Systems for forming proton beams of micron diameters

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The production and application of a nuclear microprobe using a beam of protons at MeV energies are discussed. Considerable attention is paid to matrix analytical and numerical methods for designing quadrupole beam-forming systems. Quadrupole and axisymmetric focusing systems are compared. The possibility of designing a high-frequency microprobe is considered. The effect of the space-charge distribution of the beam on its evolution in the microprobe optics is studied.

INTRODUCTION

Recently there has been intensive development of instantaneous methods for nuclear microanalysis—methods of determining the elementary composition of materials based on the detection of the secondary radiation produced when the samples are bombarded by beams of electrons, protons, and also light and heavy nuclei. This radiation might be typical x-ray radiation, elastic and inelastic scattering, radiation associated with Auger transitions, products of nuclear reactions induced by primary particles, and so on. The most well-developed of these methods is that of the detection of the characteristic x-ray radiation produced by proton (or heavy-ion) bombardment, since measurements based on excitation of samples by an electron beam obtained using an electron microprobe have a large background (about 10^3–10^4 times larger than for a proton beam), which greatly decreases the sensitivity of the method. For element analysis based on the use of the characteristic radiation emitted in a proton or nucleus beam there is increased interest in beams whose cross section at the target is several microns, since such beams can be used to study the macroscopic structure of solids and biological samples both on the surface and in the interior. This interest has been stimulated by several experimental and theoretical studies,[1–6] which demonstrate the possibility of producing microbeams of these diameters with a current of 0.1 nA. A system producing a beam with a cross section of several microns is referred to as a proton or nuclear (ionic) microprobe. The charged particles in such a microprobe are usually provided by a Van de Graaff electrostatic accelerator and, in recent years, also by a cyclotron.[7–11]

The nuclear microprobe is an auxiliary device for the microscopic analysis of surfaces, supplementing devices like the secondary-ion analyzer, the electron scanning microscope, the electron microprobe, the Auger spectrometer, and so on. Its use opens up a number of new, important possibilities for analysis, which cannot be achieved by other methods. This is due to the following features of the nuclear microprobe[12]: all the elements, including hydrogen, theoretically can be determined; the method can be "focused," especially for light elements, by selecting the nuclear reaction and the energy of the accelerated particles; the chemical content of the sample is not affected, which simplifies the calibration and renders standardized samples, with chemical content identical to that of the material being analyzed, unnecessary; it is possible to study microscopic samples (for example, biopsies) in a helium or air medium; light elements in radioactive materials can be determined (by obtaining an image of the surface distribution or the diffusion profile).

Owing to their high magnetic ridigity (three orders of magnitude larger than that of electrons in electron microscopes), ions with energies of several MeV per nucleon require strong lenses for their focusing, otherwise the devices based on them would be too large. Such beams are usually focused by systems of quadrupole lenses, although there are examples of the use of an axisymmetric superconducting magnetic lens[13] and a plasma lens[14] for these purposes.

The first theoretical studies on the determination of an entire class of systems of quadrupole lenses suitable for use in a microprobe as the equivalent of an axisymmetric lens, with detailed analysis of the dependence of their focusing properties on the field gradients in the lenses, were carried out in the USSR.[15,16] In these studies it was shown that a system of four quadrupole lenses with a particular symmetry in the positioning of the lenses and their power sources, referred to as a quadruplet of rotation, can serve as the analog of a collecting axisymmetric lens, owing to its paraxial (Gaussian) characteristics. In the foreign literature this system is referred to as the "Russian" quadruplet. Using this quadruplet, Cookson and coworkers at the Harwell laboratory in Great Britain designed the first proton microprobe setup,[1,2] which initiated the development of the new technique of quantitative microanalysis.

By now there are already several score of accelerator laboratories abroad which operate ion microprobes; their design and use are described in detail in the reviews of Refs. 17–23. In the USSR the first, and so far the only, proton microprobe was constructed by A. G. Puzyrevich at the Institute of Nuclear Physics of the Tomsk Polytechnic Institute.[24] In the existing proton microprobes the beam diameters at the target range from 0.5 μm to several tens of microns.

The system forming the microbeam is a complicated precision setup, whose optimal parameters must be found by optimization calculations. The first work in this area was carried out at the Joint Institute for Nuclear Research, and was then followed by a number of other studies,[25–31] devoted to the design of the optimal microprobe and investigation of its characteristics. These studies have been carried out at the JINR jointly with the Leningrad State University.

A small beam cross section at the target is the most important factor of the many conflicting requirements imposed on the beam. The second important factor is the current (or emittance) of the beam. The values of these parameters depend on how the ion–optical scheme of the microprobe is chosen.

The calculation of the optimal ion–optical system for

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obtaining micron and, especially, submicron beams involves the solution of a nonlinear inverse multiparameter problem, for which all types of aberration must be taken into account through fifth order. It is therefore important to trace through the entire solution of the problem, including: the selection of the coordinate system, generally a curvilinear system attached to a particle moving along the axis; writing out the equations of motion and the electromagnetic field equations in the selected coordinate system taking into account nonelectromagnetic forces, for example, gravity; the expansion of the equations of motion and field equations in Taylor series in powers of the deviation from the axial particle; the technique of solving the nonlinear problem in configuration space by reformulating it as a linear problem in phase-moment space; analysis of the structure of the coefficient matrix arising in the linear problem and choosing a definite symmetry in order to optimize the system; the compact conservative method of integrating the equations of motion, where in each step of the numerical integration the phase space of the beam is strictly conserved; theoretical analysis of the problem and determination of approximation formulas for a preliminary calculation; use of the nonderivative sliding-tolerance technique for numerical optimization of the system; comparison of different ion–optical microprobe schemes.

All of these points are discussed in this review, which is largely based on the work carried out by the authors at the JINR in 1977–1985.

In this review we use the coordinates of four-dimensional space-time. All quantities in the equations of motion and in the field equations either are dimensionless or are expressed in terms of units of length or inverse length.

1. THE EQUATIONS OF MOTION AND THE ELECTROMAGNETIC FIELD EQUATIONS

In calculations of precision ion–optical setups it is convenient to view the motion of all the particles of the beam relative to a single particle—the axial (or reference) particle. In general, the trajectory of this particle is curvilinear, and it can also be affected by nonelectromagnetic forces, such as gravity. Using Cartan's method of the moving reference vector,[32] we write the equations of motion and Maxwell's equations in a curvilinear coordinate frame attached to the axial particle. Here we write a vector either as a column matrix $\mathbf{Q}, \mathbf{M}, x, y$, or z, or as a row matrix $\tilde{\mathbf{Q}}, \tilde{\mathbf{M}}, \tilde{x}, \tilde{y}$, or $\tilde{z}$. The tilde denotes the transpose. The vector differential operator is also written as a column matrix $\nabla(\mathbf{Q})$, $\nabla(\mathbf{M})$, $\nabla(x)$, and so on, or as a row matrix $\tilde{\nabla}(\mathbf{Q})$, $\tilde{\nabla}(\mathbf{M})$, $\nabla(x)$, where $\nabla_i(\mathbf{Q}) = \partial/\partial \mathbf{Q}_i$. The expression $\mathbf{Q}\tilde{\nabla}(z)$ stands for the matrix whose ik-th element is equal to $\partial \mathbf{Q}_i/\partial z_k$. The scalar differential operator d is written as the scalar product $d = \tilde{\nabla}(z)dz = \tilde{\nabla}(\mathbf{Q})d\,\mathbf{Q}$ and can stand either in front of or behind the function on which it acts. If required for clarity, the function on which the operators d, $\nabla(x)$, or $\tilde{\nabla}(x)$ act is indicated by an arrow underneath pointing from the operator to the function. We shall use the absolute metric spaces of the vectors $\mathbf{Q}$ and $\mathbf{M}$ and the metric spaces of the vectors x, y, and z associated with the specific metric chosen. The axial particle will be characterized by the 4-vector $\mathbf{M}$, and an arbitrary particle by the 4-vector $\mathbf{Q}$. The fourth components of these vectors are proportional to the time.

The axial particle

We take the point $\mathbf{M}$ for the origin of the reference 4-vector e, where the matrix e has the form

$$e = \nabla(z_m)\,\tilde{\mathbf{M}},$$

in which (z_m) is the local Galilean coordinate system. We shall study the evolution of the vector $\mathbf{M}$ in pseudo-Euclidean 4-space (y) with orthonormal stationary reference vector given by the matrix e_0, where

$$e_0 = \nabla(y)\,\tilde{\mathbf{M}}, \quad e\tilde{e} = e_0\tilde{e}_0 = G = \begin{Vmatrix} -I_3 & 0 \\ 0 & 1 \end{Vmatrix}.$$

Here I_3 is the 3×3 unit matrix.

To the moving reference vector e we attach a local coordinate system (x), in which the motion of an arbitrary particle (point) $\mathbf{Q}$ is determined by the vector $L = \tilde{e}x$, so that $d\,\mathbf{Q} = d\,\mathbf{M} + d\,\mathbf{L}$. We use the notation of arc-length differentials in 4-dimensional space-time, $d\tau$ and $d\tau_m$, where

$$d\tau = \sqrt{d\tilde{\mathbf{Q}}\,d\mathbf{Q}}, \quad d\tau_m = \sqrt{d\tilde{\mathbf{M}}\,d\mathbf{M}},$$

the arc-length differential ds in 3-space,

$$ds = \sqrt{dz_{m1}^2 + dz_{m2}^2 + dz_{m3}^2}\,,$$

and the 4-velocity notation

$$u = d\mathbf{Q}/d\tau, \quad u_m = d\mathbf{M}/d\tau_m,$$

$$u(z) = dz/d\tau, \quad u_m(z_m) = dz_m/d\tau_m.$$

The coordinates z_m are chosen such that

$$z_{m1} = z_{m2} = 0, \; z_{m3} = s, \; z_{m4} = ct_m,$$

$$u_{m1}(z_m) = u_{m2}(z_m) = 0, \; u_{m3}(z_m) = p, \; u_{m4}(z_m) = \gamma,$$

$$e\,|_{t_m=0} = e_0, \; \gamma^2 - p^2 = 1.$$

Here p is the dimensionless momentum of the axial particle and γ is its dimensional relative total energy.

The Darboux and Frenet reference vectors

The motion of the reference vector is determined by the equation

$$\frac{de}{ds} = e' = P(k,\,l)\,e,$$

$$P(k,\,l) = \begin{Vmatrix} 0 & k_3 & -k_2 & l_1 \\ -k_3 & 0 & k_1 & l_2 \\ k_2 & -k_1 & 0 & l_3 \\ l_1 & l_2 & l_3 & 0 \end{Vmatrix},$$

where the 3-vectors k and $l = -F/p$ are functions of s, and F is the 3-vector corresponding to the gravitational force.

For the Darboux reference 4-vector, k_1 is the normal curvature, k_2 is the geodesic curvature, k_3 is the geodesic torsion, and $l = 0$.

For the Frenet reference 4-vector, $k_1 = 0$, $l = 0$, and k_2 and k_3 are, respectively, the curvature and torsion of the curve along which the particle $\mathbf{M}$ moves.

The reference vector e can be expressed in terms of e_0 using the matrizant R:

$$e = R\,(P\,(k,\,l),\,s/0)\,e_0 = Re_0,$$

satisfying the equations

$$\frac{dR}{ds}=R'=P(k,\ l)\,R,\quad R(0/0)=I_4,$$
$$RG\tilde{R}=G.$$

The equations of motion

The equations of motion in a vacuum for the axial particle **M** and an arbitrary particle **Q** are written as follows:

$$\frac{d\mathbf{M}}{d\tau_m}=\tilde{e}u_m(z_m)=\tilde{e}_0u_m(y_m);$$
$$\frac{du_m}{d\tau_m}=\tilde{e}P(B_m,\ E_m)\,u_m(z_m)=\tilde{e}P(B_m^0,\ E_m^0)\,u_m(y_m);$$
$$\frac{d\mathbf{Q}}{d\tau}=\frac{d\mathbf{M}+d\mathbf{L}}{d\tau}=\tilde{e}u(z)=u;$$
$$\frac{du}{d\tau}=\tilde{e}P(B,\ E)\,u(z).$$

Here **B** is the magnetic field induction and E is the electric field strength. The 3-vectors B, E, and F, expressed in units of inverse length in terms of dimensional quantities in SI units, denoted by a star, are defined as

$$B=\frac{q^*}{p_0^*}B^*,\quad E=\frac{q^*}{W_0^*}E^*,\quad F=\frac{F^*}{W_0^*},$$
$$W_0^*=m_0^*c^{*2},\quad p_0^*=m_0^*c^*,$$
$$\gamma=\frac{W^*}{W_0^*},\quad p=\frac{p^*}{p_0^*},\quad W^*=m^*c^{*2}.$$

Using, as above, a prime to denote the derivative with respect to s, the equations of motion can be rewritten as

$$\left(\frac{z'}{\tau'}\right)'=\check{P}z',$$
$$\check{P}=P(\check{B},\ \check{E}),\quad \tau'=\sqrt{\tilde{z}'Gz'},$$
$$\check{B}=B+k/\tau',\quad \check{E}=E-l/\tau',$$
$$z'=z_m'+x'+\tilde{P}(k,\ l)\,x,$$
$$z_m'=i(3)+\frac{\gamma}{p}i(4),\quad i(3)=\begin{Vmatrix}0\\0\\1\\0\end{Vmatrix},\quad i(4)=\begin{Vmatrix}0\\0\\0\\1\end{Vmatrix},$$
$$z_{m4}=\int_0^s\frac{\gamma}{p}ds+z_{m40}. \qquad (1)$$

The matrices $P(B,E)$ for the reference vector e and $P(B_0,E_0)$ for the reference vector e_0 are related to each other via the matrizant:

$$P(B,\ E)=\tilde{R}^{-1}P(B_0,\ E_0)\,\tilde{R}=GRGP(B_0,\ E_0)\,\tilde{R},$$
$$\nabla(z)=R\nabla(y).$$

We note that for the special case $l=0$ we have

$$P(k,\ 0)=\begin{Vmatrix}P(k)&0\\0&0\end{Vmatrix},\quad P(k)=\begin{Vmatrix}0&k_3&-k_2\\-k_3&0&k_1\\k_2&-k_1&0\end{Vmatrix},$$

$$R=\begin{Vmatrix}\bar{R}&0\\0&1\end{Vmatrix},\quad \bar{R}\tilde{\bar{R}}=\tilde{\bar{R}}\bar{R}=I_3,$$
$$B=\bar{R}B_0,\quad E=\bar{R}E_0.$$

