s and the set of the first s coefficients $\mu_m^{1,1}$ ...$m \in \overline{0,\ s-1}$, of the principal identity relation (14a) if and only if (14b) ensures that $\lambda(n)$ is non-negative.

Theorem 5. In $A(1)$ there exists a unique operator N having the required properties of an object-number operator

$$N \mid n\rangle = n \mid n\rangle, \quad n \in \overline{0,\ s}; \; [N,\ a]_- = -a, \tag{15}$$

and the normal-ordered form of this operator is

$$N = N^+ = \sum_{k=1}^{s} \nu_k \overset{+}{a}{}^k a^k; \tag{16a}$$

$$\nu_1 = \frac{1}{\lambda(1)}, \quad \nu_n = \frac{n}{\Gamma_n} - \sum_{k=1}^{n-1} \frac{\nu_k}{\Gamma_{n-k}}, \quad 2 \leqslant n \leqslant s. \tag{16b}$$

Corollary 5.1. The parameters $\lambda(n)$ of the algebra $A(1)$ are related to the coefficient ν_k of the normal-ordered form of the operator N by the recursion relations

$$\lambda(n) = n \left(\sum_{k=1}^{n} \nu_k \frac{\Gamma_{n-1}}{\Gamma_{n-k}} \right)^{-1}, \quad n \in \overline{1,\ s}. \tag{17}$$

Corollary 5.2. The algebra $A(1)$ is uniquely determined by the rank of the statistics s and the set of coefficients ν_k, $k \in \overline{1,s}$, of the normal-ordered form of the object-number operator if and only if (17) ensures that $\lambda(n)$ is finite and nonnegative.

Corollary 5.3. The algebra $A(1)$ has an irreducible matrix representation (the proper N representation) in the form

$$(a)_{kl} = \delta_{k,\,l-1} \sqrt{\lambda(k)}, \quad (\overset{+}{a})_{kl} = \delta_{k,\,l+1} \sqrt{\lambda(l)},$$

$$(N)_{kl} = \delta_{kl}(k-1), \tag{18}$$

where k, $l \in \overline{1,s+1}$, s is the rank of the statistics, and $\lambda(k)$ are the parameters of the algebra.

Theorem 6. The object-number operator can be written in the bilinear form

$$N = C_0 + \sum_{k=1}^{s} C_{1,\,k} a^k \overset{+}{a}{}^k + \sum_{k=1}^{s} C_{2,\,k} \overset{+}{a}{}^k a^k, \tag{19}$$

where the $2s+1$ real coefficients are related to the parameters of the algebra by a system of $s+1$ algebraic equations:

$$C_0 = -\sum_{k=1}^{s} C_{1,\,k}\Gamma_k; \; s - C_0 = \sum_{k=1}^{s} C_{2,\,k} \frac{\Gamma_s}{\Gamma_{s-k}}, \quad \text{if} \quad s < \infty; \tag{20a}$$

$$n - C_0 = \sum_{k=1}^{s-n} C_{1,\,k} \frac{\Gamma_{n+k}}{\Gamma_n} + \sum_{k=1}^{n} C_{2,\,k} \frac{\Gamma_n}{\Gamma_{n-k}}, \quad 1 \leqslant n < s. \tag{20b}$$

Corollary 6.1. The form (19) of the operator N is not unique.

Corollary 6.2. The specification of the rank of the statistics s and the object-number operator N in the bilinear form (19) determines a set of (admissible) algebras $A(1)$, each of which corresponds to a solution of the system (20) for the parameters $\lambda(n) > 0$.

Theorem 7. If the states $|n\rangle$ are the eigenvectors of an operator $M \in A(1)$ which is self-adjoint in $L(s)$, such that

$$M = \sum_{k=0}^{s} \xi_k \overset{+}{a}{}^k a^k; \quad M \mid n\rangle = m_n \mid n\rangle, \qquad n \in \overline{0, s}, \qquad (21)$$

then the eigenvalues m_n and the coefficients ξ_k of its normal-ordered form are related to the parameters of the algebra by the system of equations

$$m_0 = \xi_0; \ \lambda(n) = (m_n - m_0) \Big(\sum_{k=1}^{n} \xi_k \frac{\Gamma_{n-1}}{\Gamma_{n-k}} \Big)^{-1}, \qquad n \in \overline{1, s}. \qquad (22)$$

Corollary 7.1. The algebra $A(1)$ is uniquely determined by the rank of the statistics s, the set of eigenvalues m_n, $n \in \overline{0,s}$, and the coefficients ξ_k, $k \in \overline{1,s}$, of the normal-ordered form of any operator $M \in A(1)$ commuting with N if and only if (22) ensures that $\lambda(n)$ is finite and non-negative.

In summary, we again stress the fact that the quantum statistics of identical (one-level) objects of rank s corresponds to an infinite set of creation–annihilation operator algebras $A(1)$ (Corollary 2.2), and each specific algebra can be uniquely determined by at least four different methods (Corollaries 2.1, 4.1, 5.2, and 7.1).

3. $A(2)$ ALGEBRAS AND PRODUCTS OF $A(1)$ AND $A(1)$

Let us consider a system containing two types of objects, for example, α and β obeying quantum statistics of ranks s_α and s_β, respectively.

According to our initial postulates, the mathematical description of this system requires an algebra $A(\alpha, \beta)$, henceforth referred to as $A(2)$, with two pairs of generators a_α, $\overset{+}{a}_\alpha$ and a_β, $\overset{+}{a}_\beta$, acting in the space of states $L(s_\alpha, s_\beta)$.