We shall assume that the observer is located in the plane $x_3=0$, i.e., all particles Q reaching this plane at different times are detected. We then obtain the following operator equations:

$$\nabla(z)=R\nabla(y)=\frac{1}{\alpha}Z\nabla(x),$$

where

$$\nabla_3(x)=\nabla(s)=\frac{\partial}{\partial s}=\nabla_3(z);$$

$$Z=\begin{Vmatrix}1&0&0&0\\0&1&0&0\\Z_{31}&Z_{32}&1&Z_{34}\\0&0&0&1\end{Vmatrix};$$

$$Z^{-1}=\begin{Vmatrix}1&0&0&0\\0&1&0&0\\-Z_{31}&-Z_{32}&1&-Z_{34}\\0&0&0&1\end{Vmatrix};$$

$$Z_{31}=k_3x_2-l_1x_4;\ Z_{32}=-k_3x_2-l_2x_4;$$
$$Z_{34}=-\gamma/p-x_1l_1-x_2l_2;\ \alpha=1-k_2x_1+k_1x_2+l_3x_4.$$

Maxwell's equations

In matrix form these equations are written as

$$\left.\begin{aligned}&-\Delta(\mathrm{Q})\,A=\rho u;\quad \tilde{\nabla}(\mathrm{Q})\,A=0;\\&\Delta(\mathrm{Q})=\tilde{\nabla}(\mathrm{Q})\,\nabla(\mathrm{Q});\quad \nabla(\mathrm{Q})=e^{-1}\nabla(z);\quad A=\tilde{e}A(z);\\&\rho=\tilde{u}\underset{\frown}{\Delta(\mathrm{Q})}u;\\&GP(B,\ E)=\underset{\frown}{\nabla(z)\tilde{A}}\tilde{e}-eA\underset{\frown}{\tilde{\nabla}}(z).\end{aligned}\right\}\qquad(2)$$

In the coordinate system (y) with stationary reference vector e_0 the electromagnetic field equations have the form

$$P(B_0,\ E_0)\,G\nabla(y)=\rho^0\frac{dy}{dy_4},\quad \rho^0=\rho\frac{dy_4}{d\tau};$$
$$P(-E_0,\ B_0)\,G\nabla(y)=0.$$

From (2) as a special case of the Frenet reference vector ($k_1=0$, $l=0$) we obtain the Maxwell equations given in Ref. 33.

2. THE METHOD OF EMBEDDING IN PHASE-MOMENT SPACE FOR SOLVING THE NONLINEAR EQUATIONS OF MOTION

The analysis and calculation of the nonlinear systems of equations for beam formation are considerably simplified by transforming from the nonlinear differential equations of motion in the phase space (x,x') to the system of linear equations in extended phase space—the phase-moment space. This is the essence of the method of embedding in phase-moment space.

The vector moments

We define recursively the r-th power of the vector x,

$$x^r = x^{(r)}(1) = \left\| \begin{matrix} x_1 & x^{r-1} & (1) \\ \cdot & \cdot\cdot\cdot & \cdot \\ x_n & x^{r-1} & (n) \end{matrix} \right\|,$$

and call it the r-moment of the vector x or the r-th moment of the vector x, where the auxiliary vector $x^l(j)$ is defined by the recursion relation

$$x^l(j) = \left\| \begin{matrix} x_j & x^{l-1} & (j) \\ \cdot & \cdot\cdot\cdot & \cdot \\ x_n & x^{l-1} & (n) \end{matrix} \right\|, \quad j = 1, \ldots, n.$$

The vector x^r has C_{n-1+r}^{n-1} scalar elements, where

$$C_{n-1+r}^{n-1} = \frac{(n-1+r)!}{(n-1)!\,r!}$$

is the number of combinations with repetitions of n elements r at a time. For example, for $n = 2$ we have

$$x = \left\| \begin{matrix} x_1 \\ x_2 \end{matrix} \right\|, \quad x^1(1) = x, \quad x^1(2) = x_2,$$

$$x^2 = \left\| \begin{matrix} x_1 & x \\ x_2 & x_2 \end{matrix} \right\| = \left\| \begin{matrix} x_1^2 \\ x_1 \ x_2 \\ x_2^2 \end{matrix} \right\|,$$

$$x^2(1) = x^2, \quad x^2(2) = x_2 x^1(2) = x_2^2,$$

$$x^3 = \left\| \begin{matrix} x_1 & x^2 & (1) \\ x_2 & x^2 & (2) \end{matrix} \right\| = \left\| \begin{matrix} x_1^3 \\ x_1^2 \ x_2 \\ x_1 \ x_2^2 \\ x_2^3 \end{matrix} \right\|.$$

Similarly, the power of the operator $\nabla^r(x)$ is a differential r-moment operator.

Expansion of the equations of motion

We write the matrix function $F(\hat{x})$ as an m-th order polynomial:

$$F(\hat{x}) = \sum_{j=0}^{m} F^{(j)},$$

where $F^{(j)}$ is a homogeneous polynomial of degree j in $\hat{x}$,

$$F^{(j)} = \frac{1}{j!}[\tilde{\hat{x}}\nabla(\hat{x})]^j F(\hat{x}), \quad \hat{x} = \left\| \begin{matrix} x \\ x' \end{matrix} \right\|.$$

The partial derivatives of $F(\hat{x})$ with respect to $\hat{x}$ are taken at $x = 0$.

Using this notation, the equations of motion can be written as

$$(\Phi^{(0)} + \Phi^{(1)} + \Phi^{(2)}) \sum_0^m z^{(j)''}$$
$$= (\varphi^{(0)} + \varphi^{(1)} + \varphi^{(2)}) \Big[\sum_0^m \hat{\mathbf{P}}^{(j)} z'_m + \sum_0^{m-1} \hat{\mathbf{P}}^{(j)} z'^{(1)} \Big]. \qquad (3)$$

Here

$$\varphi = \tilde{z}'\mathbf{G}z', \quad \Phi = \varphi I_4 - z'\tilde{z}'\mathbf{G},$$

$$\hat{\mathbf{P}} = \mathbf{P}(\hat{\mathbf{B}}, \hat{\mathbf{E}}), \quad \hat{\mathbf{B}} = \mathbf{B}\sqrt{\varphi} + k, \quad \hat{\mathbf{E}} = \mathbf{E}\sqrt{\varphi} - l,$$

$$z'^{(0)} = z'_m, \quad z'^{(j)} = 0, \; j = 2, \ldots, m.$$

For $m = 0$ and 1 the equations of motion are written as follows:

1. $\Phi^{(0)} z''_m = \varphi^{(0)} \hat{\mathbf{P}}^{(0)} z'_m,$

$$\varphi^{(0)} = \frac{1}{p^2}, \quad \Phi = \frac{1}{p^2} I_4 - z'_m \tilde{z}'_m \mathbf{G},$$

$$\hat{\mathbf{P}}^{(0)} = \mathbf{P}(\hat{\mathbf{B}}^{(0)}, \hat{\mathbf{E}}^{(0)}) = \hat{\mathbf{P}}_m,$$

$$\hat{\mathbf{B}}^{(0)} = \frac{\mathbf{B}_m}{p} + k, \quad \hat{\mathbf{E}} = \frac{\mathbf{E}_m}{p} - l.$$

2. $\Phi^{(0)} \cdot \mathbf{z}''^{(1)} = \frac{1}{p^2} \hat{\mathbf{P}}_m \mathbf{z}'^{(1)}$

$$+ \frac{1}{p^2} \hat{\mathbf{P}}^{(1)} \cdot \mathbf{z}'_m + \varphi^{(1)} \hat{\mathbf{P}}_m \cdot \mathbf{z}'_m - \Phi^{(1)} \cdot \mathbf{z}''_m.$$

The latter equation is linear and is usually referred to as the paraxial or Gaussian equation.

The particles of the beam are detected either at the same instant of time or in a particular plane, for example, the plane $x_3 = 0$. Therefore, to the equations (3) we must add the equation for the observer, after which these equations form a system of three second-order equations:

$$x''_i = \tilde{f}^{(ir)} \hat{x}^r, \quad r = 1, \ldots, m, \; i = 1, 2, 4, \qquad (4)$$

where $f'^{(ir)}$ is a row vector of dimension equal to the number of combinations with repetitions of 6 elements r at a time and

$$\tilde{\hat{x}} = \| x_1 x_2 x_4 x'_1 x'_2 x'_4 \|.$$

The equation for the phase moments

To the nonlinear equation (4) we can associate a linear equation for the phase moments:

$$\frac{d\langle\hat{x}\rangle^{(m)}}{ds} = \mathbf{P}^{(m)} \langle\hat{x}\rangle^{(m)}, \quad \langle\hat{x}\rangle^{(m)} = \left\| \begin{matrix} \hat{x} \\ \hat{x}^2 \\ \vdots \\ \hat{x}^m \end{matrix} \right\|. \qquad (5)$$

Here the matrix function $p^{(m)}$ has the form of an upper triangular block:

$$\mathbf{P}^{(m)} = \left\| \begin{matrix} P^{11} & P^{12} & \ldots & P^{1m} \\ 0 & P^{22} & \ldots & P^{2m} \\ 0 & 0 & \ldots & P^{mm} \end{matrix} \right\|.$$

The solution of the linear equation (5) for $\hat{x}$ coincides with the solution of Eq. (4) obtained by the successive-approximation method. The method of solving Eq. (4) by reducing it to the form (5) is referred to as the method of embedding in phase-moment space.

The writing of the nonlinear equation in a linearized form makes it possible to construct its solution using the matrizant, which is independent of the initial vector $\hat{x}_0$, whereas the solution of the nonlinear equation is sought for each value of $\hat{x}_0$. We note that the use of matrices for solving nonlinear problems was first proposed in the studies by Brown.[34,35]

The solution of Eq. (5) is written in terms of the matrizant in the form

$$\hat{x} = \mathbf{X}(\mathbf{P}^{(m)}, s/s_0)\,\hat{x}_0, \quad \mathbf{X}(s_0/s_0) = \mathbf{I},$$

where, like the coefficient matrix $\mathbf{P}^{(m)}$, the matrizant has the form of an upper triangular block

$$\mathbf{X}(\mathbf{P}^{(m)}, s/s_0) = \left\| \begin{matrix} X^{11} & X^{12} & \ldots & X^{1m} \\ 0 & X^{22} & \ldots & X^{2m} \\ \cdot & \cdot & \cdot\cdot\cdot & \cdot \\ 0 & 0 & \ldots & X^{nm} \end{matrix} \right\|$$

and satisfies the differential equation

$$\mathbf{X}'(\mathbf{P}^{(m)}, s/s_0) = \mathbf{P}^{(m)}\cdot\mathbf{X}(\mathbf{P}^{(m)}, s/s_0).$$

The block matrix X^{11} is the matrizant of the linear equation in the phase space of $\hat{x}$:

$$\hat{x}' = P^{11}\hat{x}, \quad \hat{x} = X^{11}\hat{x}_0.$$

The matrix function X^{ik}, like P^{ik} $(k \geqslant i)$, is a rectangular matrix with C^{n-1}_{n-1+i} rows and C^{n-1}_{n-1+k} columns.

The matrix function X^{rr} is determined by the equation

$$(X^{11}\hat{x}_0)^r = X^{rr}\hat{x}_0^r, \quad r = 1, \ldots, m.$$

The off-diagonal block matrices X^{ij} $(j > i)$ have the form

$$X^{ij}(s/0) = \sum_{l=i+1}^{j}\int_0^s X^{ii}(s/\tau)\,P^{il}(\tau)\,X^{lj}(\tau/0)\,d\tau.$$

The matrices $X^{ij}(s/0)$ can be calculated as follows: first we calculate $X^{i,i+1}$, then $X^{i,i+2}$, and so on, up to X^{ij} $(i = 1, \ldots, m; j = i+1, \ldots, m)$. It is also possible to simultaneously calculate the entire matrizant $\mathbf{X}(\mathbf{P}^{(m)}, s/s_0)$, as will be discussed in Sec. 5.

3. THE PARAXIAL EQUATIONS OF MOTION FOR IONS IN VARIOUS TYPES OF ELECTROMAGNETIC LENSES

The paraxial equations of motion for $l = 0$ and $x_3 = 0$

For this case we obtain

$$\begin{aligned} x_1'' = &-\frac{\gamma}{p^2}\mathbf{E}_{m3}x_1' \\ &+\left(2k_3+\frac{1}{p}B_{m3}\right)x_2' + p\left(\gamma k_2+\frac{1}{p^2}E_{m1}\right)x_4' \\ &+\Big[-\frac{1}{p}\nabla_1 B_2+\frac{\gamma}{p^2}\nabla_1 E_1+p^2k_2^2 \\ &\quad+\left(\frac{1}{p}B_{m3}+k_3\right)k_3+\frac{\gamma k_2}{p^2}E_{m1}\Big]x_1 \\ &+\Big[k_3'+\frac{\gamma}{p^2}E_{m3}k_3-\frac{1}{p}\nabla_2 B_2 \\ &\quad+\frac{\gamma}{p^2}\nabla_2 E_1-p^2k_1k_2-\frac{\gamma k_1}{p^2}E_{m1}\Big]x_2 \\ &+\Big[-\frac{1}{p}\nabla_4 B_2+\frac{\gamma}{p^2}\nabla_4 E_1\Big]x_4; \end{aligned}$$

$$\begin{aligned} x_2'' = &-\left(2k_3+\frac{B_{m3}}{p}\right)x_1' \\ &-\frac{\gamma}{p^2}E_{m3}x_2'+p\left(-\gamma k_1+\frac{1}{p^2}E_{m2}\right)x_4' \\ &+\Big[-k_3'-\frac{\gamma}{p^2}E_{m3}k_3+\frac{1}{p}\nabla_1 B_1 \\ &\quad+\frac{\gamma}{p^2}\nabla_1 E_2-p^2k_1k_2+\frac{\gamma k_2}{p^2}E_{m2}\Big]x_1 \\ &+\Big[\frac{1}{p}\nabla_2 B_1+\frac{\gamma}{p^2}\nabla_2 E_2 \\ &\quad+p^2k_1^2+\left(\frac{B_{m3}}{p}+k_3\right)k_3-\frac{\gamma k_1}{p^2}E_{m2}\Big]x_2 \\ &+\Big[\frac{1}{p}\nabla_4 B_1+\frac{\gamma}{p^2}\nabla_4 E_2\Big]x_4; \end{aligned}$$

$$\begin{aligned} x_4'' = &-\frac{1}{p}\left(\gamma k_2+\frac{1}{p^2}E_{m1}\right)x_1' \\ &-\frac{1}{p}\left(-\gamma k_1+\frac{E_{m2}}{p^2}\right)x_2'-\frac{3\gamma}{p^2}E_{m3}x_4' \\ &-\frac{1}{p}\Big[\gamma k_2'+k_2\left(3+\frac{1}{p^2}\right)E_{m3} \\ &\quad+\frac{1}{p^2}\nabla_1 E_3+\frac{1}{p^2}k_3E_{m2}\Big]x_1 \\ &-\frac{1}{p}\Big[-\gamma k_1'-k_1\left(3+\frac{1}{p^2}\right)E_{m3} \\ &\quad+\frac{1}{p^2}\nabla_2 E_3-\frac{k_3}{p^2}E_{m1}\Big]x_2-\frac{1}{p^3}\nabla_4 E_3 x_4. \end{aligned}$$

Here $\nabla = \nabla(x)$; $\nabla_i B_k$ and $\nabla_i E_k$ are understood to be the derivatives along the trajectory of the axial particle and are functions of s, and the curvatures k_1 and k_2 are determined from the equations of motion of the axial particle:

$$k_1 = -\frac{B_{m1}}{p}-\frac{\gamma}{p^2}E_{m2}, \quad k_2 = -\frac{B_{m2}}{p}+\frac{\gamma}{p^2}E_{m1},$$

where

$$\gamma = \int_0^s E_{m3}\,ds+\gamma_0.$$

The phase-space volume of the beam

The phase-space volume of the beam V is determined by

$$V = V_0\exp-\int_0^s 5\frac{\gamma}{p^2}E_{m3}\,ds.$$

If we take the phase-space variables to be the elements of a vector h, where

$$h_1 = x_1, \quad h_2 = x_2, \quad h_3 = x_4,$$
$$h_4 = \frac{p}{p_0}x_1', \quad h_5 = \frac{p}{p_0}x_2', \quad h_6 = \frac{p^3}{p_0^3}x_4',$$

then in this phase space the volume remains constant during the motion.

A magnetostatic lens with rectilinear axis

For the case $k_1 = 0$, $k_2 = 0$, and $l = 0$ we obtain

$$\begin{aligned} x_1'' = &\left(\frac{k_3}{p}B_{m3}+k_3^2-\frac{1}{p}\nabla_1 B_2\right)x_1 \\ &+\left(k_3'-\frac{1}{p}\nabla_2 B_2\right)x_2+\left(2k_3+\frac{B_{m3}}{p}\right)x_2'; \\ x_2'' = &\left(-k_3'+\frac{1}{p}\nabla_1 B_1\right)x_1 \\ &+\left(\frac{k_3}{p}B_{m3}+k_3^2+\frac{1}{p}\nabla_2 B_1\right)x_2 \\ &-\left(2k_3+\frac{B_{m3}}{p}\right)x_1'. \end{aligned}$$

Choosing k_3 in the form

$$k_3 = -\frac{1}{2p}B_{m3}$$

and using the paraxial approximation of Maxwell's equations neglecting the proper space charge

$$\nabla_2 B_1 = \nabla_1 B_2 = B_{12}, \quad \nabla_1 B_1+\nabla_2 B_2 = -B_{m3}',$$

we find the equations of motion for a magnetostatic lens with rectilinear axis:

$$\left.\begin{aligned} x_1'' &= -\frac{1}{p}\left(B_{12}+\frac{B_{m3}^2}{4p}\right)x_1+\frac{1}{2p}(\nabla_1 B_1-\nabla_2 B_2)\,x_2;\\ x_2'' &= \frac{1}{2p}(\nabla_1 B_1-\nabla_2 B_2)\,x_1+\frac{1}{p}\left(B_{12}-\frac{B_{m3}^2}{4p}\right)x_2. \end{aligned}\right\} \quad (6)$$

Introducing the notation

$$k_{11}=-\frac{1}{p}\left(B_{12}+\frac{B_{m3}^2}{4p}\right),$$

$$k_{12}=\frac{1}{2p}(\nabla_1 B_1-\nabla_2 B_2),\quad \hat{x}=\begin{Vmatrix} x_1\\ x_1'\\ x_2\\ x_2' \end{Vmatrix},$$

$$k_{22}=\frac{1}{p}\left(B_{12}-\frac{B_{m3}^2}{4p}\right),$$

we rewrite (6) in matrix form:

$$\hat{x}'=\mathbf{P}\hat{x},\quad \mathbf{P}=\begin{Vmatrix} 0 & 1 & 0 & 0\\ k_{11} & 0 & k_{12} & 0\\ 0 & 0 & 0 & 1\\ k_{12} & 0 & k_{22} & 0 \end{Vmatrix}.$$

Magnetic axisymmetric ($\nabla_1 B_1=\nabla_2 B_2$, $B_{12}=0$) and quadrupole ($\nabla_1 B_1=\nabla_2 B_2=0, B_{m3}=0$) lenses are special cases of the magnetostatic lense with rectilinear axis.

The matrizant of an axisymmetric lens

The model of a bell-shaped field is usually used for an axisymmetric lens[36]:

$$B_{m3}=\frac{B_3^0}{1+\left(\frac{s-s_c}{d}\right)^2},$$

where B_3^0 is the maximum value of B_{m3} at the center of the field $s=s_c$ and d is the half-width of the field.

Introducing the notation

$$\beta_0=B_3^0/2p,\ \beta_0 d=k_0,\ \omega=\sqrt{1+k_0^2},$$

$$\frac{s-s_c}{d}=\operatorname{ctg}\vartheta,\ \frac{s_0-s_c}{d}=\operatorname{ctg}\vartheta_0,\ \alpha=\omega(\vartheta-\vartheta_0),$$

we write the expressions for the elements of the matrizant $\mathbf{R}$ as

$$\hat{x}=R\hat{x}_0,\quad \mathbf{R}=\begin{Vmatrix} \mathbf{T} & 0\\ 0 & \mathbf{T} \end{Vmatrix},\quad \mathbf{T}=\begin{Vmatrix} T_{11} & T_{12}\\ T_{21} & T_{22} \end{Vmatrix},$$

where

$$T_{11}=\frac{1}{\sin\vartheta}\left[\cos\alpha\sin\vartheta_0+\frac{\cos\vartheta_0}{\omega}\sin\alpha\right];$$

$$T_{12}=-\frac{d}{\omega}\,\frac{\sin\alpha}{\sin\vartheta_0\sin\vartheta};$$

$$T_{21}=\frac{1}{d}\left[\cos\alpha\sin(\vartheta_0-\vartheta)+\sin\alpha\,\frac{\cos\vartheta\cos\vartheta_0}{\omega}+\omega\sin\vartheta\sin\vartheta_0\right];$$

$$T_{22}=\frac{1}{\sin\vartheta_0}\left[\cos\alpha\sin\vartheta-\frac{\cos\vartheta}{\omega}\sin\alpha\right].$$

The matrizant of a quadrupole lens

For the quadrupole lens we shall use the common rectangular model of the field, with the notation

$$k=\frac{1}{p}B_{12},\quad k_c=\frac{1}{p}B_{12}^0,\quad B_{12}=\nabla_2 B_1=\nabla_1 B_2,$$

$$L=\frac{1}{k_c}\int_{-\infty}^{+\infty}k(s)\,ds,\ \beta^2=|k_c|,\ \varkappa=\beta L,$$

where B_{12}^0 is the maximum value of $\nabla_1 B_2$ on the axis of the lens, L is the effective length of the lens, β is the excitation of the lens, and $\varkappa$ is a dimensionless excitation.

With this notation the matrizant of a quadrupole lens can be written as (for $B_{12}>0$)

$$\mathbf{R}=\begin{Vmatrix} \mathbf{F} & 0\\ 0 & \mathbf{D} \end{Vmatrix},\quad \mathbf{F}=\begin{Vmatrix} F_{11} & F_{12}\\ F_{21} & F_{22} \end{Vmatrix},\quad \mathbf{D}=\begin{Vmatrix} D_{11} & D_{12}\\ D_{21} & D_{22} \end{Vmatrix},$$

where

$$F_{11}=F_{22}=\cos\varkappa;\ F_{12}=\frac{1}{\beta}\sin\varkappa;\ F_{21}=-\beta\sin\varkappa;$$

$$D_{11}=D_{22}=\operatorname{ch}\varkappa;\ D_{12}=\frac{1}{\beta}\operatorname{sh}\varkappa;\ D_{21}=\beta\operatorname{sh}\varkappa.$$

The equations of motion in deflecting magnets

We shall assume that the deflecting magnet with nonuniform field, whose boundaries at the entrance and exit are rectilinear and perpendicular to the axial trajectory, has curvature k_2, i.e., it deflects particles in the plane $(x_1 s)$. For the static case with $k_1=k_3=0$, $l=0$ the paraxial equations of motion in this magnet can be written as

$$\left.\begin{aligned} x_1'' &= \left(-\frac{1}{p}\nabla_1 B_2+p^2k_2^2\right)x_1+p\gamma k_2 x_4';\\ x_2'' &= \frac{1}{p}\nabla_2 B_1 x_2;\\ x_4'' &= -\frac{\gamma}{p}(k_2' x_1+k_2 x_1')=-\frac{\gamma}{p}(k_2 x_1)', \end{aligned}\right| \quad (7)$$

where $k_2=-B_{m2}/p=1/\rho$.

Introducing the matrizant in the space of the variables (x_1, x_1', x_4'),

$$\begin{Vmatrix} x_1\\ x_1'\\ x_4' \end{Vmatrix}=\mathbf{R}\cdot\begin{Vmatrix} x_{10}\\ x_{10}'\\ x_{40}' \end{Vmatrix},$$

let us find the elements of the matrizant for a rectilinear model of the field assuming that k_2 and $\nabla_1 B_2$ are constant along the axial trajectory of length L,

$$L=\rho\alpha=\alpha/k_2,$$

where α is the deflection angle of the axial particle. Introducing the index of field attenuation n by

$$n=\nabla_1 B_2\frac{\rho}{B_{m2}}=-\frac{\nabla_1 B_2}{pk_2^2},$$

and the dimensionless excitation φ

$$\varphi=\alpha\sqrt{1-n},$$

for $n<1$ we obtain

$$R_{11}=\cos\varphi+\frac{\gamma^2}{1-n}(1-\cos\varphi);$$

$$R_{21}=\frac{\sqrt{1-n}}{\rho}\left(\frac{\gamma^2}{1-n}-1\right)\sin\varphi;$$

$$R_{31}=-\frac{\gamma}{p\rho}\left(\frac{\gamma^2}{1-n}-1\right)(1-\cos\varphi);$$

$$R_{12}=\frac{\rho}{\sqrt{1-n}}\sin\varphi;$$

$$R_{22}=\cos\varphi;\quad R_{32}=-\frac{\gamma}{p}\frac{1}{\sqrt{1-n}}\sin\varphi;$$

$$R_{13}=\frac{p\gamma\rho}{1-n}(1-\cos\varphi);$$

$$R_{23}=\frac{p\gamma}{\sqrt{1-n}}\sin\varphi;$$

$$R_{33}=1-\frac{\gamma^2}{1-n}(1-\cos\varphi).$$

The corresponding equations for the case $n>1$ are obtained from these expressions with the substitutions

$$\sqrt{1-n}=\mathrm{i}\sqrt{n-1},\quad \varphi=\mathrm{i}\varphi^*,\quad \varphi^*=\alpha\sqrt{n-1},$$
$$\cos \mathrm{i}\varphi^*=\mathrm{ch}\,\varphi^*,\quad \sin \mathrm{i}\varphi^*=\mathrm{i}\,\mathrm{sh}\,\varphi^*.$$

We note that the relative longitudinal momentum spread of the particles, which is commonly used in studies on deflecting systems, is related to the variable x_4' as follows:

$$\delta=\Delta p/p=p\gamma x_{40}'+\gamma^2 x_{10}'.$$

The second equation in (7) is the analog of the quadrupole-lens equation, and its solution for the model of the rectilinear field is written as

$$\left\|\begin{matrix}x_2\\ x_2'\end{matrix}\right\|=\mathbf{F}\cdot\left\|\begin{matrix}x_{20}\\ x_{20}'\end{matrix}\right\|,\quad n>0;$$
$$\left\|\begin{matrix}x_2\\ x_2''\end{matrix}\right\|=\mathbf{D}\cdot\left\|\begin{matrix}x_{20}\\ x_{20}'\end{matrix}\right\|,\quad n<0,$$

where

$$F_{11}=F_{22}=\cos(\sqrt{n}\,\alpha);\quad F_{12}=\frac{\rho}{\sqrt{n}}\sin(\sqrt{n}\,\alpha);$$

$$F_{21}=-\frac{\sqrt{n}}{\rho}\sin(\sqrt{n}\,\alpha);$$

$$D_{11}=D_{22}=\mathrm{ch}(\sqrt{|n|}\,\alpha);$$

$$D_{12}=\frac{\rho}{\sqrt{|n|}}\mathrm{sh}(\sqrt{|n|}\,\alpha);$$

$$D_{21}=\frac{\sqrt{|n|}}{\rho}\mathrm{sh}(\sqrt{|n|}\,\alpha).$$

4. CANONICAL AND SYMMETRIC BEAM CONTROL SYSTEMS

The systems for focusing, forming, and deflecting the beams will be referred to as the beam control systems.

The G-canonical and T-symmetric matrix functions

In the equation for the matrizant

$$\mathbf{R}'=\mathbf{P}(s)\,\mathbf{R},\quad \mathbf{R}(s_0/s_0)=\mathbf{I}$$

the matrix function $\mathbf{P}(s)$ is a **G**-canonical function if it satisfies the equation

$$\mathbf{P}(s)\,\mathbf{G}+\mathbf{G}\tilde{\mathbf{P}}(s)=0,$$

and it is a **T**-symmetric function on the interval $[0,L]$ if

$$\mathbf{P}(L/2+s)\cdot\mathbf{T}+\mathbf{T}\cdot\mathbf{P}(L/2-s)=0.$$

In both cases the matrices **G** and **T** are understood to be real, square, constant, nonsingular matrices.