According to Theorem 1 and Corollary 1.1, the algebra $A(2)$ is uniquely determined by the values of $4s_\alpha s_\beta + 2(s_\alpha + s_\beta)$ parameters—the non-negative moduli $\lambda_\alpha(k, n)$, $\lambda_\beta(k, n)$ and real phases $\theta_\alpha(k, n)$, $\theta_\beta(k, n)$, and the action in the space $L(s_\alpha, s_\beta)$ is given by

$$a_\alpha \mid 0, k\rangle = 0; \ \overset{+}{a}_\alpha \mid s_\alpha, k\rangle = 0, \quad \text{if} \quad s_\alpha < \infty; \qquad (23a)$$

$$a_\alpha \mid n, k\rangle = \sqrt{\lambda_\alpha(n, k)}\, e^{i\theta_\alpha(n, k)} \mid n-1, k\rangle, \quad n \in \overline{1, s_\alpha}; \qquad (23b)$$

$$\overset{+}{a}_\alpha \mid n, k\rangle = \sqrt{\lambda_\alpha(n+1, k)}\, e^{-i\theta_\alpha(n+1, k)} \mid n+1, k\rangle, \quad n \in \overline{0, s_\alpha - 1}; \qquad (23c)$$

$$a_\beta \mid n, 0\rangle = 0; \ \overset{+}{a}_\beta \mid n, s_\beta\rangle = 0, \quad \text{if} \quad s_\beta < \infty; \qquad (23d)$$

$$a_\beta \mid n, k\rangle = \sqrt{\lambda_\beta(n, k)}\, e^{i\theta_\beta(n, k)} \mid n, k-1\rangle, \quad k \in \overline{1, s_\beta}; \qquad (23e)$$

$$\overset{+}{a}_\beta \mid n, k\rangle = \sqrt{\lambda_\beta(n, k+1)}\, e^{-i\theta_\beta(n, k+1)} \mid n, k+1\rangle, \quad k \in \overline{0, s_\beta - 1}. \qquad (23f)$$

The general study of $A(2)$ algebras is a quite complicated problem. Therefore, here we shall restrict ourselves to only three theorems and one specific example.

Theorem 8. The algebra $A(2)$ contains $s_\alpha + 1$ algebras $A(1)$ with rank of the statistics equal to s_β and parameters $\lambda(n) = \lambda_\beta(k, n)$ for fixed $k \in \overline{0, s_\alpha}$ and $s_\beta + 1$ algebras $A(1)$ with rank of the statistics equal to s_α and parameters $\lambda(k) = \lambda_\alpha(k, n)$ for fixed $n \in \overline{0, s_\beta}$.

The validity of this statement is obvious. For example, the relations (23a)–(23c) for each fixed k determine an algebra $A(1)$ with generators $\overset{+}{a}_\alpha$ and a_α acting in the space $L_k(s_\alpha) \subset L(s_\alpha, s_\beta)$ and spanned by a basis of $(s_\alpha + 1)$ orthonormal vectors $|n\rangle = |n,k\rangle$, $n \in \overline{0, s_\alpha}$. According to Theorem 2, this algebra is uniquely determined by the s_α parameters $\lambda(n)$, $n \in \overline{1, s_\alpha}$, and from the relations (23a)–(23c) it follows that $\lambda(n) = \lambda_\alpha(n,k)$.

Corollary 8.1. In general, the algebra $A(2)$ can contain $s_\alpha + s_\beta + 2$ different algebras $A(1)$, each of which is the creation–annihilation operator algebra of objects of a given type for a fixed number of objects of a different type.

Corollary 8.2. In general, the algebra $A(2)$ cannot be written as a product of $A(1)$ and $A(1)$.

Corollary 8.3. In general, the algebra $A(2)$ does not have any homogeneous (containing creation and annihilation operators of only one type) identity relations.

Theorem 9. The generators of the algebra $A(2)$ satisfy a system of inhomogeneous identity relations:

$$a_\alpha^{s_\alpha+1} = 0, \quad \text{if} \quad s_\alpha < \infty; \quad a_\beta^{s_\beta+1} = 0, \quad \text{if} \quad s_\beta < \infty, \qquad (24a)$$

$$a_\alpha^k \overset{+}{a}{}_\alpha^{k+l} = \sum_{m=0}^{s_\alpha - l} \sum_{n=0}^{s_\beta} \tilde{\mu}_{\alpha, mn}^{k, k+l} \overset{+}{a}{}_\alpha^{m+l} \overset{+}{a}{}_\beta^{n} a_\alpha^m a_\beta^n + (a_\alpha \leftrightarrow a_\beta, \ \overset{+}{a}_\alpha \leftrightarrow \overset{+}{a}_\beta); \qquad (24b)$$

$$a_\beta^{k'} \overset{+}{a}{}_\beta^{k'+l'} = \sum_{m=0}^{s_\beta - l'} \sum_{n=0}^{s_\alpha} \tilde{\mu}_{\beta, mn}^{k', k'+l'} \overset{+}{a}{}_\beta^{m+l'} \overset{+}{a}{}_\alpha^{n} a_\beta^m a_\alpha^n + (a_\alpha \leftrightarrow a_\beta, \ \overset{+}{a}_\alpha \leftrightarrow \overset{+}{a}_\beta); \qquad (24c)$$

$$a_\alpha^\rho \overset{+}{a}{}_\beta^\sigma = \sum_{m=0}^{s_\alpha - \rho} \sum_{n=0}^{s_\beta - \sigma} \mathfrak{L}_{\alpha\beta, mn}^{\rho\sigma} \overset{+}{a}{}_\alpha^m \overset{+}{a}{}_\beta^n a_\alpha^{m+\rho} a_\beta^{n+\sigma} + (a_\alpha \leftrightarrow a_\beta, \ \overset{+}{a}_\alpha \leftrightarrow \overset{+}{a}_\beta); \qquad (24d)$$