For example, if for the square $2k\times 2k$ matrix **P** we have the block equations

$$p_{11}=-\tilde{p}_{22},\quad p_{12}=\tilde{p}_{12},\quad p_{21}=\tilde{p}_{21},$$

then this matrix is **J**-canonical, where

$$\mathbf{J}_{2k}=\left\|\begin{matrix}0 & -I_k\\ I_k & 0\end{matrix}\right\|,\quad k=1,2\ldots$$

If for the 4×4 matrix **P** we have the matrix $\mathbf{T}=\mathbf{S}$, where

$$\mathbf{S}=\left\|\begin{matrix}\mathbf{U} & 0\\ 0 & \mathbf{U}\end{matrix}\right\|,\quad \mathbf{U}=\left\|\begin{matrix}1 & 0\\ 0 & -1\end{matrix}\right\|,$$

then the matrix **P** is termed symmetric, while if $\mathbf{T}=\mathbf{A}$, where

$$\mathbf{A}=\left\|\begin{matrix}0 & \mathbf{U}\\ \mathbf{U} & 0\end{matrix}\right\|,$$

it is termed antisymmetric on the interval $[0,L]$.

The G-canonical and T-symmetric control systems

A control system in which the linear equation of motion of particles is described by a **G**-canonical coefficient matrix $\mathbf{P}(s)$,

$$\hat{x}'=\mathbf{P}(s)\,\hat{x},$$

is termed **G**-canonical, while a control system for which **P** is a **T**-symmetric matrix is termed **T**-symmetric.

For a **G**-canonical system the matrizant **R** is a **G**-invariant:

$$\mathbf{R}\mathbf{G}\tilde{\mathbf{R}}=\mathbf{G}.$$

For example, the matrix **P** describing a quadrupole or axisymmetric lens for the vector $\hat{x}_1=\|x_1x_1'\|$ or $\hat{x}_2=\|x_2x_2'\|$ is $\mathbf{J}_2$-canonical, and the matrizant **R** of this system satisfies the condition of $\mathbf{J}_2$ invariance:

$$\left\|\begin{matrix}R_{11} & R_{12}\\ R_{21} & R_{22}\end{matrix}\right\|\left\|\begin{matrix}0 & -1\\ 1 & 0\end{matrix}\right\|\left\|\begin{matrix}R_{11} & R_{21}\\ R_{12} & R_{22}\end{matrix}\right\|=\left\|\begin{matrix}0 & -1\\ 1 & 0\end{matrix}\right\|.$$

The matrizant for a **T**-symmetric system satisfies the equation

$$\mathbf{T}\cdot\mathbf{R}(\mathbf{P},s/0)=\mathbf{R}(\mathbf{P},L-s/L)\cdot\mathbf{T}.$$

For $s=L$ and $s=L/2$ we obtain

$$\mathbf{R}(L/0)\cdot\mathbf{T}\cdot\mathbf{R}(L/0)=\mathbf{T};$$
$$\mathbf{R}\left(L\Big/\frac{L}{2}\right)\cdot\mathbf{T}\cdot\mathbf{R}\left(\frac{L}{2}\Big/0\right)=\mathbf{T}.$$

For a **G**-canonical system which is simultaneously **T**-symmetric, the following equations are valid for the matrizant:

$$\mathbf{G}\cdot\mathbf{T}\cdot\mathbf{R}(L/0)=\tilde{\mathbf{R}}(L/0)\cdot\mathbf{G}\cdot\mathbf{T};$$
$$\mathbf{G}\cdot\mathbf{T}\cdot\mathbf{R}\left(\frac{L}{2}\Big/0\right)=\tilde{\mathbf{R}}\left(L\Big/\frac{L}{2}\right)\cdot\mathbf{G}\cdot\mathbf{T}.$$

Symmetric and antisymmetric magnetostatic systems

The system of paraxial equations of motion in a magnetostatic lens has the form

$$\hat{x}'=\mathbf{P}\hat{x},\quad \hat{x}=\left\|\begin{matrix}x_1\\ x_1'\\ x_2\\ x_2'\end{matrix}\right\|,\quad \mathbf{P}=\left\|\begin{matrix}0 & 1 & 0 & 0\\ k_{11} & 0 & k_{12} & 0\\ 0 & 0 & 0 & 1\\ k_{12} & 0 & k_{22} & 0\end{matrix}\right\|. \tag{8}$$

It is easily seen that the matrix **P** is **G**-canonical, where

$$\mathbf{G}=\left\|\begin{matrix}\mathbf{J}_2 & 0\\ 0 & \mathbf{J}_2\end{matrix}\right\|.$$

It will be symmetric when the following conditions are satisfied:

$$\left.\begin{aligned}k_{11}\left(\tfrac{L}{2}+s\right)&=k_{11}\left(\tfrac{L}{2}-s\right);\\ k_{22}\left(\tfrac{L}{2}+s\right)&=k_{22}\left(\tfrac{L}{2}-s\right);\\ k_{12}\left(\tfrac{L}{2}+s\right)&=k_{12}\left(\tfrac{L}{2}-s\right),\end{aligned}\right\} \tag{9}$$

where L is the length of the system, and it will be antisymmetric if

$$\left.\begin{aligned}k_{11}\left(\tfrac{L}{2}+s\right)&=k_{22}\left(\tfrac{L}{2}-s\right);\\ k_{22}\left(\tfrac{L}{2}+s\right)&=k_{11}\left(\tfrac{L}{2}-s\right);\\ k_{12}\left(\tfrac{L}{2}+s\right)&=-k_{12}\left(\tfrac{L}{2}-s\right).\end{aligned}\right\} \tag{10}$$

From the conditions for it to be antisymmetric and **G**-canonical, it follows that the matrizant of an antisymmetric system must have the structure

$$\mathbf{R}=\left\|\begin{matrix}R_{11} & R_{12} & R_{13} & 0\\ R_{21} & R_{22} & 0 & -R_{13}\\ R_{13} & 0 & R_{22} & R_{12}\\ 0 & -R_{13} & R_{21} & R_{11}\end{matrix}\right\|. \tag{11}$$

A magnetostatic system which is the analog of an axisymmetric lens

We shall use the term "analog of an axisymmetric lens" for a system in which the matrizant **R** for length of the system L has the form

$$\mathbf{R}=\left\|\begin{matrix}R^{11} & 0\\ 0 & R^{11}\end{matrix}\right\|,\quad R^{11}=\left\|\begin{matrix}R_{11} & R_{12}\\ R_{21} & R_{22}\end{matrix}\right\|.$$

It follows from the structure of the matrizant (11) that the system (8) described by (10) will, at its exit, behave like the analog of an axisymmetric lens when the following conditions are satisfied:

$$R_{11}(L/0)=R_{22}(L/0),$$
$$R_{13}(L/0)=0.$$

In the case $k_{12}=0$ the second condition is satisfied identically, and the only requirement for a system to be the analog of an axisymmetric lens is the first condition.

In particular, a system of quadrupole lenses ($k_{22}=k_{11}=k$, $\varepsilon=0$) will be an analog of an axisymmetric lens when the following conditions hold:

$$\left.\begin{aligned}k\left(\tfrac{L}{2}+s\right)&=-k\left(\tfrac{L}{2}-s\right),\\ R_{11}(L/0)&=R_{22}(L/0).\end{aligned}\right\} \tag{12}$$

The quadruplet of rotation

The simplest focusing quadrupole system operating as the analog of an axisymmetric lens is a system of four quadrupole lenses—the quadruplet of rotation (or the "Russian" quadruplet).[15] Its properties are described in detail in Ref. 16. Equation (12) implicitly specifies the relation between $\varkappa_1$ and $\varkappa_2$ for a chosen geometry of the quadruplet of rotation. The resulting curve in the $(\varkappa_1,\varkappa_2)$ plane is referred to as the load curve.

The matrizant of the antisymmetric quadruplet in the thin-lens approximation

We shall use the following notation for a system of n quadrupole lenses. The length of the j-th lens is called L_j, s_j is the distance between the j-th and $(j+1)$-th lenses, s_0 is the distance between the location of the initial phase portrait and the first lens, s_{2n+1} is the distance between the last lens and the position of the final phase portrait (the distance between the last lens and the target), and

$$\frac{1}{p}\,|\nabla_1 B_2(j)|=\beta_j^2,\quad \varkappa_j=\beta_j L_j.$$

In the antisymmetric quadruplet $n=4$, $s_1=s_3$, $L_1=L_4$, $L_2=L_3$, $\varkappa_1=\varkappa_4$, and $\varkappa_2=\varkappa_3$. For convenience in writing out the equations, we shall use the additional notation

$$t_1=s_0+\frac{1}{2}L_1,\quad t_2=t_4=s_1+\frac{1}{2}(L_1+L_2),$$
$$t_2=s_2+L_2,\quad t_5=s_4+\frac{1}{2}L_1,$$
$$c_1=-c_4=-\frac{\varkappa_1^2}{L_1},\quad c_2=-c_3=\frac{\varkappa_2^2}{L_2},$$
$$\alpha=(c_1+c_2)\,t_3+2c_1t_2-c_1c_2^2t_2^2t_3,$$
$$\bar{L}=s_0+2s_1+s_2+s_4+2L_1+2L_2.$$

The matrizant in the $(x_1s)=(xs)$ plane is called **X**, and the matrizant in the $(x_2s)=(ys)$ plane is called **Y**.

In the thin-lens approximation the elements of the matrizant for the antisymmetric quadruplet are written as

$$X_{11}=Y_{22}=1+\alpha-c_2^2t_2t_3+t_5X_{21};$$
$$X_{12}=Y_{12}=\bar{L}+\alpha\,(t_1-t_5)$$
$$+\,t_1t_5X_{21}-c_2^2t_2t_3\,(t_1+t_2+t_5);$$
$$X_{21}=Y_{21}=t_3\,[c_1^2c_2^2t_2^2-(c_1+c_2)^2]-2c_1^2t_2;$$
$$X_{22}=Y_{11}=1-\alpha-c_2^2t_2t_3+t_1X_{21}.$$

The quadruplet of rotation in the thin-lens approximation

The equation $\alpha=0$ defines a quadruplet of rotation and is the equation of the load curve in the thin-lens approximation.

The key elements of the quadruplet of rotation are determined by the expressions

$$H'F'=HF=f=\frac{t_2\,(2-\varphi)+t_3}{\varphi\,(2-\varphi)},\quad \varphi=c_2^2t_2t_3,$$
$$S'F'=SF=f\,(1-\varphi)-\frac{1}{2}L_1.$$

Even for fairly high excitations ($\varkappa\cong 1$), the thin lens approximation gives the quadruplet parameters with an acceptable accuracy: 5–10% for HF and SF and 20% for the field gradients.

The matrix of the focusing lens, the Twiss matrix, and the equivalent length of the system

We shall use the two-parameter matrix of the focusing lens $\mathbf{F}(\mu,\nu)$ and the three-parameter Twiss matrix $\mathbf{T}(\mu,\nu,\varepsilon)$, where

$$\mathbf{F}(\mu,\nu)=\begin{Vmatrix} \cos\mu & \dfrac{\sin\mu}{\nu} \\ -\nu\sin\mu & \cos\mu \end{Vmatrix};$$

$$\mathbf{T}(\mu,\nu,\varepsilon)=\begin{Vmatrix} \dfrac{\cos(\mu-\varepsilon)}{\cos\varepsilon} & \dfrac{\sin\mu}{\nu\cos\varepsilon} \\ -\dfrac{\nu\sin\mu}{\cos\varepsilon} & \dfrac{\cos(\mu+\varepsilon)}{\cos\varepsilon} \end{Vmatrix}.$$

The matrix of the focusing lens is a special case of the Twiss matrix for $\varepsilon=0$.

We introduce the concept of the equivalent length of the focusing lens, L_{eq}, defining it as

$$L_{eq}=\mu/\nu.$$

Representation of the antisymmetric quadruplet as an equivalent focusing lens

For an antisymmetric quadruplet of length L_k, where

$$L_k=2(L_1+L_2+s_1)+s_2,$$

in each of the planes $(x_1s)=(xs)$ and $(x_2s)=(ys)$ we find the parameters of the equivalent focusing lens:

$$\cos\mu_x=\cos\mu_y=\cos\mu=\frac{1}{2}(H_{11}+H_{22})=\frac{1}{2}(V_{11}+V_{22});$$

$$\nu_x=\nu_y=\sqrt{-\frac{H_{21}}{H_{12}}}=\sqrt{-\frac{V_{21}}{V_{12}}};$$

$$\sin\varepsilon_x=\frac{H_{11}-H_{22}}{2\sqrt{-H_{12}H_{21}}};\quad \sin\varepsilon_y=-\sin\varepsilon_x.$$

Here the matrizants $\mathbf{H}$ and $\mathbf{V}$ are used to denote the values of the matrizants $\mathbf{X}$ and $\mathbf{Y}$ for $s_0=s_4=0$.

We define the coefficient of equivalence α_{eq} as

$$\alpha_{eq}=L_{eq}/L_k.$$

The quadruplet of rotation can be represented as an equivalent focusing lens of length L_{eq}, the matrizant $M=M(8/1)$ of which is the matrix of the focusing lens:

$$\mathbf{M}=\mathbf{X}(8/1)=\mathbf{Y}(8/1)=F(\mu,\nu),$$

where

$$\cos\mu=M_{11}+M_{22};\quad \alpha_{eq}=\mu/(\nu L_k).$$

Analysis of the results of numerical calculations shows that $\alpha_{eq}\cong 1$ with a very weak dependence on μ, i.e., the length of the equivalent focusing lens is approximately equal to the length of the quadruplet of rotation.

The relation between the parameters of the quadruplet of rotation and the axisymmetric lens.

We use z_F $(z_{F'})$ to denote the position of the focal point in object (image) space, z_c for the location of the center of the quadruplet of rotation or the center of the lens, and f for the focal length. Then for the quadruplet of rotation we can write

$$f=-\frac{1}{M_{21}}=\frac{1}{\nu\sin\mu},$$

$$z_c-z_F=SF+\frac{1}{2}L_k=\frac{1}{\nu}\operatorname{ctg}\mu+\frac{1}{2}L_k,$$

and for the axisymmetric lens

$$f=\frac{d}{\sin\dfrac{\pi}{\omega}},\quad z_c-z_F=-d\operatorname{ctg}\frac{\pi}{\omega},$$

from which we obtain

$$d=\sqrt{f^2-\left(SF+\frac{1}{2}L_k\right)^2},$$

$$k_0^2=\frac{\pi^2}{\bar{a}^2}-1,\quad \bar{a}=\arccos\frac{-SF+\frac{1}{2}L_k}{f}.$$

The latter equations allow us to use the known value of μ for the quadruplet of rotation and, consequently, the known values of f/L_k and SF/L_k to calculate the values of L_k/d and k_0^2 for the model of a bell-shaped lens with equal f and z_c-z.

Comparison of the magnetic fields of the quadrupole lenses in the quadruplet of rotation and the axisymmetric lens

We use B_{kj} to denote the magnetic induction at a pole of the j-th quadrupole lens, B_0 for the maximum value of the magnetic induction on the axis of the axisymmetric lens, a_j for the aperture radius of the quadrupole lens, L_j for the length of the quadrupole lens, and $\varkappa_j$ for the dimensionless excitation. Then the ratio of the fields is written as

$$\frac{B_0}{B_{kj}}=\frac{w_jL_j}{a_j},$$

where

$$w_j=\frac{2k_0}{\varkappa_j^2}\frac{L_k}{d}\frac{L_j}{L_k}.$$

Calculations show that quadrupole lenses provide a magnetic induction at a pole which is roughly 3–9 times larger than that for an axisymmetric solenoid with $L_j/a_j\cong 3$.

It follows from the above expressions that the larger the ratio of the length of the quadrupole lens to its aperture, the larger the field reached in the quadruplet of rotation in relation to the solenoid.

5. RECURSION METHODS OF CALCULATING THE MATRIZANT

Gauss brackets

In the early stage of development of geometrical optics, before the matrix apparatus existed, Gauss obtained his results on light optics using a recursion technique which later became known as the Gauss brackets.[37,38] The technique of Gauss brackets is used for two-dimensional phase transformations. It is closely related to the thin-lens representation. The use of Gauss brackets is convenient for writing down the basic equations of ray optics and the optics of phase ellipses.[39,40] A continuous generalized analog of the Gauss brackets[41] can be used to calculate the matrizant for an arbitrary coefficient matrix with rigorous conservation of the phase volume of the beam at each stage of the calculations. Effective computer programs based on this method have been written for studying the beam dynamics for an arbitrary axial field distribution through fifth order in the nonlinearity.

The multiplicative form of the matrizant

A square $\beta\times\beta$ matrix $a\langle j\rangle$,

$$I(j)=i(j)\tilde{i}(j),$$

$$a\langle j\rangle=\left\|\begin{matrix}1 & 0 & \dots & 0\\ 0 & 1 & \dots & 0\\ \cdot & \cdot & & \cdot\\ \cdot & \cdot & & \cdot\\ a_{j1} & a_{j2} & \dots & a_{j\beta}\\ 0 & 0 & \dots & 1\end{matrix}\right\|=I-I(j)+I(j)\,a,\quad j=1,\ \dots,\ \beta,$$

differing from the unit matrix in its j-th row (the generating row), is referred to as a $\langle j\rangle$-unit matrix a. The unit matrices have the following properties:

1. $a\langle j\rangle\, b\langle j\rangle=(a+b)\langle j\rangle$, if $a_{jj}=b_{jj}=1$.
2. $a^{-1}\langle j\rangle=a^*\langle j\rangle=I-I(j)\,a^{-1}I(j)\,a+I(j)\,a^{-1}I(j)$.
3. $\mathrm{Det}\ a\langle j\rangle=|a\langle j\rangle|=a_{jj},\quad j=1,\dots,\beta$.

A square $\beta\times\beta$ matrix $\mathbf{A}$ can be represented either as the product of β unit matrices (the H_1 expansion)

$$\mathbf{A}=a\langle\beta\rangle\, a\langle\beta-1\rangle\dots a\langle 1\rangle,$$

where

$$I(j)\,a\langle j\rangle=I(j)\,A(j);\quad A(1)=\mathbf{A},\quad A^{-1}(j)\neq 0;$$

$$A(j+1)=A(j)\,[I-I(j)\,A^{-1}(j)\,I(j)\,A(j)$$
$$+I(j)\,A^{-1}(j)\,I(j)],$$

or as the product of $\beta+1$ unit matrices (the H_2 expansion)

$$\mathrm{A}=b\langle\beta+1\rangle\, b\langle\beta\rangle\dots b\langle 1\rangle,$$

where

$$b\langle\beta+1\rangle=c\langle 1\rangle=\left\|\begin{matrix}b_{\beta+1,\,1} & b_{\beta+1,\,2} & \dots & b_{\beta+1,\,\beta}\\ 0 & 1 & \dots & 0\\ \dots & \dots & \dots & \dots\\ 0 & 0 & \dots & 1\end{matrix}\right\|;$$

$$b=\left\|\begin{matrix}b_{11} & \dots & b_{1\beta}\\ \dots & \dots & \dots\\ b_{\beta+1,\,1} & \dots & b_{\beta+1,\,\beta}\end{matrix}\right\|;$$

$$b_{jr}=b_{jj}B_{jr}(j),\quad r\neq j,$$

$$j=2\dots\beta+1,\quad r=1,\dots,\beta,$$

$$b_{\beta+1,1}=B_{11}(\beta+1),\quad B(1)=A;$$

$$b_{1j}=b_{j1}^{-1}\Big(A_{jj}-b_{jj}-\sum_{i=2}^{j-1}b_{ji}A_{ij}\Big),$$

$$b_{j1}^{-1}\neq 0,\quad j=2,\dots,\beta;$$

$$B_{kr}(j+1)=B_{kr}(j)-B_{kj}(j+1)\,b_{jr},$$

$$r\neq j,\quad k,\ j=1,\dots,\beta;$$

$$B_{kj}(j+1)=B_{kj}(j)\,b_{jj}^{-1},$$

$$k,\ j=1,\dots,\beta.$$

The generalized analog of the Gauss brackets

The sum of the unit matrices can be written recursively:

$$a_1+\dots+a_k=\Sigma_{k1}=a_k+\Sigma_{k-1,1},\quad \Sigma_{01}=0.$$

An example of a linear recursive operation of order 2 is the operation

$$\Sigma_{m1}=a_m\Sigma_{m-1,1}+\Sigma_{m-2,1},\quad \Sigma_{01}=1,\quad \Sigma_{-1,1}=0,$$

which defines the Gauss brackets. The generalized analog of the Gauss brackets is the linear recursive operation applied to the elements of a rectangular $n\times\beta$ matrix a:

$$S_{m,i}=\sum_{j=1}^{\beta}a_{m,\overline{m-j}}\cdot S_{m-j,i},$$

$$i=1,\ \dots,\ \beta,\quad m=1,\ \dots,\ n. \qquad (13)$$

Here $\overline{m-j}$ denotes the remainder of the division of $m-j$ by β, which takes values from 1 to β, inclusive:

$$m-j=n\beta+\overline{m-j},$$

$$\overline{m-j}=1,\ \dots,\ \beta,\quad j=1,\ \dots,\ \beta.$$

The matrix of the results of the preceding operations is defined as

$$\left\|\begin{matrix}S_{-\beta+1,\,1} & \dots & S_{-\beta+1,\,\beta}\\ S_{-\beta+2,\,1} & \dots & S_{-\beta+2,\,\beta}\\ \dots & \dots & \dots\\ S_0 & \dots & S_{0\beta}\end{matrix}\right\|=I_\beta.$$

The result of the operation of (13) on a matrix (an a-term sequence) is also referred to as the shuttle sum $S_{m,i}$ of an a-term sequence. The Gauss brackets are a special case of their generalized analog for $\beta=2$, $a_{i+2j,i}=1$, $i=1,2$, and $j=0,1\dots$.

Let us construct the $\beta\times\beta$ matrix of individual shuttle sums in the form of a matrix $S(j,a)$:

$$S(j,\ a)=\left\|\begin{matrix}S_{j-\beta+1,\,1} & \dots & S_{j-\beta+1,\,\beta}\\ \dots & \dots & \dots\\ S_{j,\,1} & \dots & S_{j,\,\beta}\end{matrix}\right\|,$$

where

$$S(0,\ a)=I_\beta.$$

The products of the unit matrices are related to the shuttle sums as

$$S(j,\ a)=a\langle j\rangle\dots a\langle 1\rangle.$$

Here the H_1 and H_2 expansions for the matrix A are written as

$$A=S(\beta,\ a)=S(\beta+1,\ b).$$

The continuous generalized analog of the Gauss brackets

Approximating the function $\mathbf{P}(s)$ in the equation for the matrizant

$$\mathbf{R}'(\mathbf{P},\ s/0) = \mathbf{P}(s)\,\mathbf{R}(\mathbf{P},\ s/0), \quad \mathbf{R}(\mathbf{P},\ 0/0) = \mathbf{I}_\beta$$

by a piecewise-constant (step) function $\mathbf{P}(j), j = 1, \ldots, n$, where

$$\sum_{j=1}^{n} \Delta s_j = s,$$

we write the matrizant as the product of the n partial matrizants $R(j)$, each of which is calculated on its smooth segment Δs_j:

$$\mathbf{R} = R(n)\,R(n-1)\ \ldots\ R(1).$$

Writing each of the partial matrizants as the product of unit matrices $a^j\langle r\rangle, r = 1, \ldots, \beta + 1$,

$$R(j) = a^j\langle\beta+1\rangle \ldots a^j\langle 1\rangle = a^j\langle\beta+1/1\rangle,$$

we obtain the matrizant for the H_1 expansion in the form of a product of $n\beta$ unit matrices and the matrizant for the H_2 expansion as a product of $n(\beta+1)$ unit matrices.

Expanding the sequence of the partial matrizant $\mathbf{R}(\mathbf{P}(j), \Delta s_j/0)$ in a series in Δs_j and keeping only the linear term, we obtain the following equations for the elements of the sequence:

$$\left.\begin{aligned}
&\check{a}_{(j-1)\beta+1,\,k} = \delta(1,\ k) + \alpha P_{1k}(j)\,\Delta s_j,\\
&j = 1,\ \ldots,\ n,\ \ k = 1,\ \ldots,\ \beta;\\
&\hat{a}_{(j-1)\beta+i,\,k} = \delta(i,\ k) + P_{ik}(j)\,\Delta s_j,\\
&i = 2,\ \ldots,\ \beta,\ \ k = 1,\ \ldots,\ \beta;\\
&\hat{a}_{j\beta+1,\,k} = \delta(1,\ k) + (1-\alpha)\,P_{1k}(j)\,\Delta s_j,\ \ k = 1,\ \ldots,\ \beta;\\
&\delta(i,\ i) = 1,\ \ \delta(i,\ j) = 0,\ \ i \neq j,\ \ i,\ j = 1,\ \ldots,\ \beta.
\end{aligned}\right\} \qquad (14)$$

Here the matrizant is approximately determined by the matrix of shuttle sums:

$$\begin{aligned}
&R_{1k} \approx S_{n(\beta+1),\,k}(\beta n + 1,\ \hat{a});\\
&R_{ik} \approx S_{(n-1)(\beta+1)+i,\,k}(\beta n + 1,\ \hat{a}); \qquad (15)\\
&R \approx S(\beta n + 1,\ \hat{a}).
\end{aligned}$$

The limit of the matrix of shuttle sums for $n \to \infty$, max $\Delta s_j \to 0$ is the matrix of the shuttle integrals of the function $\mathbf{P}(s)$:

$$\lim_{n\to\infty} S(\beta n + 1,\ \hat{a}) = \left\langle \Big| \int_0^s (I + \mathbf{P}(s)\,ds) = \mathbf{R}(\mathbf{P},\ s/0), \right.$$
$$\max \Delta s_j \to 0,$$

which is equal to the matrizant $\mathbf{R}(\mathbf{P}, s/0)$.

Equations (15) together with the elements of the sequence (14) provide an algorithm for the approximate calculation of the matrizant which is constructed by multiplying the sequence by the unit matrix I_β. Multiplication of the sequence by a matrix $\mathbf{C}$ is performed in the following manner.

We take the first row of the sequence and multiply it by the matrix $\mathbf{C} = C(1)$. This gives a new row, which is inserted into the matrix $C(1)$ in place of row I, thereby forming a new matrix $C(2)$. We then take row II of the sequence, multiply it by the matrix $C(2)$, and insert the resulting row into the matrix $C(2)$ in place of its II row. The result is a new matrix $C(3)$, and so on. In the H_1 method the final β-th row of the sequence is multiplied by the matrix $C(\beta)$ and the resulting row is inserted into the matrix $C(\beta)$ in place of its last row. The result is $C(\beta+1) = \mathbf{RC}$.

In the H_2 method the final $(\beta+1)$-th row of the sequence is multiplied by the matrix $C(\beta+1)$ and the resulting row is inserted into the matrix $C(\beta+1)$ in place of its row I. The new matrix is $C(\beta+2) = \mathbf{RC}$. These ıules are valid for each partial matrizant.

For $P_{jj} = 0$ this method of computing the matrizant is conservative with regard to the determinant of the matrizant. Therefore, the phase-space volume of the beam is rigorously conserved in each step of the calculations, which distinguishes this method from others such as the Runge–Kutta and Adams schemes.

The difference scheme of the H_1 method is a first-order scheme, and that of the H_2 method is a second-order scheme for row I of the matrizant and a first-order scheme for the other rows.

If the function $P(s)$ is $\mathbf{G}$-canonical, it is possible to construct a variant of the shuttle-sum technique for computing the matrizant in which the $\mathbf{G}$ invariance of the matrizant is strictly conserved in each stage of the calculations.

6. PROBE SYSTEMS FOR THE CONTROL OF HIGH-ENERGY BEAMS—SYSTEMS WHICH DECREASE THE PHASE DIMENSIONS OF THE BEAM

The ion probe

A control system which decreases one or more of the phase dimensions of the beam is referred to as a probe system or an ion probe.