$$a_\beta^\sigma \overset{+}{a}{}_\alpha^\rho = \sum_{m=0}^{s_\beta - \sigma} \sum_{n=0}^{s_\alpha - \rho} \mathfrak{L}_{\alpha\beta, nm}^{\rho\sigma} \overset{+}{a}{}_\alpha^n \overset{+}{a}{}_\beta^m a_\alpha^{n+\rho} a_\beta^{m+\sigma} + (a_\alpha \leftrightarrow a_\beta, \ \overset{+}{a}_\alpha \leftrightarrow \overset{+}{a}_\beta); \qquad (24e)$$

$$a_\alpha^\rho \overset{+}{a}{}_\beta^\sigma = \sum_{m=0}^{s_\alpha - \rho} \sum_{n=0}^{s_\beta - \sigma} \mathfrak{M}_{\alpha\beta, mn}^{\rho\sigma} \overset{+}{a}{}_\alpha^m \overset{+}{a}{}_\beta^{n+\sigma} a_\alpha^{m+\rho} a_\beta^n + (a_\alpha \leftrightarrow a_\beta, \ \overset{+}{a}_\alpha \leftrightarrow \overset{+}{a}_\beta), \qquad (24f)$$

where $1 \leqslant k \leqslant k \leqslant + l \leqslant s_\alpha$, $1 \leqslant k' \leqslant k' + l \leqslant s_\beta$, $1 \leqslant \rho \leqslant s_\beta$, $1 \leqslant \sigma \leqslant s_\beta$, and $\tilde{\mu}$, $\mathfrak{L}$, and $\mathfrak{M}$ are numerical, in general, complex coefficients.

The relation between the coefficients in (24) and the parameters of the algebra is easy to obtain by successive operation on the basis vectors of the space $L(s_\alpha, s_\beta)$ with the identities (24) using Eqs. (23).

Corollary 9.1. Any operator of the algebra $A(2)$ can be written in normal-ordered form.

Theorem 10. The algebra $A(2)$ with ranks of the statistics equal to s_α and s_β is a product of the algebra $A(1)$ of rank s_α and the algebra $A(1)$ of rank s_β if and only if

$$\lambda_\alpha(n, k) = \lambda_\alpha(n), \quad \lambda_\beta(n, k) = \lambda_\beta(k), \qquad (25)$$

where $\lambda_\alpha(n)$, $n \in \overline{1, s_\alpha}$, and $\lambda_\beta(n)$, $b \in \overline{1, s_\beta}$ are the parameters of the algebra $A(1)$. Here the generators a_α, $\overset{+}{a}_\alpha$ (the generators a_β, $\overset{+}{a}_\beta$) satisfy the system of identity relations (12) with the corresponding parameters in the entire space

$L(s_\alpha, s_\beta)$, and the mixed (inhomogeneous and nonlinear) identity relations can be written as

$$a_\alpha^\rho a_\beta^\sigma = \sum_{m=0}^{s_\alpha-\rho} \sum_{n=0}^{s_\beta-\sigma} \mathfrak{L}^{\sigma\rho}_{\beta\alpha,\, nm} \overset{+}{a}{}_\beta^n \overset{+}{a}{}_\alpha^m a_\beta^{n+\sigma} a_\alpha^{m+\rho}; \tag{26a}$$

$$a_\beta^\sigma a_\alpha^\rho = \sum_{m=0}^{s_\alpha-\rho} \sum_{n=0}^{s_\beta-\sigma} \mathfrak{L}^{\rho\sigma}_{\alpha\beta,\, mn} \overset{+}{a}{}_\alpha^m \overset{+}{a}{}_\beta^n a_\alpha^{m+\rho} a_\beta^{n+\sigma}; \tag{26b}$$

$$a_\alpha^\rho \overset{+}{a}{}_\beta^\sigma = \sum_{m=0}^{s_\alpha-\rho} \sum_{n=0}^{s_\beta-\sigma} \mathfrak{M}^{\sigma\rho}_{\beta\alpha,\, nm} \overset{+}{a}{}_\beta^{n+\sigma} \overset{+}{a}{}_\alpha^m a_\beta^\sigma a_\alpha^{m+\rho}; \tag{26c}$$

$$a_\beta^\sigma \overset{+}{a}{}_\alpha^\rho = \sum_{m=0}^{s_\alpha-\rho} \sum_{n=0}^{s_\beta-\sigma} \mathfrak{M}^{\rho\sigma}_{\alpha\beta,\, mn} \overset{+}{a}{}_\alpha^{\rho+m} \overset{+}{a}{}_\beta^n a_\alpha^m a_\beta^{\sigma+n}, \tag{26d}$$

where the coefficients $\mathfrak{L}$ and $\mathfrak{M}$ are uniquely determined by the parameters of the algebra $A(2)$, $1 \leqslant \rho \leqslant s_\alpha$, $1 \leqslant \sigma \leqslant s_\beta$.

The identity relations (26) make the concept of the product of algebras $A(1)$ specific by ensuring a unique linking in the space of states $L(s_\alpha, s_\beta)$.