We shall use the phase-space variables in whose space the phase volume of the beam is constant during its motion. Different types of probes can be defined, depending on which phase variable is to be decreased. In the present study we restrict ourselves to four-dimensional phase space (h), where $h_1 = x$, $h_2 = y$, $h_3 = x'$, and $h_4 = y'$. The phase dimensions at the entrance to the system $(s = s_0)$ and at its exit $(s = s_e)$ denoted by h_{j0} and h_{je}, $j = 1, \ldots, 4$. We shall consider a transverse probe $(x_e < x_0, y_e < y_0)$ and an angular probe $(x'_e < x'_0, y'_e < y'_0)$ with two diaphragms before the first lens.

A probe for which the beam at the probe exit has very small phase dimensions, for example, several microns or submicrons for a transverse probe, is referred to as a microprobe.

The problem of designing the optimal microprobe of a given length can be formulated in two variants as the inverse nonlinear problem of finding the set of lenses, their parameters, and their relative placement for a series of contraints in the form of inequalities (limits on the maximum fields at the electrodes or poles, on the apertures, and on the lengths of the lenses).

In the first variant, the initial phase set is given and the system providing the minimum phase dimension at the exit is sought. In the second variant, the phase dimensions at the exit are given and the system having maximum acceptance is sought. We shall follow the first formulation of the problem with the initial set specified parametrically.

Description of the initial phase set

The boundary of the initial phase set defined by the two diaphragms is described by the equations

$$\left.\begin{aligned}(x_0'+x_0/l)^2+(y_0'+y_0/l)^2&=(r_2/l)^2,\\ x_0^2+y_0^2&=r_1^2.\end{aligned}\right\} \qquad (16)$$

Here r_i $(i=1, 2)$ is the radius of the i-th diaphragm, and l is the distance between the diaphragms. The intersections of the contour (16) with the (xx') and (yy') planes are parallelograms, which practically become rectangles for $r_2 \gg r_1$. The area of the phase-space parallelogram is $S=4r_1r_2/l$.

The boundary of the phase set is the phase portrait of the beam. The boundary in the form of a rectangle with sides parallel to x and x' (or y and y') is the canonical phase portrait.

The paraxial system (the system in which the equations of motion are studied in the linear approximation), which takes the phase parallelogram into a rectangle, is always, owing to the conservation of the phase volume of the set $\Omega(h)$, one type of probe, since some of the phase dimensions of the beam at the exit in this system are always smaller than (or equal to) the corresponding initial dimensions.

The magnification of a probe

Henceforth, unless stated otherwise, we consider transformations of the phase portrait in the (x, x') section. Since the extremal values of x and x' are associated with the canonical phase portrait, we assume that at the target the phase portrait is a rectangle for which $x_{\max}=r$ and $x'_{\max}=\alpha$. Here the magnification of the probe is

$$m = r/x_0 = x_0'/\alpha.$$

For a transverse probe $m<1$, and for an angular probe $m>1$.

The stigmatic and *k*-step probes

A probe is termed stigmatic if the phase portrait on the target is a rectangle in the spaces (x,x') and (y,y'). A probe is a k-step stigmatic probe if it can be divided into k subsystems, each of which is a stigmatic probe. In the two-step stigmatic probe the first subsystem is referred to as the foresystem, and the second as the primary system.

Notation for the parameters of the ion-optical system (IOS) of a probe

We shall refer to the lens system of a probe as the IOS of the probe. We use the following notation for the distances in a system of quadrupole lenses, assuming the axial distribution of the field to be rectangular:

Δl is the distance between the second diaphragm and the entrance to the first lens;

s_j is the distance between the j-th and $(j+1)$-th lenses;

g is the distance between the last lens and the target;

L_i is the length of the i-th quadrupole lens;

$\overline{L}$ is the length of the probe (the distance between the first diaphragm and the target);

L is the length of the IOS (the distance between the entrance to the first lens and the exit from the last lens);

F and F' are the front and rear focal points;

H and H' are the front and rear principal planes;

S and S' are the front edge of the first lens and the back edge of the last lens;

SF and $S'F'$ are the locations of the focal points;

f is the focal length of the IOS;

r_1 and r_2 are the radii of the first and second diaphragms.

In the case of a bell-shaped field, the distances between the centers of the elements are denoted by the same symbols with a star.

We also use the following notation for the matrizant:

$$\left\|\begin{matrix}x_{2n}\\ x'_{2n}\end{matrix}\right\|=\left\|\begin{matrix}M_{x11} & M_{x12}\\ M_{x21} & M_{x22}\end{matrix}\right\|\times\left\|\begin{matrix}x_1\\ x_1'\end{matrix}\right\|,\quad \left\|\begin{matrix}x_e\\ x_e'\end{matrix}\right\|=\left\|\begin{matrix}R_{x11} & R_{x12}\\ R_{x21} & R_{x22}\end{matrix}\right\|\times\left\|\begin{matrix}x_0\\ x_0'\end{matrix}\right\|,$$

where **M** is the matrizant of the IOS, **R** is the matrizant of the probe, the index 1 refers to the plane at the entrance to the first lens, and the index 2 refers to the plane at the exit from the n-th lens. Analogous relations hold for the (y_s) plane.

For the axisymmetric lens and the quadruplet of rotation we have

$$\mathbf{M}_x=\mathbf{M}_y=\mathbf{M},\ \mathbf{R}_x=\mathbf{R}_y=\mathbf{R}.$$

Transformation of the phase portrait in the paraxial probe

Let us consider the transformations which take the initial phase parallelogram ABCD (Fig. 1) in the plane of the first diaphragm to a rectangle in the plane of the target. There are four possible variants of these transformations (Figs. 2a–2d). It can be shown that variants (a) and (c) are characteristic of the angular probe, while variants (b) and (d) are typical of the transverse probe. In obtaining the expressions for the matrizant which accomplishes these transformations, it is sufficient to consider the transformations of the points A and B into the points A' and B'.

Transformations of the phase portrait in the paraxial angular probe

For transformations (a) and (c) we obtain

$$R_{11}=\pm r/r_2;\quad R_{12}=\pm rl/r_2;\quad R_{21}=\mp r_2/rl,\quad R_{22}=0;$$
$$\alpha=r_1r_2/rl;\quad m=r/r_1=\pm fr_2/lr_1;$$

$$M_{11}=\pm\left(\frac{r}{r_2}+\frac{r_2g}{rl}\right);$$

$$M_{12}=\pm\left[\frac{r}{r_2}\Delta l+\frac{r_2g}{r}\left(1+\frac{\Delta l}{l}\right)\right];$$

$$M_{21}=\mp\frac{r_2}{rl};\quad M_{22}=\pm\frac{r_2}{r}\left(1+\frac{\Delta l}{l}\right);$$

$$SF=l+\Delta l;\quad S'F'=g+(r/r_2)^2\,l;\quad f=\pm rl/r_2.$$

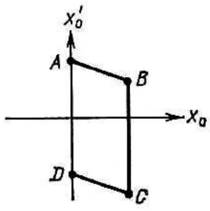

FIG. 1.

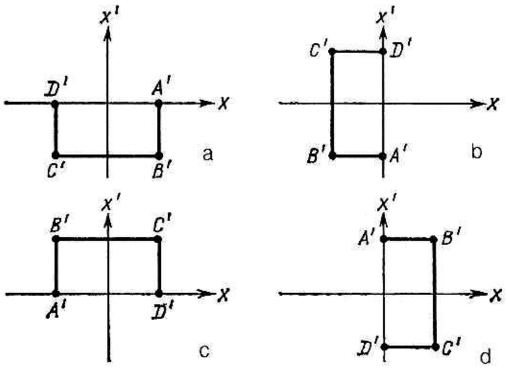

FIG. 2.

Here the upper signs refer to transformation (a) and the lower ones to transformation (c).

For a one-step probe $M_{11} = M_{22}$, $SF = S'F'$, so that

$$g = l + \Delta l - l\,(r/r_2)^2.$$

For a two-step probe consisting of two quadruplets of rotation separated by a distance λ or two axisymmetric lenses with centers separated by λ^* we have the relations

$$f = \frac{f_1 f_2}{SF_1 + SF_2 - \lambda} = \frac{f_1 f_2}{S^*F_1 + S^*F_2 - \lambda^*};$$

$$S'F' = SF_2 - \frac{f_2^2}{SF_1 + SF_2 - \lambda};$$

$$S^{*\prime}F' = S^*F_2 - \frac{f_2^2}{S^*F_1 + S^*F_2 - \lambda^*};$$

$$SF = SF_1 - \frac{f_1^2}{SF_1 + SF_2 - \lambda};$$

$$S^*F = S^*F_1 = \frac{f_1^2}{S^*F_1 + S^*F_2 - \lambda^*}.$$

Therefore, for the transformations (a) and (c) in the two-step probe we obtain

$$SF_1 = l + \Delta l + f\frac{f_1}{f_2}; \quad SF_2 = g + l\left(\frac{r}{r_2}\right)^2 + f\frac{f_2}{f_1};$$
$$\frac{f_1 f_2}{SF_1 + SF_2 - \lambda} = \pm\frac{rl}{r_2} = f.$$

Transformations of the phase portrait in the paraxial transverse probe

For the transformations (b) and (d) we find

$$R_{11} = \mp r/r_1; \quad R_{12} = 0;$$

$$R_{21} = \mp r_1/rl; \quad R_{22} = \mp r_1/r;$$

$$M_{11} = \pm\left(\frac{r_1}{rl}g - \frac{r}{r_1}\right);$$

$$M_{12} = \pm\left[\frac{r}{r_1}(l + \Delta l) - g\frac{\Delta l}{l}\frac{r_1}{r}\right];$$

$$M_{21} = \mp r_1/rl, \quad M_{22} = \pm r_1\,\Delta l/rl;$$
$$SF = \Delta l; \quad S'F' = g - l\,(r/r_1)^2 = g - f^2/l;$$
$$m = \pm f/l; \quad f = \pm rl/r_1.$$

For the one-step transverse probe we have

$$g = \Delta l + (r/r_1)^2\, l.$$

For the two-step transverse probe we have

$$SF_1 = \Delta l + ff_1/f_2; \quad SF_2 = g - f^2/l + ff_2/f_1;$$
$$f = \pm\frac{rl}{r_1} = \frac{f_1 f_2}{SF_1 + SF_2 - \lambda}.$$

Here the upper signs correspond to the transformation (b) and the lower ones to the transformation (d).

It follows from the expression for the focal length that the IOS of a microprobe must have a short focus. Therefore, if the minimum focal length of a single IOS is insufficient for obtaining the required decrease of the beam diameter, it is necessary to use a second IOS to form a compound IOS with a smaller focal length for the same length of the microprobe.

7. THE QUADRUPOLE MICROPROBE

Requirements on the IOS of a microprobe

An IOS of minimum focal length must be used to obtain a microprobe with maximal decrease of the beam diameter. The lower limit on the focal length for a one-step microprobe is determined by the lowest attainable boundary of g and the smallest possible lens length and probe length. The lower limit on the lens length is determined by the maximum possible magnetic induction at a pole or field strength at an electrode and by the lens construction. In the quadruplet of rotation with the smallest excitations, the smallest focal length can be obtained by making the free spaces between the lenses as small as possible and using lenses of equal length.

Examples of microprobes

For a quadruplet with no free spaces (quadruplet A) and $\Delta l = 0.14L_1$, where $L_1 = L_2$ is the length of the quadrupole lens, we obtain

$$f = 2.53\ L_1; \quad \varkappa_1 = \beta_1 L_1 = 1.15; \quad \varkappa_2 = \beta_2 L_2 = 0.75;$$
$$g = 0.14\ L_1 + 6.40\ L_1^2/l;$$
$$B_i = B\rho\varkappa_i^2 a/L_i^2, \quad i = 1,\ 2; \quad m = 2.53\ L_1/l.$$

Here $B\rho$ is the magnetic stiffness of a particle. For protons of energy 3 MeV, $B\rho = 0.25$ T·m.

If the free gaps in the quadruplet are taken to be $s_1 = s_2 = s_3 = 0.20L_1$ (quadruplet B), then for $\Delta l = 0.14L_1$ we obtain

$$f = 2.68\ L_1; \quad \varkappa_1 = 0.70; \quad \varkappa_2 = 1.10;$$
$$g = 0.14\ L_1 + 7.18\ L_1^2/l; \quad m = 2.68\ L_1/l.$$

For higher excitations it is possible to obtain a focal length an order of magnitude smaller. For example, for the quadruplet (quadruplet C) with parameters $L_1 = L_2$, $\Delta l = 0.056L_1$, $s_1 = s_3 = 0$, and $s_2 = 3L_1$ we have

$$f = -0.27\ L_1; \quad \varkappa_1 = 2.24; \quad \varkappa_2 = 1.27;$$
$$g = 0.056\ L_1 + 0.073\ L_1^2/l; \quad m = -0.27\ L_1/l.$$

Whereas in the first two cases the maximum beam radius in the system is somewhat larger than r_2, in the third case it is an order of magnitude larger.

The optimal microprobe

Taking into account chromatic and spherical aberrations, assuming that $r_2 \gg r_1$, we can write down an approximate expression for the beam radius at the target for $y_0 = 0$ and $y_0' = 0$:

$$x_e = R_{11} r_1 + C_p \frac{\Delta p}{p} \frac{r_2}{l} + C_s \left(\frac{r_2}{l} \right)^3,$$

where

$$C_s = (M_{13} + gM_{23})(l + \Delta l)^3 + (M_{14} + gM_{24})(l + \Delta l)^2 + (M_{15} + gM_{25})(l + \Delta l) + M_{16} + gM_{26}.$$

Here M_{ik} are the coefficients of the geometrical aberrations of the IOS of the probe, and C_p is the chromatic-aberration coefficient of the IOS.

We write the expression for $|x_e|$ in terms of the beam emittance ε and the angle $\vartheta = r_2/l$:

$$|x_e| = \frac{b_1}{\vartheta} + b_2\vartheta + b_3\vartheta^3;$$

$$b_1 = |R_{11}\varepsilon|; \quad b_2 = \left| C_p \frac{\Delta p}{p} \right|; \quad b_3 = |C_s|; \quad \varepsilon = \frac{r_1 r_2}{l}.$$

We define r_1 such that the expression for $|x_e|$ has a minimum at $|x_e| = x_m$. The values of r_1 and ϑ at $|x_e| = x_m$ are called r_{1m} and ϑ_m. We obtain

$$\vartheta_m = \sqrt{\frac{d}{6b_3}}, \quad d = \sqrt{b_2^2 + 12 b_1 b_3} - b_2,$$

$$x_m = \frac{1}{\vartheta_m} \left(\frac{4}{3} b_1 + \frac{b_2 d}{9 b_3} \right).$$

The microprobe with $\vartheta = \vartheta_m$ and $|x_e| = x_m$ is referred to as the optimal microprobe for the (xs) plane. For a given beam radius at the target, this microprobe transmits a beam with the maximum emittance, i.e., the maximum current.