Example 1. Let us consider the very simple case of a system with objects of type α and β obeying statistics of rank 1, i.e., $s_\alpha = s_\beta = 1$. Then the space of states $L(1, 1)$ is spanned by a basis of four orthonormal vectors ($|0,0\rangle$, $|0,1\rangle$, $|1,0\rangle$, and $|1,1\rangle$), and the corresponding algebra $A(2)$ which is the product of two algebras $A(1)$ is determined by six parameters [$\lambda_\alpha, \lambda_\beta, \theta_\alpha(1, 0), \theta_\beta(0, 1), \theta_\alpha(1, 1)$, and $\theta_\beta(1, 1)$] and has the system of identity relations

$$a_\alpha \overset{+}{a}_\alpha = \lambda_\alpha - \overset{+}{a}_\alpha a_\alpha, \quad a_\beta \overset{+}{a}_\beta = \lambda_\beta - \overset{+}{a}_\beta a_\beta; \tag{27a}$$

$$a_\alpha a_\beta = e^{i\Phi_{\alpha\beta}} a_\beta a_\alpha, \quad a_\alpha \overset{+}{a}_\beta = e^{-i\Phi_{\alpha\beta}} \overset{+}{a}_\beta a_\alpha. \tag{27b}$$

Here the linking (27b) of the algebras $A(1)$ is determined by a real number $\Phi_{\alpha\beta}$, related to the phase parameters of the algebra $A(2)$ as

$$\Phi_{\alpha\beta} = \theta_\beta(1,1) - \theta_\beta(0, 1) + \theta_\alpha(1, 0) - \theta_\alpha(1, 1).$$

We have actually obtained the Φ algebra for fermions, which was introduced intuitively and used in Refs. 23, 28–31, 38, and 39. In the special case $\lambda_\alpha = \lambda_\beta = 1$ and $\Phi_{\alpha\beta} = \pi$, the algebra (27) is the standard product of two Fermi algebras.

4. GENERAL PROBLEMS WITH ALGEBRAS *A(K)*

The set of statements concerning the algebras $A(1)$ and $A(2)$ given in the preceding sections shows that, in general, the algebra $A(K)$ is a rather complicated mathematical object, and a number of problems must be solved before it can be used in physical theories.

First of all, Theorem 1 and Corollary 1.1 are sufficient, but not necessary. In other words, the values of the parameters $\lambda_k(..., n_k, ...)$ and $\theta_k(..., n_k, ...)$ uniquely determine the algebra $A(K)$, but the reverse is not true. Different values of the phase parameters can therefore correspond to the same algebra [see, for example, Theorem 2 and Corollary 2.1 for $A(1)$ algebras, and also Example 1 for the algebra $A(2)$], so the problem of isolating the independent parameters must be solved.

Second, the algebra $A(K)$ contains the following: a set of, in general, distinct algebras $A(1)$, which are the creation–annihilation operator algebras for objects of one type with the numbers of objects of other types fixed, and these algebras operate in the corresponding subspaces $L(s) \subset L[(s(K)]$; a set of, in general, distinct creation–annihilation operator algebras $A(2)$ for objects of two types with the numbers of objects of other types fixed, and so on. Therefore, one of the fundamental problems with the algebra $A(K)$ is that of its "reducibility," i.e., the question of whether or not it is possible to reduce $A(K)$ to a product of two or more algebras $A(K')$ with $K' \subset K$.

Third, even when $A(K)$ is "reducible," it is necessary to study the possible linkings between the factor algebras. For example, the bilinear linking (27) and the nonlinear linking (26) of two $A(1)$ algebras as factors of the algebra $A(2)$ are binary (the identity relations of the linking involve two pairs of generators). However, it is also possible to have nonbinary linkings of factors, as, for example, in M algebras,[28,32,33,42] where each linking relation contains, in bilinear form, a restricted set of pairs of creation and annihilation operators.

These problems are purely mathematical and pertain primarily to the classification of the algebras $A(K)$.

In the use of creation–annihilation operator algebras in specific physical theories, there arises the additional problem of admissibility, i.e., determining which algebras $A(K)$ are compatible with the fundamental (imposed *a priori* or postulated) propositions of the theory. The problem of admissibility for the case of $A(1)$ algebras is studied in the following section.

5. *A*(1) ALGEBRAS FOR ACTUAL PHYSICAL OBJECTS

In the construction of an actual physical theory of systems of a variable number of objects, usually only the operators for physical quantities B are known. In the case of "free" systems (see, for example, Refs. 1–5) these operators have the form

$$B = \sum_\alpha \int dk B_{k\alpha}, \quad B_{k\alpha} = b_{k\alpha} N_{k\alpha}. \tag{28}$$

Here $b_{k\alpha}$ is the contribution of an object of type $k\alpha$ to the value of B, and $N_{k\alpha}$ is the operator corresponding to the number of identical (one-level) objects of type $k\alpha$, expressed as a bilinear combination of creation operators $\overset{+}{a}_{k\alpha}$ and annihilation operators $a_{k\alpha}$:

$$N = C_0 + C_1 a\overset{+}{a} + C_2 \overset{+}{a} a. \tag{29}$$

In (29) and below we omit the indices, since we are dealing with creation–annihilation operator algebras $A(1)$ for objects of a fixed type.

We note that an operator for physical objects (29) is written in the bilinear form (19), so, according to Theorem 6 and its corollaries, this operator determines a set of admissible algebras $A(1)$ with parameters satisfying the system of equations (20), which in the case of the operator (29) has the form

$$C_0 + C_1\lambda(1) = 0; \quad C_0 + C_2\lambda(s) = s, \quad \text{if} \quad s < \infty; \tag{30a}$$

$$C_0 + C_1\lambda(n+1) + C_2\lambda(n) = n, \quad n \in \overline{1, s}. \tag{30b}$$

If C_1 and C_2 are given, then for finite s the system of $s+1$ inhomogeneous linear equations (30) with the conditions $\lambda(n) > 0$ either uniquely determines all $s+1$ unknown C_0,

$\lambda(1),\ldots,\lambda(s)$, or has no solutions. In the former case, substitution of $\lambda(n)$ into the identity relations (12) determines an admissible algebra $A(1)$, while in the latter case such an algebra does not exist.

Let us now consider four different types of identical (one-level) physical objects (see Refs. 1–5).

1. Quantum systems (or excitations) in second-quantized nonrelativistic quantum mechanics: $C_1 = 0$, $C_2 = 1$.

2. The quanta of relativistic tensor fields: $C_1 = 1/2$, $C_2 = 1/2$.

3. The quanta of relativistic spinor fields: $C_1 = -1/2$, $C_2 = 1/2$.

4. Excitations in spin systems: $C_1 = -1$, $C_2 = 0$.

The results of investigations of the system (30) in these cases are for the most part contained in the following theorems.

Theorem 11. For quantum systems (or excitations) in second-quantized nonrelativistic quantum mechanics, statistics of any rank with a unique algebra $A(1)$ for each s are admissible:

$$\lambda(n) = n, \quad n \in \overline{1,\ s}; \quad N = \overset{+}{a}a; \tag{31a}$$

$$a\overset{+}{a} = 1 + \overset{+}{a}a - \frac{s+1}{s!}\overset{+}{a}{}^s a^s, \quad \text{if} \quad s < \infty; \tag{31b}$$

$$a\overset{+}{a} = 1 + \overset{+}{a}a, \quad \text{if} \quad s = \infty. \tag{31c}$$

The set (31) contains only the well-known algebras: the Fermi algebra for $s = 1$, the superalgebra (4) for $1 < s < \infty$, and the Bose algebra for $s = \infty$.

Theorem 12. For the quanta of relativistic tensor fields, statistics of finite rank are inadmissible, and statistics of infinite rank are admissible with a set of algebras $A_\chi(1)$ parametrized by a non-negative parameter χ:

$$\lambda(2k+1) = 2k + \varkappa, \quad \lambda(2k) = 2k, \quad \varkappa > 0, \quad k \in \overline{0,\ \infty}; \tag{32a}$$

$$N = \frac{1}{2}\left[\overset{+}{a},\ a\right]_+ - \frac{1}{2}\varkappa, \quad a\overset{+}{a} = \varkappa + \frac{2-\varkappa}{\varkappa}\overset{+}{a}a + \ldots \tag{32b}$$

In general, the identity relation (32b) contains all the operators $a^{+k}a^k$, $k \in \overline{0,\infty}$, with coefficients given by expressions (13) with the parameters (32a).

For $\chi = 1$, (32) is a Bose algebra.

Theorem 13. For the quanta of relativistic spinor fields, statistics of infinite rank are inadmissible, and statistics of any finite rank are admissible with a unique algebra $A(1)$ for each s:

$$\lambda(n) = n(s - n + 1), \quad n \in \overline{1,\ s}, \quad 1 \leqslant s < \infty; \tag{33a}$$

$$N = \frac{1}{2}\left[\overset{+}{a},\ a\right]_- + \frac{1}{2}s, \quad a\overset{+}{a} = s + \frac{s-2}{s}\overset{+}{a}a + \ldots \tag{33b}$$

In general, the identity relation (33b) contains all the operators $a^{+k}a^k$, $k \in \overline{0,s}$, with the coefficients (13) for the parameters (33a).

For $s = 1$, (33) is a Fermi algebra.

Theorem 14. For excitations in spin systems, statistics of infinite rank are inadmissible, and statistics of any finite rank are admissible with a unique algebra $A(1)$ for each s:

$$\lambda(n) = s - n + 1, \quad n \in \overline{1,\ s}, \quad 1 \leqslant s < \infty; \tag{34a}$$

$$N = s - a\overset{+}{a}, \quad a\overset{+}{a} = s - \frac{1}{s}\overset{+}{a}a + \ldots \tag{34b}$$

In general, relation (34b) contains all the operators $a^{+k}a^k$ $k \in \overline{0,s}$, with the coefficients (13) for the parameters (34a).

For $s = 1$, (34) is a Fermi algebra.

Theorem 15. Identical quanta of relativistic tensor fields (integer spin) can obey only statistics of infinite rank with the creation–annihilation operator algebra (32), a special case of which is the Bose algebra. Identical quanta of relativistic spinor fields (half-odd-integer spin) can obey only statistics of finite rank with the creation–annihilation operator algebra (33), a special case of which is the Fermi algebra.

The latter statement, the validity of which follows directly from Theorems 12 and 13, can be viewed as a generalization of the familiar Pauli theorem on the relation between spin and statistics with the *a priori* assumption that intermediate statistics are possible.

6. *A*(1) ALGEBRAS AND PARA-FERMI OPERATOR ALGEBRAS

In Sec. 5 we have already pointed out the existence of the familiar creation–annihilation operator algebras in the sets of algebras (31)–(34) for specific identical (one-level) objects: the algebra of Bose operators [the special case (31) and (32)], the algebra of Fermi operators [the special case (31), (33), and (34)], and the algebra of superoperators (4) of fixed rank [the special case (31)].

In the case of algebras of para-Fermi operators (3) of fixed rank, according to the form (3a) of the object number operator in the case of identical (one-level) objects, this algebra must coincide with the $A(1)$ algebra of the corresponding rank from the set (33). However, these algebras do not coincide for any rank, except rank 1.

Let us demonstrate this statement for the example of statistics of rank 2. For $s = 2$ the relations (33), (12), and (13) for the corresponding algebra $A(1)$ give

$$N = \frac{1}{2}\left[\overset{+}{a},\ a\right]_- + 1, \quad a^3 = 0, \quad a\overset{+}{a}{}^2 = 2\overset{+}{a} - \overset{+}{a}{}^2 a; \tag{35a}$$

$$a\overset{+}{a} = 2 - \frac{1}{2}\overset{+}{a}{}^2 a^2, \quad a^2\overset{+}{a}{}^2 = 4 - 2\overset{+}{a}a. \tag{35b}$$

It follows directly from (35) that $N^3 = 3N^2 - 2N$, i.e., $n \in \overline{0,2}$.