For the quadruplet A the geometrical-aberration coefficients are

$M_{x13} = -10.7;\quad M_{x14} = -20.8;$

$M_{x15} = -22.1;\quad M_{x16} = -8{,}7;$

$M_{x23} = -22.3;\quad M_{x24} = -63.7;$

$M_{x25} = -73.7;\quad M_{x26} = -26{,}5;$

$M_{y13} = -84.4;\quad M_{y14} = -67.0;$

$M_{y15} = -20.4;\quad M_{y16} = -2.6;$

$M_{y23} = -67.3;\quad M_{y24} = -58{,}8;$

$M_{y25} = -19.9;\quad M_{y26} = -3.8.$

Here the IOS length is taken to be unity, and $L_1 = L_2 = 0.25$.

Comparison of quadruple IOSs for microprobes

The doublet and quadruplet of rotation are used most frequently in the quadrupole IOSs of microprobes. Often the comparison of these IOSs is not completely correct, since systems with different geometries are compared. It is useful to compare different IOSs for the same microprobe length, distance between the first diaphragm and the first lens, and IOS length.

We shall consider three proton microprobes with $\overline{L} = 2.0$ and $l + \Delta l = 1.76$, having different IOSs of length L. For the first IOS, a doublet, we have $L_1 = L_2 = L_0 = 0.04$, $s_1 = 0.04$, $g = 0.12$, and $L = 0.12$. For the second, quadruplet I, we have $L = 0.12$, $g = 0.12$, $L_1 = L_2 = L_3 = L_4 = L_0 = 0.03$, and $s_1 = s_2 = s_3 = 0$. The third IOS, quadruplet II, has a slightly larger length L and, accordingly, smaller distance g: $L = 0.16$, $L_0 = 0.04$, $s_1 = s_2 = s_3 = 0$, and $g = 0.08$. The energy of the beam particles is $W = 3$ MeV. The lens aperture radius is $a = 3 \times 10^{-4}$. All lengths are given in meters.

Comparison shows (Table I) that it is possible to obtain identical magnifications in the two planes (xs) and (ys) in the quadruplet as opposed to different magnifications in the doublet because of the large number of lenses and the large field strengths. The product of the attenuations, which determines the beam emittance, is nearly identical for all three IOSs and depends only on the geometrical structure of the microprobe. Quadruplet I, and, even more so, quadruplet II have lower spherical aberration than the doublet, but in the former case this is a consequence of the larger fields, and in the latter it is due to the fact that the target is closer to the IOS. The chromatic-aberration coefficients in the quadruplets are somewhat smaller than in the doublet. We note that here the chromatic- and spherical-aberration coefficients have been calculated only for the magnetic quadrupoles.

Using the results given in Table I, let us calculate the initial parameters of a beam having a diameter of 1 μm at the target. We shall carry out the calculation for two values of

TABLE I. Parameters of quadrupole ion–optical systems with identical magnification, length, and aperture.

Parameter	Doublet	Quadruplet I	Quadruplet II
$\varkappa_1$	0.6056	0.544	0.617
$\varkappa_2$	0.7486	0.902	1.000
f_x	0.262	0.167	0.155
f_y	0.077	0.167	0.155
SF_x	0.562	0.103	0.072
SF_y	−0.021	0.103	0.072
R_{x22}	−4.6	−9.94	−10.2
R_{y22}	−23.1	−9.94	−10.2
$R_{x22}R_{y22}$	106	99	104
$C_{sx} = \langle x \mid x_0'^3 \rangle$	$-1.5\cdot 10^4$	$-1.4\cdot 10^4$	$-0.67\cdot 10^4$
$C_{sy} = \langle y \mid y_0'^3 \rangle$	$-2.4\cdot 10^4$	$-2.1\cdot 10^4$	$-1.40\cdot 10^4$
$\langle x \mid x_0' y_0'^2 \rangle$	$-7.0\ 10^4$	$-4.5\cdot 10^4$	$-2.3\cdot 10^4$
$\langle y \mid y_0' x_0'^2 \rangle$	$-4.2\cdot 10^4$	$-4.5\cdot 10^4$	$-2.3\cdot 10^4$
C_{px}	−4	−3.3	−3.2
C_{py}	−4	−3.9	−3.8
B_1, T	0.17	0.25	0.18
B_2, T	0.26	0.68	0.47
V_1, kV	11.0	8.9	6.4
V_2, kV	16.5	24.4	16.9

$\delta = \Delta p/p$: $\delta = 10^{-3}$ and $\delta = 10^{-4}$. The results are given in Table II. Comparing them, we see that all the comparable magnetic microprobes for a given momentum spread have similar four-dimensional emittances, i.e., similar currents. The quadruplet of rotation has somewhat larger emittance. In the case of the doublet, in order to obtain the maximum emittance it is necessary to use rectangular diaphragms, while in the case of the quadruplet circular diaphragms must be used. Here to obtain a beam with a diameter of several microns the width of diaphragm I in a microprobe using the doublet must be very small: 2.4 μm. In the quadruplet, $2r_1 = 5\,\mu$m. The initial angular spreads for a given δ are approximately the same for all three systems and in both planes, so that diaphragm II can be circular in all three systems.

As δ varies from 10^{-3} to 10^{-4}, the emittance $\varepsilon_x \varepsilon_y$ increases by a factor of 13–16. It is therefore very important to have the momentum spread closer to 10^{-4}.

More significant decreases can be obtained by using the IOS of minimum possible length for the maximum possible length of the microprobe. However, in systems of length 3–5 m problems arising from the mechanical stability and scattering on the residual gas in the vacuum chamber hinder the functioning of such a system as a micron microprobe. In a microprobe of length 4 m with an IOS of length 12 cm for the smallest possible fields, the decrease in each plane in the linear approximation is approximately 30. To obtain a decrease > 30 it is necessary to use a region of higher fields in the quadruplet of rotation or to use a two-step IOS. Obtaining a beam diameter of a fraction of a micron requires careful optimization of the IOS and minimization of the chromatic and spherical aberrations by using octupoles, other multipole elements, and achromatic lenses. Moreover, a submicron microprobe, or, at least, its final stage, should be oriented vertically, as is done in electron microscopes.

Comparison of the parameters of the quadrupole and axisymmetric microprobes

We shall compare the parameters of microprobes with an IOS in the form of a quadruplet of rotation and a solenoid. As before, for the quadruplet of rotation we use the model of a rectangular distribution of the axial field. For the solenoid we use the bell-shaped Glaser model.

We consider a quadrupole probe with the parameters $L_1 = L_2 = L_0$, $s_1 = s_2 = s_3 = 0$, $\varkappa_1 = 0.617$, $\varkappa_2 = 1.00$, $l + \Delta l = 17.5L_0$, $g = 0.66L_0$, $m = 0.24$, and $L = 0.2$ m. Accordingly, for the solenoid we obtain $k_0^2 = 0.23$ and $d = 1.2L_0$. For protons of energy 3 MeV we have $B\rho = 2.5$ T·m, $B_0 = 10$ T, $B_1/a = 0.625$ T/cm, $B_2/a = 0.238$ T/cm, and $d = 0.24$ m.

Comparison of the chromatic aberrations of the two types of IOS shows that they are approximately equal. The growth of the spot on the target due to the momentum spread of the particles in the beam is $\Delta r = 44L_0\delta\alpha_0$ for the solenoid and $\Delta r_x = \Delta r_y = 48L_0\delta\alpha_0$ for the quadruplet of rotation, where α_0 is the initial angle of spread of the beam.

The broadening of the spot on the target owing to geometrical aberrations for $r \ll r_2$ is mainly determined by the spherical aberration, and for the quadruplet of rotation is $\Delta r = 4C_sL_0\alpha_0^3$ and for the solenoid $\Delta r = 4C_0L_0\alpha_0^3$. For this example of a microprobe $C_s \simeq 7200$ and $C_0 = 4700$.

The calculations for bell-shaped axial field in the quadrupole system show that the difference between the values of C and C_0 arises not from the difference in the types of field, but from the difference in the form of the axial distribution of the field. For the bell-shaped distribution $C \cong 4300$.

Therefore, in this example of a microprobe all of the main optical characteristics, both linear and nonlinear, of the quadruplet of rotation and the solenoid are practically identical. The principal difference is in the values of the magnetic induction on the solenoid axis and at the poles of the quadrupole lenses.

Calculation of the allowances for adjustment

In practice, after installation and adjustment the calculated parameters of the system differ from the actual ones. It is therefore important to know the tolerances which must be adhered to in order to obtain the calculated beam diameter at the target. Assuming that the deviations of the parameters from the optimal values are small, we expand r in a Taylor series:

$$\Delta r = \frac{r - r^*}{r^*} \approx \sum_{1}^{m} \frac{\partial (r - r^*)}{\partial \alpha_k} \Delta\alpha_k = (r - r^*)\,\tilde{\nabla}(\alpha)\,\Delta\alpha,$$

where r^* is the beam radius for the calculated optimal model.

Here m is the number of parameters whose variation affects the characteristics of the beam and $\Delta\alpha_k$ is the deviation of the k-th parameter from the optimal value.

In Table III we give the values of $\partial(r - r^*)/\partial\alpha_k$ for a microprobe in a flat target, where α_x and α_y are the angles at which the lenses are rotated around the transverse axis in the x and y planes, d_x and d_y are the displacements of the lens axes relative to the system axis, and α_0 is the angle of rotation of the lens about its longitudinal axis. These values have been obtained for one of the calculated microprobe variants

TABLE II. Emittance and phase dimensions of the beam at the target for the systems of Table I with $\delta = \Delta p/p = 10^{-3}$ and $\delta = 10^{-4}$.

Exit parameter	Doublet		Quadruplet I		Quadruplet II	
	10^{-3}	10^{-4}	10^{-3}	10^{-4}	10^{-3}	10^{-4}
$10^6 r_{1x}$	1.2	1.5	2.6	3.8	2.5	3.3
$10^6 r_{1y}$	5.8	7.9	2.5	3.4	2.5	3.4
$10^5 \vartheta_{mx}$	6.1	18	7.4	19	7.7	23.4
$10^5 \vartheta_{my}$	5.9	16	6.3	16	6.3	18.6
$10^{10} \varepsilon_x$	0.7	2.8	1.9	6.6	1.95	7.8
$10^{10} \varepsilon_y$	3.5	12.5	1.6	5.6	1.6	6.3
$10^{20} \varepsilon_x \varepsilon_y$	2.5	34.4	3.0	37	3.1	49

TABLE III. Values of the tolerances for the quadruplet of rotation.

Type of deviation		Lens number 1	2	3	4
α_x		41 003	143 302	67 661	19 778
α_y		34 625	64 849	96 397	19 083
d_x		70 966	101 599	193 346	40 497
d_y		115 453	154 129	134 126	22 695
α_0	x	1130	2970	2034	275
	y	1043	2871	2239	117

with the following parameters: $r_1 = 24\ \mu$m, $r_2 = 0.22$ cm, $l = 3.35$ m, $l + \Delta l = 3.53$ m, $L_0 = 20$ cm, $g = 0.32$ m, $\varkappa_1 = 0.631$, $\varkappa_2 = 1.023$, $r = 4.8\ \mu$m, and $s_1 = s_2 = s_3 = 4$ cm. We note that for $\alpha_x \neq 0$ and $d_x \neq 0$ ($\alpha_y \neq 0$ and $d_y \neq 0$) the beam is deflected in the (xs) plane [or in the (ys) plane]. For $\alpha_0 \neq 0$ it changes its characteristics in both planes simultaneously. This is the reason why in Table III for α_0 we give two values of the derivatives for the (xs) and (ys) planes. We see from this table that the second and third lenses affect the beam characteristics most strongly, while the effect of the fourth lens is weakest.

Effect of the space charge for various emittances of a paraxial beam on the crossover dimensions in the proton quadrupole microprobe

The following paraxial equations are valid for the motion of the particles of an infinitely long beam with elliptical cross section in a quadrupole magnetic field:

$$\left.\begin{aligned} &x'' + \left[k - \frac{1}{p^3}\frac{I}{\pi r_x (r_x + r_y)}\right] x = 0;\\ &y'' - \left[k + \frac{1}{p^3}\frac{I}{\pi r_y (r_x + r_y)}\right] y = 0;\\ &k = \frac{\nabla_1 B_2}{p}, \quad I = \frac{I^*}{I_0^*}, \quad I_0^* = \frac{m_0^* c^*}{\mu_0^* q^*}, \end{aligned}\right\} \qquad (17)$$

where I^* is the current in amperes on the beam axis.

In order to determine the beam radius at the target, we split the interval $[0,\bar{L}]$ into sufficiently small segments $[s^j, s^{j+1}]$, on each of which the coefficients of Eq. (17) can be assumed constant.

We are considering a beam with elliptical cross section. We take the initial phase portrait to be an ellipse inscribed in the initial parallelogram. We use the envelope matrix $\sigma = \mathbf{R}\sigma_0\tilde{\mathbf{R}}$, where $\mathbf{R}$ is the matrizant and σ_0 is the matrix characterizing the shape of the initial phase ellipse. We note that $\sigma_{11} = r_x^2$ and $\sigma_{22} = x_m'^2$, where r_x and x_m' are the maximum values of x and x' in the phase set with elliptical boundary.

Let us consider a particular partial interval of length $h_j = s^{j+1} - s^j$ ($h_j \ll \bar{L}$). Assuming that the quantities $r_x = r_x^*$ and $r_y = r_y^*$ entering into (17) are constant on the interval h_j, we can write the expression for the σ matrix at the exit from this interval as

$$\sigma^*(j) = R(j + 1/j)\,\sigma(j)\,\tilde{R}(j + 1/j).$$

Here $R(j + 1/j)$ is the matrizant R on the interval h_j for $r = r(j)$ and $\sigma(j)$ is the envelope matrix at the point s^j.