Let us now assume that the operators a and a^+ from (35) belong to a para-Fermi operator algebra of rank 2. Then the Green's ansatz (3a) is valid, i.e.,

$$a = b + c, \quad \left[b,\ \overset{+}{b}\right]_+ = \left[c,\ \overset{+}{c}\right]_+ = 1, \quad [b,\ c]_- = \left[b,\ \overset{+}{c}\right]_- = 0. \tag{36}$$

Substituting the representation (36) into (35), we easily find that the equations (35a) are satisfied with

$$N = \overset{+}{b}b + \overset{+}{c}c, \quad N^2 = N + 2\overset{+}{b}\overset{+}{c}bc, \tag{37a}$$

while to satisfy (35b), in addition to (36) we must have

$$(\overset{+}{b} - \overset{+}{c})(b - c) - 2\overset{+}{b}\overset{+}{c}bc = 0. \tag{37b}$$

Multiplying on the left by b^+ and on the right by c, from (36) we find $b^+c^+bc = 0$. From (37a) we then arrive at a contradiction: $N^2 = N$, i.e., $n \in \overline{0,1}$.

Such algebraic contradictions arise for any rank $s\geqslant 2$ if the operators a^+ and a from the algebra (33) are formally written as Green's ansätze of the corresponding rank from the algebra (3).

Therefore, an algebra (33) of rank $s\geqslant 2$ formally does not coincide with the algebra (3) of the corresponding rank for one-level objects, in spite of the fact that expressions (33b) and (3a) for the operator N coincide. At the same time, according to Theorem 13, the set of algebras (33) contains all the creation–annihilation operator algebras for identical (one-level) objects allowed for the operator N in the form (3a) and satisfying the intuitive Definitions 1 and 2.

The point is apparently that the algebras (3) have various irreducible Fock representations, even for one-level objects (see, for example, Refs. 12 and 18). However, for the creation–annihilation operator algebras discussed in the present study, the uniqueness and irreducibility of the representation in Fock space are actually encoded in the original definitions. This leads to the appearance of identities of the type (35b) in addition to the relations (35a), which actually define[18] the algebra of para-Fermi operators of rank 2 for one-level objects.

The correspondence between the algebras which we have discussed most likely boils down to the fact that an irreducible representation of the algebra (33) is equivalent to the maximal irreducible representation of the corresponding rank of the algebra (3) for one-level objects, in which relations of the type (35b) and (37b) are transformed into identities. The existence of such irreducible representations of the algebra (3) has been demonstrated, for example, in Ref. 18.

7. AVERAGE OCCUPATION NUMBERS IN INTERMEDIATE STATISTICS

Let us consider a system of physical objects distributed among energy levels $\epsilon_k > 0$ and obeying quantum statistics of rank s_k at each level k, respectively.

Let the objects be noninteracting, and the system be located in a heat bath of temperature $\theta = kT$.

If the number of objects in the system is not fixed, then according to the general rules of statistical physics (see, for example, Refs. 40 and 41) the probability $W(n)$ of finding the state $|(n)\rangle = |...,n_k,...\rangle$ is written as

$$W(n)=\exp\left\{\frac{1}{\Theta}\left(\Omega+\sum_k(\mu-\varepsilon_k)\,n_k\right)\right\},\quad \sum_{(n)}W_{(n)}=1, \tag{38}$$

where μ is the chemical potential, Ω is the grand thermodynamical potential, and n_k is the number of objects at the level ϵ_k, $n_k\in\overline{0,s_k}$. Here the grand partition function

$$Z=\sum_{(n)}\prod_k \xi_k^{-n_k}=Z(\ldots,\xi_k,s_k,\ldots),\ \text{where}\ \xi_k=e^{\frac{\varepsilon_k-\mu}{\Theta}}, \tag{39}$$

can be used to calculate all the physical characteristics of the system, including the average occupation numbers:

$$\bar{n}_k=\sum_{(n)}n_kW_{(n)}=\partial_{\xi_k}\ln Z. \tag{40}$$

The calculation using Eqs. (38)–(40) gives

$$Z=\prod_k Z(\xi_k,s_k),\quad \bar{n}_k=\bar{n}(\xi_k,s_k), \tag{41}$$

where we have introduced the notation

$$Z(\xi,s)=\frac{\xi^{s+1}-1}{\xi^s(\xi-1)},\quad \bar{n}(\xi,s)=\frac{\xi(\xi^s-1)-s(\xi-1)}{(\xi-1)(\xi^{s+1}-1)}. \tag{42}$$

In particular, from (42) it follows that

$$\bar{n}(\xi,1)=\frac{1}{\xi+1};\ \bar{n}(\xi,2)=\frac{2+\xi}{1+\xi+\xi^2};\ \bar{n}(\xi,\infty)=\frac{1}{\xi-1}, \tag{43}$$

i.e., for $s = 1$ and $s = \infty$ the results (42) become the familiar expressions for the Fermi–Dirac and Bose–Einstein statistics, respectively, as expected.

We note that for $1 < s < \infty$ the results (42) completely coincide with the corresponding expressions obtained for parastatistics (see, for example, Ref. 41). However, this agreement exists only as long as the objects in the system do not interact and the additivity of the energy is preserved, i.e., only nondegenerate eigenvalues $n_k\in\overline{0,s_k}$ of the object-number operators N_k are actually involved.

In a quantum treatment including interactions, the statistical operator Z involves, along with the object-number operators N_k, the creation operators $a^+{}_k$ and annihilation operators a_k operators of the algebra $A(K)$, $k\in K$, corresponding to the given (fixed) quantum statistics ..., s_k, ...$\in s(K)$. Here the results of specific calculations will depend not only on the ranks s_k, as for Eq. (41), but also on the parameters of the algebra $A(K)$ chosen for describing the system [we recall that, according to Corollary 1.2, the quantum statistics $s(K)$ corresponds to the set of algebras $A(K)$].

8. QUANTUM TRANSITIONS IN A THEORY WITH INTERMEDIATE STATISTICS

Let a quantum system consist of identical (one-level) objects obeying statistics of rank s.

According to the fundamental rules of quantun theory, the evolution of a state $|\psi\rangle$ is determined by the relations

$$\begin{aligned} &i\,\partial_t\,|\psi\rangle = H\,|\psi\rangle,\quad H = EN + V,\\ &|\psi\rangle=\sum_{n=0}^{s}\psi_n(t)\,|n\rangle, \end{aligned} \tag{44}$$

where EN is the energy operator of the free system, N is the object-number operator, and V is the interaction energy operator (for self-interactions or interactions with other systems). These belong to the creation–annihilation operator algebra $A(1)$ uniquely determined (see Corollary 2.1) by the rank of the statistics s and the parameters $\lambda(n) > 0$, $n\in\overline{1,s}$.

The operators of the algebra $A(1)$ can always be written in normal-ordered form (see Corollary 3.1). Here the operator N has the form (16), and the most general form of the interaction operator V under the natural physical conditions $H^+ = H$ and $\langle 0|H|0\rangle = 0$ is

$$V=\sum_{k=1}^{s}f_k\overset{+}{a}{}^k a^k+\sum_{k=0}^{s-1}\sum_{n=1}^{s-k}g_{kn}\overset{+}{a}{}^k a^{k+n}+\text{H.c.} \tag{45}$$

Here f_k are real and g_{kn} are complex interaction constants with the dimension of energy.

Let us study the problem of quantum transitions $|n\rangle \to |n'\rangle$.

The probability $W_{nn'}$ of finding the state $|n'\rangle$ at a time $t \geqslant 0$ when at $t = 0$ the state was $|n\rangle$ is, according to (44), defined as

$$W_{nn'}(t) = |\langle n' | \psi(t)\rangle|^2$$
$$= |\psi_{n'}(t)^2| \quad \text{for} \quad |\psi(0)\rangle = |n\rangle. \qquad (46)$$

For $s = 2$ and the auxiliary condition $H|0\rangle = 0$ on the Hamiltonian of the system, the problem (44)–(46) can be solved exactly (the corresponding analytic expressions have been found in Ref. 48).

With the standard assumption that the constants g_{kn} of the interactions not commuting with N from (45) are small, in lowest-order perturbation theory the probability (46) can be rewritten as

$$W_{nn'}(t) = |\delta_{nn'} - i\int_0^t \langle n | v(\tau) | n'\rangle d\tau|^2, \qquad (47a)$$

where $v(t)$ is the perturbation operator in the interaction representation:

$$v(t) = e^{iH_0t}(H - H_0)e^{-iH_0t}, \quad H_0 = EN + 2\sum_{k=1}^{s} f_k \overset{+}{a}{}^k a^k. \qquad (47b)$$

For solving the problem (47) it is convenient to use the matrix representation (18) for the generators of the algebra $A(1)$. After calculating the matrices of the operators (47b) for the probabilities (47a) we finally obtain

$$W_{k,k+l}(t) = 4\frac{\Gamma_{k+l}}{\Gamma_k}\left|\sum_{m=0}^{k} g_{ml}\frac{\Gamma_k}{\Gamma_{k-m}}\right|^2 \frac{\sin^2\left[\frac{1}{2}(\varepsilon_{k+l} - \varepsilon_k)t\right]}{(\varepsilon_{k+l} - \varepsilon_k)^2}, \qquad (48a)$$

where $0 \leqslant k < k + l \leqslant s$ and we have introduced the notation

$$\varepsilon_n = nE + 2\sum_{m=1}^{n} f_m \frac{\Gamma_n}{\Gamma_{n-m}}, \quad \Gamma_0 = 1, \quad \Gamma_k = \prod_{n=1}^{k}\lambda(n) > 0. \qquad (48b)$$

Equations (48) show that in a system of identical (one-level) objects obeying quantum statistics of any rank a perturbation (self-interaction or interaction) can give rise to oscillations between states with given numbers of objects.

The amplitude and period of these oscillations are determined not only by the rank of the statistics and the interaction parameters, but also, according to the conclusion of Sec. 7, they have an essential dependence on the parameters of the creation–annihilation operator algebra.

9. INTERMEDIATE STATISTICS AND THE PAULI PRINCIPLE

The problem of the theoretical description of a possible (weak) violation of the Pauli principle has been discussed in Refs. 43–48. Such studies are based on the assumption that a system of identical (one-level) objects, which at the present stage of development of physics are assumed to be fermions, can in fact be found not only in the vacuum state $|0\rangle$ and the one-particle state $|1\rangle$, but also (although with low probability) in the two-particle state $|2\rangle$.

It has been proposed[44–48] that this idea be realized mathematically by assuming that the operators a and $\overset{+}{a}$ related to statistics of rank 2 act in the state space of this system spanned by the orthonormal basis $|0\rangle$, $|1\rangle$, and $|2\rangle$.

The authors of Refs. 45 and 47 have studied the algebra of paraoperators with a small parameter, which is a special case of the "para-ε-operator" algebras introduced and studied thoroughly in Refs. 17 and 18. In the opinion of the authors of Ref. 45, this algebra ensures a consistent mathematical description of weak violation of the Pauli principle in a local quantum field theory. However, in Ref. 47 it was shown that the small parameter in the paraoperator algebra leads to violation of the requirement that the metric of the state space be positive.