Substituting $r^*(j) = \sigma_{11}^*(j)$ into Eq. (17), we find the matrizant $R^*(j + 1/j)$ at $r = r^*(j)$, after which we determine the matrix $\sigma^{**}(j)$.

$$\sigma^{**}(j) = R^*[(j+1)/j]\,\sigma^*(j)\,\tilde{R}^*[(j+1)/j]$$

and the matrix $\sigma(j+1)$:

$$\sigma(j+1) = \frac{1}{2}(\sigma^*(j) + \sigma^{**}(j)).$$

The program package DENS has been developed for solving this problem. The results of computer calculations for a microprobe with the parameters

$$L_0 = 0.2 \text{ m}, \; s_1 = s_3 = 0.024 \text{ m},$$
$$s_2 = 0.5 \text{ cm}, \; l + \Delta l = 3.5 \text{ m},$$
$$g = 0.3 \text{ m}, \; l = 3.35 \text{ m}, \; r_1 = 20\ \mu\text{m},$$
$$r_2 = 0.23 \text{ cm}, \; \varkappa_1 = 0.63$$

are given in Table IV, where we have listed the values of the current in the beam which lead to the indicated increases of the beam dimensions (in units of r_m, where r_m is the beam radius at the target in the linear approximation, neglecting the space-charge distribution) for various values of the radius r_2 of the second diaphragm.

8. USE OF A HIGH-FREQUENCY ELECTRIC FIELD IN THE ION-OPTICAL SYSTEM OF A MICROPROBE

It is well known that a time-varying field can be used for transverse focusing of particles.[27] In some cases a high-frequency focusing system can be constructed more simply and cheaply than, for example, a hard-focusing system.

Here we shall restrict ourselves to first-order focusing in a high-frequency microprobe with rectilinear trajectory of the axial particle.

For the case of a rectilinear reference trajectory we have

$$z = x + z_m, \quad k = l = 0,$$
$$z_{m1} = z_{m2} = 0, \quad z_{m3} = s, \quad x_3 = 0,$$
$$z_3 = z_{m3} = s, \; \nabla_1(z) = \nabla_1(x),$$
$$\nabla_2(z) = \nabla_2(x), \; \nabla(s) = \frac{\partial}{\partial s},$$
$$\nabla_3(z) = \nabla(s) - \frac{\gamma}{p}\nabla_4(x).$$

We consider an IOS in which the longitudinal electric field E_3 is a traveling wave:

TABLE IV. Values of the current (in μA) in the beam for the quadruplet of rotation, leading to different increases of the beam dimensions at the target.

r_z, 10^{-4} m	r_x/r_m				
	1.05	1.10	1.50	2.00	3.00
1	1	1	2.5	3.5	5.0
22	8	13	21	54	69
50	19	30	70	101	225
100	40	60	140	200	324
200	96	128	270	385	640

$$E_3 = E(s) \sin\left(\frac{\omega}{c} z_4 - \bar{k} z_3\right),$$

and the equation for the trajectory of the axial particle is given by

$$z_{m4} = \frac{c}{\omega} \bar{k} s.$$

From the latter expression we find

$$\frac{\gamma}{p} = \frac{c}{\omega} \bar{k};$$

$$E_3 = E(s) \sin \frac{\omega}{c} x_4.$$

We shall assume that

$$\frac{\omega}{c} x_4 < 1, \quad \sin \frac{\omega}{c} x_4 \cong \frac{\omega}{c} x_4.$$

Writing out Maxwell's equations

$$\nabla(s) B_2 - \frac{\gamma}{p} \nabla_4 B_2 - \nabla_2 B_3 + \nabla_4 E_1 = 0;$$

$$\nabla_1 B_3 - \nabla(s) B_1 + \frac{\gamma}{p} \nabla_4 B_1 + \nabla_4 E_2 = 0;$$

$$\nabla_2 B_1 - \nabla_1 B_2 + \nabla_4 E_3 = 0;$$

$$\nabla_1 E_1 + \nabla_2 E_2 + \nabla(s) E_3 - \frac{\gamma}{p} \nabla_4 E_3 = 0;$$

$$-\nabla(s) E_2 + \frac{\gamma}{p} \nabla_4 E_2 + \nabla_2 B_3 + \nabla_4 E_1 = 0;$$

$$-\nabla_1 E_3 + \nabla(s) E_1 - \frac{\gamma}{p} \nabla_4 E_1 + \nabla_4 B_2 = 0;$$

$$-\nabla_2 E_1 + \nabla_1 E_2 + \nabla_4 B_3 = 0;$$

$$\nabla_1 B_1 + \nabla_2 B_2 + \nabla(s) B_3 - \frac{\gamma}{p} \nabla_4 B_3 = 0$$

in the lowest-order approximation, we find that the electromagnetic field in the linear approximation has the form

$$E_1 = \frac{1}{2} \frac{\omega}{c} E(s) \frac{\gamma}{p} x_1; \quad B_1 = -\frac{1}{2} E(s) \frac{\omega}{c} x_2;$$

$$E_2 = \frac{1}{2} \frac{\omega}{c} E(s) \frac{\gamma}{p} x_2; \quad B_2 = \frac{1}{2} E(s) \frac{\omega}{c} x_1;$$

$$E_3 = E(s) \frac{\omega}{c} x_4; \quad B_3 = 0.$$

The Gaussian equations of motion for this field are written as

$$x_1'' = \frac{1}{2} \psi x_1;$$

$$x_2'' = \frac{1}{2} \psi x_2; \quad \psi = \frac{E(s)}{p^3} \frac{\omega}{c}; \quad x_4'' = -\psi x_4.$$

For $E(s) < 0$ we have $\psi < 0$. In this case a high-frequency lens behaves in the transverse directions like an axisymmetric collecting lens in which

$$k_0^2 = \frac{1}{2} |\psi| d^2.$$

In this lens, along with the transverse focusing the phase and energy spread of the particles increases.

In the preceding section we compared quadrupole and axisymmetric IOSs with the parameters

$$L_1 = L_2 = L_0, \quad s_1 = s_2 = s_3 = 0,$$

$$\varkappa_1 = 0.617, \quad \varkappa_2 = 1.00,$$

$$l + \Delta l = 17.5 L_0, \quad g = 0.66 L_0,$$

$$m = 0.24, \quad L_0 = 0.2 \text{ m}, \quad k_0^2 = 0.23,$$

$$d = 1.2 L_0.$$

The product $E_m \omega$ for a high-frequency IOS equivalent to these lenses will have the following value for bell-shaped $E(s)$:

$$E_m \omega = 2 p^3 c \frac{k_0^2}{d^2} = 8 p^3 c.$$

Here E_m is the maximum value of $E(s)$. For particles of energy 3 MeV we have $p = 0.3$ and $E_m \omega = 6.47 \times 10^7$ $\text{m}^{-1} \cdot \text{sec}^{-1}$.

High-frequency IOSs can be constructed both from structures with traveling waves (a waveguide) and from structures with standing waves (a cavity resonator). The simplest example is a gap between drift tubes with the longitudinal component of the high-frequency field along the axis of the gap. The transverse focusing in this field is realized during the phase (longitudinal) defocusing.

Further investigation of the possibility of using a high-frequency field in a proton microprobe requires consideration of aberrations.

9. USE OF THE SLIDING-TOLERANCE TECHNIQUE FOR NUMERICAL OPTIMIZATION OF AN IOS

The problem of the numerical optimization of an IOS can be formulated as a problem in nonlinear programming, which amounts to minimization of a function of many variables while satisfying constraints both in the form of equalities and in the form of inequalities. The technique used for minimizing the functional is the sliding-tolerance technique, the realization of which involves the use of the deformed-polyhedron technique[42]—one of the direct methods which does not involve the calculation of derivatives.

In the numerical optimization of probe systems, the initial values of the parameters are taken to be those obtained by analytic optimization. In a number of the cases studied the numerical optimization did not give better results. The use of the sliding-tolerance technique for the optimization of quadrupole systems has been studied in Ref. 43. A program package for the calculation of probe systems has been constructed on the basis of the method of embedding in phase-moment space for the first, third, and fifth phase moments in the bell-shaped and rectangular axial-field models for quadrupole lenses, using the generalized analog of the Gauss brackets for integrating the equations of motion.[26,31]

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[1]S. A. Cookson, A. T. G. Ferguson, and F. D. Pilling, J. Radioanal. Chem. **12**, 39 (1972).
[2]J. A. Cookson and F. D. Pilling, Thin Solid Films **14**, 381 (1973).
[3]R. Nobiling *et al.*, Nucl. Instrum. Methods **130**, 325 (1975).
[4]R. Nobiling and K. Traxel, Nucl. Instrum. Methods **142**, 49 (1977).
[5]P. Horowitz and L. Grodzins, Science **189**, 795 (1975).
[6]H. P. Wilde *et al.*, Nucl. Instrum. Methods **149**, 675 (1978).
[7]A. Ito and H. Koyama-Ito, Nucl. Instrum. Methods Phys. Res. **B3**, 584 (1984).
[8]J. C. Den Ouden, A. J. J. Bos, R. D. Vis, and H. Verheul, Nucl. Instrum. Methods **181**, 131 (1982).
[9]M. Prins, L. J. B. Hoffman, and F. A. M. Lieshout, Nucl. Instrum. Methods **197**, 165 (1982).
[10]C. J. F. Legge, Nucl. Instrum. Methods **197**, 243 (1982).
[11]M. S. Al-Chazi, J. S. C. McKee, and W. D. Ramsay, IEEE Trans. Nucl. Sci. **NS-26**, 2241 (1979).
[12]K. Éngel'man, *Microanalysis and the Scanning Electron Microscope* [in Russian] (Metallurgiya, Moscow, 1985), p. 376.
[13]C. J. Maggiore, Nucl. Instrum. Methods **191**, 199 (1981).
[14]R. Booth and H. W. Lefevre, Nucl. Instrum. Methods **151**, 143 (1978).
[15]A. D. Dymnikov and S. Ya. Yavor, Zh. Tekh. Fiz. **33**, 851 (1963) [Sov. Phys. Tech. Phys. **8**, 639 (1964)].
[16]A. D. Dymnikov, T. Ya. Fishkova, and S. Ya. Yavor, Zh. Tekh. Fiz. **35**, 431 (1965) [Sov. Phys. Tech. Phys. **10**, 340 (1965)].
[17]J. A. Cookson, Nucl. Instrum. Methods **165**, 477 (1979).
[18]J. A. Cookson, J. W. McMillan, and T. B. Pierce, J. Radioanal. Chem. **48**, 337 (1979).
[19]J. A. Cookson, Nucl. Instrum. Methods **181**, 115 (1981).
[20]G. J. F. Legge, Nucl. Instrum. Methods Phys. Res. **B3**, 561 (1984).
[21]R. Prakash and J. S. C. McKee, Nucl. Instrum. Methods Phys. Res. **B10/11**, 679 (1985).
[22]R. Nobiling, Nucl. Instrum. Methods Phys. Res. **B14**, 142 (1986).
[23]B. Nens, P. Trocellier, and Ch. Engelmann, Nucl. Instrum. Methods Phys. Res. **B14**, 148 (1986).
[24]A. G. Puzyrevich and B. I. Kuznetsov, *Low- and Intermediate-Energy Ion Accelerators* [in Russian] (Naukova Dumka, Kiev, 1982), p. 148.
[25]S. N. Andrianov, A. D. Dymnikov, and G. M. Osetinskiĭ, Deposited Report B-1-9-12851 [in Russian], JINR, Dubna (1979).
[26]S. N. Andrianov, A. D. Dymnikov, and G. M. Osetinskiĭ, Preprint R9-12873 [in Russian], JINR, Dubna (1979).
[27]A. D. Dymnikov and G. M. Osetinskiĭ, Preprint R13-82-780 [in Russian], JINR, Dubna (1982).
[28]A D. Dymnikov and G. M. Osetinskiĭ, Deposited Report B-1-9-84-209 [in Russian], JINR, Dubna (1984).
[29]A. D. Dymnikov and G. M. Osetinskiĭ, Preprint 9-84-844 [in Russian], JINR, Dubna (1984).
[30]S. N. Andrianov, A. D. Dymnikov, and G. M. Osetinskiĭ, Preprint 9-85-848 [in Russian], JINR, Dubna (1985).
[31]M. F. Galochkina, A. D. Dymnikov, and G. M. Osetinskiĭ, Preprint 9-85-847 [in Russian], JINR, Dubna (1985).
[32]H. Cartan *Differential Calculus. Differential Forms* [in French] (Hermann, Paris, 1970) [Russ. transl., Mir, Moscow, 1971].
[33]A. A. Kolomensky and A. N. Lebedev, *Theory of Cyclic Accelerators* (North-Holland, Amsterdam, 1966) [Russ. original, Fizmatgiz, Moscow, 1963).
[34]K. L. Brown, R. Belbeoch, and P. Bounin, Rev. Sci. Instrum. **35**, 481 (1964).
[35]K. L. Brown, Adv. Part. Phys. **1**, 71 (1967).
[36]W. Glaser, *Grundlagen der Elektronenoptik* [in German] (Springer-Verlag, Vienna, 1952) [Russ. transl., GITL, Moscow, 1957].
[37]M. Herzberger, *Modern Geometrical Optics* (Interscience, New York, 1958) [Russ. transl., IIL, Moscow, 1962].
[38]A. D. Dymnikov, Deposited Report B1-10427 [in Russian], JINR, Dubna (1977).
[39]A. D. Dymnikov, Zh. Tekh. Fiz. **38**, 1120 (1968) [Sov. Phys. Tech. Phys. **13**, 929 (1969)].
[40]A. D. Dymnikov, in: *Proceedings of the Second All-Union Conference on Accelerators* [in Russian] (Nauka, Moscow, 1972), Vol. 2, p. 35.
[41]A. D. Dymnikov, *Problems in Mechanics and Control Processes, No. 2, Control of Dynamical Systems* [in Russian] (Leningrad State University, Leningrad, 1978), p. 64.
[42]D. M. Himmelblau, *Applied Nonlinear Programming* (McGraw-Hill, New York, 1972) [Russ. transl., Mir, Moscow, 1975].
[43]S. N. Andrianov, A. D. Dymnikov, E. M. Kulakova, *et al.*, *Programming and Mathematical Methods for Solving Physical Problems* [in Russian] (Dubna, 1978), p. 295.

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