In Refs. 44, 46, and 48 special cases of $A(1)$ algebras or products of them were actually used to theoretically describe violation of the Pauli principle, but only certain special interactions were considered. We shall give a more general treatment of this problem.

As was shown in the preceding sections, any $A(1)$ algebra with $s = 2$ is specified by two parameters, $\lambda(1) > 0$ and $\lambda(2) > 0$, and contains the system of identities

$$\overset{+}{a}a^2 = \lambda(2)\overset{+}{a} - \frac{\lambda(2)}{\lambda(1)}\overset{+}{a}{}^2 a, \quad a^3 = 0; \qquad (49a)$$

$$a\overset{+}{a} = \lambda(1) + \frac{\lambda(2) - \lambda(1)}{\lambda(1)}\overset{+}{a}a - \frac{\lambda^2(1) + \lambda^2(2) - \lambda(1)\lambda(2)}{\lambda^2(1)\lambda(2)}\overset{+}{a}{}^2 a^2; \qquad (49b)$$

$$a^2\overset{+}{a}{}^2 = \lambda(1)\lambda(2) - \lambda(2)\overset{+}{a}a + \frac{\lambda(2) - \lambda(1)}{\lambda(1)}\overset{+}{a}{}^2 a^2. \qquad (49c)$$

The object-number operator and the most general form of the interaction energy operator can be written in normal-ordered form using the parameters of the algebra and the "effective" constants g_{01} for the linear, g_{02} and $f_1 = f_1^*$ for the bilinear, g_{11} for the trilinear, and $f_2 = f_2^*$ for the quadrilinear interaction:

$$N = \frac{1}{\lambda(1)}\overset{+}{a}a + \frac{2\lambda(1) - \lambda(2)}{\lambda^2(1)\lambda(2)}\overset{+}{a}{}^2 a^2; \qquad (50)$$

$$V = g_{01}a + g_{02}a^2 + f_1\overset{+}{a}a + g_{11}\overset{+}{a}a^2 + f_2\overset{+}{a}{}^2a^2 + \text{H.c.} \qquad (51)$$

According to the results (48) for the probabilities of oscillation between states with given numbers of objects, for the interaction (51) we have

$$W_{01} = \frac{4\lambda(1)|g_{01}|^2}{E_1^2}\sin^2\frac{E_1 t}{2}, \quad E_1 = E + 2\lambda(1)f_1; \qquad (52a)$$

$$W_{02} = \frac{4\lambda(1)\lambda(2)|g_{02}|^2}{E_2^2}\sin^2\frac{E_2 t}{2};$$

$$E_2 = 2E + 2\lambda(2)f_1 + 2\lambda(1)\lambda(2)f_2; \qquad (52b)$$

$$W_{12} = \frac{4\lambda(2)|g_{01} + \lambda(1)g_{11}|^2}{E_3^2}\sin^2\frac{E_3 t}{2}; \quad E_3 = E_2 - E_1. \qquad (52c)$$

We note that the oscillation probabilities (52b) and (52c) involving the state $|2\rangle$ will be small, as required for weak violation of the Pauli principle, either if the parameter $\lambda(2)$ is small or if the quantities g_{02} and $g_{01} + \lambda(1)g_{11}$ characterizing the interaction and the algebra are small.

The first case was actually realized in Refs. 44 and 46, where algebras (49) with $\lambda(1) = 1$ and $\lambda(2) \approx 0$ were chosen. However, this choice of parameters leads to serious difficulties when generalizing to relativistic quantum field theory, as was somehow or other noticed by those authors themselves (see also Ref. 47).

The second case is realized for finite parameters of the algebra $\lambda(1) = \lambda(2) = 2$ and certain constraints on the interaction constants (51), namely, $g_{02} \approx 0$ and $g_{01} \approx -2g_{11}$. Here the relations (49) define the algebra (35), and no difficulties arise in the generalization to field theory. However, here the weak violation of the Pauli principle is dynamical, rather than algebraic in nature.

For a later comparison of the results, let us also consider the problem of oscillations in the case in which a and a^+ belong to the Fermi algebra,

$$N = \overset{+}{a}a = \frac{1}{2}\left[\overset{+}{a}, a\right]_- + \frac{1}{2}, \quad a\overset{+}{a} = 1 - \overset{+}{a}a, \quad a^2 = 0, \qquad (53)$$

and the state $|2\rangle$ does not exist. Here $s = 1$ and $\lambda(1) = 1$, so from the result (48) for the interaction (51) we have

$$W_{01}^{(F)} = \frac{4|g_{01}|^2}{E'^2} \sin^2\frac{E't}{2}, \quad E' = E + 2f_1. \qquad (54)$$

It is important to note that the results (52a) and (54) coincide for $\lambda(1) = 1$ and differ (in amplitude and period) for $\lambda(1) = 2$.

Since for dynamical weak violation of the pauli principle $\lambda(1) = 2$, Eqs. (52a) and (54) for W_{01} do not coincide even when the probabilities W_{02} and W_{12} vanish (now only on account of the specific interaction constants g_{01} and g_{02}).

Therefore, the transition process $|0\rangle \rightleftarrows |1\rangle$ "feels" the existence of the state $|2\rangle$ even when it is not involved in the transitions. This effect apparently can be used for the experimental search for objects obeying intermediate statistics (in addition to the phenomena suggested in Refs. 43 and 46).

[a] *Translation Editor's Note.* The Russian use of a bar over a pair of integers to denote an inclusive range is retained here and throughout the article.

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